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Triangulated categories with a single compact generator, and two Brown representability theorems

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Abstract

We generalize theorems of Bondal and Van den Bergh and of Rouquier. A corollary of our main results says the following. Let X be a scheme proper over a an excellent, finite-dimensional noetherian ring R. Then the Yoneda pairing, taking an object A in the category $\mathbf{D}^{\mathrm{perf}}(X)$ and an object B in the category $\mathbf{D}^b_{\mathrm{coh}}(X)$, to the finite R–module $\mathrm{Hom}(A, B)$, gives an equivalence of $\mathbf{D}^b_{\mathrm{coh}}(X)$ with the category of finite R–linear homological functors $H : \mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}} \longrightarrow R\text{–mod}$, and an equivalence of $\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}}$ with the category of finite homological functors $H : \mathbf{D}^b_{\mathrm{coh}}(X) \longrightarrow R\text{–mod}$. Recall: a homological functor H is *finite* if $\oplus_{i=-\infty}^{\infty} H^i(C)$ is a finite R–module for every $C \in \mathbf{D}^{\mathrm{perf}}(X)$. Bondal and Van den Bergh proved the special case, of the assertion about $\mathbf{D}^b_{\mathrm{coh}}(X)$ identifying with the finite homological functors on $\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}}$, as long as R is a field and X is projective over R. And the assertion about $\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}}$, identifying with the finite homological functors on $\mathbf{D}^b_{\mathrm{coh}}(X)$, again under the assumption that X is projective over a field R, is due to Rouquier. But our theorems are far more general yet. They aren't only about schemes, they work in the abstract generality of triangulated categories with coproducts and a single compact generator, satisfying a certain approximability property. At the moment I only know how to prove this approximability for the categories $\mathbf{D}_{\mathbf{qc}}(X)$ with X a quasicompact, separated scheme, for the homotopy category of spectra, for the category $\mathbf{D}(R)$ where R is a (possibly noncommutative) negatively graded dg algebra, and for certain recollements of the above. The work was inspired by Jack Hall's elegant new proof of a vast generalization of GAGA, a proof based on representability theorems of the type above. The generality of Hall's result made me wonder how far the known representability theorems could be improved.

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Contents

1 Introduction

This paper begins with the observation that triangulated categories with coproducts, and with a single compact generator, have a preferred class of *t*–structures. This allows us to define thick subcategories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$. A slightly subtler definition is that of $\mathcal{T}_c^-$. The full subcategory $\mathcal{T}_c^-\subset\mathcal{T}$ makes sense unconditionally, and it is thick as long as there exists a compact generator G and an integer $A>0$ so that $\mathrm{Hom}(G,\Sigma^i G)=0$ for all $i\geq A$. We also define a subcategory $\mathcal{T}_c^b=\mathcal{T}_c^-\cap\mathcal{T}^b$.

In the special case where $\mathcal{T}=\mathbf{D_{qc}}(X)$, with X a quasicompact, quasiseparated scheme, the preferred class of *t*–structures contains the standard *t*–structure, the subcategories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ are nothing other than the classical $\mathbf{D}^-_{\mathbf{qc}}(X)$, $\mathbf{D}^+_{\mathbf{qc}}(X)$ and $\mathbf{D}^b_{\mathbf{qc}}(X)$, and if X is noetherian the subcategories $\mathcal{T}_c^b\subset\mathcal{T}_c^-$ can be proved to be $\mathbf{D}^b_{\mathrm{coh}}(X)\subset\mathbf{D}^-_{\mathrm{coh}}(X)$. What we have learned so far is that these standard categories have an intrinsic description. There is a method to construct them out of $\mathcal{T}$ in purely triangulated-category terms.

Still in the world of triangulated categories with coproducts and a single compact generator: the category $\mathcal{T}$ may be *weakly approximable* or even *approximable*. We will define these concepts later in the introduction, and study their properties in the body of the paper. For now we note that the category $\mathbf{D_{qc}}(X)$ is weakly approximable as long as X is a quasicompact, quasiseparated scheme, and approximable if X is separated. The homotopy category of spectra is also approximable.

To show that this abstraction can be useful we will prove representability theorems. To state them we begin with

Definition 1.1 Let R be a commutative ring, let $\mathcal{T}$ be an R–linear triangulated category and let $\mathcal{B} \subset \mathcal{T}$ be a full, replete subcategory with $\Sigma\mathcal{B} = \mathcal{B}$. A *$\mathcal{B}$–homological functor* is an R–linear functor $H : \mathcal{B} \longrightarrow R$–Mod which takes triangles to long exact sequences. Dually, a *$\mathcal{B}$–cohomological functor* is an R–linear functor $\widetilde{H} : \mathcal{B}^{\mathrm{op}} \longrightarrow$ R–Mod which takes triangles to long exact sequences. This means that, if we have a triangle $x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x$ with all three of x, y, z belonging to $\mathcal{B}$, then H (respectively $\widetilde{H}$) takes it to a long exact sequence in R–Mod.

Suppose the ring R is noetherian, and let $G \in \mathcal{B} \subset \mathcal{T}$ be an object. The $\mathcal{B}$–(co)homological functor H is called *G–locally finite* if

- (i) $H(\Sigma^i G)$ is a finite R–module for all $i \in \mathbb{Z}$.
- (ii) $H\big(\Sigma^i G\big) = 0$ for $i \ll 0$.

The $\mathcal{B}$–(co)homological functor H is *G–finite* if, in addition to the above, we have

- (iii) $H\big(\Sigma^i G\big) = 0$ for $i \gg 0$.

Remark 1.2 Let $\mathcal{T}$ be an R–linear triangulated category, $\mathcal{B}$ a full, replete subcategory with $\Sigma\mathcal{B} = \mathcal{B}$, and H a $\mathcal{B}$–(co)homological functor. If H is G–locally finite (respectively finite) for every $G \in \mathcal{B}$ we leave out the G, and just say that H is locally finite (respectively finite).

Note that if H is G–locally finite (respectively finite) then it is also G'–locally finite for any G' obtainable from G by forming in $\mathcal{B}$ finite direct sums, direct summands, suspensions or triangles. Thus local finiteness (respectively finiteness) can be checked on any classical generator.

Remark 1.3 The careful reader will observe that Definition 1.1 is *not* self-dual. If we replace $\mathcal{T}$ by $\mathcal{T}^{\mathrm{op}}$, and take $\mathcal{B} \subset \mathcal{T}^{\mathrm{op}}$ to be a full, replete subcategory with $\Sigma\mathcal{B} = \mathcal{B}$, then for a (co)-homological functor H on $\mathcal{B}$ to be G–finite is of course equivalent to the (co)homological functor H being G–finite on $\mathcal{B}^{\mathrm{op}} \subset \mathcal{T}$. But local finiteness changes, because on $\mathcal{T}^{\mathrm{op}}$ the shift functor Σ is replaced by Σ^{-1}.

There exists a way to remedy this, by viewing local finiteness as continuity with respect to certain metrics. This will be discussed extensively in subsequent articles.

Our main theorem says that

Theorem 1.4 *Let R be a commutative, noetherian ring, and $\mathcal{T}$ an R–linear triangulated category with coproducts. Assume $\mathcal{T}$ is approximable, and suppose further that there exists in $\mathcal{T}$ a compact generator G such that* $\mathrm{Hom}\big(G, G[n]\big)$ *is a finite R–module for all $n \in \mathbb{Z}$. Consider the two functors*

$$\mathcal{Y} : \mathcal{T}_c^- \longrightarrow \underset{R}{\mathrm{Hom}}\big([\mathcal{T}^c]^{\mathrm{op}}, \textit{R–Mod}\big), \qquad \widetilde{\mathcal{Y}} : \big[\mathcal{T}_c^-\big]^{\mathrm{op}} \longrightarrow \underset{R}{\mathrm{Hom}}\big(\mathcal{T}_c^b, \textit{R–Mod}\big)$$

defined by the formulas $\mathcal{Y}(B) = \mathrm{Hom}(-, B)$ *and* $\widetilde{\mathcal{Y}}(A) = \mathrm{Hom}(A, -)$. *Note that, in these formulas, we permit all $A, B \in \mathcal{T}_c^-$.*

But the $(-)$ in the formula $\mathcal{Y}(B) = \mathrm{Hom}(-, B)$ *is assumed to belong to $\mathcal{T}^c$, whereas the $(-)$ in the formula* $\widetilde{\mathcal{Y}}(A) = \mathrm{Hom}(A, -)$ *must lie in $\mathcal{T}_c^b$. Now consider* *the following composites*

$$\begin{tikzcd}
\mathcal{T}_c^b \arrow[r, hook, "i"] & \mathcal{T}_c^- \arrow[r, "\mathcal{Y}"] & \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}\,,\ R\text{–}\mathit{Mod}\big) \\
{[\mathcal{T}^c]^{\mathrm{op}}} \arrow[r, hook, "\widetilde{\imath}"] & {[\mathcal{T}_c^-]^{\mathrm{op}}} \arrow[r, "\widetilde{\mathcal{Y}}"] & \mathrm{Hom}_R\big(\mathcal{T}_c^b\,,\ R\text{–}\mathit{Mod}\big)
\end{tikzcd}$$

We assert:

- (i) *The functor $\mathcal{Y}$ is full, and the essential image consists of the locally finite cohomological functors. The composite $\mathcal{Y} \circ i$ is fully faithful, and the essential image consists of the finite cohomological functors.*
- (ii) *With the notation as in Reminder* 1.12(*xii*), *assume further that there exists an integer $N > 0$ and an object $G' \in \mathcal{T}_c^b$ with $\mathcal{T} = \overline{\langle G' \rangle}_N^{(-\infty,\infty)}$. Then the functor $\widetilde{\mathcal{Y}}$ is full, and the essential image consists of the locally finite homological functors. The composite $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$ is fully faithful, and the essential image consists of the finite homological functors.*

Remark 1.5 The original versions of the result were posted on the archive as two articles, with the first proving Theorem 1.4(i) and the second devoted to Theorem 1.4(ii). The reason they could not be merged, at the time, was that the proof of Theorem 1.4(ii) depended on very useful little results in joint work with Jesse Burke and with Bregje Pauwels, see [8, Lemma 3.6 and 3.9(iv)]. And the joint article with Burke and Pauwels built in turn on the theory developed to prove Theorem 1.4(i). This issue has since been solved, through the inscrutable vissicitudes of the publication process: as it happens the joint work with Burke and Pauwels appeared first, and the two articles for which I happen to be the sole author could therefore be merged.

But I should take this opportunity to highlight the value of the joint work with Burke and Pauwels, especially [8, Lemma 3.9(iv)]. At the time we proved it seemed, to the three of us, to be a small, technical lemma. But by now this little lemma has found diverse and surprising applications.

From Theorem 1.4(i) we deduce:

Corollary 1.6 *Let $\mathcal{T}$ be as in Theorem* 1.4(*i*), *but assume further that $\mathcal{T}^c$ is contained in $\mathcal{T}_c^b$. Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be one of the preferred t–structures.*

Let $\mathcal{L} : \mathcal{T}_c^b \longrightarrow \mathcal{S}$ be an R–linear triangulated functor. Then the functor $\mathcal{L}$ has a right adjoint if and only if the following three conditions hold:

- (i) *For any pair of objects (t, s), with $t \in \mathcal{T}^c$ and $s \in \mathcal{S}$, the R–module* $\mathrm{Hom}\big(\mathcal{L}(t), s\big)$ *is finite.*
- (ii) *For any object $s \in \mathcal{S}$ there exists an integer $A > 0$ with* $\mathrm{Hom}\big(\mathcal{L}(\mathcal{T}_c^b \cap \mathcal{T}^{\leq -A})\,,\ s\big) = 0.$
- (iii) *For any object $t \in \mathcal{T}^c$ and any object $s \in \mathcal{S}$ there exists an integer A so that* $\mathrm{Hom}\big(\mathcal{L}(\Sigma^m t), s\big) = 0$ *for all $m \leq -A$.*

In the special example of $\mathcal{T} = \mathbf{D}_{\mathbf{qc}}(X)$, Theorem 1.4 specializes to

Example 1.7 Let X be a scheme proper over a noetherian ring R. Then it is separated and quasicompact, hence the category $\mathcal{T} = \mathbf{D_{qc}}(X)$ is approximable. But properness also guarantees that, for any compact generator $G \in \mathcal{T}$ and any $i \in \mathbb{Z}$, the R–module $\mathrm{Hom}(\Sigma^i G, G)$ is finite. The conditions of Theorem 1.4(i) are satisfied, and the conclusion is:

- (i) The functor $\mathcal{Y} \circ i$ gives an equivalence from the category $\mathbf{D}^b_{\mathrm{coh}}(X)$ to the category of finite cohomological functors $\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}} \longrightarrow R$–Mod.
- (ii) On the larger category $\mathbf{D}^-_{\mathrm{coh}}(X)$, the functor $\mathcal{Y}$ is full and the essential image is the category of locally finite cohomological functors $\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}} \longrightarrow R$–Mod.

If we furthermore assume that the ring R is of finite Krull dimension and excellent, then from Aoki [2, Theorem 5.1] we know that there exists an object $G' \in \mathcal{T}^b_c = \mathbf{D}^b_{\mathrm{coh}}(X)$ and an integer $N > 0$ with $\mathbf{D_{qc}}(X) = \overline{\langle G' \rangle}_N$. Thus the hypotheses of Theorem 1.4(ii) are also satisfied, and we obtain:

- (iii) The functor $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$ gives an equivalence from the category $\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}}$ to the category of finite homological functors $\mathbf{D}^b_{\mathrm{coh}}(X) \longrightarrow R$–Mod.
- (iv) On the larger category $\mathbf{D}^-_{\mathrm{coh}}(X)^{\mathrm{op}}$, the functor $\widetilde{\mathcal{Y}}$ is full and the essential image is the category of locally finite homological functors $\mathbf{D}^b_{\mathrm{coh}}(X) \longrightarrow R$–Mod.

Problem 1.8 The results in Theorem 1.4 and of Example 1.7 have an obvious symmetry, which leads one to wonder if there is a way to understand and unify them.

Only very recently have I made any substantial progress on this, and the work is still very much in progress.

Remark 1.9 If R is a field and X is projective over R, then the part of Example 1.7 concerning the objects in the image of $\mathcal{Y} \circ i$ and of $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$ is known—see Bondal and Van den Bergh [5, Theorem A.1] for $\mathcal{Y} \circ i$ and Rouqier [22, Corollary 7.51(ii)] for $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$. Neither of the old theorems says anything about the functors $\mathcal{Y} \circ i$ and $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$ being fully faithful.

About the existing proofs: what Rouquier presents in [22, Corollary 7.51(ii)] is the outline of how a proof might go, which unfortunately I haven't been able fill out. And aside from this one published sketch, there is nothing in the existing literature resembling a proof of Example 1.7(iv).

The existing proofs of variants of Example 1.7(ii), including the current one, proceed in two steps. Starting with a finite $\mathcal{T}^c$–cohomological functor H one first proves that $H \cong \mathcal{Y}(t)$ for some $t \in \mathcal{T}$, and then shows that t must actually belong to $\mathcal{T}^b_c$. Bondal and Van den Bergh [5, Theorem A.1] and Jack Hall [12, Proposition 4.1] rely on suitable special features that allow the functor $H : \mathcal{T}^c = \mathbf{D}^{\mathrm{perf}}(X) \longrightarrow R$–Mod to extend to a cohomological functor on all of $\mathcal{T} = \mathbf{D_{qc}}(X)$, and then use the usual Brown representability theorem for $\mathbf{D_{qc}}(X)$. For Bondal and Van den Bergh the key is forming the double dual—this works since R is assumed a field, and a finite-dimensional vector space over R is canonically isomorphic to its double dual. Jack Hall relies on the fact that his functors come from morphisms of ringed spaces

$c : \mathfrak{X} \longrightarrow X$, and formal properties then provide adjoints

$$\begin{tikzcd} \mathbf{D_{qc}}(X) \arrow[r, shift left, "\text{natural}"] & \mathbf{D}(X) \arrow[l, shift left, "\mathbf{R}Q_X"] \arrow[r, shift left, "\mathbf{L}c^*"] & \mathbf{D}(\mathfrak{X}) \arrow[l, shift left, "\mathbf{R}c_*"] \end{tikzcd}$$

We should recall one more result in the literature: although Ben-Zvi, Nadler and Preygel [4, Sect. 3] is not technically either a special case or a generalization of Theorem 1.4, the reader is nonetheless encouraged to look at it—there are interesting parallels. Enhancements play a role in [4], as well as the construction of an explicit generator and estimates similar to those of [17, Theorem 4.1].

What's different here is the generality. Let H be *any* locally finite $\mathcal{T}^c$–cohomological functor. Under hypotheses weaker than approximability (see Proposition 9.10 for the precise statement) we prove that $H \cong \mathcal{Y}(t)$ where $t \in \mathcal{T}$ is some object—the existence of t is formal, not special to narrow classes of $\mathcal{T}$'s or H's. And by combining a careful analysis of the proof of Proposition 9.10, with the theory developed in Sect. 3, we will deduce—*under only the approximability hypothesis*—that t must belong to $\mathcal{T}_c^-$.

Remark 1.10 The work was inspired at the time by the lovely new proof of a vast generalization of GAGA, to be found in Jack Hall [12]. More precisely: it was inspired by the original idea, which is to be found in [12, Sect. 2, the section labeled "A simple case"]. As Hall's paper became more general it developed a different tack and, by the published version [11], the short section with the simple, central idea went missing. One of the points of the current paper is that our representability theorems obviate the need to do much to pass from Hall's original, simple idea to a full-blown proof. In Example A.2 the reader can find this spelt out: the Appendix is all of two pages long and gives a full proof of GAGA.

The condensed summary of the Appendix is as follows. Let X be a scheme proper over the complex numbers $\mathbb{C}$. With $R = \mathbb{C}$ we apply Corollary 1.6, with $\mathcal{T} = \mathbf{D_{qc}}(X)$ [and hence with $\mathcal{T}_c^b = \mathbf{D}^b_{\text{coh}}(X)$], with $\mathcal{S} = \mathbf{D}^b_{\text{coh}}(X^{\text{an}})$, and where the functor $\mathcal{L} : \mathbf{D}^b_{\text{coh}}(X) \longrightarrow \mathbf{D}^b_{\text{coh}}(X^{\text{an}})$ is the analytification. The hypotheses of Corollary 1.6 are trivial to check, and hence $\mathcal{L}$ has a right adjoint $\mathcal{R} : \mathbf{D}^b_{\text{coh}}(X^{\text{an}}) \longrightarrow \mathbf{D}^b_{\text{coh}}(X)$. And then, proving that $\mathcal{L}$ and $\mathcal{R}$ are quasi-inverses, reduces to checking that the unit $\eta : \text{id} \longrightarrow \mathcal{R}\mathcal{S}$ and counit $\varepsilon : \mathcal{L}\mathcal{R} \longrightarrow \text{id}$ of adjunction are isomorphisms, and standard-enough techniques make this an easy exercise. For details see the Appendix.

It isn't often that one achieves such a thing: find a more elegant, much shorter proof of a theorem by a giant like Jean-Pierre Serre. After all: Serre isn't only a giant as a mathematician, he is also a master of crisp and elegant exposition. Of course most of the credit goes to Jack Hall, he had the beautiful key idea. In this article we provide the technical, triangulated category framework allowing for a straightforward and direct passage from the simple idea to a complete proof.

Remark 1.11 In the time since the first versions of the current manuscript appeared on the archive, people have found applications of the representability theorems presented in Example 1.7 quite different from the one that originally motivated the author. Perhaps the most spectacular of these is Bondarko [6], which uses the full strength of the respresentability theorems to extablish a bijection between semiorthogonal decompositions of $\mathbf{D}^{\mathrm{perf}}(X)$ and of $\mathbf{D}^b_{\mathrm{coh}}(X)$. This amounts to a major improvement on earlier, wonderful results by Karmazyn, Kuznetsov and Shinder [16].

We have already mentioned that part of the interest of the paper is that natural objects, like the subcategories $\mathbf{D}^b_{\mathrm{coh}}(X) \subset \mathbf{D}^-_{\mathrm{coh}}(X)$ of the category $\mathcal{T} = \mathbf{D_{qc}}(X)$, have an intrinsic description. The definitions are not hard to give, we include them in the Introduction. Before all else we recall some standard notation.

Reminder 1.12 Let $\mathcal{T}$ be a triangulated category. We define

(i) If $\mathcal{A} \subset \mathcal{T}$ is a full subcategory, then $\mathrm{smd}(\mathcal{A})$ is the full subcategory of all direct summands of objects of $\mathcal{A}$.

(ii) If $\mathcal{A} \subset \mathcal{T}$ is a full subcategory, then $\mathrm{add}(\mathcal{A})$ is the full subcategory of all finite direct sums of objects of $\mathcal{A}$.

(iii) If $\mathcal{T}$ has small coproducts and $\mathcal{A} \subset \mathcal{T}$ is a full subcategory, then $\mathrm{Add}(\mathcal{A})$ is the full subcategory of all coproducts of objects of $\mathcal{A}$.

(iv) If $\mathcal{A}, \mathcal{B}$ are two full subcategories of $\mathcal{T}$, then $\mathcal{A} \star \mathcal{B}$ is the full subcategory of all objects $y \in \mathcal{T}$ such that there exists a triangle $a \longrightarrow y \longrightarrow b \longrightarrow$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

(v) Given an object $G \in \mathcal{T}$ and two integers $A \le B$, let $\mathcal{C} \subset \mathcal{T}$ be the full subcategory with objects $\{\Sigma^{-i}G \mid A \le i \le B\}$. For integers $n > 0$ we define the subcategories $\mathrm{coprod}_n\big(G[A,B]\big)$, inductively on the integer n, by the formulas

$$\begin{aligned} \mathrm{coprod}_1\big(G[A,B]\big) &= \mathrm{add}(\mathcal{C})\ , \\ \mathrm{coprod}_{n+1}\big(G[A,B]\big) &= \mathrm{coprod}_1\big(G[A,B]\big) \star \mathrm{coprod}_n\big(G[A,B]\big)\ , \\ \mathrm{coprod}\big(G[A,B]\big) &= \textstyle\bigcup_{n=1}^{\infty} \mathrm{coprod}_n\big(G[A,B]\big)\ . \end{aligned}$$

(vi) Given an object $G \in \mathcal{T}$ and three integers $A \le B$ and $n > 0$, we define the subcategories $\langle G\rangle_n^{[A,B]}$ by the formula $\langle G\rangle_n^{[A,B]} = \mathrm{smd}\big[\mathrm{coprod}_n\big(G[A,B]\big)\big]$.

(vii) We adopt the following conventions:

$$\langle G\rangle_n^{(-\infty,B]} = \bigcup_A \langle G\rangle_n^{[A,B]}, \qquad \langle G\rangle_n^{[A,\infty)} = \bigcup_B \langle G\rangle_n^{[A,B]},$$

$$\langle G\rangle_n = \bigcup_{A\le B} \langle G\rangle_n^{[A,B]}, \qquad \langle G\rangle = \bigcup_{n>0} \langle G\rangle_n,$$

$$\langle G\rangle^{[A,B]} = \bigcup_{n>0} \langle G\rangle_n^{[A,B]}, \qquad \langle G\rangle^{(-\infty,B]} = \bigcup_A \langle G\rangle^{[A,B]},$$

$$\langle G\rangle^{[A,\infty)} = \bigcup_B \langle G\rangle^{[A,B]}.$$

(viii) Suppose $\mathcal{T}$ has coproducts, let G be an object, and let $A \le B$ be two integers. We define $\mathcal{C} \subset \mathcal{T}$ to be the full subcategory with objects $\{\Sigma^{-i}G \mid A \le i \le B\}$. For integers $n > 0$ we define the subcategories $\mathrm{Coprod}_n\big(G[A,B]\big)$, inductively on the integer n, by the formulas

$$\begin{aligned}\operatorname{Coprod}_1\big(G[A,B]\big) &= \operatorname{Add}(\mathcal{C})\ ,\\ \operatorname{Coprod}_{n+1}\big(G[A,B]\big) &= \operatorname{Coprod}_1\big(G[A,B]\big)\star\operatorname{Coprod}_n\big(G[A,B]\big)\ .\end{aligned}$$

In other words the difference between Coprod and coprod is that in Coprod we allow infinite coproducts in the formation of Coprod_1. The inductive procedure is unaltered.

(ix) We allow A and B to be infinite in (viii). For example $\operatorname{Coprod}_1\big(G(-\infty,B]\big)$ is defined to be $\operatorname{Add}(\mathcal{C})$ with $\mathcal{C}=\{\Sigma^{-i}G\mid i\leq B\}$.

(x) Let $A\leq B$ be integers, possibly infinite. Then $\operatorname{Coprod}\big(G[A,B]\big)$ is the smallest full subcategory $\mathcal{S}\subset\mathcal{T}$, closed under coproducts, with $\mathcal{S}\star\mathcal{S}\subset\mathcal{S}$, and with $\Sigma^{-i}G\in\mathcal{S}$ for $A\leq i\leq B$.

(xi) For triples of integers $A\leq B$ and $n>0$, we let
$\overline{\langle G\rangle}_n^{[A,B]}=\operatorname{smd}\big[\operatorname{Coprod}_n\big(G[A,B]\big)\big]$.
In this formula we also allow A and B to be infinite.

(xii) For pairs of integers $A\leq B$ we let $\overline{\langle G\rangle}^{[A,B]}=\operatorname{smd}\big[\operatorname{Coprod}\big(G[A,B]\big)\big]$. In this formula we also allow A and B to be infinite, but as it happens for infinite A we obtain nothing new. The categories

$$\operatorname{Coprod}\big(G(-\infty,B]\big),\qquad \operatorname{Coprod}\big(G(-\infty,\infty)\big)$$

are closed under coproducts and (positive) suspensions, and therefore contain all direct summands of their objects.

The following lemma is an easy consequence of the definitions.

Lemma 1.13 *Suppose G, H are objects in a triangulated category $\mathcal{T}$. We show*

(i) *If $H\in\langle G\rangle$ then there exists an integer $A>0$ with $H\in\langle G\rangle_A^{[-A,A]}$.*

(ii) *If $\langle G\rangle=\langle H\rangle$ then there exists an integer $A>0$ with $H\in\langle G\rangle_A^{[-A,A]}$ and with $G\in\langle H\rangle_A^{[-A,A]}$.*

Proof For (i) the assumption is $H\in\langle G\rangle=\cup_{A>0}\langle G\rangle_A^{[-A,A]}$, hence H belongs to one of the sets in the union. For (ii) observe that $\langle G\rangle=\langle H\rangle$ implies that $H\in\langle G\rangle$ and $G\in\langle H\rangle$ and apply (i). ☐

Now we come to the first new definition.

Definition 1.14 Suppose we are given two t–structures on a triangulated category $\mathcal{T}$, that is we are given two pairs of subcategories $(\mathcal{T}_1^{\leq 0},\mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0},\mathcal{T}_2^{\geq 0})$ satisfying the conditions in [3, Définition 1.3.1]. These t–structures are *equivalent* if and only if there exists an integer $A>0$ with $\mathcal{T}_1^{\leq -A}\subset\mathcal{T}_2^{\leq 0}\subset\mathcal{T}_1^{\leq A}$.

Observation 1.15 *For any t–structure $(\mathcal{T}^{\leq 0},\mathcal{T}^{\geq 0})$ we have $\mathcal{T}^{\leq 0}=\Sigma^{-1}\big({}^\perp\mathcal{T}^{\geq 0}\big)$ and $\mathcal{T}^{\geq 0}=(\Sigma\mathcal{T}^{\leq 0})^\perp$. It immediately follows that two t–structures $(\mathcal{T}_1^{\leq 0},\mathcal{T}_1^{\geq 0})$*

and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ *are equivalent if and only if there exists an integer* $A > 0$ *with* $\mathcal{T}_1^{\geq A} \subset \mathcal{T}_2^{\geq 0} \subset \mathcal{T}_1^{\geq -A}$.

Observation 1.16 *Recall that, for any t–structure* $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, *the categories* $\mathcal{T}^-$, $\mathcal{T}^+$ *and* $\mathcal{T}^b$ *are defined by*

$$\mathcal{T}^- = \cup_{m>0}\mathcal{T}^{\leq m}\ , \qquad \mathcal{T}^+ = \cup_{m>0}\mathcal{T}^{\geq -m}\ , \qquad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+\ .$$

If $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ *and* $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ *are equivalent t–structures we note*

- (i) $\mathcal{T}_1^- = \mathcal{T}_2^-$, $\mathcal{T}_1^+ = \mathcal{T}_2^+$ *and* $\mathcal{T}_1^b = \mathcal{T}_2^b$.
- (ii) *If* $\mathcal{T}^-$ [*respectively* $\mathcal{T}^+$, *respectively* $\mathcal{T}^b$] *contains a compact generator* $G \in \mathcal{T}^c$, *then* $\mathcal{T}^-$ [*respectively* $\mathcal{T}^+$, *respectively* $\mathcal{T}^b$] *contains all of* $\mathcal{T}^c$.

Proof We prove (i) and (ii) for $\mathcal{T}^-$ and leave $\mathcal{T}^+$ and $\mathcal{T}^b$ to the reader. To prove (i) observe that the inclusions $\mathcal{T}_1^{\leq -A} \subset \mathcal{T}_2^{\leq 0} \subset \mathcal{T}_1^{\leq A}$ imply

$$\bigcup_{m>0} \mathcal{T}_1^{\leq -A+m} \qquad \subset \qquad \bigcup_{m>0} \mathcal{T}_2^{\leq m} \qquad \subset \qquad \bigcup_{m>0} \mathcal{T}_1^{\leq A+m}$$

that is $\mathcal{T}_1^- \subset \mathcal{T}_2^- \subset \mathcal{T}_1^-$.

For the proof of (ii) the assumption is that $G \in \mathcal{T}^-$. This makes $\mathcal{T}^- \subset \mathcal{T}$ a thick subcategory containing G, hence $\mathcal{T}^c = \langle G \rangle \subset \mathcal{T}^-$. $\square$

Example 1.17 Let $\mathcal{T}$ be a triangulated category with coproducts. Given any compact object $G \in \mathcal{T}$, from Alonso, Jeremías and Souto [25, Theorem A.1] we learn that $\mathcal{T}$ has a unique t–structure $(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0})$ *generated* by G. In the notation of Reminder 1.12, the aisle $\mathcal{T}_G^{\leq 0}$ of this t–structure is nothing other than $\mathcal{T}_G^{\leq 0} = \overline{\langle G \rangle}^{(-\infty,0]}$. It follows formally that both $\mathcal{T}_G^{\leq 0}$ and $\mathcal{T}_G^{\geq 0}$ are closed under coproducts and direct summands—the closure under direct summands is true for any aisle and co-aisle of a t–structure, the closure of $\mathcal{T}_G^{\leq 0}$ under coproducts is also true for any aisle, while the fact that $\mathcal{T}_G^{\geq 0}$ is closed under coproducts may be found in [25, Proposition A.2]; it comes from the compactness of the object G.

If G, H are two compact objects of $\mathcal{T}$ with $\langle G \rangle = \langle H \rangle$, Lemma 1.13(ii) tells us that there exists an integer $A > 0$ with $H \in \langle G \rangle_A^{[-A,A]}$ and $G \in \langle H \rangle_A^{[-A,A]}$. Hence $\overline{\langle H \rangle}^{(-\infty,-A]} \subset \overline{\langle G \rangle}^{(-\infty,0]} \subset \overline{\langle H \rangle}^{(-\infty,A]}$, that is $\mathcal{T}_H^{\leq -A} \subset \mathcal{T}_G^{\leq 0} \subset \mathcal{T}_H^{\leq A}$. Thus the t–structures generated by G and H are equivalent. This leads us to

Definition 1.18 If the compactly generated triangulated category $\mathcal{T}$ has a single compact object G that generates it, then the *preferred equivalence class of t–structures* is the one containing the t–structure $\left(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\right)$ generated by G.

Remark 1.19 For any compact generator G we have that $\langle G \rangle = \mathcal{T}^c$, the full subcategory of all compact objects. Any two compact generators G, H satisfy $\langle G \rangle = \mathcal{T}^c = \langle H \rangle$, and Example 1.17 says that G and H generate equivalent t–structures. Thus the preferred equivalence class of t–structures does not depend on the choice of compact generator.

Now [25, Proposition A.2] guarantees that, in the preferred equivalence class, there will exist some t–structures with $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ both closed under coproducts—just take $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ for a compact generator G. The reader should note that this property is *not* stable under equivalence. In general there will be t–structures in the preferred equivalence class where $\mathcal{T}^{\geq 0}$ is not closed in $\mathcal{T}$ under coproducts.

From Observation 1.16(i) we learn that, as long as we stick to the preferred equivalence class of t–structures, the categories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ are intrinsic.

And now for the next formal construction.

Definition 1.20 Suppose $\mathcal{T}$ is a triangulated category with coproducts and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t–structure.

An object F belongs to the subcategory $\mathcal{T}_c^- \subset \mathcal{T}$ if, for any integer $m > 0$, there exists a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\leq -m}$.

The subcategory $\mathcal{T}_c^b$ is defined by $\mathcal{T}_c^b = \mathcal{T}_c^- \cap \mathcal{T}^b$.

Remark 1.21 Note that the definition of $\mathcal{T}_c^-$ depends on the choice of a t–structure, but not much—equivalent t–structures lead to the same $\mathcal{T}_c^-$. For any choice of t–structure the category $\mathcal{T}_c^-$ contains $\mathcal{T}^c$. After all if F is compact then the triangle $F \xrightarrow{\text{id}} F \longrightarrow 0$ has $F \in \mathcal{T}^c$ and $0 \in \mathcal{T}^{\leq -m}$, for every m and every t–structure.

Remark 1.22 Assume the t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is such that there is a compact generator G contained in $\mathcal{T}^-$; any t–structure in the preferred equivalence class is an example, after all $G \in \overline{\langle G \rangle}^{(-\infty,0]} = \mathcal{T}_G^{\leq 0} \subset \mathcal{T}^-$. Observation 1.16(ii) gives that $\mathcal{T}^c \subset \mathcal{T}^-$, and Definition 1.20 tells us that, for any integer $m > 0$,

$$\mathcal{T}_c^- \quad \subset \quad \mathcal{T}^c * \mathcal{T}^{\leq -m} \quad \subset \quad \mathcal{T}^- * \mathcal{T}^- \quad = \quad \mathcal{T}^- .$$

Still in gorgeous generality we will prove

Proposition 1.23 *Let $\mathcal{T}$ be a triangulated category with coproducts, and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t–structure. If there exists an integer $A > 0$ and a compact generator $G \in \mathcal{T}$ with $\operatorname{Hom}\big(\Sigma^{-A}G, \mathcal{T}^{\leq 0}\big) = 0$ then $\mathcal{T}_c^b \subset \mathcal{T}_c^-$ are triangulated subcategories of $\mathcal{T}$. If furthermore $G \in \mathcal{T}^-$, then $\mathcal{T}_c^b \subset \mathcal{T}_c^- \subset \mathcal{T}$ are thick subcategories of $\mathcal{T}^-$.*

Remark 1.24 We are most interested in the special case where the t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is in the preferred equivalence class and $\mathcal{T}_c^b \subset \mathcal{T}_c^-$ are independent of choices.

Suppose there exists a compact generator G and an integer $A > 0$, so that $\operatorname{Hom}(G, \Sigma^i G) = 0$ for all $i \geq A$. Define the full subcategory $\mathcal{S}$ by

$$\mathcal{S} \quad = \quad \{S \in \mathcal{T} \mid \operatorname{Hom}(\Sigma^{-A}G, S) = 0\}.$$

The compactness of G says that $\mathcal{S}$ is closed under coproducts, by hypothesis $\mathcal{S}$ contains $\Sigma^i G$ for all $i \geq 0$, while obviously $\mathcal{S}$ is closed under direct summands and $\mathcal{S} * \mathcal{S} \subset \mathcal{S}$. Therefore $\mathcal{S}$ contains $\overline{\langle G \rangle}^{(-\infty,0]} = \mathcal{T}_G^{\leq 0}$. We deduce that

$\mathrm{Hom}\big(\Sigma^{-A}G, \mathcal{T}_G^{\leq 0}\big) = 0$. Since G is obviously in $\mathcal{T}_G^{\leq 0} \subset \mathcal{T}^-$, Proposition 1.23 informs us that $\mathcal{T}_c^b \subset \mathcal{T}_c^-$ are thick subcategories of $\mathcal{T}^-$.

For the structure defined so far we needed very little. To go further it turns out to be useful to estimate how much effort it takes to approximate an object in $\mathcal{T}^-$ by a compact generator G. This leads us to

Definition 1.25 Let $\mathcal{T}$ be a triangulated category with coproducts. The category $\mathcal{T}$ is called *weakly approximable* if there exists a compact generator G, a *t*–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and an integer $A > 0$ so that

(i) $\Sigma^A G \in \mathcal{T}^{\leq 0}$ and $\mathrm{Hom}\big(\Sigma^{-A}G, \mathcal{T}^{\leq 0}\big) = 0$.
(ii) Every object $F \in \mathcal{T}^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \overline{\langle G\rangle}^{[-A,A]}$ and $D \in \mathcal{T}^{\leq -1}$.

The category $\mathcal{T}$ is called *approximable* if the integer A can be chosen to further satisfy

(iii) In the triangle $E \longrightarrow F \longrightarrow D$ of (ii) above we may strengthen the condition on E, we may assume $E \in \overline{\langle G\rangle}_A^{[-A,A]} \subset \overline{\langle G\rangle}^{[-A,A]}$.

The following are easy to prove, they will be part of a string of formal consequences of approximability, see Sect. 3.

Facts 1.26 *Let $\mathcal{T}$ be a triangulated category with coproducts. If $\mathcal{T}$ is weakly approximable then*

(i) *The t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, which is part of Definition 1.25 and is assumed to satisfy some hypotheses, must belong to the preferred equivalence class.*
(ii) *For any compact generator G and any t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class there must exist an integer A, depending on G and on the t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, which satisfies Definition 1.25(i) and (ii). If $\mathcal{T}$ is approximable the integer A may be chosen to satisfy (iii) as well.*

Thus in proving that $\mathcal{T}$ is (weakly) approximable we can choose our compact generator and t–structure to suit our convenience. Once we know the category is approximable, it follows that the convenient t–structure is in the preferred class, and any compact generator and any t–structure in the preferred equivalence class fulfill the approximability criteria.

Facts 1.27 *As stated in the first few paragraphs of the introduction [before we presented the definitions] we will prove that, if X is a quasicompact, separated scheme, then $\mathcal{T} = \mathbf{D}_{qc}(X)$ is approximable and the standard t–structure is in the preferred equivalence class. If X is only quasiseparated, then the reader is referred to [21, Theorem 3.2 (iii) and (iv)] for the fact that $\mathcal{T} = \mathbf{D}_{qc}(X)$ is weakly approximable and the standard t–structure is in the preferred equivalence class. In the special case where X is noetherian then $\mathcal{T}_c^b \subset \mathcal{T}_c^-$ are just $\mathbf{D}^b_{\mathrm{coh}}(X) \subset \mathbf{D}^-_{\mathrm{coh}}(X)$. For non-noetherian X the description of $\mathcal{T}_c^b \subset \mathcal{T}_c^-$ is slightly more complicated, but still classical—see*

Remark 5.8 *for details. The fact that the standard t–structure is in the preferred equivalence class tells us that* $\mathcal{T}^- = \mathbf{D}^-_{\boldsymbol{qc}}(X)$, $\mathcal{T}^+ = \mathbf{D}^+_{\boldsymbol{qc}}(X)$ *and* $\mathcal{T}^b = \mathbf{D}^b_{\boldsymbol{qc}}(X)$.

Another example is the homotopy category $\mathcal{T}$ *of spectra. In this case we can take* $\mathcal{T}^{\leq 0} \subset \mathcal{T}$ *to be the subcategory of connective spectra—the t–structure this defines is in the preferred equivalence class. The category* $\mathcal{T}$ *turns out to be approximable, and the subcategory* $\mathcal{T}^-_c$ *is the category of spectra* X *whose stable homotopy groups* $\pi_i(X)$ *are finitely generated* $\mathbb{Z}$*–modules and* $\pi_i(X) = 0$ *for* $i \ll 0$. *And* $\mathcal{T}^b_c \subset \mathcal{T}^-_c$ *is the subcategory where all but finitely many of the* $\pi_i(X)$ *vanish.*

The representability we prove in Theorem 1.4 *applies to this example but the result is not new. There is a theorem of Adams* [1] *which says that every cohomological functor* H *on* $\mathcal{T}^c$ *is the restriction of a representable one on* $\mathcal{T}$, *and it is easy to show that finiteness or local finiteness of* H *translate to saying that the representing object must lie in* $\mathcal{T}^b_c$ *or* $\mathcal{T}^-_c$. *But the theorem of Adams does not generalize to* $\mathbf{D}^{\mathrm{perf}}(X) \subset \mathbf{D}_{\boldsymbol{qc}}(X)$; *see* [10, 19].

If X *is a quasicompact, quasiseparated scheme and* $Z \subset X$ *is a closed subset with quasicompact complement, then the category* $\mathcal{T} = \mathbf{D}_{\boldsymbol{qc},Z}(X)$, *the subcategory of* $\mathbf{D}_{\boldsymbol{qc}}(X)$ *of all complexes supported on* Z, *turns out to be weakly approximable but not* (*in general*) *approximable. The proof of the weak approximability is in* [21, *Theorem* 3.2(*iv*)], *and an example showing that approximability fails in general is given in* [21, *Remark* 8.1]. *In* [21, *Theorem* 3.2(*iii*)] *the reader will learn that the standard t–structure is in the preferred equivalence class. If* X *is noetherian the categories* $\mathcal{T}^-_c$ *and* $\mathcal{T}^b_c$ *are* (*respectively*) *the intersections of* $\mathbf{D}^-_{\mathrm{coh}}(X)$ *and* $\mathbf{D}^b_{\mathrm{coh}}(X)$ *with the category* $\mathbf{D}_{\boldsymbol{qc},Z}(X)$; *the reader can find this in* [21, *second paragraph of Sect.* 8].

The definitions have all been made and the reader can go back to the statements of Theorem 1.4 and Corollary 1.6, which are now precise. Note that in both results $\mathcal{T}$ has to be approximable, weakly approximable is not enough.

We have discussed what we know, but should point out that there are many more potential examples. After all: let R be a commutative ring and let T be a dg R–algebra. Then the category $\mathcal{T} = \mathbf{D}(T\text{–Mod})$ is a triangulated category with coproducts and a single compact generator T. It has a preferred equivalence class of t–structures, one can define the intrinsic subcategories $\mathcal{T}^-$, $\mathcal{T}^+$, $\mathcal{T}^b$, $\mathcal{T}^-_c$ and $\mathcal{T}^b_c$, and in general I have no idea what they are. If $H^i(T) = 0$ for $i \gg 0$ then the subcategories $\mathcal{T}^-_c$ and $\mathcal{T}^b_c$ are thick, this follows from Remark 1.24. If $H^i(T) = 0$ for all $i \geq 1$ we are in the trivial case (see Remark 5.3), where it's easy to prove the category $\mathbf{D}(T\text{–Mod})$ to be approximable and work out explicitly what are $\mathcal{T}^-$, $\mathcal{T}^+$, $\mathcal{T}^b$, $\mathcal{T}^-_c$ and $\mathcal{T}^b_c$. More generally: if T is a dg algebra, with $H^i(T) = 0$ for all $i \geq 2$, then Bondarko and Vostokov [7, Corollary 4.3] proves that the derived category $\mathbf{D}(T\text{–Mod})$ is weakly approximable. So far the only other general result, producing further examples of approximable triangulated categories, is [8, Theorem 4.1]. It says that, under reasonable hypotheses, the recollement of two approximable triangulated categories is approximable. But for T a general dga, satisfying $H^i(T) = 0$ for $i \gg 0$, I have no idea when the categories $\mathbf{D}(T\text{–Mod})$ are approximable or weakly approximable. In view of Theorem 1.4 and Corollary 1.6 it would be interesting to find out, especially since the categories $\mathbf{D}(T\text{–Mod})$ are of so much current active interest—their study is at the core of noncommutative algebraic geometry. Who knows, there might be a noncommutative generalization of GAGA.

2 Basics

Since t–structures will play a big part in the article we begin with a quick reminder of some elementary facts.

Reminder 2.1 In this section $\mathcal{T}$ will be a triangulated category and $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ will be a t–structure on $\mathcal{T}$. The category $\mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is abelian, it is called the heart of the t–structure. The functor $\mathcal{H}(-) = \big[(-)^{\leq 0}\big]^{\geq 0}$ is a homological functor $\mathcal{H} : \mathcal{T} \longrightarrow \mathcal{A}$. We will let $\mathcal{H}^\ell$ be the functor $\mathcal{H}^\ell(-) = \mathcal{H}\big[\Sigma^\ell(-)\big] = \Sigma^\ell\big[(-)^{\leq \ell}\big]^{\geq \ell}$.

Lemma 2.2 *Let $\mathcal{T}$ be a triangulated category and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t–structure on $\mathcal{T}$. If F is an object of $\mathcal{T}^-$, and $\mathcal{H}^\ell(F) = 0$ for all $\ell > -i$, then F belongs to $\mathcal{T}^{\leq -i}$.*

Proof We are given that F belongs to $\mathcal{T}^- = \cup_n \mathcal{T}^{\leq n}$, hence $F \in \mathcal{T}^{\leq n}$ for some n and the map $F^{\leq n} \longrightarrow F$ is an isomorphism. But now the triangle $F^{\leq \ell-1} \longrightarrow F^{\leq \ell} \longrightarrow \Sigma^{-\ell}\mathcal{H}^\ell(F)$ informs us that, as long as $\ell > -i$, the map $F^{\leq \ell-1} \longrightarrow F^{\leq \ell}$ is also an isomorphism. Composing the string of isomorphisms $F^{\leq -i} \longrightarrow F^{\leq -i+1} \longrightarrow \cdots \longrightarrow F^{\leq n} \longrightarrow F$ we have that $F^{\leq -i} \longrightarrow F$ is an isomorphism—therefore $F \in \mathcal{T}^{\leq -i}$. ☐

Lemma 2.3 *If there is an integer A and a generator $G \in \mathcal{T}$ with* $\mathrm{Hom}\big(G, \mathcal{T}^{\leq -A}\big) = 0$, *then*

- (i) *Any object $F \in \mathcal{T}^-$, with $\mathcal{H}^\ell(F) = 0$ for all ℓ, must vanish.*
- (ii) *If $f : E \longrightarrow F$ is a morphism in $\mathcal{T}^-$ such that $\mathcal{H}^\ell(f)$ is an isomorphism for every $\ell \in \mathbb{Z}$, then f is an isomorphism.*

Proof To prove (i) assume $\mathcal{H}^\ell(F) = 0$ for all ℓ; Lemma 2.2 says that F belongs to $\cap_\ell \mathcal{T}^{\leq \ell}$. But then $\mathrm{Hom}(\Sigma^i G, F) = 0$ for all $i \in \mathbb{Z}$, and as G is a generator this implies $F = 0$.

(ii) follows by applying (i) to the mapping cone of f. ☐

Lemma 2.4 *Suppose the category $\mathcal{T}$ has coproducts, and the t–structure is such that both $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ are closed under the coproducts of $\mathcal{T}$. Then:*

- (i) *The functors $(-)^{\leq 0}$ and $(-)^{\geq 0}$ both respect coproducts.*
- (ii) *The heart $\mathcal{A} \subset \mathcal{T}$ is closed in $\mathcal{T}$ under coproducts, and the functor $\mathcal{H} : \mathcal{T} \longrightarrow \mathcal{A}$ respects coproducts.*
- (iii) *The abelian category $\mathcal{A}$ satisfies* [*AB*4], *that is coproducts are exact.*
- (iv) *If $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ is a sequence of objects and morphisms in $\mathcal{T}$, then there is a short exact sequence in the heart $\mathcal{A}$ of the t–structure*

$$0 \to \underrightarrow{\mathrm{colim}}\,\mathcal{H}^\ell(E_i) \longrightarrow \mathcal{H}^\ell\Big(\underrightarrow{\mathit{Hocolim}}\ E_i\Big) \longrightarrow \underrightarrow{\mathrm{colim}}^1\,\mathcal{H}^{\ell+1}(E_i) \to 0$$

Proof Suppose we are given in $\mathcal{T}$ a collection of objects $\{E_\lambda,\ \lambda\in\Lambda\}$. For each λ we have a canonical triangle $E_\lambda^{\leq 0}\longrightarrow E_\lambda\longrightarrow E_\lambda^{\geq 1}\longrightarrow \Sigma E_\lambda^{\leq 0}$. The coproduct of these triangles is a triangle

$$\bigoplus_{\lambda\in\Lambda} E_\lambda^{\leq 0}\longrightarrow \bigoplus_{\lambda\in\Lambda} E_\lambda\longrightarrow \bigoplus_{\lambda\in\Lambda} E_\lambda^{\geq 1}\longrightarrow \bigoplus_{\lambda\in\Lambda} \Sigma E_\lambda^{\leq 0}$$

By hypothesis $\oplus_{\lambda\in\Lambda} E_\lambda^{\leq 0}$ belongs to $\mathcal{T}^{\leq 0}$ and $\oplus_{\lambda\in\Lambda} E_\lambda^{\geq 1}$ belongs to $\mathcal{T}^{\geq 1}$, and the triangle above must be canonically isomorphic to

$$\left(\bigoplus_{\lambda\in\Lambda} E_\lambda\right)^{\leq 0}\longrightarrow \bigoplus_{\lambda\in\Lambda} E_\lambda\longrightarrow \left(\bigoplus_{\lambda\in\Lambda} E_\lambda\right)^{\geq 1}\longrightarrow \left(\bigoplus_{\lambda\in\Lambda} \Sigma E_\lambda\right)^{\leq 0}$$

This proves (i).

Since $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ are closed in $\mathcal{T}$ under coproducts so is their intersection $\mathcal{A}=\mathcal{T}^{\leq 0}\cap\mathcal{T}^{\geq 0}$. By (i) we know that the functors $(-)^{\leq 0}$ and $(-)^{\geq 0}$ both respect coproducts, hence so does their composition $\mathcal{H}(-)=\big[(-)^{\leq 0}\big]^{\geq 0}$. This proves (ii).

The category $\mathcal{T}$ has coproducts and its subcategory $\mathcal{A}$ is closed under these coproducts, hence $\mathcal{A}$ has coproducts—it satisfies [AB3]. Now suppose we are given a set $\{f_\lambda : A_\lambda\to B_\lambda,\ \lambda\in\Lambda\}$ of morphisms in $\mathcal{A}$. Complete these to triangles $A_\lambda\longrightarrow B_\lambda\longrightarrow C_\lambda\longrightarrow \Sigma A_\lambda$ and form the coproduct

$$\bigoplus_{\lambda\in\Lambda} A_\lambda\xrightarrow{\ \bigoplus_{\lambda\in\Lambda} f_\lambda\ }\bigoplus_{\lambda\in\Lambda} B_\lambda\longrightarrow \bigoplus_{\lambda\in\Lambda} C_\lambda\longrightarrow \bigoplus_{\lambda\in\Lambda} \Sigma A_\lambda$$

which is a triangle. The long exact sequence obtained by applying $\mathcal{H}$ to this triangle tells us that the kernel of the map $\bigoplus_{\lambda\in\Lambda} f_\lambda$ is $\mathcal{H}^{-1}\big(\bigoplus_{\lambda\in\Lambda} C_\lambda\big)$, but (ii) informs us that this is $\bigoplus_{\lambda\in\Lambda}\mathcal{H}^{-1}(C_\lambda)$, which is $\bigoplus_{\lambda\in\Lambda}\mathrm{Ker}(f_\lambda)$. The right exactness of coproducts is formal, completing the proof of (iii).

Finally (iv) follows by applying the functor $\mathcal{H}$ to the triangle

$$\bigoplus_{i=1}^{\infty} E_i\longrightarrow \bigoplus_{i=1}^{\infty} E_i\longrightarrow \underrightarrow{\mathrm{Hocolim}}\ E_i\longrightarrow \bigoplus_{i=1}^{\infty} \Sigma E_i$$

and using (ii) to compute the long exact sequence. ☐

Remark 2.5 Remark 1.19 tells us that, if $\mathcal{T}$ is a triangulated category with coproducts and a single compact generator, then the preferred equivalence class contains t–structures $(\mathcal{T}^{\leq 0},\mathcal{T}^{\geq 0})$ with $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ both closed under coproducts. This is the situation in which we will apply Lemma 2.4. Note also that Remark 1.19 warns us that not every t–structure in the preferred equivalence class needs to satisfy this property.

We will mostly use Lemma 2.4(iv) in the special case where the sequences $\mathcal{H}^\ell(E_1) \longrightarrow \mathcal{H}^\ell(E_2) \longrightarrow \mathcal{H}^\ell(E_3) \longrightarrow \cdots$ eventually stabilize for every ℓ. When this happens the $\underrightarrow{\mathrm{colim}}^1$ terms all vanish, and the natural map is an isomorphism $\underrightarrow{\mathrm{colim}}\, \mathcal{H}^\ell(E_i) \longrightarrow \mathcal{H}^\ell\big(\, \underrightarrow{\mathrm{Hocolim}}\ E_i\big)$.

Remark 2.6 We should note that, in the special case where the *t*–structure is compactly generated (for example the *t*–structure generated by a single compact object), then much more is now known. As the reader can see in Saorín and Šťovíček [23, Theorem 8.31], the heart of a compactly generated *t*–structure has to be a locally finitely presented Grothendieck abelian category.

The proof of this recent theorem, by Saorín and Šťovíček, is substantially more involved than the simple-minded, short argument we presented in this section. And for us, in this article, the results of the current section amply cover what we will need.

3 The fundamental properties of approximability

Lemma 3.1 *Let $\mathcal{T}$ be a triangulated category with a t–structure $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$, and let $\mathcal{S} \subset \mathcal{T}$ be a full subcategory with $\Sigma\mathcal{S} = \mathcal{S}$. Assume $\mathcal{A}$ is also a full subcategory of $\mathcal{T}$, and define $\mathcal{A}(m)$ inductively by*

(i) $\mathcal{A}(1) = \mathcal{A}$.
(ii) $\mathcal{A}(m+1) = \mathcal{A}(m) \star \Sigma^m\mathcal{A}$.

Suppose every object in $F \in \mathcal{S} \cap \mathcal{T}^{\le 0}$ admits a triangle $E_1 \longrightarrow F \longrightarrow D_1$, with $E_1 \in \mathcal{A}$ and $D_1 \in \mathcal{S} \cap \mathcal{T}^{\le -1}$. Then we can construct a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$, with a map from the sequence to F and so that, if we complete $E_m \longrightarrow F$ to a triangle $E_m \longrightarrow F \longrightarrow D_m$, then $E_m \in \mathcal{A}(m)$ and $D_m \in \mathcal{S} \cap \mathcal{T}^{\le -m}$.

Proof We are given the case $m = 1$; assume we have constructed the sequence as far as an integer $m > 0$, and we want to extend it to $m + 1$. Take any object $F \in \mathcal{S} \cap \mathcal{T}^{\le 0}$, and by the inductive hypothesis construct the sequence up to m. In particular choose a triangle $E_m \longrightarrow F \longrightarrow D_m$ with $E_m \in \mathcal{A}(m)$ and $D_m \in \mathcal{S} \cap \mathcal{T}^{\le -m}$. Now apply the case $m = 1$ to $\Sigma^{-m} D_m$; we produce a triangle $E' \longrightarrow D_m \longrightarrow D_{m+1}$ with $D_{m+1} \in \mathcal{S} \cap \mathcal{T}^{\le -m-1}$ and $E' \in \Sigma^m\mathcal{A}$. Form an octahedron from the composable morphisms $F \longrightarrow D_m \longrightarrow D_{m+1}$, that is

$$\begin{tikzcd}
E_m \arrow[r, equal] \arrow[d] & E_m \arrow[d] & \\
E_{m+1} \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & D_{m+1} \arrow[d, equal] \\
E' \arrow[r] & D_m \arrow[r] & D_{m+1}
\end{tikzcd}$$

The object D_{m+1} belongs to $\mathcal{S} \cap \mathcal{T}^{\leq -m-1}$ by construction. The triangle $E_m \longrightarrow E_{m+1} \longrightarrow E'$ tells us that $E_{m+1} \in \mathcal{A}(m) \star \Sigma^m \mathcal{A} = \mathcal{A}(m+1)$, and we have factored the map $E_m \longrightarrow F$ as $E_m \longrightarrow E_{m+1} \longrightarrow F$ so that, in the triangle $E_{m+1} \longrightarrow F \longrightarrow D_{m+1}$, we have $E_{m+1} \in \mathcal{A}(m+1)$ and $D_{m+1} \in \mathcal{S} \cap \mathcal{T}^{\leq -m-1}$. ☐

Corollary 3.2 *Let $\mathcal{T}$ be a triangulated category with coproducts, let $G \in \mathcal{T}$ be an object, and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t–structure. The following is true.*

3.2.1 *Suppose every object $F \in \mathcal{T}^{\leq 0}$ admits a triangle $E_1 \longrightarrow F \longrightarrow D_1$, with $E_1 \in \overline{\langle G \rangle}^{[-A,A]}$ and $D_1 \in \mathcal{T}^{\leq -1}$. Then we can extend to a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$, with a map from the sequence to F and so that, if we complete $E_m \longrightarrow F$ to a triangle $E_m \longrightarrow F \longrightarrow D_m$, then $E_m \in \overline{\langle G \rangle}^{[1-m-A,A]}$ and $D_m \in \mathcal{T}^{\leq -m}$.*

3.2.2 *Suppose every object $F \in \mathcal{T}^{\leq 0}$ admits a triangle $E_1 \longrightarrow F \longrightarrow D_1$, with $E_1 \in \overline{\langle G \rangle}_A^{[-A,A]}$ and $D_1 \in \mathcal{T}^{\leq -1}$. Then we can extend to a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$, with a map from the sequence to F and so that, if we complete $E_m \longrightarrow F$ to a triangle $E_m \longrightarrow F \longrightarrow D_m$, then $E_m \in \overline{\langle G \rangle}_{mA}^{[1-m-A,A]}$ and $D_m \in \mathcal{T}^{\leq -m}$.*

3.2.3 *For a full subcategory $\mathcal{S} \subset \mathcal{T}$ with $\Sigma\mathcal{S} = \mathcal{S}$, suppose every object $F \in \mathcal{S} \cap \mathcal{T}^{\leq 0}$ admits a triangle $E_1 \longrightarrow F \longrightarrow D_1$, with $E_1 \in \langle G \rangle^{[-A,A]}$ and $D_1 \in \mathcal{S} \cap \mathcal{T}^{\leq -1}$. Then we can extend to a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$, with a map from the sequence to F and so that, if we complete $E_m \longrightarrow F$ to a triangle $E_m \longrightarrow F \longrightarrow D_m$, then $E_m \in \langle G \rangle^{[1-m-A,A]}$ and $D_m \in \mathcal{S} \cap \mathcal{T}^{\leq -m}$.*

3.2.4 *For a full subcategory $\mathcal{S} \subset \mathcal{T}$ with $\Sigma\mathcal{S} = \mathcal{S}$, suppose every object $F \in \mathcal{S} \cap \mathcal{T}^{\leq 0}$ admits a triangle $E_1 \longrightarrow F \longrightarrow D_1$, with $E_1 \in \langle G \rangle_A^{[-A,A]}$ and $D_1 \in \mathcal{S} \cap \mathcal{T}^{\leq -1}$. Then we can extend to a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$, with a map from the sequence to F and so that, if we complete $E_m \longrightarrow F$ to a triangle $E_m \longrightarrow F \longrightarrow D_m$, then $E_m \in \langle G \rangle_{mA}^{[1-m-A,A]}$ and $D_m \in \mathcal{S} \cap \mathcal{T}^{\leq -m}$.*

Proof In each case we apply Lemma 3.1 with a suitable choice of $\mathcal{A}$ and $\mathcal{S}$.

To prove (3.2.1) let $\mathcal{S} = \mathcal{T}$ and let $\mathcal{A} = \overline{\langle G \rangle}^{[-A,A]}$. By induction we see that $\mathcal{A}(m) \subset \overline{\langle G \rangle}^{[1-m-A,A]}$ and the result follows.

To prove (3.2.2) let $\mathcal{S} = \mathcal{T}$ and let $\mathcal{A} = \overline{\langle G \rangle}_A^{[-A,A]}$. By induction we see that $\mathcal{A}(m) \subset \overline{\langle G \rangle}_{mA}^{[1-m-A,A]}$ and the result follows.

To prove (3.2.3) let $\mathcal{A} = \langle G \rangle^{[-A,A]}$. By induction we see that $\mathcal{A}(m) \subset \langle G \rangle^{[1-m-A,A]}$ and the result follows.

To prove (3.2.4) let $\mathcal{A} = \langle G \rangle_A^{[-A,A]}$. By induction we see that $\mathcal{A}(m) \subset \langle G \rangle_{mA}^{[1-m-A,A]}$ and the result follows. ☐

Lemma 3.3 *Suppose $\mathcal{T}$ is a compactly generated triangulated category, G is a compact generator and $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ a t–structure. Suppose there exists an integer B with* $\mathrm{Hom}(\Sigma^{-B} G, \mathcal{T}^{\leq 0}) = 0$.

With any sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ mapping to F, and such that in the triangles $E_m \longrightarrow F \longrightarrow D_m$ we have $D_m \in \mathcal{T}^{\leq -m}$, the (non-canonical) map $\underrightarrow{\mathrm{Hocolim}}\, E_m \longrightarrow F$ is an isomorphism.

Proof For any $n \geq 0$ we have $\mathcal{T}^{\leq -n} \subset \mathcal{T}^{\leq 0}$, hence $\operatorname{Hom}(\Sigma^{-B}G, \mathcal{T}^{\leq -n}) = 0$. By shifting we deduce that $\operatorname{Hom}(\Sigma^{-\ell}G, \mathcal{T}^{\leq -m}) = 0$ as long as $m + \ell \geq B$.

The triangle $E_m \longrightarrow F \longrightarrow D_m$, with $D_m \in \mathcal{T}^{\leq -m}$, tells us that if $m > \max(1, B - \ell)$ then the functor $\operatorname{Hom}(\Sigma^{-\ell}G, -)$ takes the map $E_m \longrightarrow F$ to an isomorphism. Now [18, Lemma 2.8], applied to the compact object $G \in \mathcal{T}$ and the map from the sequence $\{E_m\}$ to F, tells us that $\operatorname{Hom}(\Sigma^{-\ell}G, -)$ takes the map $\underrightarrow{\operatorname{Hocolim}}\, E_m \longrightarrow F$ to an isomorphism. But G is a generator, hence the map $\underrightarrow{\operatorname{Hocolim}}\, E_m \longrightarrow F$ must be an isomorphism. ☐

Proposition 3.4 *Suppose the triangulated category $\mathcal{T}$, the generator G and the t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ are as in the hypotheses of weakly approximable categories of Definition 1.25. We remind the reader: $\mathcal{T}$ has coproducts, G is a compact generator, and there is an integer $A > 0$ so that*

(i) $\Sigma^A G \in \mathcal{T}^{\leq 0}$ *and* $\operatorname{Hom}(\Sigma^{-A}G, \mathcal{T}^{\leq 0}) = 0$.
(ii) *Every object $F \in \mathcal{T}^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \overline{\langle G\rangle}^{[-A,A]}$ and $D \in \mathcal{T}^{\leq -1}$.*

Then the t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is in the preferred equivalence class.

Proof By (i) we have $\Sigma^A G \in \mathcal{T}^{\leq 0}$, hence $\Sigma^m G \in \mathcal{T}^{\leq 0}$ for all $m \geq A$. Therefore $\mathcal{T}^{\leq 0}$ contains $\overline{\langle G\rangle}^{(-\infty,-A]} = \mathcal{T}_G^{\leq -A}$. It remains to show an inclusion in the other direction.

But (3.2.1) constructed, for every object $F \in \mathcal{T}^{\leq 0}$, a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ with $E_m \in \overline{\langle G\rangle}^{[1-m-A,A]} \subset \overline{\langle G\rangle}^{(-\infty,A]}$. In Lemma 3.3 we proved that F is isomorphic to $\underrightarrow{\operatorname{Hocolim}}\, E_m$. There exists a triangle

$$\begin{tikzcd}
\displaystyle\bigoplus_{m=1}^{\infty} E_m \arrow[r] & F \arrow[r] & \Sigma\left[\displaystyle\bigoplus_{m=1}^{\infty} E_m\right]
\end{tikzcd}$$

where the outside terms obviously lie in $\overline{\langle G\rangle}^{(-\infty,A]} = \mathcal{T}_G^{\leq A}$. Hence $F \in \mathcal{T}_G^{\leq A}$, and since $F \in \mathcal{T}^{\leq 0}$ is arbitrary we conclude that $\mathcal{T}^{\leq 0} \subset \mathcal{T}_G^{\leq A}$. ☐

Lemma 3.5 *Let $\mathcal{T}$ be a compactly generated triangulated category, let G be a compact generator, and let $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ be two equivalent t–structures. Let $A > 0$ be an integer so that, with $k = 1$, the conditions*

(i) $\Sigma^A G \in \mathcal{T}_k^{\leq 0}$ *and* $\operatorname{Hom}(\Sigma^{-A}G, \mathcal{T}_k^{\leq 0}) = 0$.
(ii) *Every object $F \in \mathcal{T}_k^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \overline{\langle G\rangle}^{[-A,A]}$ and $D \in \mathcal{T}_k^{\leq -1}$.*

both hold. Then, after increasing the integer A if necessary, (i) and (ii) will also hold for $k = 2$. Furthermore if (iii) below holds for $k = 1$

(iii) *In the triangle $E \longrightarrow F \longrightarrow D$ of (ii) above we may strengthen the condition on E, we may assume $E \in \overline{\langle G\rangle}_A^{[-A,A]} \subset \overline{\langle G\rangle}^{[-A,A]}$.*

then the integer A may be chosen large enough so that (iii) will hold for $k = 2$.

Proof Because the t–structures are equivalent we may choose an integer B so that $\mathcal{T}_2^{\leq -B} \subset \mathcal{T}_1^{\leq 0} \subset \mathcal{T}_2^{\leq B}$. Hence $\mathrm{Hom}\big(\Sigma^{-A-B}G, \mathcal{T}_2^{\leq 0}\big) \cong \mathrm{Hom}\big(\Sigma^{-A}G, \mathcal{T}_2^{\leq -B}\big) = 0$, where the vanishing is because $\mathcal{T}_2^{\leq -B} \subset \mathcal{T}_1^{\leq 0}$ and $\mathrm{Hom}\big(\Sigma^{-A}G, \mathcal{T}_1^{\leq 0}\big) = 0$. Also $\Sigma^A G \in \mathcal{T}_1^{\leq 0} \subset \mathcal{T}_2^{\leq B}$ implies $\Sigma^{A+B}G \in \mathcal{T}_2^{\leq 0}$. This proves (i) for $k = 2$, as long as we replace A by $A + B$.

If F is an object in $\mathcal{T}_2^{\leq 0} \subset \mathcal{T}_1^{\leq B}$ we may, using (ii) in combination with (3.2.1) applied to $\Sigma^B F \in \mathcal{T}_1^{\leq 0}$, construct a triangle $E_{2B+1} \longrightarrow F \longrightarrow D_{2B+1}$ with $E_{2B+1} \in \overline{\langle G\rangle}^{[-B-A,B+A]}$ and $D_{2B+1} \in \mathcal{T}_1^{\leq -B-1} \subset \mathcal{T}_2^{\leq -1}$. Thus (ii) also holds for $k = 2$, as long as A is replaced by $A + B$.

It remains to prove the assertion (iii) for $k = 2$, assuming it holds for $k = 1$. By (3.2.2) applied to $\Sigma^B F \in \mathcal{T}_1^{\leq 0}$, we may construct the triangle $E_{2B+1} \longrightarrow F \longrightarrow D_{2B+1}$ with $E_{2B+1} \in \overline{\langle G\rangle}_{(2B+1)A}^{[-B-A,B+A]}$ and $D_{2B+1} \in \mathcal{T}_1^{\leq -B-1} \subset \mathcal{T}_2^{\leq -1}$. Thus assertion (iii) holds, but we must replace A by $\widetilde{A} = \max\big[A + B, A(2B + 1)\big]$. □

Proposition 3.6 *Suppose $\mathcal{T}$ is a weakly approximable triangulated category, H is a compact generator, and $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ is any t–structure in the preferred equivalence class. Then there exists an integer $A > 0$ so that*

(i) $\Sigma^A H \in \mathcal{T}_1^{\leq 0}$ *and* $\mathrm{Hom}(\Sigma^{-A}H, \mathcal{T}_1^{\leq 0}) = 0$.

(ii) *Every object $F \in \mathcal{T}_1^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \overline{\langle H\rangle}^{[-A,A]}$ and $D \in \mathcal{T}_1^{\leq -1}$.*

If the category $\mathcal{T}$ is approximable then the integer A may be chosen to further satisfy

(iii) *In the triangle $E \longrightarrow F \longrightarrow D$ of* (*ii*) *above we may strengthen the condition on E, we may assume $E \in \overline{\langle H\rangle}_A^{[-A,A]} \subset \overline{\langle H\rangle}^{[-A,A]}$.*

Proof The definition of weakly approximable categories gives us a compact generator G, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and an integer A satisfying (i) and (ii), plus (iii) if $\mathcal{T}$ is approximable. Proposition 3.4 guarantees that $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is in the preferred equivalence class of t–structures. By assumption so is $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$, hence the t–structures $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ are equivalent. By Lemma 3.5 we can, by modifying the integer A, also have the conditions (i), (ii) and [when appropriate] (iii) hold for the t–structure $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and the compact generator G. Thus we may assume that the t–structures are the same. We have a single t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) = (\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$, and two compact generators G and H. There exists an integer A that works for G and the t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, and we need to produce an integer that works for H and the t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$.

We are given that G and H are compact generators of $\mathcal{T}$, hence $\langle G\rangle = \mathcal{T}^c = \langle H\rangle$, and Lemma 1.13(ii) allows us to choose an integer $B > 0$ with $G \in \langle H\rangle_B^{[-B,B]}$ and $H \in \langle G\rangle_B^{[-B,B]}$. By (i) for G we know that $\Sigma^A G \in \mathcal{T}^{\leq 0}$ and $\mathrm{Hom}\big(\Sigma^{-A}G, \mathcal{T}^{\leq 0}\big) = 0$. It immediately follows that $\langle G\rangle_B^{[-A-2B,-A]} \subset \mathcal{T}^{\leq 0}$ and that $\mathrm{Hom}\Big(\langle G\rangle_B^{[A,A+2B]}, \mathcal{T}^{\leq 0}\Big) = 0$, and as $\Sigma^{A+B}H \in \langle G\rangle_B^{[-A-2B,-A]}$ and $\Sigma^{-A-B}H \in \langle G\rangle_B^{[A,A+2B]}$ we deduce that $\Sigma^{A+B}H \in \mathcal{T}^{\leq 0}$ and that $\mathrm{Hom}\big(\Sigma^{-A-B}H, \mathcal{T}^{\leq 0}\big) = 0$. This established (i) for H, if we replace A by $A + B$.

Now for (ii) and (iii): for any $F \in \mathcal{T}^{\leq 0}$ we know that there exists a triangle $E \longrightarrow F \longrightarrow D$ with $D \in \mathcal{T}^{\leq -1}$, with $E \in \overline{\langle G\rangle}^{[-A,A]}$, and if $\mathcal{T}$ is approximable we may even choose E to lie in $\overline{\langle G\rangle}_A^{[-A,A]}$. But G belongs to $\langle H\rangle_B^{[-B,B]}$, and therefore

$$\overline{\langle G\rangle}^{[-A,A]} \subset \overline{\langle H\rangle}^{[-A-B,A+B]} \qquad \text{while} \qquad \overline{\langle G\rangle}_A^{[-A,A]} \subset \overline{\langle H\rangle}_{AB}^{[-A-B,A+B]} .$$

Thus (ii) and [when appropriate] (iii) hold for H if A is replaced by $\max(A+B, AB)$. ☐

Remark 3.7 We have so far proved Facts 1.26: Proposition 3.4 amounts to 1.26(i) and Proposition 3.6 to 1.26(ii). The remainder of the section will be devoted to the basic properties of the subcategory $\mathcal{T}_c^-$ of Definition 1.20.

Lemma 3.8 *Suppose $\mathcal{T}$ is a triangulated category with coproducts and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t–structure. Assume there exists a compact generator G and an integer $A > 0$ so that* $\mathrm{Hom}\left(\Sigma^{-A}G, \mathcal{T}^{\leq 0}\right) = 0$.

Then for any compact object $H \in \mathcal{T}$ there exists an integer $B > 0$, depending on H, with $\mathrm{Hom}\left(\Sigma^{-B}H, \mathcal{T}^{\leq 0}\right) = 0$.

Proof Let $H \in \mathcal{T}$ be a compact object. The fact that G is a compact generator gives the equality in $H \in \mathcal{T}^c = \langle G\rangle$; Lemma 1.13(i) allows us to deduce that $H \in \langle G\rangle^{[-C,C]}$ for some $C > 0$. Thus $\Sigma^{-A-C}H \in \langle G\rangle^{[A,A+2C]}$, and as $\mathrm{Hom}\left(\langle G\rangle^{[A,A+2C]}, \mathcal{T}^{\leq 0}\right) = 0$ the Lemma follows, with $B = A + C$. ☐

Lemma 3.9 *Suppose $\mathcal{T}$ is a compactly generated triangulated category and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t–structure. Assume there exists a compact generator G and an integer $A > 0$ so that* $\mathrm{Hom}\left(\Sigma^{-A}G, \mathcal{T}^{\leq 0}\right) = 0$.

Then the subcategory $\mathcal{T}_c^- \subset \mathcal{T}$ is triangulated.

Proof It is clear that $\mathcal{T}_c^-$ is closed under all suspensions and is additive. We must show that, if $R \longrightarrow S \longrightarrow T \longrightarrow \Sigma R$ is a triangle so that R and T belong to $\mathcal{T}_c^-$, then S must also belong to $\mathcal{T}_c^-$.

Choose any integer $m > 0$. Because T belongs to $\mathcal{T}_c^-$ we may choose a triangle $T' \longrightarrow T \longrightarrow T''$ with $T' \in \mathcal{T}^c$ and $T'' \in \mathcal{T}^{\leq -m}$. Since T' is compact, Lemma 3.8 says that we may choose an integer $B > 0$ with $\mathrm{Hom}\left(T', \mathcal{T}^{\leq -B}\right) = 0$.

Now R belongs to $\mathcal{T}_c^-$, allowing us to choose a triangle $R' \longrightarrow R \longrightarrow R''$ with $R' \in \mathcal{T}^c$ and $R'' \in \mathcal{T}^{\leq -m-B}$. We have a diagram

$$\begin{tikzcd}
T' \arrow[r] & T \arrow[r] \arrow[d] & T'' \\
\Sigma R' \arrow[r] & \Sigma R \arrow[r] & \Sigma R''
\end{tikzcd}$$

The composite from top left to bottom right is a map $T' \longrightarrow \Sigma R''$, with $\Sigma R'' \in \mathcal{T}^{\leq -m-B-1} \subset \mathcal{T}^{\leq -B}$. Since $B > 0$ was chosen so that $\mathrm{Hom}\left(T', \mathcal{T}^{\leq -B}\right) = 0$ the map $T' \longrightarrow \Sigma R''$ must vanish, hence the composite $T' \longrightarrow \Sigma R$ must factor through $\Sigma R' \longrightarrow \Sigma R$. We produce a commutative square

$$\begin{tikzcd}
T' \arrow[r] \arrow[d] & T \arrow[d] \\
\Sigma R' \arrow[r] & \Sigma R
\end{tikzcd}$$

which we may complete to a 3×3 diagram where the rows and columns are triangles

$$\begin{tikzcd}
R' \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R'' \arrow[r] \arrow[d] & \Sigma R' \arrow[d] \\
S' \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & S'' \arrow[r] \arrow[d] & \Sigma R' \arrow[d] \\
T' \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & T'' \arrow[r] \arrow[d] & \Sigma T' \arrow[d] \\
\Sigma R' \arrow[r] & \Sigma R \arrow[r] & \Sigma R'' \arrow[r] & \Sigma^2 R''
\end{tikzcd}$$

Because R' and T' are compact, the triangle $R' \longrightarrow S' \longrightarrow T'$ tells us that S' must be compact. Also $T'' \in \mathcal{T}^{\leq -m}$ and $R'' \in \mathcal{T}^{\leq -m-B} \subset \mathcal{T}^{\leq -m}$, and the triangle $R'' \longrightarrow S'' \longrightarrow T''$ implies that $S'' \in \mathcal{T}^{\leq -m}$. The triangle $S' \longrightarrow S \longrightarrow S''$ now does the job for S. □

Proposition 3.10 *Suppose $\mathcal{T}$ is a compactly generated triangulated category and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t–structure. Assume there exists a compact generator G and an integer $A > 0$ so that $\Sigma^A G \in \mathcal{T}^{\leq 0}$ and $\operatorname{Hom}\big(\Sigma^{-A} G, \mathcal{T}^{\leq 0}\big) = 0$.*

Then the subcategory $\mathcal{T}_c^-$ is thick.

Proof We already know that $\mathcal{T}_c^-$ is triangulated, we need to prove it closed under direct summands. Suppose therefore that $S \oplus T$ belongs to $\mathcal{T}_c^-$, we must prove that so does S.

Consider the map $0 \oplus \mathrm{id} : S \oplus T \longrightarrow S \oplus T$. Complete to a triangle

$$\begin{tikzcd}
S \oplus T \arrow[r, "0\oplus\mathrm{id}"] & S \oplus T \arrow[r] & S \oplus \Sigma S
\end{tikzcd}$$

By Lemma 3.9 we deduce that $S \oplus \Sigma S$ belongs to $\mathcal{T}_c^-$. Induction on n allows us to prove that, for any $n \geq 0$, the object $S \oplus \Sigma^{2n+1} S$ belongs to $\mathcal{T}_c^-$. To spell it out: we have proved the case $n = 0$ above. For any n we know that $\Sigma^{2n+1}(S \oplus \Sigma S) \cong$

$\Sigma^{2n+2}S \oplus \Sigma^{2n+1}S$ lies in $\mathcal{T}_c^-$, and induction on n allows us to assume that so does $S \oplus \Sigma^{2n+1}S$. The triangle

$$\begin{tikzcd}
\Sigma^{2n+2}S \oplus \Sigma^{2n+1}S \arrow[r, "0\oplus\mathrm{id}"] & S \oplus \Sigma^{2n+1}S \arrow[r] & S \oplus \Sigma^{2n+3}S
\end{tikzcd}$$

then informs us that $S \oplus \Sigma^{2n+3}S$ belongs to $\mathcal{T}_c^-$.

By Remark 1.22 the category $\mathcal{T}_c^-$ is contained in $\mathcal{T}^-$, and the object $S \oplus \Sigma S$ must belong to $\mathcal{T}^{\leq \ell}$ for some $\ell > 0$. Hence S belongs to $\mathcal{T}^{\leq \ell}$ and, for every integer $m > 0$, we have that $\Sigma^{\ell+m}S \in \mathcal{T}^{\leq -m}$. Choose an integer $n \geq 0$ with $2n+2 \geq \ell+m$; then $\Sigma^{2n+2}S \in \mathcal{T}^{\leq -m}$. Since the object $S \oplus \Sigma^{2n+1}S$ belongs to $\mathcal{T}_c^-$ we may choose a triangle $K \longrightarrow S \oplus \Sigma^{2n+1}S \longrightarrow P$ with $K \in \mathcal{T}^c$ and $P \in \mathcal{T}^{\leq -m}$. Now form the octahedron on the composable morphisms $K \longrightarrow S \oplus \Sigma^{2n+1}S \longrightarrow S$. We obtain

$$\begin{tikzcd}
K \arrow[r] \arrow[d, equal] & S \oplus \Sigma^{2n+1}S \arrow[r] \arrow[d] & P \arrow[d] \\
K \arrow[r] & S \arrow[r] \arrow[d] & Q \arrow[d] \\
 & \Sigma^{2n+2}S \arrow[r, equal] & \Sigma^{2n+2}S
\end{tikzcd}$$

The triangle $P \longrightarrow Q \longrightarrow \Sigma^{2n+2}S$, together with the fact that both P and $\Sigma^{2n+2}S$ belong to $\mathcal{T}^{\leq -m}$, tell us that Q must belong to $\mathcal{T}^{\leq -m}$. Now the triangle $K \longrightarrow S \longrightarrow Q$ does the trick for S. □

The next few results work out how $\mathcal{T}_c^-$ behaves when $\mathcal{T}$ is approximable or weakly approximable.

Lemma 3.11 *Let us fix a weakly approximable* [*or approximable*] *triangulated category $\mathcal{T}$. Choose a compact generator G and a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class. Choose an integer A as in Proposition 3.6, and let $m > 2A+1$ be an integer.*

Then for any $K \in \mathcal{T}^c \cap \mathcal{T}^{\leq 0}$ there exists an object L and a triangle $E \longrightarrow K \oplus L \longrightarrow D$ with $E \in \langle G\rangle^{[1-m-A,A]}$ and $D \in \langle G\rangle^{(-\infty,-m+A]}$.

If $\mathcal{T}$ is approximable we may further assume $E \in \langle G\rangle_{mA}^{[1-m-A,A]}$.

Proof Because K belongs to $\mathcal{T}^{\leq 0}$ the result (3.2.1) permits us to construct, for every integer $m > 0$, a triangle $E_m \longrightarrow K \longrightarrow D_m$ with $E_m \in \overline{\langle G\rangle}^{[1-m-A,A]}$ and $D_m \in \mathcal{T}^{\leq -m}$. If the category $\mathcal{T}$ is approximable we may assume $E_m \in \overline{\langle G\rangle}_{mA}^{[1-m-A,A]}$.

The object K is assumed compact and Lemma 3.8 produces for us a positive integer B, which we may assume $\geq m+2A$, with $\mathrm{Hom}\big(K, \mathcal{T}^{\leq -B}\big) = 0$. Since we chose $B \geq m+2A$ we have $\mathrm{Hom}\Big(\Sigma\overline{\langle G\rangle}^{[1-m-A,A]}, \mathcal{T}^{\leq -B}\Big) = 0$, and in particular $\mathrm{Hom}\big(\Sigma E_m, \mathcal{T}^{\leq -B}\big) = 0$. From the triangle $K \longrightarrow D_m \longrightarrow \Sigma E_m$ and the fact that $\mathrm{Hom}\big(K,\mathcal{T}^{\leq -B}\big)$ and $\mathrm{Hom}\big(\Sigma E_m,\mathcal{T}^{\leq -B}\big)$ both vanish we deduce that $\mathrm{Hom}\big(D_m,\mathcal{T}^{\leq -B}\big)=0$.

From $D_m\in\mathcal{T}^{\leq -m}$ we construct a triangle $E'\longrightarrow D_m\longrightarrow Q$ with $E'\in\overline{\langle G\rangle}^{[1-B-A,-m+A]}$ and $Q\in\mathcal{T}^{\leq -B}$. Since $\mathrm{Hom}\big(D_m,\mathcal{T}^{\leq -B}\big)=0$ the map $D_m\longrightarrow Q$ must vanish, hence the map $E'\longrightarrow D_m$ must be a split epimorphism. Since E' belongs to $\overline{\langle G\rangle}^{[1-B-A,-m+A]}\subset\overline{\langle G\rangle}^{(-\infty,-m+A]}$ so does its direct summand D_m.

We have learned that K belongs to $\overline{\langle G\rangle}^{[1-m-A,A]}*\overline{\langle G\rangle}^{(-\infty,-m+A]}$, and if $\mathcal{T}$ is approximable K even belongs to the smaller $\overline{\langle G\rangle}_{mA}^{[1-m-A,A]}*\overline{\langle G\rangle}^{(-\infty,-m+A]}$. Now set

$$\begin{aligned}\mathcal{X}_1&=\mathrm{Coprod}\big(G[1-m-A,A]\big)\ ,\\ \mathcal{X}_2&=\mathrm{Coprod}_{mA}\big(G[1-m-A,A]\big)\ ,\\ \mathcal{Z}&=\mathrm{Coprod}\big(G(-\infty,-m+A]\big)\ .\end{aligned}$$

Then $\mathcal{Z}=\mathrm{smd}(\mathcal{Z})$ is closed under direct summands so $\mathcal{Z}=\overline{\langle G\rangle}^{(-\infty,-m+A]}$, while

$$\overline{\langle G\rangle}^{[1-m-A,A]}=\mathrm{smd}(\mathcal{X}_1)\qquad\text{and}\qquad\overline{\langle G\rangle}_{mA}^{[1-m-A,A]}=\mathrm{smd}(\mathcal{X}_2)\ .$$

We are given that K belongs to $\mathrm{smd}(\mathcal{X}_i)*\mathcal{Z}\subset\mathrm{smd}(\mathcal{X}_i*\mathcal{Z})$ with $i=1$ or 2, depending on whether $\mathcal{T}$ is approximable. Choose an object K' in one of the categories $\mathcal{X}_i*\mathcal{Z}$ above, so that K is a direct summand and $K\xrightarrow{f}K'\longrightarrow K$ is a pair of morphisms composing to the identity. Now put

$$\begin{aligned}\mathcal{A}_1&=\mathrm{coprod}\big(G[1-m-A,A]\big)\ ,\\ \mathcal{A}_2&=\mathrm{coprod}_{mA}\big(G[1-m-A,A]\big)\ ,\\ \mathcal{C}&=\mathrm{coprod}\big(G(-\infty,-m+A]\big)\ .\end{aligned}$$

By [20, Lemma 1.8] any morphism from an object in $\mathcal{T}^c$, to any of $\mathcal{X}_1$, $\mathcal{X}_2$ or $\mathcal{Z}$, factors (respectively) through an object in $\mathcal{A}_1$, $\mathcal{A}_2$ or $\mathcal{C}$. Furthermore: we have that $\mathcal{T}^c\subset\mathcal{T}$ is a triangulated subcategory and contains $\mathcal{C}$. Now the map $f:K\longrightarrow K'$ is a morphism from $K\in\mathcal{T}^c$ to an object $K'\in\mathcal{X}_i*\mathcal{Z}$, with $i=1$ or $i=2$. By [20, Lemma 1.6 and Remark 1.7] the map f factors as $K\longrightarrow K''\longrightarrow K'$ where $K''\in\mathcal{A}_i*\mathcal{C}$, with $i=1$ or 2. Since the composite $K\longrightarrow K''\longrightarrow K'\longrightarrow K$ is the identity we deduce that K is a direct summand of the object in $K''\in\mathcal{A}_i*\mathcal{C}$, proving the Lemma. ☐

Lemma 3.12 *Let us fix a weakly approximable* [*or approximable*] *triangulated category* $\mathcal{T}$. *Choose a compact generator* G *and a t–structure* $(\mathcal{T}^{\leq 0},\mathcal{T}^{\geq 0})$ *in the preferred equivalence class.*

There exists an integer $B>0$ *so that, for any object* $K\in\mathcal{T}^c\cap\mathcal{T}^{\leq 0}$, *there exists a triangle* $E\longrightarrow K\longrightarrow D$ *with* $E\in\langle G\rangle^{[-B,B]}$ *and* $D\in\mathcal{T}^{\leq -1}$.

If $\mathcal{T}$ *is approximable we may further assume* $E\in\langle G\rangle_B^{[-B,B]}$.

Proof Choose an integer A as in Proposition 3.6. We apply Lemma 3.11 to the object K, with $m = 4A + 1$, and obtain a triangle $E \longrightarrow K \oplus L \longrightarrow D$ with $E \in \langle G\rangle^{[-5A,A]}$ and $D \in \langle G\rangle^{(-\infty,-3A-1]}$; if the category is approximable we may even assume $E \in \langle G\rangle^{[-5A,A]}_{(4A+1)A}$. Now $\langle G\rangle^{(-\infty,-3A-1]} \subset \mathcal{T}^c$ hence the object D is compact, and it belongs to $\mathcal{T}^{\le -2A-1}$ since $G \in \mathcal{T}^{\le A}$. Applying Lemma 3.11 to the object $\Sigma^{-2A-1}D$ and with $m = 6A$ we obtain a triangle $E' \longrightarrow D \oplus M \longrightarrow D'$ with $E' \in \langle G\rangle^{[-9A,-A-1]}$ and $D' \in \langle G\rangle^{(-\infty,-7A-1]}$; if the category is approximable we may even assume $E' \in \langle G\rangle^{[-9A,-A-1]}_{6A^2}$. Now complete the composable maps $K \oplus L \oplus M \longrightarrow D \oplus M \longrightarrow D'$ to an octahedron

$$\begin{tikzcd}
E \arrow[r] \arrow[d, equal] & E'' \arrow[r] \arrow[d] & E' \arrow[d] \\
E \arrow[r] & K \oplus L \oplus M \arrow[r] \arrow[d] & D \oplus M \arrow[d] \\
& D' \arrow[r, equal] & D'
\end{tikzcd}$$

We know that $E \in \langle G\rangle^{[-5A,A]}$ and $E' \in \langle G\rangle^{[-9A,-A-1]}$, and the triangle $E \longrightarrow E'' \longrightarrow E'$ tells us that E'' belongs to

$$\langle G\rangle^{[-5A,A]} * \langle G\rangle^{[-9A,-A-1]} \qquad \subset \qquad \langle G\rangle^{[-9A,A]} ;$$

if $\mathcal{T}$ is approximable E'' belongs to $\langle G\rangle^{[-9A,A]}_{10A^2+A}$.

Now the object D' belongs to $\langle G\rangle^{(-\infty,-7A-1]} \subset \mathcal{T}^{\le -6A-1}$. The object E' belongs to $\langle G\rangle^{[-9A,-A-1]} \subset \mathcal{T}^{\le -1}$ and the triangle $E' \longrightarrow D \oplus M \longrightarrow D'$ guarantees that $D \oplus M$ and therefore its direct summand M belongs to $\mathcal{T}^{\le -1}$. Summarizing we have

(i) The object E belongs to $\langle G\rangle^{[-5A,A]}$, the object E'' belongs to $\langle G\rangle^{[-9A,A]}$, the object D belongs to $\langle G\rangle^{(-\infty,-3A-1]} \subset \mathcal{T}^{-2A-1}$, the object M belongs to $\mathcal{T}^{\le -1}$ and the object D' belongs to $\mathcal{T}^{\le -6A-1}$.

(ii) If the category $\mathcal{T}$ is approximable then the objects E and E'' were chosen so that $E \in \langle G\rangle^{[-5A,A]}_{4A^2+A}$ and $E'' \in \langle G\rangle^{[-9A,A]}_{10A^2+A}$.

Now consider the following diagram

$$\begin{tikzcd}
E \arrow[r] & K \oplus L \arrow[r] \arrow[d] & D \\
E'' \arrow[r] & K \oplus L \oplus M \arrow[r] & D'
\end{tikzcd}$$

where the vertical map is the direct sum of $\mathrm{id} : L \longrightarrow L$ with the zero map. The composite from top left to bottom right is a morphism $E \longrightarrow D'$, with $E \in \langle G\rangle^{[-5A,A]}$ and $D' \in \mathcal{T}^{\le -6A-1}$, hence must vanish. Therefore the composite $E \longrightarrow K \oplus L \longrightarrow$

$K \oplus L \oplus M$ must factor through $E'' \longrightarrow K \oplus L \oplus M$. We deduce a commutative square

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & K \oplus L \arrow[d] \\
E'' \arrow[r] & K \oplus L \oplus M
\end{tikzcd}$$

which we may complete to a 3×3 diagram whose rows and columns are triangles

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & K \oplus L \arrow[r] \arrow[d] & D \arrow[d] \\
E'' \arrow[r] \arrow[d] & K \oplus L \oplus M \arrow[r] \arrow[d] & D' \arrow[d] \\
\widetilde{E} \arrow[r] & K \oplus \Sigma K \oplus M \arrow[r] & D''
\end{tikzcd}$$

The triangle $D' \longrightarrow D'' \longrightarrow \Sigma D$, together with the fact that $D \in \mathcal{T}^{-2A-1}$ and $D' \in \mathcal{T}^{-6A-1}$, tell us that $D'' \in \mathcal{T}^{\leq -2A-2}$. The triangle $E'' \longrightarrow \widetilde{E} \longrightarrow \Sigma E$, combined with the fact that $E \in \langle G \rangle^{[-5A,A]}$ and $E'' \in \langle G \rangle^{[-9A,A]}$, tell us that $\widetilde{E} \in \langle G \rangle^{[-9A,A]}$; if $\mathcal{T}$ is approximable we have that $E \in \langle G \rangle^{[-5A,A]}_{4A^2+A}$ and $E'' \in \langle G \rangle^{[-9A,A]}_{10A^2+A}$ and therefore $\widetilde{E}$ belongs to $\langle G \rangle^{[-9A,A]}_{14A^2+2A}$. Now complete the composable maps $\widetilde{E} \longrightarrow K \oplus \Sigma K \oplus M \longrightarrow K$ to an octahedron

$$\begin{tikzcd}
\widetilde{E} \arrow[r] \arrow[d, equal] & K \oplus \Sigma K \oplus M \arrow[r] \arrow[d] & D'' \arrow[d] \\
\widetilde{E} \arrow[r] & K \arrow[r] \arrow[d] & \widetilde{D} \arrow[d] \\
 & \Sigma^2 K \oplus \Sigma M \arrow[r, equal] & \Sigma^2 K \oplus \Sigma M
\end{tikzcd}$$

We have that $\Sigma^2 K$ and ΣM both belong to $\mathcal{T}^{\leq -2}$ and D'' belongs to $\mathcal{T}^{\leq -2A-2}$. Hence $\widetilde{D} \in \mathcal{T}^{\leq -2}$, and the triangle $\widetilde{E} \longrightarrow K \longrightarrow \widetilde{D}$ satisfies the assertion of the Lemma. □

Proposition 3.13 *Let us fix a weakly approximable [or approximable] triangulated category $\mathcal{T}$. Choose a compact generator G and a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class. Choose an integer $B > 0$ as in Lemma 3.12.*

Then for any object $F \in \mathcal{T}_c^- \cap \mathcal{T}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \langle G \rangle^{[-B,B]}$ and $D \in \mathcal{T}^{\leq -1}$.

If $\mathcal{T}$ is approximable we may further assume $E \in \langle G \rangle^{[-B,B]}_B$.

Proof Because F belongs to $\mathcal{T}_c^-$ we may choose a triangle $K \longrightarrow F \longrightarrow D_1$ with $K \in \mathcal{T}^c$ and $D_1 \in \mathcal{T}^{\leq -1}$. The triangle $\Sigma^{-1}D_1 \longrightarrow K \longrightarrow F$, coupled with the fact that both $\Sigma^{-1}D_1$ and F belong to $\mathcal{T}^{\leq 0}$, tell us that $K \in \mathcal{T}^{\leq 0}$. Thus $K \in \mathcal{T}^c \cap \mathcal{T}^{\leq 0}$.

We may therefore apply Lemma 3.12; there exists a triangle $E \longrightarrow K \longrightarrow D_2$ with $E \in \langle G \rangle^{[-B,B]}$ and $D_2 \in \mathcal{T}^{\leq -1}$. If $\mathcal{T}$ is approximable the object E may be chosen in $\langle G \rangle_B^{[-B,B]}$. Now complete the composable maps $E \longrightarrow K \longrightarrow F$ to an octahedron

$$\begin{tikzcd}
E \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & D_2 \arrow[d] \\
E \arrow[r] & F \arrow[r] \arrow[d] & D \arrow[d] \\
& D_1 \arrow[r, equal] & D_1
\end{tikzcd}$$

The triangle $D_2 \longrightarrow D \longrightarrow D_1$, coupled with the fact that D_2 and D_1 both lie in $\mathcal{T}^{\leq -1}$, tell us that $D \in \mathcal{T}^{\leq -1}$. And the triangle $E \longrightarrow F \longrightarrow D$ satisfies the assertion of the Proposition. □

Corollary 3.14 *Let $\mathcal{T}$ be a weakly approximable triangulated category. Let G be a compact generator and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t–structure in the preferred equivalence class. Choose an integer $B > 0$ as in Lemma 3.12.*

For any object $F \in \mathcal{T}_c^- \cap \mathcal{T}^{\leq 0}$ there exists a sequence of objects $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ mapping to F, and so that $E_m \in \langle G \rangle^{[1-m-B,B]}$ and in each triangle $E_m \longrightarrow F \longrightarrow D_m$ we have $D_m \in \mathcal{T}^{\leq -m}$. For any such sequence the non-canonical map $\underrightarrow{\operatorname{Hocolim}}\, E_m \longrightarrow F$ is an isomorphism.

If the category is approximable we may construct the E_m to lie in $\langle G \rangle_{mB}^{[1-m-B,B]}$.

Proof The fact that any such sequence would deliver a non-canonical isomorphism $\underrightarrow{\operatorname{Hocolim}}\, E_m \longrightarrow F$ is contained in Lemma 3.3. We need to prove the existence of the sequence.

In Proposition 3.13 we constructed a triangle $E_1 \longrightarrow F \longrightarrow D_1$ with $E_1 \in \langle G \rangle^{[-B,B]}$ and $D_1 \in \mathcal{T}^{\leq -1}$. But $\langle G \rangle^{[-B,B]} \subset \mathcal{T}^c$, and in Remark 1.21 we noted that $\mathcal{T}^c \subset \mathcal{T}_c^-$. In the triangle $E_1 \longrightarrow F \longrightarrow D_1$ we have that both E_1 and F lie in $\mathcal{T}_c^-$, while Lemma 3.9 proved that the category $\mathcal{T}_c^-$ is triangulated. Therefore $D_1 \in \mathcal{T}_c^- \cap \mathcal{T}^{\leq -1}$. If we let $\mathcal{S} = \mathcal{T}_c^-$ we are in the situation of Corollary 3.2, more specifically the hypotheses of (3.2.3) hold; if $\mathcal{T}$ is approximable the hypotheses of (3.2.4) hold. The current Corollary is simply the conclusions of (3.2.3) and (3.2.4). □

4 Products and homotopy inverse limits in weakly approximable triangulated categories

This short section assembles together a few facts about products an homotopy inverse limits in weakly approximable triangulated categories. We are ready to prove these facts now, but will not use them until much later in the article, in the proof of Theorem 1.4(ii). See Sect. 11.

It should perhaps be highlighted that [8, Lemmas 3.6 and 3.9(iv)] play a key role in the proofs of the present section. It was thus the joint work, with Burke and Pauwels, that produced the key lemmas allowing me to prove Theorem 1.4(ii). See also Remark 1.5.

Lemma 4.1 *Suppose $\mathcal{T}$ is a weakly approximable triangulated category, and let $\big(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0}\big)$ be a t–structure in the preferred equivalence class. Suppose $\cdots \longrightarrow Z_3 \longrightarrow Z_2 \longrightarrow Z_1 \longrightarrow Z_0$ is an inverse sequence of objects in $\mathcal{T}$ so that, for any integer n, the functor $(-)^{\ge n}$ takes it to a sequence that is eventually stable.*

Let $Z = \underleftarrow{\mathrm{Holim}}\, Z_i$ and consider the natural map $f_i : Z \longrightarrow Z_i$. Then for $i \gg 0$ the functor $(-)^{\ge n}$ takes the map f_i to an isomorphism.

More precisely: there exists an integer $L > 0$ so that, whenever the sequence $\cdots \longrightarrow Z_3^{\ge -L} \longrightarrow Z_2^{\ge -L} \longrightarrow Z_1^{\ge -L} \longrightarrow Z_0^{\ge -L}$ is constant, the map $f_i^{\ge 0} : Z^{\ge 0} \longrightarrow Z_i^{\ge 0}$ is an isomorphism.

Proof Let G be a compact generator for $\mathcal{T}$ and suppose $\big(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0}\big)$ is a t–structure in the preferred equivalence class. By [8, Lemma 3.9(iv)] there exists an integer $A > 0$ so that, if $\mathrm{Hom}(\Sigma^i G, X) = 0$ for all $i \le A$, then X must belong to $\mathcal{T}^{\le 0}$. Choose and fix such an integer A. Then choose an integer $B > 0$ with $\mathrm{Hom}\big(G, \mathcal{T}^{\le -B}\big) = 0$.

By shifting and passing to a subsequence it suffices to prove the “moreover” assertion, and we assert that $L = A + B + 1$ works. Suppose therefore that the sequence $\cdots \longrightarrow Z_3^{\ge -A-B-1} \longrightarrow Z_2^{\ge -A-B-1} \longrightarrow Z_1^{\ge -A-B-1} \longrightarrow Z_0^{\ge -A-B-1}$ is constant, and consider the commutative diagram in which the rows are triangles

$$\begin{tikzcd}
\prod_{i=0}^{\infty} Z_i^{\le -A-B-2} \arrow[r] & \prod_{i=0}^{\infty} Z_i \arrow[r] \arrow[d, "1-\text{shift}"'] & \prod_{i=0}^{\infty} Z_i^{\ge -A-B-1} \arrow[d, "1-\text{shift}"] \\
\prod_{i=0}^{\infty} Z_i^{\le -A-B-2} \arrow[r] & \prod_{i=0}^{\infty} Z_i \arrow[r] & \prod_{i=0}^{\infty} Z_i^{\ge -A-B-1}
\end{tikzcd}$$

We may complete it to a 3 × 3 diagram of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Y \arrow[d] \\
\prod_{i=0}^{\infty} Z_i^{\le -A-B-2} \arrow[r] \arrow[d] & \prod_{i=0}^{\infty} Z_i \arrow[r] \arrow[d, "1-\text{shift}"'] & \prod_{i=0}^{\infty} Z_i^{\ge -A-B-1} \arrow[d, "1-\text{shift}"] \\
\prod_{i=0}^{\infty} Z_i^{\le -A-B-2} \arrow[r] & \prod_{i=0}^{\infty} Z_i \arrow[r] & \prod_{i=0}^{\infty} Z_i^{\ge -A-B-1}
\end{tikzcd}$$

where the middle column is the definition of $Z = \underleftarrow{\text{Holim}}\ Z_i$. The inverse sequence $Z_i^{\ge -A-B-1}$ is constant, hence the object Y is canonically isomorphic to $Z_i^{\ge -A-B-1}$. What we know about the object X is that it sits in the triangle given by the left column

$$\prod_{i=0}^{\infty} (\Sigma^{-1} Z_i)^{\le -A-B-1} \longrightarrow X \longrightarrow \prod_{i=0}^{\infty} Z_i^{\le -A-B-2}$$

By the choice of B we have that $\mathrm{Hom}(\Sigma^i G, -)$ kills the two outside terms whenever $i \le A+1$, hence the choice of A gives that the two outside terms lie in $\mathcal{T}^{\le -1}$. Therefore $X \in \mathcal{T}^{\le -1}$, and it follows that the truncation $(-)^{\ge 0}$ takes the map $Z \longrightarrow Y = Z_i^{\ge -A-B-1}$ to an isomorphism. □

Proposition 4.2 *Suppose $\mathcal{T}$ is a weakly approximable triangulated category, and let $\left(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0}\right)$ be a t–structure in the preferred equivalence class. Let Y be any object in $\mathcal{T}$, assume $\cdots \longrightarrow Z_3 \longrightarrow Z_2 \longrightarrow Z_1 \longrightarrow Z_0$ is an inverse sequence of objects in $\mathcal{T}$, and let $f_* : Y \longrightarrow Z_*$ be a map from Y to the inverse system. Suppose that, for any integer $i > 0$, there exists an integer $N > 0$ so that*

$$n \ge N \quad \Longrightarrow \quad f_n^{\ge -i} : Y^{\ge -i} \longrightarrow Z_n^{\ge -i} \text{ is an isomorphism.}$$

If $Z = \underleftarrow{\text{Holim}}\ Z_n$ is the homotopy inverse limit, then the (non-canonical) map $f : Y \longrightarrow Z$ is an isomorphism.

Applying to the inverse system $Z_n = Y^{\ge -n}$, we learn that $\mathcal{T}$ is left-complete with respect to any t–structure in the preferred equivalence class.

Proof By [8, Lemma 3.6] the t–structure is nondegenerate, hence it suffices to prove that the morphism $\varphi^{\ge -i} : Y^{\ge -i} \longrightarrow Z^{\ge -i}$ is an isomorphism for every $i \in \mathbb{Z}$. By shifting it suffices to prove this for $i = 0$. Let $L > 0$ be the integer in the "moreover" part of Lemma 4.1, choose an integer N so large that $f_n^{\ge -L} : Y^{\ge -L} \longrightarrow Z_n^{\ge -L}$ is an isomorphism whenever $n \geq N$, and apply the functor $(-)^{\geq 0}$ to the commutative triangle below

$$\begin{tikzcd}
Y \arrow[dd, "f"'] \arrow[dr, "f_N"] & \\
& Z_N^{\geq -L} \\
Z \arrow[ur, "\varphi_N"'] &
\end{tikzcd}$$

Lemma 4.1 teaches us that $\varphi_N^{\geq 0}$ is an isomorphism. Since $f_N^{\geq 0}$ is also an isomorphism, the commutativity forces $f^{\geq 0}$ to be an isomorphism. □

Lemma 4.3 *Let $\mathcal{T}$ be a weakly approximable triangulated category, and choose a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ in the preferred equivalence class. Suppose $\cdots \longrightarrow Z_3 \longrightarrow Z_2 \longrightarrow Z_1 \longrightarrow Z_0$ is an inverse sequence of objects in $\mathcal{T}_c^-$ so that, for any integer n, the functor $(-)^{\geq n}$ takes it to a sequence that is eventually stable.*

Then $Z = \underleftarrow{\mathrm{Holim}}\; Z_i$ belongs to $\mathcal{T}_c^-$.

Proof Since $\mathcal{T}$ is weakly approximable, the "moreover" part of Lemma 4.1 provides an integer $L > 0$ such that, given an inverse sequence $\cdots \longrightarrow Z_3 \longrightarrow Z_2 \longrightarrow Z_1 \longrightarrow Z_0$ in $\mathcal{T}$ with $\cdots \longrightarrow Z_3^{\geq -L} \longrightarrow Z_2^{\geq -L} \longrightarrow Z_1^{\geq -L} \longrightarrow Z_0^{\geq -L}$ all isomorphisms, we have that the map $\big[\, \underleftarrow{\mathrm{Holim}}\; Z_*\big]^{\geq 0} \longrightarrow Z_0^{\geq 0}$ is an isomorphism. Also: since $\mathcal{T}$ is weakly approximable it has a compact generator G, and Corollary 3.14 permits us to choose an integer $B > 0$ such that

(i) $\mathrm{Hom}\big(G, \mathcal{T}^{\leq -B}\big) = 0$.
(ii) Every object $X \in \mathcal{T}_c^-$ admits a triangle $W \longrightarrow X \longrightarrow D$, with $W \in \langle G \rangle^{[-B,\infty)}$ and $D \in \mathcal{T}^{\leq 0}$.

OK: we have chosen the integers L and B and it's time to get to work. Suppose we are given in $\mathcal{T}_c^-$ a sequence $\cdots \longrightarrow Z_3 \longrightarrow Z_2 \longrightarrow Z_1 \longrightarrow Z_0$ satisfying the hypotheses, put $Z = \underleftarrow{\mathrm{Holim}}\; Z_*$, we need to show that $Z \in \mathcal{T}_c^-$. Choose any integer $m > 0$. We choose an integer $i > 0$ so that the maps in the subsequence $\cdots \longrightarrow Z_{i+2}^{\geq -m-L-2B+1} \longrightarrow Z_{i+1}^{\geq -m-L-2B+1} \longrightarrow Z_i^{\geq -m-L-2B+1}$ are all isomorphisms. By (ii) we may choose a triangle $W \longrightarrow Z_i \longrightarrow D$ with $W \in \langle G \rangle^{[-m-B,\infty)}$ and $D \in \mathcal{T}^{\leq -m}$. By the choice of L the map $Z^{\geq -m-2B+1} \longrightarrow Z_i^{\geq -m-2B+1}$ is an isomorphism, and in the triangle $Z \longrightarrow Z_i \longrightarrow \widetilde{D}$ we have $\widetilde{D} \in \mathcal{T}^{\leq -m-2B}$. Therefore the composite $W \longrightarrow Z_i \longrightarrow \widetilde{D}$ is a map from $W \in \langle G \rangle^{[-m-B,\infty)}$ to $\widetilde{D} \in \mathcal{T}^{\leq -m-2B}$ and vanishes by (i). We may therefore factor $W \longrightarrow Z_i$ as $W \longrightarrow Z \longrightarrow Z_i$. Completing to an octahedron

$$\begin{tikzcd}
W \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d] & D' \arrow[d] \\
W \arrow[r] & Z_i \arrow[r] \arrow[d] & D \arrow[d] \\
& \widetilde{D} \arrow[r, equal] & \widetilde{D}
\end{tikzcd}$$

produces a triangle $W \longrightarrow Z \longrightarrow D'$, with $W \in \mathcal{T}^c$. The triangle $\Sigma^{-1}\widetilde{D} \longrightarrow D' \longrightarrow D$, coupled with the facts that $\Sigma^{-1}\widetilde{D} \in \mathcal{T}^{\leq -m-2B+1}$ and $D \in \mathcal{T}^{\leq -m}$, guarantee that $D' \in \mathcal{T}^{\leq -m}$. ☐

5 Examples

In Sects. 3 and 4 we developed some abstraction, and it's high time to look at examples. We begin with the trivial ones.

Example 5.1 Let R be an associative (possibly noncommutative) ring, and put $\mathcal{T} = \mathbf{D}(R)$. Note: our $\mathbf{D}(R)$ is an abbreviation for what, in the more explicit notation standard in representation theory, usually goes by the name $\mathbf{D}(R\text{–Mod})$. It is the derived category whose objects are the (possibly unbounded) cochain complexes of left R-modules. The category $\mathcal{T}$ has coproducts and R is a compact generator. Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be the standard t–structure. Then $\mathrm{Hom}\left(\Sigma^{-1}R, \mathcal{T}^{\leq 0}\right) = 0$ and $\Sigma R \in \mathcal{T}^{\leq 0}$.

Let F be any object in $\mathcal{T}^{\leq 0}$. That is, we take a cochain complex F with $H^\ell(F) = 0$ for all $\ell > 0$. Such a complex has a free resolution; it is isomorphic in $\mathcal{T}$ to the cochain complex

$$\cdots \to F^{-3} \to F^{-2} \to F^{-1} \to F^0 \to 0 \to 0 \to 0 \to \cdots$$

with F^i free R–modules. The brutal truncation produces for us a short exact sequence of cochain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & F^0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & F^{-3} \arrow[r] \arrow[d, equal] & F^{-2} \arrow[r] \arrow[d, equal] & F^{-1} \arrow[r] \arrow[d, equal] & F^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & F^{-3} \arrow[r] & F^{-2} \arrow[r] & F^{-1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

and this is a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \overline{\langle R\rangle}_1^{[0,0]}$ and $D \in \mathcal{T}^{\leq -1}$. The category is approximable, the standard t–structure is in preferred equivalence class, and $\mathcal{T}_c^-$ is just $\mathbf{D}^-(R\text{–proj})$, the category of bounded-above complexes which admit projective resolutions by finitely generated projectives.

The category $\mathcal{T}_c^b$ is the intersection $\mathcal{T}_c^- \cap \mathcal{T}^b$; it consists of the objects in $\mathbf{D}^-(R\text{–proj})$ with only finitely many nonzero cohomology groups. We have an inclusion $\mathbf{D}^b(R\text{–proj}) \subset \mathcal{T}_c^b$, and for R general I don't know much about the difference $\mathcal{T}_c^b - \mathbf{D}^b(R\text{–proj})$. When R is noetherian we have $\mathcal{T}_c^b = \mathbf{D}^b(R\text{–mod})$, which is usually much larger than $\mathbf{D}^b(R\text{–proj})$.

Example 5.2 A very similar analysis works when $\mathcal{T}$ is the homotopy category of spectra. The sphere S^0 is a compact generator. Consider the t–structure where $\mathcal{T}^{\leq 0}$ is the category of connective spectra—these are the spectra F with $\pi_i(F) = 0$ when $i < 0$. Then $\mathrm{Hom}\left(\Sigma^{-1}S^0, \mathcal{T}^{\leq 0}\right) = 0$ and $\Sigma S^0 \in \mathcal{T}^{\leq 0}$. And any object $F \in \mathcal{T}^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \overline{\langle S^0\rangle}_1^{[0,0]}$ and $D \in \mathcal{T}^{\leq -1}$; this just says that we may choose a bouquet of zero-spheres E and a map $E \longrightarrow F$ which is surjective on π_0. The category is approximable and the t–structure above is in the preferred equivalence class.

Remark 5.3 Examples 5.1 and 5.2 should be viewed as the baby case. If $\mathcal{T}$ has a compact generator G, such that $\mathrm{Hom}(G, \Sigma^i G) = 0$ for all $i > 0$, then $\mathcal{T}$ is approximable. Just take the t–structure $\left(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0})\right)$; then $\Sigma G \in \mathcal{T}_G^{\leq 0}$ and $\mathrm{Hom}\left(\Sigma^{-1}G, \mathcal{T}_G^{\leq 0}\right) = 0$, and every object $F \in \mathcal{T}_G^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \overline{\langle G\rangle}_1^{[0,0]}$ and $D \in \mathcal{T}_G^{\leq -1}$.

Now we turn to more complicated examples of approximable and weakly approximable categories. We begin with

Reminder 5.4 If X is a quasicompact, quasiseparated scheme then $\mathcal{T} = \mathbf{D_{qc}}(X)$ has a single compact generator, see Bondal and Van den Bergh [5, Theorem 3.1.1(ii)]. Let G be any such compact generator; [5, Theorem 3.1.1(i)] tells us that G is a perfect complex.

Since the categories $\mathbf{D_{qc}}(X)$ have single compact generators, one can wonder which of them is approximable or weakly approximable. To address this question we begin with

Lemma 5.5 *Let X be a separated, quasicompact scheme, let $\mathcal{T} = \mathbf{D_{qc}}(X)$ be its derived category, and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be the standard t–structure. Then there is a compact generator $G' \in \mathcal{T}$ and an integer $A > 0$, so that every object $F \in \mathcal{T}^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \overline{\langle G'\rangle}_A^{[-A,A]}$ and $D \in \mathcal{T}^{\leq -1}$.*

Proof Absolute noetherian approximation, that is Thomason and Trobaugh [27, Theorem C.9] or [26, Tags 01ZA combined with 01YX], allows us to choose a separated scheme Y, of finite type over $\mathbb{Z}$, and an affine map $f : X \longrightarrow Y$. From [20, Definition 5.2 and Theorem 5.8], in the special case where $\mathcal{S} = \mathbf{D_{qc}}(Y)^{\leq 0}$ for the standard t–structure and $m = 0$, we learn:

(i) There exists a compact generator $G \in \mathbf{D_{qc}}(Y)$ and an integer $A > 0$, so that every $F' \in \mathbf{D_{qc}}(Y)^{\leq 0}$ admits a triangle $E' \longrightarrow F' \longrightarrow D'$ with $E' \in \overline{\langle G\rangle}_A^{[-A,A]}$ and $D' \in \mathbf{D_{qc}}(Y)^{\leq -1}$.

Hall and Rydh [13, Lemma 8.2] tells us that $G' = \mathbf{L}f^*G$ is a compact generator for $\mathbf{D_{qc}}(X)$; this is our choice of G' for the Lemma. Now take any object $F \in \mathbf{D_{qc}}(X)^{\leq 0}$. Since $f : X \longrightarrow Y$ is affine we have $\mathbf{R}f_*F \cong f_*F \in \mathbf{D_{qc}}(Y)^{\leq 0}$, and (i) above permits us to find a triangle $E' \longrightarrow f_*F \longrightarrow D'$ with $E' \in \overline{\langle G\rangle}_A^{[-A,A]}$ and $D' \in \mathbf{D_{qc}}(Y)^{\leq -1}$. Applying the functor $\mathbf{L}f^*$, and remembering that $\mathbf{L}f^*\mathbf{D_{qc}}(Y)^{\leq 0} \subset \mathbf{D_{qc}}(X)^{\leq 0}$, we deduce

(ii) There is in $\mathbf{D_{qc}}(X)$ a triangle $\mathbf{L}f^*E' \longrightarrow \mathbf{L}f^*f_*F \longrightarrow \mathbf{L}f^*D'$, with $\mathbf{L}f^*D' \in \mathbf{D_{qc}}(X)^{\leq -1}$ and $\mathbf{L}f^*E' \in \mathbf{L}f^*\overline{\langle G\rangle}_A^{[-A,A]} \subset \overline{\langle \mathbf{L}f^*G\rangle}_A^{[-A,A]}$.

But the counit of adjunction gives a map $\varepsilon : \mathbf{L}f^*f_*F \longrightarrow F$, and the fact that the maps $f_*F \xrightarrow{\eta f_*} f_*\mathbf{L}f^*f_*F \xrightarrow{f_*\varepsilon} f_*F$ compose to the identity tells us that the functor f_* takes $\varepsilon : \mathbf{L}f^*f_*F \longrightarrow F$ to a split epimorphism. In particular $f_*\varepsilon$ induces an epimorphism on cohomology sheaves and, because f is affine, this means that ε induces an epimorphism of cohomology sheaves already over X. We have a morphism $\varepsilon : \mathbf{L}f^*f_*F \longrightarrow F$ in $\mathbf{D_{qc}}(X)^{\leq 0}$ and, if we complete it to a triangle, the long exact sequence of cohomology sheaves gives

(iii) In the triangle $\mathbf{L}f^*f_*F \xrightarrow{\varepsilon} F \longrightarrow D''$ we have $D'' \in \mathbf{D_{qc}}(X)^{\leq -1}$.

Next we form the octahedron

$$\begin{tikzcd}
\mathbf{L}f^*E' \arrow[r, equal] \arrow[d] & \mathbf{L}f^*E' \arrow[d] & \\
\mathbf{L}f^*f_*F \arrow[r, "\varepsilon"] \arrow[d] & F \arrow[r] \arrow[d] & D'' \arrow[d, equal] \\
\mathbf{L}f^*D' \arrow[r] & D \arrow[r] & D''
\end{tikzcd}$$

and (ii) tells us that $\mathbf{L}f^*E' \in \overline{\langle \mathbf{L}f^*G\rangle}_A^{[-A,A]}$ and $\mathbf{L}f^*D' \in \mathbf{D_{qc}}(X)^{\leq -1}$, while (iii) gives that $D'' \in \mathbf{D_{qc}}(X)^{\leq -1}$. The triangle $\mathbf{L}f^*D' \longrightarrow D \longrightarrow D''$ tells us that $D \in \mathbf{D_{qc}}(X)^{\leq -1}$, and the triangle $\mathbf{L}f^*E' \longrightarrow F \longrightarrow D$ does the trick. □

Example 5.6 Assume X is separated and quasicompact, and let the *t*–structure on $\mathcal{T} = \mathbf{D_{qc}}(X)$ be the standard one. Lemma 5.5 finds a generator G' and an integer $A > 0$ so that, for every object $F \in \mathcal{T}^{\leq 0}$, there exists a triangle $E \longrightarrow F \longrightarrow D$ with $E \in \overline{\langle G'\rangle}_A^{[-A,A]}$ and $D \in \mathcal{T}^{\leq -1}$. From Reminder 5.4 we know that the compact generator $G' \in \mathbf{D_{qc}}(X)$ is a perfect complex and we may, after increasing the integer A if necessary, guarantee that $\mathrm{Hom}\left(\Sigma^{-A}G', \mathcal{T}^{\leq 0}\right) = 0$ and $\Sigma^A G' \in \mathcal{T}^{\leq 0}$. Putting this together we have that $\mathcal{T}$ is approximable; it satisfies Definition 1.25 for the compact generator G', the standard *t*–structure, and the integer $A > 0$.

And now Proposition 3.4 informs us that the standard t–structure is in the preferred equivalence class.

Remark 5.7 In the recent [21, Theorem 3.2 (iii) and (iv)] the reader can find a proof that, in the generality where X is only assumed quasicompact and quasiseparated, the category $\mathcal{T} = \mathbf{D_{qc}}(X)$ is weakly approximable and the standard t–structure is in the preferred equivalence class.

In the present article we care much more about the case where $\mathcal{T}$ is not only weakly approximable, but genuinely approximable. And, at least with the current machinery, to prove the approximability of $\mathbf{D_{qc}}(X)$ we need to assume X separated. The methods used in [21], to prove the weak approximability of $\mathbf{D_{qc}}(X)$ for general X, do not modify in a straightforward way to prove approximability. In [21, Remarks 8.1 and 8.2] there is a discussion: these techniques prove the weak approximability of categories that are decidedly not approximable.

Remark 5.8 Let X be a quasicompact, quasiseparated scheme, and let $\mathcal{T} = \mathbf{D_{qc}}(X)$. The category $\mathcal{T}$ has a single compact generator by Reminder 5.4, and we now know that the standard t–structure is in the preferred equivalence class. One immediately computes that the preferred $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ are given by

$$\mathcal{T}^- = \mathbf{D}^-_{\mathbf{qc}}(X), \qquad \mathcal{T}^+ = \mathbf{D}^+_{\mathbf{qc}}(X), \qquad \mathcal{T}^b = \mathbf{D}^b_{\mathbf{qc}}(X).$$

Next we turn our attention to the computation of $\mathcal{T}^-_c$ and its subcategory $\mathcal{T}^b_c$.

To facilitate the discussion it might be helpful to imagine two classes of objects in $\mathcal{T} = \mathbf{D_{qc}}(X)$.

(i) The objects belonging to $\mathcal{T}^-_c = \mathbf{D_{qc}}(X)^-_c$, for the $\mathcal{T}^-_c$ defined using the standard t–structure.

(ii) The objects locally with this property. That is the objects $P \in \mathbf{D_{qc}}(X)$ such that, for any open immersion $j : \mathrm{Spec}(R) \longrightarrow X$, the object $\mathbf{L}j^*P \in \mathbf{D_{qc}}\big(\mathrm{Spec}(R)\big) \cong \mathbf{D}(R)$ is in $\mathbf{D}(R)^-_c$. See Example 5.1 for a description of $\mathbf{D}(R)^-_c$.

The objects satisfying (ii) are classically called *pseudocoherent,* they were first studied in Illusie's exposés [14, 15] in SGA6. Now [17, Theorem 4.1] is precisely the statement that the objects satisfying (ii) all satisfy (i). It is trivial to check that the objects satisfying (i) must satisfy (ii); this means that, for the standard t–structure on $\mathcal{T} = \mathbf{D_{qc}}(X)$, the subcategory $\mathcal{T}^-_c$ is just $\mathbf{D}^p_{\mathbf{qc}}(X) \subset \mathbf{D_{qc}}(X)$, the subcategory of pseudocoherent complexes. If X happens to be noetherian then pseudocoherence simplifies to something more familiar: for noetherian X we have $\mathcal{T}^-_c = \mathbf{D}^p_{\mathbf{qc}}(X) = \mathbf{D}^-_{\mathrm{coh}}(X)$. And the subcategory $\mathcal{T}^b_c \subset \mathcal{T}^-_c$ is also explicit, it is the full subcategory $\mathbf{D}^{p,b}_{\mathbf{qc}}(X) \subset \mathbf{D}^p_{\mathbf{qc}}(X)$ whose objects are the pseudocoherent complexes with bounded cohomology. When X is noetherian this simplifies to $\mathcal{T}^b_c = \mathbf{D}^{p,b}_{\mathbf{qc}}(X) = \mathbf{D}^b_{\mathrm{coh}}(X)$.

Summary 5.9 Until now we have shown

(i) The category $\mathbf{D_{qc}}(X)$ is weakly approximable for all quasicompact, quasiseparated X, and approximable as long as X is separated.

(ii) The standard t–structure is in the preferred equivalence class.

(iii) We have computed the intrinsic subcategories $\mathbf{D_{qc}}(X)^-$, $\mathbf{D_{qc}}(X)^+$, $\mathbf{D_{qc}}(X)^b$, $\mathbf{D_{qc}}(X)^-_c$ and $\mathbf{D_{qc}}(X)^b_c$.

The computations in (iii) yielded no big surprises, the intrinsic subcategories turned out to be classical, old friends.

When we apply Corollary 3.14 we discover something new.

(iv) Let X be a quasicompact, quasiseparated scheme, and let G be a compact generator of $\mathbf{D_{qc}}(X)$. There exists an integer $B > 0$ so that, for any integer $m > 0$ and any object $F \in \mathbf{D}^p_{\mathbf{qc}}(X) \cap \mathbf{D_{qc}}(X)^{\leq 0}$, there is a triangle $E_m \longrightarrow F \longrightarrow D_m$ with $E_m \in \langle G\rangle^{[1-m-B,B]}$ and $D_m \in \mathbf{D_{qc}}(X)^{\leq -m}$.

(v) If X is separated we may choose B to guarantee that $E_m \in \langle G\rangle^{[1-m-B,B]}_{mB}$.

6 An easy representability theorem

It's time to start proving representability theorems. The main theorems in the article are a little technical to prove—they hinge on taking homotopy colimits carefully. For this reason I thought it best to illustrate the methods in a simple case, which involves no homotopy colimits. In this section we give a simple proof of an old result of Rouquier, generalizing an even older result of Bondal and Van den Bergh.

But first we recall the following little lemma, well-known in some circles.

Reminder 6.1 Let $\mathcal{S}$ be a triangulated category, let $\mathcal{B} \subset \mathcal{S}$ be a full, replete subcategory with $\Sigma\mathcal{B} = \mathcal{B}$, and let $\mathcal{A}$ be an abelian category. Assume we are given a short exact sequence $0 \longrightarrow H' \longrightarrow H \longrightarrow H'' \longrightarrow 0$ of functors $\mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{A}$.

If two of the three functors $H', H, H'' : \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{A}$ are $\mathcal{B}$–cohomological as in Definition 1.1, then so is the third.

Proof Let $x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x$ be a triangle in $\mathcal{S}$, such that the objects x, y, z all belong to $\mathcal{B}$. Then with $\widetilde{H}$ being any of H', H or H'', the sequence

$$\cdots \to \widetilde{H}(\Sigma x) \to \widetilde{H}(z) \to \widetilde{H}(y) \to \widetilde{H}(x) \to \widetilde{H}(\Sigma^{-1}z) \to \cdots$$

is a cochain complex in the abelian category $\mathcal{A}$. Thus we have three cochain complexes, one for each of H', H or H''. Because two of the functors are $\mathcal{B}$–cohomological, two of these three cochain complexes must be acyclic. But then the short exact sequence $0 \longrightarrow H' \longrightarrow H \longrightarrow H'' \longrightarrow 0$ tells us that the third complex is also acyclic. ☐

Next we prove

Lemma 6.2 *Let R be a commutative, noetherian ring, let $\mathcal{S}$ be an R–linear triangulated category, let $G \in \mathcal{S}$ be an object and, with the notation of Definition 1.1 and Remark 1.2, assume* $\mathrm{Hom}(-, G)$ *is a G–finite cohomological functor. Let $k \geq 0$ be an integer and H a finite $\langle G\rangle_{2^k}$–cohomological functor. Then there exists an object $F \in \langle G\rangle_{2^k}$ and an epimorphism* $\mathrm{Hom}(-, F)|_{\langle G\rangle_{2^k}} \longrightarrow H(-)$.

Proof The proof is by induction on k. Suppose $k = 0$: by hypothesis $H(\Sigma^{-i}G)$ is a finite R–module, and vanishes for i outside a bounded interval $[-A, A]$. For each i with $-A \le i \le A$ choose a finite set of generators $\{f_{ij},\ j \in J_i\}$ for the R–module $H(\Sigma^{-i}G)$. Yoneda tells us that each f_{ij} corresponds to a natural transformation $\varphi_{ij} : \mathrm{Hom}(-, \Sigma^{-i}G)|_{\langle G\rangle_1} \longrightarrow H(-)$. Define F to be $F = \oplus_{i=-A}^{A} \oplus_{j\in J_i} \Sigma^{-i}G$, and let $\varphi : \mathrm{Hom}(-, F)|_{\langle G\rangle_1} \longrightarrow H(-)$ be the composite

$$\mathrm{Hom}(-, F)|_{\langle G\rangle_1} = \bigoplus_{i=-A}^{A} \bigoplus_{j\in J_i} \mathrm{Hom}(-,\ \Sigma^{-i}G)|_{\langle G\rangle_1} \xrightarrow{(\varphi_{ij})} H(-)$$

Obviously F belongs to $\langle G\rangle_1$ and φ is surjective.

Now suppose we know the Lemma for $k \ge 0$, and let H be a finite $\langle G\rangle_{2^{k+1}}$–cohomological functor. Then the restriction of H to $\langle G\rangle_{2^k}$ is a finite $\langle G\rangle_{2^k}$–cohomological functor, and induction permits us to find an object $F_1 \in \langle G\rangle_{2^k}$ and an epimorphism $\varphi : \mathrm{Hom}(-, F_1)|_{\langle G\rangle_{2^k}} \longrightarrow H(-)|_{\langle G\rangle_{2^k}}$. Complete the map φ to a short exact sequence

$$0 \longrightarrow H'(-) \xrightarrow{\sigma} \mathrm{Hom}(-, F_1)|_{\langle G\rangle_{2^k}} \xrightarrow{\varphi} H(-)|_{\langle G\rangle_{2^k}} \longrightarrow 0\,.$$

By Reminder 6.1 the functor H' is $\langle G\rangle_{2^k}$–cohomological, it is clearly finite, and induction applies again to tell us that there exists an object $F_2 \in \langle G\rangle_{2^k}$ and an epimorphism $\rho : \mathrm{Hom}(-, F_2)|_{\langle G\rangle_{2^k}} \longrightarrow H'(-)$. Combining the results we deduce an exact sequence of functors

$$\mathrm{Hom}(-, F_2)|_{\langle G\rangle_{2^k}} \xrightarrow{\sigma\rho} \mathrm{Hom}(-, F_1)|_{\langle G\rangle_{2^k}} \xrightarrow{\varphi} H(-)|_{\langle G\rangle_{2^k}} \longrightarrow 0\,.$$

Because F_1 and F_2 both lie in $\langle G\rangle_{2^k}$ the functors $\mathrm{Hom}(-, F_i)|_{\langle G\rangle_{2^k}}$ are representable for $i \in \{1, 2\}$—Yoneda's lemma applies. The natural transformation $\sigma\rho : \mathrm{Hom}(-, F_2)|_{\langle G\rangle_{2^k}} \longrightarrow \mathrm{Hom}(-, F_1)|_{\langle G\rangle_{2^k}}$ is $\mathrm{Hom}(-, \alpha)|_{\langle G\rangle_{2^k}}$ for some morphism $\alpha : F_2 \longrightarrow F_1$, the natural transformation $\varphi : \mathrm{Hom}(-, F_1)|_{\langle G\rangle_{2^k}} \longrightarrow H(-)|_{\langle G\rangle_{2^k}}$ corresponds to some element $y \in H(F_1)$, and the vanishing of the composite $\varphi\sigma\rho$ says that $H(\alpha) : H(F_1) \longrightarrow H(F_2)$ must take y to zero.

Complete $\alpha : F_2 \longrightarrow F_1$ to a triangle $F_2 \xrightarrow{\alpha} F_1 \xrightarrow{\beta} F \xrightarrow{\gamma} \Sigma F_2$. As F_1 and ΣF_2 belong to $\langle G\rangle_{2^k}$, the triangle tells us that F must be in $\langle G\rangle_{2^k} * \langle G\rangle_{2^k} \subset \langle G\rangle_{2^{k+1}}$. And now we remember that H is actually a $\langle G\rangle_{2^{k+1}}$–cohomological functor. The sequence $H(F) \xrightarrow{H(\beta)} H(F_1) \xrightarrow{H(\alpha)} H(F_2)$ is exact, and the vanishing of $H(\alpha)(y)$ says that there exists an element $x \in H(F)$ so that $H(\beta) : H(F) \longrightarrow H(F_1)$ takes $x \in H(F)$ to $y \in H(F_1)$. By Yoneda x corresponds to a natural transformation $\psi : \mathrm{Hom}(-, F)|_{\langle G\rangle_{2^{k+1}}} \longrightarrow H(-)$. The fact that $H(\beta)x = y$ translates, via Yoneda, to the assertion that the composite

$$\begin{tikzcd}\mathrm{Hom}(-,F_1)|_{\langle G\rangle_{2^{k+1}}} \arrow[r, "\mathrm{Hom}(-,\beta)|_{\langle G\rangle_{2^{k+1}}}"] & \mathrm{Hom}(-,F)|_{\langle G\rangle_{2^{k+1}}} \arrow[r, "\psi"] & H(-)\end{tikzcd}$$

restricts to be φ on the category $\langle G\rangle_{2^k}$.

We assert that ψ is surjective. Take any object $C' \in \langle G\rangle_{2^{k+1}}$; we need to show the surjectivity of the map $\psi : \mathrm{Hom}(C', F) \longrightarrow H(C')$. Now by definition $\langle G\rangle_{2^{k+1}} = \mathrm{smd}\big[\langle G\rangle_{2^k} * \langle G\rangle_{2^k}\big]$, and applying this gives an object $C'' \in \langle G\rangle_{2^{k+1}}$ with $C' \oplus C'' \in \langle G\rangle_{2^k} * \langle G\rangle_{2^k}$. Put $C = C' \oplus C''$ and it clearly suffices to prove the surjectivity of $\psi : \mathrm{Hom}(C, F) \longrightarrow H(C)$. Choose a triangle $A \xrightarrow{\alpha'} B \xrightarrow{\beta'} C \xrightarrow{\gamma'} \Sigma A$ with $A, B \in \langle G\rangle_{2^k}$.

Take any $z \in H(C)$. The map $H(\beta') : H(C) \longrightarrow H(B)$ takes z to an element $H(\beta')z \in H(B)$. But the map $\varphi : \mathrm{Hom}(B, F_1) \longrightarrow H(B)$ is surjective, hence there is an element $g \in \mathrm{Hom}(B, F_1)$ with $\varphi(g) = H(\beta')(z)$. Now $0 = H(\alpha')H(\beta')z = H(\alpha')\varphi(g) = \varphi\,\mathrm{Hom}(\alpha', F_1)g$, where the last equality is by the naturality of the map $\varphi : \mathrm{Hom}(-, F_1)|_{\langle G\rangle_{2^k}} \longrightarrow H(-)|_{\langle G\rangle_{2^k}}$. Therefore $\mathrm{Hom}(\alpha', F_1)g = g\alpha' \in \mathrm{Hom}(A, F_1)$ lies in the kernel of $\varphi : \mathrm{Hom}(A, F_1) \longrightarrow H(A)$. The exact sequence

$$\begin{tikzcd}\mathrm{Hom}(-,F_2)|_{\langle G\rangle_{2^k}} \arrow[r, "\mathrm{Hom}(-,\alpha)|_{\langle G\rangle_{2^k}}"] & \mathrm{Hom}(-,F_1)|_{\langle G\rangle_{2^k}} \arrow[r, "\varphi"] & H(-)|_{\langle G\rangle_{2^k}}\end{tikzcd}$$

tells us that there is an $f \in \mathrm{Hom}(A, F_2)$ with $g\alpha' = \mathrm{Hom}(A, \alpha)f = \alpha f$.

In concrete terms we have produced a commutative diagram

$$\begin{tikzcd}A \arrow[r, "\alpha'"] \arrow[d, "f"] & B \arrow[r, "\beta'"] \arrow[d, "g"] & C \arrow[r, "\gamma'"] & \Sigma A \\ F_2 \arrow[r, "\alpha"] & F_1 \arrow[r, "\beta"] & F \arrow[r, "\gamma"] & \Sigma F_2\end{tikzcd}$$

where the rows are triangles, which we may complete to a morphism of triangles

$$\begin{tikzcd}A \arrow[r, "\alpha'"] \arrow[d, "f"] & B \arrow[r, "\beta'"] \arrow[d, "g"] & C \arrow[r, "\gamma'"] \arrow[d, "h"] & \Sigma A \arrow[d, "\Sigma f"] \\ F_2 \arrow[r, "\alpha"] & F_1 \arrow[r, "\beta"] & F \arrow[r, "\gamma"] & \Sigma F_2\end{tikzcd}$$

We now have an element $h \in \mathrm{Hom}(C, F)$ with $h\beta' = \beta g$, or in more complicated terms $\mathrm{Hom}(\beta', F)h = \mathrm{Hom}(B, \beta)g$. This buys us the third equality below

$$\begin{aligned} H(\beta')z &= \varphi(g) \\ &= \psi\,\mathrm{Hom}(B,\beta)(g) \\ &= \psi\,\mathrm{Hom}(\beta',F)h \\ &= H(\beta')\psi(h) \end{aligned}$$

The first equality is the choice of g, the second is because on the category $\langle G\rangle_{2^k}$ we have $\varphi = \psi \circ \mathrm{Hom}(-,\beta)$, and the fourth equality is the naturality of ψ. We deduce that $H(\beta') : H(C) \longrightarrow H(B)$ annihilates $z - \psi(h)$.

But the exact sequence $H(\Sigma A) \xrightarrow{H(\gamma')} H(C) \xrightarrow{H(\beta')} H(B)$ tells us there exists a $z' \in H(\Sigma A)$ with $H(\gamma')z' = z - \psi(h)$. Since ΣA belongs to $\langle G\rangle_{2^k}$ the map $\varphi : \mathrm{Hom}(\Sigma A, F_1) \longrightarrow H(\Sigma A)$ must be surjective; there is a $\lambda \in \mathrm{Hom}(\Sigma A, F_1)$ with $\varphi(\lambda) = z'$. Therefore

$$\begin{aligned} z - \psi(h) &= H(\gamma')\varphi(\lambda) \\ &= H(\gamma')\psi\,\mathrm{Hom}(\Sigma A,\beta)(\lambda) \\ &= \psi\,\mathrm{Hom}(\gamma',F)\,\mathrm{Hom}(\Sigma A,\beta)(\lambda) \\ &= \psi(\beta\lambda\gamma') \end{aligned}$$

where the second equality is the fact that on the category $\langle G\rangle_{2^k}$ we have $\varphi = \psi \circ \mathrm{Hom}(-,\beta)$, the third is the naturality of ψ, and the fourth is obvious. Hence $z = \psi(h + \beta\lambda\gamma')$ is in the image of ψ. ☐

When R is a field the theorem below is due to Bondal and Van den Bergh [5, Theorem 1.3], and in the generality below it may be found in Rouquier [22, Theorem 4.16 and Corollary 4.18]. We include this new proof because it contains the simple ideas, whose more technical adaptation will yield the theorems of Sects. 9 and 12.

Theorem 6.3 *Let R be a noetherian, commutative ring, let $\mathcal{S}$ be an R–linear triangulated category, and assume that $G \in \mathcal{S}$ is a strong generator—we remind the reader, this means that there exists some integer $n > 0$ with $\langle G\rangle_n = \mathcal{S}$. Assume H is a finite cohomological functor, as is* $\mathrm{Hom}(-,G)$. *Then there exists a cohomological functor H' with $H \oplus H'$ representable. If $\mathcal{S}$ is Karoubian, meaning idempotents split, then H is representable.*

Proof Choose an integer k with $\langle G\rangle_{2^k} = \mathcal{S}$. Applying Lemma 6.2 to the $\langle G\rangle_{2^k}$–cohomological functor H we can find an epimorphism $\mathrm{Hom}(-,F) \longrightarrow H(-)$. Complete to an exact sequence $0 \longrightarrow H'(-) \longrightarrow \mathrm{Hom}(-,F) \longrightarrow H(-) \longrightarrow 0$, and it follows that H' is also a finite cohomological functor. Applying Lemma 6.2 again we have a surjection $\mathrm{Hom}(-,F') \longrightarrow H'(-)$, and this assembles to an exact sequence $\mathrm{Hom}(-,F') \longrightarrow \mathrm{Hom}(-,F) \longrightarrow H(-) \longrightarrow 0$.

Yoneda's lemma tells us that the natural transformation $\mathrm{Hom}(-,F') \longrightarrow \mathrm{Hom}(-,F)$ must be of the form $\mathrm{Hom}(-,\alpha)$ for some $\alpha : F' \longrightarrow F$, that the natural transformation $\mathrm{Hom}(-,F) \longrightarrow H(-)$ corresponds to an element $y \in H(F)$, and that the vanishing of the composite $\mathrm{Hom}(-, F') \longrightarrow \mathrm{Hom}(-, F) \longrightarrow H(-)$ means that the map $H(\alpha) : H(F) \longrightarrow H(F')$ must take $y \in H(F)$ to zero. Now complete $\alpha : F' \longrightarrow F$ to a triangle $F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \longrightarrow$. The exactness of $H(F'') \xrightarrow{H(\beta)} H(F) \xrightarrow{H(\alpha)} H(F')$, coupled with the fact that $H(\alpha)y = 0$, means that there must be an element $x \in H(F'')$ with $y = H(\beta)x$. By Yoneda this means that we obtain a natural transformation $\mathrm{Hom}(-, F'') \longrightarrow H(-)$ so that the diagram below commutes

$$\begin{tikzcd}
\mathrm{Hom}(-, F') \arrow[r] \arrow[d, equal] & \mathrm{Hom}(-, F) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(-, F'') \arrow[d] & \\
\mathrm{Hom}(-, F') \arrow[r] & \mathrm{Hom}(-, F) \arrow[r] & H(-) \arrow[r] & 0
\end{tikzcd}$$

and the rows are exact. It immediately follows that the map $\mathrm{Hom}(-, F'') \longrightarrow H(-)$ is a split epimorphism.

Choose a splitting $H(-) \longrightarrow \mathrm{Hom}(-, F'')$, for example the one coming from the diagram above. The composite $\mathrm{Hom}(-, F'') \longrightarrow H(-) \longrightarrow \mathrm{Hom}(-, F'')$ is an idempotent natural endomorphism of a representable functor, therefore of the form $\mathrm{Hom}(-, e)$ where $e : F'' \longrightarrow F''$ is idempotent. If e splits then H is representable. ☐

7 Approximating systems

In this short section we assemble together a few fairly routine, elementary facts about countable direct limits of representable functors. We work in the generality of R–linear functors between R–linear categories, where R is a commutative ring. What is slightly unusual is that we allow three interacting categories, and hence need a moment's reflection to make sure that the standard arguments still work.

Definition 7.1 Let R be a commutative ring, let $\mathcal{T}$ be an R–linear category, let $\mathcal{A}$, $\mathcal{B}$ be full subcategories of $\mathcal{T}$, and let $H : \mathcal{B}^{\mathrm{op}} \longrightarrow R\text{–Mod}$ be an R–linear functor. An *$\mathcal{A}$–approximating system for H* is a sequence in $\mathcal{A}$ of objects and morphisms $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$, so that

(i) There is a cofinal subsequence of E_* whose objects belong to $\mathcal{A} \cap \mathcal{B}$.
(ii) We are given an isomorphism $\underrightarrow{\mathrm{colim}}\, \mathrm{Hom}(-, E_i) \longrightarrow H(-)$ of functors $\mathcal{B}^{\mathrm{op}} \longrightarrow R\text{–Mod}$.

In the proof of Theorem 1.4(i) we will mostly consider the case where $\mathcal{A}$ is contained in $\mathcal{B}$, but in the proof of Theorem 1.4(ii) we will need the more flexible notion.

Since we will freely use approximating systems in our constructions, it is comforting to know that they are all the same up to subsequences. More precisely we have

Lemma 7.2 *Suppose we have an R–linear functor $H:\mathcal{B}^{\mathrm{op}}\longrightarrow R$–Mod, and two $\mathcal{A}$–approximating systems E_* and F_* for H. Then the systems E_* and F_* are ind-isomorphic. We remind the reader: this means that there exists an $\mathcal{A}$–approximating system L_* for the functor H, more explicitly $L_1\longrightarrow L_2\longrightarrow L_3\longrightarrow\cdots$, and subsequences $E'_*\subset E_*$, $F'_*\subset F_*$ and $L'_*, L''_*\subset L_*$ with $E'_*=L'_*$ and $F'_*=L''_*$.*

Proof Since we may pass to subsequences we will assume that both E_* and F_* belong to $\mathcal{A}\cap\mathcal{B}$. To slightly compress the argument we will extend the sequences by zero; we will set $E_0=F_0=L_0=0$, look at the sequences $E_0\longrightarrow E_1\longrightarrow E_2\longrightarrow\cdots$ and $F_0\longrightarrow F_1\longrightarrow F_2\longrightarrow\cdots$, and out of them construct the sequence $L_0\longrightarrow L_1\longrightarrow L_2\longrightarrow\cdots$.

Let $L_0=0$ and $L_1=E_1$, and inductively proceed as follows. Assume that, for some $m\geq 1$, the sequence $L_0\longrightarrow L_1\longrightarrow\cdots L_{m-1}\longrightarrow L_m$ and the map from the sequence to $H(-)$ have been defined, in such a way that $L_1\longrightarrow L_3\longrightarrow L_5\longrightarrow\cdots$ is a subsequence of E_* and $L_0\longrightarrow L_2\longrightarrow L_4\longrightarrow L_6\longrightarrow\cdots$ is a subsequence of F_*. We wish to extend to $m+1$. There are two cases, m can be odd or even, but up to interchanging E_* and F_* in the argument below they are the same. We will therefore assume m odd and leave to the reader the even case.

Then L_{m-1} is F_I for some $I\geq 0$, while L_m may be assumed equal to E_J for some $J>I$. Viewed in the category of functors $(\mathcal{A}\cap\mathcal{B})^{\mathrm{op}}\longrightarrow R$–Mod, the map $\mathrm{Hom}(-,L_m)=\mathrm{Hom}(-,E_J)\longrightarrow H(-)=\underrightarrow{\mathrm{colim}}\,\mathrm{Hom}(-,F_i)$ is a natural transformation from the representable functor $\mathrm{Hom}(-,E_J)$ to the colimit. Yoneda tells us that it corresponds to an element in $H(E_J)=\underrightarrow{\mathrm{colim}}\,\mathrm{Hom}(E_J,F_i)$, where the colimit is over i. We may therefore choose an $I'>J$ and a morphism $L_m=E_J\longrightarrow F_{I'}$ which delivers the right element in the colimit. The composite $F_I=L_{m-1}\longrightarrow L_m\longrightarrow F_{I'}$ does not have to agree with the map $F_I\longrightarrow F_{I'}$ of the sequence F_*, but they have the same image in $H(F_I)=\underrightarrow{\mathrm{colim}}\,\mathrm{Hom}(F_I,F_i)$ [where the colimit is again over i]. That is: after composing with some $F_{I'}\longrightarrow F_{I''}$ in the sequence F_* they become equal. Set $L_{m+1}=F_{I''}$. ☐

Lemma 7.3 *Let $\mathcal{T}$ be an R–linear category, let $\mathcal{A},\mathcal{B}$ be full subcategories, let $H,H':\mathcal{B}^{\mathrm{op}}\longrightarrow R$–Mod be two R–linear functors, and assume we are given for H an $\mathcal{A}$–approximating system $E_1\longrightarrow E_2\longrightarrow E_3\longrightarrow\cdots$. Replacing the sequence E_* by a subsequence belonging to $\mathcal{A}\cap\mathcal{B}$, there is a natural isomorphism* $\mathrm{Hom}(H,H')\cong\underleftarrow{\lim}\,H'(E_m)$ *in the category of functors $\mathcal{B}^{\mathrm{op}}\longrightarrow R$–Mod.*

Proof We have isomorphisms

$$\begin{aligned}\mathrm{Hom}(H,H')&=\mathrm{Hom}\big(\underrightarrow{\mathrm{colim}}\,\mathrm{Hom}(-,E_m)\,,\ H'(-)\big)\\&=\underleftarrow{\lim}\ \mathrm{Hom}\big(\mathrm{Hom}(-,E_m)\,,\ H'(-)\big)\\&=\underleftarrow{\lim}\ H'(E_m)\end{aligned}$$

where the last isomorphism is by Yoneda. ☐

Corollary 7.4 *Suppose we are given R–linear categories* $\mathscr{A} \subset \mathscr{B}$ *and two R–linear functors* $H, H' : \mathscr{B}^{\mathrm{op}} \longrightarrow R\text{–}Mod$*. If* H *has an* $\mathscr{A}$*–approximating system then restriction to the subcategory* $\mathscr{A} \subset \mathscr{B}$ *is a natural bijection* $\mathrm{Hom}(H, H') \longrightarrow \mathrm{Hom}\big(H|_{\mathscr{A}}, H'|_{\mathscr{A}}\big)$.

Proof Choose an $\mathscr{A}$–approximating system $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ for H. Lemma 7.3 tells us that both sets are in bijection with $\varprojlim H'(E_m)$, and the bijection commutes with the restriction map. □

Lemma 7.5 *Let* $\mathscr{B}$ *be an R–linear category, let* $H, H' : \mathscr{B}^{\mathrm{op}} \longrightarrow R\text{–}Mod$ *be two R–linear functors, and let* $\varphi : H \longrightarrow H'$ *be a natural transformation. If each of* H, H' *has a* $\mathscr{B}$*–approximating system, let's say* $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ *for* H *and* $E'_1 \longrightarrow E'_2 \longrightarrow E'_3 \longrightarrow \cdots$ *for* H' *then, after replacing* $E'_1 \longrightarrow E'_2 \longrightarrow E'_3 \longrightarrow \cdots$ *by a subsequence, we can produce a map of sequences* $f_* : E_* \longrightarrow E'_*$ *so that* $\varphi : H \longrightarrow H'$ *is the colimit of the image of* f_* *under Yoneda.*

Moreover: assume we specify in advance subsequences $\widetilde{E}_* \subset E_*$ *and* $\widetilde{E}'_* \subset E'_*$*. Then the replacement of* E'_* *by a subsequence on which the morphism* f *is defined, as in the last paragraph, can be made in such a way that if* E_n *belongs to* $\widetilde{E}_*$ *then* E'_n *will belong to* $\widetilde{E}'_n$*. And we can even handle more than one subsequence: we can specify in advance a finite set of matching subsequences, pairs*

$$\Big(\{\textit{subsequence of } E_*\} \quad , \quad \{\textit{subsequence of } E'_*\}\Big) \; ,$$

and, in the replacement of E'_* *by a subsequence as in the previous paragraph, we can guarantee that if* E_n *belongs to one of the specified subsequences of* E_* *then* E'_n *is chosen to belong to the matching subsequence of* E'_**.*

Proof By Lemma 7.3 we have an isomorphism

$$\mathrm{Hom}(H, H') \cong \varprojlim H'(E_i) = \varprojlim{}_i \varinjlim{}_j \mathrm{Hom}(E_i, E'_j).$$

Our element $\varphi \in \mathrm{Hom}(H, H')$ must therefore correspond to an inverse system of elements $\varphi_i \in \varinjlim{}_j \mathrm{Hom}(E_i, E'_j)$. We proceed inductively.

(i) Choose some integer j_1 and a preimage in $\mathrm{Hom}(E_1, E'_{j_1})$ of $\varphi_1 \in \varinjlim{}_j \mathrm{Hom}(E_1, E_j)$. Call this map $f_1 : E_1 \longrightarrow E'_{j_1}$. If E_1 belongs to one of the prescribed subsequences of E_* then choose j_1 so that E'_{j_1} belongs to the matching subsequence of E'_*.

(ii) Suppose the sequence has been constructed up to an integer $m \geq 1$. In particular we have a map $E_m \longrightarrow E'_{j_m}$, whose image under the natural map $\mathrm{Hom}(E_m, E'_{j_m}) \longrightarrow \varinjlim{}_j \mathrm{Hom}(E_m, E'_j)$ is φ_m.

We have the element $\varphi_{m+1} \in \varinjlim{}_j \mathrm{Hom}(E_{m+1}, E'_j)$, we can choose a preimage in $\mathrm{Hom}(E_{m+1}, E_J)$ for some integer J, and we may assume $J > j_m$.

This gives us a map $f' : E_{m+1} \longrightarrow E'_J$. Now the square

$$\begin{tikzcd}
E_m \arrow[r, "f_m"] \arrow[d] & E'_{j_m} \arrow[d] \\
E_{m+1} \arrow[r, "f'"] & E'_J
\end{tikzcd}$$

need not commute, but the two composites both go, via the map $\mathrm{Hom}(E_m, E'_J) \longrightarrow \underrightarrow{\mathrm{colim}}_j \mathrm{Hom}(E_m, E'_j)$, to the same element φ_m. Hence replacing E'_J by some $E'_{j_{m+1}}$ with $j_{m+1} > J$, we may assume the square commutes. And if E_{m+1} belongs to one of the prescribed subsequences of E_*, choose $j_{m+1} > J$ so that $E'_{j_{m+1}}$ belongs to the matching subsequence of E'_*.

We have replaced E'_* by a subsequence and produced a map of sequences $f_* : E_* \longrightarrow E'_*$. The reader can check that, if we apply Yoneda to the map of sequences $f_* : E_* \longrightarrow E'_*$ and then take colimits, we recover $\varphi : H \longrightarrow H'$. ☐

Remark 7.6 In the remainder of the paper we will use approximating systems in the following situation. We will work in some ambient R–linear triangulated category $\mathcal{T}$, and will assume that $\mathcal{T}$ has coproducts. What is special in this case is that, given a functor $H : \mathcal{B}^{\mathrm{op}} \longrightarrow R$–Mod and an $\mathcal{A}$–approximating system $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ for H, we can construct in $\mathcal{T}$ the homotopy colimit $F = \underrightarrow{\mathrm{Hocolim}}\ E_i$. For $(-)$ in the category $\mathcal{B}$ we have a natural map

$$H(-) \Longleftarrow\!\!\Longrightarrow \underrightarrow{\mathrm{colim}}\ \mathrm{Hom}(-, E_i) \longrightarrow \mathrm{Hom}(-, F)$$

and we will be interested in approximating sequences for which this map $H(-) \longrightarrow \mathrm{Hom}(-, F)$ is an isomorphism of functors $\mathcal{B}^{\mathrm{op}} \longrightarrow R$–Mod.

In this situation we will say that $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ is an $\mathcal{A}$–approximating system for F over $\mathcal{B}$.

Remark 7.7 In the proof of Theorem 1.4(i), the case of interest is where $\mathcal{B} \subset \mathcal{T}^c$, that is the objects of $\mathcal{B}$ are all compact. From [18, Lemma 2.8] we know that, for any compact object $K \in \mathcal{T}$ and any sequence of objects of $\mathcal{T}$, the natural map is an isomorphism $H(K) \cong \underrightarrow{\mathrm{colim}}\ \mathrm{Hom}(K, E_i) \longrightarrow \mathrm{Hom}\big(K, \underrightarrow{\mathrm{Hocolim}}\ E_i\big)$. Thus we're automatically in the situation of Remark 7.6; any sequence E_* in $\mathcal{A}$ is an $\mathcal{A}$–approximating system for $F = \underrightarrow{\mathrm{Hocolim}}\ E_i$, over any $\mathcal{B} \subset \mathcal{T}^c$. In Sect. 9 we will therefore allow ourselves to occasionally leave unspecified the category $\mathcal{B} \subset \mathcal{T}^c$, and just say that E_* is an $\mathcal{A}$–approximating system for F.

But here we are careful to specify $\mathcal{B}$, because this will come up in the proof of Theorem 1.4(ii).

In the generality of Remark 7.6 we note the following little observation.

Lemma 7.8 *Let $\mathcal{T}$ be a triangulated category with coproducts, let $\mathcal{A}$, $\mathcal{B}$ be subcategories, and assume $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ is a $\mathcal{A}$–approximating system for $F \in \mathcal{T}$ over $\mathcal{B}$.*

If G is another object of $\mathcal{T}$, and if $\varphi : \mathrm{Hom}(-, F)|_{\mathcal{B}} \longrightarrow \mathrm{Hom}(-, G)|_{\mathcal{B}}$ is a natural transformation of functors on $\mathcal{B}$, then there exists in $\mathcal{T}$ a (non-unique) morphism $f : F \longrightarrow G$ with $\varphi = \mathrm{Hom}(-, f)|_{\mathcal{B}}$.

Proof By Lemma 7.3 the natural transformation $\varphi : \mathrm{Hom}(-, F) \longrightarrow \mathrm{Hom}(-, G)$ corresponds to an element in $\varprojlim \mathrm{Hom}(E_i, G)$. Thus for each i we are given in $\mathcal{T}$ a morphism $f_i : E_i \longrightarrow G$, compatibly with the sequence maps $E_i \longrightarrow E_{i+1}$. The compatibility means that the composite

$$\begin{tikzcd}
\bigoplus_{i=1}^{\infty} E_i \arrow[r, "1-\text{shift}"] & \bigoplus_{i=1}^{\infty} E_i \arrow[r, "{(f_1, f_2, f_3, \ldots)}"] & G
\end{tikzcd}$$

must vanish. Hence the map $\bigoplus_{i=1}^{\infty} E_i \longrightarrow G$ factors (non-uniquely) through $F = \underrightarrow{\mathrm{Hocolim}}\, E_i$, which is the third edge in the triangle

$$\begin{tikzcd}
\bigoplus_{i=1}^{\infty} E_i \arrow[r, "1-\text{shift}"] & \bigoplus_{i=1}^{\infty} E_i \arrow[r, "{(f_1, f_2, f_3, \ldots)}"] & \underrightarrow{\mathrm{Hocolim}}\, E_i .
\end{tikzcd}$$

□

8 A couple of technical lemmas

In Sect. 9 we will prove Theorem 1.4(i), and in Sect. 12 we will prove Theorem 1.4(ii). Both proofs will rely heavily on a couple of technical lemmas—in this section we state and prove these lemmas, in sufficient generality to cover both applications. Let us therefore set up a little notation.

Notation 8.1 Throughout this section R will be a commutative ring, $\mathcal{T}$ will be an R–linear triangulated category with coproducts, and $\mathcal{S} \subset \mathcal{T}$ will be a triangulated subcategory. The Yoneda functor $\mathcal{Y} : \mathcal{T} \longrightarrow \mathrm{Hom}_R\left[\mathcal{S}^{\mathrm{op}}, R\text{–Mod}\right]$ will be the map taking $t \in \mathcal{T}$ to $\mathrm{Hom}(-, t)$, where $\mathrm{Hom}(-, t)$ is viewed as an R–linear functor $\mathcal{S}^{\mathrm{op}} \longrightarrow R$–Mod.

We remind the reader of Definition 7.1: suppose $\mathcal{A}$ is a full subcategory of $\mathcal{T}$ closed under direct summands, finite coproducts and suspensions, and H is an $\mathcal{S}$–cohomological functor, meaning $H : \mathcal{S}^{\mathrm{op}} \longrightarrow R$–Mod is an R–linear cohomological functor. Then an $\mathcal{A}$–approximating system for H is a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ in $\mathcal{A}$, with a subsequence in $\mathcal{A} \cap \mathcal{S}$, and an isomorphism $\varinjlim \mathcal{Y}(E_i) \longrightarrow H(-)$.

Definition 8.2 Let $\mathcal{T}$ be an R–linear triangulated category with coproducts and let $\mathcal{S}$ be a triangulated subcategory. If H is a $\mathcal{S}$–cohomological functor, we define ΣH by the rule $\Sigma H(s) = H(\Sigma^{-1} s)$.

A *weak triangle* in the category $\mathrm{Hom}_R\big[\mathcal{S}^{\mathrm{op}}, R\text{–Mod}\big]$ is a sequence of cohomological functors $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$ such that any rotation of the following is true: given any triangle $a \xrightarrow{u'} b \xrightarrow{v'} c \xrightarrow{w'} \Sigma a$ in the category $\mathcal{S}$ and a commutative diagram

$$\begin{tikzcd}
\mathcal{Y}(a) \arrow[r, "\mathcal{Y}(u')"] \arrow[d, "f"] & \mathcal{Y}(b) \arrow[r, "\mathcal{Y}(v')"] \arrow[d, "g"] & \mathcal{Y}(c) \arrow[r, "\mathcal{Y}(w')"] & \mathcal{Y}(\Sigma a) \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & \Sigma A
\end{tikzcd}$$

there is an extension to a commutative diagram

$$\begin{tikzcd}
\mathcal{Y}(a) \arrow[r, "\mathcal{Y}(u')"] \arrow[d, "f"] & \mathcal{Y}(b) \arrow[r, "\mathcal{Y}(v')"] \arrow[d, "g"] & \mathcal{Y}(c) \arrow[r, "\mathcal{Y}(w')"] \arrow[d, "h"] & \mathcal{Y}(\Sigma a) \arrow[d, "\Sigma f"] \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & \Sigma A
\end{tikzcd}$$

A diagram $\widehat{A} \xrightarrow{\widehat{u}} \widehat{B} \xrightarrow{\widehat{v}} \widehat{C} \xrightarrow{\widehat{w}} \Sigma\widehat{A}$ in the category $\mathcal{T}$ is called a weak triangle if the functor $\mathcal{Y}$ takes it to a weak triangle in $\mathrm{Hom}_R\big[\mathcal{S}^{\mathrm{op}}, R\text{–Mod}\big]$.

Remark 8.3 We remind the reader of Remark 7.6 and Lemma 7.8: if A, B and C have $\mathcal{T}$–approximating systems $\mathfrak{A}_*$, $\mathfrak{B}_*$ and $\mathfrak{C}_*$, we may form in $\mathcal{T}$ the homotopy colimits $\widehat{A} = \underrightarrow{\mathrm{Hocolim}}\ \mathfrak{A}_*$, $\widehat{B} = \underrightarrow{\mathrm{Hocolim}}\ \mathfrak{B}_*$ and $\widehat{C} = \underrightarrow{\mathrm{Hocolim}}\ \mathfrak{C}_*$. Remark 7.6 tells us that there are canonical maps $\alpha : A \longrightarrow \mathcal{Y}(\widehat{A})$, $\beta : B \longrightarrow \mathcal{Y}(\widehat{B})$ and $\gamma : C \longrightarrow \mathcal{Y}(\widehat{C})$. Since our plan is to apply the lemmas in this section to prove representability theorems, we will mostly be interested in cases where α, β and γ are isomorphisms. In this case Lemma 7.8 says that the maps u, v and w may be lifted (non-uniquely) to $\mathcal{T}$; we may form in $\mathcal{T}$ a diagram $\widehat{A} \xrightarrow{\widehat{u}} \widehat{B} \xrightarrow{\widehat{v}} \widehat{C} \xrightarrow{\widehat{w}} \Sigma\widehat{A}$ whose image under $\mathcal{Y}$ is (canonically) isomorphic to $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$.

Lemma 8.4 *Suppose* $A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$ *is a weak triangle in* $\mathrm{Hom}_R\big[\mathcal{S}^{\mathrm{op}}, R\text{–}Mod\big]$. *For any* $s \in \mathcal{S}$ *the functor* $\mathrm{Hom}\big(\mathcal{Y}(s), -\big)$ *takes it to an exact sequence.*

Proof Given any map $f : \mathcal{Y}(s) \longrightarrow A$, we can consider the commutative diagram

$$\begin{tikzcd}
\mathcal{Y}(s) \arrow[r, equal] \arrow[d, "f"'] & \mathcal{Y}(s) \arrow[r] \arrow[d, "uf"] & 0 \arrow[r] & \mathcal{Y}(\Sigma s) \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & \Sigma A
\end{tikzcd}$$

and the fact that this commutative diagram can be extended gives the vanishing of vuf.

A morphism $g : \mathcal{Y}(s) \longrightarrow B$ so that $vg = 0$ gives a commutative diagram

$$\begin{tikzcd}
\mathcal{Y}(s) \arrow[r, equal] & \mathcal{Y}(s) \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d] & \mathcal{Y}(\Sigma s) \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & \Sigma A
\end{tikzcd}$$

and the existence of an extension gives a morphism $f : \mathcal{Y}(s) \longrightarrow A$ with $g = uf$. $\square$

Lemma 8.5 *With the conventions of Notation 8.1 and Definition 8.2 suppose we are given*:

(i) *A morphism $\alpha : A \longrightarrow B$ in the category* $\mathrm{Hom}_R\left[\mathcal{S}^{\mathrm{op}}, R\text{–}Mod\right]$.
(ii) *A morphism $\alpha_* : \mathfrak{A}_* \longrightarrow \mathfrak{B}_*$ of sequences in $\mathcal{T}$, and an isomorphism in* $\mathrm{Hom}_R\left[\mathcal{S}^{\mathrm{op}}, R\text{–}Mod\right]$ *of $\alpha : A \longrightarrow B$ with the colimit of $\mathcal{Y}(\alpha_*) : \mathcal{Y}(\mathfrak{A}_*) \longrightarrow \mathcal{Y}(\mathfrak{B}_*)$.*
(iii) *The sequence α_* is assumed to have a subsequence in $\mathcal{S}$.*

With just these hypotheses we may complete $\alpha_* : \mathfrak{A}_* \longrightarrow \mathfrak{B}_*$ to a sequence $\mathfrak{A}_* \xrightarrow{\alpha_*} \mathfrak{B}_* \xrightarrow{\beta_*} \mathfrak{C}_* \xrightarrow{\gamma_*} \Sigma\mathfrak{A}_*$ of triangles in $\mathcal{T}$, and the colimit of $\mathcal{Y}(\mathfrak{A}_*) \xrightarrow{\mathcal{Y}(\alpha_*)} \mathcal{Y}(\mathfrak{B}_*) \xrightarrow{\mathcal{Y}(\beta_*)} \mathcal{Y}(\mathfrak{C}_*) \xrightarrow{\mathcal{Y}(\gamma_*)} \Sigma\mathcal{Y}(\mathfrak{A}_*)$ is a weak triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$.

Suppose we add the following assumptions:

(iv) *We are given two subcategories $\mathcal{A} \subset \mathcal{B} \subset \mathcal{S}$, closed under finite coproducts, direct summands and suspensions.*
(v) *There is a subsequence of $\alpha_* : \mathfrak{A}_* \longrightarrow \mathfrak{B}_*$ such that the $\mathfrak{A}_i$ belongs to $\mathcal{A}$ and the $\mathfrak{B}_i$ belong to $\mathcal{B}$. Put $\mathcal{C} = \mathrm{smd}(\mathcal{B} * \mathcal{A})$.*
(vi) *Assume furthermore that we are given a $\mathcal{C}$–cohomological functor H and a natural transformation of $\mathcal{B}$–cohomological functors $\varphi : B|_{\mathcal{B}} \longrightarrow H|_{\mathcal{B}}$. Assume that, on the category $\mathcal{A} \subset \mathcal{B}$, the composite*

$$\begin{tikzcd}
A|_{\mathcal{A}} \arrow[r, "\alpha|_{\mathcal{A}}"] & B|_{\mathcal{A}} \arrow[r, "\varphi|_{\mathcal{A}}"] & H|_{\mathcal{A}}
\end{tikzcd}$$

vanishes.
(vii) *Assume further that the approximating system $\mathfrak{A}_*$ for A is such that each morphism $\mathfrak{A}_i \longrightarrow \mathfrak{A}_{i+1}$ is a split monomorphism.*

Then there exists a map $\psi : C|_{\mathcal{C}} \longrightarrow H$ so that $\varphi : B|_{\mathcal{B}} \longrightarrow H|_{\mathcal{B}}$ is equal to the composite

$$\begin{tikzcd}
B|_{\mathcal{B}} \arrow[r, "\beta|_{\mathcal{B}}"] & C|_{\mathcal{B}} \arrow[r, "\psi|_{\mathcal{B}}"] & H|_{\mathcal{B}} .
\end{tikzcd}$$

Proof We are given a morphism of sequences $\alpha_* : \mathfrak{A}_* \longrightarrow \mathfrak{B}_*$, meaning for each $m > 0$ we have a commutative square

$$\begin{tikzcd}
\mathfrak{A}_m \arrow[r, "\alpha_m"] \arrow[d] & \mathfrak{B}_m \arrow[d] \\
\mathfrak{A}_{m+1} \arrow[r, "\alpha_{m+1}"] & \mathfrak{B}_{m+1}
\end{tikzcd}$$

We extend this to a morphism of triangles

$$\begin{tikzcd}
\mathfrak{A}_m \arrow[r, "\alpha_m"] \arrow[d] & \mathfrak{B}_m \arrow[r, "\beta_m"] \arrow[d] & \mathfrak{C}_m \arrow[r, "\gamma_m"] \arrow[d] & \Sigma\mathfrak{A}_m \arrow[d] \\
\mathfrak{A}_{m+1} \arrow[r, "\alpha_{m+1}"] & \mathfrak{B}_{m+1} \arrow[r, "\beta_{m+1}"] & \mathfrak{C}_{m+1} \arrow[r, "\gamma_{m+1}"] & \Sigma\mathfrak{A}_{m+1}
\end{tikzcd}$$

This produces for us in $\mathcal{T}$ the sequence of triangles $\mathfrak{A}_* \xrightarrow{\alpha_*} \mathfrak{B}_* \xrightarrow{\beta_*} \mathfrak{C}_* \xrightarrow{\gamma_*} \Sigma\mathfrak{A}_*$, with a subsequence in $\mathcal{S}$, and it is easy to see that the colimit of $\mathcal{Y}(\mathfrak{A}_*) \xrightarrow{\mathcal{Y}(\alpha_*)} \mathcal{Y}(\mathfrak{B}_*) \xrightarrow{\mathcal{Y}(\beta_*)} \mathcal{Y}(\mathfrak{C}_*) \xrightarrow{\mathcal{Y}(\gamma_*)} \Sigma\mathcal{Y}(\mathfrak{A}_*)$ is a weak triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$.

It remains to prove the part with the further assumptions added. Note that, by passing to a subsequence, we may assume $\mathfrak{A}_i \in \mathcal{A}$ and $\mathfrak{B}_i \in \mathcal{B}$, and hence $\mathfrak{C}_i \in \mathcal{B} * \mathcal{A} \subset \mathcal{C}$. As H is a cohomological functor on $\mathcal{C}$ and $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$, we have that, for each integer m, the sequence $H(\Sigma\mathfrak{A}_m) \xrightarrow{H(\gamma_m)} H(\mathfrak{C}_m) \xrightarrow{H(\beta_m)} H(\mathfrak{B}_m) \xrightarrow{H(\alpha_m)} H(\mathfrak{A}_m)$ must be exact. As m increases this gives an inverse system of exact sequences, which we now propose to analyze. The short exact sequences $0 \longrightarrow \operatorname{Ker}\big(H(\gamma_m)\big) \longrightarrow H(\Sigma\mathfrak{A}_m) \longrightarrow \operatorname{Im}\big(H(\gamma_m)\big) \longrightarrow 0$ give an exact sequence

$$\begin{tikzcd}
\varprojlim{}^1 H(\Sigma\mathfrak{A}_m) \arrow[r] & \varprojlim{}^1 \operatorname{Im}\big(H(\gamma_m)\big) \arrow[r] & \varprojlim{}^2 \operatorname{Ker}\big(H(\gamma_m)\big)
\end{tikzcd}$$

In (vii) we assumed that the maps $\mathfrak{A}_m \longrightarrow \mathfrak{A}_{m+1}$ are split monomorphisms, hence the maps $H(\Sigma\mathfrak{A}_{m+1}) \longrightarrow H(\Sigma\mathfrak{A}_m)$ are split epimorphisms, making the sequence Mittag-Leffler. Therefore $\varprojlim{}^1 H(\Sigma\mathfrak{A}_m) = 0$. We have $\varprojlim{}^2 \operatorname{Ker}\big(H(\gamma_m)\big) = 0$ just because we're dealing with a countable limit. We conclude that $\varprojlim{}^1 \operatorname{Im}\big(H(\gamma_m)\big) = 0$.

Now consider the inverse system of short exact sequences $0 \longrightarrow \operatorname{Im}\big(H(\gamma_m)\big) \longrightarrow H(\mathfrak{C}_m) \longrightarrow \operatorname{Im}\big(H(\beta_m)\big) \longrightarrow 0$. Passing to the limit we obtain an exact sequence

$$\begin{tikzcd}
\varprojlim H(\mathfrak{C}_m) \arrow[r] & \varprojlim \operatorname{Im}\big(H(\beta_m)\big) \arrow[r] & \varprojlim{}^1 \operatorname{Im}\big(H(\gamma_m)\big)
\end{tikzcd}$$

We have proved the vanishing of $\varprojlim{}^1 \operatorname{Im}\big(H(\gamma_m)\big)$, allowing us to conclude that the map $\varprojlim H(\mathfrak{C}_m) \longrightarrow \varprojlim \operatorname{Im}\big(H(\beta_m)\big)$ is an epimorphism. Finally we observe the exact sequences $0 \longrightarrow \operatorname{Im}\big(H(\beta_m)\big) \longrightarrow H(\mathfrak{B}_m) \longrightarrow H(\mathfrak{A}_m)$ and, since inverse limit is left exact, we deduce the exactness of

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim \operatorname{Im}\big(H(\beta_m)\big) \arrow[r] & \varprojlim H(\mathfrak{B}_m) \arrow[r] & \varprojlim H(\mathfrak{A}_m)
\end{tikzcd}$$

Combining the results we have the exactness of

$$\begin{tikzcd}
\varprojlim H(\mathfrak{C}_m) \arrow[r] & \varprojlim H(\mathfrak{B}_m) \arrow[r] & \varprojlim H(\mathfrak{A}_m) .
\end{tikzcd}$$

Now Lemma 7.3 tells us that $\varphi : B|_{\mathcal{B}} \longrightarrow H|_{\mathcal{B}}$ corresponds to an element $f \in \varprojlim H(\mathfrak{B}_m)$, and the vanishing of the composite $A|_{\mathcal{A}} \longrightarrow B|_{\mathcal{A}} \longrightarrow H|_{\mathcal{A}}$ translates to saying that the image of f under the map $\varprojlim H(\mathfrak{B}_m) \longrightarrow \varprojlim H(\mathfrak{A}_m)$ vanishes. The exactness tells us that f is in the image of the map $\varprojlim H(\mathfrak{C}_m) \longrightarrow \varprojlim H(\mathfrak{B}_m)$. This exactly says that there is a natural transformation $\psi : C|_{\mathcal{C}} \longrightarrow H$ with $\varphi = \psi \circ \beta$. We have proved the "extra assumptions" part. ☐

Lemma 8.6 *With the conventions of Notation 8.1 and Definition 8.2 suppose we are given:*

(i) *Two full subcategories $\mathcal{A} \subset \mathcal{B}$ of the category $\mathcal{S}$, closed under finite coproducts, direct summands and suspensions.*
(ii) *Put $\mathcal{C} = \operatorname{smd}(\mathcal{B} * \mathcal{A})$. Assume we are also given a $\mathcal{C}$–cohomological functor H.*
(iii) *We are given a weak triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$, and a natural transformation of $\mathcal{C}$–cohomological functors $\psi : C|_{\mathcal{C}} \longrightarrow H$.*
(iv) *The composite $(\psi\beta)|_{\mathcal{B}} : B|_{\mathcal{B}} \longrightarrow H|_{\mathcal{B}}$ is surjective.*
(v) *The sequence*

$$\begin{tikzcd}
A|_{\mathcal{A}} \arrow[r, "\alpha|_{\mathcal{A}}"] & B|_{\mathcal{A}} \arrow[r, "(\psi\beta)|_{\mathcal{A}}"] & H|_{\mathcal{A}} \arrow[r] & 0
\end{tikzcd}$$

is exact.

Then the map $\psi : C|_{\mathcal{C}} \longrightarrow H$ an epimorphism.

Proof We need to show the surjectivity of the map $\psi : C(c) \longrightarrow H(c)$ for every $c \in \mathcal{C} = \operatorname{smd}(\mathcal{B} * \mathcal{A})$; without loss of generality we may assume $c \in \mathcal{B} * \mathcal{A}$. Choose a triangle $a \xrightarrow{\alpha'} b \xrightarrow{\beta'} c \xrightarrow{\gamma'} \Sigma a$ with $b \in \mathcal{B}$ and $a \in \mathcal{A}$. Given any element $y \in H(c)$, the map $H(\beta') : H(c) \longrightarrow H(b)$ takes y to an element $H(\beta')(y)$ which must be in the image of the surjective map $\psi\beta : B(b) \longrightarrow H(b)$. After all b is an object in $\mathcal{B}$, and the map $\psi\beta : B(b) \longrightarrow H(b)$ is an epimorphism on objects $b \in \mathcal{B}$. Choose an element $g \in B(b)$ mapping under $\psi\beta$ to $H(\beta')(y)$. The naturality of $\psi\beta$ means that the square below commutes

$$\begin{tikzcd}
 & B(b) \arrow[r, "B(\alpha')"] \arrow[d, "\psi\beta"] & B(a) \arrow[d, "\psi\beta"] \\
H(c) \arrow[r, "H(\beta')"] & H(b) \arrow[r, "H(\alpha')"] & H(a)
\end{tikzcd}$$

If we apply the equal composites in the square to $g \in B(b)$ we discover that it goes to $H(\alpha')H(\beta')(y) = 0$, where the vanishing is because $\beta'\alpha' = 0$. Therefore the map $B(\alpha')$ takes $g \in B(b)$ to an element in the kernel of $\psi\beta : B(a) \longrightarrow H(a)$. As a belongs to $\mathcal{A}$ the map $\alpha : A(a) \longrightarrow B(a)$ surjects onto this kernel; there is an element $f \in A(a)$ with $\alpha(f) = B(\alpha')(g)$.

We have produced elements $f \in A(a)$ and $g \in B(b)$, and Yoneda allows us to view them as natural transformations $f : \mathcal{Y}(a) \longrightarrow A$ and $g : \mathcal{Y}(b) \longrightarrow B$. The equality $\alpha(f) = B(\alpha')(g)$ transforms into the assertion that the square below commutes

$$\begin{tikzcd}
\mathcal{Y}(a) \arrow[r, "\mathcal{Y}(\alpha')"] \arrow[d, "f"] & \mathcal{Y}(b) \arrow[r, "\mathcal{Y}(\beta')"] \arrow[d, "g"] & \mathcal{Y}(c) \arrow[r, "\mathcal{Y}(\gamma')"] & \mathcal{Y}(\Sigma a) \\
A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r, "\gamma"] & \Sigma A
\end{tikzcd}$$

The top row is the image under Yoneda of a triangle in $\mathcal{S}$, while the bottom row is a weak triangle; hence we may complete to a commutative diagram

$$\begin{tikzcd}
\mathcal{Y}(a) \arrow[r, "\mathcal{Y}(\alpha')"] \arrow[d, "f"] & \mathcal{Y}(b) \arrow[r, "\mathcal{Y}(\beta')"] \arrow[d, "g"] & \mathcal{Y}(c) \arrow[r, "\mathcal{Y}(\gamma')"] \arrow[d, "h"] & \mathcal{Y}(\Sigma a) \arrow[d, "\Sigma f"] \\
A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r, "\gamma"] & \Sigma A
\end{tikzcd}$$

We have produced a morphism $h : \mathcal{Y}(c) \longrightarrow C$, which we may view as an element $h \in C(c)$. And the commutativity of the middle square translates, under Yoneda, to the statement that $C(\beta') : C(c) \longrightarrow C(b)$ takes $h \in C(c)$ to $\beta(g) \in C(b)$. Applying $\psi : C \longrightarrow H$ we obtain the second equality below

$$\begin{aligned}
H(\beta')(y) &= \psi\beta(g) \\
&= \psi C(\beta')(h) \\
&= H(\beta')\psi(h) \ .
\end{aligned}$$

The first equality is by construction of $g \in B(b)$, and the third is the naturality of ψ. Therefore the map $H(\beta') : H(c) \longrightarrow H(b)$ annihilates $y - \psi(h)$. Because H

is cohomological there is an element $x \in H(\Sigma a)$ with $H(\gamma')(x) = y - \psi(h)$. But $a \in \mathcal{A} \subset \mathcal{B}$ and we may choose a $\theta \in B(\Sigma a)$ with $\psi\beta(\theta) = x$. We have

$$\begin{aligned} y - \psi(h) &= H(\gamma')\psi\beta(\theta) \\ &= \psi C(\gamma')\beta(\theta) \end{aligned}$$

where the first equality is the construction of θ, and the second is the naturality of $\psi : C \longrightarrow H$. These equalities combine to the formula $y = \psi\big[h + C(\gamma')\beta(\theta)\big]$, which exihibits $y \in H(c)$ as lying in the image of $\psi : C(c) \longrightarrow H(c)$. ☐

The next lemma will not be needed until the proof of Theorem 12.6.

Lemma 8.7 *With the conventions of Notation 8.1 and Definition 8.2 suppose we are given*:

(i) *Two full subcategories $\mathcal{A} \subset \mathcal{B}$ of the category $\mathcal{S}$, closed under finite coproducts, direct summands and suspensions.*
(ii) *Put* $\mathcal{C} = \mathrm{smd}(\mathcal{B} * \mathcal{A})$.
(iii) *In the category* $\mathrm{Hom}_R\big[\mathcal{S}^{\mathrm{op}}, R\text{–}Mod\big]$ *we are given a diagram of cohomological functors*

$$\begin{tikzcd} & \widetilde{B} \arrow[d, "\delta"] & & \\ A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r, "\gamma"] \arrow[d, "\psi"] & \Sigma A \\ & & H & \end{tikzcd}$$

where the middle row is a weak triangle.
(iv) *The composite* $(\psi\beta)|_{\mathcal{B}} : B|_{\mathcal{B}} \longrightarrow H|_{\mathcal{B}}$ *is surjective.*
(v) *The kernel of the morphism* $(\psi\beta\delta)|_{\mathcal{B}} : \widetilde{B}|_{\mathcal{B}} \longrightarrow H|_{\mathcal{B}}$ *is annihilated by* $\delta : \widetilde{B} \longrightarrow B$.
(vi) *The sequence*

$$\begin{tikzcd} A|_{\mathcal{A}} \arrow[r, "\alpha|_{\mathcal{A}}"] & B|_{\mathcal{A}} \arrow[r, "(\psi\beta)|_{\mathcal{A}}"] & H|_{\mathcal{A}} \arrow[r] & 0 \end{tikzcd}$$

is exact.

Then the map $(\beta\delta) : \widetilde{B} \longrightarrow C$ *annihilates the kernel of* $(\psi\beta\delta)|_{\mathcal{C}} : \widetilde{B}|_{\mathcal{C}} \longrightarrow H|_{\mathcal{C}}$.

Proof We need to show that, if $c \in \mathcal{C} = \mathrm{smd}(\mathcal{B} * \mathcal{A})$ and $y \in \widetilde{B}(c)$ is annihilated by the map $\psi\beta\delta : \widetilde{B}(c) \longrightarrow H(c)$, then y is already annihilated by the shorter map $\beta\delta : \widetilde{B}(c) \longrightarrow C(c)$. Note that without loss of generality we may assume that

$c \in \mathcal{B} * \mathcal{A}$. Choose therefore a triangle $a \xrightarrow{\alpha'} b \xrightarrow{\beta'} c \xrightarrow{\gamma'} \Sigma a$ with $b \in \mathcal{B}$ and $a \in \mathcal{A}$, and consider the commutative diagram with exact rows

$$\begin{tikzcd}
 & & \widetilde{B}(c) \arrow[r, "\widetilde{B}(\beta')"] \arrow[d, "\delta"] & \widetilde{B}(b) \arrow[d, "\delta"] \\
B(\Sigma b) \arrow[r, "B(\Sigma\alpha')"] \arrow[d, "\psi\beta"] & B(\Sigma a) \arrow[r, "B(\gamma')"] \arrow[d, "\psi\beta"] & B(c) \arrow[r, "B(\beta')"] \arrow[d, "\psi\beta"] & B(b) \arrow[d, "\psi\beta"] \\
H(\Sigma b) \arrow[r, "H(\Sigma\alpha')"] & H(\Sigma a) \arrow[r, "H(\gamma')"] & H(c) \arrow[r, "H(\beta')"] & H(b)
\end{tikzcd}$$

We are given an element $y \in \widetilde{B}(c)$ such that the vertical composite in the third column annihilates it. Therefore $\widetilde{B}(\beta')(y)$ is an element of $\widetilde{B}(b)$ annihilated by the vertical composite in the fourth column. By assumption (v) the element $\widetilde{B}(\beta')(y)$ is already killed by $\delta : \widetilde{B}(b) \longrightarrow B(b)$, and we conclude that the equal composites in the top-right square annihilate y. Hence the map $\delta : \widetilde{B}(c) \longrightarrow B(c)$ must take $y \in \widetilde{B}(c)$ to an element in the image of $B(\gamma')$, and we may therefore

(vii) Choose an $x \in B(\Sigma a)$ with $B(\gamma')(x) = \delta(y)$.

Recall that the vertical composite in the third column kills y, hence the equal composites in the middle square at the bottom must annihilate x. Therefore $\psi\beta(x) \in H(\Sigma a)$ lies in the kernel of $H(\gamma')$, which is the image of $H(\Sigma\alpha') : H(\Sigma b) \longrightarrow H(\Sigma a)$. Now by (iv) the map $\psi\beta : B(\Sigma b) \longrightarrow H(\Sigma b)$ is surjective, and we may lift further to $B(\Sigma b)$; we can choose an element $w \in B(\Sigma b)$ whose image under the equal composites in the bottom-left square are equal to $\psi\beta(x)$. Therefore we have that, with $x \in B(\Sigma a)$ as in (vii) and $w \in B(\Sigma b)$ as above, the element $x - B(\Sigma\alpha')(w)$ is annihilated by $\psi\beta : B(\Sigma a) \longrightarrow H(\Sigma a)$. Now (vi) tells us that

(viii) We may choose an element $v \in A(\Sigma a)$ whose image under $\alpha : A(\Sigma a) \longrightarrow B(\Sigma a)$ is equal to $x - B(\Sigma\alpha')(w)$.

To complete the proof consider the commutative diagram with vanishing horizontal and vertical composites

$$\begin{tikzcd}
 & A(\Sigma a) \arrow[r, "A(\gamma')"] \arrow[d, "\alpha"] & A(c) \arrow[d, "\alpha"] \\
B(\Sigma b) \arrow[r, "B(\Sigma\alpha')"] & B(\Sigma a) \arrow[r, "B(\gamma')"] \arrow[d, "\beta"] & B(c) \arrow[d, "\beta"] \\
 & C(\Sigma a) \arrow[r, "C(\gamma')"] & C(c)
\end{tikzcd}$$

The horizontal map in the top row takes the element $v \in A(\Sigma a)$ constructed in (viii) to $A(\gamma')(v)$, which must be annihilated by the vertical composite in the third column. By the commutativity of the top square coupled with (viii), this means that $x - B(\Sigma\alpha')(w)$ is an element of $B(\Sigma a)$ annihilated by the equal composites in the bottom square. In particular the horizontal map $B(\gamma')$ takes $x - B(\Sigma\alpha')(w)$ to an element of the kernel of $\beta : B(c) \longrightarrow C(c)$. But the map $B(\gamma')$ annihilates $B(\Sigma\alpha')(w)$, and by (vii) it takes x to $\delta(y)$. We conclude that $\beta\delta(y) = 0$. ☐

9 The proof of Theorem 1.4 (i)

It's time to prove Theorem 1.4(i); we should focus the general lemmas of Sect. 8 on the situation at hand. Thus in this section we make the following global assumptions:

Notation 9.1 We specialize the conventions of Notation 8.1 by setting $\mathcal{S} = \mathcal{T}^c$, that is $\mathcal{S}$ is the subcategory of compact objects in $\mathcal{T}$. Thus in this section the functor $\mathcal{Y}$ of Notation 8.1 specializes to $\mathcal{Y} : \mathcal{T} \longrightarrow \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}, R\text{–Mod}\big)$, which takes an object $t \in \mathcal{T}$ to $\mathcal{Y}(t) = \mathrm{Hom}(-, t)|_{\mathcal{T}^c}$.

In the generality of Notation 8.1 we considered $\mathcal{S}$–cohomological functors H and $\mathcal{B}$–approximating systems $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$. Because we are now in the special case where $\mathcal{S} = \mathcal{T}^c$ Remark 7.7 applies: if $F = \underrightarrow{\mathrm{Hocolim}}\, E_i$ then the natural map $H \longrightarrow \mathcal{Y}(F)$ must be an isomorphism.

We will furthermore assume that we have chosen in $\mathcal{T}$ a single compact generator G. We will suppose given a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class, with $\mathcal{T}^{\geq 0}$ closed under coproducts. For example we could let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be equal to $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$, see Remark 1.19. Let $\mathcal{A}$ be the heart of the t–structure, and $\mathcal{H} : \mathcal{T} \longrightarrow \mathcal{A}$ the homological functor of Reminder 2.1. We will assume given an integer $A > 0$ with $\mathrm{Hom}\big(\Sigma^{-A}G, \mathcal{T}^{\leq 0}\big) = 0$. The existence of such an A is equivalent to the hypothesis that $\mathrm{Hom}(G, \Sigma^i G) = 0$ for $i \gg 0$. And what is important for us is that this guarantees that the category $\mathcal{T}_c^-$ is a thick subcategory of $\mathcal{T}$.

Finally and most importantly: as in Notation 8.1 the triangulated category $\mathcal{T}$ is assumed to be R–linear for some commutative ring R. But from now on we add the assumption that the ring R is noetherian and, with G as in the last paragraph, the R–module $\mathrm{Hom}(G, \Sigma^i G)$ is finite for every $i \in \mathbb{Z}$. Combining this paragraph with the last: the functor $\mathcal{Y}(G)$ is G–locally finite.

Under some additional approximability assumptions, Theorem 1.4 describes the essential image of the functor $\mathcal{Y}$ taking $F \in \mathcal{T}_c^-$ to $\mathcal{Y}(F) = \mathrm{Hom}(-, F)|_{\mathcal{T}^c}$, and tells us that the functor $\mathcal{Y}$ is full. To show that $\mathcal{Y}(F)$ lies in the expected image one doesn't need any hypotheses beyond the ones above, we prove

Lemma 9.2 *With the assumptions of Notation 9.1, for any $F \in \mathcal{T}_c^-$ the functor $\mathcal{Y}(F) : [\mathcal{T}^c]^{\mathrm{op}} \longrightarrow R$–Mod is a locally finite $\mathcal{T}^c$–cohomological functor.*

Proof We are given that the functor $\mathcal{Y}(G)$ is G–locally finite. In particular: for $i \ll 0$ we have $\mathrm{Hom}(\Sigma^i G, G) = 0$. With $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ as in Notation 9.1, that is our fixed t–structure in the preferred equivalence class, Remark 1.24 coupled with Lemma 3.8 tell us that, for any object $K \in \mathcal{T}^c$, there is an integer $B > 0$ so that $\mathrm{Hom}\left(\Sigma^{-B}K, \mathcal{T}^{\leq 0}\right) = 0$. Remark 1.22 gives the inclusion in $F \in \mathcal{T}_c^- \subset \mathcal{T}^-$, hence we may choose an integer $A > 0$ with $\Sigma^A F \in \mathcal{T}^{\leq 0}$. We deduce that $\mathrm{Hom}(\Sigma^i K, F) = 0$ for all $i \leq -A - B$.

The fact that the functor $\mathcal{Y}(G)$ is G–locally finite also means that, for every integer $i \in \mathbb{Z}$, the R–module $\mathrm{Hom}(\Sigma^i G, G)$ is finite. The full subcategory $\mathcal{L} \subset \mathcal{T}^c$ defined by

$$\mathcal{L} \quad = \quad \{L \in \mathcal{T}^c \mid \mathrm{Hom}(\Sigma^i G, L) \text{ is a finite } R\text{–module for all } i \in \mathbb{Z}\}$$

is thick and contains G, hence $\mathcal{T}^c = \langle G \rangle \subset \mathcal{L}$. Now take any $L \in \mathcal{T}^c$ and define the full subcategory $\mathcal{K}(L) \subset \mathcal{T}^c$ by

$$\mathcal{K}(L) \quad = \quad \{K \in \mathcal{T}^c \mid \mathrm{Hom}(\Sigma^i K, L) \text{ is a finite } R\text{–module for all } i \in \mathbb{Z}\}$$

Then $\mathcal{K}(L)$ is thick and contains G, hence $\mathcal{T}^c = \langle G \rangle \subset \mathcal{K}(L)$. We conclude that $\mathrm{Hom}(\Sigma^i K, L)$ is a finite R–module for all $K, L \in \mathcal{T}^c$ and all $i \in \mathbb{Z}$.

Now fix the integer i, the object $K \in \mathcal{T}^c$ and the object $F \in \mathcal{T}_c^-$, and we want to prove that $\mathrm{Hom}(\Sigma^i K, F)$ is a finite R–module. The first paragraph of the proof produced an integer $B > 0$ with $\mathrm{Hom}\left(\Sigma^{-B}K, \mathcal{T}^{\leq 0}\right) = 0$, and since F belongs to $\mathcal{T}_c^-$ there exists a triangle $L \longrightarrow F \longrightarrow D$ with $L \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\leq -i-B-1}$. In the exact sequence

$$\mathrm{Hom}(\Sigma^i K, \Sigma^{-1}D) \to \mathrm{Hom}(\Sigma^i K, L) \to \mathrm{Hom}(\Sigma^i K, F) \to \mathrm{Hom}(\Sigma^i K, D)$$

we have that $\mathrm{Hom}(\Sigma^i K, D) = 0 = \mathrm{Hom}(\Sigma^i K, \Sigma^{-1}D)$, and $\mathrm{Hom}(\Sigma^i K, F) \cong \mathrm{Hom}(\Sigma^i K, L)$ must be a finite R–module by the second paragraph of the proof. ☐

In Corollary 3.14 we learned that, under some approximability hypotheses, objects in $\mathcal{T}_c^-$ can be well approximated by sequences with special properties. We don't need all these properties yet; for the next few lemmas we formulate what we will use.

Definition 9.3 Adopting the conventions of Notation 9.1, a *strong* $\langle G \rangle_n$*–approximating system* is a sequence of objects and morphisms $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$

(i) Each E_m belongs to $\langle G \rangle_n$.
(ii) The map $\mathcal{H}^i(E_m) \longrightarrow \mathcal{H}^i(E_{m+1})$ is an isomorphism whenever $i \geq -m$.

In this definition we also allow $n = \infty$, we simply declare $\langle G \rangle_\infty = \langle G \rangle = \mathcal{T}^c$.

Suppose we are also given an object $F \in \mathcal{T}$, together with

(iii) A map of the approximating system E_* to F.
(iv) The map in (iii) is such that $\mathcal{H}^i(E_m) \longrightarrow \mathcal{H}^i(F)$ is an isomorphism whenever $i \geq -m$.

Then we declare E_* to be a *strong* $\langle G \rangle_n$*–approximating system for* F.

Remark 9.4 Although Definition 9.3 was phrased in terms of the particular choice of t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, made in Notation 9.1, it is robust—up to passing to subsequences a strong $\langle G\rangle_n$–approximating system for F will work for any equivalent t–structure.

Lemma 9.5 *With the conventions of Definition 9.3 we have*

(i) *Given an object $F \in \mathcal{T}^-$ and a strong $\langle G\rangle_n$–approximating system E_* for F, then the (non-canonical) map $\underrightarrow{\text{Hocolim}}\, E_i \longrightarrow F$ is an isomorphism.*
(ii) *Any object $F \in \mathcal{T}_c^-$ has a strong $\mathcal{T}^c$–approximating system.*
(iii) *Any strong $\langle G\rangle_n$–approximating system $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ is a strong $\langle G\rangle_n$–approximating system of the homotopy colimit $F = \underrightarrow{\text{Hocolim}}\, E_i$. Moreover F belongs to $\mathcal{T}_c^-$.*

Proof We begin by proving (iii). Suppose $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ is a strong $\langle G\rangle_n$–approximating system, and let $F = \underrightarrow{\text{Hocolim}}\, E_i$. The objects E_i all belong to $\langle G\rangle_n \subset \mathcal{T}^c \subset \mathcal{T}^-$. Choose an integer $\ell > 0$ with $E_1 \in \mathcal{T}^{\leq \ell}$. The fact that $\mathcal{H}^i(E_1) \longrightarrow \mathcal{H}^i(E_m)$ is an isomorphism for all $i \geq -1$ means that $\mathcal{H}^i(E_m) = 0$ for all $i > \ell$ and all m, and Lemma 2.2 gives that the E_m all lie in $\mathcal{T}^{\leq \ell}$. Hence the homotopy colimit F also belongs to $\mathcal{T}^{\leq \ell}$.

Now Remark 2.5 tells us that the map $\underrightarrow{\text{colim}}\, \mathcal{H}^i(E_m) \longrightarrow \mathcal{H}^i(F)$ is an isomorphism for every $i \in \mathbb{Z}$. By the previous paragraph the triangle $E_m \longrightarrow F \longrightarrow D_m$ lies in $\mathcal{T}^-$, and as $\mathcal{H}^i(E_m) \longrightarrow \mathcal{H}^i(F)$ is an isomorphism for $i \geq -m$ we deduce that $\mathcal{H}^i(D_m) = 0$ for all $i \geq -m$. Lemma 2.2 guarantees that $D_m \in \mathcal{T}^{\leq -m-1}$, and as $E_m \in \langle G\rangle_n \subset \mathcal{T}^c$ and $m > 0$ is arbitrary we have that F satisfies the criterion for belonging to $\mathcal{T}_c^-$.

Next we prove (i). By (iii) the map $\underrightarrow{\text{Hocolim}}\, E_i \longrightarrow F$ is a morphism from $\underrightarrow{\text{Hocolim}}\, E_i \in \mathcal{T}_c^-$ to $F \in \mathcal{T}^-$, hence it is a morphism in $\mathcal{T}^-$. The hypothesis of (i), coupled with Remark 2.5, tell us that $\mathcal{H}^i\big(\underrightarrow{\text{Hocolim}}\, E_i\big) \longrightarrow \mathcal{H}^i(F)$ is an isomorphism for every $i \in \mathbb{Z}$. By Lemma 2.3(ii) the map $\underrightarrow{\text{Hocolim}}\, E_i \longrightarrow F$ is an isomorphism.

It remains to prove (ii). Note that, if we assume more approximability hypotheses on $\mathcal{T}$, then (ii) is immediate from Corollary 3.14. But let us see that we don't yet need any strong assumptions.

Take any $F \in \mathcal{T}_c^-$. There exists a triangle $E_1 \longrightarrow F \longrightarrow D_1$ with $E_1 \in \mathcal{T}^c$ and $D_1 \in \mathcal{T}^{\leq -3}$. When $i \geq -1$ exact sequence $\mathcal{H}^{i-1}(D_1) \longrightarrow \mathcal{H}^i(E_1) \longrightarrow \mathcal{H}^i(F) \longrightarrow \mathcal{H}^i(D_1)$ has $\mathcal{H}^{i-1}(D_1) = 0 = \mathcal{H}^i(D_1)$, starting the construction of E_*.

Suppose now that we have constructed the sequence up to an integer $n > 0$, that is we have a map $f_m : E_m \longrightarrow F$, with $E_m \in \mathcal{T}^c$, and so that $\mathcal{H}^i(f_m)$ is an isomorphism for all $i \geq -m$. Lemma 3.8 allows us to choose an integer $N > 0$ so that $\text{Hom}\big(E_m, \mathcal{T}^{\leq -N}\big) = 0$. Because F belongs to $\mathcal{T}_c^-$ we may choose a triangle $E_{m+1} \longrightarrow F \longrightarrow D_{m+1}$ with $E_{m+1} \in \mathcal{T}^c$ and $D_{m+1} \in \mathcal{T}^{\leq -N-m-3}$. As in the paragraph above we show that the map $\mathcal{H}^i(E_{m+1}) \longrightarrow \mathcal{H}^i(F)$ is an isomorphism for all $i \geq -m-1$. And since the composite $E_m \xrightarrow{f_n} F \longrightarrow D_{m+1}$ vanishes, the map f_n must factor as $E_m \longrightarrow E_{m+1} \longrightarrow F$. ☐

Remark 9.6 Lemma 9.5(i) and Remark 7.7 combine to tell us that a strong $\langle G\rangle_n$–approximating system for F, in the sense of Definition 9.3, is in fact an approximating system for F as defined in Remark 7.6. Our terminology isn't misleading.

Remark 9.7 Let us now specialize Lemma 8.5 to the framework of this section. Assume we are given

(i) A morphism $\widehat{\alpha}:\widehat{A}\longrightarrow\widehat{B}$ in the category $\mathcal{T}_c^-$.
(ii) Two integers n' and n, as well as a strong $\langle G\rangle_{n'}$–approximating system $\mathfrak{A}_*$ for $\widehat{A}$ and a strong $\langle G\rangle_n$–approximating system $\mathfrak{B}_*$ for $\widehat{B}$.

Lemma 7.5 allows us to choose a subsequence of $\mathfrak{B}'_*\subset\mathfrak{B}_*$ and a map of sequences $\alpha_*:\mathfrak{A}_*\longrightarrow\mathfrak{B}'_*$ compatible with $\widehat{\alpha}:\widehat{A}\longrightarrow\widehat{B}$. A subsequence of a strong $\langle G\rangle_n$–approximating sequence is clearly a strong $\langle G\rangle_n$–approximating sequence, hence $\mathfrak{B}'_*$ is a strong $\langle G\rangle_n$–approximating sequence for $\widehat{B}$. Now as in Lemma 8.5 we extend $\alpha_*:\mathfrak{A}_*\longrightarrow\mathfrak{B}'_*$ to a sequence of triangles, in particular for each $m>0$ this gives a morphism of triangles

$$\begin{tikzcd}
\mathfrak{A}_m \arrow[r, "\alpha_m"] \arrow[d] & \mathfrak{B}'_m \arrow[r, "\beta_m"] \arrow[d] & \mathfrak{C}_m \arrow[r, "\gamma_m"] \arrow[d] & \Sigma\mathfrak{A}_m \arrow[r, "\Sigma\alpha_m"] \arrow[d] & \Sigma\mathfrak{B}'_m \arrow[d] \\
\mathfrak{A}_{m+1} \arrow[r, "\alpha_{m+1}"] & \mathfrak{B}'_{m+1} \arrow[r, "\beta_{m+1}"] & \mathfrak{C}_{m+1} \arrow[r, "\gamma_{m+1}"] & \Sigma\mathfrak{A}_{m+1} \arrow[r, "\Sigma\alpha_{m+1}"] & \Sigma\mathfrak{B}'_{m+1}
\end{tikzcd}$$

Applying the functor $\mathcal{H}^i$ with $i\geq -m$ yields a commutative diagram in the heart of $\mathcal{T}$ where the rows are exact, and where the vertical maps away from the middle are isomorphisms. By the 5-lemma the middle vertical map, i.e. the map $\mathcal{H}^i(\mathfrak{C}_m)\longrightarrow\mathcal{H}^i(\mathfrak{C}_{m+1})$, must also be an isomorphism when $i\geq -m$. We conclude that $\mathfrak{C}_*$ is a strong $\langle G\rangle_{n'+n}$–approximating system. Put $\widehat{C}=\underrightarrow{\operatorname{Hocolim}}\,\mathfrak{C}_*$. By Lemma 9.5(iii) the object $\widehat{C}$ belongs to $\mathcal{T}_c^-$ and $\mathfrak{C}_*$ is a strong $\langle G\rangle_{n'+n}$–approximating system for $\widehat{C}$, while Remark 8.3 guarantees that the weak triangle $A\xrightarrow{u}B\xrightarrow{v}C\xrightarrow{w}\Sigma A$ of Lemma 8.5 is isomorphic to the image under $\mathcal{Y}$ of a weak triangle $\widehat{A}\xrightarrow{\widehat{u}}\widehat{B}\xrightarrow{\widehat{v}}\widehat{C}\xrightarrow{\widehat{w}}\Sigma\widehat{A}$ in the category $\mathcal{T}_c^-$.

Furthermore: the homological functor $\mathcal{H}$ takes each of the triangles $\mathfrak{A}_m\xrightarrow{\alpha_m}\mathfrak{B}'_m\xrightarrow{\beta_m}\mathfrak{C}_m\xrightarrow{\gamma_m}\Sigma\mathfrak{A}_m$ to a long exact sequence, and by Remark 2.5 the (eventually stable) colimit is $\mathcal{H}$ of the weak triangle $\widehat{A}\xrightarrow{\widehat{u}}\widehat{B}\xrightarrow{\widehat{v}}\widehat{C}\xrightarrow{\widehat{w}}\Sigma\widehat{A}$. Hence $\mathcal{H}$ takes the weak triangle $\widehat{A}\xrightarrow{\widehat{u}}\widehat{B}\xrightarrow{\widehat{v}}\widehat{C}\xrightarrow{\widehat{w}}\Sigma\widehat{A}$ to a long exact sequence.

Lemma 9.8 *Let the conventions be as in Notation 9.1. Assume H is a locally finite $\langle G\rangle_n$–cohomological functor. Then there exists an object $F\in\mathcal{T}_c^-$ and an epimorphism of $\langle G\rangle_n$–cohomological functors $\varphi:\mathcal{Y}(F)|_{\langle G\rangle_n}\longrightarrow H$. Furthermore the object F may be chosen to have a strong $\langle G\rangle_n$–approximating system.*

We will in fact prove a refinement of the above. Since H is assumed to be a locally finite $\langle G\rangle_n$–cohomological functor its restriction to $\langle G\rangle_m$, for any integer $m<n$, is a

locally finite $\langle G\rangle_m$*–cohomological functor. Hence for any* $m < n$ *the first paragraph delivers an object* $F_m \in \mathcal{T}_c^-$, *with a strong* $\langle G\rangle_m$*–approximating system, and a surjective natural transformation* $\varphi_m : \mathcal{Y}(F_m)|_{\langle G\rangle_m} \longrightarrow H|_{\langle G\rangle_m}$. *We will actually construct these* F_m*'s compatibly. We will produce in* $\mathcal{T}_c^-$ *a sequence* $F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_{n-1} \longrightarrow F_n$, *with compatible maps* $\varphi_m : \mathcal{Y}(F_m)|_{\langle G\rangle_n} \longrightarrow H$, *so that*

(i) *For each* $m > 0$ *the object* F_m *has a strong* $\langle G\rangle_m$*–approximating system, and the map* $\varphi_m|_{\langle G\rangle_m} : \mathcal{Y}(F_m)|_{\langle G\rangle_m} \longrightarrow H|_{\langle G\rangle_m}$ *is an epimorphism.*
(ii) *The sequence is such that the kernel of the map* $(\varphi_m)|_{\langle G\rangle_1} : \mathcal{Y}(F_m)|_{\langle G\rangle_1} \longrightarrow H|_{\langle G\rangle_1}$ *is annihilated by the map* $\mathcal{Y}(F_m)|_{\langle G\rangle_1} \longrightarrow \mathcal{Y}(F_{m+1})|_{\langle G\rangle_1}$.

Proof The proof is by induction on n. In the case $n = 1$ we prove the refinement that allows the induction to proceed

(iii) Suppose H is a locally finite $\langle G\rangle_1$–cohomological functor. Then we may construct an object $F \in \mathcal{T}_c^-$ and an epimorphism $\mathcal{Y}(F)|_{\langle G\rangle_1} \longrightarrow H$. Furthermore the object F can be chosen to have a strong $\langle G\rangle_1$–approximating system $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ *in which every morphism* $E_i \longrightarrow E_{i+1}$ *is a split monomorphism.*

The proof of (iii) is easy: we have that $H(\Sigma^i G)$ is a finite R–module for every $i \in \mathbb{Z}$, and vanishes if $i \ll 0$. For each i with $H(\Sigma^i G) \neq 0$ choose a finite number of generators $\{f_{ij},\ j \in J_i\}$ for the R–module $H(\Sigma^i G)$. By Yoneda every $f_{ij} \in H(\Sigma^i G)$ corresponds to a morphism $\varphi_{ij} : \mathcal{Y}(\Sigma^i G)|_{\langle G\rangle_1} \longrightarrow H$. Let F be defined by

$$F = \coprod_{i\in\mathbb{Z}} \bigoplus_{j\in J_i} \Sigma^i G$$

and let the morphism $\varphi : \mathcal{Y}(F)|_{\langle G\rangle_1} \longrightarrow H$ be given by

$$\begin{tikzcd}
\mathcal{Y}(F)|_{\langle G\rangle_1} \arrow[r, equal] & \bigoplus_{i\in\mathbb{Z}} \bigoplus_{j\in J_i} \mathcal{Y}(\Sigma^i G)|_{\langle G\rangle_1} \arrow[r, "(\varphi_{ij})"] & H
\end{tikzcd}$$

where (φ_{ij}) stands for the row matrix with entries φ_{ij}; on the i, j summand the map is φ_{ij}. Finally: because the t–structure is in the preferred equivalence class there is an integer $B > 0$ with $\Sigma^B G \in \mathcal{T}^{\le 0}$. For $m > 0$ we define

$$E_m = \bigoplus_{i\le m+B} \bigoplus_{j\in J_i} \Sigma^i G$$

The sum is finite by hypothesis, making E_m an object of $\langle G\rangle_1$. The obvious map $E_m \longrightarrow E_{m+1}$ is a split monomorphism, and in the decomposition $F \cong E_m \oplus \widetilde{F}$ we have that $\widetilde{F}$, being the coproduct of $\Sigma^i G$ for $i \ge m + B + 1$, belongs to $\mathcal{T}^{\le -m-1}$. Therefore the map $\mathcal{H}^i(E_m) \longrightarrow \mathcal{H}^i(F)$ is an isomorphism when $i \ge -m$, making E_* is a strong $\langle G\rangle_1$–approximating system for F.

Now for the induction step. Suppose $n \ge 1$ is an integer and we know the Lemma for all integers $\le n$. We wish to show it holds for $n + 1$. Let H be a locally finite

$\langle G\rangle_{n+1}$–cohomological functor. Then the restriction of H to $\langle G\rangle_n$ is a locally finite $\langle G\rangle_n$–cohomological functor, and we may apply the induction hypothesis to produce in $\mathcal{T}_c^-$ a sequence $F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_{n-1} \longrightarrow F_n$, with compatible surjections $\varphi_m : \mathcal{Y}(F_m)|_{\langle G\rangle_m} \longrightarrow H|_{\langle G\rangle_m}$. In particular the map $\varphi_n : \mathcal{Y}(F_n)|_{\langle G\rangle_n} \longrightarrow H|_{\langle G\rangle_n}$ is an epimorphism. Complete the natural transformation φ_n to a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H' \arrow[r] & \mathcal{Y}(F_n)|_{\langle G\rangle_n} \arrow[r, "\varphi_n"] & H|_{\langle G\rangle_n} \arrow[r] & 0 \end{tikzcd}$$

of functors on $\langle G\rangle_n$. Since $H|_{\langle G\rangle_n}$ and $\mathcal{Y}(F_n)|_{\langle G\rangle_n}$ are locally finite $\langle G\rangle_n$–cohomological functors so is H', and induction applies. For any $m \le n$ we may choose a surjection $\varphi'_m : \mathcal{Y}(F')|_{\langle G\rangle_m} \longrightarrow H'|_{\langle G\rangle_m}$, as in (i) and (ii). We wish to consider the special case $m = 1$, where we can assume our F' is as in (iii). That is we choose an object $F' \in \mathcal{T}_c^-$, which admits a strong $\langle G\rangle_1$–approximating system and a surjection $\varphi' : \mathcal{Y}(F')|_{\langle G\rangle_1} \longrightarrow H'|_{\langle G\rangle_1}$. And we may further assume that our approximating system $E'_1 \longrightarrow E'_2 \longrightarrow E'_3 \longrightarrow \cdots$ for F' is such that every morphism $E'_i \longrightarrow E'_{i+1}$ is a split monomorphism.

We have natural transformations $\mathcal{Y}(F')|_{\langle G\rangle_1} \longrightarrow H'|_{\langle G\rangle_1} \longrightarrow \mathcal{Y}(F_n)|_{\langle G\rangle_1}$, and as F' admits a $\langle G\rangle_1$–approximating system Lemma 7.8 tells us that the composite is equal to $\mathcal{Y}(\alpha_n)|_{\langle G\rangle_1}$ for some morphism $\alpha_n : F' \longrightarrow F_n$ in the category $\mathcal{T}$. And now Lemma 8.5 applies; see Remark 9.7 for an elaboration of how it specializes to the current context. We learn that

(iv) There exists a weak triangle $F' \xrightarrow{\alpha_n} F_n \xrightarrow{\beta_n} F_{n+1}$ in the category $\mathcal{T}_c^-$, with F_{n+1} admitting a strong $\langle G\rangle_{n+1}$–approximating system.

(v) There is natural transformation $\varphi_{n+1} : \mathcal{Y}(F_{n+1})|_{\langle G\rangle_{m+1}} \longrightarrow H$, such that $\varphi_n : \mathcal{Y}(F_n)|_{\langle G\rangle_n} \longrightarrow H|_{\langle G\rangle_n}$ is equal to the composite

$$\begin{tikzcd} \mathcal{Y}(F_n)|_{\langle G\rangle_n} \arrow[r, "\mathcal{Y}(\beta_n)|_{\langle G\rangle_n}"] & \mathcal{Y}(F_{n+1})|_{\langle G\rangle_n} \arrow[r, "\varphi_{n+1}|_{\langle G\rangle_n}"] & H|_{\langle G\rangle_n} . \end{tikzcd}$$

Comparing the two exact sequences

$$\begin{tikzcd} \mathcal{Y}(F')|_{\langle G\rangle_1} \arrow[r, "\mathcal{Y}(\alpha_n)|_{\langle G\rangle_1}"] & \mathcal{Y}(F_n)|_{\langle G\rangle_1} \arrow[r, "\varphi_n|_{\langle G\rangle_1}"] & H|_{\langle G\rangle_1} \\ \mathcal{Y}(F')|_{\langle G\rangle_1} \arrow[r, "\mathcal{Y}(\alpha_n)|_{\langle G\rangle_1}"] & \mathcal{Y}(F_n)|_{\langle G\rangle_1} \arrow[r, "\mathcal{Y}(\beta_n)|_{\langle G\rangle_1}"] & \mathcal{Y}(F_{n+1})|_{\langle G\rangle_1} \end{tikzcd}$$

we conclude that $\varphi_n|_{\langle G\rangle_1}$ and $\mathcal{Y}(\beta_n)|_{\langle G\rangle_1}$ have the same kernel, that is the map $\beta_n : F_n \longrightarrow F_{n+1}$ satisfies (ii).

To finish the proof of (i) it remains to show that $\varphi_{n+1} : \mathcal{Y}(F_{n+1})|_{\langle G\rangle_{n+1}} \longrightarrow H$ is an epimorphism, but this is now immediate from Lemma 8.6. ☐

Remark 9.9 Let the conventions be as in Notation 9.1 and assume H is a locally finite $\mathcal{T}^c$–cohomological functor. For every integer $n > 0$ the restriction of H to $\langle G\rangle_n$ is a locally finite $\langle G\rangle_n$–homological functor, and Lemma 9.8 permits us to construct a sequence $F_1 \xrightarrow{\beta_1} F_2 \xrightarrow{\beta_2} F_3 \xrightarrow{\beta_3} \cdots$ in the category $\mathcal{T}_c^-$, together with compatible epimorphisms $\varphi_n : \mathcal{Y}(F_n)|_{\langle G\rangle_n} \longrightarrow H|_{\langle G\rangle_n}$. Since F_n is constructed to have a $\langle G\rangle_n$–approximating system, Corollary 7.4 says that each φ_n lifts uniquely to a natural transformation defined on all of $\mathcal{T}^c$, which we denote $\varphi_n : \mathcal{Y}(F_n) \longrightarrow H$. Note the abuse of notation, where φ_n is allowed to stand both for the natural transformation of functors on $\mathcal{T}^c$ and for its restriction to $\langle G\rangle_n$. The triangle

$$\begin{tikzcd}
 & \mathcal{Y}(F_{n+1}) \arrow[d, "\varphi_{n+1}"] \\
\mathcal{Y}(F_n) \arrow[ur, "\mathcal{Y}(\beta_n)"] \arrow[r, "\varphi_n"'] & H
\end{tikzcd}$$

commutes when restricted to the subcategory $\langle G\rangle_n \subset \langle G\rangle$, and the fact that F_n has a $\langle G\rangle_n$–approximating system coupled with the uniqueness assertion of Corollary 7.4 tells us that the triangle commutes on the nose, in the category $\mathrm{Hom}\big([\mathcal{T}^c]^{\mathrm{op}}, R\text{–Mod}\big)$.

Proposition 9.10 *Let the conventions be as in Notation 9.1 and assume H is a locally finite $\mathcal{T}^c$–cohomological functor. Then there exists an object $F \in \mathcal{T}$ and an isomorphism $\varphi : \mathcal{Y}(F) \longrightarrow H$.*

Now let $\mathcal{G} = \oplus_{C \in \mathcal{T}^c} C$; for those worried about set theoretic issues this means that $\mathcal{G}$ is the coproduct, over the isomorphism classes of objects in $\mathcal{T}^c$, of a representative in each isomorphism class. Then F may be chosen to lie in $\overline{\langle \mathcal{G}\rangle}_4$.

Proof In Remark 9.9 we noted that Lemma 9.8 constructs for us a sequence $F_1 \xrightarrow{\beta_1} F_2 \xrightarrow{\beta_2} F_3 \xrightarrow{\beta_3} \cdots$ in the category $\mathcal{T}$, together with compatible maps $\varphi_n : \mathcal{Y}(F_n) \longrightarrow H$; there is an induced map $\underrightarrow{\mathrm{colim}}\, \mathcal{Y}(F_n) \longrightarrow H$. If we define F to be $F = \underrightarrow{\mathrm{Hocolim}}\, F_n$ then we have an object $F \in \mathcal{T}$, and [18, Lemma 2.8] tells us that the natural map $\underrightarrow{\mathrm{colim}}\, \mathcal{Y}(F_n) \longrightarrow \mathcal{Y}(F)$ is an isomorphism. We have constructed a map $\varphi : \mathcal{Y}(F) \longrightarrow H$ and will prove that φ is an isomorphism.

Let us consider the restriction of the natural transformation φ to the subcategory $\langle G\rangle_1 \subset \mathcal{T}^c$. The natural transformation $\varphi|_{\langle G\rangle_1} : \mathcal{Y}(F)|_{\langle G\rangle_1} \longrightarrow H|_{\langle G\rangle_1}$ is the map to $H|_{\langle G\rangle_1}$ from the colimit of the sequence

$$\begin{tikzcd}
\mathcal{Y}(F_1)|_{\langle G\rangle_1} \arrow[r, "\mathcal{Y}(\beta_1)|_{\langle G\rangle_1}"] & \mathcal{Y}(F_2)|_{\langle G\rangle_1} \arrow[r, "\mathcal{Y}(\beta_2)|_{\langle G\rangle_1}"] & \cdots
\end{tikzcd}$$

and Lemma 9.8(ii) says the sequence is such that each map $\mathcal{Y}(\beta_n)|_{\langle G\rangle_1}$ factors as $\mathcal{Y}(F_n)|_{\langle G\rangle_1} \longrightarrow H|_{\langle G\rangle_1} \longrightarrow \mathcal{Y}(F_{n+1})|_{\langle G\rangle_1}$. Hence the colimit agrees with the colimit of the ind-isomorphic constant sequence $H|_{\langle G\rangle_1} \longrightarrow H|_{\langle G\rangle_1} \longrightarrow H|_{\langle G\rangle_1} \longrightarrow$, and this proves that the restriction of $\varphi : \mathcal{Y}(F) \longrightarrow H$ to the category $\langle G\rangle_1$ is an isomorphism. Concretely: for every $i \in \mathbb{Z}$ the map $\varphi : \mathrm{Hom}(\Sigma^i G, F) \longrightarrow H(\Sigma^i G)$ is an isomorphism. The full subcategory $\mathcal{K} \subset \mathcal{T}^c$ defined by

$$\mathcal{K} \quad = \quad \left\{ K \in \mathcal{T}^c \;\middle|\; \begin{array}{c} \forall i \in \mathbb{Z} \text{ the map} \\ \varphi : \mathrm{Hom}(\Sigma^i K, F) \longrightarrow H(\Sigma^i K) \\ \text{is an isomorphism} \end{array} \right\}$$

is thick and contains G, hence $\mathcal{K} \subset \mathcal{T}^c = \langle G \rangle \subset \mathcal{K}$.

It remains to prove that the F we constructed belongs to $\overline{\langle \mathcal{G} \rangle}_4$. We begin with the observation that each F_n in the sequence $F_1 \xrightarrow{\beta_1} F_2 \xrightarrow{\beta_2} F_3 \xrightarrow{\beta_3} \cdots$ of Remark 9.9 has a strong $\langle G \rangle_n$–approximating system. This means that $F_n = \underrightarrow{\mathrm{Hocolim}}\; E_i^n$, with each $E_i^n \in \langle G \rangle_n$. The triangle

$$\begin{tikzcd} \displaystyle\bigoplus_{i=1}^{\infty} E_i^n \arrow[r] & \displaystyle\bigoplus_{i=1}^{\infty} E_i^n \arrow[r] & F_n \arrow[r] & \displaystyle\bigoplus_{i=1}^{\infty} \Sigma E_i^n \end{tikzcd}$$

tells us that F_n must belong to $\overline{\langle \mathcal{G} \rangle}_1 * \overline{\langle \mathcal{G} \rangle}_1 \subset \overline{\langle \mathcal{G} \rangle}_2$. But now $F = \underrightarrow{\mathrm{Hocolim}}\; F_n$, and the triangle

$$\begin{tikzcd} \displaystyle\bigoplus_{n=1}^{\infty} F_n \arrow[r] & \displaystyle\bigoplus_{n=1}^{\infty} F_n \arrow[r] & F \arrow[r] & \displaystyle\bigoplus_{n=1}^{\infty} \Sigma F_n \end{tikzcd}$$

gives that F belongs to $\overline{\langle \mathcal{G} \rangle}_2 * \overline{\langle \mathcal{G} \rangle}_2 \subset \overline{\langle \mathcal{G} \rangle}_4$. ☐

Notation 9.11 This is as far as we get with the assumptions of Notation 9.1. From now on we will assume further that $\mathcal{T}$ is weakly approximable.

Lemma 9.12 *Let the conventions be as in Notation 9.11. Assume H is a locally finite $\mathcal{T}^c$–cohomological functor. There exists an integer $\widetilde{A} > 0$ and, for any $n > 0$, an object $F_n \in \mathcal{T}_c^- \cap \mathcal{T}^{\leq \widetilde{A}}$ as well as a natural transformation $\varphi_n : \mathcal{Y}(F_n) \longrightarrow H$ which is surjective when restricted to $\langle G \rangle_n$.*

Proof Remark 9.9 produced for us an object $F_n \in \mathcal{T}_c^-$ and a natural transformation $\varphi_n : \mathcal{Y}(F_n) \longrightarrow H$, so that the restriction to $\langle G \rangle_n$ of φ_n is surjective. The new assertion is that we may choose F_n to lie in $\mathcal{T}^{\leq \widetilde{A}}$ for some $\widetilde{A} > 0$ independent of n.

Because H is locally finite there exists an integer $B' > 0$ with $H(\Sigma^{-i} G) = 0$ for all $i \geq B'$. Since H is cohomological it follows that $H(E) = 0$ for all $E \in \langle G \rangle^{[B',\infty)}$. And Corollary 3.14 gives us an integer $B > 0$ so that, for every integer $m > 0$, every object $F \in \mathcal{T}_c^- \cap \mathcal{T}^{\leq 0}$ admits a triangle $E_m \longrightarrow F \longrightarrow D_m$ with $D_m \in \mathcal{T}^{\leq -m}$ and $E_m \in \langle G \rangle^{[1-m-B,B]} \subset \langle G \rangle^{[1-m-B,\infty)}$. I assert that $\widetilde{A} = B + B'$ works.

Let us begin with the F_n provided by Remark 9.9. Because it belongs to $\mathcal{T}_c^- \subset \mathcal{T}^-$ there exists an integer $\ell > B + B'$ with $F_n \in \mathcal{T}^{\leq \ell}$. Applying Corollary 3.14 to the object $\Sigma^\ell F_n \in \mathcal{T}_c^- \cap \mathcal{T}^{\leq 0}$, with $m = \ell - B - B'$, we learn that there exists a triangle $E \xrightarrow{\alpha} F_n \longrightarrow D$ with $D \in \mathcal{T}^{\leq \ell - m} = \mathcal{T}^{\leq B+B'} = \mathcal{T}^{\leq \widetilde{A}}$ and with

$E \in \langle G\rangle^{[\ell+1-m-B,\infty)} = \langle G\rangle^{[1+B',\infty)}$. In particular $E \in \mathcal{T}^c$, and $H(E) = 0$ by the choice of B'. Hence the composite

$$\begin{tikzcd} \mathcal{Y}(E) \arrow[r, "\mathcal{Y}(\alpha)"] & \mathcal{Y}(F_n) \arrow[r, "\varphi"] & H \end{tikzcd}$$

vanishes.

Now we apply Lemma 8.5 with $\mathcal{A} = \mathcal{B} = \mathcal{T}^c$. The object $F_n \in \mathcal{T}_c^-$, of Lemma 9.8 and Remark 9.9, comes with a strong $\langle G\rangle_n$–approximating system, which is certainly a strong $\mathcal{T}^c$–approximating system. The object $E \in \mathcal{T}^c$ comes with the trivial strong $\mathcal{T}^c$–approximating system $E \xrightarrow{\mathrm{id}} E \xrightarrow{\mathrm{id}} E \xrightarrow{\mathrm{id}} \cdots$. In this system the connecting maps are all identities, which are split monomorphisms. The hypotheses of Lemma 8.5 and Remark 9.7 hold and the Lemma produces for us, in $\mathcal{T}_c^-$, a weak triangle $E \xrightarrow{\alpha} F_n \xrightarrow{\beta} \widetilde{D} \longrightarrow \Sigma E$ and a factorization of $\varphi : \mathcal{Y}(F_n) \longrightarrow H$ as a composite

$$\begin{tikzcd} \mathcal{Y}(F_n) \arrow[r, "\mathcal{Y}(\beta)"] & \mathcal{Y}(\widetilde{D}) \arrow[r, "\psi"] & H\,. \end{tikzcd}$$

The surjectivity of the restriction to $\langle G\rangle_n$ of φ implies the surjectivity of the restriction to $\langle G\rangle_n$ of ψ.

It remains to show that $\widetilde{D} \in \mathcal{T}_c^-$ belongs to $\mathcal{T}_c^- \cap \mathcal{T}^{\leq \widetilde{A}}$. We know that, in the triangle $E \longrightarrow F_n \longrightarrow D \longrightarrow \Sigma E$, the object D belongs to $\mathcal{T}^{\leq \widetilde{A}}$. The long exact sequence $\mathcal{H}^{i-1}(D) \longrightarrow \mathcal{H}^i(E) \longrightarrow \mathcal{H}^i(F_n) \longrightarrow \mathcal{H}^i(D)$ tells us that $\mathcal{H}^i(\alpha) : \mathcal{H}^i(E) \longrightarrow \mathcal{H}^i(F)$ is surjective if $i = 1 + \widetilde{A}$ and is an isomorphism when $i > 1 + \widetilde{A}$. The long exact sequence $\mathcal{H}^i(E) \longrightarrow \mathcal{H}^i(F_n) \longrightarrow \mathcal{H}^i(\widetilde{D}) \longrightarrow \mathcal{H}^{i+1}(E) \longrightarrow \mathcal{H}^{i+1}(F_n)$ says that $\mathcal{H}^i(\widetilde{D}) = 0$ if $i \geq 1 + \widetilde{A}$. By Lemma 2.2 we conclude that $\widetilde{D} \in \mathcal{T}^{\leq \widetilde{A}}$. ☐

Notation 9.13 This is as far as we get with the assumptions of Notation 9.11, from now on we assume further that $\mathcal{T}$ is approximable—weak approximability will no longer be enough.

Lemma 9.14 *Let the conventions be as in Notation 9.13, and assume H is a locally finite $\mathcal{T}^c$–cohomological functor. Choose an integer $B > 0$ as in Lemma 3.12.*

Suppose $\widetilde{F}$, F' are objects in $\mathcal{T}_c^- \cap \mathcal{T}^{\leq 0}$, E is an object in $\mathcal{T}^c \cap \mathcal{T}^{\leq 0}$, and we have a morphism $\alpha : E \longrightarrow \widetilde{F}$ in $\mathcal{T}_c^-$. Assume we are given an integer $m > 0$, as well as natural transformations $\widetilde{\varphi} : \mathcal{Y}(\widetilde{F}) \longrightarrow H$ and $\varphi' : \mathcal{Y}(F') \longrightarrow H$, so that $\widetilde{\varphi}$ restricts to an epimorphism on $\langle G\rangle_{mB}$ and φ' restricts to an epimorphism on $\langle G\rangle_{(m+1)B}$.

Then there exists in $\mathcal{T}_c^- \cap \mathcal{T}^{\leq 0}$ a commutative diagram

$$\begin{tikzcd} E \arrow[rr, "\varepsilon"] \arrow[dr, "\alpha"'] & & E' \arrow[dl, "\gamma"] \arrow[dr, "\alpha'"] & & F' \arrow[dl, "\beta"'] \\ & \widetilde{F} & & \widetilde{F}' & \end{tikzcd}$$

and there exists a natural transformation $\widetilde{\varphi}':\mathcal{Y}(\widetilde{F}')\longrightarrow H$ *so that*

- (i) *The object* E' *belongs to* $\mathcal{T}^c\subset\mathcal{T}_c^-$.
- (ii) *The maps* $\mathcal{H}^i(\alpha')$ *and* $\mathcal{H}^i(\gamma)$ *are isomorphisms for all* $i\geq -m+2$.
- (iii) *The triangle*

$$\begin{tikzcd}
 & \mathcal{Y}(\widetilde{F}') \arrow[d, "\widetilde{\varphi}'"] \\
\mathcal{Y}(F') \arrow[ur, "\mathcal{Y}(\beta)"] \arrow[r, "\varphi'"'] & H
\end{tikzcd}$$

commutes; the surjectivity of the restriction to $\langle G\rangle_{(m+1)B}$ *of* φ' *therefore implies the surjectivity of the restriction to* $\langle G\rangle_{(m+1)B}$ *of* $\widetilde{\varphi}'$.

- (iv) *The square below commutes*

$$\begin{tikzcd}
\mathcal{Y}(E') \arrow[r, "\mathcal{Y}(\gamma)"] \arrow[d, "\mathcal{Y}(\alpha')"] & \mathcal{Y}(\widetilde{F}) \arrow[d, "\widetilde{\varphi}"] \\
\mathcal{Y}(\widetilde{F}') \arrow[r, "\widetilde{\varphi}'"'] & H
\end{tikzcd}$$

Proof Corollary 3.14, applied to the object $F'\in\mathcal{T}_c^-\cap\mathcal{T}^{\leq 0}$ and the integer m, permits us to construct in $\mathcal{T}_c^-$ a triangle $E'_m\xrightarrow{a}F'\xrightarrow{b}D_m$ with $D_m\in\mathcal{T}^{\leq -m}$ and $E'_m\in\langle G\rangle_{mB}^{[1-m-B,B]}\subset\langle G\rangle_{mB}$. Because E'_m belongs to $\mathcal{T}^c$ the composite

$$\begin{tikzcd}
\mathcal{Y}(E'_m) \arrow[r, "\mathcal{Y}(a)"] & \mathcal{Y}(F') \arrow[r, "\varphi'"] & H
\end{tikzcd}$$

is a natural transformation from a representable functor on $\mathcal{T}^c$ to H, and corresponds to an element $x\in H(E'_m)$. As E'_m belongs to $\langle G\rangle_{mB}$ the morphism $\widetilde{\varphi}:\operatorname{Hom}(E'_m,\widetilde{F})\longrightarrow H(E'_m)$ is surjective; there is a map $f:E'_m\longrightarrow\widetilde{F}$ with $\widetilde{\varphi}(f)=x$. Yoneda translates this to mean that the square below commutes

$$\begin{tikzcd}
\mathcal{Y}(E'_m) \arrow[r, "\mathcal{Y}(f)"] \arrow[d, "\mathcal{Y}(a)"] & \mathcal{Y}(\widetilde{F}) \arrow[d, "\widetilde{\varphi}"] \\
\mathcal{Y}(F') \arrow[r, "\varphi'"'] & H
\end{tikzcd}$$

We now have a morphism $(\alpha,f):E\oplus E'_m\longrightarrow\widetilde{F}$. The object $E\oplus E'_m$ belongs to $\mathcal{T}^c$ while the object $\widetilde{F}\in\mathcal{T}_c^-$ has a strong $\mathcal{T}^c$–approximating system—see

Lemma 9.5(ii). There exists a sequence $\widetilde{E}_1 \longrightarrow \widetilde{E}_2 \longrightarrow \widetilde{E}_3 \longrightarrow \cdots$ in $\mathcal{T}^c$, and a map of $\widetilde{E}_*$ to $\widetilde{F}$, and so that $\mathcal{H}^i(\widetilde{E}_j) \longrightarrow \mathcal{H}^i(\widetilde{F})$ is an isomorphism whenever $i \geq -j$. The morphism $E \oplus E'_m \longrightarrow \widetilde{F}$ must factor through some $\widetilde{E}_j$—see Lemma 9.5(i) and [18, Lemma 2.8]. Choose such an $\widetilde{E}_j$ with $j \geq m$, declare $E' = \widetilde{E}_j$, and let $\varepsilon : E \longrightarrow E'$, $g : E'_m \longrightarrow E'$ and $\gamma : E' \longrightarrow \widetilde{F}$ be the obvious maps. Then $\gamma\varepsilon = \alpha$ and $\gamma g = f$.

By construction the map $\mathcal{H}^\ell(\gamma) : \mathcal{H}^\ell(E') \longrightarrow \mathcal{H}^\ell(\widetilde{F})$ is an isomorphism for all $\ell \geq -m$. Recalling the triangle $E'_m \xrightarrow{a} F' \longrightarrow D'_m$ with $D'_m \in \mathcal{T}^{\leq -m}$, the exact sequence

$$\begin{tikzcd}
\mathcal{H}^{\ell-1}(D'_m) \arrow[r] & \mathcal{H}^\ell(E'_m) \arrow[r, "\mathcal{H}^\ell(a)"] & \mathcal{H}^\ell(F') \arrow[r] & \mathcal{H}^\ell(D'_m)
\end{tikzcd}$$

teaches us that

(v) The maps $\mathcal{H}^\ell(\gamma)$ and $\mathcal{H}^\ell(a)$ are isomorphisms for all $\ell \geq -m+2$.

Since $m \geq 1$ by assumption, we learn in particular that if $\ell \geq 1$ then $\mathcal{H}^\ell(E') \cong \mathcal{H}^\ell(\widetilde{F}) = 0$ and $\mathcal{H}^\ell(E'_m) \cong \mathcal{H}^\ell(F') = 0$. As both E' and E'_m are objects of $\mathcal{T}^c \subset \mathcal{T}^-$, Lemma 2.2 informs us that

(vi) E', E'_m both lie in $\mathcal{T}^{\leq 0}$.

Now consider the commutative square

$$\begin{tikzcd}
\mathcal{Y}(E'_m) \arrow[r, "\mathcal{Y}(g)"] \arrow[d, "\mathcal{Y}(a)"] & \mathcal{Y}(E') \arrow[d, "\rho=\widetilde{\varphi}\circ\mathcal{Y}(\gamma)"] \\
\mathcal{Y}(F') \arrow[r, "\varphi'"'] & H
\end{tikzcd}$$

In other words: we have in $\mathcal{T}_c^- \cup \mathcal{T}^{\leq 0}$ a morphism $\sigma = \begin{pmatrix} -g \\ a \end{pmatrix} : E'_m \longrightarrow E' \oplus F'$, as well as a natural transformation $(\rho, \varphi') : \mathcal{Y}(E' \oplus F') \longrightarrow H$, and the composite

$$\begin{tikzcd}
\mathcal{Y}(E'_m) \arrow[r, "\mathcal{Y}(\sigma)"] & \mathcal{Y}(E' \oplus F') \arrow[r, "(\rho,\varphi')"] & H
\end{tikzcd}$$

vanishes. The object $E' \oplus F' \in \mathcal{T}_c^-$ has a strong $\mathcal{T}^c$–approximating system by Lemma 9.5(ii), and the object $E'_m \in \mathcal{T}^c$ has the trivial strong $\mathcal{T}^c$–approximating system $E'_m \xrightarrow{\text{id}} E'_m \xrightarrow{\text{id}} E'_m \xrightarrow{\text{id}} \cdots$. We may apply Lemma 8.5 as specialized in Remark 9.7, with $\mathcal{A} = \mathcal{B} = \mathcal{T}^c$, to deduce that $E'_m \xrightarrow{\sigma} E' \oplus F'$ may be completed in $\mathcal{T}_c^-$ to a weak triangle

$$\begin{tikzcd}
E'_m \arrow[r, "\sigma"] & E' \oplus F' \arrow[r, "(\alpha',\beta)"] & \widetilde{F}' \arrow[r, "\tau"] & \Sigma E'_m
\end{tikzcd}$$

in such a way that the morphism $(\rho, \varphi') : \mathrm{Hom}(-, E' \oplus F')|_{\mathcal{T}^c} \longrightarrow H(-)$ factors as $\widetilde{\varphi}' \circ \mathrm{Hom}\big(-, (\alpha', \beta)\big)$ for some natural transformation $\widetilde{\varphi}' : \mathrm{Hom}(-, \widetilde{F}')|_{\mathcal{T}^c} \longrightarrow H(-)$. We have constructed in $\mathcal{T}_c^-$ a commutative square

$$\begin{tikzcd}
E'_m \arrow[r, "g"] \arrow[d, "a"] & E' \arrow[d, "\alpha'"] \\
F' \arrow[r, "\beta"] & \widetilde{F}'
\end{tikzcd}$$

and a natural transformation $\widetilde{\varphi}' : \mathrm{Hom}(-, \widetilde{F}')|_{\mathcal{T}^c} \longrightarrow H(-)$ so that the diagram below commutes

$$\begin{tikzcd}
\mathcal{Y}(E'_m) \arrow[r, "\mathcal{Y}(g)"] \arrow[d, "\mathcal{Y}(a)"] & \mathcal{Y}(E') \arrow[r, "\mathcal{Y}(\gamma)"] \arrow[d, "\mathcal{Y}(\alpha')"] & \mathcal{Y}(\widetilde{F}) \arrow[dd, "\widetilde{\varphi}"] \\
\mathcal{Y}(F') \arrow[r, "\mathcal{Y}(\beta)"] \arrow[drr, bend right, "\varphi'"'] & \mathcal{Y}(\widetilde{F}') \arrow[dr, "\widetilde{\varphi}'"] & \\
 & & H
\end{tikzcd}$$

This finishes our construction of the diagram

$$\begin{tikzcd}
E \arrow[rr, "\varepsilon"] \arrow[dr, "\alpha"'] & & E' \arrow[dl, "\gamma"] \arrow[dr, "\alpha'"] & & F' \arrow[dl, "\beta"'] \\
 & \widetilde{F} & & \widetilde{F}' &
\end{tikzcd}$$

and the natural transformation $\widetilde{\varphi}' : \mathcal{Y}(\widetilde{F}') \longrightarrow H$. The assertions (i), (iii) and (iv) of the Lemma have already been proved, as well as half of (ii). It remains to show that $\widetilde{F}' \in \mathcal{T}_c^-$ also belongs to $\mathcal{T}^{\le 0}$, and that the morphisms $\mathcal{H}^\ell(\alpha')$ are isomorphisms for $\ell \ge -m+2$.

To prove this we recall that the commutative square

$$\begin{tikzcd}
\mathcal{H}^\ell(E'_m) \arrow[r, "\mathcal{H}^\ell(g)"] \arrow[d, "\mathcal{H}^\ell(a)"'] & \mathcal{H}^\ell(E') \arrow[d, "\mathcal{H}^\ell(\alpha')"] \\
\mathcal{H}^\ell(F') \arrow[r, "\mathcal{H}^\ell(\beta)"] & \mathcal{H}^\ell(\widetilde{F}')
\end{tikzcd}$$

comes from applying $\mathcal{H}^\ell$ to the weak triangle

$$\begin{tikzcd}
E'_m \arrow[r, "\sigma"] & E' \oplus F' \arrow[r, "(\alpha',\beta)"] & \widetilde{F}' \arrow[r, "\tau"] & \Sigma E'_m
\end{tikzcd}$$

The morphism $\mathcal{H}^\ell(\tau) : \mathcal{H}^\ell(\widetilde{F}') \longrightarrow \mathcal{H}^{\ell+1}(E'_m)$ fits in a long exact sequence. In (v) we proved that $\mathcal{H}^\ell(a) : \mathcal{H}^\ell(E'_m) \longrightarrow \mathcal{H}^\ell(F')$ is an isomorphism for $\ell \geq -m+2$, which makes the map $\mathcal{H}^\ell(\sigma) : \mathcal{H}^\ell(E'_m) \longrightarrow \mathcal{H}^\ell(E' \oplus F')$ a split monomorphism for all $\ell \geq -m+2$. The exactness tells us first that $\mathcal{H}^\ell(\tau) = 0$ for all $\ell \geq -m+1$, and then that the square

$$\begin{tikzcd}
\mathcal{H}^\ell(E'_m) \arrow[r, "\mathcal{H}^\ell(g)"] \arrow[d, "\mathcal{H}^\ell(a)"'] & \mathcal{H}^\ell(E') \arrow[d, "\mathcal{H}^\ell(\alpha')"] \\
\mathcal{H}^\ell(F') \arrow[r, "\mathcal{H}^\ell(\beta)"] & \mathcal{H}^\ell(\widetilde{F}')
\end{tikzcd}$$

is bicartesian for all $\ell \geq -m+2$. As long as $\ell \geq -m+2$, the fact that $\mathcal{H}^\ell(a)$ is an isomorphism forces $\mathcal{H}^\ell(\alpha')$ to also be.

In particular: since $m \geq 1$ we have that $-m+2 \leq 1$, and deduce that $\mathcal{H}^\ell(\widetilde{F}') \cong \mathcal{H}^\ell(E') = 0$ for all $\ell \geq 1$. Lemma 2.2 now gives that $\widetilde{F}'$ belongs to $\mathcal{T}^{\leq 0}$. ☐

Lemma 9.15 *Let the conventions be as in Notation 9.13, and assume H is a locally finite $\mathcal{T}^c$–cohomological functor. There exists an object $F \in \mathcal{T}_c^-$ and an epimorphism* $\mathrm{Hom}(-, F)|_{\mathcal{T}^c} \longrightarrow H(-)$.

Proof By Lemma 9.12 we may assume given: an integer $\widetilde{A} > 0$, objects $F_n \in \mathcal{T}_c^- \cap \mathcal{T}^{\leq \widetilde{A}}$ and natural transformations $\varphi_n : \mathcal{Y}(F_n) \longrightarrow H$ which restrict to epimorphisms on $\langle G \rangle_n$. Replacing the functor H by $H(\Sigma^{\widetilde{A}}-)$ we may assume $\widetilde{A} = 0$. Let $B > 0$ be the integer whose existence is given by Lemma 3.12. Next we need to make our construction.

We will proceed inductively, using Lemma 9.14, to construct in $\mathcal{T}_c^- \cap \mathcal{T}^{\leq 0}$ a sequence

$$\begin{tikzcd}
E_1 \arrow[rr, "\varepsilon_2"] \arrow[dr, "\alpha_1"'] & & E_2 \arrow[rr, "\varepsilon_3"] \arrow[dl, "\gamma_2"] \arrow[dr, "\alpha_2"'] & & E_3 \arrow[dl, "\gamma_3"] \arrow[dr, "\alpha_3"'] & & \cdots \\
& \widetilde{F}_1 & & \widetilde{F}_2 & & \widetilde{F}_3 &
\end{tikzcd}$$

as well as natural transformations $\widetilde{\varphi}_i : \mathcal{Y}(\widetilde{F}_i) \longrightarrow H$, satisfying the following

(i) The restriction of $\widetilde{\varphi}_i$ to the subcategory $\langle G \rangle_{(i+3)B}$ is surjective.

(ii) For each $i > 0$ the square

$$\begin{tikzcd}
\mathcal{Y}(E_{i+1}) \arrow[r, "\mathcal{Y}(\alpha_{i+1})"] \arrow[d, "\mathcal{Y}(\gamma_{i+1})"'] & \mathcal{Y}(\widetilde{F}_{i+1}) \arrow[d, "\widetilde{\varphi}_{i+1}"] \\
\mathcal{Y}(\widetilde{F}_i) \arrow[r, "\widetilde{\varphi}_i"] & H
\end{tikzcd}$$

commutes.

(iii) The morphisms $\mathcal{H}^\ell(\gamma_i)$ and $\mathcal{H}^\ell(\alpha_i)$ are isomorphisms whenever $\ell \geq -i$.

To start the induction we declare $\widetilde{F}_1 = F_{4B}$ and $\widetilde{\varphi}_1 = \varphi_{4B}$. Choose a triangle $E_1 \xrightarrow{\alpha_1} \widetilde{F}_1 \longrightarrow D_1$ with $D_1 \in \mathcal{T}^{\leq -3}$; we immediately have that $\mathcal{H}^\ell(\alpha_1)$ is an isomorphism for $\ell \geq -1$. In particular for $\ell \geq 1$ we have $\mathcal{H}^\ell(E_1) \cong \mathcal{H}^\ell(F_{4B}) = 0$; hence $E_1 \in \mathcal{T}^c \cap \mathcal{T}^{\leq 0}$.

Suppose our induction has proceeded as far as n. In particular: we have produced in $\mathcal{T}_c^- \cap \mathcal{T}^{\leq 0}$ a morphism $\alpha_n : E_n \longrightarrow \widetilde{F}_n$, with $E_n \in \mathcal{T}^c$, as well as a natural transformation $\widetilde{\varphi}_n : \mathcal{Y}(\widetilde{F}_n) \longrightarrow H$ which is surjective when restricted to $\langle G \rangle_{(n+3)B}$. And we have done it in such a way that $\mathcal{H}^\ell(\alpha_n)$ is an isomorphism for $\ell \geq -n$. We wish to go on to $n+1$.

Now the first paragraph of the proof gives us an object $F_{(n+4)B} \in \mathcal{T}_c^- \cap \mathcal{T}^{\leq 0}$, as well as a natural transformation $\varphi_{(n+4)B} : \mathcal{Y}(F_{(n+4)B}) \longrightarrow H$ whose restriction to $\langle G \rangle_{(n+4)B}$ is surjective. Lemma 9.14, with $m = n+3$, allows us to construct in $\mathcal{T}_c^- \cap \mathcal{T}^{\leq 0}$ the diagram

$$\begin{tikzcd}
E_n \arrow[rr, "\varepsilon_{n+1}"] \arrow[dr, "\alpha_n"'] & & E_{n+1} \arrow[dl, "\gamma_{n+1}"] \arrow[dr, "\alpha_{n+1}"] & & F_{(n+4)B} \arrow[dl, "\beta"'] \\
& \widetilde{F}_n & & \widetilde{F}_{n+1} &
\end{tikzcd}$$

as well as the natural transformation $\widetilde{\varphi}_{n+1} : \mathcal{Y}(\widetilde{F}_{n+1}) \longrightarrow H$ satisfying lots of properties: Lemma 9.14(i) tells us that E_{n+1} belongs to $\mathcal{T}^c$. Lemma 9.14(ii) gives that $\mathcal{H}^\ell(\alpha_{n+1})$ and $\mathcal{H}^\ell(\gamma_{n+1})$ are isomorphisms if $\ell \geq -n-3+2 = -n-1$. Lemma 9.14(iii) says that $\widetilde{\varphi}_{n+1}$ is an epimorphism when restricted to $\langle G \rangle_{(n+4)B}$, and Lemma 9.14(iv) says that the square in (ii) above commutes. This finishes the induction.

It remains to see how to deduce the Lemma. We have produced a sequence $E_1 \xrightarrow{\varepsilon_2} E_3 \xrightarrow{\varepsilon_3} E_3 \xrightarrow{\varepsilon_4} \cdots$ and, for each $i > 0$

(iv) We define $\psi_i : \mathcal{Y}(E_i) \longrightarrow H$ to be the composite

$$\begin{tikzcd}
\mathcal{Y}(E_i) \arrow[r, "\mathcal{Y}(\alpha_i)"] & \mathcal{Y}(\widetilde{F}_i) \arrow[r, "\widetilde{\varphi}_i"] & H
\end{tikzcd}$$

With these definitions we will prove

(v) The following triangles commute

$$\begin{tikzcd}
 & \mathcal{Y}(E_{i+1}) \arrow[d, "\psi_{i+1}"] \\
\mathcal{Y}(E_i) \arrow[ur, "\mathcal{Y}(\varepsilon_{i+1})"] \arrow[r, "\psi_i"'] & H
\end{tikzcd}$$

This permits us to make the next definitions

(vi) We put $F = \underrightarrow{\text{Hocolim}}\ E_i$, and let $\psi : \mathcal{Y}(F) \longrightarrow H$ be

$$\begin{tikzcd}
\mathcal{Y}(F) \arrow[r, equal] & \underrightarrow{\operatorname{colim}}\, \mathcal{Y}(E_i) \arrow[r] & H
\end{tikzcd}$$

that is the colimit of the maps ψ_i.

And with the definitions made, we will prove

(vii) The object F belongs to $\mathcal{T}_c^-$.
(viii) The map $\psi : \mathcal{Y}(F)|_{\mathcal{T}^c} \longrightarrow H$ is an epimorphism.

Together, (vii) and (viii) contain the assertion of the Lemma. All that remains is to prove (v), (vii) and (viii).

To prove (v) the reader should consider the diagram

$$\begin{tikzcd}
 & \mathcal{Y}(E_{i+1}) \arrow[r, "\mathcal{Y}(\alpha_{i+1})"] \arrow[d, "\mathcal{Y}(\gamma_{i+1})"] & \mathcal{Y}(\widetilde{F}_{i+1}) \arrow[d, "\widetilde{\varphi}_{i+1}"] \\
\mathcal{Y}(E_i) \arrow[ur, "\mathcal{Y}(\varepsilon_{i+1})"] \arrow[r, "\mathcal{Y}(\alpha_i)"'] & \mathcal{Y}(\widetilde{F}_i) \arrow[r, "\widetilde{\varphi}_i"] & H
\end{tikzcd}$$

We wish to prove the commutativity of the perimeter. The square commutes by (ii), and the triangle by applying the functor $\mathcal{Y}$ to the commutative triangle

$$\begin{tikzcd}
E_i \arrow[rr, "\varepsilon_{i+1}"] \arrow[dr, "\alpha_i"'] & & E_{i+1} \arrow[dl, "\gamma_{i+1}"] \\
 & \widetilde{F}_i &
\end{tikzcd}$$

Also: we may apply $\mathcal{H}^\ell$ to the commutative triangle. From (iii) we know that $\mathcal{H}^\ell(\alpha_i)$ is an isomorphism for $\ell \geq -i$ and also that $\mathcal{H}^\ell(\gamma_{i+1})$ is an isomorphism for $\ell \geq -i-1$. Since $\mathcal{H}^\ell(\alpha_i) = \mathcal{H}^\ell(\gamma_{i+1})\mathcal{H}^\ell(\varepsilon_{i+1})$ we learn that $\mathcal{H}^\ell(\varepsilon_{i+1})$ is an isomorphism when $\ell \geq -i$. Therefore E_* is a strong $\mathcal{T}^c$–approximating system as in Definition 9.3, and Lemma 9.5 informs us that $F = \underrightarrow{\mathrm{Hocolim}}\, E_i$ belongs to $\mathcal{T}_c^-$—that is we have proved (vii). It remains only to prove (viii).

Suppose therefore that C is an object $\mathcal{T}^c$. There exists an $n > 0$ with $C \in \langle G \rangle_{(n+3)B}$, and Lemma 3.8 says that we may also choose n so that $\mathrm{Hom}(C, \mathcal{T}^{-n-1}) = 0$. Because $\widetilde{\varphi}_n : \mathcal{Y}(\widetilde{F}_n) \longrightarrow H$ is surjective on $\langle G \rangle_{(n+3)B}$ we have that the map $\widetilde{\varphi}_n : \mathrm{Hom}(C, \widetilde{F}_n) \longrightarrow H(C)$ is surjective. In the triangle $E_n \xrightarrow{\alpha_n} \widetilde{F}_n \longrightarrow \widetilde{D}_n$ we have that $\mathcal{H}^\ell(\alpha_n)$ is an isomorphism for $\ell \geq -n$, therefore $\mathcal{H}^\ell(\widetilde{D}_n) = 0$ for $\ell \geq -n$, therefore $\widetilde{D}_n \in \mathcal{T}^{\leq -n-1}$. In the exact sequence

$$\mathrm{Hom}(C, E_n) \xrightarrow{\mathrm{Hom}(C,\alpha_n)} \mathrm{Hom}(C, \widetilde{F}_n) \longrightarrow \mathrm{Hom}(C, \widetilde{D}_n)$$

we have $\mathrm{Hom}(C, \widetilde{D}_n) = 0$ since $\widetilde{D}_n \in \mathcal{T}^{\leq -n-1}$. The map $\psi_n : \mathrm{Hom}(C, E_n) \longrightarrow H(C)$ of (iv) is the composite of the two epimorphisms

$$\mathrm{Hom}(C, E_n) \xrightarrow{\mathrm{Hom}(C,\alpha_n)} \mathrm{Hom}(C, \widetilde{F}_n) \xrightarrow{\widetilde{\varphi}_n} H(C)$$

But it factors through $\psi : \mathrm{Hom}(C, F) \longrightarrow H(C)$, which must therefore be epi. ☐

Reminder 9.16 Let $\mathcal{T}$ be a triangulated category with coproducts. A morphism $f : D \longrightarrow E$ is called *phantom* if, for every compact object $C \in \mathcal{T}$, the induced map $\mathrm{Hom}(C, f) : \mathrm{Hom}(C, D) \longrightarrow \mathrm{Hom}(C, E)$ vanishes. The phantom maps form an ideal: if $f, f' : D \longrightarrow E$ are phantom then so is $f + f'$, and if $D' \xrightarrow{e} D \xrightarrow{f} E \xrightarrow{g} E'$ are composable morphisms with f phantom, then $gfe : D' \longrightarrow E'$ is also phantom.

Corollary 9.17 *Let the conventions be as in Notation 9.13. Let $F' \in \mathcal{T}$ be an object such that the functor $H = \mathcal{Y}(F')$ is a locally finite $\mathcal{T}^c$–cohomological functor. There exists an object $F \in \mathcal{T}_c^-$ and a triangle $F \xrightarrow{f} F' \xrightarrow{g} D$ with g phantom.*

Proof Lemma 9.15 gives us an object $F \in \mathcal{T}_c^-$ and an epimorphism $\varphi : \mathcal{Y}(F) \longrightarrow H = \mathcal{Y}(F')$. Since F belongs to $\mathcal{T}_c^-$ Lemma 9.5(ii) produces for F a (strong) $\mathcal{T}^c$–approximating system. Lemma 7.8 allows us to realize the natural transformation φ as $\mathcal{Y}(f) : \mathcal{Y}(F) \longrightarrow \mathcal{Y}(F')$ for some (non-unique) $f : F \longrightarrow F'$. Complete f to a triangle $F \xrightarrow{f} F' \xrightarrow{g} D$. For every object $C \in \mathcal{T}$ we have an exact sequence

$$\mathrm{Hom}(C, F) \xrightarrow{\mathrm{Hom}(C,f)} \mathrm{Hom}(C, F') \xrightarrow{\mathrm{Hom}(C,g)} \mathrm{Hom}(C, D)$$

and if C is compact the map $\mathrm{Hom}(C, f) = \mathcal{Y}(f)(C)$ is surjective. It follows that $\mathrm{Hom}(C, g)$ is the zero map. ☐

Now that it's time to state the main theorem we include all the hypotheses explicitly.

Theorem 9.18 *Let R be a noetherian, commutative ring. Let $\mathcal{T}$ be an R–linear triangulated category with coproducts, and suppose it has a compact generator G such that* $\mathrm{Hom}(-, G)$ *is a G–locally finite cohomological functor. Assume further that $\mathcal{T}$ is approximable.*

Let $\mathcal{T}_c^- \subset \mathcal{T}$ be the category of Definition 1.20, where the t–structure with respect to which we define it is in the preferred equivalence class. Then the functor $\mathcal{Y} : \mathcal{T}_c^- \longrightarrow \mathrm{Hom}\big((\mathcal{T}^c)^{\mathrm{op}}, R\text{–}Mod\big)$, taking $F \in \mathcal{T}_c^-$ to $\mathcal{Y}(F) = \mathrm{Hom}(-, F)|_{\mathcal{T}^c}$, satisfies

(i) *The objects in the essential image of $\mathcal{Y}$ are the locally finite $\mathcal{T}^c$–cohomological functors.*
(ii) *The functor $\mathcal{Y}$ is full.*

Proof The fact that, for any object $F \in \mathcal{T}_c^-$, the functor $\mathcal{Y}(F)$ is a locally finite $\mathcal{T}^c$–cohomological functor was proved in Lemma 9.2. In Lemma 9.5(ii) we saw that any $F \in \mathcal{T}_c^-$ admits a $\mathcal{T}^c$–approximating system, and Lemma 7.8 guarantees that any natural transformation $\varphi : \mathcal{Y}(F) \longrightarrow \mathcal{Y}(F')$ can be expressed as $\varphi = \mathcal{Y}(f)$ for some $f : F \longrightarrow F'$; that is the functor is full. It remains to show that any locally finite $\mathcal{T}^c$–cohomological functor $H : (\mathcal{T}^c)^{\mathrm{op}} \longrightarrow R$–Mod is in the essential image; we must show it isomorphic to $\mathcal{Y}(F)$ for some $F \in \mathcal{T}_c^-$.

Proposition 9.10 produced a candidate F; we have an $F \in \overline{\langle \mathcal{G} \rangle}_4$ and an isomorphism $H \cong \mathcal{Y}(F)$. We wish to show that F belongs to $\mathcal{T}_c^-$. We proceed by induction to prove

(iii) Let $\mathcal{I}$ be the ideal of phantom maps. For each integer $n > 0$ there exists a triangle $F_n \longrightarrow F \xrightarrow{\beta_n} D_n$ with $F_n \in \mathcal{T}_c^-$ and $\beta_n \in \mathcal{I}^n$.

We prove (iii) by induction on n. The case $n = 1$ is given by Corollary 9.17. Now for the inductive step: assume that, for some $n \geq 1$, we are given a triangle $F_n \longrightarrow F \xrightarrow{\beta_n} D_n$ with $F_n \in \mathcal{T}_c^-$ and $\beta_n \in \mathcal{I}^n$. We know that both $\mathcal{Y}(F_n)$ and $\mathcal{Y}(F)$ are locally finite $\mathcal{T}^c$–cohomological functors, and the exact sequence

$$0 \longrightarrow \mathcal{Y}(\Sigma^{-1} D_n) \longrightarrow \mathcal{Y}(F_n) \longrightarrow \mathcal{Y}(F) \longrightarrow 0$$

says that so is $\mathcal{Y}(\Sigma^{-1} D_n)$. Corollary 9.17 permits us to construct a triangle $F' \longrightarrow D_n \xrightarrow{\gamma} D_{n+1}$ with $F' \in \mathcal{T}_c^-$ and $\gamma \in \mathcal{I}$. Let $\beta_{n+1} : F \longrightarrow D_{n+1}$ be the composite $F \xrightarrow{\beta_n} D_n \xrightarrow{\gamma} D_{n+1}$. Since $\beta_n \in \mathcal{I}^n$ and $\gamma \in \mathcal{I}$ we deduce that $\beta_{n+1} \in \mathcal{I}^{\mathrm{n+1}}$. If we complete β_{n+1} to a triangle $F_{n+1} \longrightarrow F \xrightarrow{\beta_{n+1}} D_{n+1}$, the octahedral axiom allows us to find a triangle $F_n \longrightarrow F_{n+1} \longrightarrow F'$. Since F_n and F' both lie in $\mathcal{T}_c^-$ so does F_{n+1}. This completes the proof of (iii).

Now consider the triangle $F_4 \longrightarrow F \longrightarrow D_4$. The morphism $F \longrightarrow D_4$ is in $\mathcal{I}^4$, but F belongs to $\overline{\langle \mathcal{G} \rangle}_4$. One easily shows that $\big(\overline{\langle \mathcal{G} \rangle}_1, \mathcal{I}\big)$ is a projective class as in Christensen [9, Definition 2.2], and [9, Theorem 1.1] tells us that so is $\big(\overline{\langle \mathcal{G} \rangle}_4, \mathcal{I}^4\big)$. The map $F \longrightarrow D_4$ is a morphism in $\mathcal{I}^4$ out of an object in $\overline{\langle \mathcal{G} \rangle}_4$ and must vanish, making F a direct summand of $F_4 \in \mathcal{T}_c^-$. Proposition 3.10 tells us that $\mathcal{T}_c^-$ is thick, and therefore $F \in \mathcal{T}_c^-$. □

Lemma 9.19 *Let the assumptions be as in Theorem 9.18. Suppose* $f : F \longrightarrow F'$ *is a morphism in* $\mathcal{T}_c^-$ *and assume* F' *belongs to* $\mathcal{T}_c^b$*. Then* $\mathcal{Y}(f) = 0$ *implies* $f = 0$.

Proof Because F' belongs to $\mathcal{T}_c^b$ there must be an integer ℓ with $F' \in \mathcal{T}^{\geq \ell}$; without loss of generality we may assume $\ell = 0$. Now F belongs to $\mathcal{T}_c^-$, hence there must exist a triangle $E \xrightarrow{g} F \xrightarrow{h} D$ with $E \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\leq -1}$. Choose such a triangle.

The vanishing of $\mathcal{Y}(f)$ means that $\mathrm{Hom}(E, f) : \mathrm{Hom}(E, F) \longrightarrow \mathrm{Hom}(E, F')$ must take $g \in \mathrm{Hom}(E, F)$ to zero; that means $fg = 0$. But the triangle tells us that $f : F \longrightarrow F'$ must factor as $F \xrightarrow{h} D \longrightarrow F'$. As $D \in \mathcal{T}^{\leq -1}$ and $F' \in \mathcal{T}^{\geq 0}$ we have $\mathrm{Hom}(D, F') = 0$, hence $f = 0$. □

Theorem 9.20 *Let the assumptions be as in Theorem 9.18. The restriction of the functor* $\mathcal{Y}$ *to the subcategory* $\mathcal{T}_c^b$ *is fully faithful, and the essential image is the class of finite* $\mathcal{T}^c$*–cohomological functors.*

Proof The functor is full on all of $\mathcal{T}_c^-$, and Lemma 9.19 guarantees that on the subcategory $\mathcal{T}_c^b$ it is faithful. It remains to identify the essential image. Let F be an object in $\mathcal{T}_c^-$, we need to show that $\mathcal{Y}(F)$ is finite if and only if $F \in \mathcal{T}_c^b$.

Suppose $F \in \mathcal{T}_c^b \subset \mathcal{T}^+$ and $C \in \mathcal{T}^c \subset \mathcal{T}^-$. We can choose an integer $\ell > 0$ so that $\mathrm{Hom}(\Sigma^i C, F) = 0$ for all $i \geq \ell$, which implies that $\mathrm{Hom}(-, F)|_{\mathcal{T}^c}$ is C–finite. Since this is true for every $C \in \mathcal{T}^c$ we have that $\mathcal{Y}(F)$ is finite.

Conversely: suppose $\mathcal{Y}(F)$ is finite and choose a compact generator G. Because $\mathcal{Y}(F) = \mathrm{Hom}(-, F)|_{\mathcal{T}^c}$ is G–finite there is an integer ℓ so that $\mathrm{Hom}(\Sigma^i G, F) = 0$ for all $i \geq \ell$. But then $\mathrm{Hom}(T, F) = 0$ for all $T \in \overline{\langle G \rangle}^{(-\infty, -\ell]} = \mathcal{T}_G^{\leq -\ell}$ and F must belong to $\mathcal{T}_c^- \cap \mathcal{T}_G^{\geq -\ell+1} \subset \mathcal{T}_c^b$. □

10 Implications of the hypothesis $\mathcal{T} = \overline{\langle G' \rangle}_N$, with $G' \in \mathcal{T}_c^b \subset \mathcal{T}^b$

We have proved Theorem 1.4(i), and the time has now come to turn our attention to Theorem 1.4(ii). And, as the reader can easily check, in Theorem 1.4(ii) we added the hypothesis that there exist an object $G' \in \mathcal{T}_c^b$ with $\mathcal{T} = \overline{\langle G' \rangle}_N$. This section will focus on finding out what this extra hypothesis buys us.

But before all else we prove the easy part of Theorem 1.4(ii), showing that the images of the functors $\widetilde{\mathcal{Y}}$ and $\widetilde{\mathcal{Y}} \circ \widetilde{\iota}$ are contained where the theorem asserts they should be. For this the extra hypothesis, regarding the existence of an object $G' \in \mathcal{T}_c^b$ with $\mathcal{T} = \overline{\langle G' \rangle}_N$, is unnecessary.

Lemma 10.1 *Let* R *be a commutative, noetherian ring. Suppose* $\mathcal{T}$ *is an* R*–linear triangulated category. Suppose* $\mathcal{T}$ *has a single compact generator* G*, such that* $\mathrm{Hom}(G, \Sigma^i G)$ *is a finite* R*–module for every* $i \in \mathbb{Z}$*, and vanishes when* $i \gg 0$.

Then $\mathrm{Hom}(A, \Sigma^i B)$ *is a finite* R*–module whenever* $A \in \mathcal{T}_c^-$ *and* $B \in \mathcal{T}_c^b$*. For fixed* A *and* B *it vanishes when* $i \ll 0$.

If A *belongs to* $\mathcal{T}^c \subset \mathcal{T}_c^-$*, then* $\mathrm{Hom}(A, \Sigma^i B)$ *also vanishes for* $i \gg 0$.

Proof Fix A and B. Since $A \in \mathcal{T}^-$ and $B \in \mathcal{T}^+$ there will be some integer $m > 0$ such that $\operatorname{Hom}(A, \Sigma^i B) = 0$ for $i < -m$.

We need to prove the finiteness for fixed i, and without loss we may assume $i = 0$. Shifting if necessary we may assume $B \in \mathcal{T}^{\geq 0}$. But $A \in \mathcal{T}_c^-$ means that there must exist a triangle $E \longrightarrow A \longrightarrow D$ with $E \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\leq -2}$. In the exact sequence

$$\operatorname{Hom}(D, B) \longrightarrow \operatorname{Hom}(A, B) \longrightarrow \operatorname{Hom}(E, B) \longrightarrow \operatorname{Hom}(\Sigma^{-1} D, B)$$

we have $\operatorname{Hom}(D, B) = 0 = \operatorname{Hom}(\Sigma^{-1} D, B)$, hence $\operatorname{Hom}(A, B) \cong \operatorname{Hom}(E, B)$. The finiteness of $\operatorname{Hom}(E, B)$ is contained in Lemma 9.2.

Now assume $A \in \mathcal{T}^c$. The vanishing of $\operatorname{Hom}(A, \Sigma^i B)$ for $i \gg 0$ follows from Lemma 3.8. □

And now we turn our attention to understanding categories of the form $\overline{\langle G' \rangle}_N$, with G' first assumed to belong to $\mathcal{T}^b$, and as we progress we will make the more restrictive hypothesis that G' belongs to $\mathcal{T}_c^b \subset \mathcal{T}^b$.

On a matter of notation: we will focus mostly on the object $G' \in \mathcal{T}_c^b \subset \mathcal{T}$, but on occasion we will recall that the entire theory assumes the existence of a compact generator $G \in \mathcal{T}$. Because the symbols G and G' are easy to confuse, in the remainder of this section the compact generator will be $H \in \mathcal{T}^c$ and the object in $\mathcal{T}_c^b \subset \mathcal{T}$ will be called G.

Lemma 10.2 *Let $\mathcal{T}$ be a triangulated category with coproducts, and let $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ be a t–structure on $\mathcal{T}$. Let $G \in \mathcal{T}^b$ be an object, and let $N > 0$ be an integer.*

Then there exists an integer $M > 0$ so that any map $A \longrightarrow C$, with $A \in \mathcal{T}^{\leq 0}$ and with $C \in \langle G \rangle_N^{[-m,\infty)}$, must factor through some object $B \in \langle G \rangle_N^{[-m,M]}$.

If we further assume that $\mathcal{T}^{\geq 0}$ is closed under coproducts, then the integer $M > 0$ may be chosen so that any map $A \longrightarrow C$, with $A \in \mathcal{T}^{\leq 0}$ and $C \in \overline{\langle G \rangle}_N^{[-m,\infty)}$, will factor through some object $B \in \overline{\langle G \rangle}_N^{[-m,M]}$.

In both statements we allow $m > 0$ to be (possibly) infinite.

Proof Because $G \in \mathcal{T}^b$ we may choose an integer $K > 0$ so that $G \in \mathcal{T}^{\geq -K} \cap \mathcal{T}^{\leq K}$. Fix such a K; for any integer $n > 0$ we clearly have $\langle G \rangle_n^{[K,\infty)} \subset \mathcal{T}^{\geq 0}$, and if $\mathcal{T}^{\geq 0}$ is closed under coproducts then we also have $\overline{\langle G \rangle}_n^{[K,\infty)} \subset \mathcal{T}^{\geq 0}$.

Next we proceed to prove the assertion of the lemma beginning with "if we further assume". We leave to the reader the case with $C \in \langle G \rangle_N^{[-m,\infty)}$. We are given that $C \in \overline{\langle G \rangle}_N^{[-m,\infty)}$, and by [20, Corollary 1.12] we have

$$\overline{\langle G \rangle}_N^{[-m,\infty)} = \operatorname{smd}\Big[\operatorname{Coprod}_N\big(G[-m, \infty)\big)\Big].$$

Therefore we may choose an object C' with $C \oplus C' \in \operatorname{Coprod}_N\big(G[-m, \infty)\big)$, and any map $f : A \longrightarrow C$ obviously factors as the composite $A \xrightarrow{f} C \xrightarrow{i} C \oplus C' \xrightarrow{\pi} C$, meaning f factors through $C \oplus C'$. It therefore suffices to show that, with $K > 0$ as in the paragraph above, we have

(i) Any map $f : A \longrightarrow C$, with $A \in \mathcal{T}^{\leq 0}$ and $C \in \operatorname{Coprod}_N\big(G[-m,\infty)\big)$, must factor through some object $B \in \operatorname{Coprod}_N\big(G[-m,(2K+1)N]\big)$.

Now we proceed by induction on N. If $N = 1$ we are given a map $A \longrightarrow C$, with $A \in \mathcal{T}^{\leq 0}$ and $C \in \operatorname{Coprod}_1\big(G[-m,\infty)\big)$. But

$$\operatorname{Coprod}_1\big(G[-m,\infty)\big) \quad = \quad \operatorname{Coprod}_1\big(G[-m,K]\big) \bigoplus \operatorname{Coprod}_1\big(G[K+1,\infty)\big)$$

As $\operatorname{Coprod}_1\big(G[K+1,\infty)\big)$ is contained in $\mathcal{T}^{\geq 1}$ and $A \in \mathcal{T}^{\leq 0}$, the map $A \longrightarrow C$ must factor through $B \in \operatorname{Coprod}_1\big(G[-m,K]\big)$. We have proved an improvement on (i) in the case $N = 1$.

Next assume we know (i) for all integers $\leq N$, and keep in mind that, for $N = 1$, we proved an improvement on (i) in the last paragraph. Now let $\mathcal{S} = \mathcal{T}^{\leq 0}$ and put

$$\mathcal{X} = \operatorname{Coprod}_N\big(G[-m,\infty)\big) \qquad \mathcal{A} = \operatorname{Coprod}_N\big(G[-m,(2K+1)(N+1)]\big)$$

$$\mathcal{Z} = \operatorname{Coprod}_1\big(G[-m,\infty)\big) \qquad \mathcal{C} = \operatorname{Coprod}_1\big(G[-m,K]\big)$$

By the induction any pair of maps $s \longrightarrow x$ and $s \longrightarrow z$, with $s \in \mathcal{S}$, $x \in \mathcal{X}$ and $z \in \mathcal{Z}$, factor (respectively) as $s \longrightarrow a \longrightarrow x$ and $s \longrightarrow c \longrightarrow z$, with $a \in \mathcal{A}$ and $c \in \mathcal{C}$.

Now let d be an object of $\big(\Sigma^{-1}\mathcal{C}\big) * \mathcal{S}$. As $\Sigma^{-1}\mathcal{C} \subset \mathcal{T}^{\leq 2K+1}$ and $\mathcal{S} = \mathcal{T}^{\leq 0}$ we deduce that $d \in \mathcal{T}^{\leq 2K+1} = \Sigma^{-2K-1}\mathcal{S}$, and induction tells us that any map $d \longrightarrow x$, with $x \in \mathcal{X}$, must factor as $d \longrightarrow a \longrightarrow x$ with $a \in \mathcal{A}$. The hypotheses of [20, Lemma 1.6] are satisfied, hence any morphism
$s \longrightarrow \mathcal{X} * \mathcal{Z} = \operatorname{Coprod}_{N+1}\big(G[-m,\infty)\big)$ factors through
$\mathcal{A} * \mathcal{C} \subset \operatorname{Coprod}_{N+1}\big(G[-m,(2K+1)(N+1)]\big)$. ☐

Lemma 10.3 *Let $\mathcal{T}$ be a triangulated category with coproducts, as well as a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$. Let $G \in \mathcal{T}^b$ be an object, and let $N > 0$ be an integer. Then there exists an integer $M > 0$ so that*

(i) *Any map $C \longrightarrow A$, with $A \in \mathcal{T}^{\geq 0}$ and $C \in \langle G\rangle_N^{(-\infty,m]}$, must factor through some object $B \in \langle G\rangle_N^{[-M,m]}$.*
(ii) *Suppose $\mathcal{T}^{\geq 0}$ is closed under coproducts. Then any map $C \longrightarrow A$, with $A \in \mathcal{T}^{\geq 0}$ and $C \in \overline{\langle G\rangle}_N^{(-\infty,m]}$, must factor through some object $B \in \overline{\langle G\rangle}_N^{[-M,m]}$.*

In both statements the integer $m > 0$ is possibly infinite.

Proof Because $G \in \mathcal{T}^b$ we may choose an integer $K > 0$ so that $G \in \mathcal{T}^{\geq -K} \cap \mathcal{T}^{\leq K}$. Fix such a K. The fact that $\mathcal{T}^{\leq 0} = {}^{\perp}\mathcal{T}^{\geq 1}$ tells us that $\mathcal{T}^{\leq 0}$ is (automatically) closed under coproducts, and hence for any integer $n > 0$ we have $\langle G\rangle_n^{(-\infty,-K]} \subset \overline{\langle G\rangle}_n^{(-\infty,-K]} \subset \mathcal{T}^{\leq 0}$.

The rest of the proof is just the dual of the proof of Lemma 10.2; we sketch the proof of assertion (ii) and leave (i) to the reader. Any map $C \longrightarrow A$, with $A \in \mathcal{T}^{\geq 0}$ and C in

$$\operatorname{Coprod}_1\big(G(-\infty,m]\big) \quad = \quad \operatorname{Coprod}_1\big(G(-\infty,-K-1]\big) \bigoplus \operatorname{Coprod}_1\big(G[-K,m]\big)$$

must factor through some $B \in \mathrm{Coprod}_1\big(G[-K,m]\big)$. Now proceeding dually to the proof of Lemma 10.2 one shows, inductively on N, that

(iii) Any map $f : C \longrightarrow A$, with $A \in \mathcal{T}^{\geq 0}$ and $C \in \mathrm{Coprod}_N\big(G(-\infty,m]\big)$, must factor through some object $B \in \mathrm{Coprod}_N\big(G[-(2K+1)N,m]\big)$.

More precisely: let $\mathcal{S} = \mathcal{T}^{\geq 0}$ and

$$\mathcal{X} = \mathrm{Coprod}_N\big(G(-\infty,m]\big) \qquad \mathcal{A} = \mathrm{Coprod}_N\big(G[-(2K+1)(N+1),m]\big)$$

$$\mathcal{Z} = \mathrm{Coprod}_1\big(G[-\infty,m]\big) \qquad \mathcal{C} = \mathrm{Coprod}_1\big(G[-K,m]\big)$$

The hypothesis that $\mathcal{T}^{\geq 0}$ is closed under coproducts gives that $\Sigma\mathcal{C} \subset \mathcal{T}^{\geq -2K-1} = \Sigma^{2K+1}\mathcal{S}$, and hence $\mathcal{S} * \Sigma\mathcal{C} \subset \mathcal{S} * \Sigma^{2K+1}\mathcal{S} \subset \Sigma^{2K+1}\mathcal{S}$. Now induction coupled with [20, Lemma 1.6] guarantee that any map $y \longrightarrow s$, with $y \in \mathcal{Z} * \mathcal{X}$ and $s \in \mathcal{S} = \mathcal{T}^{\geq 0}$, must factor through $B \in \mathcal{C} * \mathcal{A} \subset \mathrm{Coprod}_{N+1}\big(G[-(2K+1)(N+1),m]\big)$. ☐

Lemma 10.4 *Let $\mathcal{T}$ be a triangulated category with coproducts, as well as a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ with $\mathcal{T}^{\geq 0}$ is closed under coproducts. Let $G \in \mathcal{T}^b$ be an object, let $N > 0$ be an integer, and assume $\mathcal{T} = \overline{\langle G\rangle}_N$.*

Then there exists an integer $M > 0$ such that, for any integers $a \leq b$,

$$\mathcal{T}^{\geq 0} \subset \overline{\langle G\rangle}_N^{[-M,\infty)}, \qquad \mathcal{T}^{\leq 0} \subset \overline{\langle G\rangle}_N^{(-\infty,M]}, \qquad \mathcal{T}^{\geq a} \cap \mathcal{T}^{\leq b} \subset \overline{\langle G\rangle}_N^{[a-M,b+M]}.$$

Proof For the given integer N and object G, pick an integer M so that Lemmas 10.2 and 10.3 both hold.

Let a be an integer, and suppose A is an object in $\mathcal{T}^{\geq a}$. The identity map $A \longrightarrow A$ is a morphism from $A \in \overline{\langle G\rangle}_N$ to $A \in \mathcal{T}^{\geq a}$, and Lemma 10.3 (with $m = \infty$) guarantees that the map factors through an object $B \in \overline{\langle G\rangle}_N^{[a-M,\infty)}$. As A is a direct summand of B it must also lie in $\overline{\langle G\rangle}_N^{[a-M,\infty)}$. The case $a = 0$ gives the first assertion of the Lemma.

Now assume $A \in \mathcal{T}^{\geq a} \cap \mathcal{T}^{\leq b}$ with a possibly equal to $-\infty$. By the first half of the Lemma, already proved, A must lie in $\overline{\langle G\rangle}_N^{[a-M,\infty)}$. The identity map $A \longrightarrow A$ is a morphism from $A \in \mathcal{T}^{\leq b}$ to $A \in \overline{\langle G\rangle}_N^{[a-M,\infty)}$, and Lemma 10.2 guarantees that it factors through some $B \in \overline{\langle G\rangle}_N^{[a-M,b+M]}$. Therefore A, being a direct summand of B, must also belong to $\overline{\langle G\rangle}_N^{[a-M,b+M]}$. ☐

Lemma 10.5 *Let $\mathcal{T}$ be a triangulated category with coproducts and a single compact generator H. Assume* $\mathrm{Hom}(H, \Sigma^n H) = 0$ *for $n \gg 0$.*

Let $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ be a t–structure on $\mathcal{T}$, in the preferred equivalence class. Given an integer $K > 0$ and a collection of objects $\{X_i \in \mathcal{T}^{\leq K} \cap \mathcal{T}^{\geq -K} \mid i \in \mathbb{Z}\}$, then the map

$$\coprod_{i=-\infty}^{\infty} \Sigma^i X_i \xrightarrow{\quad\varphi\quad} \prod_{i=-\infty}^{\infty} \Sigma^i X_i$$

is an isomorphism.

Proof Because the t–structure is in the preferred equivalence class and H is compact, Observation 1.16 tells us that there is some integer $A > 0$ with $H \in \mathcal{T}^{\leq A-1}$, and therefore $\mathrm{Hom}(H, -)$ vanishes on $\mathcal{T}^{\geq A} = \big[\mathcal{T}^{\leq A-1}\big]^{\perp}$. And Lemma 3.8 allows us to assume, possibly after increasing A, that $\mathrm{Hom}\big(H, \mathcal{T}^{\leq -A}\big) = 0$. Therefore the functor $\mathrm{Hom}(H, -)$ vanishes on the union $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$. As X_i is assumed to belong to $\mathcal{T}^{\leq K} \cap \mathcal{T}^{\geq -K}$, we have that $\mathrm{Hom}\big(H, \Sigma^n X_i\big) = 0$ whenever $|n| > A + K$.

Any morphism $\Sigma^n H \longrightarrow \coprod_{i=-\infty}^{\infty} \Sigma^i X_i$ will factor through a finite subcoproduct by the compactness of H, and any map $\Sigma^n H \longrightarrow \prod_{i=-\infty}^{\infty} \Sigma^i X_i$ will factor through a finite subproduct by the vanishing of $\mathrm{Hom}\big(H, \Sigma^{i-n} X_i\big)$ for all but finitely many i. Therefore the functor $\mathrm{Hom}(\Sigma^n H, -)$ takes the map φ to an isomorphism, for every $n \in \mathbb{Z}$, and as H is a generator the map φ must be an isomorphism. ☐

Proposition 10.6 *Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator H with $\mathrm{Hom}(H, \Sigma^i H) = 0$ for $i \gg 0$. Suppose G is an object in $\mathcal{T}^b$, where we mean the $\mathcal{T}^b$ that comes from the preferred equivalence class of t–structures. Assume that there exists an integer $N > 0$ with $\overline{\langle G \rangle}_N = \mathcal{T}$.*

Suppose $F \in \mathcal{T}_c^-$ is an object such that $\mathrm{Hom}(F, \Sigma^i G) = 0$ for $i \gg 0$. Then F is compact.

Proof In the preferred equivalence class choose a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ so that $\mathcal{T}^{\geq 0}$ is closed under coproducts. We are given that $G \in \mathcal{T}^b$, hence we may choose an integer $K > 0$ with $G \in \mathcal{T}^{\geq -K} \cap \mathcal{T}^{\leq K}$. Because $F \in \mathcal{T}_c^-$, for any integer $i > 0$ we may form a triangle $E_i \xrightarrow{\alpha_i} F \longrightarrow D_i \longrightarrow$ with $E \in \mathcal{T}^c$ and $D_i \in \mathcal{T}^{\leq -i-K-2}$. For any $X \in \mathcal{T}$ we have the exact sequence

$$\mathrm{Hom}(D_i, X) \to \mathrm{Hom}(F, X) \xrightarrow{\mathrm{Hom}(\alpha_i, X)} \mathrm{Hom}(E_i, X) \to \mathrm{Hom}(\Sigma^{-1} D_i, X)$$

and if $X \in \mathcal{T}^{\geq -i-K}$ then $\mathrm{Hom}(D_i, X) = 0 = \mathrm{Hom}(\Sigma^{-1} D_i, X)$. Thus $\mathrm{Hom}(\alpha_i, X)$ is an isomorphism for all $X \in \mathcal{T}^{\geq -i-K}$.

Now choose an integer $A > 0$ so that $\mathrm{Hom}(F, \Sigma^i G) = 0$ for all $i \geq A$. Pick an integer $i \geq A$; the vanishing of $\mathrm{Hom}(F, \Sigma^i G)$ implies the vanishing of $\mathrm{Hom}(E_i, \Sigma^i G)$, and the compactness of E_i implies the vanishing of $\mathrm{Hom}\big(E_i, \Sigma^i \mathrm{Coprod}_1(G)\big)$. But $\mathcal{T}^{\geq -i-K}$ is closed under coproducts and contains $\Sigma^i G$, hence $\Sigma^i \mathrm{Coprod}_1(G)$ is contained in $\mathcal{T}^{\geq -i-K}$, and the vanishing of $\mathrm{Hom}\big(E_i, \Sigma^i \mathrm{Coprod}_1(G)\big)$ implies the vanishing of $\mathrm{Hom}\big(F, \Sigma^i \mathrm{Coprod}_1(G)\big)$.

Thus $\mathrm{Hom}(F, -)$ vanishes on $\Sigma^i \mathrm{Coprod}_1(G)$ for any $i \geq A$. Now every object $X \in \mathrm{Coprod}_1\big(G(-\infty, -A]\big)$ can be written as $\coprod_{i=A}^{\infty} \Sigma^i X_i$ with $X_i \in \mathrm{Coprod}_1(G) \subset \mathcal{T}^{\geq -K} \cap \mathcal{T}^{\leq K}$. By Lemma 10.5 there is an isomorphism $\coprod_{i=A}^{\infty} \Sigma^i X_i \cong \prod_{i=A}^{\infty} \Sigma^i X_i$, and hence $\mathrm{Hom}(F, -)$ vanishes on the category $\mathrm{Coprod}_1\big(G(-\infty, -A]\big)$.

It immediately follows that $\mathrm{Hom}(F, -)$ also vanishes on the category $\overline{\langle G \rangle}_N^{(-\infty, -A]}$. But now Lemma 10.4 establishes the existence of some integer $m > 0$ with $\mathcal{T}^{\leq -m} \subset \overline{\langle G \rangle}_N^{(-\infty, -A]}$, and hence $\mathrm{Hom}(F, -)$ vanishes on $\mathcal{T}^{\leq -m}$. On the other hand F belongs to $\mathcal{T}_c^-$, hence we may choose a triangle $E \longrightarrow F \longrightarrow D \longrightarrow$ with $E \in \mathcal{T}^c$ and

$D \in \mathcal{T}^{\leq -m}$, and as the map $F \longrightarrow D$ must vanish we have that F is a direct summand of $E \in \mathcal{T}^c$. ☐

Until now, the object G with $\mathcal{T} = \overline{\langle G \rangle}_N$ was only assumed to lie in $\mathcal{T}^b$. From now on we strengthen the hypotheses on G, and assume it belongs to $\mathcal{T}_c^b \subset \mathcal{T}^b$.

Lemma 10.7 *Let $\mathcal{T}$ be a triangulated category with coproducts and a single compact generator H. Assume* $\mathrm{Hom}(H, \Sigma^n H) = 0$ *for $n \gg 0$. Suppose G is an object in $\mathcal{T}_c^b$, where we mean the $\mathcal{T}_c^b$ that comes from the preferred equivalence class of t–structures.*

For any integer $N > 0$, any map $F \longrightarrow Y$, with $F \in \mathcal{T}_c^-$ and $Y \in \overline{\langle G \rangle}_N^{[a,b]}$, factors through an object in $\langle G \rangle_N^{[a,b]}$. We allow either or both of b, N to be infinite.

For the sake of clarity: allowing N to be infinite means that any map $F \longrightarrow Y$, with $F \in \mathcal{T}_c^-$ and $Y \in \overline{\langle G \rangle}^{[a,b]}$, factors through an object in $\langle G \rangle^{[a,b]}$.

Proof Choose a t–structure in the preferred equivalence class, and pick it so that $\mathcal{T}^{\geq 0}$ is closed under coproducts. By hypothesis G is contained in $\mathcal{T}^b$, hence we may choose and fix an integer $K > 0$ with $G \in \mathcal{T}^{\geq -K}$. Then, for any (finite or infinite) integer $N > 0$, we have

$$\langle G \rangle_N^{[a,b]} \qquad \subset \qquad \overline{\langle G \rangle}_N^{[a,b]} \qquad \subset \qquad \mathcal{T}^{\geq a-K} .$$

Next we observe

(i) For any object $F \in \mathcal{T}_c^-$ we may choose triangle $\Sigma^{-1} D \longrightarrow E \longrightarrow F \longrightarrow D$ with $E \in \mathcal{T}^c$ and $D \in \mathcal{T}^{\leq a-K-2}$, and hence

$$\mathrm{Hom}\left(\Sigma^{-1} D, \mathcal{T}^{\geq a-K}\right) \quad = \quad 0 \quad = \quad \mathrm{Hom}\left(D, \mathcal{T}^{\geq a-K}\right) .$$

Thus, for any object $Y \in \overline{\langle G \rangle}_N^{[a,b]}$, the natural map $\mathrm{Hom}(F, Y) \longrightarrow \mathrm{Hom}(E, Y)$ is an isomorphism.

Now we prove the case $N < \infty$ of the lemma by induction on N. Suppose first that $N = 1$; then any object $Y \in \overline{\langle G \rangle}_1^{[a,b]}$ is a direct summand of an object in $\mathrm{Coprod}_1\big(G[a, b]\big)$, hence we may assume Y belongs to $\mathrm{Coprod}_1\big(G[a, b]\big) \subset \overline{\langle G \rangle}_1^{[a,b]}$. Because $F \in \mathcal{T}_c^-$ we may choose a triangle $\Sigma^{-1} D \longrightarrow E \longrightarrow F \longrightarrow D$ as in (i). But E is compact and Y is a coproduct, hence any map $E \longrightarrow Y$ factors through a finite subcoproduct. Therefore so does any map $F \longrightarrow Y$, completing the proof in the case $N = 1$.

Now assume we know the Lemma for all integers i with $0 < i \leq N$. Put

$$\mathcal{S} = \mathcal{T}_c^- , \quad \mathcal{A} = \langle G \rangle_1^{[a,b]}, \quad \mathcal{C} = \langle G \rangle_N^{[a,b]}, \quad \mathcal{X} = \overline{\langle G \rangle}_1^{[a,b]}, \quad \mathcal{Z} = \overline{\langle G \rangle}_N^{[a,b]} .$$

By the hypotheses of the lemma $\mathcal{S} = \mathcal{T}_c^-$ is triagulated and contains $\mathcal{C}$. The inductive hypothesis, coupled with [20, Lemma 1.6 and Remark 1.7], give that any map $s \longrightarrow y$, with $s \in \mathcal{S}$ and $y \in \mathcal{X} * \mathcal{Z}$, must factor through an object $b \in \mathcal{A} * \mathcal{C}$. The case of the lemma where $N < \infty$ follows immediately.

To prove the statement with $N=\infty$, let $\mathcal{R}\subset\overline{\langle G\rangle}^{[a,b]}$ be the full subcategory of all objects $Y\in\overline{\langle G\rangle}^{[a,b]}$ such that any map $F\longrightarrow Y$, with $F\in\mathcal{T}_c^-$, factors through an object in $\langle G\rangle^{[a,b]}$. We need to prove that $\mathcal{R}=\overline{\langle G\rangle}^{[a,b]}$.

By the case where $N<\infty$ we already know that $\overline{\langle G\rangle}_1^{[a,b]}\subset\mathcal{R}$, and it is obvious that any direct summand of an object in $\mathcal{R}$ belongs to $\mathcal{R}$. It remains to prove that $\mathcal{R}$ is closed under coproducts and extensions.

Therefore let $Y=\coprod_{\lambda\in\Lambda}Y_\lambda$ be a coproduct of objects in $\mathcal{R}$, and let $F\longrightarrow Y$ be a morphism with $F\in\mathcal{T}_c^-$. By (i) we may choose a morphism $E\longrightarrow F$, with $E\in\mathcal{T}^c$, and such that $\operatorname{Hom}(F,Z)\longrightarrow\operatorname{Hom}(E,Z)$ is an isomorphism for any $Z\in\overline{\langle G\rangle}^{[a,b]}$. In particular it is an isomorphism for $Z=Y=\coprod_{\lambda\in\Lambda}Y_\lambda$. But E is compact and hence the morphism $E\longrightarrow\coprod_{\lambda\in\Lambda}Y_\lambda$ factors through a finite subcoproduct. And for each λ in the resulting finite set, the fact that $Y_\lambda\in\mathcal{R}$ allows us to factor $E\longrightarrow Y_\lambda$ further through an object $X_\lambda\in\langle G\rangle^{[a,b]}$. We deduce

(ii) The subcategory $\mathcal{R}\subset\overline{\langle G\rangle}^{[a,b]}$ is closed under coproducts.

Next put

$$\mathcal{S}=\mathcal{T}_c^-,\quad \mathcal{A}=\langle G\rangle^{[a,b]},\quad \mathcal{C}=\langle G\rangle^{[a,b]},\quad \mathcal{X}=\mathcal{R},\quad \mathcal{Z}=\mathcal{R}.$$

By the hypotheses of the lemma $\mathcal{S}=\mathcal{T}_c^-$ is triagulated and contains $\mathcal{C}$. By the definition of $\mathcal{R}$, any map $s\longrightarrow x$ with $s\in\mathcal{S}$ and $x\in\mathcal{X}$ factors through an object $a\in\mathcal{A}$, and any map $s\longrightarrow z$ with $s\in\mathcal{S}$ and $z\in\mathcal{Z}$ factors through an object $c\in\mathcal{C}$. We are in a position to apply [20, Lemma 1.6 and Remark 1.7], which gives that any map $s\longrightarrow y$, with $s\in\mathcal{S}$ and $y\in\mathcal{X}*\mathcal{Z}$, must factor through an object $b\in\mathcal{A}*\mathcal{C}=\langle G\rangle^{[a,b]}$. Thus

(iii) The inclusion $\mathcal{R}*\mathcal{R}\subset\mathcal{R}$ holds.

This completes the proof of the Lemma. □

Proposition 10.8 *Suppose $\mathcal{T}$ is a triangulated category, and assume it has a compact generator H with $\operatorname{Hom}(H,\Sigma^i H)=0$ for $i\gg 0$. Let $\mathcal{T}_c^b$ be the one corresponding to a preferred t–structure.*

Suppose there is an object $G\in\mathcal{T}_c^b$ and an integer $N>0$ with $\mathcal{T}=\overline{\langle G\rangle}_N$. Then $\mathcal{T}_c^b=\langle G\rangle_N$.

Proof Let $\left(\mathcal{T}^{\le 0},\mathcal{T}^{\ge 0}\right)$ be a *t*–structure in the preferred equivalence class, and choose it so that $\mathcal{T}^{\ge 0}$ is closed under coproducts.

Take any object $A\in\mathcal{T}_c^b$. Because A lies in $\mathcal{T}^b$ is must belong to $\mathcal{T}^{\ge a}\cap\mathcal{T}^{\le b}$ for some integers $a\le b$, and Lemma 10.4 guarantees that it must belong to $\overline{\langle G\rangle}_N^{[a-M,b+M]}$ for some $M>0$. But then the identity $A\longrightarrow A$ is a morphism from the object $A\in\mathcal{T}_c^b\subset\mathcal{T}_c^-$ to the object $A\in\overline{\langle G\rangle}_N^{[a-M,b+M]}$, and Lemma 10.7 gives that it must factor through an object $B\in\langle G\rangle_N^{[a-M,b+M]}\subset\langle G\rangle_N$. As $\langle G\rangle_N$ is closed under direct summands it must contain A. □

Lemma 10.9 *Suppose $\mathcal{T}$ is a triangulated category with coporoducts, and assume that $\mathcal{T}$ has a compact generator H with $\operatorname{Hom}(H,\Sigma^i H)=0$ for $i\gg 0$. Suppose further that there is an object $G\in\mathcal{T}_c^b$ and an integer $N>0$ with $\mathcal{T}=\overline{\langle G\rangle}_N$.*

Assume $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ *is a t–structure in the preferred equivalence class. Then there exists an integer* $M > 0$ *such that, for any object* $Y \in \mathcal{T}_c^-$ *and any integer* $i \in \mathbb{Z}$*, the t–structure truncation morphism* $Y \longrightarrow Y^{\geq -i}$ *may be factored as* $Y \longrightarrow Y^{\geq -i-2M} \longrightarrow C \longrightarrow Y^{\geq -i}$ *with* $C \in \langle G\rangle_N^{[-i-M,\infty)}$.

Proof Replacing the t–structure by an equivalent one if necessary, we may assume $\mathcal{T}^{\geq 0}$ is closed under coproducts. Choose an integer $M > 0$ as in Lemma 10.4; in particular $\mathcal{T}^{\geq -i} \subset \overline{\langle G\rangle}_N^{[-i-M,\infty)}$. Assume further that M is large enough so that $G \in \mathcal{T}_c^b$ belongs to $\mathcal{T}^{\geq -M}$.

We are given the map $Y \longrightarrow Y^{\geq -i}$, with $Y \in \mathcal{T}_c^-$ and $Y^{\geq -i}$ in $\mathcal{T}^{\geq -i}$, and by the choice of M we have that $\mathcal{T}^{\geq -i}$ is contained in $\overline{\langle G\rangle}_N^{[-i-M,\infty)}$. Now Lemma 10.7 permits us to factor $Y \longrightarrow Y^{\geq -i}$ through an object $C \in \langle G\rangle_N^{[-i-M,\infty)}$. This far we have composites $Y \longrightarrow C \longrightarrow Y^{\geq -i}$.

But G is an object of $\mathcal{T}^{\geq -M}$, hence $\langle G\rangle_N^{[-i-M,\infty)}$ must be contained in $\mathcal{T}^{\geq -i-2M}$. The map $Y \longrightarrow C$ must therefore factor canonically through the t–structure truncation, and we have our factorization $Y \longrightarrow Y^{\geq -i-2M} \longrightarrow C \longrightarrow Y^{\geq -i}$. ☐

Corollary 10.10 *Suppose* $\mathcal{T}$ *is a triangulated category with coporoducts, and assume there is an object* $G \in \mathcal{T}_c^b$ *and an integer* N *with* $\mathcal{T} = \overline{\langle G\rangle}_N$. *Assume further that* $\mathcal{T}$ *has a compact generator* H *with* $\mathrm{Hom}(H, \Sigma^i H) = 0$ *for* $i \gg 0$. *For any object* $Y \in \mathcal{T}_c^-$ *we may choose an inverse sequence* $\cdots \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0$ *so that the subsequence* $\cdots \longrightarrow E_7 \longrightarrow E_5 \longrightarrow E_3 \longrightarrow E_1$ *lies in* $\mathcal{T}_c^b$ *while the subsequence* $\cdots \longrightarrow E_8 \longrightarrow E_6 \longrightarrow E_2 \longrightarrow E_0$ *is a subsequence of* $\cdots \longrightarrow Y^{\geq -3} \longrightarrow Y^{\geq -2} \longrightarrow Y^{\geq -1} \longrightarrow Y^{\geq 0}$.

Proof The construction of the sequence E_i is just by iterating Lemma 10.9. ☐

In view of Corollary 10.10, the next lemma becomes interesting.

Lemma 10.11 *Suppose* $Y \in \mathcal{T}_c^-$ *is an object, mapping to an inverse system* $\cdots \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0$ *in* $\mathcal{T}_c^b$. *Assume this inverse system is pro-isomorphic to* $\cdots \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0$, *and what we know about* F_* *is that for any* $n > 0$ *there exists an* $m > 0$ *so that the map* $Y^{\geq -n} \longrightarrow F_i^{\geq -n}$ *is an isomorphism for all* $i \geq m$.

When we view $\mathrm{Hom}(Y, -)$ *as a functor on* $\mathcal{T}_c^b$, *it is equal to the colimit of* $\mathrm{Hom}(E_i, -)$.

Proof Any object $Z \in \mathcal{T}_c^b$ belongs to $\mathcal{T}^{\geq -n}$ for some $n > 0$, and the sequence $\mathrm{Hom}(F_i, Z)$ becomes stable and isomorphic to $\mathrm{Hom}(Y, Z)$ for $i \gg 0$. Hence $\mathrm{Hom}(Y, -)$ is the colimit of the sequence $\mathrm{Hom}(F_i, -)$. Therefore it must also be the colimit of the ind-isomorphic sequence $\mathrm{Hom}(E_i, -)$. ☐

11 Combining the hypothesis $\mathcal{T} = \overline{\langle G\rangle}_N$ with approximability

The careful reader might have noticed that, in Sect. 10, we didn't assume much about the triangulated category $\mathcal{T}$. Let us perhaps rephrase this more accurately: we made the strong assumption that there exists an object $G \in \mathcal{T}_c^b$ with $\overline{\langle G \rangle}_N = \mathcal{T}$. But, beyond that, the only assumption was that there exist a compact generator $H \subset \mathcal{T}$ satisfying $\mathrm{Hom}(H, \Sigma^i H) = 0$ for $i \gg 0$.

This is about to change, in this section we combine this with the approximability hypotheses.

Notation 11.1 It is time to begin recalling the setup in Theorem 1.4. In the next lemmas R will be a commutative ring and $\mathcal{T}$ will be an R–linear, weakly approximable triangulated category. Let $\mathcal{T}_c^b$ and $\mathcal{T}_c^-$ be understood with respect to the preferred equivalence class of t–structures. We will be considering two functors

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\mathcal{Y}"] & \mathrm{Hom}_R\left(\left[\mathcal{T}^c\right]^{\mathrm{op}}, R\text{–Mod}\right) \\
\mathcal{T}^{\mathrm{op}} \arrow[r, "\widetilde{\mathcal{Y}}"] & \mathrm{Hom}_R\left(\mathcal{T}_c^b, R\text{–Mod}\right)
\end{tikzcd}$$

The object $A \in \mathcal{T}$ goes under the functors, respectively, to

$$\mathcal{Y}(A) = \mathrm{Hom}(-, A), \qquad \widetilde{\mathcal{Y}}(A) = \mathrm{Hom}(A, -).$$

For $\mathcal{Y}(A)$ the variable $(-)$ takes its values in $\mathcal{T}^c$, while in the case of $\widetilde{\mathcal{Y}}(A)$ the variable $(-)$ lies in $\mathcal{T}_c^b$.

We will mostly be concerned with the restrictions of $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ to the subcategory $\mathcal{T}_c^-$.

Proposition 11.2 *Let the conventions be as in Notation 11.1, and assume there is an object $G \in \mathcal{T}_c^b$ and an integer N with $\mathcal{T} = \overline{\langle G \rangle}_N$.*

Then restriction to $\mathcal{T}_c^-$ of the map $\widetilde{\mathcal{Y}}$ is full. More generally: any morphism $\varphi : \widetilde{\mathcal{Y}}(b) \longrightarrow \widetilde{\mathcal{Y}}(a)$, with $b \in \mathcal{T}_c^-$ and $a \in \mathcal{T}$, is equal to $\widetilde{\mathcal{Y}}(f)$ for some $f : a \longrightarrow b$.

Proof By Corollary 10.10 we may construct for b an inverse sequence $\cdots \longrightarrow C_3 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0$ pro-isomorphic to the sequence $\cdots \longrightarrow b^{\geq -3} \longrightarrow b^{\geq -2} \longrightarrow b^{\geq -1} \longrightarrow b^{\geq 0}$, and do it in such a way that C_i all belong to $\mathcal{T}_c^b$. Lemma 10.11 tells us, moreover, that $\widetilde{\mathcal{Y}}(b)$ is the colimit of $\widetilde{\mathcal{Y}}(C_i)$.

But $\widetilde{\mathcal{Y}}(C_i)$ is representable, hence the composite $\widetilde{\mathcal{Y}}(C_i) \longrightarrow \widetilde{\mathcal{Y}}(b) \xrightarrow{\varphi} \widetilde{\mathcal{Y}}(a)$ is a morphism from a representable functor. Yoneda tells us that it must be $\widetilde{\mathcal{Y}}$ of a unique map $f_i : a \longrightarrow C_i$. These maps are compatible, and hence all factor through a morphism $f : a \longrightarrow z$ with $z = \underleftarrow{\mathrm{Holim}}\, C_i$. Because the sequence C_i is pro-isomorphic to the sequence $b^{\geq -i}$ we have that $z = \underleftarrow{\mathrm{Holim}}\, b^{\geq -i}$, and Proposition 4.2 gives an isomorphism $b \cong z$. We have produced a morphism $f : a \longrightarrow b$, and it's now obvious that $\widetilde{\mathcal{Y}}(f) = \varphi$. ☐

Lemma 11.3 *Let the conventions be as in Notation 11.1, and assume there is an object $G \in \mathcal{T}_c^b$ and an integer N with $\mathcal{T} = \overline{\langle G \rangle}_N$.*

Given two morphisms $f, g : X \longrightarrow Y$ in the category $\mathcal{T}_c^-$, we have $\mathcal{Y}(f) = \mathcal{Y}(g)$ if and only if $\widetilde{\mathcal{Y}}(f) = \widetilde{\mathcal{Y}}(g)$.

Proof Choose and fix a t–structure $\big(\mathcal{T}^{\leq 0},\mathcal{T}^{\geq 0}\big)$ in the preferred equivalence class.

Suppose $\mathcal{Y}(f)=\mathcal{Y}(g)$, let $S\in\mathcal{T}^b_c$ be an object, and let $h:Y\longrightarrow S$ be an element of $\widetilde{\mathcal{Y}}(Y)(S)=\mathrm{Hom}(Y,S)$. Because $S\in\mathcal{T}^b$ we may choose an integer $m>0$ with $S\in\mathcal{T}^{\geq -m+1}$. Now the object X belongs to $\mathcal{T}^-_c$, hence there is a triangle $E\xrightarrow{\alpha}X\longrightarrow D$ with $E\in\mathcal{T}^c$ and $D\in\mathcal{T}^{\leq -m}$. Because $\mathcal{Y}(f)=\mathcal{Y}(g)$ the two composites

$$\begin{tikzcd} E \arrow[r, "\alpha"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

must be equal. Hence so are the longer composites

$$\begin{tikzcd} E \arrow[r, "\alpha"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] & S \end{tikzcd}$$

Therefore the map $h(f-g):X\longrightarrow S$ must factor as $X\longrightarrow D\longrightarrow S$, but as $D\in\mathcal{T}^{\leq -m}$ and $S\in\mathcal{T}^{\geq -m+1}$ the map $D\longrightarrow S$ must vanish. Therefore $hf=hg$. Since this is true for every h we have $\widetilde{\mathcal{Y}}(f)=\widetilde{\mathcal{Y}}(g)$.

Next suppose $\widetilde{\mathcal{Y}}(f)=\widetilde{\mathcal{Y}}(g)$, let $E\in\mathcal{T}^c$ be an object, and let $\alpha:E\longrightarrow X$ be an element in $\mathcal{Y}(X)(E)=\mathrm{Hom}(E,X)$. By Lemma 3.8 we may choose an integer $m>0$ with $\mathrm{Hom}\big(E,\mathcal{T}^{\leq -m}\big)=0$. By Lemma 10.9 the map $Y\longrightarrow Y^{\geq -m+1}$ factors as $Y\xrightarrow{h}S\longrightarrow Y^{\geq -m+1}$ with $S\in\mathcal{T}^b_c$. Now the two composites

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] & S \end{tikzcd}$$

are equal because $\widetilde{\mathcal{Y}}(f)=\widetilde{\mathcal{Y}}(g)$, hence the longer composites

$$\begin{tikzcd} E \arrow[r, "\alpha"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] & S \arrow[r] & Y^{\geq -m+1} \end{tikzcd}$$

must also be equal. In other words: the map $Y\longrightarrow Y^{\geq -m+1}$ annihilates the map $(f-g)\alpha:E\longrightarrow Y$, and hence $(f-g)\alpha$ must factor as $E\longrightarrow Y^{\leq -m}\longrightarrow Y$. But $\mathrm{Hom}\big(E,\mathcal{T}^{\leq -m}\big)=0$, and we deduce that $f\alpha=g\alpha$. As this is true for every α we have $\mathcal{Y}(f)=\mathcal{Y}(g)$. □

Because the symbols G and G' are similar and easily confused with each other, in Sect. 10 we tried to avoid using both in any one statement. But in the next lemma we cannot escape the conundrum; the letter H it taken, it means a homological functor.

Lemma 11.4 *Let R be a noetherian ring. Let the conventions be as in Notation 11.1, except that weak approximability is no longer enough—for this Lemma we assume $\mathcal{T}$ approximable. Suppose also that there is an object $G\in\mathcal{T}^b_c$ and an integer N*

with $\mathcal{T}=\overline{\langle G\rangle}_N$*. Assume further that there is a compact generator* $G'\in\mathcal{T}$ *with* $\mathrm{Hom}(G',\Sigma^i G')$ *a finite* R*–module for every* $i\in\mathbb{Z}$*.*

Suppose we are given an R*–linear cohomological functor* $H:\mathcal{T}_c^b\longrightarrow R$*–Mod, as well as an object* $Y\in\mathcal{T}_c^-$ *such that* H *is a direct smmand of* $\widetilde{\mathcal{Y}}(Y)$*. Then there exists an object* $Y'\in\mathcal{T}_c^-$ *and an isomorphism* $H\cong\widetilde{\mathcal{Y}}(Y')$*.*

Proof We are given that H is a direct summand of $\widetilde{\mathcal{Y}}(Y)$, and the composite $\widetilde{\mathcal{Y}}(Y)\longrightarrow H\longrightarrow\widetilde{\mathcal{Y}}(Y)$ is an idempotent endomorphism $\varphi:\widetilde{\mathcal{Y}}(Y)\longrightarrow\widetilde{\mathcal{Y}}(Y)$. By Proposition 11.2 there is a morphism $f:Y\longrightarrow Y$ in the category $\mathcal{T}_c^-$ with $\widetilde{\mathcal{Y}}(f)=\varphi$. Because $\widetilde{\mathcal{Y}}(f)$ is idempotent we have that $\widetilde{\mathcal{Y}}(f)=\widetilde{\mathcal{Y}}(f^2)$, and Lemma 11.3 informs us that $\mathcal{Y}(f)=\mathcal{Y}(f^2)$. Therefore $\mathcal{Y}(f)$ is idempotent, and corresponds to the projection to a direct summand of $\mathcal{Y}(Y)$. The finiteness hypotheses on $\mathrm{Hom}(G',\Sigma^i G')$ guarantee that $\mathcal{Y}(Y)$ is a locally finite cohomological functor, hence so is any direct summand, and by Theorem 9.20 there exists an object $Y'\in\mathcal{T}_c^-$ and morphisms $Y\xrightarrow{\alpha}Y'\xrightarrow{\beta}Y$ with $\mathcal{Y}(f)=\mathcal{Y}(\beta\alpha)$ and $\mathcal{Y}(\alpha\beta)=\mathrm{id}_{\mathcal{Y}(Y')}=\mathcal{Y}(\mathrm{id}_{Y'})$.

Lemma 11.3 informs us that $\widetilde{\mathcal{Y}}(f)=\widetilde{\mathcal{Y}}(\beta\alpha)$ and $\widetilde{\mathcal{Y}}(\alpha\beta)=\widetilde{\mathcal{Y}}(\mathrm{id}_{Y'})$. Thus $\widetilde{\mathcal{Y}}(f)$ factors as

$$\begin{tikzcd}\widetilde{\mathcal{Y}}(Y) \arrow[r, "\widetilde{\mathcal{Y}}(\alpha)"] & \widetilde{\mathcal{Y}}(Y') \arrow[r, "\widetilde{\mathcal{Y}}(\beta)"] & \widetilde{\mathcal{Y}}(Y)\end{tikzcd}$$

while

$$\begin{tikzcd}\widetilde{\mathcal{Y}}(Y') \arrow[r, "\widetilde{\mathcal{Y}}(\beta)"] & \widetilde{\mathcal{Y}}(Y) \arrow[r, "\widetilde{\mathcal{Y}}(\alpha)"] & \widetilde{\mathcal{Y}}(Y')\end{tikzcd}$$

composes to the identity. The current lemma follows. ☐

12 The proof of Theorem 1.4 (ii)

The proof of the main theorems will be by applying to the current situation the lemmas of Sect. 8. More precisely

Notation 12.1 With the conventions of Notation 11.1, assume given an object $G\in\mathcal{T}_c^b$ and an integer $N>0$ such that $\mathcal{T}=\overline{\langle G\rangle}_N$. Assume also that we have fixed a t–structure $(\mathcal{T}^{\le 0},\mathcal{T}^{\ge 0})$ in the preferred equivalence class, with $\mathcal{T}^{\ge 0}$ closed under coproducts. Let $\mathcal{A}$ be the heart of the t–structure, and $\mathcal{H}:\mathcal{T}\longrightarrow\mathcal{A}$ the standard homological functor.

The lemmas of Sect. 8 will be applied to the category $\mathcal{T}^{\mathrm{op}}$. The subcategory $\mathcal{S}$ of Notation 8.1 will be $\mathcal{S}=\left[\mathcal{T}_c^b\right]^{\mathrm{op}}$.

The next definition and lemma are similar to what works in Sect. 9. The reader might wish to compare the definition below, and the lemma that follows, with Definition 9.3 and Lemma 9.5.

Definition 12.2 A *powerful $\langle G\rangle_n$–approximating sequence* is an inverse system $\cdots \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0$ in $\mathcal{T}$, so that

(i) Each E_m belongs to $\langle G\rangle_n$.
(ii) The map $\mathcal{H}^i(E_{m+1}) \longrightarrow \mathcal{H}^i(E_m)$ is an isomorphism whenever $i \geq -m$.

Suppose we are also given an object $F \in \mathcal{T}$, together with

(iii) A map from F to the approximating system E_*.
(iv) The map in (iii) is such that $\mathcal{H}^i(F) \longrightarrow \mathcal{H}^i(E_m)$ is an isomorphism whenever $i \geq -m$.

Then we declare E_* to be a *powerful $\langle G\rangle_n$–approximating system for F*.

Lemma 12.3 *With the conventions of Definition 12.2 we have*

(i) *Given an object $F \in \mathcal{T}$ and a powerful $\langle G\rangle_n$–approximating system E_* for F, then the (non-canonical) map $F \longrightarrow \underleftarrow{\mathrm{Holim}}\; E_i$ is an isomorphism.*
(ii) *Any $\langle G\rangle_n$–powerful approximating system $\cdots \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0$ has a subsequence which is a powerful $\langle G\rangle_n$–approximating system of the homotopy limit $F = \underleftarrow{\mathrm{Holim}}\; E_i$. Moreover F belongs to $\mathcal{T}_c^-$.*

Proof Assertion (i) is contained in Proposition 4.2, and the "moreover" part of (ii) is contained in Lemma 4.3.

Let $L > 0$ be the integer of Lemma 4.1. Then Lemma 4.1 says that, for any $\langle G\rangle_n$–powerful approximating system $\cdots \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0$ and with $F = \underleftarrow{\mathrm{Holim}}\; E_m$, the maps $\mathcal{H}^i(F) \longrightarrow \mathcal{H}^i(E_{m+L})$ are isomorphisms whenever $i \geq -m$. In other words the subsequence $\cdots \longrightarrow E_{3+L} \longrightarrow E_{2+L} \longrightarrow E_{1+L} \longrightarrow E_L$ is a powerful approximating sequence for F. This proves the first half of (ii). ☐

Remark 12.4 Let us now explain how to specialize Lemma 8.5 to the framework of this section. Suppose we are given an object $\widehat{B} \in \mathcal{T}_c^-$ and a powerful $\langle G\rangle_n$–approximating system $\mathfrak{B}_*$ for $\widehat{B}$. Lemma 10.11 informs us that the natural map $\underrightarrow{\mathrm{colim}}\, \widetilde{\mathcal{Y}}(\mathfrak{B}_i) \longrightarrow \widetilde{\mathcal{Y}}(\widehat{B})$ is an isomorphism. Thus

(i) A powerful $\langle G\rangle_n$–approximating system $\mathfrak{B}_*$ for $\widehat{B}$ is an approximating system for $\widetilde{\mathcal{Y}}(\widehat{B})$ in the sense of Definition 7.1. Moreover: Lemma 12.3(i) tells us that the map $\widehat{B} \longrightarrow \underleftarrow{\mathrm{Holim}}\; \mathfrak{B}_m$ is an isomorphism, hence a powerful $\langle G\rangle_n$–approximating system for $\widehat{B}$, as in Definition 12.2, is also an approximating system for $\widehat{B}$ as in Remark 7.6.

Note also that in Lemma 12.3(ii) we learned that any powerful $\langle G\rangle_n$–approximating system $\mathfrak{B}_*$ has a subsequence which is a powerful $\langle G\rangle_n$–approximating system of $\underleftarrow{\mathrm{Holim}}\; \mathfrak{B}_i$.

Next assume we are given

(ii) A morphism $\widehat{\beta} : \widehat{B} \longrightarrow \widehat{C}$ in the category $\mathcal{T}_c^-$.
(iii) Two integers n and n', as well as a powerful $\langle G\rangle_n$–approximating system $\mathfrak{B}_*$ for $\widehat{B}$ and a powerful $\langle G\rangle_{n'}$–approximating system $\mathfrak{C}_*$ for $\widehat{C}$.

The dual of Lemma 7.5 allows us to choose a subsequence of $\mathfrak{B}'_* \subset \mathfrak{B}_*$ and a map of sequences $\beta_* : \mathfrak{B}'_* \longrightarrow \mathfrak{C}_*$ compatible with $\widehat{\beta} : \widehat{B} \longrightarrow \widehat{C}$. A subsequence of a powerful $\langle G\rangle_n$–approximating sequence is clearly a powerful $\langle G\rangle_n$–approximating sequence, hence $\mathfrak{B}'_*$ is a powerful $\langle G\rangle_n$–approximating sequence for $\widehat{B}$. Now as in Lemma 8.5 we extend $\beta_* : \mathfrak{B}'_* \longrightarrow \mathfrak{C}_*$ to a sequence of triangles, in particular for each $m > 0$ this gives a morphism of triangles

$$\begin{tikzcd}
\Sigma^{-1}\mathfrak{B}'_{m+1} \arrow[r, "\Sigma^{-1}\beta_{m+1}"] \arrow[d] & \Sigma^{-1}\mathfrak{C}_{m+1} \arrow[r, "\Sigma^{-1}\gamma_{m+1}"] \arrow[d] & \mathfrak{A}_{m+1} \arrow[r, "\alpha_{m+1}"] \arrow[d] & \mathfrak{B}'_{m+1} \arrow[r, "\beta_{m+1}"] \arrow[d] & \mathfrak{C}_{m+1} \arrow[d] \\
\Sigma^{-1}\mathfrak{B}'_{m} \arrow[r, "\Sigma^{-1}\beta_{m}"] & \Sigma^{-1}\mathfrak{C}_{m} \arrow[r, "\Sigma^{-1}\gamma_{m}"] & \mathfrak{A}_{m} \arrow[r, "\alpha_{m}"] & \mathfrak{B}'_{m} \arrow[r, "\beta_{m}"] & \mathfrak{C}_{m}
\end{tikzcd}$$

Applying the functor $\mathcal{H}^i$ with $i \geq -m+1$ yields a commutative diagram in the heart of $\mathcal{T}$ where the rows are exact, and where vertical maps away from the middle are isomorphisms. By the 5-lemma the middle vertical map, i.e. the map $\mathcal{H}^i(\mathfrak{A}_{m+1}) \longrightarrow \mathcal{H}^i(\mathfrak{A}_m)$, must also be an isomorphism when $i \geq -m+1$. We conclude that a subsequence of $\mathfrak{A}_*$ is a powerful $\langle G\rangle_{n'+n}$–approximating system. Put $\widehat{A} = \underleftarrow{\mathrm{Holim}}\ \mathfrak{A}_*$. By Lemma 12.3(ii) the object $\widehat{A}$ belongs to $\mathcal{T}_c^-$, and Lemma 10.11 coupled with Proposition 11.2 guarantee that the weak triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$ of Lemma 8.5 is isomorphic to the image under $\widetilde{\mathcal{Y}}$ of a weak triangle $\widehat{A} \xrightarrow{\widehat{u}} \widehat{B} \xrightarrow{\widehat{v}} \widehat{C} \xrightarrow{\widehat{w}} \Sigma\widehat{A}$ in the category $\mathcal{T}_c^-$.

Lemma 12.5 *Suppose H is a locally finite $\langle G\rangle_1$–homological functor. Then there is a surjection $\widetilde{\mathcal{Y}}(F)|_{\langle G\rangle_1} \longrightarrow H$, where $F \in \mathcal{T}_c^-$ has a powerful $\langle G\rangle_1$–approximating system $\cdots \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0$. Moreover: the system may be chosen so that the maps $E_{m+1} \longrightarrow E_m$ are split epimorphisms.*

Proof We have that $H(\Sigma^i G)$ is a finite R–module for every $i \in \mathbb{Z}$, and vanishes when $i \ll 0$. For each i with $H(\Sigma^i G) \neq 0$ choose a finite number of generators $\{f_{ij},\ j \in J_i\}$ for the R–module $H(\Sigma^i G)$. By Yoneda every $f_{ij} \in H(\Sigma^i G)$ corresponds to a morphism $\varphi_{ij} : \widetilde{\mathcal{Y}}(\Sigma^i G) \longrightarrow H$. Let F be defined by

$$F \;=\; \coprod_{i\in\mathbb{Z}} \bigoplus_{j\in J_i} \Sigma^i G \;\cong\; \prod_{i\in\mathbb{Z}} \bigoplus_{j\in J_i} \Sigma^i G$$

where the isomorphism of the coproduct and the product is by Lemma 10.5. Let the morphism $\varphi : \widetilde{\mathcal{Y}}(F) \longrightarrow H$ be given by

$$\begin{tikzcd}
\widetilde{\mathcal{Y}}(F)|_{\langle G\rangle_1} \arrow[r, equal] & \displaystyle\bigoplus_{i\in\mathbb{Z}} \bigoplus_{j\in J_i} \widetilde{\mathcal{Y}}(\Sigma^i G) \arrow[r, "(\varphi_{ij})"] & H
\end{tikzcd}$$

where (φ_{ij}) stands for the row matrix with entries φ_{ij}; on the i, j summand the map is φ_{ij}. Finally: because $G \in \mathcal{T}_c^b$ there is an integer $B > 0$ with $\Sigma^B G \in \mathcal{T}^{\leq -1}$. For

$m > 0$ we define

$$E_m = \bigoplus_{i<m+B} \bigoplus_{j \in J_i} \Sigma^i G$$

The sum is finite by hypothesis, making E_m an object of $\langle G\rangle_1$. The obvious map $E_{m+1} \longrightarrow E_m$ is a split epimorphism, and in the decomposition $F \cong E_m \oplus \widetilde{F}$ we have that $\widetilde{F}$, being the coproduct of $\Sigma^i G$ for $i \geq m+B$, belongs to $\mathcal{T}^{\leq -m-1}$. Therefore the map $\mathcal{H}^i(F) \longrightarrow \mathcal{H}^i(E_m)$ is an isomorphism if $i \geq -m$, making the E_* a powerful $\langle G\rangle_1$–approximating system for F. ☐

And now the time has come to prove the main results.

Theorem 12.6 *Let R be a noetherian, commutative ring. Let $\mathcal{T}$ be an R–linear, approximable triangulated category, and assume there is a compact generator $G' \in \mathcal{T}$ such that* $\mathrm{Hom}(G', \Sigma^i G')$ *is a finite R–module for all $i \in \mathbb{Z}$. Let $\mathcal{T}_c^-$ and $\mathcal{T}_c^b$ be the ones corresponding to the preferred equivalence class of t–structures, and assume there is an object $G \in \mathcal{T}_c^b$ and an integer $N > 0$ with $\mathcal{T} = \overline{\langle G\rangle}_N$.*

Then the functor $\widetilde{\mathcal{Y}} : \mathcal{T}_c^- \longrightarrow \mathrm{Hom}\left[\mathcal{T}_c^{b}, R\text{–}Mod\right]$ if full, and the essential image consists of the locally finite homological functors.

Proof The fact that the functor $\widetilde{\mathcal{Y}}$ is full was proved in Proposition 11.2, and the fact that its image is contained in the locally finite homological functors was shown in Lemma 10.1. What needs proof is that every locally finite homological functor can be realized as $\widetilde{\mathcal{Y}}(F)$ for some $F \in \mathcal{T}_c^-$.

Suppose therefore that H is a locally finite $\mathcal{T}_c^b$–homological functor. Therefore $H|_{\langle G\rangle_1}$ is a locally finite $\langle G\rangle_1$–homological functor, and Lemma 12.5 produces an object $F_1 \in \mathcal{T}_c^-$, with a powerful $\langle G\rangle_1$–approximating system, and an epimorphism $\widetilde{\mathcal{Y}}(F_1)|_{\langle G\rangle_1} \longrightarrow H|_{\langle G\rangle_1}$. From Corollary 7.4 if follows that we may lift the natural transformation to all of $\mathcal{T}_c^b$; there is a natural transformation $\varphi_1 : \widetilde{\mathcal{Y}}(F_1) \longrightarrow H$ so that $\varphi_1|_{\langle G\rangle_1} : \widetilde{\mathcal{Y}}(F_1)|_{\langle G\rangle_1} \longrightarrow H|_{\langle G\rangle_1}$ is surjective.

Next we proceed inductively. Suppose we have constructed $F_n \in \mathcal{T}_c^-$, with a powerful $\langle G\rangle_n$–approximating system, and a natural transformation $\varphi_n : \widetilde{\mathcal{Y}}(F_n) \longrightarrow H$, and assume that $\varphi_n|_{\langle G\rangle_1} : \widetilde{\mathcal{Y}}(F_n)|_{\langle G\rangle_1} \longrightarrow H|_{\langle G\rangle_1}$ is surjective. Since both $H|_{\langle G\rangle_1}$ and $\widetilde{\mathcal{Y}}(F_n)|_{\langle G\rangle_1}$ are locally finite and the ring R is noetherian, the kernel of $\varphi_n|_{\langle G\rangle_1}$ is also locally finite. Lemma 12.5 permits us to find a surjection to the kernel: there is an object $F' \in \mathcal{T}_c^-$, with a powerful $\langle G\rangle_1$–approximating system $\cdots \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0$ in which all the connecting maps $E_{m+1} \longrightarrow E_m$ are split epimorphisms, and an exact sequence $\widetilde{\mathcal{Y}}(F')|_{\langle G\rangle_1} \longrightarrow \widetilde{\mathcal{Y}}(F_n)|_{\langle G\rangle_1} \longrightarrow H|_{\langle G\rangle_1}$. Now Corollary 7.4 allows us to lift the map to $\mathcal{T}_c^-$. We deduce:

(i) There is a morphism $\alpha : F_n \longrightarrow F'$ so that the sequence below is exact

$$\begin{tikzcd}
\widetilde{\mathcal{Y}}(F')|_{\langle G\rangle_1} \arrow[r, "\widetilde{\mathcal{Y}}(\alpha)|_{\langle G\rangle_1}"] & \widetilde{\mathcal{Y}}(F_n)|_{\langle G\rangle_1} \arrow[r, "\varphi_n|_{\langle G\rangle_1}"] & H|_{\langle G\rangle_1}
\end{tikzcd}$$

Forget for a second the exactness; the vanishing of the composite in (i), coupled with Lemma 8.5, allows us to construct

(ii) With the notation of Definition 8.2, and working in the category $\mathcal{T}^{\mathrm{op}}$ and with $\mathcal{S} = \left[\mathcal{T}_c^b\right]^{\mathrm{op}}$, there is an object $F_{n+1} \in \mathcal{T}_c^-$ with a powerful $\langle G\rangle_{n+1}$–approximating system, a weak triangle in $\mathcal{T}_c^-$ of the form $F_{n+1} \xrightarrow{\beta_n} F_n \xrightarrow{\alpha} F' \longrightarrow \Sigma F_{n+1}$, and a morphism $\varphi_{n+1} : \widetilde{\mathcal{Y}}(F_{n+1}) \longrightarrow H$ so that the following triangle commutes

$$\begin{tikzcd}
 & \widetilde{\mathcal{Y}}(F_{n+1}) \arrow[d, "\varphi_{n+1}"] \\
\widetilde{\mathcal{Y}}(F_n) \arrow[ur, "\widetilde{\mathcal{Y}}(\beta_n)"] \arrow[r, "\varphi_n"'] & H
\end{tikzcd}$$

This inductively constructs an inverse sequence in $\mathcal{T}_c^-$ of the form $\cdots \longrightarrow F_4 \xrightarrow{\beta_3} F_3 \xrightarrow{\beta_2} F_2 \xrightarrow{\beta_1} F_1$, as well as compatible maps $\varphi_n : \widetilde{\mathcal{Y}}(F_n) \longrightarrow H$.

Now the map $\varphi_1 : \widetilde{\mathcal{Y}}(F_1) \longrightarrow H$ restricts to an epimorphism on $\langle G\rangle_1$ by construction, and the exactness of the sequence in (i) coupled with Lemma 8.6 informs us, by induction, that $\varphi_n : \widetilde{\mathcal{Y}}(F_n) \longrightarrow H$ restricts to an epimorphism on $\langle G\rangle_n$. By Proposition 10.8 we have $\mathcal{T}_c^b = \langle G\rangle_N$, hence φ_N is an epimorphism.

Now apply Lemma 8.7 to the diagram

$$\begin{tikzcd}
 & \widetilde{\mathcal{Y}}(F_N) \arrow[d] & & \\
\widetilde{\mathcal{Y}}(F') \arrow[r, "\widetilde{\mathcal{Y}}(\alpha)"] & \widetilde{\mathcal{Y}}(F_{N+n}) \arrow[r, "\widetilde{\mathcal{Y}}(\beta_{N+n})"] & \widetilde{\mathcal{Y}}(F_{N+n+1}) \arrow[r] \arrow[d, "\varphi_{n+N+1}"] & \widetilde{\mathcal{Y}}(\Sigma^{-1}F') \\
 & & H &
\end{tikzcd}$$

Induction on $n \geq 0$ teaches us that the map $\widetilde{\mathcal{Y}}(F_N)|_{\langle G\rangle_n} \longrightarrow \widetilde{\mathcal{Y}}(F_{N+n})|_{\langle G\rangle_n}$ annihilates the kernel of $\widetilde{\mathcal{Y}}(F_N)|_{\langle G\rangle_n} \longrightarrow H|_{\langle G\rangle_n}$. If we put $n = N$ and remember that $\langle G\rangle_N = \mathcal{T}_c^b$, we have that the map $\widetilde{\mathcal{Y}}(F_N) \longrightarrow \widetilde{\mathcal{Y}}(F_{2N})$ and the epimorphism $\widetilde{\mathcal{Y}}(F_N) \longrightarrow H$ have the same kernel. Thus $\widetilde{\mathcal{Y}}(F_N) \longrightarrow \widetilde{\mathcal{Y}}(F_{2N}) \longrightarrow H$ factors as $\widetilde{\mathcal{Y}}(F_N) \longrightarrow H \longrightarrow \widetilde{\mathcal{Y}}(F_{2N}) \longrightarrow H$, making H is a direct summand of $\widetilde{\mathcal{Y}}(F_{2N})$. Lemma 11.4 produces an object $Y \in \mathcal{T}_c^-$ with $H = \widetilde{\mathcal{Y}}(Y)$. □

Theorem 12.7 *Let the notation be as in Theorem 12.6. The essential image under $\widetilde{\mathcal{Y}}$ of the subcategory $\mathcal{T}^c \subset \mathcal{T}_c^-$ is precisely the finite homological functors. Moreover the restriction of $\widetilde{\mathcal{Y}}$ to $\mathcal{T}^c$ is fully faithful: it induces an equivalence of $\left[\mathcal{T}^c\right]^{\mathrm{op}}$ with the category of finite homological functors $\mathcal{T}_c^b \longrightarrow R$–mod .*

The "moreover" part can even be strengthened as follows: for any pair of objects $a \in \mathcal{T}^c$ *and* $b \in \mathcal{T}_c^-$ *the natural map is an isomorphism*

$$\mathrm{Hom}(a, b) \xrightarrow{\qquad\qquad} \mathrm{Hom}\left[\widetilde{\mathcal{Y}}(b), \widetilde{\mathcal{Y}}(a)\right]$$

Proof The fact that $\widetilde{\mathcal{Y}}(A)$ is finite when $A \in \mathcal{T}^c$ follows from Lemma 10.1—the essential image under $\widetilde{\mathcal{Y}}$ of the subcategory $\mathcal{T}^c \subset \mathcal{T}_c^-$ is contained in the finite functors.

Now suppose $H : \mathcal{T}_c^b \longrightarrow R$–mod is a finite homological functor. Since finite homological functors are locally finite Theorem 12.6 tells us that there exists an object $A \in \mathcal{T}_c^-$ and an isomorphism $\widetilde{\mathcal{Y}}(A) \cong H$. It suffices to prove that $A \in \mathcal{T}^c$. But the finiteness tells us that, for the object $G \in \mathcal{T}_c^b$ with $\mathcal{T} = \overline{\langle G \rangle}_N$ of the hypotheses of Theorem 12.6, we must have that $H^i(G) \cong \mathrm{Hom}(A, \Sigma^i G) = 0$ for $i \gg 0$. By Proposition 10.6 A must be compact.

It remains to prove the full faithfulness, or rather the strengthened version. We already know the surjectivity of the map

$$\mathrm{Hom}(a, b) \xrightarrow{\qquad\qquad} \mathrm{Hom}\left[\widetilde{\mathcal{Y}}(b), \widetilde{\mathcal{Y}}(a)\right],$$

that was part of Theorem 12.6. Suppose therefore that we have two morphisms $f, g : a \longrightarrow b$ with $\widetilde{\mathcal{Y}}(f) = \widetilde{\mathcal{Y}}(g)$. Then Lemma 11.3 informs us that $\mathcal{Y}(f) = \mathcal{Y}(g)$, and as $\mathcal{Y}(a) = \mathrm{Hom}(a, -)$ is representable we deduce from Yoneda that $f = g$. ☐

13 Applications: the construction of adjoints

We prove Corollary 1.6, a restricted version of which was the key tool in Jack Hall's original, simple proof of GAGA—see Remark 1.10. Hall's later proofs of more general results, see [11, 12], sidestep the representability theorems presented here.

Theorem 13.1 *Let R be a noetherian, commutative ring. Let $\mathcal{T}$ be an R–linear triangulated category with coproducts, and assume that it is approximable. Let $\mathcal{T}_c^b \subset \mathcal{T}_c^-$ be the subcategories of Definition 1.20, constructed using a t–structure in the preferred equivalence class. Assume the category $\mathcal{T}^c$ is contained in $\mathcal{T}_c^b$. Assume further that $\mathcal{T}$ has a compact generator G so that* $\mathrm{Hom}_{\mathcal{T}}(-, G)$ *is a G–locally finite cohomological functor.*

Let $\mathcal{L} : \mathcal{T}_c^b \longrightarrow \mathcal{S}$ be an R–linear triangulated functor, and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be any t–structure in the preferred equivalence class. Then the functor $\mathcal{L}$ has a right adjoint if and only if the following three conditions hold:

- (i) *For any pair of objects (t, s), with $t \in \mathcal{T}^c$ and $s \in \mathcal{S}$, the R–module* $\mathrm{Hom}\big(\mathcal{L}(t), s\big)$ *is finite.*
- (ii) *For any object $s \in \mathcal{S}$ there exists an integer $A > 0$ with* $\mathrm{Hom}\big(\mathcal{L}(\mathcal{T}_c^b \cap \mathcal{T}^{\leq -A}), s\big) = 0$.
- (iii) *For any object $t \in \mathcal{T}^c$ and any object $s \in \mathcal{S}$ there exists an integer A so that* $\mathrm{Hom}\big(\mathcal{L}(\Sigma^m t), s\big) = 0$ *for all $m \leq -A$.*

Proof We begin by proving the necessity of the three conditions. Assume that $\mathcal{L}$ has a right adjoint $\mathcal{R}:\mathcal{S}\longrightarrow\mathcal{T}_c^b$, and we will prove that (i), (ii) and (iii) hold.

For any object $s\in\mathcal{S}$ we have that $\mathcal{R}(s)\in\mathcal{T}_c^b$. If $t\in\mathcal{T}^c$ is any object, then $\mathrm{Hom}_{\mathcal{T}}\big(t,\mathcal{R}(s)\big)$ is a finite R–module by Lemma 9.2. The isomorphism

$$\mathop{\mathrm{Hom}}_{\mathcal{T}}\big(t,\mathcal{R}(s)\big)\cong\mathop{\mathrm{Hom}}_{\mathcal{S}}\big(\mathcal{L}(t),s\big)$$

informs us that $\mathrm{Hom}_{\mathcal{S}}\big(\mathcal{L}(t),s\big)$ is also a finite R–module, giving (i).

With s still an object of $\mathcal{S}$, we have that $\mathcal{R}(s)\in\mathcal{T}_c^b\subset\mathcal{T}^b$. There must therefore exist an integer $A>0$ with $\mathcal{R}(s)\in\mathcal{T}^{\geq -A}\cap\mathcal{T}^{\leq A}$. The fact that $\mathcal{R}(s)$ belongs to $\mathcal{T}^{\geq -A}$ tells us that $\mathrm{Hom}_{\mathcal{T}}\big(-,\mathcal{R}(s)\big)$ annihilates $\mathcal{T}^{\leq -A-1}$, and hence its subcategory $\mathcal{T}_c^b\cap\mathcal{T}^{\leq -A-1}$. But by adjunction $\mathrm{Hom}(-,s)$ must annihilate $\mathcal{L}\big(\mathcal{T}_c^b\cap\mathcal{T}^{\leq -A-1}\big)$, proving (ii).

Now let $t\in\mathcal{T}^c$ be an object. Remark 1.24 combined with Lemma 3.8 allow us to choose an integer $B>0$ with $\mathrm{Hom}\big(t,\Sigma^B\mathcal{T}^{\leq 0}\big)=0$. But now $\Sigma^m\mathcal{R}(s)$ belongs to $\mathcal{T}^{\leq 0}$ for all $m\geq A$, and hence $\mathrm{Hom}\big(t,\Sigma^{m+B}\mathcal{R}(s)\big)=0$ for all $m\geq A$. And now the isomorphism

$$\mathop{\mathrm{Hom}}_{\mathcal{T}}\big(t\,,\ \Sigma^{m+B}\mathcal{R}(s)\big)\cong\mathop{\mathrm{Hom}}_{\mathcal{S}}\big(\mathcal{L}(\Sigma^{-m-B}t)\,,\ s\big)$$

gives the vanishing of $\mathrm{Hom}_{\mathcal{S}}\big(\mathcal{L}(\Sigma^{-m-B}t)\,,\ s\big)$ for all $m\geq A$, completing the proof of (iii).

Now for the sufficiency: assume the three conditions hold, and we need to produce the right adjoint $\mathcal{R}$. For any pair of objects $t\in\mathcal{T}^c$, $s\in\mathcal{S}$ and any integer $m\in\mathbb{Z}$, from (i) we learn that $\mathrm{Hom}\big(\mathcal{L}(\Sigma^m t),s\big)$ is a finite R–module. Now (ii) and (iii) guarantee that it vanishes whenever $m\gg 0$ or $m\ll 0$. Thus $\mathrm{Hom}\big(\mathcal{L}(-),s\big)$ is a finite $\mathcal{T}^c$–cohomological functor. The assignment taking $s\in\mathcal{S}$ to the functor $\mathrm{Hom}\big(\mathcal{L}(-),s\big)$ is a functor from $\mathcal{S}$ to the category of finite $\mathcal{T}^c$–cohomological functors; by Theorem 9.18 we can lift it through the equivalence of categories $\mathcal{Y}$. There is a functor $\mathcal{R}:\mathcal{S}\longrightarrow\mathcal{T}_c^b$ so that, for all objects $t\in\mathcal{T}^c$ and all objects $s\in\mathcal{S}$, we have a natural isomorphism

$$\mathrm{Hom}\big(\mathcal{L}(t),s\big)\xrightarrow{\ \ \varphi\ \ }\mathrm{Hom}\big(t,\mathcal{R}(s)\big)\ .$$

Fix $t'\in\mathcal{T}_c^b$ and consider the following composite, which is natural in $t\in\mathcal{T}^c$, $t'\in\mathcal{T}_c^b$

$$\mathrm{Hom}(t,t')\xrightarrow{\ \ \mathcal{L}\ \ }\mathrm{Hom}\big(\mathcal{L}(t),\mathcal{L}(t')\big)\xrightarrow{\ \ \varphi\ \ }\mathrm{Hom}\big(t,\mathcal{R}\mathcal{L}(t')\big)\ .$$

We have objects $t',\mathcal{R}\mathcal{L}(t')\in\mathcal{T}_c^b$ and a natural transformation $\mathcal{Y}(t')\longrightarrow\mathcal{Y}\big(\mathcal{R}\mathcal{L}(t')\big)$, and Theorem 9.20 allows us to express it as $\mathcal{Y}(\alpha_{t'})$ for a unique morphism $\alpha_{t'}:t'\longrightarrow\mathcal{R}\mathcal{L}(t')$ in the category $\mathcal{T}_c^b$. We leave it to the reader to check that $\alpha_{t'}$ is natural in t'; it gives a natural transformation $\alpha:\mathrm{id}\longrightarrow\mathcal{R}\mathcal{L}$.

For a general $t'\in\mathcal{T}_c^b$ we have no idea how to compute $\alpha_{t'}$; it is a mysterious morphism in $\mathcal{T}_c^b$ that comes from a representability theorem applied to

$\mathcal{Y}(t') \longrightarrow \mathcal{Y}\big(\mathcal{R}\mathcal{L}(t')\big)$. However: when t' belongs to $\mathcal{T}^c \subset \mathcal{T}_c^b$ then $\mathcal{Y}(t') = \mathrm{Hom}(-, t')$ is a representable functor on $\mathcal{T}^c$, and Yoneda's lemma tells us how to compute $\alpha_{t'}$; it is the image of $1 \in \mathrm{Hom}(t', t')$ under $\mathcal{Y}(t') \longrightarrow \mathcal{Y}\big(\mathcal{R}\mathcal{L}(t')\big)$. But this is explicit: it is the image of $1 \in \mathrm{Hom}(t', t')$ under the composite

$$\begin{tikzcd}\mathrm{Hom}(t', t') \arrow[r, "\mathcal{L}"] & \mathrm{Hom}\big(\mathcal{L}(t'), \mathcal{L}(t')\big) \arrow[r, "\varphi"] & \mathrm{Hom}\big(t', \mathcal{R}\mathcal{L}(t')\big)\end{tikzcd},$$

which agrees with the image of $1 : \mathcal{L}(t') \longrightarrow \mathcal{L}(t')$ under the map $\varphi : \mathrm{Hom}\big(\mathcal{L}(t'), \mathcal{L}(t')\big) \longrightarrow \mathrm{Hom}\big(t', \mathcal{R}\mathcal{L}(t')\big)$.

Now we define a natural transformation $\psi : \mathrm{Hom}\big(\mathcal{L}(-), -\big) \longrightarrow \mathrm{Hom}\big(-, \mathcal{R}(-)\big)$. For objects $t \in \mathcal{T}_c^b$, $s \in \mathcal{S}$ the map is

$$\begin{tikzcd}\mathrm{Hom}\big(\mathcal{L}(t), s\big) \arrow[r, "\mathcal{R}"] & \mathrm{Hom}\big(\mathcal{R}\mathcal{L}(t), \mathcal{R}(s)\big) \arrow[r, "\mathrm{Hom}\left(\alpha_t, \mathcal{R}(s)\right)"] & \mathrm{Hom}\big(t, \mathcal{R}(s)\big)\end{tikzcd}.$$

Once again: for a general $t \in \mathcal{T}_c^b$ we don't understand this map, but when we restrict t to lie in $\mathcal{T}^c \subset \mathcal{T}_c^b$ then we can compute. If $\beta : \mathcal{L}(t) \longrightarrow s$ is a morphism in $\mathcal{S}$, then the fact that φ is a natural transformation gives the commutativity of the square

$$\begin{tikzcd}\mathrm{Hom}\big(\mathcal{L}(t), \mathcal{L}(t)\big) \arrow[r, "\mathrm{Hom}\left(\mathcal{L}(t), \beta\right)"] \arrow[d, "\varphi"'] & \mathrm{Hom}\big(\mathcal{L}(t), s\big) \arrow[d, "\varphi"] \\ \mathrm{Hom}\big(t, \mathcal{R}\mathcal{L}(t)\big) \arrow[r, "\mathrm{Hom}\left(t, \mathcal{R}(\beta)\right)"] & \mathrm{Hom}\big(t, \mathcal{R}(s)\big)\end{tikzcd}$$

and, computing the image of $1 : \mathcal{L}(t) \longrightarrow \mathcal{L}(t)$ under the two equal composites, we obtain the first equality in

$$\varphi(\beta) = \mathcal{R}(\beta) \circ \alpha_t = \psi(\beta)\ ,$$

where the second equality is by the definition of the natural transformation ψ.

Hence when restricted to $t \in \mathcal{T}^c \subset \mathcal{T}_c^b$ the map ψ agrees with φ and is an isomorphism. It suffices to prove that ψ is an isomorphism for all $t \in \mathcal{T}_c^b$ and all $s \in \mathcal{S}$.

Fix $t \in \mathcal{T}_c^b$ and $s \in \mathcal{S}$. By (ii) we can choose an integer $A > 0$ with $\mathrm{Hom}\big(\mathcal{L}(\mathcal{T}_c^b \cap \mathcal{T}^{\le -A}),\ s\big) = 0$. Because $\mathcal{R}(s)$ belongs to $\mathcal{T}_c^b \subset \mathcal{T}^+$ we may choose an integer $A' > 0$ so that $\mathrm{Hom}\big(\mathcal{T}^{\le -A'}, \mathcal{R}(s)\big) = 0$. Now take $m \ge 1 + \max(A, A')$, and choose a triangle $\Sigma^{-1}d \longrightarrow e \longrightarrow t \longrightarrow d$ with $e \in \mathcal{T}^c$ and $d \in \mathcal{T}^{\le -m}$. Because $t \in \mathcal{T}_c^b$ and $e \in \mathcal{T}^c \subset \mathcal{T}_c^b$ we have that $d \in \mathcal{T}_c^b \cap \mathcal{T}^{\le -m}$. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
\operatorname{Hom}\big(\mathscr{L}(d), s\big) \arrow[r] \arrow[d] & \operatorname{Hom}\big(\mathscr{L}(t), s\big) \arrow[r, "a"] \arrow[d, "b"] & \operatorname{Hom}\big(\mathscr{L}(e), s\big) \arrow[r] \arrow[d, "c"] & \operatorname{Hom}\big(\mathscr{L}(\Sigma^{-1}d), s\big) \arrow[d] \\
\operatorname{Hom}\big(d, \mathscr{R}(s)\big) \arrow[r] & \operatorname{Hom}\big(t, \mathscr{R}(s)\big) \arrow[r, "a'"] & \operatorname{Hom}\big(e, \mathscr{R}(s)\big) \arrow[r] & \operatorname{Hom}\big(\Sigma^{-1}d, \mathscr{R}(s)\big)
\end{tikzcd}$$

By our choice of m we know that

$$\operatorname{Hom}\big(\mathscr{L}(d), s\big) = 0 = \big(\mathscr{L}(\Sigma^{-1}d), s\big), \quad \operatorname{Hom}\big(d, \mathscr{R}(s)\big) = 0 = \operatorname{Hom}\big(\Sigma^{-1}d, \mathscr{R}(s)\big).$$

Hence a, a' are isomorphisms. But c is an isomorphism by the compactness of e, and therefore b is an isomorphism. $\square$

Appendix: A criterion for checking that a triangulated functor is an equivalence

Lemma A.1 *Let $\mathscr{L} : \mathcal{U} \longrightarrow \mathscr{S}$ be a triangulated functor with right adjoint $\mathscr{R} : \mathscr{S} \longrightarrow \mathcal{U}$. Suppose P is a class of objects in $\mathcal{U}$ satisfying $P = \Sigma P$, and such that*

(i) $P^{\perp} = \{0\}$, *which means that if $u \in \mathcal{U}$ is an object and* $\operatorname{Hom}(P, u) = 0$ *then* $u = 0$.

(ii) $\mathscr{L}(P)^{\perp} = \{0\}$, *meaning that if $s \in \mathscr{S}$ is an object and* $\operatorname{Hom}\big(\mathscr{L}(P), s\big) = 0$ *then* $s = 0$.

(iii) *The map* $\operatorname{Hom}(p, u) \longrightarrow \operatorname{Hom}\big(\mathscr{L}(p), \mathscr{L}(u)\big)$ *is an isomorphism for objects $p \in P$ and $u \in \mathcal{U}$.*

Then $\mathscr{L}$ and $\mathscr{R}$ are quasi-inverses.

Proof Let $\eta : \mathrm{id} \longrightarrow \mathscr{R}\mathscr{L}$ and $\varepsilon : \mathscr{L}\mathscr{R} \longrightarrow \mathrm{id}$ be (respectively) the unit and counit of adjunction—it suffices to prove that η and ε are isomorphisms.

Let us begin with η. For objects $p \in P$ and $u \in \mathcal{U}$ the natural maps

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{U}}(p, u) \arrow[r, "\alpha"] & \operatorname{Hom}_{\mathscr{S}}\big(\mathscr{L}(p), \mathscr{L}(u)\big) \arrow[r, "\beta"] & \operatorname{Hom}_{\mathcal{U}}\big(p, \mathscr{R}\mathscr{L}(u)\big)
\end{tikzcd}$$

are both isomorphisms, α by (iii) and β by the adjunction. Hence the composite, which is the map $\operatorname{Hom}(p, \eta) : \operatorname{Hom}_{\mathcal{U}}(p, u) \longrightarrow \operatorname{Hom}_{\mathcal{U}}\big(p, \mathscr{R}\mathscr{L}(u)\big)$, must be an isomorphism. Thus $\operatorname{Hom}(p, -)$ annihilates the mapping cone of $\eta : u \longrightarrow \mathscr{R}\mathscr{L}(u)$, and by (iii) η must be an isomorphism.

We have proved that η is an isomorphism, and the fact that the composite $\mathscr{R} \xrightarrow{\eta\mathscr{R}} \mathscr{R}\mathscr{L}\mathscr{R} \xrightarrow{\mathscr{R}\varepsilon} \mathscr{R}$ is the identity tells us that $\mathscr{R}\varepsilon : \mathscr{R}\mathscr{L}\mathscr{R} \longrightarrow \mathscr{R}$ must also be an isomorphism. Hence for any $p \in P$ and any object $s \in \mathscr{S}$ we have that $\operatorname{Hom}_{\mathcal{U}}(p, -)$ takes that map $\mathscr{R}\varepsilon(s) : \mathscr{R}\mathscr{L}\mathscr{R}(s) \longrightarrow \mathscr{R}(s)$ to an isomorphism, and adjunction tells us that $\operatorname{Hom}_{\mathscr{S}}\big(\mathscr{L}(p), -\big)$ must take the map $\varepsilon(s) : \mathscr{L}\mathscr{R}(s) \longrightarrow s$ to an isomorphism. Applying (ii) to the mapping cone of $\varepsilon(s)$, for every $s \in \mathscr{S}$, we deduce that ε must be an isomorphism. $\square$

Example A.2 Let X be a scheme proper over the field $\mathbb{C}$ of complex numbers, and let X^{an} be the analytification of X. The category $\mathcal{T} = \mathbf{D_{qc}}(X)$ is approximable and $\mathbb{C}$–linear. Now let $\mathcal{L} : \mathcal{T}_c^b \longrightarrow \mathcal{S}$ be the analytification functor $\mathcal{L} : \mathbf{D}^b_{\mathrm{coh}}(X) \longrightarrow \mathbf{D}^b_{\mathrm{coh}}(X^{\mathrm{an}})$. Then the hypotheses of Theorem 13.1 are satisfied, hence the functor $\mathcal{L}$ has a right adjoint $\mathcal{R}$.

Next we apply Lemma A.1: for every closed point $x \in X$ choose a nonzero perfect complex $p(x)$ supported at x, and we set

$$P \quad = \quad \left\{ \Sigma^n p(x) \;\middle|\; n \in \mathbb{Z}, \text{ while } x \in X \text{ is a closed point} \right\} .$$

It's an easy exercise to show that this choice of P satisfies the hypotheses of Lemma A.1, hence the functor $\mathcal{L}$ must be an equivalence.

The idea at the core of the argument above is due to Jack Hall, the technical problem he faced was that the representability theorems available to him were less powerful than Theorem 13.1. The reader is also referred to Serre [24] for the first proof of the version of GAGA in the couple of paragraphs above, and to Hall [11, 12] for generalizations of his core idea that go in a direction different from the one of this article.

Remark A.3 The reader should note that the proof in Example A.2 depends on substantial structural theorems about $\mathbf{D_{qc}}(X)$, but all we need to know about the category $\mathcal{S} = \mathbf{D}^b_{\mathrm{coh}}(X^{\mathrm{an}})$ and the functor $\mathcal{L} : \mathbf{D}^b_{\mathrm{coh}}(X) \longrightarrow \mathbf{D}^b_{\mathrm{coh}}(X^{\mathrm{an}})$ is the minimal data that goes into showing that the hypotheses of Theorem 13.1 and Lemma A.1 are satisfied. This is the reason that all the GAGA theorems in algebraic geometry are special cases of general theorems as in Hall [11, Theorems A and B]; in fact the $P \subset \mathbf{D}^b_{\mathrm{coh}}(X)$ of the proof of Example A.2 works for all of them.

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