Journal of Pure and Applied Algebra 227 (2023) 107348

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

On lax protomodularity for Ord-enriched categories [☆]

Maria Manuel Clementino [a,*], Andrea Montoli [b], Diana Rodelo [c,a]

[a] University of Coimbra, CMUC, Department of Mathematics, 3000-143 Coimbra, Portugal
[b] Dipartimento di Matematica "Federigo Enriques", Università degli Studi di Milano, Via Saldini 50, 20133 Milano, Italy
[c] Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal

A R T I C L E   I N F O

Article history:
Received 26 July 2022
Received in revised form 5 January 2023
Available online 8 February 2023
Communicated by J. Adámek

MSC:
18E13; 06F20; 18D20; 18N10

Keywords:
Preordered abelian groups
Protomodular categories
Categories enriched in the category of preorders

A B S T R A C T

Our main focus concerns a possible lax version of the algebraic property of protomodularity for Ord-enriched categories. Having in mind the role of comma objects in the enriched context, we consider some of the characteristic properties of protomodularity with respect to comma objects instead of pullbacks. We show that the equivalence between protomodularity and certain properties on pullbacks also holds when replacing conveniently pullbacks by comma objects in any finitely complete category enriched in Ord, and propose to call lax protomodular such Ord-enriched categories. We conclude by studying this sort of lax protomodularity for the category OrdAb of preordered abelian groups, equipped with a suitable Ord-enrichment, and show that OrdAb fulfills the equivalent lax protomodular properties with respect to the weaker notion of *precomma object*; we call such categories lax preprotomodular.

0. Introduction

The notion of a protomodular category [6] has proved to be central in the developments of categorical algebra of the last decades. A finitely complete category X is protomodular when the pullback functors $\alpha^*\colon \mathsf{Pt}_Y(\mathsf{X}) \to \mathsf{Pt}_A(\mathsf{X})$ are conservative for any morphism $\alpha\colon A \to Y$. When, in addition, X is pointed, X is protomodular if and only if the Split Short Five Lemma holds. We recall in Theorem 2.1 well-known equivalent conditions expressed by properties on pullbacks which characterise protomodularity.

In the first part of this paper we introduce and study a lax version of protomodularity in the context of Ord-enriched categories. For a finitely complete Ord-enriched category $\mathbb{C}$ with comma objects and 2-pullbacks, we replace pullbacks conveniently with comma objects or 2-pullbacks and the pullback change-of-

☆ The first and third authors acknowledge partial financial assistance by the Centre for Mathematics of the University of Coimbra – UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.

* Corresponding author.
E-mail addresses: mmc@mat.uc.pt (M.M. Clementino), andrea.montoli@unimi.it (A. Montoli), drodelo@ualg.pt (D. Rodelo).

https://doi.org/10.1016/j.jpaa.2023.107348

base functors with comma object change-of-base functors in order to analyse lax versions of the properties of Theorem 2.1. Since comma objects are defined on ordered pairs of morphisms, we have two possible change-of-base functors, given a morphism $\alpha\colon A \to Y$: the *vertical* comma object functor $V_\alpha\colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_A(\mathbb{C})$ and the *horizontal* comma object functor $H_\alpha\colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_A(\mathbb{C})$. We show that the lax version of the equivalences in Theorem 2.1 hold for $\mathbb{C}$; see Theorem 2.5, Corollary 2.6, Theorem 2.11 and Corollary 2.12.

Due to the above characterisations, we propose to call a finitely complete Ord-enriched category $\mathbb{C}$ which admits comma objects and 2-pullbacks *lax protomodular* when the comma object functor V_α is conservative for any morphism α in $\mathbb{C}$. We shall call $\mathbb{C}$ *colax protomodular* when $\mathbb{C}^{\mathrm{co}}$ is lax protomodular, that is, when H_α is conservative for any morphism α in $\mathbb{C}$. The interplay between the different ingredients encoded on protomodularity is worked out in Section 2.

While the categories of internal groups and of internal abelian groups in Ord are protomodular, and hence lax and colax protomodular for any compatible Ord-enrichment by Corollary 2.7, the *classical* categories OrdGrp of preordered groups (studied e.g. in the recent work [8]) and OrdAb of preordered abelian groups and monotone homomorphisms are not protomodular. (In these categories the inversion morphism of the group structure is not necessarily monotone; it is, in fact, anti-monotone.) One can hence pose the following question:

Is there a non-degenerate Ord-enrichment of these categories so that protomodularity is recovered in a lax sense?,

which remains unanswered.

Nevertheless in the second part of the paper, in Section 3, we present an Ord-enrichment of OrdAb, denoted by $\mathbb{O}$rdAb, which yet does not admit comma objects but admits *precomma objects*, and therefore allows the study of change-of-base functors with respect to precomma objects. Since the results for lax protomodularity can be carried out naturally to this weaker setting, we call a finitely complete Ord-enriched category $\mathbb{C}$ with precomma objects *lax preprotomodular* (resp. *colax preprotomodular*) when the precomma object functor V_α (resp. H_α) is conservative for any morphism α in $\mathbb{C}$. We show in Section 3 that $\mathbb{O}$rdAb is lax preprotomodular but not colax preprotomodular.

In addition, we analyse the behaviour of two different factorisation systems in $\mathbb{O}$rdAb, which correspond to (bijective on objects, full and faithful) and (surjective on objects, monic, full and faithful) factorisation, showing that they are not pullback-stable, and so $\mathbb{O}$rdAb fails to be Ord-regular.

1. The Ord-enriched change-of-base functors

In the following $\mathbb{C}$ denotes a finitely complete category enriched in the category Ord of preordered sets and monotone maps. (Recall that a *preorder* is a reflexive and transitive relation.) This means that for any objects X and Y of $\mathbb{C}$, $\mathbb{C}(X,Y)$ is equipped with a preorder such that (pre)composition preserves it. We will denote this preorder of morphisms by $\preccurlyeq$. If we consider in $\mathbb{C}$ the reverse preorder we obtain again an Ord-enriched category which we denote, as usual, by $\mathbb{C}^{\mathrm{co}}$.

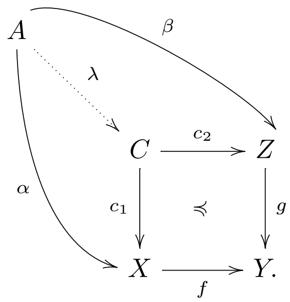
A morphism $f\colon X \to Y$ is said to be *full* when: given morphisms $a, a'\colon A \to X$ such that $fa \preccurlyeq fa'$, then $a \preccurlyeq a'$; equivalently, $fa \preccurlyeq fa'$ if and only if $a \preccurlyeq a'$. Note that, in the Ord-enriched context all morphisms are faithful.

Given an ordered pair of morphisms $(f\colon X \to Y, g\colon Z \to Y)$ in $\mathbb{C}$ with common codomain, the *(strict) comma object* of (f, g) is defined by an object C and morphisms $c_1\colon C \to X$, $c_2\colon C \to Z$ such that

(C1) $fc_1 \preccurlyeq gc_2$;
(C2) it has the universal property: given morphisms $\alpha\colon A \to X$ and $\beta\colon A \to Z$ with $f\alpha \preccurlyeq g\beta$, there exists a unique morphism $\lambda\colon A \to C$ such that $c_1\lambda = \alpha$ and $c_2\lambda = \beta$, as in diagram (1.i) below;

(C3) for morphisms $\alpha, \alpha' \colon A \to X$, $\beta, \beta' \colon A \to Z$ such that $f\alpha \preccurlyeq g\beta$, $f\alpha' \preccurlyeq g\beta'$, $\alpha \preccurlyeq \alpha'$ and $\beta \preccurlyeq \beta'$, the corresponding unique morphisms $\lambda, \lambda' \colon A \to C$ verify $\lambda \preccurlyeq \lambda'$;

(1.i)



The comma object of (f, g) will be denoted by f/g, its "projections" by $\pi_1 \colon f/g \to X$ and $\pi_2 \colon f/g \to Z$, and the induced morphism as above by $\lambda = \langle \alpha, \beta \rangle$. Note that, if $\mathbb{C}$ admits 2-products then condition (C3) is equivalent to the fact that the morphism $\langle c_1, c_2 \rangle \colon C \to X \times Z$ is full and faithful. Although the faithfulness of morphisms comes for free in this context, this is no longer the case in the general enriched context. For this reason, we chose to refer to both conditions throughout the text.

We call a construction as above the *precomma object* of (f, g) when only conditions (C1) and (C2) are required to hold.

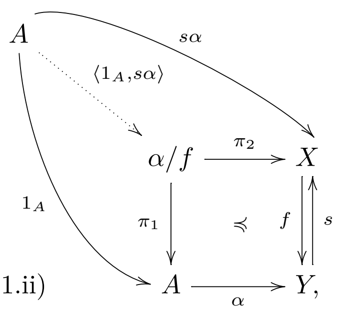
Given a comma object diagram as in (1.i), if $fc_1 = gc_2$, then it is easy to check that f/g is the 2-pullback of (f, g); similarly the precomma object of (f, g) coincides with the pullback of (f, g).

From now on, $\mathbb{C}$ will denote a finitely complete category enriched in Ord which admits comma objects.

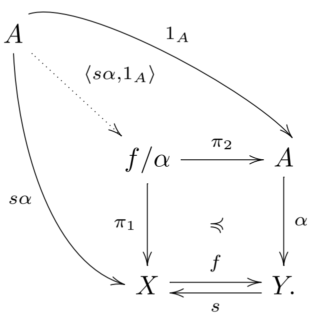
Given an object Y, as usual we denote by $\mathbb{C}/Y$ the slice category of $\mathbb{C}$ over Y. Our main change-of-base functors will be defined on points over Y. Here by *point over* Y we mean a morphism from the terminal object $1_Y \colon Y \to Y$ into an arbitrary object $f \colon X \to Y$ of $\mathbb{C}/Y$; that is, a $\mathbb{C}$-morphism $s \colon Y \to X$ so that $fs = 1_Y$. Hence a point in $\mathbb{C}$ is given by a split epimorphism $f \colon X \to Y$ with a chosen splitting $s \colon Y \to X$. We denote by $\mathsf{Pt}_Y(\mathbb{C})$ the category of points over Y in $\mathbb{C}$.

Given a morphism $\alpha \colon A \to Y$ in $\mathbb{C}$, we can define two possible functors by taking comma objects along points. For a point $X \underset{s}{\overset{f}{\rightleftarrows}} Y$, we form the comma objects

(1.ii)

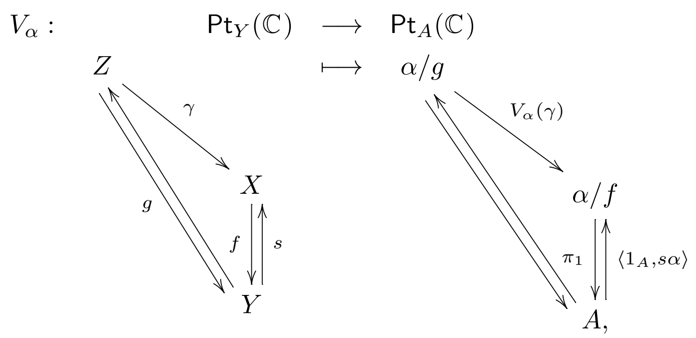


and

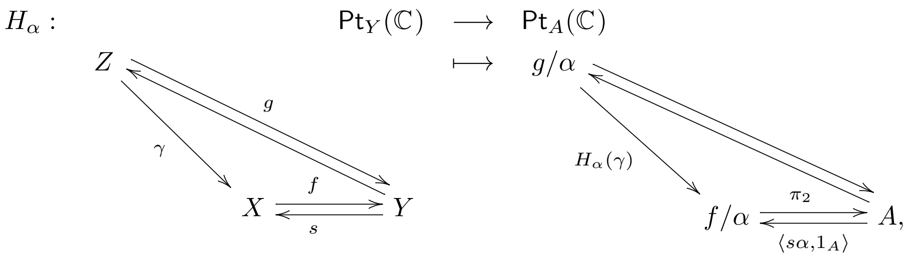


(1.iii)

These constructions define respectively the *vertical* and *horizontal comma objects change-of-base functors*:



where $V_\alpha(\gamma)$ is induced by the universal property of α/f, and



where $H_\alpha(\gamma)$ is induced by the universal property of f/α.

As in any Ord-enriched category, the notion of adjoint pair of morphisms allows us to consider the following particular points, that play special roles in comma objects and consequently in the change-of-base functors, as explained next. A point (f, s) is called *rali* (short for right adjoint, left inverse) when $sf \preccurlyeq 1_X$; it is called *lali* (short for left adjoint, left inverse) when $1_X \preccurlyeq sf$. We denote by $\mathsf{Ptr}_Y(\mathbb{C})$ (resp. $\mathsf{Ptl}_Y(\mathbb{C})$) the category of rali (resp. lali) points over Y.

Remark 1.1. It is easy to check that, in diagram (1.ii), with $X \underset{s}{\overset{f}{\rightleftarrows}} Y$ a rali, also the point $\alpha/f \underset{\langle 1_A, s\alpha\rangle}{\overset{\pi_1}{\rightleftarrows}} A$ is a rali since $\langle \pi_1, \pi_2\rangle\colon \alpha/f \to A \times X$ is full and faithful, $\pi_1\langle 1_A, s\alpha\rangle\pi_1 = \pi_1$ and $\pi_2\langle 1_A, s\alpha\rangle\pi_1 = s\alpha\pi_1 \preccurlyeq sf\pi_2 \preccurlyeq \pi_2$. Analogously, if $X \underset{s}{\overset{f}{\rightleftarrows}} Y$ is a lali, in diagram (1.iii) the point $f/\alpha \underset{\langle s\alpha, 1_A\rangle}{\overset{\pi_2}{\rightleftarrows}} A$ is lali. In particular, for any morphism $\alpha\colon A \to Y$, the point $\alpha/1_Y \underset{\langle 1_A, \alpha\rangle}{\overset{\pi_1}{\rightleftarrows}} A$ is always rali, while the point $1_Y/\alpha \underset{\langle \alpha, 1_A\rangle}{\overset{\pi_2}{\rightleftarrows}} A$ is always lali.

Therefore, *for any morphism $\alpha\colon A \to Y$, V_α (co)restricts to a functor on rali points; similarly, H_α (co)restricts to a functor on lali points.*

2. Lax protomodularity

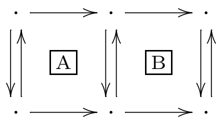
Several algebraic properties in a category X with pullbacks can be expressed by properties of the pullback functors $\alpha^*\colon \mathsf{Pt}_Y(\mathsf{X}) \to \mathsf{Pt}_A(\mathsf{X})$, for morphisms $\alpha\colon A \to Y$. One of the key results in this direction is the following (a proof can be found e.g. in [3]):

Theorem 2.1. *For a category X with pullbacks, the following conditions are equivalent:*

(i) *the pullback functors $\alpha^*\colon \mathsf{Pt}_Y(\mathsf{X}) \to \mathsf{Pt}_A(\mathsf{X})$ are conservative for every $\alpha\colon A \to Y$;*

(ii) *for any commutative diagram of points*

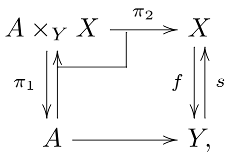
(2.i)



(also commuting with the upward sections) where $\boxed{\text{A}}$ *and* $\boxed{\text{A}|\text{B}}$ *are pullbacks, then* $\boxed{\text{B}}$ *is also a pullback;*

(iii) *for any pullback of a point* (f, s) *along an arbitrary morphism*

(2.ii)



the pair (π_2, s) *is jointly extremally epimorphic.*

A finitely complete category X is called *protomodular*[1] [6] if the equivalent conditions of the previous theorem hold in X.

Example 2.2. Some examples of protomodular categories are (see [3] for more examples):

- The variety Grp of groups and, more generally, of Ω-groups (the corresponding theory has, among its operations, a unique constant and the group operations). In fact, in [7] the varieties which form a protomodular category were characterized as those for which there exist $n \in \mathbb{N}$ and
  - constants $e_1, \dots, e_n$;
  - binary operations $\alpha_1, \dots, \alpha_n$ such that $\alpha_i(x, x) = e_i$ for all $i = 1, \dots, n$;
  - an $(n + 1)$-ary operation θ such that $\theta(\alpha_1(x, y), \dots, \alpha_n(x, y), y) = x$.
- Any additive category with finite limits.
- $\mathsf{Set}^{\mathrm{op}}$ and, more generally, the dual of any elementary topos.

If, moreover, the category X is pointed, then it is immediate to see that the conservativeness of all pullback functors $\alpha^* \colon \mathsf{Pt}_Y(\mathsf{X}) \to \mathsf{Pt}_A(\mathsf{X})$ is implied by (and hence equivalent to) the conservativeness of the pullback functors induced by the morphisms whose domain is the zero object. This last property is equivalent to the classical Split Short Five Lemma (see, for example, Proposition 3.1.2 in [3]). Hence, for pointed categories, *protomodularity is equivalent to the validity of the Split Short Five Lemma.*

Our goal now is to replace the protomodularity condition with a lax version of it, where we look at the conservativeness of the comma object functors. We can consider a "vertical" version of Theorem 2.1 where the α^* are replaced by V_α and the points are vertical arrows as in diagrams (2.i) and (2.ii). Or we can consider the "horizontal" case with H_α and where the points appear horizontally in those diagrams. Despite this distinction, the equivalences obtained in $\mathbb{C}$ in one direction give immediately the corresponding results in the other direction when applied to $\mathbb{C}^{\mathrm{co}}$.

We chose vertical as our "priority case" with detailed proofs and simply state the equivalences for the horizontal case. Note that an Ord-enriched category may fulfill the equivalent vertical properties and fail to fulfill the horizontal ones, or vice-versa. We propose the names *lax protomodular category* for an Ord-enriched finitely complete category $\mathbb{C}$ with comma objects and 2-pullbacks such that the above mentioned

[1] The original definition only asks for X to admit the existence of pullbacks of points along arbitrary morphisms. When X is such, the equivalences of Theorem 2.1 still hold.

equivalences hold in the vertical direction and *colax protomodular category* with respect to the horizontal direction (see Definition 2.13).

Replacing pullbacks with comma objects to get similar equivalences as those in Theorem 2.1 is not straightforward. Properties on pullbacks do not give similar properties on comma objects; e.g. gluing comma objects together does not give a comma object in general. However, there are some well-known properties combining comma objects and 2-pullbacks, whose proof can be found, for instance, in [13]:

Lemma 2.3. *Let $\mathbb{C}$ be a finitely complete Ord-enriched category which admits comma objects. Consider the diagram where the right square is a comma object and the left square is commutative*

$$\begin{array}{ccccc} P & \xrightarrow{p_2} & f/g & \xrightarrow{\pi_2} & Z \\ {\scriptstyle p_1}\big\downarrow & & {\scriptstyle \pi_1}\big\downarrow & \preccurlyeq & \big\downarrow{\scriptstyle g} \\ X' & \xrightarrow[x]{} & X & \xrightarrow[f]{} & Y. \end{array}$$

The outer rectangle is a comma object if and only if the left square is a 2-pullback.

A similar result holds by stacking squares vertically:

Lemma 2.4. *Let $\mathbb{C}$ be a finitely complete Ord-enriched category which admits comma objects. Consider the diagram where the bottom square is a comma object and the top square is commutative*

$$\begin{array}{ccc} P & \xrightarrow{p_2} & Z' \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle z} \\ f/g & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\big\downarrow & \preccurlyeq & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

The outer rectangle is a comma object if and only if the top square is a 2-pullback.

We now prove that when we replace the pullback functor with a comma object functor and the pullbacks with comma objects we obtain enriched versions of the equivalence Theorem 2.1 (i) ⇔ (ii).

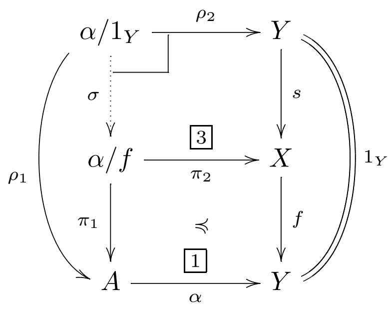
Theorem 2.5. *Let $\mathbb{C}$ be a finitely complete Ord-enriched category which admits comma objects and 2-pullbacks. The following statements are equivalent:*

(i) *$V_\alpha \colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_A(\mathbb{C})$ is conservative for any morphism $\alpha \colon A \to Y$.*
(ii) *In any diagram where the left square $\boxed{1}$ and total rectangle $\boxed{1}\boxed{2}$ are comma diagrams and the right square $\boxed{2}$ is commutative*

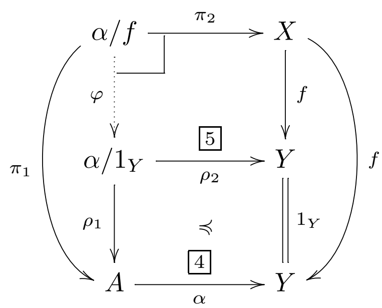
$$\begin{array}{ccccc} \alpha/f & \xrightarrow{\pi_2} & X & \xrightarrow{\chi} & U \\ {\scriptstyle \pi_1}\big\downarrow\big\uparrow{\scriptstyle i_1} & \preccurlyeq \;\; \boxed{1} & {\scriptstyle f}\big\downarrow\big\uparrow{\scriptstyle s} & \boxed{2} & {\scriptstyle g}\big\downarrow\big\uparrow{\scriptstyle t} \\ A & \xrightarrow[\alpha]{} & Y & \xrightarrow[\beta]{} & V, \end{array} \qquad (2.\mathrm{iii})$$

that is, $\alpha\pi_1 \preccurlyeq f\pi_2$, $i_1 = \langle 1_A, s\alpha\rangle$, $fs = 1_Y$, $gt = 1_V$, $\beta f = g\chi$, $\chi s = t\beta$, *the right square* $\boxed{2}$ *is a 2-pullback.*

Proof. (i) $\Rightarrow$ (ii). In the following diagram

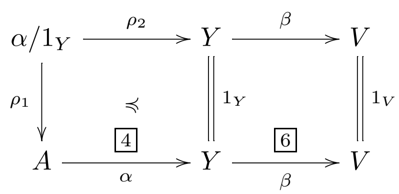


we get an induced morphism $\sigma\colon \alpha/1_Y \to \alpha/f$ such that the bottom square and total rectangle are comma objects; thus the top square $\boxed{3}$ is a 2-pullback (Lemma 2.4). Similarly, we get an induced morphism $\varphi\colon \alpha/f \to \alpha/1_Y$



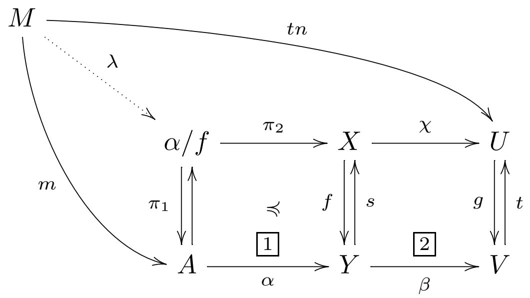
such that $\boxed{5}$ is a 2-pullback. It is easy to check that $\varphi\sigma = 1_{\alpha/1_Y}$.

Next we prove that the following diagram



is a comma object diagram. Indeed:

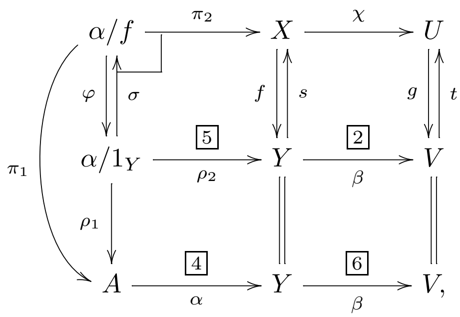
(C1) $\beta\alpha\rho_1 \preccurlyeq \beta\rho_2$;
(C2) if $m\colon M \to A$ and $n\colon M \to V$ are such that $\beta\alpha m \preccurlyeq n$, then $\beta\alpha m \preccurlyeq gtn$. We get an induced morphism $\lambda\colon M \to \alpha/f$



such that $\pi_1\lambda = m$ and $\chi\pi_2\lambda = tn$. We consider the morphism $\varphi\lambda\colon M \to \alpha/1_Y$ which is such that $\rho_1\varphi\lambda = \pi_1\lambda = m$ and $\beta\rho_2\varphi\lambda = \beta f\pi_2\lambda = g\chi\pi_2\lambda = gtn = n$. To prove the uniqueness of such a morphism, suppose that $\xi\colon M \to \alpha/1_Y$ is such that $\rho_1\xi = m$ and $\beta\rho_2\xi = n$. Then $t\beta\rho_2\xi = tn$ or, equivalently, $\chi s\rho_2\xi = tn$, from which we get $\chi\pi_2\sigma\xi = tn$. Since also, $\pi_1\sigma\xi = \pi_1\xi = m$, we conclude that $\lambda = \sigma\xi$, from the universal property of the comma object of $(\beta\alpha, g)$. It follows that $\varphi\lambda = \varphi\sigma\xi = \xi$;

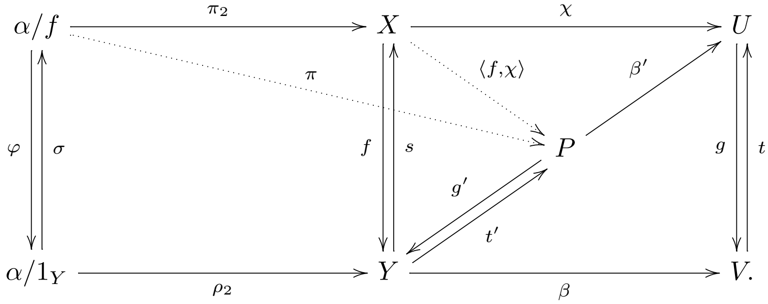
(C3) consider morphisms $c, c'\colon B \to A, d, d'\colon B \to V$ such that $\beta\alpha c \preccurlyeq d$, $\beta\alpha c' \preccurlyeq d'$, $c \preccurlyeq c'$ and $d \preccurlyeq d'$. Let $b = \langle c, d\rangle, b' = \langle c', d'\rangle$ be the induced morphisms from B to $\alpha/1_Y$. With respect to the comma object diagram $\boxed{1}\boxed{2}$, we have morphisms c, c', td, td' such that $\beta\alpha c \preccurlyeq gtd$, $\beta\alpha c' \preccurlyeq gtd'$, $c \preccurlyeq c'$ and $td \preccurlyeq td'$. Thus the induced morphisms $\sigma b, \sigma b'\colon B \to \alpha/f$ are such that $\sigma b \preccurlyeq \sigma b'$. Then $b = \varphi\sigma b \preccurlyeq \varphi\sigma b' = b'$.

In the following diagram

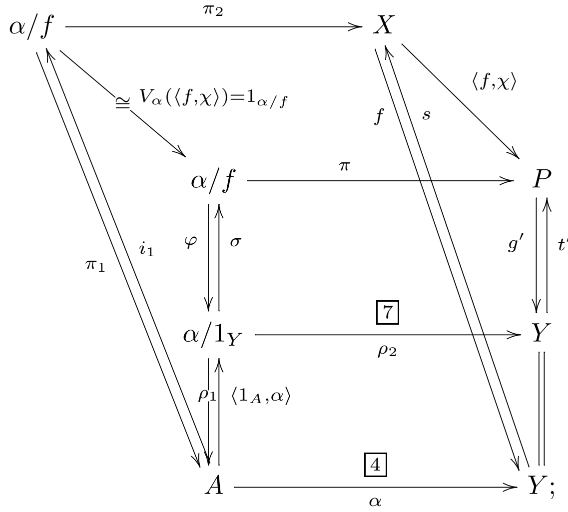


the bottom rectangle and the total diagram are comma objects, so that the top rectangle is a 2-pullback (Lemma 2.4).

Next, we take the 2-pullback of β and g to get the following diagram

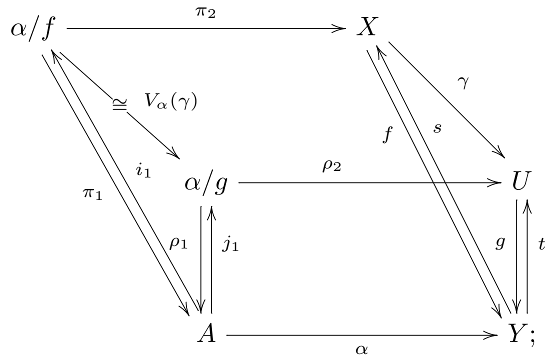


Since the whole rectangle is a 2-pullback, the bottom left quadrangle is a 2-pullback; let us call it $\boxed{7}$. The diagram composed by $\boxed{4}$ on the bottom and $\boxed{7}$ on top is a comma object diagram (Lemma 2.4). This is the front face in

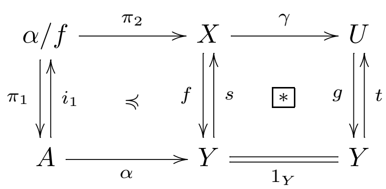


the left point being $(\rho_1\varphi, \sigma\langle 1_A, \alpha\rangle) = (\pi_1, i_1)$. Note that, $V_\alpha(\langle f, \chi\rangle) = 1_{\alpha/f}$, so that $\langle f, \chi\rangle$ is an isomorphism by assumption. This proves that $\boxed{2}$ is indeed a 2-pullback.

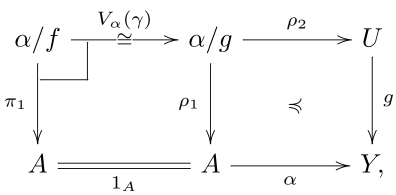
(ii) $\Rightarrow$ (i). Consider an arbitrary morphism $\alpha\colon A \to Y$ and a morphism $\gamma\colon (f, s) \to (g, t)$ in $\mathsf{Pt}_Y(\mathbb{C})$. Suppose that $V_\alpha(\gamma)$ is an isomorphism



here $i_1 = \langle 1_A, s\alpha\rangle$ and $j_1 = \langle 1_A, t\alpha\rangle$. The diagram



is of the type (2.iii). The whole rectangle is a comma object diagram because it is the same as



where the left commutative square is obviously a 2-pullback (see Lemma 2.3). By assumption $\boxed{*}$ is a 2-pullback; thus γ is an isomorphism. $\square$

The horizontal version of the previous result is stated next.

Corollary 2.6. *Let $\mathbb{C}$ be a finitely complete* Ord*-enriched category which admits comma objects and 2-pullbacks. The following statements are equivalent:*

(i) $H_\alpha \colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_A(\mathbb{C})$ *is conservative for any morphism $\alpha \colon A \to Y$.*
(ii) *In any diagram where the top square $\boxed{1}$ and total rectangle are comma diagrams and the bottom square $\boxed{2}$ is commutative*

$$\begin{array}{ccc} f/\alpha & \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} & A \\ {\scriptstyle \pi_1}\downarrow & \preccurlyeq \quad \boxed{1} & \downarrow{\scriptstyle \alpha} \\ X & \underset{s}{\overset{f}{\rightleftarrows}} & Y \\ {\scriptstyle \chi}\downarrow & \boxed{2} & \downarrow{\scriptstyle \beta} \\ U & \underset{t}{\overset{g}{\rightleftarrows}} & V, \end{array}$$

that is, $f\pi_1 \preccurlyeq \alpha\pi_2$, $i_2 = \langle s\alpha, 1_A \rangle$, $fs = 1_Y$, $gt = 1_V$, $\beta f = g\chi$, $\chi s = t\beta$, the bottom square $\boxed{2}$ is a 2-pullback.

Corollary 2.7. *Let* X *be a protomodular category. Then any* Ord*-enrichment $\mathbb{X}$ of* X *that admits comma objects and* 2*-pullbacks fulfills the equivalent conditions of Theorem 2.5 and of Corollary 2.6.*

Proof. Note that pullbacks coincide with 2-pullbacks in this setting. Consider a diagram as in Theorem 2.5.(ii). Following its proof above, we deduce that $\boxed{5}\boxed{2}$ is a (2-)pullback. Since $\boxed{5}$ is also a (2-)pullback, we conclude that $\boxed{2}$ is a (2-)pullback from protomodularity (Theorem 2.1). $\square$

Remark 2.8. The statements of Theorem 2.5 and Corollary 2.6 still hold when we replace 2-pullbacks with (1-)pullbacks. Indeed, in the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 2.5, we can take the pullback of β and g and still conclude that $\boxed{7}$ is a 2-pullback. The rest of that proof is the same. For (ii) $\Rightarrow$ (i) in Theorem 2.5, we get that $\boxed{*}$ is a pullback and can still conclude that γ is an isomorphism.

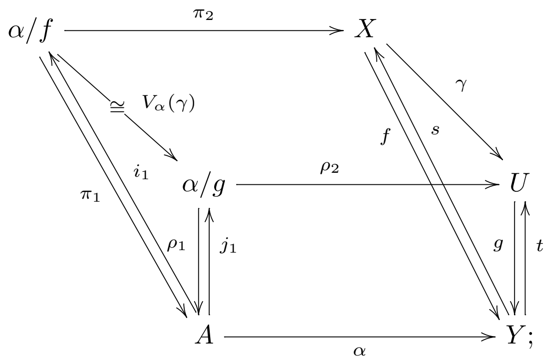
In the 1-dimensional context, in Theorem 2.1, the proof of the equivalence between condition (iii) and conditions (i) and (ii) uses the following well-known fact: the conservativeness property of a left exact functor F is equivalent to its conservativeness on monomorphisms. A functor F is said to be conservative on monomorphisms if, for every monomorphism f, if $F(f)$ is an isomorphism, then so is f. Any pullback functor $\alpha^* \colon \mathsf{Pt}_Y(\mathsf{X}) \to \mathsf{Pt}_A(\mathsf{X})$ preserves finite limits, as soon as the base category X admits pullbacks along split epimorphisms. In the enriched context, such a property fails to hold for the comma objects functors. In fact, given a morphism $\alpha \colon A \to Y$ and the terminal object $(1_Y, 1_Y)$ of $\mathsf{Pt}_Y(\mathbb{C})$, the comma object $\alpha/1_Y$ does not necessarily give rise to the terminal object $(1_A, 1_A)$ of $\mathsf{Pt}_A(\mathbb{C})$ (nor does $1_Y/\alpha$). Despite this setback, we still obtain a similar result with respect to the conservativeness of V_α and H_α.

Proposition 2.9. *Let $\mathbb{C}$ be a finitely complete* Ord*-enriched category which admits comma objects. The following statements are equivalent:*

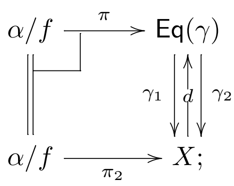
(i) $V_\alpha \colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_A(\mathbb{C})$ *is conservative, for any morphism* $\alpha \colon A \to Y$.
(ii) $V_\alpha \colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_A(\mathbb{C})$ *is conservative on monomorphisms, for any morphism* $\alpha \colon A \to Y$.

Proof. (i) $\Rightarrow$ (ii). Obvious

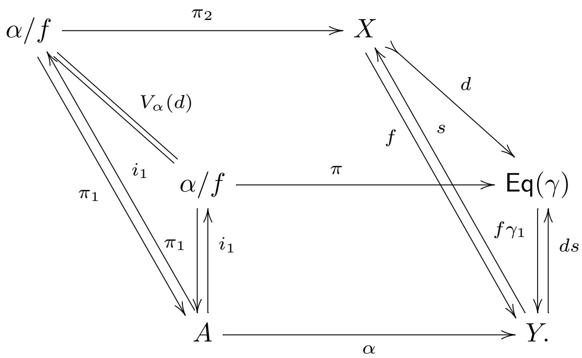
(ii) $\Rightarrow$ (i). Consider an arbitrary morphism $\alpha \colon A \to Y$ and a morphism $\gamma \colon (f, s) \to (g, t)$ in $\mathsf{Pt}_Y(\mathbb{C})$. Suppose that $V_\alpha(\gamma)$ is an isomorphism



here $i_1 = \langle 1_A, s\alpha \rangle$ and $j_1 = \langle 1_A, t\alpha \rangle$. From Lemma 2.4, the top quadrangle is a 2-pullback. Taking kernel pairs vertically we obtain a diagram where both downward squares are 2-pullbacks



here $d = \langle 1_X, 1_X \rangle$. Combining the 2-pullback with first projections and the comma object of (α, f), we get a comma object (Lemma 2.4) which is the front face in



By assumption, the monomorphism d is an isomorphism. This implies that $\gamma_1 = \gamma_2$ and, consequently, γ is a monomorphism. Applying our assumption again, we conclude that γ is an isomorphism. $\square$

Corollary 2.10. *Let* $\mathbb{C}$ *be a finitely complete* Ord*-enriched category which admits comma objects. The following statements are equivalent:*

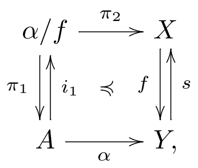
(i) $H_\alpha \colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_A(\mathbb{C})$ *is conservative, for any morphism* $\alpha \colon A \to Y$.

(ii) *$H_\alpha\colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_A(\mathbb{C})$ is conservative on monomorphisms, for any morphism $\alpha\colon A \to Y$.*

Thanks to these facts, we get the following:

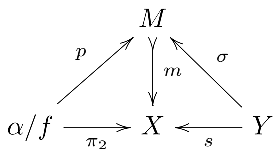
Theorem 2.11. *Let $\mathbb{C}$ be a finitely complete* Ord*-enriched category which admits comma objects. The following statements are equivalent:*

(i) *$V_\alpha\colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_A(\mathbb{C})$ is conservative for any morphism $\alpha\colon A \to Y$.*
(ii) *for any comma object of the form*

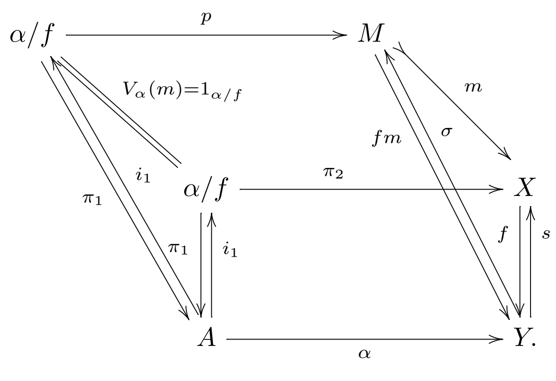


the pair (π_2, s) is jointly extremally epimorphic; here $i_1 = \langle 1_A, s\alpha\rangle$.

Proof. (i) ⇒ (ii). Let m be a monomorphism for which the diagram



commutes. We get the following diagram



Note that the top quadrangle is a 2-pullback, so that the back face is a comma object diagram (Lemma 2.4). Since m is a monomorphism and $mpi_1 = \pi_2 i_1 = s\alpha = m\sigma\alpha$, it follows that $pi_1 = \sigma\alpha$. From the assumption we conclude that m is an isomorphism.

(ii) ⇒ (i) Conversely, for any morphism $\alpha\colon A \to Y$, a diagram as above gives a factorisation of π_2 and s through the monomorphism m, which is then an isomorphism by assumption. This proves that V_α is conservative on monomorphisms, which is equivalent to being conservative (Proposition 2.9). □

Corollary 2.12. *Let $\mathbb{C}$ be a finitely complete* Ord*-enriched category which admits comma objects. The following statements are equivalent:*

(i) *$H_\alpha\colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_A(\mathbb{C})$ is conservative for any morphism $\alpha\colon A \to Y$.*

(ii) *for any comma object of the form*

$$\begin{array}{ccc} f/\alpha & \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} & A \\ {\scriptstyle \pi_1}\downarrow & \preccurlyeq & \downarrow{\scriptstyle \alpha} \\ X & \underset{s}{\overset{f}{\rightleftarrows}} & Y, \end{array}$$

the pair (π_1, s) *is jointly extremally epimorphic; here* $i_2 = \langle s\alpha, 1_A \rangle$.

Having recovered the lax versions of the equivalences in Theorem 2.1 for Ord-enriched categories, we can now propose the following:

Definition 2.13. A finitely complete Ord-enriched category $\mathbb{C}$ which admits comma objects and 2-pullbacks is called *lax protomodular* when the comma object functor V_α is conservative for any morphism $\alpha\colon A \to Y$ in $\mathbb{C}$. We say that $\mathbb{C}$ is *colax protomodular* when $\mathbb{C}^{\mathrm{co}}$ is lax protomodular, that is when the comma object functor H_α is conservative for any morphism $\alpha\colon A \to Y$ in $\mathbb{C}$.

Exactly as in the classical case, it is immediate to see that, in a pointed Ord-enriched category $\mathbb{C}$ with comma objects, the conservativeness of all comma object functors V_α is equivalent to the conservativeness of the functors V_{i_A}, where $i_A\colon 0 \to A$ is the only arrow from the zero object (and the same holds for the H's). Then, in the pointed context, $\mathbb{C}$ is lax protomodular if and only if the following *lax version of the Split Short Five Lemma* holds:

Theorem 2.14. *Let* $\mathbb{C}$ *be a pointed finitely complete* Ord*-enriched category which admits comma objects and 2-pullbacks.* $\mathbb{C}$ *is lax protomodular if and only if, given a commutative diagram of split sequences of the form*

$$\begin{array}{ccccc} 0/f & \overset{k}{\longrightarrow} & X & \underset{s}{\overset{f}{\rightleftarrows}} & A \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \gamma} \\ 0/f' & \overset{k}{\longrightarrow} & X & \underset{s'}{\overset{f'}{\rightleftarrows}} & A', \end{array}$$

if α *and* γ *are isomorphisms, then* β *is, too.*

A similar result holds for pointed colax protomodular categories.

Remark 2.15. All the results of this section still hold when replacing comma objects with precomma objects and when replacing 2-pullbacks with (1-)pullbacks. We use the same notation V_α and H_α for the precomma object functors. A finitely complete Ord-enriched category $\mathbb{C}$ which admits precomma objects is called *lax preprotomodular* when the precomma object functor V_α is conservative for any morphism α in $\mathbb{C}$. We say that $\mathbb{C}$ is *colax preprotomodular* when $\mathbb{C}^{\mathrm{co}}$ is lax preprotomodular.

Example 2.16.

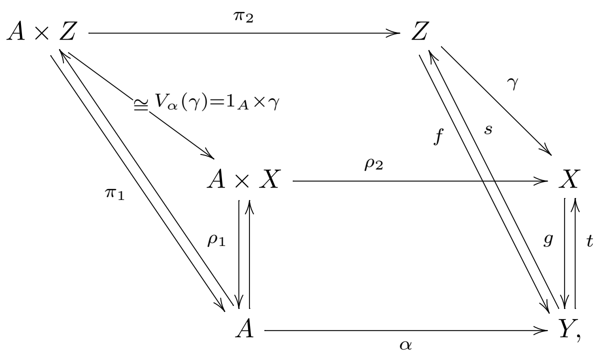
(1) According to Corollary 2.7, an Ord-enriched protomodular category with comma objects and 2-pullbacks is both lax and colax protomodular.

(2) If $\mathscr{T}$ is the theory of a protomodular variety, then the category $\mathbb{C}^{\mathscr{T}}$ of internal $\mathscr{T}$-algebras in $\mathbb{C}$ is also protomodular, and so it is both lax protomodular and colax protomodular, for any compatible Ord-enrichment which admits 2-pullbacks and comma objects.
This applies in particular to $\mathsf{Ord}^{\mathscr{T}}$. We point out, however, that, for any algebraic theory $\mathscr{T}$ which contains a Mal'tsev operation, the preorder of any internal $\mathscr{T}$-algebra in Ord is symmetric: if $X \in \mathsf{Ord}^{\mathscr{T}}$ and $p\colon X^3 \to X$ is a monotone Mal'tsev operation, then, for $x, y \in X$ with $x \leqslant y$ one obtains that $y = p(x,x,y) \leqslant p(x,y,y) = x$. This is the case of every theory $\mathscr{T}$ of a protomodular variety since it has a Mal'tsev operation defined by

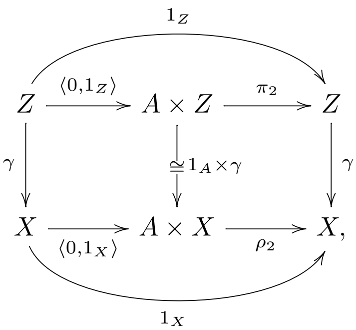
$$p(x,y,z) = \theta(\alpha_1(x,y), \ldots, \alpha_n(x,y), z)$$

(using the operations described in Example 2.2). Therefore, the obvious Ord-enrichment inherited from Ord – the pointwise Ord-enrichment – will be also symmetric.
Throughout by *degenerate* Ord-enrichment we mean one whose hom-sets are symmetric.

(3) Let $\mathbb{D}$ be a finitely complete category with the (degenerate) Ord-enrichment where $f \preccurlyeq g$ for all parallel morphisms f and g. It is easy to check that $\mathbb{D}$ admits comma objects, which are simply binary products and their projections. When $\mathbb{D}$ is pointed, then it gives an example of a lax protomodular category, since V_α is conservative for any morphism $\alpha\colon A \to Y$. Indeed, consider the following commutative diagram



where $V_\alpha(\gamma) = 1_A \times \gamma$ is an isomorphism. Using the top quadrangle pullback above (Lemma 2.3) in the commutative diagram



we conclude that the left square is a pullback; this proves that γ is an isomorphism.

(4) The former two examples raise the question

Is there a protomodular category with a non-degenerate Ord-enrichment with comma objects and 2-pullbacks?

Since the dual of an elementary topos is protomodular, one can more specifically ask:

Is there an elementary topos with a non-degenerate Ord-enrichment admitting cocomma objects and 2-pushouts?

We have put this question to Peter Johnstone, who collected some interesting results on the subject in [10]. Namely:

– *The only poset-enrichment of a localic topos over* Set *is the discrete one.*
– *The* Ord*-enrichment of a topos for which equalizers and exponential adjunctions are* Ord*-enriched is degenerate.*

Moreover, in [10] an example of a topos with a non-degenerate Ord-enrichment is presented, but it does not fulfil our conditions since it does not have cocomma objects.

(5) In the next section we will study the behaviour of OrdAb equipped with a suitable Ord-enrichment. It is not an example of a (co)lax protomodular category since it does not admit comma objects (although it admits precomma objects). Hence we pose the more general open question

Is there any lax protomodular or colax protomodular non-degenerate Ord-enriched category?

3. The 2-category $\mathbb{O}$rdAb of preordered abelian groups

Both the categories Grp of groups and Ab of abelian groups are protomodular, while OrdGrp and OrdAb are not (see Theorem 4.6 in [8]). In this section we introduce an enriched preorder structure on morphisms which does work in favour of lax protomodularity for OrdAb.

We start by analysing possible Ord-enrichments for the category OrdGrp of preordered groups and monotone homomorphisms. We recall that a preordered group is a (not necessarily abelian) group $(X, +, 0)$ equipped with a preorder $\leqslant$ such that the group operation is monotone

$$x \leqslant y, u \leqslant v \quad \Rightarrow \quad x + u \leqslant y + v,$$

for any elements $x, y, u, v \in X$; their morphisms are the monotone group homomorphisms. The preorder of a group $(X, +, 0)$ is completely determined by its *positive cone*, which is the submonoid of X, closed under conjugation, given by its positive elements, $P_X = \{x \in X : 0 \leqslant x\}$.

In OrdGrp the pointwise preorder on morphisms trivialises; that is, if one defines, for morphisms $f, g\colon X \to Y$, $f \leqslant g$ if, for all $x \in X$, $f(x) \leqslant g(x)$, then also $f(-x) \leqslant g(-x)$, and consequently, $\leqslant$ is symmetric. So, instead we use the pointwise order restricted to positive elements, and define, for morphisms $f, g\colon X \to Y$ of OrdGrp,

$$f \preccurlyeq g \quad \Leftrightarrow \quad \forall x \in P_X, f(x) \leqslant g(x). \tag{3.i}$$

It is straightforward to check that (pre)composition preserves the preorder of OrdGrp(X, Y), for any preordered groups X and Y, and so this defines an Ord-enriched category $\mathbb{O}$rdGrp.

$\mathbb{O}$rdGrp does not have precomma objects in general. In order to prove this assertion first note that, if (1.i) is a precomma object in $\mathbb{O}$rdGrp, then C is isomorphic to $X \times Z$, as a group, and c_1, c_2 are the product projections. This follows easily from the following inequality

$$\begin{array}{ccc} (X\times Z,0) & \xrightarrow{\pi_Z} & Z \\ {\scriptstyle \pi_X}\downarrow & \preccurlyeq & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

and the universal properties of products and precomma objects. So what remains to be studied is the existence of a positive cone for $X \times Z$ that makes (1.i) a precomma object. Let Y be a preordered group, $y \in P_Y$ and $\varphi\colon \mathbb{Z} \to Y$ with $\varphi(1) = y$, where $\mathbb{Z}$ is the usual ordered group of integers, and assume that the precomma object of $(\varphi, 1_Y)$ exists in $\mathbb{O}\mathsf{rdGrp}$:

$$\begin{array}{ccc} (\mathbb{Z}\times Y,P) & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\downarrow & \preccurlyeq & \downarrow{\scriptstyle 1_Y} \\ \mathbb{Z} & \xrightarrow[\varphi]{} & Y. \end{array}$$

Then it is easy to check that $(1, y) \in P$, and so, by conjugation, $(1, a + y - a)$ belongs also to P, for any $a \in Y$. Therefore $y = (\varphi\pi_1)(1, a + y - a) \leqslant \pi_2(1, a + y - a) = a + y - a$. Since this inequality is not valid in general, we conclude that the precomma object may not exist. As an example of this failure, consider the group Y of monotone bijective (and therefore continuous) endomaps of the real line $y\colon \mathbb{R} \to \mathbb{R}$ with the operation given by composition, ordered by $y \leqslant y'$ if, for every $x \in \mathbb{R}$, $y(x) \leqslant y'(x)$. Then, for instance, for $y, a\colon \mathbb{R} \to \mathbb{R}$ defined by $y(x) = x + 1$ and $a(x) = \dfrac{x}{2}$, y is positive, since, for every x, $x \leqslant y(x)$, but $y \nleqslant a \circ y \circ a^{-1}$.

To overcome this absence of precomma objects, we focus on its full subcategory $\mathbb{O}\mathsf{rdAb}$ of preordered abelian groups. Here the preorder of an abelian group $(X, +, 0)$ is completely determined by a submonoid of X of its positive elements, $P_X = \{x \in X : 0 \leqslant x\}$, since closedness under conjugation comes for free. We consider in $\mathbb{O}\mathsf{rdAb}$ the Ord-enrichment inherited from $\mathbb{O}\mathsf{rdGrp}$.

Then $\mathbb{O}\mathsf{rdAb}$ does not admit comma objects in general, but it admits precomma objects. As for $\mathbb{O}\mathsf{rdGrp}$, if (1.i) is a (pre)comma object in $\mathbb{O}\mathsf{rdAb}$, then C is isomorphic to $X \times Z$, as a group, and c_1, c_2 are the product projections as in the diagram below:

$$\begin{array}{ccc} f/g = (X\times Z,P_{f/g}) & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\downarrow & \preccurlyeq & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y. \end{array} \tag{3.ii}$$

The positive cone of $f/g = X \times Z$ must be

$$P_{f/g} = \{(x, z) \in P_{X\times Z} : f(x) \leqslant g(z)\}.$$

Indeed, if $\alpha\colon A \to X$, $\beta\colon A \to Z$ are such that $f\alpha \preccurlyeq g\beta$, then $\langle\alpha, \beta\rangle\colon A \to X \times Z$ is monotone: for every $a \in A$, if $a \geqslant 0$ then both $\alpha(a) \geqslant 0$ and $\beta(a) \geqslant 0$, and, moreover, $f\alpha(a) \leqslant g\beta(a)$, hence $\langle\alpha, \beta\rangle(a) \in P_{f/g}$.

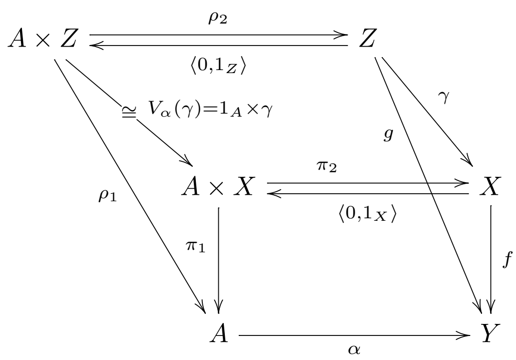
To check that (3.ii) is not always a comma object, consider $f = g = 1_{\mathbb{Z}}$, where again $\mathbb{Z}$ is the usual ordered group of integers, so that $P_{f/g} = \{(n, m) \in \mathbb{Z} \times \mathbb{Z}\ ;\ n \geqslant 0, m \geqslant 0, n \leqslant m\}$. The homomorphisms $t, t'\colon \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, defined by $t(n) = (n, 4n)$ and $t'(n) = (3n, 5n)$ for every $n \in \mathbb{Z}$, are monotone and such that $\pi_1 t \preccurlyeq \pi_1 t'$ and $\pi_2 t \preccurlyeq \pi_2 t'$, but $t \not\preccurlyeq t'$: $t(1) \leqslant t'(1)$ would imply $(2, 1) = t'(1) - t(1) \in P_{f/g}$, which is false.

Note that, in the precomma object above, π_2 is split by the morphism $\langle 0, 1_Z\rangle : Z \to X \times Z$ (as usual), while π_1 is not split by the group homomorphism $\langle 1_X, 0\rangle : X \dashrightarrow X \times Z$, since it is not monotone (denoted $\dashrightarrow$ to emphasize it is not a morphism in $\mathbb{O}$rdAb). Actually, $(\pi_2, \langle 0, 1_Z\rangle)$ is a rali point: for any $(x, z) \in P_{f/g}$, we get $(0, z) \leqslant (x, z)$, because $x \in P_X$; thus $\langle 0, 1_Z\rangle\pi_2 \preccurlyeq 1_{f/g}$.

The following result proves the conservativeness of the vertical precomma object change-of-base functor with respect to the slice category, which implies the same property with respect to points. Thus $\mathbb{O}$rdAb *is a lax preprotomodular category.*

Proposition 3.1. *For any morphism* $\alpha\colon A \to Y$ *in* $\mathbb{O}$rdAb*, the precomma object functor* $V_\alpha\colon \mathbb{O}\mathsf{rdAb}/Y \to \mathbb{O}\mathsf{rdAb}/A$ *is conservative.*

Proof. We build the diagram

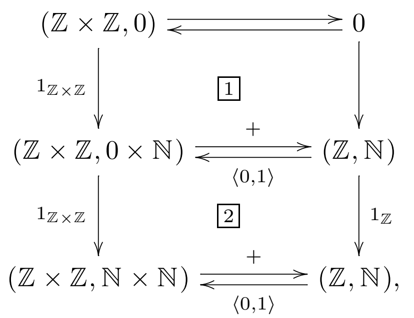


where the front and back faces are precomma diagrams (in particular, $\alpha\pi_1 \preccurlyeq f\pi_2$, $\alpha\rho_1 \preccurlyeq g\rho_2$) and $f\gamma = g$. Note that the upper trapezoid also commutes with the precomma projection splittings, i.e. $\langle 0, 1_X\rangle\gamma = V_\alpha(\gamma)\langle 0, 1_Z\rangle$. If $V_\alpha(\gamma)$ $(= 1_A \times \gamma)$ is an isomorphism, then γ is also an isomorphism. The inverse of γ is given by the composite of morphisms $\rho_2 V_\alpha(\gamma)^{-1}\langle 0, 1_X\rangle\colon\ X \to Z$, since

$$\gamma\rho_2 V_\alpha(\gamma)^{-1}\langle 0, 1_X\rangle = \pi_2 V_\alpha(\gamma)V_\alpha(\gamma)^{-1}\langle 0, 1_X\rangle = \pi_2\langle 0, 1_X\rangle = 1_X$$
$$\rho_2 V_\alpha(\gamma)^{-1}\langle 0, 1_X\rangle\gamma = \rho_2 V_\alpha(\gamma)^{-1}V_\alpha(\gamma)\langle 0, 1_Z\rangle = \rho_2\langle 0, 1_Z\rangle = 1_Z. \quad \square$$

The corresponding horizontal result does not hold for $\mathbb{O}$rdAb, as the next example shows.

Example 3.2. In the following diagram



it is easily checked that both $\boxed{1}$ and the outer rectangle are precomma objects. Indeed the positive cone of $\mathbb{Z} \times \mathbb{Z}$ for both precomma objects is given by

$$\{(z, z') \in \mathbb{Z} \times \mathbb{Z}\,;\, z \geqslant 0,\, z' \geqslant 0,\, z + z' \leqslant 0\} = \{(0, 0)\}.$$

However, $\boxed{2}$ is commutative but it is not a pullback, showing that $\mathbb{O}\mathsf{rdAb}$ does not satisfy condition (ii) of Corollary 2.6, that is, $\mathbb{O}\mathsf{rdAb}$ *is not colax preprotomodular.*

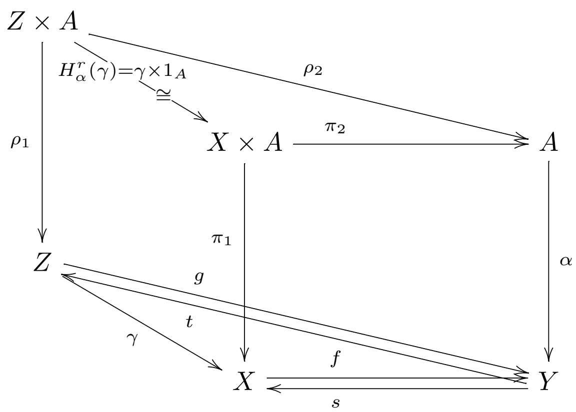
With respect to the horizontal precomma object change-of-base functor and its conservativeness for $\mathbb{O}\mathsf{rdAb}$, we can only prove that each H_α is conservative when applied to rali points. As mentioned above, the top projection of a precomma object of a pair of morphisms $(f\colon X \to Y, g\colon Z \to Y)$ always gives rise to a rali point $(\pi_2, \langle 0, 1_Z\rangle)$. However, taking the precomma object of (f, g), when (f, s) is a rali point, gives rise to a point $(\pi_2, \langle sg, 1_Z\rangle)$ which is not necessarily rali. For instance, in the precomma object of $(1_\mathbb{Z}, 1_\mathbb{Z})$, $(\pi_2, \langle 1_\mathbb{Z}, 1_\mathbb{Z}\rangle)$ is not a rali. So, the functor below goes from rali points to ordinary points.

Proposition 3.3. *For any morphism $\alpha\colon A \to Y$ in $\mathbb{O}\mathsf{rdAb}$, the functor*

$$H^r_\alpha\colon \mathsf{Ptr}_Y(\mathbb{O}\mathsf{rdAb}) \to \mathsf{Pt}_A(\mathbb{O}\mathsf{rdAb})$$

is conservative.

Proof. We build the diagram



where the front and back faces are precomma diagrams (in particular, $f\pi_1 \preccurlyeq \alpha\pi_2$, $g\rho_1 \preccurlyeq \alpha\rho_2$); also $f\gamma = g$, $\gamma t = s$ and $sf \preccurlyeq 1_X$. If $H^r_\alpha(\gamma)(= \gamma \times 1_A)$ is an isomorphism, then it easily follows that γ is a monomorphism (=injective) and an epimorphism (=surjective). We cannot proceed as in the previous proof since the vertical projections of precomma objects need not be split epimorphisms. To conclude that γ is an isomorphism, it suffices to show that γ is a regular epimorphism, i.e. that $\gamma(P_Z) \supseteq P_X$, which gives $\gamma(P_Z) = P_X$.

For $x \in P_X$, we get $x - sf(x) \in P_X$, because f is rali, thus $(x - sf(x), 0) \in P_{X\times A}$. Since $f(x - sf(x)) = 0 \leqslant \alpha(0)$, then $(x - sf(x), 0) \in P_{f/\alpha}$. It follows that $\rho_1 H^r_\alpha(\gamma)^{-1}(x - sf(x), 0) \in P_Z$. We also have $tf(x) \in P_Z$. So, there exists an element $\overline{z} = \rho_1 H^r_\alpha(\gamma)^{-1}(x - sf(x), 0) + tf(x) \in P_Z$, such that $\gamma(\overline{z}) = x - sf(x) + \gamma tf(x) = x - sf(x) + sf(x) = x$. $\square$

Finally we remark that Ord-enriched regularity does not follow from the corresponding1-dimensional property. Indeed, as an epireflective subcategory of the regular category OrdGrp (see [8] and [4]), OrdAb is a regular category in the sense of [2], but $\mathbb{O}\mathsf{rdAb}$ is not Ord-regular (cf. [5] and [11]), neither in the case we consider as right factor $\mathscr{M}$ of the factorisation the full and faithful morphisms nor the monic, full and faithful morphisms. Indeed, as we show next, in both cases there is a class of morphisms $\mathscr{E}$ such that $(\mathscr{E}, \mathscr{M})$ is a *non-stable* orthogonal factorisation system.

Proposition 3.4. *Let $\mathscr{M}$ (respectively $\mathscr{M}'$) be the class of (respectively monic and) full and faithful morphisms in $\mathbb{O}\mathsf{rdAb}$ and let*

$$\mathscr{E} = \{h\colon A \to B \text{ bijective};\ \text{for all } b \in P_B,\ b = h(a' - a),\ \text{for } a', a \in P_A\},\ \text{and}$$

$$\mathscr{E}' = \{h\colon A \to B \text{ surjective};\ \text{for all } b \in P_B,\ b = h(a' - a),\ \text{for } a', a \in P_A\}.$$

(1) *A morphism $f\colon X \to Y$ in $\mathbb{O}\mathsf{rdAb}$ belongs to $\mathscr{M}$ if, and only if, for all $x, x' \in P_X$, $x \leqslant x' \;\Leftrightarrow\; f(x) \leqslant f(x')$; a full and faithful morphism f belongs to $\mathscr{M}'$ if, in addition, it is an injective map.*

(2) *Given a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[v]{} & Y, \end{array} \tag{3.iii}$$

with $h \in \mathscr{E}'$ and $f \in \mathscr{M}'$, there exists a unique morphism $d\colon B \to X$ such that $dh = u$ and $fd = v$.

(3) *Given a commutative diagram (3.iii), with $h \in \mathscr{E}$ and $f \in \mathscr{M}$, there exists a unique morphism $d\colon B \to X$ such that $dh = u$ and $fd = v$.*

(4) *Every morphism in $\mathbb{O}\mathsf{rdAb}$ factors through a morphism in $\mathscr{E}'$ followed by a morphism in $\mathscr{M}'$.*

(5) *Every morphism in $\mathbb{O}\mathsf{rdAb}$ factors through a morphism in $\mathscr{E}$ followed by a morphism in $\mathscr{M}$.*

(6) *$(\mathscr{E}, \mathscr{M})$ and $(\mathscr{E}', \mathscr{M}')$ are non-stable factorisation systems.*

Proof. (1) If $f\colon X \to Y$ is full and faithful and $x, x' \in P_X$ are such that $f(x) \leqslant f(x')$, then $\varphi, \varphi'\colon \mathbb{Z} \to X$ defined by $\varphi(1) = x$ and $\varphi(1) = x'$ are morphisms in $\mathbb{O}\mathsf{rdAb}$ such that $f\varphi \preccurlyeq f\varphi'$, and so $\varphi \preccurlyeq \varphi'$, or, equivalently, $x \leqslant x'$. Conversely, if for all $x, x' \in P_X$, $x \leqslant x' \;\Leftrightarrow\; f(x) \leqslant f(x')$ and $fg \preccurlyeq fh$, for $g, h\colon W \to X$, then, for every $w \in P_W$, $f(g(w)) \leqslant f(h(w))$ and so $g(w) \leqslant h(w)$, that is $g \preccurlyeq h$.

(2) Given (3.iii) with $h \in \mathscr{E}'$ and $f \in \mathscr{M}'$, since h is surjective and f is injective, we define the homomorphism $d\colon B \to X$ as usual in Ab: $d(b) = u(a)$ for any $a \in h^{-1}(b)$. Then d is the unique map such that $dh = u$ and $fd = v$. It remains to show that it is monotone: if $b \in P_B$, then $b = h(a' - a)$ for some $a, a' \in P_A$, and $d(b) = u(a' - a)$. From $v(b) = f(u(a' - a)) \in P_Y$, we get $u(a' - a) \in P_X$, since f is full and faithful; thus $d(b) = u(a' - a) \in P_X$.

(3) An analogous argument shows that morphisms in $\mathscr{E}$ are orthogonal to full and faithful morphisms: given (3.iii) with $h \in \mathscr{E}$ and $f \in \mathscr{M}$, since h is bijective we define $d\colon B \to X$ by $d(b) = u(a)$, where a is the unique element of $h^{-1}(b)$. Monotonicity of d follows from arguments similar to those used above.

(4) Every morphism $g\colon Z \to Y$ factors as $Z \xrightarrow{g'} (g(Z), P') \xrightarrow{m} Y$, where $P' = \{y \in P_Y;\ y = g(z' - z) \text{ for } z, z' \in P_Z\}$, with, by construction, the corestriction g' of g in $\mathscr{E}'$ and the inclusion m in $\mathscr{M}'$.

(5) Analogously, every morphism $g\colon Z \to Y$ factors as $Z \xrightarrow{1_Z} (Z, P) \xrightarrow{\tilde{g}} Y$, where $P = \{z \in Z; g(z) \in P_Y \text{ and } z = z'' - z' \text{ for } z', z'' \in P_Z\}$, with the identity 1_Z in $\mathscr{E}$, and $\tilde{g}$ defined as g, which is full and faithful due to the way P is defined.

(6) Since all these classes are closed under composition with isomorphisms, we may conclude that both pairs are factorisation systems in $\mathbb{O}\mathsf{rdAb}$ (cf. [1, Definition 14.1]).

To show that they are not stable, we consider the following pullback

$$\begin{array}{ccc} (\mathbb{Z}, \{0\}) & \xrightarrow{1} & (\mathbb{Z}, -\mathbb{N}) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ (\mathbb{Z}, \mathbb{N}) & \xrightarrow[1]{} & (\mathbb{Z}, \mathbb{Z}), \end{array}$$

where $(\mathbb{Z}, \mathbb{N}) \to (\mathbb{Z}, \mathbb{Z})$ belongs to both $\mathscr{E}$ and $\mathscr{E}'$ but $(\mathbb{Z}, \{0\}) \to (\mathbb{Z}, -\mathbb{N})$ does not belong to either of them. □

Remark 3.5. In [9] the category V-Grp of V-groups and V-homomorphisms, for a commutative and unital quantale V, was studied. (The reader may want to give a look at [9], and the subsequent paper [12], to know more on V-groups.)

Very briefly, we point out that what we have done for OrdAb can be generalized for the category V-Ab of abelian V-groups and V-homomorphisms, for a commutative and unital quantale V. The Ord-enrichment in V-Ab is defined, for V-homomorphisms $f, g\colon (X, a) \to (Y, b)$ by $f \preccurlyeq g$ if, for all $x \in X$, $a(0, x) \leqslant b(f(x), g(x))$ in V; as for OrdAb, we denote this Ord-enriched category by $\mathbb{V}$-Ab. This category has precomma objects: given morphisms $f\colon (X, a) \to (Y, b)$ and $g\colon (Z, c) \to (Y, b)$ in $\mathbb{V}$-Ab, the precomma object is defined as in the following diagram

$$\begin{array}{ccc} (X \times Z, d) & \xrightarrow{\pi_2} & (Z, c) \\ {\scriptstyle \pi_1}\downarrow & \preccurlyeq & \downarrow{\scriptstyle g} \\ (X, a) & \xrightarrow[f]{} & (Y, b), \end{array}$$

where $d((x, z), (x', z')) = a(x, x') \wedge c(z, z') \wedge b(f(x' - x), g(z' - z))$, for every $(x, z), (x', z') \in X \times Z$. Since d makes $X \times Z$ a V-category and it is invariant under shifting, $(X \times Z, d)$ is a V-group by [9, Proposition 3.1]. Moreover, from the definition of d it follows that the projections π_1 and π_2 are V-homomorphisms, and that $f\pi_1 \preccurlyeq g\pi_2$. The universal property of this diagram is easily checked, that is, conditions (C1) and (C2) are satisfied.

Now it is clear that the inclusion

$$\mathsf{OrdAb} \longrightarrow V\text{-}\mathsf{Ab}$$

becomes Ord-enriched and preserves precomma objects, and so we may conclude directly that $\mathbb{V}$-Ab does not have comma objects in general.

From the failure of lax preprotomodularity for $\mathbb{O}$rdAb$^{\mathrm{co}}$ it follows immediately that $\mathbb{V}$-Ab$^{\mathrm{co}}$ is not lax preprotomodular. Still, using exactly the arguments of the proof of Proposition 3.1 and the description of precomma objects above, it is straightforward to conclude that $\mathbb{V}$-Ab is lax preprotomodular.

Acknowledgements

We are grateful to John Bourke for his interesting tips on 2-dimensional regularity, Graham Manuell for useful discussions on Ord-enrichments on toposes, and mostly to Peter T. Johnstone for his thorough thoughts on topos Ord-enrichments, which gave rise to the notes [10].

We are also grateful to the referee for several suggestions that helped to improve the final version of the paper.

References

[1] J. Adámek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories: The Joy of Cats, reprint of the 1990 original Wiley, New York, 1990, Repr. Theory Appl. Categ. 17 (2006) 1–507.
[2] M. Barr, P.A. Grillet, D.H. van Osdol, Exact Categories and Categories of Sheaves, Lecture Notes in Math., vol. 236, Springer-Verlag, 1971.
[3] F. Borceux, D. Bourn, Mal'cev, Protomodular, Homological and Semi-Abelian Categories, Math. Appl., vol. 566, Kluwer Acad. Publ., 2004.

[4] F. Borceux, M.M. Clementino, M. Gran, L. Sousa, Protolocalisations of homological categories, J. Pure Appl. Algebra 212 (2008) 1898–1927.

[5] J. Bourke, R. Garner, Two-dimensional regularity and exactness, J. Pure Appl. Algebra 218 (2014) 1346–1371.

[6] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, Springer Lect. Notes Math. 1488 (1991) 43–62.

[7] D. Bourn, G. Janelidze, Characterization of protomodular varieties of universal algebras, Theory Appl. Categ. 11 (2003) 143–147.

[8] M.M. Clementino, N. Martins-Ferreira, A. Montoli, On the categorical behaviour of preordered groups, J. Pure Appl. Algebra 223 (2019) 4226–4245.

[9] M.M. Clementino, A. Montoli, On the categorical behaviour of V-groups, J. Pure Appl. Algebra 225 (2021) 106550.

[10] P.T. Johnstone, Notes on Ord-enrichment of Toposes, June 2022, manuscript.

[11] A. Kurz, J. Velebil, Quasivarieties and varieties of ordered algebras: regularity and exactness, Math. Struct. Comput. Sci. 27 (2017) 1153–1194.

[12] A. Michel, Torsion theories and coverings of V-groups, Appl. Categ. Struct. 30 (2022) 659–684.

[13] V. Aravantinos-Sotiropoulos, The exact completion for regular categories enriched in posets, J. Pure Appl. Algebra 226 (2022) 106885.