



MEDA

Mathematics Education in the Digital Age

Proceedings
of the 17th ERME Topic Conference
MEDA 4

University of Bari Aldo Moro, Italy
3-6 September 2024

Edited by
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Proceedings of the 17th ERME Topic Conference MEDA 4

held on 3 – 6 September 2024 in Bari, Italy

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Introduction

There have been three successful ERME Topic Conferences (ETCs) on the topic Mathematics Education in the Digital Age (MEDA). The first event (ETC 5, September 2018 in Copenhagen) was inspired by the contributions to the Thematic Working Groups 15 and 16 in the CERME 10 in Dublin, highlighting the diversity of research at that time and its overlaps with other TWG themes. It was an interdisciplinary, multifaceted collaboration that brought together participants who would normally attend a range of CERME Thematic Working Groups to provide the opportunity for further in-depth discussion and debate. The successful experience, which involved intensive communication and collaboration during the Conference, resulted in collegial work towards a publication of a post-conference book “Mathematics Education in the Digital Age. Learning, Practice and Theory” in the ERME Series published at Routledge in 2021. Inspired by the contributions to the Thematic Working Groups 15 and 16 in the last CERME 11 in Utrecht and CERME 12, and the success of the ETC 10 MEDA 2 organized in September 2020 (which was held online, hosted in Linz), a third edition of the conference (ETC 13 MEDA 3, September 2022 in Nitra) offered the opportunity for further in-depth discussion and debate. Two special issues within the journals *ZDM – Mathematics Education* and *International Journal of Research in Undergraduate Mathematics Education* are in preparation for publication, based on the works presented and discussed at MEDA3.

Teaching, learning and assessment practices within all phases of mathematics education have dramatically changed due to the impacts of global pandemics and the increasing dependences on digital technologies across education systems. Meanwhile, the field has witnessed significant changes in research foci regarding the diverse applications of digital technologies; from supporting active engagement in rich STEM activities to enabling new assessment practices, communication, and collaboration opportunities. More than ever before, it seems that intensive research is now needed on the wide spectrum of roles that technologies play in mathematics education at all phases.

Alongside, rapid developments in Artificial Intelligence (AI) technologies are poised to impact education in, as yet, unseen ways. There is a lack of timely and responsive research that can grasp, document, and acknowledge the positive and negative impacts of digital technologies with respect to its potential to both innovate, and seek to address existing and emerging educational and societal challenges within the context of mathematics education.

‘New Digital Experiences and Perspectives in Mathematics Education’ is a theme that captures the goals for MEDA 4, which is a natural continuation of the three previous MEDA conferences, which offered a space for discussing future directions in the research in mathematics education post-pandemic, and in the age of Artificial Intelligence.

Informed by the call for papers, we received theoretical, methodological, empirical or developmental papers and poster proposals in relation to the following three main conference themes:

Theme 1: Mathematics educators’ practices, teacher education and professional learning

- Shared virtual/hybrid spaces and resources for teacher education and professional learning;
- Formative and summative assessment in remote and hybrid conditions at all levels of education;
- Teachers’ experiences and practices in STEM contexts, with Artificial Intelligence, or regarding the promotion of computational thinking.

Theme 2: Design, implementations and evaluations of digital resources and environments

- Design and implementation of resources with emerging technologies such as 3D printers, Virtual Reality, and Augmented Reality;
- Applications of Learning Management Systems, Learning Analytics and Artificial Intelligence in practices, assessment and design of resources;
- Impact of emerging technologies on curriculum transformation in STEM contexts, or regarding the promotion of computational thinking within mathematics education.

Theme 3: Students' experience and learning with digital technologies

- Impact of digital technologies on students' engagement, learning and assessment;
- Development and investigation of students' learning with emerging technologies;
- Role of technologies in developing mathematical competencies in STEM contexts;
- Integration of Artificial Intelligence technologies within formal, high-stakes examination and assessment systems.

All of the submitted papers and posters were peer-reviewed by two academics (selected from the authors who had submitted academic work to the conference) and one member of the International Program Committee.

Fifty-two papers and twenty-three posters were accepted to be presented at the conference and are included in the conference proceedings.

The poster session was organised in collaboration with the museum MuMA (Museo della Matematica) of the Department of Mathematics, University of Bari Aldo Moro.

Four invited plenary talks also offered participants opportunities to reflect and discuss the topics. The respective abstracts are provided below.

Plenary 1: Transformation of mathematics education environments by digital resources (by Birgit Pepin)

The increase in technology and digitalization (e.g., in schools, universities) and the advent of more and more access to Digital Resources (DRs) raises high expectations to be able to better educate students with the use of educational technology and DRs because of their perceived affordances. However, education practitioners and technology developers not only suggest new opportunities but also warn of their constraints, and of the problematic issues associated with their (development and) use. In this presentation, referring to chapters in the Handbook of Digital Resources in Mathematics Education, I take up particular issues that in my view need special attention when considering the transformation of mathematics education learning environments by DRs. Moreover, I discuss the (likely) influence of recent resources such as ChatGPT on the learning process.

Plenary 2: Augmented reality to discover the derivative. A dialogue crossing the perspectives of semiotic bundles and embodied instrumentation (by Ferdinando Arzarello and Paul Drivers)

Digital technology offers exciting opportunities to the learning and teaching of mathematics. Still, the math education research community is struggling to identify theoretically founded and research informed ways to best benefit from these opportunities. As for the theoretical foundations, different perspectives have been offered. In this duo-presentation, we will address the notions of semiotic bundles and embodied instrumentation to analyse a short videoclip of a student using augmented reality technology to explore the notion of derivative. The differences, similarities and affordances of each of the two approaches will be discussed and some possible theoretical and practical outcomes that these perspectives offer will be pointed out.

Plenary 3: The prevalence and use of emerging technologies in the thematic working group on “Mathematics in the context of STEM education” at CERME (by Behiye Ubuz)

“Mathematics in the context of STEM Education” thematic working group (TWG26) was introduced in the 11th Congresses of the European Society for Research in Mathematics Education. Science, Technology, Engineering and Mathematics (STEM) education merges a variety of subjects in the fields of science, technology, engineering and mathematics to solve real world problems. In this study, my goal was to engage in a qualitative content analysis of all papers (n = 38) and posters (n = 12) that have been presented in TWG26 in the last three Congresses of the European Society for Research in Mathematics Education (CERME) to identify the prevalence and application of emerging technologies within the landscape of STEAM Education through asking a main question: “What is “T” in STEAM representing?” together with some sub-questions. I adopt a broad definition of STEAM education in this study since a movement towards STEAM was fuelled in this group, where “A” represents the arts and humanities.

Plenary 4: Aligning AI with MathEd Values and Practices (by Manolis Mavrikis)

In an era where artificial intelligence (AI) is increasingly hyped, it is important to learn from the past and align the design and use of AI with our core values and practices in mathematics education. This talk explores the intersection of AI and MathEd, going beyond the generative AI hype to research-based AI approaches that can support the teaching and learning of K-12 mathematics. Drawing on several research projects, I will highlight the potential of intelligent support in fostering mathematical meaning making in constructionist, exploratory learning. I will also discuss how AI and learning analytics can be utilised as ‘teacher assistant’ tools, highlighting the benefits and challenges faced by educators. I will conclude with insights and practical recommendations for educators, researchers, and ‘edtech’ developers working at the intersection of AI and MathEd, and invite the audience to contribute to ongoing work towards achieving alignment.

More than thirty early-career researchers submitted their work to the conference and the award of “best paper” was given to three best papers by an early career researcher, one per each theme of the conference.

The following two specific activities for early-career researcher participants were included in the program.

Workshop 1: "Giving a Talk" - an Art in itself! (by Hans-Georg Weigand)

Presenting one's own research results or other (scientific) findings, i.e. "Giving a Talk", is a recurring part of everyday life for scientists of all ages. Presenting a talk professionally is an art that needs to be learnt. There are rules and principles that at least make it easier to get started with this art. This workshop will use positive and negative examples, especially in connection with Power Point Presentations, to establish and discuss general principles that are fundamental for a successful presentation. Participants can and should also bring - short or parts of - own presentations, that will be constructively evaluated during the workshop.

Workshop 2: How to establish professional connections? (by Michal Tabach)

One of the aims of international conferences is meeting researchers with similar research interests. We will use this workshop to increase the chances of creating such connections, by a structured activity. To this end, we aimed at having two rounds of small group discussions. In every round, each participant will provide a concise description of his research focus in 3 min. Then, some more time will be devoted to questions and answers among the small group members.

The variety of contributions to the MEDA4 conference highlight the challenge to describe and understand better new ways of working with new or newly interpreted concepts like digital competences, digital resources, digital design or computational thinking in the era of Artificial Intelligence. The MEDA4 conference and its accompanying proceedings mark a step forward in furthering human understanding of the complex and diverse interrelationship between mathematics (and STEM) education and the new digital technologies.

The Organising Committee

Hans-Georg Weigand, Alison Clark-Wilson, Eleonora Faggiano, Michal Tabach

Contributions per themes

Theme 1: 3, 5, 8, 11, 12, 13, 16, 17, 24, 27, 30, 31, 36, 39, 43, 46, 49, 58, 59, 64, 69, 71

Theme 2: 2, 3, 6, 11, 13, 15, 18, 19, 20, 21, 23, 25, 29, 32, 38, 39, 42, 44, 45, 48, 53, 54, 55, 56, 57, 60, 61, 62, 63, 65, 66, 67, 68, 70, 72, 74

Theme 3: 1, 4, 7, 8, 9, 10, 14, 19, 22, 26, 28, 33, 34, 35, 36, 37, 38, 40, 41, 44, 47, 50, 51, 52, 53, 56, 61, 62, 63, 66, 67, 68, 69, 72, 73, 74, 75

Best paper award

Theme 1: *The role of teachers' knowledge and the use of different technologies in the classroom*, by Maria do Carmo Botelho and Helena Rocha

Theme 2: *Feedback from E-textbook as a Resource for Mathematics Teachers to Re-design*, by Ismael Almahdi and Osama Swidan

Theme 3: *Dialectical chatbots: AI-driven debate in mathematical education*, by Amedeo Matteucci, Francesco Saverio Tortoriello, Ilaria Veronesi

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Matlab LiveScript for promoting high-level problem solving processes in engineering students

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In this study, we compare engineering students' problem solving processes when they face linear algebra tasks in a pen and pencil environment and Matlab LiveScript environment. By a qualitative research method, we investigate if and how the devolution of computations to the software allows students to focus on significant aspects of the problem solving process, shifting from the procedures to strategic and communicative aspects. The results suggest that the tool promotes an aware view of the problem solving process at different levels, depending on students' knowledge and competency.

Keywords: engineering, Matlab, linear algebra, problem solving processes, shift of attention

Conceptual background and research question

Recent research investigated the potential of using domain-specific software in teaching mathematics to engineering students (Kanwal, 2018, van der Wal et al., 2017, 2019, Pepin et al., 2021). This paper sets in this mainstream and is part of a wider research (Albano et al., 2023, Albano & Telloni, 2023) concerning the educational opportunities of making engineering freshmen solve tasks using the Matlab LiveScript. Matlab is a software widely used by engineers, both in training and work. The LiveScript environment allows to combine code and text and produce justified problem solving processes (PSP), leaving the software to perform the computations. We chose to focus on engineering freshmen because typically these students have an operational view of Mathematics and perceive the correctness of the procedures as the core of the discipline (Jaworski, 2012). Moreover, we submitted them tasks of linear algebra because this subject requires combining conceptual and procedural understanding, and acquiring a structural vision of the mathematical objects (Stewart et al., 2019, Turgut, 2023). The general hypothesis underlying our research is that this approach should favor the emergence of a high-level view of PSP, i.e. as a sequence of conceptual steps, where the procedures are managed as objects, in a “black-box” fashion, without going into their details. This should be linked to students' shifts of attention (Mason, 1998), from procedural aspects to strategical and communicational ones. To test the hypothesis, we organized our course of Linear Algebra so that students were assigned weekly tasks that became the subject of collective discussion during the lectures. During the course, the students have been introduced to the use of Matlab LiveScript, being guided to move from paper and pencil resolution to Matlab LiveScript one. The final exam has been designed coherently with the organization of the course. Together with the exam, the students have been provided with a survey to collect their perceptions about the use of Matlab LiveScript to solve problems. The students' answers confirmed our hypotheses about the possible shifts of attention (Albano & Telloni, 2023), i.e.: *shift toward the theory*: the students describe their use of Matlab highlighting the opportunity to focus on theoretical aspects; *shift toward the solving strategy*: the students describe their use of Matlab highlighting the opportunity to focus on thinking about the overall solving strategy of the problem; *shift toward the reasoning*: the students describe their use of Matlab highlighting the opportunity to concentrate on the reasoning needed to identify how to proceed

in the PSP from the procedural point of view; *shift toward the justification*: the students describe their use of Matlab highlighting the opportunity to focus on the explanation to justify the procedural steps of the PSP. In this study, we investigated if the shifts of attention arising from the students' answers to the survey also emerge from their solving processes, i.e. if on average the students' perception corresponds to their way of solving the tasks. We analyzed the students' PSP relying on revised Polya's steps of problem solving. In particular, we referred to: strategic planning, justifying (referring to the theoretical results underlying procedures), performing (carrying out the plan), interpreting, and verifying (checking correctness). Our specific research question is (RQ) Do the previous shifts of attention declared by the students emerge from their PSP?

Methodology

The study involved 101 Computer Engineering freshmen, attending a course in Linear Algebra. In the first phase of the course, the students could solve the tasks and upload a photo of their solving process; in the central part of the course, they were asked to solve the task in a Moodle text-box, so that they would be forced to explain their PSP verbally; finally, in the last part of the course, they should solve the tasks by using the Matlab LiveScript, so that all the computational load was left to the software and the students could focus on the strategy or monitoring of the PSP. At the final exam, each student has been required to solve in Moodle platform three open-ended questions (selected among those in Table 1), two reporting in a Moodle text-box the solving process done in PP environment, and one sharing the solving process in ML environment. All the tasks have the same premise: *Solve the following task in a correct, complete, clear, and effective way, progressively explaining the solving process and justifying it by means of appropriate theoretical references*. This premise is the same as proposed in all the assignments provided to the students during the course.

Table 1. The tasks submitted to the students

<p>TASK PP1. Let us consider the following linear system:</p> $\begin{cases} (h-4)x_1 + 3x_2 + (3-h)x_4 + x_5 = 0 \\ 2x_2 + 4x_3 + x_4 + 2x_5 = h-3 \\ 3x_1 + 2x_3 + x_5 = h-3 \end{cases}$ <p>a) find for which values of h in \mathbb{R} the solution space W is a vector subspace of \mathbb{R}^5;</p> <p>b) for the values in a), establish if $u=(1,-3,2,0,1)$ belongs to W;</p> <p>c) for the values in a), compute the dimension and a basis of W^\perp.</p>	<p>TASK ML1. Let $f: \mathbb{R}^5 \rightarrow \mathbb{R}^7$ be the homomorphism such that:</p> <p>$f(-1,2,0,1,3)=(2,8,-7,-8,-1,-5,-14)$</p> <p>$f(-3,0,3,-2,-1)=(-9,-7,19,-10,-6,8,-10)$</p> <p>$f(0,2,3,-2,1)=(-4,7,10,-2,3,-2,-1)$</p> <p>$f(-1,-2,0,-1,0)=(-7,-3,9,-3,-2,-3,-8)$</p> <p>$f(0,3,-3,2,-3)=(16,-8,-25,-1,-10,20,14)$</p> <p>a) Establish if the linear closure of f exists and if it is unique;</p> <p>b) in case of existence, compute $f(3,5,0,-2,1)$.</p>
<p>TASK PP2. Let f be the homomorphism having the following matrix as canonical associated matrix</p> $A = \begin{pmatrix} 1 & 2 & h & -1 \\ h & 0 & 2h & 0 \\ 0 & -h & -1 & h+3 \end{pmatrix}$ <p>For the value of h such that f is not surjective, find the dimension and a cartesian representation of $(\text{Im}f)^\perp$.</p>	<p>TASK ML2. Let $f: \mathbb{R}^7 \rightarrow \mathbb{R}^5$ be the homomorphism such that:</p> <p>$f(x,y,z,t,l,m,n)=(x-2y+4z-3t-l-4m,-y+2z-3t-l-4m-2n,2x+2y+4z-2t-2m+2n,-y+2z+3t-l+2m+4n,-2x+y-2z+3t-l+2m)$</p> <p>a) Compute the dimension and a basis of $\text{Im}f$.</p> <p>b) Compute an orthonormal basis of $\text{Im}f$, starting from the basis computed in a).</p>

The students are familiar with the given tasks. They are essentially based on the management and the interpretation of basic operations on matrices and linear systems. The PP tasks and the ML tasks differ for the size of the data, which in the second case are difficult to handle manually. We made

this choice to foster the students' awareness of the Matlab LiveScript potential and to make them focus on strategic and communicative aspects. We analyzed the students' productions in the first exam session by using a qualitative research method, consisting in a content analysis guided by the possible shifts of attention when students move from PP environment to ML environment. The researchers separately analyzed the students' productions, identifying the structure of the PSP and those excerpts that highlight some shifts of attention and other relevant themes. Among the latter ones, communicative and metacognitive aspects arose, together with an unusual use of the software to check the correctness of the procedures. Then the researchers compared and discussed their analysis, reaching an agreement.

Data analysis

We analyze some paradigmatic cases, showing the structure of the PSP, in tune with Rott et al. (2021), and the emerging shifts of attention. Two cases are extensively described, and some further excerpts are presented to show additional features. Let us consider the productions of Toni, who is a low-medium achiever, on the basis of the mark obtained at the final exam: his PSP in the PP environment is essentially the execution of procedures (Figure 1), not always justified. Moreover, the PSP is not concluded: Toni does not obtain the final solution to the problem, probably because he does not interpret the outcomes of the procedures. His protocol suggests that all his resources and efforts are devoted to the computations.

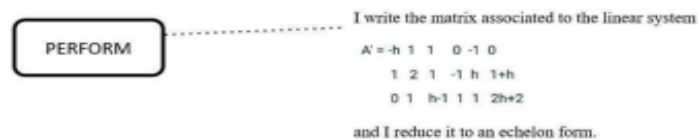


Figure 1. Scheme of Toni's PSP of task PP1 and corresponding excerpt

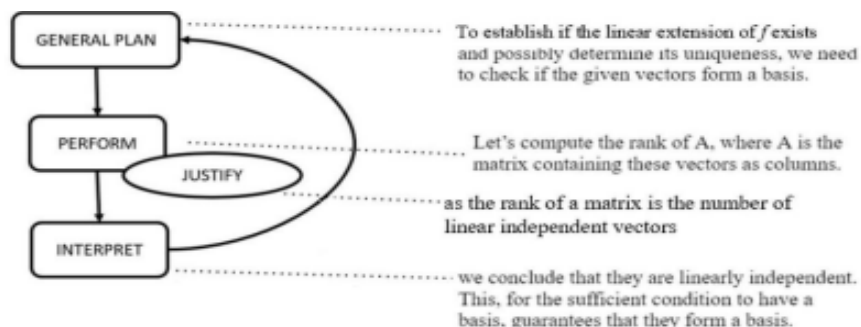


Figure 2. Scheme of Toni's PSP of task ML1 and examples of corresponding excerpts

Toni's PSP in the ML environment is very different (Figure 2): at the beginning, the student clarifies the aim of the PSP, identifying the strategy to solve the problem. The general plan develops in the execution of a sequence of steps justified by the underlying theory. Finally, the outcomes of the computations are interpreted in light of the problem's request. The final sentence, answering the question, linguistically reproduces the general plan initially declared. Toni's PSP in the ML environment is more complete and effective. Indeed, the student initially plans the PSP, passing from the mere implementation of procedural steps to the fruitful integration of the goal of the PSP with the theoretical justification and the computations (Figure 2). The comparison between the two PSP

suggests that the opportunity to leave the computations to the software allows the student a) to focus not on the calculations, but on the consequences of the calculations for the goal; and b) to pay attention to the planning (*shift toward the strategy*) and the theory (*shift toward the theory*).

Let us consider the productions of Paolo, who is a high achiever for the mark obtained in the final exam. He solves the given problem in PP environment through different cycles of local planning and the execution of computations theoretically justified (Figure 3). In the PSP, the interpretation of the computations in light of the request is not always explicit (we indicate this fact with a dotted line for the “interpret” step in Figure 3).

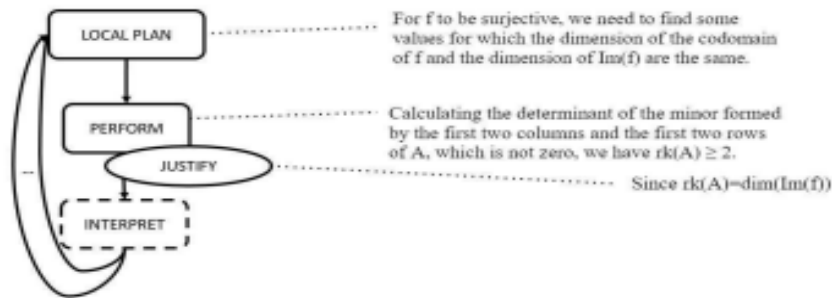


Figure 3. Scheme of Paolo’s PSP of task PP2 and examples of corresponding excerpts

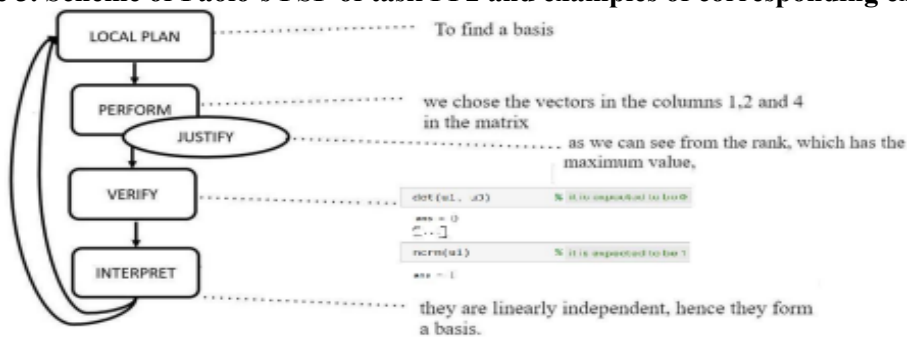


Figure 4. Scheme of Paolo’s PSP of task ML2 and examples of corresponding excerpts

The structure of Paolo’s PSP in the ML environment is similar to that in the PP environment, except for two aspects: 1) a new phase appears in the PSP, when the correctness of the results is not trivial; we call it “verify” since it aims at guaranteeing the correctness of the obtained result (Figure 4 and Table 2, second column); 2) the interpretation of the outcomes of the computations is always explicit; moreover, the theoretical justification underlying the procedures is more accurate and outspoken (Figure 4). Aspect 1) suggests an aware use of the tool at disposal, allowing easy verifications, which are not possible in the PP environment. Aspect 2) suggests that the release of calculations allows Paolo to focus on the justifications and implications of the procedures rather than on the procedures themselves. This contributes to a view of the PSP as a sequence of high-level steps, essentially characterized by the goal (local plan), the motivation (justify) for the procedures, and the consequences for the set problem (interpret). The communication of the PSP is clearer and more detailed than that in the PP environment: in particular, the student enriched his PSP with remarks and comments on the code lines to make the algorithm understandable for the reader (Table 2). In the Remark in column 1 of Table 2, Paolo provides pieces of information emerging from the computations, even if they are not required from the problem, e.g. concerning the injectivity of the function. Moreover, in column 2, he points out that he verifies that the found vectors of the

orthonormal basis are orthogonal (Table 2). The structure of the PSP does not significantly change, but the ML environment reveals more awareness and it seems that the devolution of procedures to the tool induces the student's *shifts toward the theory and the justification*.

Table 2. Remark and comments in Paolo's PSP in ML environment

<p><i>Remark:</i> f is not surjective since the dimension of Im(f) is not equal to that of the codomain and it is not injective, since, for the dimension theorem, the dimension of Ker(f) is 4</p>	<pre>dot(u1, u2) % verification of the computations ans = 0 dot(u1, u3) % it is expected to be 0</pre>
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In some cases, we observe a *shift toward the theory* at a higher level. Indeed, some students do not just only accurately justify the procedural steps but use the theoretical results as an a-priori guarantee of confidence in the results. An example of this aspect is the PSP of Luca, who facing the task ML2, wrote: “To find an orthonormal basis from that found at the previous step (it always exists for spaces with finite dimension for Gram-Schmidt’s Theorem), we use the proof of such theorem as an algorithm to calculate the basis”. Then Luca implemented the Gram-Schmidt orthonormalization and he did not need to verify the computations, since for him the correctness of the result is a consequence of the proof of the theorem. In this case, the PSP and the comments by the student suggest the logical priority of the theoretical result on the outcomes of the computations. Namely, the safety that from a basis we can always constructively obtain an orthonormal one is more important than the actual vectors of that basis. Another example in this sense is the PSP by Giorgio, who similarly to Paolo and Luca shifted toward the theory but used the theory itself to deeply justify the interpretation of the computations which he was going to do. Facing the task ML1, he stated that the dimension of the vector subspace Im(f) is equal to the rank of an associated matrix, then he proceeded to prove this statement: “The dimension of Im(f) is equal to the rank of A, where A is the canonical matrix representing the homomorphism. This [is] because we know, from the dimension theorem, that $\dim(R^7) = \dim(Ker(f)) + \dim(Im(f))$. But $\dim(Ker(f)) = \#unknowns - rk(A)$ (from its Cartesian representation), hence $\dim(Ker(f)) = 7 - rk(A)$, where A is the canonical matrix representing the homomorphism. Hence, by substitution: $\dim(R^7) = 7 - rk(A) + \dim(Im(f)) \rightarrow \dim(Im(f)) = rk(A)$.”

Let us now consider the PSP of Pietro, who is a low-achiever student. The PSP in the PP environment is the mere execution of procedures, without theoretical justification, which ends with the interpretation of the outcomes (Figure 5).

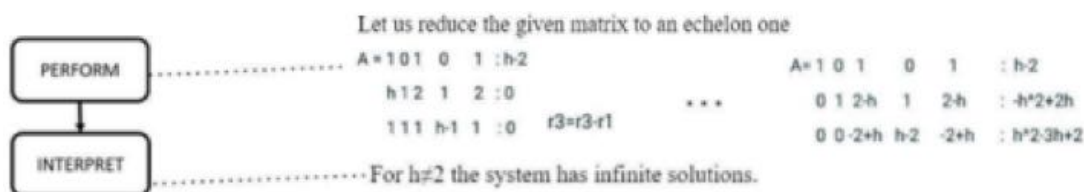


Figure 5. Scheme of Pietro's PSP of task PP1 and examples of corresponding excerpts

Pietro's solving process in the ML environment is essentially different (Figure 6): at the beginning, the student clarified the aim of the PSP, identifying the general plan. Then he implemented procedures previously justified and then interpreted toward the solution of the problem. The reduction of the cognitive load for the computation seems to allow the student to *shift toward the reasoning*, giving for each procedural step adequate justification and interpretation. In some cases, the students' PSP in the ML environment does not simply reproduce the PSP in PP environment, but deeply exploits the

potential of the tool. We just observed this fact from the prototypical behaviour of Paolo, who added the “verify” phase to his PSP, induced by the potential of Matlab. Another example of this phenomenon is the case in which the choice of the procedures to be implemented to achieve a specific goal is tailored to the environment where the problem is solved.

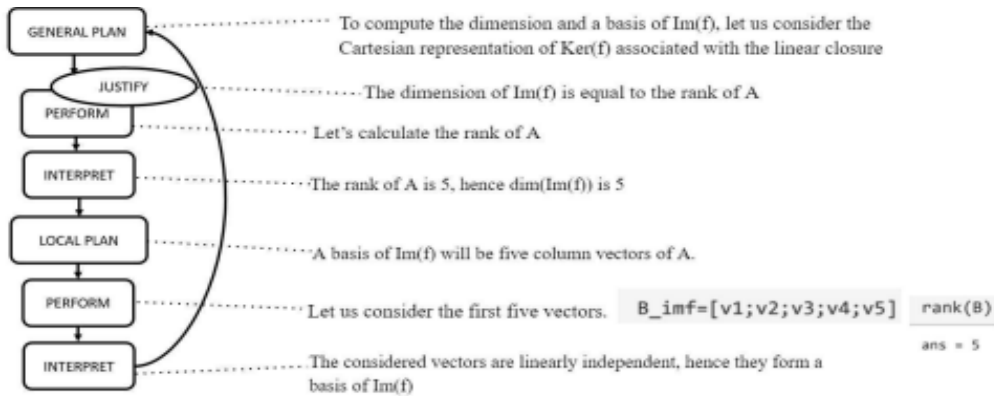


Figure 6. Scheme of Pietro’s PSP of task ML2 and examples of corresponding excerpts

Let us compare the procedures that Patrizia and Daniele used in facing task ML2 to extract linearly independent column vectors from a matrix to form a basis.

Table 3. Two procedures to extract linearly independent column vectors in ML environment

<p>Since the dimension of $\text{Im}f$ is 3, a basis will be formed from 3 independent vectors. In particular, let us see what are the linearly independent columns by reducing the matrix to an echelon matrix:</p> <pre> r3=r3-2*r1 r3 = 1:7 0 2 -4 4 2 6 2 r5=r5+2*r1 r5 = 1:7 0 -3 6 -3 -3 -6 0 * [...]</pre> <pre> B = 5:7 1 -2 4 -3 -1 -4 0 0 -1 2 -3 -1 -4 -2 0 0 0 -2 0 -2 -2 0 0 0 0 0 0 0 0 0 0 0 0 0 0</pre> <p>The columns containing a pivot are those linearly independent. Hence the basis is formed by:</p> <pre> Base=[1 0 2 0 -2; -2 -1 -2 -1 1; -3 -3 -2 3 3]</pre>	<p>The rank is 3, hence I have a basis formed by 3 vectors. I take 3 column vectors $\{c_1, c_2, c_4\}$ and I verify the rank by using the function rank.</p> <pre> B = [c1; c2; c4]; rank(B) ans = 3</pre> <p>The rank is still 3, hence, being these vectors some generators of $\text{Im}f$, they form a basis of $\text{Im}f$. I can deduce that $\dim(\text{Im}f)=3$ and that $B \text{Im}f = \{c_1, c_2, c_4\}$.</p>
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In the first column of Table 3, we have the procedure implemented by Patrizia. She proceeds like she would do in a PP environment, applying the operations needed to reduce the matrix to echelon form to identify the linearly independent vectors and choosing the columns corresponding to pivots. Although she does not perform computations, the only list of the needed commands takes one page and a half. In the second column of Table 3, we have the procedure implemented by Daniele to achieve the same goal. He chooses to compute the rank of the submatrix containing columns 1, 2 and 4, displaying the mastery of the characterization of the rank of a matrix as the number of its linearly independent rows/columns. His procedures take a few lines. The comparison between the PSP suggests different levels of awareness of the mathematical meanings involved, bringing to a different exploitation of the potential of the tool. Indeed, Patrizia makes computations for choosing the linearly independent vectors, whilst Daniele chooses the expected number of linearly independent vectors

and then uses the computations to verify the appropriateness of the choice. In the second case, the potential of the tool is highly exploited (i.e. use of the built-in function “*rank*” of Daniele versus a long list of separated commands reproducing PP operations of Patrizia). Also from a linguistic point of view, Patrizia searches “*what are the linearly independent columns*”, as they were the only ones (and actually the applied procedure produces a unique set); whilst, Daniele writes “*I take 3 column vectors*”, suggesting his awareness of the fact that it is not the unique choice.

Final remarks

In this paper, we analyzed the students’ PSP in PP and ML environments to highlight possible shifts of attention from computations to other aspects of problem solving. These shifts of attention emerged in a previous study focused on the students’ perception of the use of Matlab LiveScript to solve problems (Albano & Telloni, 2023). As an answer to our research question, the comparison between the students’ productions in PP and ML environments made evident all the expected shifts of attention:

- *toward the theory*: the students, like Paolo, Luca and Giorgio, focus on the theoretical results underlying the procedural steps, rather than on the computations; this happens at different levels, according to the students’ competencies and mastery of the subject and the tool;
- *toward the solving strategy*: the release of computations induces a better overall view of the PSP and typically students, like Paolo, clarify their plan to solve the problem before executing procedures;
- *toward the reasoning*: students, like Pietro, pay more attention to the logical reasons supporting the procedures to be implemented;
- *toward the justification*: students, like Paolo and Giorgio, communicate their PSP more effectively, displaying the PSP as a sequence of connected and justified high-level steps.

Besides, further findings come out. Most students, like Toni and Pietro, change their approach to the problem, passing from the mere execution of procedures, possibly justified, to a richer and more complete structure of PSP, including planning and interpretation of the outcomes of the computations in light of the problem to be solved. Moreover, in some cases, a new phase of PSP appears, that is the “verification” of the results, when not trivial, which is made possible by the software. Other students, like Paolo, do not significantly change their approach but enrich their PSP with theoretical justifications and interpretations of computations. Moreover, often in the ML environment, the order between the “perform” and “justify” steps is reversed with respect to the typical PSP in PP environment. Indeed, the students, like Luca, seem to give more importance to the theory guaranteeing the correctness of the results, whatever they are, rather than on the results themselves. In some sense, it seems they move forward the typical view of an expert mathematician. Various new issues emerge that are induced by the exploitation of the tool linked to the student’s knowledge. First, differing from what happens for the PP tasks, in the ML environment the students are willing to verify the correctness of their results, when not trivial, by fruitfully integrating the exploitation of the potential of the tools and their theoretical knowledge. As just remarked, a new phase appears within the PSP, as in the case of Paolo (Figure 4), and this improvement can be considered a monitoring activity on the solving process, which is a crucial aspect of the success of the PSP. Second, some students behave differently in the two environments, not simply reproducing in the ML environment the procedures they would do in the PP environment, but saving resources by exploiting in-built functions, as in the case of Daniele. Finally, we would point out a remarkable opportunity of the tool

from the designer and tutor perspective: as emerged from the data analysis, making students solve mathematical tasks within the Matlab LiveScript under the described conditions allows the teacher to pursue at a time individualization and personalization of learning (Shemshack & Spector, 2020). Indeed, it enables some students, typically low-achievers, to reach common minimal formative goals, i.e. the awareness of the PSP as a sequence of directed and justified steps; on the other hand, it induces other students, typically high-achievers, to go behind, acquiring a view of PSP which is typical of an expert mathematician, who smartly integrates internal and external resources toward the goal. In other terms, the same tool allows the remediation for some students (i.e. individualization, as the opportunity to tailor didactical strategies to enable all the students to reach the same goals) and the promotion of excellence (i.e. personalization, as the differentiation of didactical strategies, paths, and goals). In this sense, the tool should steer the student in each phase of his/her learning path.

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Developing computational thinking in college students with digital technologies

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To contribute to the need for more prepared professionals to face the increasing challenges of modern societies, universities should find strategies that provide college students with 21st century skills that are necessary to solve real world problems. Computational thinking is a cognitive process that requires problem-solving skills, and also can help learning mathematics, often accused of being responsible for the lack of professionals in STEM areas. This paper research computational thinking dimensions used by college students from two math courses of a Portuguese University in the context of a project aimed to calculate the annual percentage rate of charge, using STEM contents. With a qualitative methodology using an interpretive approach, and based on documental analysis applied to projects developed by the students, it is concluded that the project promoted several dimensions of computational thinking in the students as stated in the literature.

Keywords: Computational thinking, college students, mathematics education, interdisciplinarity, STEM.

Introduction

In modern societies with complex and increasing challenges, it is crucial to prepare students to the real-world and future careers (Tedre & Denning, 2022). In fact, more and stronger skills are required to solve real-life problems, and digital technology has a relevant role in this matter (Shute et al., 2017). Computational Thinking (CT) is critical, since it is a cognitive process that is related to the capability of solving problems, designing, projecting and understanding the human behaviour (Shute et al., 2017; Wing, 2006). In addition, it is expected that students, in particular undergraduates in STEM (Science, Technology, Engineering and Mathematics) bachelor courses have an interdisciplinary knowledge related to several subject matters (Costa et al., 2023). Moreover, CT can be applied to various fields requiring problem-solving skills, namely to help learning mathematics (Barr & Stephenson, 2011; Lu et al., 2022; Wing, 2014).

This paper aims to research CT in college students in the context of a project proposed to students with the goal of developing a function to calculate the APR (Annual percentage rate of charge) using data on the financial costs involved in a credit agreement (EPCEU, 2008), applying the bisection method (Burden and Faires, 2011). The APR project was implemented with students from the discipline of Numerical Analysis of two Maths bachelor courses (Maths and Applied Maths) of a Portuguese university in Lisbon. The teacher intended to implement an interdisciplinary approach to solve a mathematical problem to better prepare students to real world applications. Since CT is related to problem solving skills, to guide the study the following research question is raised:

What dimensions of CT are used by maths students when solving the APR project?

Because there is lack of research in higher education related to CT (Lu et al., 2022), in particular pedagogical experiences in math courses that help teachers implement this approach in the classroom, we believe our work is an important contribution for the literature in this matter. Specially because this is an opportunity to innovate mathematics teaching in math courses, which is not usual in Portugal. Indeed, higher education organizations have the responsibility to prepare their students for the labor market (McGunagle & Zizka, 2020). With this in mind, this research represents an effort to prepare college students, in particular from STEM courses, for the professional challenges that await them when they finish their courses, where CT and 21st century skills are crucial in this direction.

Theoretical framework and literature review

The literature states that students have shortage of skills in STEM fields and that students' difficulties with maths contribute to the problem of insufficient STEM graduates (Beswick & Fraser, 2019). Thus, it is crucial to find approaches to tackle these issues, notably by highlighting the role of maths in STEM, namely in what concerns real world applications (Costa et al., 2023; Stohlmann, 2018).

Usually, mathematical disciplines in higher education are taught traditionally, with contents lectured by the teacher mainly based in theoretical constructs (McGunagle & Zizka, 2020). However, it is recommended to resort to approaches such as problem-based learning as they enable students to develop skills like creativity, logical and critical thinking, and CT, which are essential to meet the growing challenges of modern societies (Costa et al., 2019; Costa et al., 2023; de Vere, 2009; Wing, 2014). Programming is related to CT that involves reasoning with analytical and algorithmic approaches (Grover & Pea, 2018; Wing, 2006). In this context, mathematics and computer science are linked through algorithms with sequences of instructions created in a structured way and respecting various steps to fulfil objectives (Albuquerque, 2022; Barr & Stephenson, 2011).

There are several definitions of CT (Barr & Stephenson, 2011), but in this paper, we will use the framework of Grover and Pea (2018), who consider CT concepts and practices. The CT concepts are: logic and logical thinking, algorithms and algorithmic thinking, patterns and pattern recognition, abstraction and generalization, evaluation, and automation. The CT practices are problem decomposition, creating computational artefacts, testing and debugging, iterative refinement (incremental development), collaboration and creativity.

Tedre and Denning (2022) present an historical overview from CT for beginners until a more professional perspective and claim that CT is important to understand the increasing technological world and therefore brings social benefits, namely helps to prepare students to future careers. Moreover, it is critical to implement CT across disciplines and various areas of learning, as well as to provide teachers with resources to help them develop this approach (Barr & Stephenson, 2011).

Thus, examples that highlight this approach can be helpful in this direction. Jong and Jeuring (2020) present an overview of interventions used to teach CT in higher education. They conclude that most interventions are targeted at students in Computer Science and Teacher Education. Our study stands out from this one because the APR project was implemented in math courses where CT is not taught to students. But, because solving the project requires problem-solving skills, we research what dimensions of CT are developed by the students.

Methodology

To answer the research question, we used a qualitative methodology with an interpretive approach and participant observation (Cohen et al., 2018). Data was collected from the projects developed and delivered by the students, as well as from participant observation by the math course teacher during the development of the project. Documental analysis was carried out based on the projects, seeking to assess their feedback and learning, in particular identifying CT dimensions as stated in Grover and Pea (2018). Triangulation of the data was carried out together with the authors of this article, and several meetings were held to cross-check the perceptions of the authors, in order to achieve the greatest possible accuracy in interpreting the data.

The participants in this study are 129 students of the discipline of Numerical Analysis from second grade of Maths and Applied Maths courses of a Portuguese university in Lisbon, who carried out the project in the academic year 2022/23. The students had two weeks to work in groups of three, and delivered 43 projects. From those, we selected a few that are representative of full marks and others that did not achieve all objectives of the project.

Data analysis and discussion

This section is organized in two subsections. In a first stage the APR project delivered to students is presented, and in a second stage discussion is focused in CT used by students to solve the project.

The project delivered to students

A 4 page file was delivered to students with instructions to develop the project. The problem was described: in consumer credit advertising, the European Union legislation imposes the display of the annual percentage rate of charge (APR) (EPCEU, 2008). The work consists in producing several Octave (software) functions that can be combined to compute the APR from the data defined by a credit agreement. Some examples are given to be used as test cases for debugging. The APR is defined as the solution of an equation that relates specified financial flows between creditor and consumer (drawdowns, repayments and payments of charges). These payments are made on certain dates. The European directive that specifies the APR states that the rate is the solution X of the equation:

$$\sum_{k=1}^n A_k (1 + X)^{-t_k} = 0,$$

where A_k represents the amounts of financial flows (the sign is used to define if the consumer is paying or receiving) and where t_k represents the times at which these flows occur, expressed in years (or fractions). For this work, data was modelled as a pair of vectors: a vector with the moments when financial flows occur, and a vector with the signed values of those flows. As many credit agreements involve a certain number of periodic payments (for example, monthly payments) with a fixed amount, students were asked to also consider part of the flows to be defined by four parameters: the fixed value of the payments, the number of payments, the moment of the first payment and the fixed time interval between payments. The parameters should be used to generate the adequate values of A_k and t_k and append them to the respective vectors. This subtask can be solved through an algorithm (using a cycle, for example) or through vector operations in Octave. Part of the decomposition of the problem is already given, as three functions must be programmed by students: a function that receives all data

for some credit agreement and returns the computed APR, a function to apply the bisection method to a given function and a function that implements the function defined by the APR (EPCEU, 2008)

$$f(X) = \sum_{k=1}^n A_k (1 + X)^{-t_k} .$$

A numerical method like the bisection method may be applied in a general and abstract situation, provided that we have a continuous computable real function and the extremes of an interval where the function is defined but changes its sign. Therefore, the students are asked to implement a general version of the method using as arguments only an Octave function, the extremes of the interval and a tolerance to be used as a stopping criterion. A consequence of this is that the function f must be defined as an independent function. The students were asked to implement f as an anonymous Octave function, using the vectors with the values of A_k and t_k . The APR function then calls the bisection function with the adequate arguments and returns the result given by the bisection method.

Each groups of 3 students should deliver Octave .m files with all the code they produced and a written report in a pdf file. The report should describe the development of the work, the tests that were executed and the results of the tests. The criteria for the assessment were given to students. There were three criteria related to the coding products (the code should be fully operational, should follow the specifications given, and should include inline comments) and four criteria related to the written report on the work (scientific correction, clarity of the text, organization, and inclusion of references).

Analysis of the projects developed by students

In this subsection, we had in account the reports developed and delivered by students and discuss the dimensions of CT used by students in the APR project. Most groups used the test cases included in the assignment. Some groups did not try different tests and sometimes missed some of the specifications. Other groups presented a discussion of the implementation of the algorithms in the project. As not all the algorithms were specified in the assignment, some groups developed their creativity on algorithms, and described the details in the report.

The code

Most groups achieved almost all the objectives of the assignment. Students created algorithms to perform some small tasks and coded them. The most common fails were operational fails (syntax errors and programs that freeze), and algorithmic errors (in the creation of the vectors with the values of A_k and t_k , and in the control of the bisection method when the middle of the interval is a root of the function). Below are some code examples relating to the bisection method.

The code from the bisection cycle (figures 1 to 3) illustrates different levels of development of CT in students. Group 1 codes exactly what was asked as the while cycle continues if $r \geq \text{tol}$ therefore implementing a stopping criterion that can be expressed as $r < \text{tol}$, and checks if an exact solution was found: $f(x) = 0$. Group 2 ignores that the criterion should be in the argument space and instead checks if the image of the last estimate is larger than the tolerance. Group 3 does not check if an exact solution is found which may lead to situations where the cycle never stops and uses a strict inequality for the while control which is not completely correct. This reveals that although almost all groups delivered code that worked in many situations, several groups did not fully understand the logic of the control

of the execution or did not consider all the possible paths of the execution of the algorithm that was implemented, even when that had been explicitly asked for.

```

while r >= tol
    if f(x) == 0
        return
    elseif f(a) * f(x) < 0
        b = x;
    elseif f(a) * f(x) > 0
        a = x;
    endif
    x = (a + b) / 2;
    r = x - a;
endwhile

```

Figure 1: The bisection cycle: group 1

```

while abs(f(c))>tol
    if f(a) * f(c) > 0
        a=c;
    endif
    if f(a)*f(c) < 0
        b=c;
    endif
    c = (a+b)/2;
endwhile

```

Figure 2: The bisection cycle: group 2

```

while abs(a-b) > tol
    m=(a+b)/2;
    if f(a)*f(m)<0
        b=m;
    else
        a=m;
    endif
endwhile

```

Figure 3: The bisection cycle: group 3

Although almost all students were able to use logical thinking in the control of the while cycle, only some of the students fully understood the logical implications of the specifications on the code to be produced. From the algorithmic point of view some students implemented only a simpler version of the bisection algorithm, while others developed the code to include all the cases that were described in the specifications.

The concepts and practices of CT

From the point of view of the concepts of CT, logical thinking was present in the control of the computational flow, especially in the bisection method and most of the students were able to use it. The algorithmic thinking was present, and students needed to be creative to generate the vectors with the times and financial flows. A possible approach to generate the vectors was to recognize a pattern and use Octave vector operations. This is illustrated in figure 4.

```

t_fim = t_inicio + intervalo_tempo*(n_pagamentos-1);
tempos = [tempos, t_inicio:intervalo_tempo:t_fim];
fluxos = [fluxos, montante * ones(1, n_pagamentos)];

```

Figure 4: Completing the time and flow vectors with the periodic payments data

A combined logical and algorithmic thinking challenge was to properly control the case when the root of the function is exactly at the middle point of the interval and several groups did not control for this situation, as was seen before. The main abstractions were already given in the assignment,

namely the use of vectors for the data and the use of a function to implement the bisection method. The evaluation was specifically requested in the assignment by giving test cases and asking for a written report, and the automation was the main goal of the project.

The practices of CT were already embedded in the assignment. The decomposition was implicit in the request to implement three different functions. Computational artifacts were some of the products that students should deliver, together with the report, and some tests were explicitly requested. An iterative refinement was not explicitly asked but it naturally occurs, as it was easier to start by implementing the case without the periodic payments and only later add this possibility. The collaboration was implicit as this was a group project.

In summary, most students were able to solve the problem posed in the project creating code to respond to the proposed tasks. During the execution of the project, several students asked for clarifications on the specifications of the code to be developed. Students verbally expressed that they were excited about working with a real-life problem and that this is a way of observing the importance of maths. One of the students noticed that the APR has a formal definition that, from the mathematical point of view, is the same as the definition of the internal rate of return of an investment. This shows critical thinking, which is one dimension of CT. In fact, several dimensions of CT as defined in Grover and Pea (2018) were observed in the project developed by students.

Conclusions

The project delivered to students includes a real-life problem that requires analytical and computational approaches, which are related to CT (Grover & Pea, 2018; Wing, 2006, 2014). In addition, students were able to use mathematics in a programming context, and to develop almost all of the CT dimensions as described in Grover and Pea (2018), namely abstraction, logical and algorithmic thinking, decomposition, automation, and evaluation. For example, logical thinking was present in the control of the computational flow, and algorithmic thinking was also present (Figures 1 to 3). Also, pattern recognition was needed to generate the vectors, as illustrated in Figure 4. Moreover, several mathematical contents were used such as Bolzano's theorem and the bisection method. Indeed, to develop the APR project that was proposed in two Maths bachelor courses of a Portuguese university, the students revealed problem-solving skills and developed several dimensions of CT (Barr & Stephenson, 2011; Lu et al., 2022; Wing, 2014).

Several students verbally expressed that they were excited with this approach that made them use mathematics and programming to develop the project, which means that the implementation of this approach besides developing CT in students, also contributes for their motivation to learn. Thus, it is recommended to keep developing this approach in higher education, which is not usual in Portugal. In fact, according to Barr and Stephenson (2011), it is critical to implement CT across disciplines and to provide teachers with resources to help them develop this approach.

Finally, it is concluded that the project promoted CT and 21st century skills in the students because they showed creativity and were able to work in groups and to develop different computational artifacts to effectively solve a real-life problem. Despite some projects did not fully accomplish all the goals, all the students used most of the CT dimensions as required, which are critical to correspond to the increasing complexity of the modern world (Costa et al., 2023; Jong & Jeurig, 2020; Wing,

2014). In fact, CT is important to understand the increasing technological world and thus it contributes for the preparation of future careers (Tedre & Denning, 2022).

However, there are some limitations regarding this study. In the future, and in similar projects, we intend to conduct interviews with some students to better understand their performance in the development of the project. Also, more studies should be conducted with students from other universities, namely it would be interesting to compare results from CT dimensions used by math students and engineering students.

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Feedback from E-textbook as a Resource for Mathematics Teachers to Re-design

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It is common to find reports on how teachers use e-textbooks. However, reports on teachers as designers of e-textbooks are less common. This study focuses on a middle school teacher who participated in a professional development course to design a chapter of an e-textbook. We analyzed the decisions made by the teacher using Schoenfeld's Resources, Orientations, and Goals framework and how this framework influenced his design decision-making. Semi-structured interviews and observations of the teacher in his classes were used to collect data. We discuss how the teacher can use feedback from the e-textbook to modify the chapter content. Our findings suggest that feedback from the e-textbook helps the teacher become more aware of their resources, orientations, and goals during the design process, which in turn impacts the design decisions of teachers who participate in the e-textbook design process.

Keywords: Teacher as designer, E-textbook, Feedback, Decision-making.

Introduction

In this paper, we are examining how teachers design and redesign e-textbooks, as well as the role of feedback in the redesign process. E-textbooks can vary in the level of interaction they offer to users. According to Pepin et al. (2016), there are four types of e-textbooks: a) scanned PDF versions of printed textbooks, b) PDF books with hyperlinks, c) interactive e-textbooks with tasks and linked interactive objects, and d) evolving e-textbooks, which allow teachers to author and modify content. They can also provide the teachers with feedback, such as a summative analysis of students' academic performance. Our study used evolving e-textbooks to observe the teachers' design and redesign processes.

The flexibility of the evolving e-textbooks may allow teachers to redesign and customize the content for their students. Involving teachers in this process can lead to modifying the content and structure of their e-textbooks to better meet the educational needs of their students (Trouche et al., 2018). Additionally, the feedback provided by e-textbooks can give teachers insights into their students' learning processes and challenges. This feedback may include statistics on students' performance, such as the number of exercises completed and academic achievements. Hence, our premise is that the feedback that the teachers receive can help them redesign their e-textbooks based on their understanding of the students' progression.

In this contribution, we aim to identify how the feedback that e-textbooks provide may change the teachers' decisions regarding the design of their e-textbooks. To do this, we utilized Schoenfeld's (2010) decision-making framework (DMF) to track a teacher's process of designing and redesigning a chapter aimed at teaching triangle properties and the calculation of their areas. Schoenfeld's framework comprises three interconnected elements: Resources, Orientation, and Goals, which inform the decisions teachers make. By taking these elements into account during the design and redesign of e-textbooks, we hope to gain a better understanding of how teachers adjust e-textbooks

and the reasons behind these changes. We are particularly interested in the impact of feedback provided by e-textbooks on teachers' decision-making processes when it comes to revising mathematical content in a chapter.

The study was guided by two research questions:

1. What are the resources, orientations, and goals, initially expressed by the teacher when designing a chapter in an e-textbook?
2. How do the initial resources, orientations, and goals of teachers change after receiving feedback from the e-textbook?

Feedback in an E-textbook

Digital learning environments are capable of collecting data about students' learning, which can be used to adjust the learning path and help teachers make informed choices and decisions. One of the most prevalent features of e-textbooks is the ability to provide feedback on students' solutions (Brnic et al., 2024; Rezat, 2021). In this paper, we are referring to Hattie and Timperley's definition of feedback, which is defined as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (Hattie & Timperley, 2007, p. 81). According to them, the information conveyed to the feedback receivers is meant to bring about change by highlighting discrepancies between the actual and intended performance.

Feedback is considered a crucial factor in teaching and is often viewed as a benefit of e-textbooks. After working with an e-textbook in class, teachers can receive information about their students' solutions (Rezat, 2021). This feedback from e-textbooks can be used by teachers to assess their students' progress and adjust their teaching objectives accordingly. Therefore, when designing an e-textbook, teachers should consider how their designs are adjusted to the students who learn by e-textbook. We hypothesized that the feedback obtained from an e-textbook through the automatic elaboration of students' solutions can enable teachers to discuss, reflect, and consider their design from alternate perspectives and make informed decisions.

Resources, Orientations, and Goals Framework

This study is guided by Schoenfeld's (2010) decision-making framework (DMF), which illustrates how a teacher's resources, orientations, and goals are linked to the decisions they make. According to this framework, teachers make decisions in order to accomplish their desired goals. Schoenfeld (2010) suggested that understanding a teacher's decision-making process involves identifying their resources, orientations, and goals. Therefore, a teacher's decision-making is influenced by these three factors:

1. Resources: primarily their knowledge, but also the tools available to them.
2. Orientations: these are beliefs, values, and preferences that guide their decisions.
3. Goals: these are often selected based on their orientation and the resources available to them.

Resources refer to not only materials and tools but also the knowledge possessed by a teacher, such as their familiarity with specific classroom routines, their past experiences with how students will respond to particular teaching methods, and their understanding of the importance of time management. Orientation, on the other hand, encompasses a teacher's beliefs, values, preferences, and disposition, which influence how they interpret and react to situations, including the setting and

prioritization of their pedagogical goals. According to Schoenfeld (2010), a teacher's goals are something they strive to achieve, and are often influenced by the resources available to them.

When designing an e-textbook, the teacher's beliefs, attitudes, and confidence in using resources will likely influence the design considerations. This is because the teacher's goals will be a part of the design process. In routine situations, decision-making processes are automated. However, in the design of an e-textbook, the teacher's decision-making may be updated, especially after receiving feedback. Since designing an e-textbook involves making several decisions, such as selecting resources, types of mathematical content, and representations, the decision-making framework has the potential to provide insights about their decisions.

Methodology

This report is about a small part of a larger research project that aims to explore the role of teachers as designers of e-textbooks. The project involved a 30-hour professional development course, held once a week for two months, that guided teachers on how to design e-textbooks. It was the first time these teachers were involved in the process of designing an e-textbook. In this case study, we focus on a middle school teacher named Abbas, who volunteered to participate in the research project. Abbas demonstrated seriousness during the professional development course and agreed to implement the material he designed in his classrooms. Additionally, Abbas identified himself as a teacher who integrates digital tools into his classes. We followed him during both the design process and the implementation of what he designed in the classroom. Abbas is 26 years old with five years of experience teaching mathematics at a middle school. He teaches 7th and 8th grades. The classroom was located in southern Israel and consisted of 7th-grade (age 12-13) high-ability mathematics students in a low socioeconomic environment. All the students in the class were native Arabic speakers. It was the first time the students were studying using an e-textbook designed by the teacher, who specifically chose this class for the new teaching experience.

Data collection and analysis

The data was obtained by conducting interviews with a teacher and observing the design process in the classroom. The interviews were conducted to gather information about the decision-making process that the teacher used after implementing a lesson. During the interviews, the teacher described his experience of the lesson and explained the reasoning behind certain design decisions. Furthermore, the teacher was asked to create a new task in an e-textbook for future lessons and to provide an explanation for the new design. The interviews were transcribed and coded using Schoenfeld's framework to identify the decisions that influenced the teacher's design process. The observations were focused on the teachers' design of e-textbooks. The data from the observation was recorded, transcribed, and annotated with descriptions of the teachers' decision-making processes during the design process.

After collecting the data, we analyzed it using Schoenfeld's (2010) DMF. We looked for instances where the teacher's responses mentioned resources, orientations, or goals. We identified resources when the teacher mentioned things like the curriculum, their knowledge of technology, or their teaching experience. We inferred the teacher's orientation when they described a particular classroom environment as being helpful for learning. Finally, we identified the teacher's goals from their comments. Sometimes, the teacher stated a specific outcome they wanted to achieve in their lesson.

Findings

In this section, we will discuss the outcomes of Abbas' DMF before he receives any feedback. To explain this, we will describe a case in which Abbas was involved as a designer of a chapter in an e-textbook. We then present our findings that demonstrate how Abbas's design was affected by receiving feedback from an e-textbook.

Table 1: Selected data about Abbas's initial DMF

Resources	Orientations	Goals
R1: Teacher's understanding of students' motivation for exploring triangle properties and their seriousness in solving geometrical tasks.	O1: He believes that most of his students are capable of solving most of the geometrical tasks he has designed.	G1: To offer appropriate tasks to help students explore triangle properties using GeoGebra.
R2: Teacher recognition that most of the students in this class excel in mathematics.	O2: He believes that designing tasks in an e-textbook with the GeoGebra tool adds value to teaching and learning geometry.	G2: To make the geometry lesson interesting and engaging for students.
R3: Teacher awareness that the students are familiar with using the GeoGebra tool to solve and explore geometrical shapes.	O3: He believes that interacting with GeoGebra in the e-textbook enriches students' geometrical thinking and their mathematical skills.	G3: Provide the students with tasks to calculate the area of a triangle.
	O4: He believes that the ten tasks he designed in the e-textbook correspond to the level of the students in the class.	G4: Providing students with challenging geometrical tasks that promote critical thinking and the application of problem-solving skills.

In the first interview, the teacher reflected on the considerations that guided him in the initial design. The following transcript illustrates the teacher's orientation and the resources that are available to him.

The tasks that include digital tools are supposed to help students' thinking during the solution of the geometric tasks.

I want to try to combine tasks with GeoGebra, which was introduced during the course (professional development course), so that my students will explore properties... using GeoGebra.

These utterances suggest that the resources available to Abbas referred to e-textbook affordance, which enables the design of geometric tasks that utilize digital tools, such as GeoGebra applets. Abbas's orientation was that incorporating digital tools into the chapter may enhance the students' thinking, which may help the students improve their problem-solving skills. In his own words, he identified himself as a teacher who effectively incorporates technology into his teaching methods.

Thus, his goal is to provide his students with mathematical tasks incorporating GeoGebra, aiming at helping them to explore geometrical properties and to develop thinking skills.

In the second phase, Abbas teaches his classroom with the chapter he designed for a 45-minute lesson. During this experiment, Abbas received feedback from the e-textbook platform. The feedback includes correct and incorrect solutions (green or red pattern, respectively), items the students completed (gray pattern) or did not (gray pattern with forbidden icon), and items that the student did not solve at all (white pattern). The numbers in the first row correspond to the item numbers. A green check mark indicates that the student has opened all the items, while a gray circle indicates that the student has not.

סך הכל		1	2	3	4	5	6	7	8	9
✓	100%	Green	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
○	100%	Green	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
✓	0%	Red	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Red with icon
✓	100%	Green	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
○	100%	Green	Gray	Gray	Gray	Gray with icon	Gray	Gray	Gray	Gray
○	0%	Red	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
○	100%	Green	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
○	100%	Green	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
○	100%	Green	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
○	100%	Green	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
○	100%	Green	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
○	0%	Red	Red with icon	Red with icon	Red with icon	Red with icon	Red with icon	Red with icon	Red with icon	Red with icon
○	100%	Green	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
○	0%	Red	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
○	0%	Red	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray
○	0%	Red	Gray	Gray	Gray	Gray	Gray	Gray	Gray	Gray

Figure 1: Feedback that the teacher received through automatic processing of the students' solutions.

Abbas reviewed the feedback from his e-textbooks and noticed that some of his students did not complete the tasks he had assigned. In the interview, he mentioned his desire to create a limited number of tasks in the revised chapter, saying, "I want them to solve everything." However, he observed from the feedback that many of the tasks remained unanswered.

Abbas shared his insights about this feedback with the other teachers who were participating in the professional development course. Following the discussion, he re-designed the chapter, trying to put the insights he received from the feedback into action. Abbas revised the design of the chapter by deleting some of the items and using different wording for the remaining items. Moreover, he alternated open items with multiple-choice or true/false items.

I thought it [receiving feedback] was good ... because it shows what your students solved and what was difficult for them during the lesson.

In addition, he says:

I saw it [feedback] several times...you can see here that the students did not solve some of the questions; perhaps they found using the technological tools difficult.

The feedback the teacher receives is considered a significant resource that leads to a change in the teacher's knowledge about his students. It seems that this resource has influenced the teacher's orientation as he mentioned, "Perhaps they found using the GeoGebra tool difficult." It seems that the shift in the teacher's orientation causes him to adjust his goal. This was reflected in the decisions he made regarding the re-design of the chapter in the e-textbook. The modification of the resources, orientations, and goals are summarized in Table 2.

Table 2: Abbas's DMK follows stage 2

Resources	Orientations	Goals
R1: Abbas experience in redesigning the chapter after the implementation in the classroom.	O1: He believes in the importance of preparing students to know how to use the GeoGebra tool.	G1: Assigning multiple assignments related to the topics covered in previous lessons.
R2: Abbas awareness of the students' limitations in using the GeoGebra tool.	O2: He believes that mathematical content should not be too conceptually heavy for his students.	G2: Providing hints and explanations to help students with geometry-related tasks.
R3: Abbas experience in redesigning the chapter after discussing the feedback he received with colleagues.	O3: He empathizes with struggling students who take time to learn to use e-textbooks, and they didn't complete most of the tasks.	G3: Designing tasks for students that closely resemble those in the printed book they use.

The following transcript aims to illustrate the new resources, orientations, and goals that emerged after Abbas discussed with his colleagues the feedback received from the e-textbook platform.

The objective of my design is to enable the student to understand the content of the lesson and solve some tasks correctly. In my opinion, this is better than designing several tasks that the students might not be able to solve most of them.

The feedback Abbas received from the e-textbook platform changed his mind about the ability of his students. After the discussion he conducted with his colleagues, Abbas argued that "in my opinion, this is better than designing several tasks that the students might not be able to solve most of them." This utterance suggests that Abbas gained new knowledge about his students' abilities, which led him to change the design of the tasks in the chapter. The ideas of Abbas's colleagues that emerged during the discussion about this feedback were resources that affected Abbas's decision to reduce the number of items. This change also affected Abbas's goals, which emphasized that he needed to provide the students with explanations and hints so that they could help them when needed.

The change in Abbas's orientation is reflected in the fact that he re-designed the chapter by choosing topics from the curriculum and a limited number of tasks. He believes that his role as a designer of the e-textbook is to adjust the mathematical contents to the students' intellectual level.

Final remarks

This paper focused on the teacher's decision-making as he designed a chapter in an e-textbook for teaching geometry. In the first round of design, the teacher's goal was to enable his students to solve all the items successfully. His orientation was that the tasks he designs are apt to the student's mathematical level. In fact, after reading carefully the feedback received from the e-textbook platform, Abbas's knowledge about the ability of his students has changed. As a result of these changes, Abbas redesigned his chapter by providing more explanations and hints to facilitate its reading and understanding of the items.

This case has shown that feedback received from the e-textbooks platform has prompted Abbas to change his DMF. The re-design of the chapter was not based on spontaneous decisions but rather a consequence of conscious and deliberate decisions made by the teacher as a result of observing and discussing the feedback with his colleagues.

By means of this case study, the use of the feedback received from the e-textbook platform for post-implementation discussion may assist the teachers who designed the e-textbook in becoming aware of their design process. The teacher's design decisions in terms of resources, orientations, and goals add richness to understanding his design process. It seems that teachers can benefit from the potential of e-textbooks in the context of design; the fact that they offer immediate feedback suggests that the effectiveness of this feature can support the teachers' roles as designers of e-textbooks when they refer to its content; this is particularly relevant because the design process is considered challenging for teachers. The findings of this study thus join the initial evidence on the effectiveness of features of e-textbooks in motivating and helping teachers in the design process (Pepin et al., 2016). Finally, it is worth mentioning that this study focused on one teacher with a specific cultural background. The results discussed here are firmly related to the teacher's background. Hence, a deep and extensive research study is needed to enrich our understanding of the role of feedback on decision-making regarding the design of e-textbooks.

Acknowledgment

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Prompting embodied instrumented covariation with the digital Tracer

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This study explores a learning environment that involves tasks designed by exploiting a digital artifact called digital Tracer. Such an artifact enables an embodied design of tasks that may prompt a covariational perspective on (functional) relationships traditionally taught by adopting a static approach. The paper aims to investigate how peers work together with the digital Tracer (what can be called an intercorporeal functional dynamic system) in order to solve some tasks elaborated with an embodied design. Employing the framework of embodied instrumentation, the preliminary analysis presented in this study examines a group of three university students, one of whom acted as an observer, while solving tasks that required them to follow two given traces (a broken line and a circumference) using the digital Tracer.

Keywords: Embodied instrumentation, covariation, intercorporeal functional dynamic system, Tracer.

Introduction

The ability to interpret the graphical representation of a function or a curve is a fundamental goal in mathematics that should be developed from the last years of primary school. However, this is far from easy for students, especially if they should understand such graphs as a representation of “a continuum of states of covarying quantities” (Saldanha & Thompson, 1998, p. 305) as it might happen in the conceptualization of dynamic situations. Indeed, the points on the graph should be understood as a covariation between the co-varying quantities, mathematically representable as an abscissa and an ordinate, which might be in a relation of functional dependence.

In this paper, we investigate the potentialities of a digital-embodied design involving the use of an artifact named *digital Tracer* to approach covariational reasoning, realized within a multi-touch dynamic geometry environment. Such an artifact displays a Cartesian plane with two bars that can be moved using the touch of fingers. By moving these bars, users can move and control their intersection point to draw or trace lines in the given plane, engaging their bodies in the activity.

Literature in Mathematics Education has shown how perceptuomotor integration in solving tasks can benefit mathematical learning and understanding (e.g., Nemirovsky et al., 2013). In order to incorporate embodied learning experiences with the digital Tracer into educational contexts, the five design principles (DPs) of Palatnik et al. (2023, pp. 7-8), which we briefly recall here, can be considered. DP1: *Involve students' bodies in the learning process*, either through perception-based design artifacts or action-based design; DP2: *Offer immediate sensorimotor interactions with artifacts* through which create stable forms of engagement with the environment and promote mathematical understanding through various means of communication, including verbal and nonverbal language; DP3: *Attend to the semiotic sensitivity of the design* by reflecting on the signs included in artifacts that can be produced by students or teachers while solving the task and their relationship with mathematical knowledge, as well as how students interpret and attribute meanings to these signs; DP4: *Include a variety of semiotic registers and artifacts that potentiate mathematical perception*

and discourse, recognizing that learning and mathematics understanding happens when learners are able to convert between different semiotic registers and to treat within one register; DP5: *Foster multimodal engagement and “linguaging”* by integrating various modes of communication, such as speech, gestures, and drawings, into the learning process, acknowledging that learning new concepts often involves discovering new ways of acting in the environment and participating in cultural discourse. The five principles are reminiscent of Papert’s constructionism (1980), according to which learning and understanding are more effective through hands-on experience with technologies and through the exploration of individuals. Indeed, the interaction with the environment plays a central role, and scaffolding is used to support learning.

This research aims to investigate the efficacy of an embodied design of tasks, elaborated with the 5 DPs in mind, using the digital Tracer, for supporting a covariational perspective. This exploratory study allows us as researchers to describe the process of resolution of a small group of university students involved in the experiment, by adopting embodied instrumentation as the theoretical framework, and to evaluate the embodied design of the tasks.

Theoretical framework: Embodied instrumentation

The theoretical framework of embodied instrumentation stems from a radical embodied perspective (Shvarts et al., 2021): learning can only happen through the body, and such a body meets the *affordances* of the environment (Gibson, 1986) including various digital tools, culturally-established artifacts, and even other individuals acting as observers. Body potentialities help to perceive the affordances of the environment and then act on them. *Perception* and *action* are connected in loops driven by intentionality and oriented towards the anticipation of the environment’s affordances. Fulfilling the intentionality, an individual (brain and body) and an artifact form what is called *body-artifact functional system* (Shvarts et al., 2021). When solving a mathematical task, a body-artifact functional system activates action-perception loops to recognize and act on mathematical affordances of cultural artifacts, leading to the emergence of *stabilized behaviors*. But what about when more than one individual is involved in the resolution of a task? Shvarts and Abrahamson (2020) refer to a student and a teacher as forming an *intercorporeal functional dynamic system* (IFDS), in which one student is a sub-system. When individuals are engaged in a coupled coordination of motor actions, various *attentional anchors* (Abrahamson & Sanchez-Garcia, 2016), i.e. foci of attention, may pop up as new perceptual structures and then contribute to stable performances. Within an IFDS, the two bodies attune to perform a common action and work together toward the resolution and conceptualisation of a mathematical task. The construct of *embodied instrumented covariation* (Bagossi, 2022) has already been introduced to condense the idea that a bodily-sensorial experience by using an interactive multimodal technology may mediate the elaboration of an intuitive idea of covariation enabling the learners to perceive the mathematical processes. How such an instrumented approach that integrates bodily involvement and coordination can aid in understanding covariational relationships, specifically learning about variations that occur simultaneously, is still scarcely explored. In light of the theoretical framework outlined and the insights coming from an explorative study like ours, the research question we are willing to focus on can be formulated as: How does an intercorporeal functional dynamic system engage with the digital Tracer to explore covariational relationships?

The digital Tracer

The digital Tracer (Figure 1) is the corresponding digital version of the physical Tracer (made of a wooden panel and two moving bars) introduced in Ferretti et al. (2024). Inspired by the mechanism of the game Etch A Sketch, the digital Tracer is an artifact conceived using the dynamic geometry software GC/htlm5, an updated version of Geometric Constructor, one of the free dynamic geometry software used in Japan since 1989 and developed by Prof. Y. Iijima. It is compatible with both iPad and



Figure 1: The digital Tracer

Android tablets and supports multi-touch interaction. The digital Tracer consists of two perpendicular axes (black lines in Figure 1), two points (A and B) belonging to these lines, and the two bars of the Tracer represented by other lines (brown lines in Figure 1) that pass through these points and are perpendicular to the previous lines. By dragging points A and B, it is possible to control the point of intersection of the bars so that it can follow a specific path. By activating the “Trace” command, clicking on the fourth lowest button placed at the bottom of the screen, the intersection point of the bars will leave a colored trace when moved. The digital Tracer can be used by a single user, but in our experiment, as we will explain in more detail, we had the students work in pairs on the screen showing the digital Tracer. To describe a given curve or function, A should be moved simultaneously with B, because it is through the combination of these movements that the trace is left on the path or not. In order to move the intersection point along the given path when there are two users, it is essential that they coordinate their movements.

Method

Embodied design with the Tracer

We designed three tasks involving the use of the physical and digital Tracer:

1. Follow four given traces (in pairs and then individually): steps, a broken line, a circumference, and a parabola (see Figures 2a and 2b).
2. Reproduce the trace (in pairs): two broken lines with different slopes and two parabolas with different widths.
3. Produce two parallel lines (in pairs).

Such mathematical tasks meet those embodied design principles outlined by Palatnik et al. (2023). Indeed, students’ bodies are involved in the learning process (DP1) through the movement of the two bars by means of their arms or fingers and the attunement made possible by the eye-hand coordination of the bodies involved. It offers sensorimotor interaction with the Tracer (DP2) through the movement of the two bars. The result of such interaction can be made visible by using a marker with the physical Tracer or by activating the Trace option on the digital Tracer (DP3). The formulation of the tasks mainly relies on the graphical register, but students are also invited to elaborate verbally on their actions (DP4), hence the activity fosters languaging (DP5). Eventually, we believe that the multimodal engagement prompted by such an embodied design is coherent with the mathematical

content of the activities. In fact, through the movement of the bars and the resulting intersection, students can feel the co-variation of the two bars, which may be associated with variables x and y .

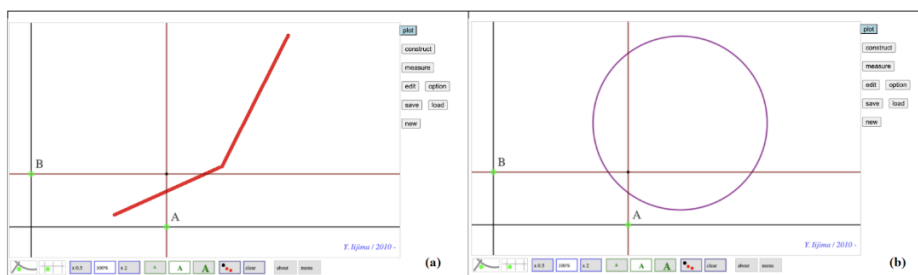


Figure: 2a. Broken line trace; 2b. Circumference trace

Case study: participants

In this paper, we focus on a case study involving three students from a Master degree program in Mathematics, Anna, Bea, and Chiara. During the experiment, the researcher gave the students three tasks, initially executed using the physical Tracer, and then performed with the digital Tracer. In what follows we will focus on the resolution in pairs of task 1 where students were asked to follow the traces of a broken line and a circumference (Figures 2a and 2b): this intentionality determines the assembly of the IFDS that we will analyze. During the experiment, Anna and Bea worked in pairs to solve the tasks using the digital Tracer: these three elements form the IFDS we are willing to investigate (Figure 3). Concerning the affordances of the environment we should consider that the two students had previously worked with the physical Tracer to solve the same tasks. Moreover, Chiara acted as an external observer who sometimes prompted some reflections at the end of the task.

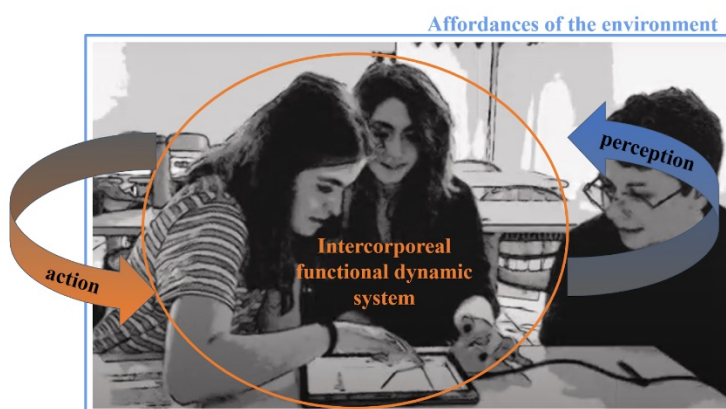


Figure 3: Intercorporeal functional dynamic system and environment's affordances. From left to right, Anna, Bea, and Chiara

Data collection and analysis

The entire experiment (90 minutes more or less) was videorecorded. In this contribution, a preliminary analysis of two episodes is presented to outline some considerations related to our research question. The episodes were analyzed qualitatively by focusing on the key elements of the embodied instrumentation framework: *intentionality* (Which is the aim of the IFDS?); *bodily experience* (Which embodied attunements between people emerge? Which attentional anchors emerge within the IFDS? Which action-perception loops emerge?); *stabilized behaviors* (Which stable techniques are adopted by the IFDS when using the digital Tracer?); *conceptualization* (Which

verbal formalizations are elaborated by the IFDS when solving the task?). The qualitative analysis of the two selected episodes was conducted through descriptive coding based on the four elements previously outlined. The authors analyzed the episodes independently and discordant interpretations were discussed and revised until reaching an agreement.

Results

Episode 1: Broken line

The two students were requested to follow the given trace of a broken line, and this is the goal they try to achieve by leaving a nice trace (*intentionality*). The legacy of the two girls' use of the physical Tracer is evident. In fact, they immediately know that one of them will have to consider only one bar and therefore point A and the other will consider the other bar and, therefore, point B (*intentionality*).



Figure: 4a. Chiara attuning with the IFDS; 4b. Different behaviors of Anna and Bea while moving the bars; 4c. Final trace produced

Their bodies attune to perform this action by positioning their finger on the screen and adjusting their posture. Even Chiara who is not involved in performing actions with the Tracer attunes herself, so becoming part of the IFDS (Figure 4a). Having tacitly agreed on the points to be moved by each of them (and this derives from their previous experience with the physical Tracer, in which Anna moved the abscissa bar and Bea the ordinate bar) (*stabilized behavior*), the two girls silently and with concentration perform the movements. Anna and Bea agree on the starting point (bottom left of the broken line), and after positioning the intersection point of the two bars in that extreme, they start moving the two bars, and they direct their attention to the intersection point of the two bars (*bodily experience*). While Anna moves her finger along the x -axis, the finger of Bea tends to follow the curve and does not stay strictly along the axis, as can be observed in Figure 4b (*bodily experience*). The hand-eye coordination is evident: each person observes the trace progressively left by the intersection point, while simultaneously moving points A and B on the axes with their index fingers (another *stabilized behavior* with which they move the points). Only at the end, they hint a smile, a little bitterly, an expression that can be observed even on Chiara's face: they have finished this performance, but the result is not optimal (Figure 4c). Anna says "the first trait ... um ... I can hardly see because I put my finger on it" (meaning that she did not have a full view of the line to trace because her hand and wrist partially covered the line - *bodily experience*).

Episode 2: Circumference

In this episode, Anna and Bea work together to trace the given circumference, and they agree on starting from the lowest point of the circumference and moving in a counterclockwise sense (*intentionality*). Anna perceives a lack of fluidity in tracing the circumference, especially comparing the previous experience she had with Bea with the physical Tracer. When approaching the extreme

right point of the circumference, she perceives from the left blue trace (Figure 5a) that she is mistakenly progressing in the positive direction of the x -axis (*bodily experience*), and the two girls' attunement is interrupted (Figure 5b). Hence, a disruption of the IFDS happens, which is made evident by a joint laugh of the girls breaking the tension of the previous coordination. Anna says, "Before, I was used to stopping when we had to change direction, I'm not used to doing everything in one go", and again: "I'm not used to changing direction quickly" (*conceptualization*). These different behaviors emerge even though both students are directing their attention to the intersection point rather than on their fingers (*intentionality* and *bodily experience*). Bea is the first who observes that she and Anna behaved differently: "We do two completely different things: you [Anna] always keep your finger in the same place, I on the other hand, I don't know why, but with my finger, I follow the progression [of the trace] a bit" (*conceptualization* on the *stabilized behavior*). Anna reacts by saying, "I always keep my finger on the same point because first I go in one direction and then I have to change; I move within a very small range." Throughout this dialogue, the eyes of the three girls, even those of Chiara who is silent, all look at the tablet screen and follow the finger movements on it. Chiara has an attentive and focused gaze; sometimes she smiles, and this is a tacit confirmation of agreement with the colleagues' statements (*bodily experience*).

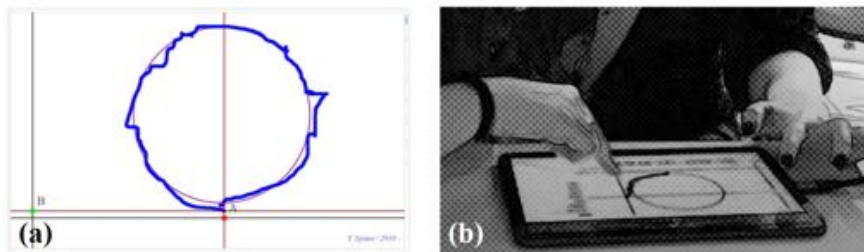


Figure: 5a. Final trace produced by Anna and Bea; 5b. Disruption of the IFDS

Discussion

The intentionality of the IFDS is strongly focused on drawing the traces as precisely as possible. Attention remains concentrated on responding to this intentionality: Anna and Bea do not look at their hands as they follow the trace, nor at the colored trace that gradually emerges with their movements. Rather, they look at the trace assigned to them. While performing the tasks, Anna and Bea attune and engage in loops of action-perception in which they go back and forth, move together, and stop, elaborating on stabilized behaviors. However, little conceptualisation emerges. In previous experiments, in which we had first involved primary school children (Ferretti et al., 2024) and then school teachers in training, free observations and then conceptualisation of the lived experience spontaneously emerged. In this experimentation with undergraduate mathematics students, we realized that conceptualisation does not naturally arise, but rather needs to be stimulated. The embodied design should certainly provoke some initial insights, but then a reflective stage is needed to trigger conceptualization (Alberto et al., 2022). Considering the qualitative nature of the tasks here proposed, such a reflective stage could be achieved through open questions, warmed up by a researcher or a teacher, asking for the what, how and why of the sensory-motor experience.

It is worth drawing attention to another aspect of our environment. Shvarts and Abrahamson (2020), define the IFDS as consisting of a student, working with an artifact, and a tutor. We, on the other hand, have two peers, Anna and Bea, interacting simultaneously with the artifact. Notably, Anna and Bea in enacting their actions with the artifact do not speak. They feel and act. Then, in our

environment, there is also Chiara. She has the role of observer but becomes somehow part of the IFDS: she attunes, nods, smiles, and follows with her gaze all the movements of her colleagues. Moreover, at the end of the performance of tracing the circumference, all three burst out laughing: all three knew how tense they were in their effort to trace the circumference as smoothly as possible.

Final remarks

The IFDS proves to be a valid construct for investigating how the digital Tracer can be used to explore covariational relationships. Indeed, the interactivity of an artifact like the Tracer may help the learner give meaning to functions, or relationships, as dynamic entities, as opposed to a static view of functions supported by the other approaches to functional thinking, such as the correspondence one. Moreover, a specific task design such as the one proposed here may support a smooth image of change, in contrast to a chunky one: indeed, the focus is on the sensorial perception of the variations of the quantities at stake, favouring a qualitative approach. However, retrospectively analyzing the videos, we realized that the university students created windows for conceptualization, but the proposed task design did not allow full use of them. Even though in this paper we present two episodes focused on the digital Tracer, the whole sequence allows for a comparison between the physical and digital Tracers and for reasoning as to which of the two artifacts has more potential to achieve the goal of the task. For example, when Anna and Bea observe their failure in making a smooth trace with the digital Tracer, we could have explicitly asked them to compare their experience with the one previously had with the physical Tracer: the concepts of smoothness and approximation might have come out more. Therefore, we reiterate that conceptualisation needs to be triggered and more could be done in the design to prompt a reflection on covariational aspects. We believe that such an embodied approach may help students not only perceive, given the different speeds with which one can move the bars, what variation means, but also conceptualize the given trace as a dynamic object. In this way, it is possible to see the covariation between A and B not only as a result of the coordination of the two users' actions, expressed as a colored trace, but also to feel and perceive the covariation through the bodies and their movements' coordination. Eventually, our embodied task design, elaborated to support the conceptualization of covariational relationships, focuses on a qualitative approach so far. Hence, a quantitative stage (Alberto et al., 2022) could be introduced to reinforce the idea of covariation between quantities, for example by inserting a grid in the digital Tracer. These will be points of future reflection.

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Principles for professional development to improve teachers' PCK for teaching mathematics with technology

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This paper proposes principles (CERICS: Creativity, Explicit PCK, Reflection, Investigate student thinking, Collaboration, and Self-efficacy) for professional development (PD) aimed at increasing teachers' PCK for teaching mathematics with technology, specifically general purpose tools such as CAS or dynamic geometry systems. Given the ubiquity of technologies and affordances for teaching and learning, principles to underpin PD may support design, evaluation, and research of PD programs.

Keywords: Technology, mathematics teachers, professional development, PCK

Introduction

Worldwide considerable resources have been devoted to mathematics teacher PD. For example, National Council of Teachers of Mathematics institutes (USA)¹, National Centre for Excellence in the Teaching of Mathematics PD (UK)² and technology focused PD through departments of education and private providers. This range of structures is also present in our countries, Germany and Australia. One example, the Deutsches Zentrum für Lehrkräftebildung Mathematik (DZLM)³ is a nationwide organization in Germany run by a formal consortium of universities collaborating with government to deliver, and conduct research about, mathematics teacher PD. There are several motivations for teachers to undertake PD related to teaching mathematics with technology. These include:

- curriculum documents that include references to technology (e.g., VCAA, n.d.)
- teachers' desire to develop pedagogical content knowledge to take advantage of the new opportunities that technologies offer for teaching and learning (e.g., Thurm & Barzel, 2020)
- assessment or examinations that allow technology use (e.g., Wiliam & Stacey, 2013)

The place of technologies in curriculum documents varies, with several countries (or states) referencing technologies (or technology). For example, the mathematics curriculum in Victoria, Australia states what students should do with technology (e.g., “find percentages of quantities and express one quantity as a percentage of another, with and without digital tools”; VCAA, n.d.). Similar statements exist in some German curricula. An implication is that teaching should support students' understanding of mathematics and their ability to use technology to do mathematics.

In addition to curricula obligations, the intention to use digital tools can be impacted by teachers' perceptions about what it means to teach and learn mathematics with such tools (Pierce & Ball, 2009). Technologies can impact the epistemic value of techniques (Artigue, 2002), with Dick (2008)

¹ <http://www.nctm.org/Conferences-and-Professional-Development/Institutes/>

² <https://www.ncetm.org.uk/>

³ German Centre of teacher education in mathematics, www.dzlm.de

stressing the need for cognitive fidelity of technologies so they are “illuminating mathematical thinking processes [rather] than simply arriving at “black box” final results as efficiently as possible” (p. 337). Rethinking teaching with technologies is not new, with Pierce and Stacey (2010) suggesting teachers “re-balance emphasis on skills, concepts, applications” (p.6), alongside other pedagogical opportunities. The process of instrumental genesis (Artigue, 2002), where a teacher learns to use technologies and has their work shaped by such tools, can support teachers to use technologies to achieve pedagogical outcomes. Given instrumental genesis can take time and teacher perceptions play a role in uptake of technology, the evolution of pedagogy is complex, and PD is needed.

PD on technologies and mathematics: Recent findings

Teacher PD to improve teaching and learning mathematics with technologies has been reported extensively. For example, Thurm et al. (2024) reviewed research on PD for teaching mathematics with digital technology from 2010 – 2021, resulting in the need for consolidated PD principles to realize good PD programs. The review noted a wide range of different use and explication of principles. The need for collaboration was a key principle for PD across several studies, with Mouza et al. (2014) promoting learning-by-design where teams address pedagogical problems through technology solutions and Ratnayake et al. (2020) proposing teachers collaborate to design tasks and share learning. Emergent design principles call for technology tasks that present lesson “hiccups” for teachers, followed by consideration and practice of possible classroom responses (Clark-Wilson & Noss, 2015); this can result in teachers dealing in-the-moment with occurrences in lessons, with Pierce et al. (2011) noting that technologies can be idiosyncratic and work differently to pen-and-paper maths. Reiten (2021) noted that teacher development of mathematics content knowledge for teaching should also be a consideration for PD. Thurm et al. (2024) reported the need for consideration of the TPACK model by Mishra & Koehler (2006), highlighting the principle that PD should consider the interplay between content knowledge, pedagogy and technology to build teachers’ competences. Stressing authentic experiences (e.g. Mouza et al, 2014), supported the principle of trialing and reflecting, and that PD should be over time. In addition, teachers’ perceptions play a role in uptake of technology in teaching (Pierce and Ball, 2009). Thus, active learning, connections between theory and practice and providing opportunities for reflection are important.

Goldsmith, Doerr and Lewis (2014), in a review of 106 refereed journal articles from the mid-eighties to 2008, reported that teacher learning “is often incremental, nonlinear, and iterative, proceeding through repeated cycles of inquiry outside the classroom and experimentation inside the classroom” (p. 20). This supports learning fostered through PD programs where the external learning (i.e., groups of teachers meet with a PD provider) is integrated with phases where teachers are involved in trialing new approaches in the classroom (e.g., Thurm & Barzel, 2020). Integral to this is the necessity to reflect on classroom implementation and student learning and to incorporate this reflection into subsequent external learning phases.

Internationally there have been studies on the effectiveness of general PD programs (e.g. Timperley et al., 2007). A meta-study of literature (Barzel & Selter, 2015) provided the basis for the DZLM principles for PD in mathematics:

- competence-orientation (e.g. aims focussed on mathematics CK and PCK; aims clearly articulated to teachers)

- participant-orientation (e.g. cater for different teacher expertise and beliefs; foster teachers' active participation in sessions)
- various instruction formats (e.g. multi-day programs where teachers implement new ideas in classes between sessions; individual work and presentations within a session)
- stimulating cooperation (e.g. promote group work within sessions and professional learning teams between and after sessions)
- fostering reflection (e.g. on tasks, classroom arrangements, student solutions and teacher self-reflection)
- case-relatedness (e.g. use of videos of classrooms and sample student work)

PD related to teaching mathematics with technologies should follow the same principles; however, there are extra dimensions to be considered related to teaching with technology.

- Development of a teacher's individual technological skills and knowledge (e.g., Mishra & Koehler, 2006)
- Pedagogical considerations associated with teaching with technologies (e.g., Pierce & Stacey, 2010)

Both dimensions should be intertwined in each phase of the PD process. In both cases, selection of tasks and the features of good tasks to promote development of good practices are crucial.

Given the rapid change in technologies over the last two decades, PD should be focused on adaptable principles and be responsive to new opportunities that we cannot even imagine in the current climate. We now propose principles for teacher PD related to technologies and mathematics that may guide conceptualisation, evaluation or research of PD programs for teaching mathematics with technology.

Methodology

On the basis of Thurm et al. (2024), we have elaborated on missing principles in teacher professional development for teaching mathematics with technology. The CERICS principles (refer to Table 1) were developed using an iterative process by the two authors, each of whom has more than two decades of experience in research-based design, delivery and evaluation of PD programs. Theme analysis (Strauss & Corbin, 1990) informed the approach used to identify principles within the framework. The approach is outlined below.

Stage 1: First we discussed and documented initial drafts of key principles and approaches for PD with technology, based on experience and literature (see above).

Stage 2: To refine the list each author made individual lists of key aspects of PD programs designed. Two examples of key aspects are "Learn technology in the context of a topic which is relevant to the participants....curriculum documents provide guidance here" and "Mix tasks, features and PCK-reflection from the very beginning".

Stage 3: The authors discussed the lists to clarify any phrases and to determine where use of terms was different in the cultural context of the two authors.

Stage 4: Phrases were placed in clusters aligned with the initial drafts of key principles developed in stage 1. Any phrases that did not fit into a cluster were collected and reconsidered separately, with the clusters being reworded where required.

Stage 5: The draft principles were analysed for clarity and completeness by a third mathematics education researcher with decades of experience in conceptualising and developing PD.

Overview of the CERICS Principles

Table 1: CERICS - Principles for Professional Development for Teaching Mathematics with Technology

Focus	Principle
Creativity	Promote teachers' creativity in the design of lessons and /or assessments that balance technology, pen-and-paper and mental strategies
Explicit PCK	Make pedagogical content knowledge for teaching mathematics with technology explicit
Reflection	Plan focused reflection by teachers on the role and impact of technology
Investigate student thinking	Include discussions of classroom scenarios and/or student work samples to highlight affordances and constraints of technology
Collaboration	Plan for interaction and exchange between teacher participants to share pedagogical approaches and experiences of teaching with technology
Self-efficacy	Reduce teachers' anxiety about using technology and promote their risk taking in use of technology

Creativity - Promote teachers' creativity in design of lessons with technology

As teachers create tasks and lessons they need an appreciation that technologies have a place alongside pen-and-paper (p&p) and mental strategies in mathematics classrooms. At different times, and in different situations, it is beneficial to use (or not use) technologies. PD sessions must develop insight into an appropriate balance between technologies, p&p and mental strategies, which is unlikely to be an equal share for each. This can be achieved through careful selection of activities that highlight the power of technologies, p&p and mental strategies for different purposes. Considerations such as this focus teachers' attention on finding a balance to optimize students' understanding of mathematics. Discussion of features of carefully selected examples can highlight features of examples to enrich mathematics classes. Development of creativity involves teachers being able to identify general principles for teaching from examples and to create specific examples from general principles. This can be achieved by having teachers adapt new ideas and tasks from a PD session to achieve a given goal or for use in their own teaching practice. PD sessions should encourage creativity, to help teachers to explore and experiment in their teaching.

Explicit PCK - Make PCK for teaching mathematics with technology explicit

Having technology present will not necessarily affect the teaching and learning of mathematics; appropriate pedagogical approaches are essential for change. PD should focus on the influence, benefits and constraints of the use of technologies for teaching and learning mathematics, rather than just teach technical skills. Learning of technical skills should be targeted at the appropriate level for

the specific teachers participating in the PD. For novices start with an instructional phase on technical skills so that teachers develop some familiarity and confidence with syntax whilst starting to reflect on implications for teaching. Initial tasks should be designed to show the benefit of the technology and to promote discussion of pedagogical advantages; this can provide a model for teachers to use with students in learning technologies. For experts there is scope to start with consideration of PCK, rather than technical skills. Teachers should critique pedagogical approaches and compare different pedagogical use of digital technologies. They should identify affordances and limitations of use of technologies for teaching and for developing students mathematical understanding.

Reflection - Reflection by teachers on the role and impact of technology

This principle relates to two aspects of reflection by teachers when learning to use and to teach with technology; PD formally plan to address both and that research findings are discussed with teachers. Firstly, teachers should be encouraged to reflect on the displays produced by a technology; this builds knowledge of conventions, limitations, and issues that students might face (e.g. Pierce, et al, 2011). The second aspect of reflection relates to identification of pedagogical issues; this can encourage consideration of new teaching approaches, as well as rethinking the role of technologies in teaching and learning (e.g. Clark-Wilson et al., 2014). The following examples elaborate this principle.

Limitations of technologies: Include examples to highlight limitations of the technology. Raise teachers' awareness that it is useful to investigate limitations of a particular technology. For example, an applet to scaffold student learning of 'do the same to both sides' to solve equations might be programmed so that it is only possible to use whole numbers, rather than fractions or decimals.

Syntactical requirements: Technology syntax may be different to p&p mathematics. For example, there can be challenges posed through conventions in variable naming using multiple representations with CAS (Pierce et al, 2011). Discussion can help teachers make links between mathematical work and syntax requirements, informing pedagogical decisions. Teacher reflection on syntactical constraints can help teachers to predict potential classroom difficulties related to programming, rather than mathematical understanding.

Expect unexpected displays: Reflection on unexpected or interesting technology displays can prompt teacher learning. Examples where unexpected results occur prompt discussion of the need to expect the unexpected and can prompt teachers to include such discussion in their own classes to make links between p&p maths and technology displays.

Pedagogical reflection: Planning for reflection should be included in all aspects of PD to highlight technical and pedagogical considerations that arise as technology is used (e.g., consideration of functional and pedagogical use of technologies in discovery work). Pedagogical choices about when, where and how technologies should be used provides impetus for teacher reflection, as well as consideration of the role of technologies in tasks, assessment and for learning mathematics.

Investigate student thinking - Include discussions of classroom scenarios and/or student work samples

Integrating cases of sample student work and classroom scenarios enables the benefits and constraints of technologies to be visible and concrete (Thurm, et al., 2023). Consideration of sample student work and classroom scenarios assists teachers in anticipating typical situations and supports them in

dealing with the challenges in learning to teach with a new technology. Student work samples, either pen-and-paper or screen-captured through technology, can initiate discussions about PCK and show that the range of solutions is extended with technologies available. Classroom scenarios, through video of real classes or via simulations of virtual classrooms, can support understanding and development of teaching approaches and classroom interactions to highlight the specific complexities of teaching with technologies. These resources promote discussion of responses to given scenarios, assisting teachers to be prepared to react appropriately in different situations.

Collaboration □ Plan for interaction and exchange between teacher participants to share pedagogical approaches and experiences of teaching with technology

One characteristic of effective PD is to foster collaboration during, after and between sessions (e.g., Gräsel et al., 2007). For PD related to technology in mathematics this can be achieved by initiating teamwork when designing lessons and assessments. Drawing on teachers' experiences by asking them to share and discuss interesting examples, as well as assist each other to address technical or pedagogical questions and ideas can promote collaboration and reduce teachers' anxiety.

Self-efficacy - Reduce teachers' anxiety about using technology and promote their risk taking in use of technology

PD should empower teachers to develop their technical expertise and ability to consider pedagogical issues associated with learning to teach with technologies, and improve their self-efficacy (Thurm & Barzel, 2020). Fundamental to this is that teachers are not anxious about learning to use, or teach with, technologies. Anxiety of novices can be reduced if unexpected displays are not seen as problematic, but instead as interesting to discuss (Thurm & Barzel, 2020). Emphasise that teachers' knowledge of mathematics helps to understand and navigate technologies, including idiosyncrasies of syntax (Pierce, et al., 2011), and that teaching with tools does not require memorization of key sequences. Encourage teachers to implement a small activity with their students as a starting point, to build confidence in using technologies more extensively. This principle relies on the previous principle of collaboration, as discussing classroom trials will provide support for PD participants.

Concluding remarks

This paper proposes, CERICS, which provides principles for teacher PD related to teaching mathematics with technologies. As discussed previously, CERICS builds on principles for effective PD outlined in the literature, but endeavours to show the special nature of PD where technologies are concerned. For example, the interplay between technologies, pen-and-paper and mental strategies is a unique consideration to PD for teaching mathematics with technologies.

One goal of CERICS is to capture the complexity of PD related to teaching mathematics with technologies; this PD requires much more than just the teaching of syntax or provision of lessons for participants to use with their classes. If teachers are to develop the PCK to incorporate technologies into their teaching, then we propose the need for considerable reflection and discussion about the role of technologies in teaching and learning mathematics, as well as the impact of technology availability on task design. We believe CERICS will provide useful principles for PD designers and researchers endeavoring to design or critique PD programs.

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How learning platforms support mathematical modelling: An exemplary approach via the ASYMPTOTE system

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Mathematical modelling (MM) aims to investigate and explore real-world situations. It can be supported by digital tools in various ways, for example by the provision of communication and feedback functionalities. Since these functionalities are provided by most learning platforms, one might expect that these tools facilitate MM teaching and learning. However, this is not the case, as learning platforms hardly offer rich and complex MM tasks. Against this background, this paper presents the ASYMPTOTE system as an exemplary learning platform and examines how it can support MM activities. Our analysis shows that ASYMPTOTE provide functionalities to enhance the student's work at a whole as well as multiple modelling steps of MM tasks. Considering that several of the discussed functionalities can also be found in other learning platforms, we conclude that these tools can generally support MM – as far as learning platforms provide rich and complex MM tasks.

Keywords: ASYMPTOTE system, intelligent tutoring systems, learning platforms, mathematics education, mathematical modelling.

Introduction

Mathematical modelling (MM) is characterized as “solving real world problems by means of mathematics” (Blum, 2015, p. 77). This competency describes the ability of students to apply their mathematical knowledge in order to deal with real-world situations (Blum, 2015; Niss & Højgaard, 2019). In doing so, MM aims to show the relevance of learning mathematics (Siller et al., 2023) and paves the way to linked mathematics education to other STEM disciplines (Goos et al., 2023).

Despite the huge potential of MM, the integration of MM into mathematics lessons is reported in the literature as a major challenge for teachers and students: teachers experience a number of issues related the preparation, conduct or evaluation of MM activities, such as the selection of MM tasks, appropriate interventions on the student's work process as well as its formative assessment (Borromeo Ferri & Blum, 2010). Also for the learners, a variety of issues are well-known in literature, e.g., dealing with the openness of MM tasks, finding an appropriate model of the real-world situation and validating the calculated solution (Klock & Siller, 2020). In view of the named issues of teaching and learning, it is not surprising that several authors claim the high potential of supporting MM activities by the means of digital technology (e.g., Cevikbas et al., 2023; Siller et al., 2023). In this paper, we aim to investigate how digital learning platforms can support the teaching and learning of MM. To this end, the ASYMPTOTE system is presented and examined as an example.

Digital Support of Mathematical Modelling

Based on the work of Blum and Leiss (2007), the process of MM is commonly depicted as an idealized cycle of seven steps that have to be taken in order to solve a real-world problem (see Figure 1, black numbers). Starting with the *understanding* a real situation, one constructs a situation model. By its *simplifying and structuring* and its *mathematization*, i.e., its transfer from the rest of the world to the world of mathematics, a mathematical model is set up. After *working mathematically*.

the mathematical results are *interpreted, validated* and *exposed* in the light of the given real-world situation (Blum and Leiss, 2007).

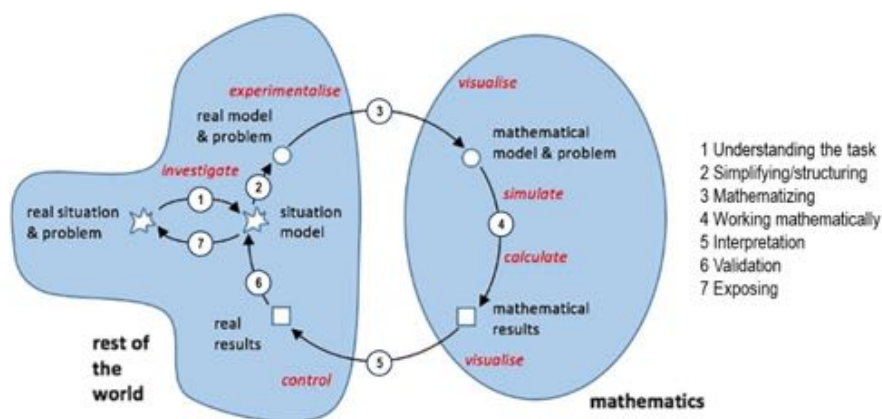


Figure 1: The MM cycle by Blum and Leiss (2007) and the support of MM steps by digital technology according to Cevikbas et al. (2023); Figure adapted from Cevikbas et al. (2023, p. 4)

In their literature reviews, Cevikbas et al. (2023) highlight that the MM steps can be supported digitally in various ways (see Figure 1, red activities). For example, digital technology might help to *investigate* the given real-world situation. In addition, it can support *visualizing* the given data as well as the mathematical results. Further, *experiments* and *simulations* can be carried out with digital tools, while mathematical *calculations* can be outsourced to technology. Lastly, the *control* and validation function can be taken over by digital tools, giving students an idea of their result's correctness.

Table 1: Categories of digital technology for MM activities based on Siller et al. (2023)

Aim	Category	Example	Support of MM
Extension of mathematical capacities	digital resources specific to mathematics	Computer algebra systems & dynamic geometry software	Calculation Visualization Control
	digital resources not specific to mathematics	Spreadsheets	Calculation Visualization
Access to extra-mathematical knowledge	digital resources for communication and information gathering	Encyclopedia & platforms providing communication and feedback functionalities	Investigation Visualization Control
	digital resources for simulated, virtual and augmented worlds	Digital simulations & virtual reality applications	Experimentation Visualization Simulation

Facing the huge variation of digital tools that might support MM, Siller et al. (2023) differentiate digital technologies from an instrumental perspective. In a first place, they distinguish the aim of the technology use, namely to extend mathematical capacities or to access extra-mathematical

knowledge. In a second place, the functionality of the digital technology is used to form four categories in view of MM. These are summarized in Table 1.

Methods and Research Question

Based on the previous considerations, it can be assumed that digital tools are in general conducive to MM education. However, recent publications such as Cevikbas et al. (2023) and Siller et al. (2023) do not explicitly focus on learning platform. To make an initial contribution to the role of these digital tools for the teaching and learning of MM, we raise the following research question: *To what extent can digital learning platforms support mathematical modelling (MM)?*

To answer this question, we first take a look at learning platforms in general and argue how learning platforms in general can support MM activities. Secondly, we present the ASYMPTOTE system as an exemplary learning platform and describe how MM activities can be prepared, implemented, and evaluated with this particular tool.

In this paper, learning platforms, such as (intelligent) tutoring systems, are understood as digital tools that provide an interactive learning environment, in which students process digital tasks by the help or adaptive support and feedback (Thurm & Graewert, 2022; Steenbergen-Hu & Cooper, 2013).

Mathematical Modelling and Learning Platforms

As outlined by Siller et al. (2023), digital technology can support MM activities in multiple ways. For example, digital tools can facilitate communication and information gathering (Siller et al., 2023). This support can, hypothetically, be provided by most digital learning platforms.

In view of the modelling cycle by Blum and Leiss (2007) and the possibilities to support MM steps (cf. Cevikbas et al., 2023; Siller et al., 2023), digital learning platforms can support the students' MM activities as follows: For *investigating* a real situation, a learning platform can provide information on the particular topic of the MM task, e.g., by storing or linking information such as texts or videos or data sets. In view of *visualizing* data, illustrative diagrams, tables or videos can be offered. The *control* and validation of MM results can be supported by learning platforms by the provision of feedback on the student's task process and the calculated solution. In addition, support by the embedding of tools for *calculations*, *simulations* and *experimentations*, such as spreadsheets or dynamic geometry software, is well imaginable for learning platforms in order to extent students' mathematical capabilities (cf. Siller et al., 2023).

Based on these considerations, one can assume that learning platforms provide promising functionalities to support MM. However, as stated by Thurm and Graewert (2022), current learning platforms hardly offer any MM tasks: In a study on the three German-speaking learning platforms *Anton*, *bettermarks* and *Mathegym*, the authors investigated 106 tasks on the multiplication of fractions as well as the Pythagorean theorem. 15 % of the 106 tasks analyzed showed a reality-based situation but could be solved by only one solution step. Only three more tasks demanded for a multiple-step processing, while not a single task demanded for the development of an appropriate mathematical model or the validation of the model.⁴ In view of this situation, we aim to exemplarily

⁴ In particular, the selection of the topics does not seem to effect this result: A wide range of MM tasks is offered in the literature for the Pythagorean Theorem (e.g., Rellensmann, & Schukajlow, 2017).

illustrate how digital learning platforms can support MM activities. In order to do so, we present the ASYMPTOTE system in the following.

Mathematical Modelling and the ASYMPTOTE System

ASYMPTOTE stands for *Adaptive Synchronous Mathematics Learning Paths for Online Teaching in Europe*. It is the acronym of both an Erasmus+ Strategic Partnership (2021–2023; www.asymptote-project.eu) and the resulting learning platform developed in view of Covid-19 pandemic. The ASYMPTOTE system contains two components, namely a web portal for teachers and a mobile app for students: the web portal enables teachers to select or create samples of tasks, a so-called *learning graph*. Via a smartphone app, students can access the learning graph (cf. Barlovits et al., 2022).

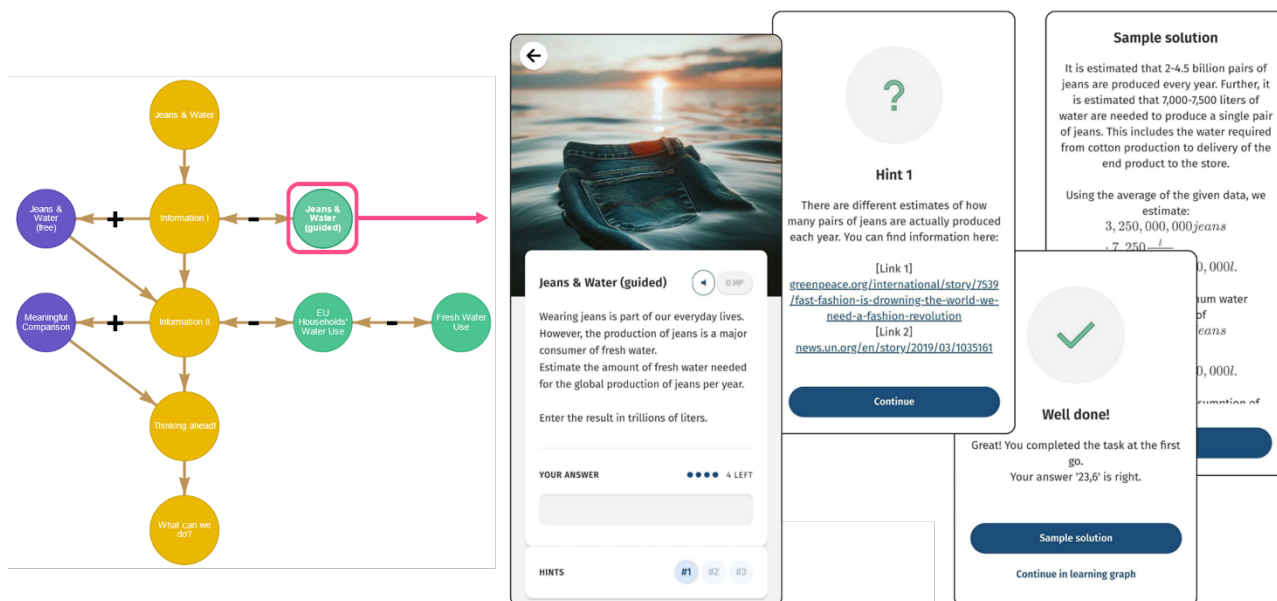


Figure 2, left side: The structure of the learning graph “Jeans & Water”; right side: Processing the support task “Jeans & Water (guided)” in the ASYMPTOTE app

An ASYMPTOTE learning graph contains several main tasks (yellow) that are arranged as a linear sequence. For each main task, both support tasks (green) and challenge tasks (purple) can be added: Whereas the support tasks provide related tasks on a lower level and can be seen as further guidance or repetition of contents, challenge tasks provide related tasks on a more difficult level. They aim to further encourage students who solved the related main task. Based on this categorization of tasks, the ASYMPTOTE system proposes students to work on tasks on a more appropriate level depending on their performance in a previous task, i.e., a basic form of adaptivity is implemented (cf. Plass & Pawar, 2020). Simultaneously, the system only advises to students and thus promotes self-regulated learning (cf. Greene et al., 2011). An exemplary learning graph is shown in Figure 2 (left side).

Besides this guidance through the learning graph and its different task categories, the ASYMPTOTE app (smartphone/tablet for Android and iOS each) contains several functionalities that support students in solving the tasks. The app displays the task formulation and, if applicable, a picture related to the task. Students can recall up to three hints for each task and enter their solution to receive immediate feedback on its correctness (see Figure 2, right side).

The students’ progress on the learning graph can be monitored by the teacher via the so-called *Digital Classroom*. This includes the logging of the students’ interaction with the app, such as the use of hints

and the entered solution. In addition, a student-teacher-chat for text, audio and video messages is provided. For more information on the ASYMPTOTE system, we refer to Barlovits et al. (2022).

Task Example “Jeans & Water”

In order to demonstrate the realization of MM in ASYMPTOTE, the task example "Jeans & Water" is presented below, which deals with the immense water consumption of global jeans production. This MM task is pre-structured in the learning graph shown Figure 2 (left side). It can be accessed via the ASMYPTOTE app by entering the code *g34928*. The task is formulated as follows:

Wearing jeans is part of our everyday lives. However, the production of jeans is a major consumer of fresh water. Estimate the amount of fresh water needed for the global production of jeans per year. Find a meaningful comparison to illustrate the amount of water.

This task can clearly be divided into two parts. Part (a) aims at calculating and estimating the water consumption of the global jeans production. In view of the modelling cycle by Blum and Leiss (2007), this part aims in particular to *simplify*, *structure* and *mathematize* a real situation. Part (b) asks for finding a meaningful comparison to illustrate the amount of fresh water used by the global jeans production. In doing so, it aims at facilitating students to *interpret* and *expose* their mathematical results in reality. Both parts of the task are represented in the learning graph. We provide a free and a guided version for the two parts each, which can be accessed by the students on their own choice. This allows students to independently decide if they want to conduct the process in the free version of part (a) or the guided version of part (b).

On a contextual level, the *guided version* of part (a) offers support by three content-specific hints. They contain links to different websites giving information about the number of jeans produced per year (hint 1) as well as the water consumption per jeans (hint 2). In addition, hint 3 emphasizes as content-specific strategy is to handle the varying information from the different websites, namely to average the information found on the websites, e.g. in relation to the number of jeans produced per year. The first and second hints thus focus on the *simplifying and structuring* as well as *mathematizing*, while the third hint refers to step of *working mathematically*, i.e., to the inner-mathematical calculation process in order to derive a mathematical result. On the other hand, the *free version* of part (a) only provides a strategic hint as proposed by Schukajlow et al. (2015), namely to research relevant data on the internet. Here, neither specific assumptions nor inner-mathematical procedures for deriving a mathematical result from the given real situation are proposed.

For part (b), the *free version* only asks for finding a meaningful comparison. Here, related to Schukajlow et al. (2015) only the strategic hint to *interpret* this comparison in the real situation, i.e., the task's context, is given. The *guided version* of part (b) proposes a specific comparison, namely, to compare the yearly fresh water consumption of the global jeans production and of private households in the European Union. In the learning graph, this is guided step-by-step by two support tasks (see Figure 2, left side). They offer links for online recherche (daily private water consume in the EU), indicate relevant information (number of households in the EU), and provide context-related information on how to deal with the relevant data. It leads to the surprising result that with 24 trillion liters per year, the global jeans production consumes approximately consumes as much water as all

private households of the European Union!⁵ The *guided version* thus paves the way to *interpret* the results a part (a) by illustrating the almost unimaginably large consumption of fresh water in global jeans production using a concrete and tangible real-world example.

To *expose* the gained knowledge and to critically reflect on it, the learning graph includes two further tasks (see Figure 2, left side). The first one aims at a global perspective, as it asks to connect the calculated results with the United Nation's (2015) Sustainable Development Goals of *clean water and sanitation* as well as *responsible consumption and production*. The second raises the question "What can we do?" to focus on personal and emotional engagement with the mathematical results.

Mathematical Modelling and ASYMPTOTE

In the given task example, ASYMPTOTE can support MM in multiple ways: With a reference to Cevikbas et al. (2023) as well as Siller et al. (2023), it can be stated that the presentation of the task formulation together with a picture can introduce and *investigate* the real situation, which is the starting point of the MM cycle (cf. Blum & Leiss, 2007). In addition, the *investigation* of the task's context is supported by the hints. They either provide references on specific websites [part (a), guided version] or propose internet research on a strategic level [part (a), free version].

Furthermore, the ASYMPTOTE app supports students in *controlling* their mathematical results by two functionalities. Firstly, the answer validation checks the students' mathematical results on a numerical level. Secondly, the sample solution allows students to compare their assumptions, their mathematical work process as well as real-world interpretation of the mathematical results with a possible solution. Lastly, the learning platform ASYMPTOTE allows to store an image of students' notes in the Digital Classroom. This also can be seen as contribution of the tool for the *visualization* of the mathematical results in order to control, validate and debrief them in a plenary discussion with the teacher. Furthermore, the conduct and debriefing of MM tasks carried out with ASYMPTOTE is supported by the tool as the teacher can monitor the students' work process on the task in real-time.

The learning graph structure enables students to either apply on a solution strategy to a specific model [parts (a) and (b), guided versions] or to develop an own strategy and model [parts (a) and (b), free version]. For the *guided versions*, the ASYMPTOTE learning graph pre-structures the mathematical process [part (a) and (b), guided version]. It particular addresses lower-achieving students by providing the needed guidance in dealing with the open-ended MM task. With the *free versions*, the learning graph allows the use of different models of the students' choice but still does not accept arbitrary results since the students' solution is automatically validated by the ASYMPTOTE app based on a pre-defined interval of suitable answers. In doing so, it particularly facilitates higher-achieving students to work independently on the given MM task. Regardless of whether students choose to work on the free or guided parts, they are encouraged by the structure of the learning graph to undergo the idealized modelling cycle (cf. Blum and Leiss, 2007) as described above. In particular, the task places emphasis on the MM steps of *simplifying and structuring*, *mathematizing*, and *interpreting*.

⁵ To enable effective debriefing in class – especially in view of the free version of part (b) – students are asked to upload an image of their calculations via the chat tool. Thus, their results are stored in the ASYMPTOTE Digital Classroom.

Discussion

Recent research underlines the potential of digital tools for mathematical modelling (MM), e.g., by collaboration and feedback functionalities (cf. Siller et al., 2023). On the one hand, most learning platforms offer such functionalities, while on the other hand current learning platforms hardly offer MM tasks (Thurm & Graewert, 2021). Facing this situation, this paper examined from a theoretical perspective in how far these digital tools can support MM teaching and learning.

To do so, we designed and analyzed the MM task “Jeans & Water” in the example of the learning platform ASYMPTOTE. Our results show that the ASYMPTOTE system – as other learning platforms – could provide various digital supports for MM activities, such as *visualizing*, *investigating* and *controlling* (cf. Cevikbas et al., 2023; Siller et al., 2023). Particularly, the *controlling* of MM results can be outsourced to learning platforms, e.g., to functionalities such as answer validation, hints and sample solutions. Furthermore, the case of ASYMPTOTE indicates that learning platforms may contribute (i) to support students in completing the idealized modelling cycle of Blum and Leiss (2007) and (ii) to promote internal differentiation in MM tasks in order to students to work at their own performance level.

Based on these considerations, we see ASYMPTOTE as one example of a learning platform that can promote students’ MM processes from a theoretical perspective. However, an empirical validation of this hypothesis, e.g. by case studies, is required and an open task for future research. Lastly, we highlight the role of an appropriate task design and availability as one sufficient condition for the successful use of learning platforms as ASYMPTOTE for the teaching and learning of MM.

Acknowledgment

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“Math is all about symbols” – Developing students’ beliefs about mathematics through an app-based outdoor program

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Students' beliefs about mathematics include static (schema and formalism) and dynamic (application and process) aspects and influence students' motivation, emotions, and learning. Students often hold static beliefs, which emphasize the memorization of symbols and procedures and potentially hinder learning. We investigated how students' beliefs developed from beginning to end of an app-based outdoor program, which connects mathematical concepts with real life. Twenty-six fifth and sixth graders participated in the outdoor program for 6 months. A qualitative content analysis of students' drawings of how they viewed mathematics before and after the program revealed a reduction in the schema aspect. No significant developments were observed in the application and process aspects, although a trend toward everyday applications and less emphasis on exact solutions emerged. Thus, outdoor interventions, supported by digital tools, might foster more nuanced mathematical beliefs in young learners.

Keywords: Beliefs; Outdoor learning; Digital tools; Mathematics

Introduction

Students' beliefs about mathematics contribute to their learning, motivation, and enjoyment of the subject (Gattermann et al., 2012; Østergaard, 2024; Roesken et al., 2011). Formed early in life, these beliefs act as a filter through which students process new experiences and information (Østergaard, 2022). Existing research has indicated that middle school students' beliefs are often limited to static beliefs about mathematics (e.g., they emphasize calculations and memorization) and thus overlook dynamic aspects of mathematics (e.g., applying math in everyday life; Østergaard, 2024). Consequently, there is a need to further explore which learning experiences might contribute to diversifying students' beliefs (Østergaard, 2022). In this study, we investigated whether participating in an app-based outdoor mathematics intervention would contribute to students' beliefs about mathematics (possibly by increasing their dynamic beliefs about mathematics).

Theoretical Background

Beliefs about mathematics

Beliefs have not been clearly defined in mathematics education (e.g., Di Martino & Zan, 2011; Furinghetti & Pehkonen, 2002; Østergaard, 2022), but some commonalities can be identified across the various definitions (e.g., Furinghetti & Pehkonen, 2002; Hannula et al., 2012): Beliefs are mental constructs that reflect a student's enduring subjective understanding of and emotions regarding a particular object or concept (e.g., mathematics). Importantly, these convictions might not always be grounded in objective reasoning or verifiable evidence (Pehkonen, 1998). According to Grigutsch et al. (1998), beliefs about mathematics (or world views) refer to four aspects of mathematics: The *schema aspect* focuses on the rules and procedures applied in mathematics, the *formalism aspect* focuses on the use of logical and precise thinking in mathematics, the *application aspect* focuses on

the usefulness of mathematics in everyday life, and the *process aspect* stresses the importance of mathematics as a method for considering, understanding, and solving problems. Whereas the schema and formalism aspects emphasize mathematics as a static body that needs to be deciphered (i.e., static beliefs), the application and process aspects emphasize mathematics as a body that is constantly created and evolving (i.e., dynamic beliefs).

Middle school students' beliefs about mathematics

Previous research on students' beliefs about mathematics at an early age (e.g., in middle school) has indicated that students predominantly hold static beliefs about mathematics (e.g., Grigutsch et al., 1998; Grootenboer & Marshman, 2016, Østergaard, 2022). For example, Halverscheid and Rolka (2011) asked fifth and sixth graders to make a drawing of what mathematics is to them, reporting that drawings were dominated by a view of mathematics that emphasizes numbers and calculations. Such a view is of concern, as perceiving mathematics as a static set of symbols, rules, and procedures to memorize might impede students' learning and enjoyment (Østergaard, 2024). Conversely, viewing mathematics as a dynamic and applicable system is positively related to performance, enthusiasm, and a positive self-concept in mathematics (e.g., Gattermann et al., 2012). As beliefs can be formed by learning experiences (Grootenboer & Marshman, 2016), and beliefs that result from repeated experiences have been found to be rather enduring (Østergaard, 2024), research is needed on the best ways to develop and sustain middle school students' beliefs about mathematics. Previous research indicates that outdoor math activities support dynamic beliefs, such as the application aspect (Jablonski, 2023), and that digital tools can be used to help students apply mathematics to the real world (Jablonski et al., 2023).

An app-based outdoor mathematics program

We report on a 15-hr extracurricular outdoor mathematics program, conducted across a 6-month period, for fifth- and sixth-grade students. The program was aimed at connecting mathematical concepts with real-life scenarios by asking students to develop a *math trail* through their hometown (Schoenherr, 2024). During math trails, students visit a certain number of places where they can perceive mathematics in the environment or answer and ask mathematical questions (Barbosa et al., 2022). For example, students walk around the city looking for symmetrical objects and create the task of identifying the types of symmetry in the church's ornament. Students implemented the tasks in the BIPARCOURS app and played tasks developed by peers using their tablets (Schoenherr, 2024). Since students were unfamiliar with the app, they received prior instruction and support when needed.

The BIPARCOURS app (Bildungspartner NRW, 2015) played a crucial role in guiding students through the process of creating and testing tasks for the math trail. For task creation, the BIPARCOURS creator serves as a user-friendly interface that guides students through the process of task development. It provides templates for various task formats, including free-text answers, multiple-choice questions, and scale estimation tasks. Students are prompted to input information and attach photographs of real-world objects relevant to the tasks they are creating. Figure 1 illustrates parts of the step-by-step process, illustrating how students can input information, attach relevant images, and design tasks with different formats. Importantly, the app requires students to determine the criteria for rating different solutions, adding a critical thinking element to the task creation process. Once the tasks are created, they are consolidated into a cohesive math trail. This math trail can then be experienced by users through the BIPARCOURS app on their smartphones or tablets as depicted in Figure 2. In summary, the

BIPARCOURS app provided a structured and interactive environment for students to independently create, refine, and test math tasks. Its browser-based creator facilitates the creation of real-world tasks, and the app itself allows students to engage with the tasks in the real-world context.

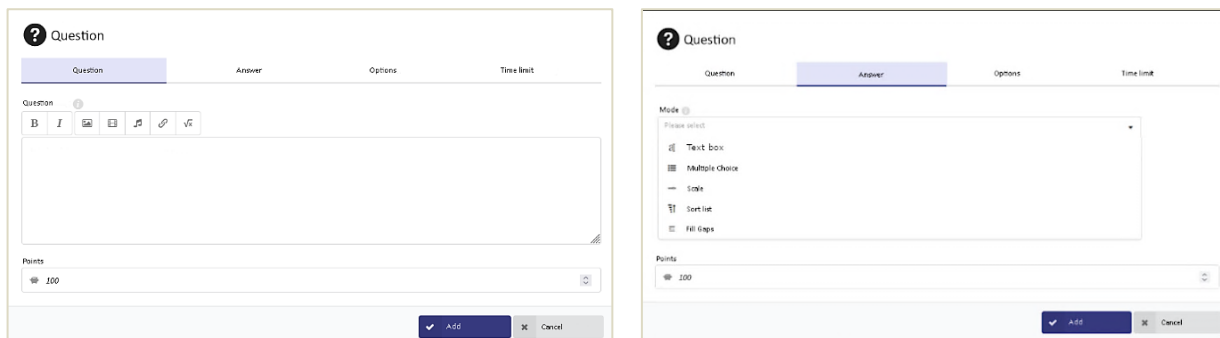


Figure 1: Design interface of the BIPARCOURS creator (Schoenherr, 2024)

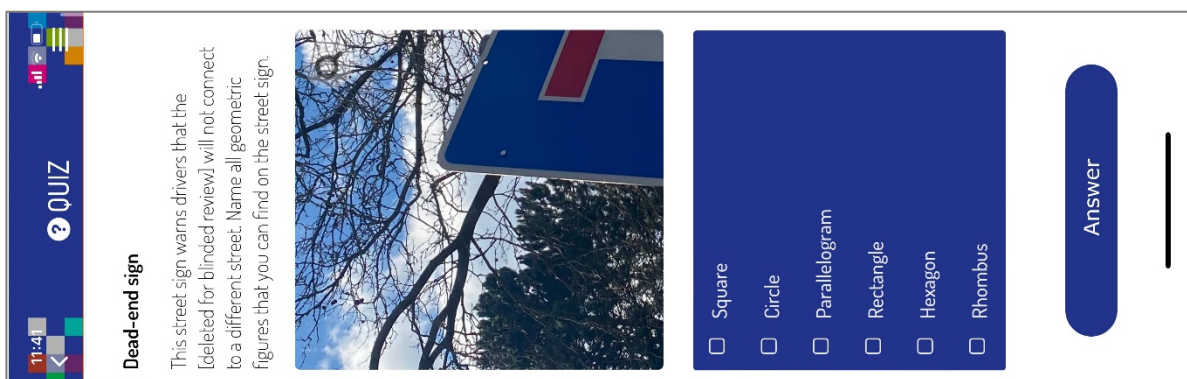


Figure 2: Quiz interface of the BIPARCOURS app (Schoenherr, 2024)

Research questions and expectations

In this study, we investigated the following research question: How do students' static and dynamic beliefs about mathematics develop from the beginning to the end of the outdoor mathematics program? On the basis of prior research and the design of the app-based outdoor mathematics program, we expected that, initially, students would predominantly report the schema and formalism aspects of mathematics beliefs but that students' schema and formalism beliefs would then decrease and that application and process beliefs would increase from before to after the program.

Method

Participants and procedure

Participants were the same participants included in Schoenherr (2024). A total of 26 fifth- and sixth-grade students (18 boys, 8 girls, mean age = 10.85, $SD = 0.73$) voluntarily participated in the after-school outdoor mathematics program and were present when pre- and posttesting took place. Before the program began, consent was obtained from the children's legal guardians for the use of the data for research purposes. In the first and last sessions of the app-based outdoor math program, we assessed students' mathematical beliefs.

Instruments

Following Halverscheid and Rolka (2011), we asked students to draw a picture of how they see mathematics and to provide a brief written explanation of their drawing. Then, we applied a qualitative content analysis with a deductive-inductive procedure to the drawings and explanations (Mayring, 2022), expanding prior research by taking a related belief system (Grigutsch et al., 1998). One third of the drawings were independently double-coded by two trained coders. The intercoder reliability indicated substantial agreement between coders (Cohen's $\kappa > .74$). We coded the presence (code = 1) and absence (code = 0) of subaspects of schema, formalism, application, and process aspects, which were extrapolated from the data, as follows.

The *static schema aspect* focuses on the rules and algorithms that are applied in mathematics (Grigutsch et al., 1998; Østergaard, 2022), including the aspects of *incoherence* (e.g., arbitrary arrangement of symbols and objects), *focus on procedures* (e.g., depiction of computations and algorithms), and *distance* (e.g., lack of everyday references). The *static formalism aspect* focuses on the internal consistency of mathematics and the importance of logical and precise thinking in mathematics. Its sub-aspects are *general quality* (e.g., attempts to use proofs as a means of deriving new results within the system), *completeness* (e.g., a mathematics text book that provides a complete summary of the mathematics to be learned), and a focus on *accuracy and precision* (e.g., question marks and check marks for correct solutions). The *dynamic application aspect* refers to the usefulness of mathematics in everyday life and is composed of the sub-aspects *utility* (e.g., mathematics as useful for describing real-world phenomena, e.g., finance), *everyday life* (e.g., the relevance of mathematics in present and future everyday life, e.g., career opportunities), and *diversity* (e.g., making connections between different mathematical domains, e.g., algebra and geometry). The *dynamic process aspect* emphasizes mathematics as a method that can be applied to understand, investigate, and solve problems, including the sub-aspects *idea* (i.e., highlighting the roles of ideas and invention in mathematics), *dynamics* (e.g., changing representations of one object), and *freedom* (e.g., viewing mathematics on a metalevel). As one example, code 1 was given to the left drawing in Figure 4 in the schema aspects incoherence, procedure, and distance, and code 0 in all other aspects.

Results

To answer the research question, we present quantitative findings on how students' beliefs developed from before to after the program and illustrate these in depth by presenting typical and atypical student drawings. Paired *t* tests indicated that, as expected, the schema aspect decreased from before to after the program with a small effect size, $t(24) = 2.29, p = .031, d = 0.458$. Contrary to our expectations, no differences were found for the formalism aspect, $t(24) = -0.78, p = .444$; the application aspect, $t(24) = -1.56, p = .131$; or the process aspect, $t(24) = -1.81, p = .083$. Still, regarding the sub-aspects, our findings suggest that the formalism sub-aspect of *accuracy* decreased, and the application aspect of relevance in *everyday life* increased (Figure 3).

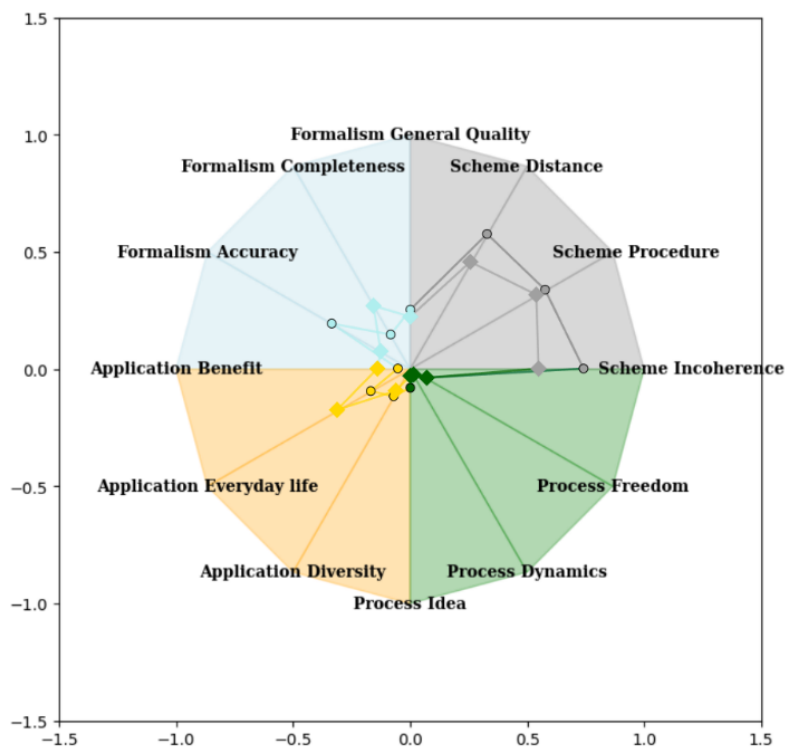


Figure 3: Mean values on the belief scales before (circles) and after (squares) the program

We illustrate these findings with a typical student: Before the program began, the student predominantly held static beliefs about mathematics, as represented in a disconnected collection of numbers, symbols, operations, calculations, and geometric shapes (as illustrated in the left part of Figure 4). After the program, the schema and formalism aspects were still present, but other aspects were added (e.g., connections between mathematics and everyday life) via verbal expressions (e.g., “Math is part of life”) and by referring to the context of money and the symmetry of a manhole cover (as illustrated in the right part of Figure 4).

We rarely identified the process aspect in students’ drawings; however, a few students emphasized this aspect of mathematics in particular. One student drew a blank sheet of paper and added verbally that “math is a blank sheet of paper because there are always new ways to design it and new things to discover” (as shown in the left part of Figure 5) and another student illustrated the various forms that mathematical ideas can take in our brains (as shown in the right part of Figure 5).

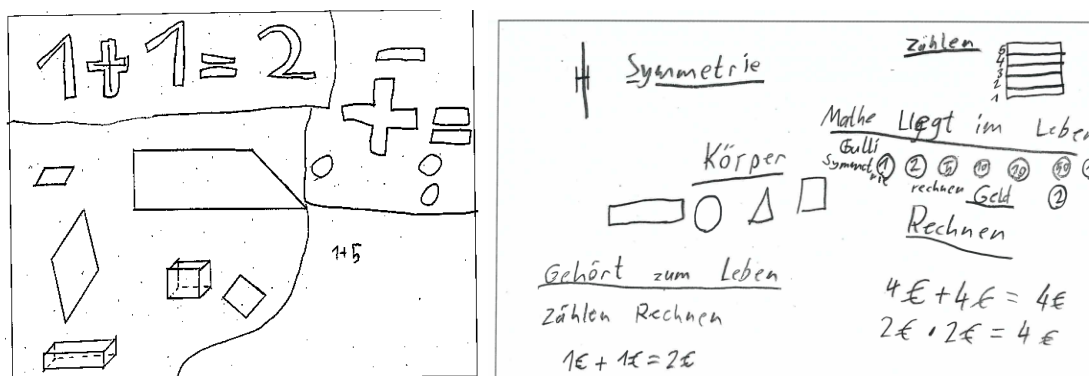


Figure 4: Drawings made by one student before (left) and after (right) the program

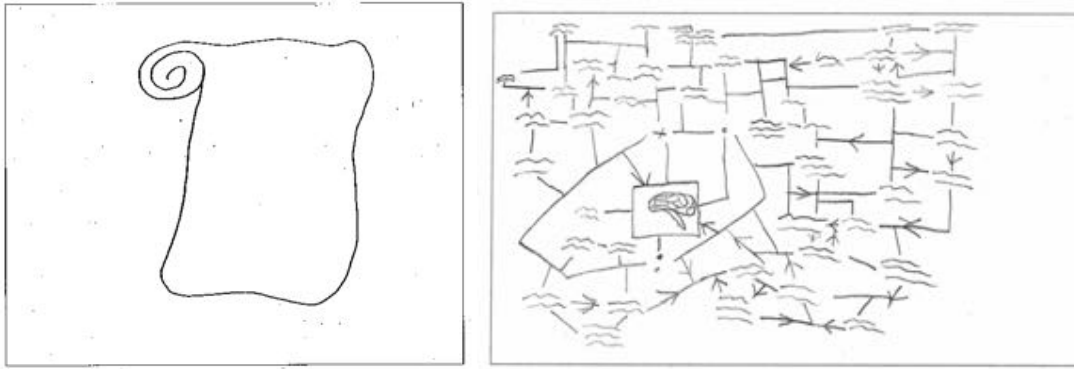


Figure 5: Exemplary student drawings illustrating dynamic beliefs about mathematics

Discussion

Our aim was to describe students' mathematical beliefs and examine how students' beliefs develop from before to after an app-based outdoor mathematics program. Our analysis of students' drawings revealed that, before the program, students most often expressed the schema aspect, followed by the formalism aspect. These findings mean that, initially, the students had more static than dynamic beliefs about mathematics, that is, they saw mathematics as an incoherent collection of rules, algorithms, and calculations that have a deductive-logical and formal structure. The dynamic character of mathematics (e.g., the application and process aspects of mathematics) were seldom present in students' drawings. Most students did not mention how mathematics is present in their everyday lives or that doing mathematics involves inductive and creative processes. Although this result might be biased by the methodological limitation that static aspects are more easily represented by drawing, it is in line with previous findings (Østergaard, 2024; Halverscheid & Rolka, 2011). In future research, the analysis of student drawings should be supplemented with questionnaires or interviews to describe the beliefs of young learners as comprehensively as possible.

Comparing students' drawings from before and after the app-based outdoor mathematics program showed that the schema and formalism aspects were still predominant in students' drawings. However, as expected, the schema aspect was less present in students' drawings after the program. One explanation is that beliefs built on repeated experiences are rather enduring, but they can be complemented by new beliefs that arise after new experiences (Østergaard, 2024). Another explanation is that, by the end of the program, students included more application-related representations in their drawings, believing these were what the educators expected (i.e., the didactic contract effect). Contrary to our expectations, participating in the outdoor program did not result in a statistically significant increase in the application aspect in students' drawings, although a trend could be seen. Future studies should use larger sample sizes to investigate the effects of outdoor mathematics programs on students' beliefs with higher statistical power. Still, in our analysis of sub-aspects, accuracy (i.e., a sub-aspect of formalism) was reduced, and the everyday application aspect of mathematics increased from before to after the program. One hypothesis is that participating in the app-based outdoor program helped students realize that mathematics is connected to their everyday lives and that, beyond unconnected symbols, rules, and algorithms that result in exact solutions, different methods and solutions exist. The use of the BIPARCOURS app to create and play the tasks games from the math trail may have contributed to this develop in beliefs through its technical requirements. For example, the application aspect was present because students were asked to upload

a photograph of the real-world object that their math task referred to. As another example, the assumption that mathematics is always precise and exact was challenged because students had to discuss and create a rating scale to rate different solutions to a problem rather than indicating only one correct solution.

Given the design of this study—particularly the lack of control group—the results need to be interpreted with caution. We cannot rule out other explanations for the developments in students' beliefs (e.g., their regular math class). But the current study contributes to the existing body of research on students' beliefs about mathematics, and the findings support the hypothesis that learning experiences, such as from the app-based outdoor mathematics program, can contribute to diversifying students' beliefs (Østergaard, 2022). Future research should use experimental study designs with follow-up tests to investigate whether curricular outdoor math learning experience can equally contribute to diversifying students' beliefs and whether changes in students' beliefs are enduring beyond the program. Regarding the design of the outdoor program, more research is needed on the specific app features, which are beneficial for students' learning and their beliefs about mathematics.

To conclude, the current study suggests that the outdoor mathematics program helped students develop more diverse beliefs about mathematics, and these beliefs were reflected in their drawings of how they see mathematics. The effects can be traced back to the design of the outdoor learning activities and the specific opportunities the BIPARCOURS app provided for students in developing and playing the tasks from their math trail.

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Leveraging ChatGPT for problem posing: an exploratory study of pre-service teachers' professional use of AI

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The use of AI tools in the professional development of teachers is evolving into an important part of education. Despite growing interest, there is a gap in understanding how AI, specifically ChatGPT, can support mathematics teachers, for example, when posing problems of varying degrees of difficulty. This study aims to explore how interactions with ChatGPT assist pre-service mathematics teachers (PSTs) in posing problems of various levels of difficulty. Five PSTs were asked to create one easy, moderate, and difficult problem based on a given problem. The interactions with ChatGPT were exploratively analyzed using qualitative methods. The analysis identified three themes (collaboration, pedagogical content knowledge, and support in formulation), offering initial insights into nuances of lesson planning and contributing to our understanding of human-AI interaction in education.

Keywords: problem posing, ChatGPT, Large Language Models, pre-service teacher

Introduction

AI is becoming more important in education, ranging from simple use in lessons to complex mathematical problem-solving. One specific use of AI in teacher education can be in supporting teachers in lesson planning (Karaman & Goksu, 2024). Traditionally, when planning lessons, teachers aim to provide tasks of different levels of difficulty for students. This activity can be described as problem posing, that is, the process of formulating new tasks based on a given task or mathematical structure. Despite its importance, problem posing is complex and creatively demanding, especially for PSTs (Baumanns & Rott, 2024) as they often solve problems but are unfamiliar with posing them. To mitigate these challenges, collaboration, such as pair work, can foster a richer brainstorming process and a broader array of ideas (Wang et al., 2022).

Current developments in AI enable this kind of collaboration through a dialogical exchange. ChatGPT is an AI-based chatbot that fairly accurately understands and generates human language using an advanced large language model (LLM). Insights into ChatGPT's impact on supporting teachers in lesson preparation (Karaman & Goksu, 2024) and task development (Küchemann et al., 2023) are at early stages of both research and teaching. However, a significant gap in understanding how these technologies influence the specific skill of mathematical problem posing remains.

Therefore, this study aims to delve into the potential of ChatGPT as a supportive tool for pre-service mathematics teachers, specifically in the context of problem posing for middle school students. By analyzing the interactions between these PSTs and ChatGPT in modifying and extending a given problem, we aim to determine the extent to which this AI tool can stimulate creative ideas, activate pedagogical content knowledge, and facilitate formulating tasks of various difficulty levels. By focusing on this nuanced aspect of lesson planning, the study endeavors to shed light on the practical

applications of ChatGPT in educational settings, thereby contributing to a deeper understanding of its role in enhancing the quality of teaching and learning of mathematics.

Theoretical Background

ChatGPT in teacher education

ChatGPT enables a human-like dialogic exchange with an AI, as it understands and generates human language. As Spannagel (2023) pointed out, this dialogic exchange exemplifies the concept of *distributed cognition* (Salomon, 1993). The perspective of *distributed cognition* understands cognitive processes as spread across a network of interactions that may involve both humans and artifacts. In practical terms, when a teacher uses ChatGPT to plan a lesson, they are in a cognitive partnership with the AI, each contributing to the task interactively.

The use of LLM in educational settings is currently being discussed intensely and controversially due to the challenges they present (Buchholtz et al., 2023, Whalen & Mouza, 2023). Findings from a systematic review show that ChatGPT is considered a fundamental educational support for improving teaching and learning (Montenegro-Rueda et al., 2023). However, the quality of LLMs' output differs greatly, e.g., regarding the correctness of mathematical problem solutions. Plevris et al. (2023) found that performance decreases for more complex mathematical and logic tasks, while simpler tasks such as arithmetic or algebraic expressions are more often solved correctly. Therefore, the respective AI-generated outputs should always be checked carefully. Regarding lesson planning, Karaman and Goksu (2024) observed in a control group design study that students' achievements increased significantly in mathematics lessons taught according to lesson plans developed with ChatGPT. Therefore, ChatGPT might be useful for teachers in planning their lessons.

Problem Posing

Problem posing from a teacher-oriented perspective refers to “an accomplishment that consists of constructing a problem that satisfies the following three conditions: (a) it somehow differs from the problems that appear in the resources available to the teacher; (b) it has not been approached by the students; and (c) it can be used in order to fulfill teaching needs that otherwise could be difficult to fulfill” (Klinshtern et al., 2015, p. 463). Studies on problem posing often used tasks that *explicitly* require either generating new problems or reformulating given ones. A problem-posing task consists of a situation and a prompt. Problem-posing situations can be differentiated between unstructured and structured situations (Baumanns & Rott, 2021). In structured situations, people are given an initial task on which problem posing should be based. This study focuses on structured situations and uses the following initial task: “How many divisors does the number 60 have?” Problem-posing prompts are the trigger initiating problem-posing activities (Cai et al., 2023). Prompts can lead problem posing in various directions. For example, “pose a math problem” gives fewer constraints than “pose a problem for a math contest.” In this study, we want the participants to pose one easy, one moderate, and one difficult problem.

Küchemann et al. (2023) compared the use of ChatGPT to classical textbooks when pre-service physics teachers are asked to develop text-based physics tasks (i.e., problem posing). They found no difference in task correctness between students using a textbook or ChatGPT. However, textbook users achieved better clarity and meaningful contexts. Both groups adjusted task difficulty but struggled with providing enough specific information.

Despite these insights in physics education, qualitative insights into the nuances of PSTs' interactions with ChatGPT in the context of mathematical problem posing still need to be addressed. Based on this gap, the following research question arises: *What aspects of interaction are evident when pre-service mathematics teachers collaborate with ChatGPT in the problem-posing process?*

Methods

Participants, Problem-Posing Task, and Data Collection

Five pre-service middle school and mathematics teachers working in specialist provisions in their early Bachelor's program from Germany participated in the study. The participants were asked to take part in the study voluntarily. Pre-service mathematics teachers were chosen because, as future teachers, they are not fully qualified yet and, therefore, are suitable for preparing them for their future profession, especially in the context of posing problems for their learners (Baumanns & Rott, 2024).

The data collection included an instruction video that explained problem posing and how to use ChatGPT. The video informed the participants that they were asked to pose one easy, one medium, and one difficult problem while using ChatGPT based on a given problem. The video ended with a link to a website with a predefined prompt that all participants entered into ChatGPT 3.5:

Your task, ChatGPT, is to act as a supportive assistant, helping me develop ideas for adapting the difficulty level of mathematical task for middle school students. Instead of proposing your own ideas, ask me targeted questions that encourage me to think about different aspects and approaches. Start with the problem 'How many divisors does the number 60 have?' and ask me about possible ways to modify or extend this problem to adjust the difficulty level. Your questions should aim to utilize my creativity and professional knowledge to enhance the student's learning process. Remember to only provide hints to me and not to pose your own tasks. The conversation ends only when I have decided on an easy, a moderate, and a difficult problem. Once I have made these decisions, you should end the conversation with the word 'END'.

Your first response should be: 'Got it! Let's begin. How could we adjust the problem "How many divisors does the number 60 have?" to make it easier for middle school students?'

Fig 1: Prompt used for ChatGPT to initiate problem posing (translated from German)

The PSTs' were given just the first prompt so that every interaction with ChatGPT started the same way. In previous tests, a free choice of the starting prompt (e.g., "I need to pose an easy, a moderate, and a difficult problem based on the following problem: How many divisors does the number 60 have?" to make it easier for middle school students?) resulted in ChatGPT simply posing three tasks without real problem-posing process or interaction being created. This prompt contains the given task "How many divisors does the number 60 have?" from which new problems ought to be posed. Therefore, we used a structured problem-posing situation. The task for the participants was to pose one easy, one moderate, and one difficult problem. ChatGPT automatically starts with: "How could we adjust the problem to make it easier for middle school students?"

After the participants had posed three problems, ChatGPT ended the conversation with "END." This marks a clear beginning and end to the interactions for research. The participants shared their conversation with ChatGPT with the authors. ChatGPT does not contain any log data that allows

statements to be made about the exact time the participants used ChatGPT. However, after about 20 minutes, the authors asked the participants to formulate their final thoughts.

Data Analysis

We employed a Grounded Theory approach to explore the potential of ChatGPT in assisting pre-service mathematics teachers in posing problems for middle school students. Grounded Theory, as articulated by Clarke (2007), emphasizes the systematic analysis of qualitative data to uncover key forms of action of participants within a specific context. This approach is particularly suited to our study, aiming to generate insightful concepts and theoretical frameworks rooted in empirical evidence. Following Strauss and Corbin (1990), data analysis is structured around three core procedures: open, axial, and selective coding. Open coding aims to break down the conversations between each participant with ChatGPT and categorize the data. Axial coding aims at making connections between the categories. Selective coding aims at selecting core categories and relating them to other categories. In this study, we focus on open coding, in which we specifically coded which goal the PSTs pursued using ChatGPT in problem posing. This approach led to three aspects of interaction which we refer to as *themes*.

Results

Three themes have emerged from the qualitative analysis regarding the research question: (1) Collaboration, (2) Pedagogical Content Knowledge (PCK), and (3) Support in Formulation. We develop the themes along selected transcript excerpts from the interaction between the PSTs and ChatGPT. In these transcripts, P labels the prompts of the respective PST.

Theme 1: Collaboration

The collaboration between PSTs and ChatGPT emerged as a theme, characterized by the mutual use of collaborative language and an interactive exchange of ideas. Inclusive language is notably facilitated through the consistent use of the first-person plural form (“we”) by ChatGPT (e.g., “ChatGPT: Do you have any ideas on how *we* can make the question about the divisor of 15 a bit more challenging?”), which served to create a sense of collaboration in the joint problem-posing process. On their end, participants occasionally mirrored this collaborative language, suggesting ideas with statements like, “P4: *We* could ask specifically about the prime factors.” However, it was observed that participants often adopted a more general or impersonal stance in their contributions, using phrases such as, “P2: *One* could consider how many divisors the number 30 has?” This may be rooted in the German language, which often uses “one”, whereas in English “you” would be used.

Furthermore, participants showed interest in collaborative dialogue with ChatGPT, often resembling a natural conversational flow:

- | | | |
|---|---------|--|
| 1 | ChatGPT | How could we use this idea to create a task for middle school students that focuses on the divisors of 60 while reducing the level of difficulty? |
| 2 | P3 | Do you have any ideas? |
| 3 | ChatGPT | Of course! How about asking students to find the divisors of 60 by first identifying the divisors of 30 and then understanding how these can be combined with the divisors of 2? This could take the process step by step. What do you think about this? |
| 4 | P3 | Yes, I think that’s a good idea. So now the students first look at which numbers have to be multiplied to get to 30. |

Turn 2 and the last question in turn 3 reflect an informal yet personal and dialogic interaction, illustrating the participants' openness to consider ChatGPT as a collaborative partner.

Theme 2: Pedagogical Content Knowledge (PCK)

An essential theme identified in the study was the discussion of Pedagogical Content Knowledge (PCK). ChatGPT attempted participants to draw upon their PCK. For instance, ChatGPT proposed that students could explore the divisors of 30, prompting to formulate questions to encourage investigation: "ChatGPT: Students could start by finding the divisors of 30 by investigating which numbers can be multiplied by 2 to get 30. What question could we ask to emphasize this aspect and encourage this specific learning process?" However, these prompts of ChatGPT are often not picked up by the participants. In almost all cases, there is no in-depth exchange on PCK.

On the other hand, the participants actively ask ChatGPT for a didactic assessment of students' knowledge, for example, by asking: "P5: How much do middle school students normally already know about the divisors of a number?" However, ChatGPT's response, "Good consideration! To adjust the difficulty level, we could use the student's prior knowledge," did not address the participants' question. Another example in this regard is:

- | | | |
|---|---------|---|
| 1 | ChatGPT | [...] How could we adapt the problem "How many divisors does the number 60 have?" to make it easier for middle school students? |
| 2 | P2 | You could ask how many prime factors the 60 has. Would this be suitable for middle school? |
| 3 | ChatGPT | Good consideration! That could indeed be an adjustment. Now, how could we further modify the problem to make it more moderate for middle school students? |
| 4 | P2 | I have no idea. |
| 5 | ChatGPT | No problem! Instead of asking for the number of prime factors, how about asking students to identify the prime factors of 60? This could be a step-by-step approach. What do you think? |
| 5 | P2 | That's a good idea. Then the children can work out the divisors of 60 step by step. |

ChatGPT does not answer P2's question in turn 2. Instead, it only commended P2's posed problem. This indicates a potential improvement regarding the prompt entered into ChatGPT in this study that may hinder ChatGPT from bringing in its own ideas. This also hindered the collaboration (see theme 1) between ChatGPT and the participants, as ChatGPT did not help PSTs make informed decisions based on PCK.

The participants' active engagement with the idea of scaffolding students' learning is evident in their didactic assessments in response to ChatGPT's suggestions. When ChatGPT proposed a step-by-step approach to identifying the prime factors of 60, P2 acknowledged the value of this method for facilitating understanding.

Theme 3: Support in formulation

Another theme that emerged was *supporting formulation*. This theme encapsulates the dual role of ChatGPT in facilitating the process of refining and adapting task formulations to meet pedagogical objectives. On one hand, participants often sought ChatGPT's assistance to simplify the language of their tasks. Requests such as "P4: We could put it in simpler language" and "P3: I think the question is too difficult, can you put it in simpler terms" indicate a conscious effort by the PSTs to make

problems more accessible to students. However, ChatGPT does not always achieve this simplification as intended, as the following example shows:

- 1 ChatGPT How about the question: “Express the number 60 as a product of prime factors with the usage of powers, if possible.” [...]
- 2 P3 I think the question is too difficult, can you put it in simpler terms.
- 3 ChatGPT Of course! How about the question: “Write the number 60 as the product of two numbers, where one of the numbers is a power of 2 and the other is divisible by 2.”

On the one hand, it can be argued whether the second formulation is simpler. We should add that the German-language output uses unusual words (e.g., “Expressiere [...]”). On the other hand, ChatGPT not only tries to formulate the task more simply but also changes its content considerably. Instead of asking for the general prime factorization, the focus is now on a very specific decomposition.

Conversely, ChatGPT itself played a proactive role in challenging the PSTs to formulate tasks that meet specific criteria: “ChatGPT: How could we guide students to identify the prime numbers that divide the number 120? Think about how we could formulate the question to subtly encourage this,” and challenges to increase the difficulty level, like “ChatGPT: Do you have any ideas on how we could formulate this to challenge the students?” This theme was also identified because formulating the task is an elementary part of problem posing.

Discussion & Conclusion

This exploratory study, with a limited sample size of five pre-service mathematics teachers, aims to explore the potential of ChatGPT as support for problem posing for PSTs. This limited sample size also underscores the need for a cautious interpretation of the findings.

In our qualitative analysis, three themes emerged from the interactions between PSTs and ChatGPT. *Collaboration* was a theme, where both ChatGPT and participants utilized collaborative language, engaging in an interactive idea exchange. ChatGPT’s consistent use of “we” fostered a sense of partnership, although participants sometimes retained a more impersonal tone. This theme underscores the cognitive partnership ChatGPT may enable in terms of the concept of distributed cognition (Salomon, 1993). Although collaboration is not problem-posing specific per se, this topic suggests that ChatGPT has the potential for pair work that can be beneficial for problem-posing outcomes (Wang et al., 2020).

Pedagogical Content Knowledge (PCK) emerged as another theme. ChatGPT aimed to activate participants’ PCK by formulating questions to encourage students’ investigation. However, the participants often did not engage deeply with these prompts, indicating a missed opportunity for richer pedagogical discussions. Despite attempts to inquire about students’ knowledge levels, responses from ChatGPT sometimes lacked direct answers, suggesting a need for improved prompt engineering (chtcht et al., 2023) to better utilize ChatGPT’s potential in activating PCK.

The third theme, *Support in Formulation*, highlighted ChatGPT’s dual role in assisting teachers to refine and adapt problem statements. Participants sought ChatGPT’s support in simplifying task language, reflecting an effort to make problems more accessible. Yet, the effectiveness of these simplifications varied, with ChatGPT occasionally altering the problem’s content significantly. Additionally, ChatGPT proactively challenged teachers to craft problems that met specific

educational objectives, indicating its utility in encouraging thoughtful task formulation as an integral part of the problem-posing process (Baumanns & Rott, 2022).

One of the notable challenges in the interactions between participants and ChatGPT were instances where ChatGPT failed to directly answer participants' queries. This highlights the critical importance of prompt engineering (Schorcht et al., 2023)—for the design of our study and also for the participants using ChatGPT. Prompt engineering becomes crucial in educational contexts when we want to consider LLMs as genuine partners of a cognitive network (Salomon, 1993). Furthermore, the absence of specific training for the participants on how to leverage ChatGPT's capabilities may have contributed to an underutilization of the tool's potential. This aligns with findings from the systematic review of Montenegro-Rueda et al. (2023), indicating a gap in harnessing the full educational benefits of LLMs due to a lack of user familiarity and proficiency. Training PSTs to effectively use ChatGPT could enhance the quality of their problem-posing tasks. In this context, the differentiation between artifacts and instruments hinges on the process of instrumental genesis, where artifacts, like ChatGPT, undergo transformation by users into instruments, reflecting their strategies and educational objectives in their cognitive network (Guin & Trouche, 2002).

Despite these challenges, the study provides valuable preliminary insights into the application of ChatGPT in mathematics education, particularly in the realm of problem posing. The interactions between PSTs and ChatGPT suggest that AI can play a supportive role in stimulating pedagogical content knowledge and enhancing creating ideas for lesson planning. There is potential for future research to build upon the findings of this study. A control group design study with a bigger sample could provide a more robust insight for evaluating ChatGPT's impact on problem-posing skills among pre- and in-service mathematics teachers. By comparing outcomes between groups with and without access to ChatGPT or with and without training in the use of ChatGPT, we could assess the tool's effectiveness in improving the quality of posed problems and its utility in teacher education. Such research could significantly contribute to our understanding of AI's role in enhancing educational practices and supporting teacher professional development in the digital age.

Declaration

While preparing this work, the authors used ChatGPT 4 for language processing. After using this tool, they reviewed and made necessary edits, taking full responsibility for the publication's content.

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Function Art: Analyzing Popular Functions and Strategies

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This study explores the integration of mathematics and visual arts through "function art" within the STEAM education framework. Utilizing GeoGebra software, 335 students from Grades 8 to 12 in the Philippines participated in an online webinar, resulting in the creation of 235 function art pieces. The analysis identified quadratic, linear, and sine/cosine functions as the predominant choices for symmetry, straight lines, and waves, respectively. These findings align with constructionist principles, emphasizing the significance of foundational functions and practical considerations in art creation. Function art presents a promising avenue for engaging students in meaningful mathematical exploration while nurturing artistic expression. Nonetheless, further research is necessary to evaluate its impact on learning within the STEAM context.

Keywords: STEAM education, function art, GeoGebra, math-art integration, constructionism

Introduction

The studies on the integration of Science, Technology, Engineering, Arts, and Mathematics (STEAM) have been increasing in the past decade (Li et al., 2020). However, many of such integrated studies have been dominated by science and engineering (Martín-Páez et al., 2019), and mathematics and arts have been marginalized (Lajevic, 2013). The purpose of this study is to give more attention to the integration of these two disciplines in the context of STEAM learning. It also aims to investigate the unexplored territory of integrating mathematics and visual art using “function art.”

Function art is defined as “art constructions whose components include graphs or parts/segments of graphs of mathematical functions” (Bautista et al., in press). We selected functions because of their rich mathematical properties. Firstly, they allow the exploration of variable relationships, domains, ranges, and symmetry. Secondly, working with functions in art enables students to perform transformations, viewing functions as objects rather than individual points. Lastly, functions offer multiple representations (Bautista et al., 2023), fostering inventiveness and deepening comprehension of fundamental concepts (Yerushalmy, 1991).

In our study, we utilized GeoGebra, a freely available mathematics software, to create function art constructions by plotting function graphs. GeoGebra integrates Algebra, Geometry, Calculus, and spreadsheets into a single package. We selected GeoGebra due to its accessibility, being cost-free and compatible with major operating systems and usable across various devices.

Literature Review

Earlier studies asked students to create art constructions, but none of those studies defined that activity. Those studies were mainly non-empirical. Among the 17 journal articles we examined

(Bautista et al., forthcoming), 15 presented descriptive accounts of the implementation process.

Fan Disher (Disher, 1995) asked students to use grid paper to draw a picture, analyze the curves within these pictures, and determine their corresponding equations. Notably, during the period of this study, graphing applications had not gained widespread popularity. Subsequent studies such as those conducted by (Lee, 2002) and (Black, 2011) continued to use grid papers for similar activities. In contrast, an adaptation of Disher's study incorporated graphing calculators (Avila, 2013). The challenge associated with manual graphing, as highlighted by Avila (2013, p.7), involves the laborious task of checking functions one by one—an issue alleviated using graphing calculators.

Several notable studies have explored the use of digital images to teach mathematical concepts such as slope and symmetry (Sharp, 2007), tracing outlines of animals to link mathematics and biology (Barry, 2021), and using function graphs to draw letters (Rebholz, 2017).

In the studies mentioned above, constant and linear functions are commonly utilized for drawing straight lines, while quadratic functions are favored for creating curves. Notably, students often employ the quadratic function in vertex form, represented as $f(x) = a(x - h)^2 + k$. Additionally, some students incorporate absolute value functions into their artwork (e.g. (Avila, 2013). Beigie's study (Beigie, 2014) stands out for its diverse range of functions, including linear, quadratic, and radical functions, as well as conic sections and inequalities to generate colors within regions. Remarkably, none of these studies are dedicated to exploring the quantitative measures of functions employed by learners in the creation of art through the application of mathematical concepts. The quantification of the frequency of these functions can prove instrumental in elucidating patterns, identifying trends, and cultivating a more profound comprehension of the descriptive narratives. This research endeavors to fill this void by systematically identifying and tallying the various functions utilized by learners in their artistic endeavors.

The study presented in this paper is an extension of the study conducted by Bautista et al. (in press). While that study focused on Grade 11 students, ours included samples from Grades 8 to 12. Our aim is to address the gap identified in the existing non-empirical studies mentioned earlier. The investigation endeavors to elucidate the nuanced levels of familiarity, exposure, and depth of knowledge and understanding exhibited by learners concerning diverse categories of functions.

Theoretical Framework

Constructionism, as defined by Papert (Papert, 1980), serves as a theoretical framework for our study on function art. This framework, rooted in constructivism (Piaget, 1970), offers valuable insights into the learning process, particularly in the context of creating and enhancing mathematical artwork.

Constructionism emphasizes the notion that learning is most effective when students are actively involved in the creation of meaningful artifacts and knowledge. In the context of function art, students actively experiment with mathematical functions, investigate different transformations, combine functions with other elements, and apply principles of symmetry and reflection (Bautista, in press). These hands-on experiences enable students to internalize mathematical concepts in a tangible way. As they actively create, manipulate, and experiment with these functions, students are not merely passive recipients of knowledge; they become active constructors of their understanding, a key feature of constructionist learning.

To further delve into the integration of mathematics and visual art in STEAM education, we now pose our research question: How do students utilize mathematical functions, specifically linear, quadratic, and trigonometric functions, in the creation of function art using GeoGebra software, and what are their underlying strategies and rationales?

Methodology

Our research methodology was influenced by the constructionist framework. We have designed our study to involve students actively in the process of creating mathematical art constructions and exploring the underlying graphs or parts/segments of graphs of mathematical functions. This methodology aligns with constructionism's emphasis on experiential learning and the creation of meaningful artifacts (Papert, 1980).

Sampling and Intervention

We employed a purposive sampling technique (Patton, 2014) to select 335 students from Grades 8 to 12 across 4 schools in the Philippines. The selection criteria ensured that students had access to technology, allowing them to actively participate in the study. This must be taken into careful consideration because many of the students in the Philippines do not have gadgets or a good internet connection.

The intervention involved a 2.5-hour online webinar conducted in each school by the first author via video conferencing software. It covered an introduction to GeoGebra's graphical user interface, tutorials on plotting graphs, a review of function transformations, discussions on defining domains of functions, and a demonstration of creating function art. Following the webinar, students were given a month to create artwork using GeoGebra. We encouraged them to use mathematical functions, but it was not mandatory. After completing their art construction, they uploaded them to their GeoGebra accounts and submitted the artwork's URL via Google Form. We surveyed a subset of students about the functions they used and interviewed some for clarification, using these insights to corroborate our findings.

Data Analysis

Initially, we collected 335 artworks, but 48 were excluded due to the absence of functions, and 18 duplicates were filtered out, resulting in 269 unique pieces. Initially, we considered removing artworks with fewer than 5 functions, but intriguing artworks with as few as 2 functions led us to adjust our criteria. As a result, artworks containing only a single function were excluded, totaling 34 such artworks. After these selection procedures, our final sample consisted of 235 function art pieces for analysis. The distribution of artworks by grade level is as follows: Grade 8 - 32 (13.6%), Grade 9 - 26 (11.1%), Grade 10 - 13 (5.5%), Grade 11 - 144 (61.3%), and Grade 12 - 20 (8.5%). While the sample's disparity per grade level is noted, we included all applets to enrich the dataset.

In our data analysis, we adopted a convergent design mixed-methods approach (Creswell & Creswell, 2018) to gain a comprehensive understanding of how students employed mathematical functions and associated strategies in their work. For the quantitative part, we tallied the type and number of functions used in each artwork using GeoGebra's Algebra and Graphics views. For qualitative analysis, we used the survey data about some students' preferred functions and their reasons for selecting them.

For quantitative data, we used content analysis and applied a systematic coding procedure inspired by Krippendorff (2018) to delve deeper by identifying and categorizing the distinct types of functions used in each artwork. While coding for function types is inherently explicit, given the vastness of the dataset, we employed Cohen’s Kappa test to ensure both the accuracy and consistency of the codes. The calculated Cohen’s Kappa value of 0.956 signifies a substantial level of agreement between the coders. Since classifications of functions are clear and explicit, all disagreements were effectively resolved.

In categorizing functions, we strategically grouped them to gain a comprehensive understanding of how students use them in their artwork. We considered multiple factors, including the mathematical properties of functions, their alignment with the curriculum, and insights gathered from our literature review. For instance, despite linear, quadratic, and polynomial functions all falling under polynomials, we distinguished them due to their introduction in different grade levels. These considerations led to the creation of nine distinct categories of functions: absolute value, constant, linear, quadratics, other polynomials, radical, sine/cosine, exponential/logarithm, and other functions. This categorization of functions and examining the individual artwork helped us understand how students used functions and GeoGebra in creating their artwork, showcasing the active role of students in constructing their understanding of mathematical concepts within the framework of constructionism.

Results

In what follows, we discuss the results of our analysis. We examined the number and type of functions students used in their artwork and how they utilized them in creating their artwork.

Number and Type of Functions

Figure 1 illustrates the categories of functions and the number of applets containing them. Notably, the quadratic function emerged as the predominant choice among the artworks, followed by linear function, and then by the combination of sine and cosine functions. Specifically, 129 (54.9%) artworks featured quadratic, 111 (47.2%) incorporated linear functions, and 93 (39.6%) utilized sine function, cosine function, or both.

Among the 235 artworks examined, a total of 8,453 functions were identified. There are 4,320 (51.1%) linear functions, 1,791 (21.1%) quadratic functions, and 681 (8.1%) are combinations of sine and cosine functions. Other polynomial functions accounted for 432 (5.1%), followed by constant functions at 344 (4.1%).

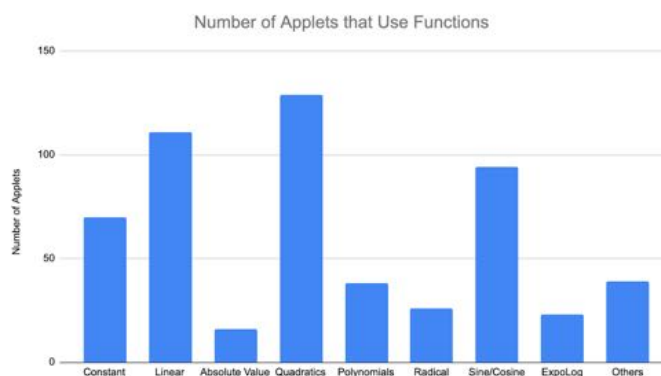


Figure 1: Number of applets that use functions.

The maximum number of functions observed in a single artwork was 400, whereas the minimum was 2. As mentioned in the Methodology section, the initial intention was to incorporate artworks with more than 5 functions; however, we found many intriguing pieces that contained only two functions. Consequently, the only exclusion criterion applied was artworks with only one function. The average number of functions per artwork is 35.9. Note that 198 of these artworks have fewer than 50 functions, while only 26 have more than 100 functions.

The Most Popular Functions

In this section, we discuss the three most popular functions that students used and how students used them in their artwork. These are quadratic, linear, and sine/cosine functions. Figure 2 shows an example of the artwork that uses these functions: linear for straight lines, and quadratic and sine functions for curves.

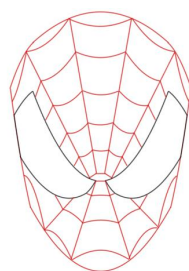


Figure 2: Sample artwork that uses linear, quadratic, and sine functions.

The quadratic function is contained in 129 (54.9%) artworks. Among these artworks, 101 used the vertex form $f(x) = a(x - h)^2 + k$, 58 used the general form $f(x) = ax^2 + bx + c$, and 30 used both. Additionally, 28 of the artworks exclusively used quadratic functions.

Many of the artworks used quadratic functions for symmetric curves. Out of the total 129 artworks that used quadratic functions, 56 of them utilized it to create at least one symmetric curve. Most of these artworks used the form $f(x) = ax^2$ and $f(x) = ax^2 + k$ to exhibit symmetry with respect to the y-axis. In other cases, students graphed curves on one side of the y-axis and then applied reflection across the y-axis to achieve symmetry. It is also worth noting that there were symmetric curves in these artworks whose axis of symmetry is not the y-axis. Achieving this type of symmetry is most challenging, requiring a higher level of mathematical knowledge.

Despite our intention to encourage students to use linear functions to create straight line segments, a substantial number of them opted to use the “Line segment” tool of GeoGebra. There are 138 artworks with line segments, 119 with linear functions, and 68 with constant functions. Furthermore, several artworks combine two or three of these mathematical objects, with 26 artworks incorporating all three in their designs. Some students diligently used linear functions whenever feasible. They used linear functions for slanted lines, constant functions for horizontal line segments, and reserving the Line segment tool exclusively for creating vertical lines.

The sine and cosine functions were also popular among students appearing in 93 (39.6%) artworks. There were 66 artworks that used sine, 56 used cosine, and 29 used both. Out of this total number of artworks containing sine or cosine, 26 used sine or cosine functions at least once for symmetric curves. Students also used these functions to depict waves, edges of objects (e.g. flags, dresses), as well as movement (e.g. swooshing of a flying object).

Students' Explanation of their Choice of Functions

In our survey, we asked students to identify the mathematical functions they utilized prominently and explain their choices. Respondents who selected quadratic, linear, and sine/cosine functions cited reasons such as aesthetic appeal, ease of use and manipulation, suitability for the artwork, and efficiency. Below are some of their specific responses.

- Diane: "I used quadratic function the most because it was easy to plot and move."
Albert: "Quadratic functions mainly because the graph of a parabola has been very useful in tracing the curves of the image I used as a reference."
Charles: "The linear function is the most used function in my artwork because all of the equations are linear and straight lines, so I did not use curve edges..."
Mia: "I used the linear function the most because it was the easiest function for me."
Freya: "Trigonometric functions. My art composed several layers of ocean."
Gina: "I used the function sin, because it was useful for curves."

In summary, the quantitative analysis reveals that the students' most popular functions were linear, quadratic, sine, and cosine functions. These functions not only appeared in the highest number of artworks but were also utilized extensively within each artwork. Specifically, quadratic, sine, and cosine functions were primarily employed to craft symmetric curves, whereas sine and cosine functions found popularity in representing waves and simulating motions.

Discussion

The study's findings, which reveal quadratic, linear, and sine/cosine functions as the most popular among students for function art, resonate with the constructionist notion that learning is most effective when students are actively involved in creating meaningful artifacts and knowledge. This suggests that students gravitate towards functions they know and find easy to use and emphasizes the practical factors influencing their choices in art creation. Educators should note this preference, as it underscores the importance of teaching these foundational functions within a constructionist framework, enabling students to express mathematical concepts confidently through art.

The widespread use of quadratic functions, particularly in their vertex form $f(x) = a(x - h)^2 + k$, suggests a strong recognition of their unique advantages. These functions facilitate the creation of symmetric curves and align with students' artistic intentions in connecting mathematical concepts with visual aesthetics. This preference aligns with prior research by Avila (2013) and Beigie (2014) that emphasize the pedagogical significance of fostering students' proficiency in manipulating function parameters. Therefore, educators can strategically integrate teaching methods that emphasize the vertex form to empower students to explore and manipulate functions with greater flexibility.

The second most popular function used by the participants of this study is the linear function, which differs from the findings of Bautista (in press) where sine/cosine functions ranked second. This variation could be attributed to differences in sample composition: they analyzed Grade 11 students' artwork, while our study involved Grade 8 students who had learned only linear functions and exclusively used them.

Lastly, sine and cosine functions also enjoyed popularity among the students. These functions were commonly utilized for achieving symmetry in artwork, while others employed them for more specialized purposes, such as representing waves and creating a sense of movement.

The three most popular functions mentioned above were further substantiated by the responses obtained from the student survey regarding their preferred functions for use. Students consistently

find these three functions easy to understand and graph, with linear and quadratic functions standing out in particular. This aligns with prior research that has indicated that students tend to encounter fewer challenges when graphing and manipulating linear and quadratic functions (Baker et al., 2001).

Conclusion

This study aimed to investigate the use of functions in the creation of function art and to delineate the strategies used by students in this creative process. Our findings highlight the prominence of linear, quadratic, sine, and cosine functions among students as the most favored choices for their artistic endeavors.

The implications drawn from our study suggest that function art can serve as a valuable project for students to develop their skills in utilizing functions and their properties. However, it is important to acknowledge a limitation in our approach; we did not measure the direct impact of function art on student learning. To address this gap, a structured experimental study could be designed and conducted to assess the benefits of integrating function art into the learning process.

This episode is chosen to answer the first research question by describing the development and evolution of the instrumental genesis. It shows how ChatGPT became an instrument to foster the discussion. More precisely we discuss this episode to show the emergence and evolution of the artifact to support the activity and the gradual formation and evolution of schemes for using the instrument in practice.

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What makes project-based STEAM tasks motivating? How do they enhance learning?

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Literature and our experience provide evidence for the motivating effect of real-life STEAM projects. This study tries to unveil factors that may explain this effect. This study is based on a case study involving two lower secondary school classes engaged in a house modelling project including the stages: drawing the plans, modelling and 3D printing the house and putting the activity in the context by networking with another school and a local company. Data were collected from questionnaires, interviews and classroom observations to study students' motivation, and the results of the Austrian National Competency Test (IKM Plus)⁶ are considered to refer to changes in learning results. Data analysis shows that working in teams, having more time to work on a meaningful task, and creativity seem to be the most important. The results of IKM Plus show a significant increase in comparison with the Austrian national average, however, defining the reasons needs further research.

Keywords: Motivation, 3D - printing, modelling, meaningful task, collaboration.

Introduction

In the literature, numerous studies provide evidence that the STEAM (Science, Technology, Engineering, Art, Mathematics) approach can increase students' motivation (El Bedewy et al., 2022; Liljedahl & Zager, 2021; Ulbrich et al., 2020) and enhance learning (Conradty & Bogner, 2020; Suryaningsih et al., 2022). Conradty and Bogner (2020) claim that creativity in STEAM tasks can be beneficial for motivation. Suryaningsih et al. (2022) found that STEAM activities referring to everyday life can enhance motivation. In this paper, we will define further factors that may make STEAM tasks motivating and beneficial for students' learning. Research showed that cooperative learning may increase students' motivation to learn (van Tran, 2019), and STEAM activities are cooperative. The research focus regarding STEAM activities and their motivating effects was more on real-life reference and creativity. There seems to be a research gap concerning STEAM activities from the social aspect, that is, if STEAM tasks are motivating as they are cooperative tasks. We intend to find factors increasing motivation by analysing students' self-reports.

Theoretical Framework

As this study focuses on a project-based STEAM task and its motivating effect on students, we rely on project-based learning (PBL) concerning the task design and on Self Determination Theory (SDT)

⁶ The Austrian National Competency Test, IKM Plus, is carried out by the Institut des Bundes für Qualitätssicherung im österreichischen Schulwesen. You can read more about the guidelines on the onlineplatform: <https://www.iqs.gv.at/themen/nationale-kompetenzerhebung/onlineplattform-ikm-plus>.

while studying students' motivation. Shin and colleagues (Shin et al., 2021) define PBL as students participating in science practices in collaborative activities and actively constructing a tangible product over a longer time. PBL activities have five key features: (1) a driving question to be solved; (2) students exploring the question; (3) collaboration among students and teachers; (4) students' self-management and scaffolding by teachers and technology; (5) students creating a tangible product (Krajcik & Blumenfeld, 2006). These features underpin our task design as described in the following section. Krajcik and Blumenfeld (2005) also stated that students learn by doing and claimed that students benefit twofold: they get a more thorough understanding, and their motivation increases as they develop a positive attitude towards mathematics. SDT is an approach to human motivation. To be motivated means that one is moved to act. Ryan and Deci (2000) differentiate between intrinsic and extrinsic motivation depending on the source of motivation. The first comes from the person and his curiosity, whereas the second has external reasons. Intrinsic motivation has a positive impact on learning outcomes and students' creativity (Ryan & Deci, 2000); hence it raised educators' interest. However, as Ryan and Deci claim, extrinsic motivation can result not only in resentment but also in willingness and acceptance of values, which educators may benefit from. This willingness may increase engagement meaning time and effort put into solving a problem. This may result in the enjoyable feeling of success, which may change extrinsic motivation into intrinsic motivation as students want to experience success again. Thus, we aim to motivate students by raising their curiosity. However, we also hope to reinforce this motivation by increased engagement originating from PBL as the project-based STEAM activities require more time and in-depth inquiry, hence increased engagement. For the scope of this study, we define motivation as readiness to participate in an activity. Researchers at MIT also studied the role of motivation, interest and positive affect concerning learning (Picard et al., 2004) and found that positive affect can enhance the learning effect. Framed by the depicted theories, PBL and SDT, the following research questions guide our study:

RQ1: What factors seem to increase students' motivation while working on a STEAM task designed following the 5 features of PBL?

RQ2: How does the depicted STEAM task affect students' mathematical competencies?

The two questions are of different natures; hence we need different methods.

Methods

This mixed-method research is built on a case study focusing on the 5 features of PBL. To study students' motivation and answer the first question, we collected data from self-reports via questionnaires, focus groups, and observations. We apply grounded analysis, as we define the motivating effects based on the data. Further, to answer the second question, we refer to the results of the informal competence test IKM Plus to study changes in students' achievement. IKM Plus is carried out yearly in Grades 3, 4, 7 and 8 in Austria starting in 2022. The study started in 2022, in the school year 2022/23 and stretched over more than a year. The last phase, printing some of the models, took place the next school year, at the beginning of 2024 as the involved teacher needed to gain experience with 3D printing and the school received the printers just in January.

The Project

The project was carried out in two Grade 7 classes involving 48 students starting in 2022. Students had to design the house of their dreams in smaller groups. The task design followed the 5 features of

PBL: (1) students learned about the problem and had to draw a plan; (2) students explored the problem by designing a model using cardboard paper and modelling by Tinkercad (Figure 1); (3) students worked in groups and we collaborated with another school and a local company to learn about 3D-printing (Figure 2) at small scale and the industrial use of 3D-printing; (4) students used Tinkercad to design a more detailed model; (5) students built their houses using various materials and we 3D-printed one of the models. As we needed to gain experience with 3D-printing and access to a 3D printer, this last step was carried out at the beginning of 2024. The learning aim was to practice measuring, proportions, scaled drawing, and enhance spatial imagination. During the whole project, students worked in teams, they could select the team members and had no limitations on what kind of a house they should design.



Figure 1: Cardboard and Tinkercad models

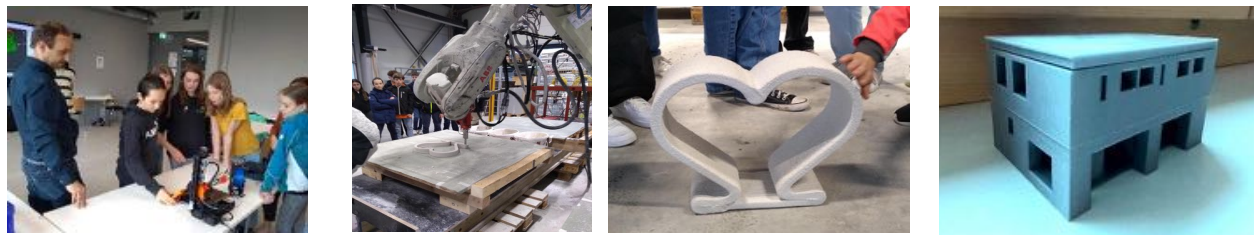


Figure 2: 3D-printing: workshop, visiting a local company, the printed model

Data Collection

To answer the first research question, in 2022 we collected data in both participating classes containing 48 Grade 7 students using a questionnaire based on a 5-point Likert scale and some open-ended questions. We aimed to receive feedback on more projects in general and the open-ended questions served to shed light on students' likes and dislikes. Some questions focused on perceived usefulness and the learning effect of the projects. We observed students' engagement in project activities, and how elaborate the models were. Students received the questionnaires right after completing the houses and participating in the 3D-printing workshop. A year later, after 3D-printing one of the models, we collected data from the same two classes containing only 43 students as some left the school. We again used a questionnaire based on a 5-point Likert scale and open-ended questions. This time, the questions were more specific and focused on the house modelling project. The open-ended questions aimed to detect any changes in students' views. We also interviewed a group of students selected based on the elaborateness of their models as an indicator of their engagement and motivation.

To answer the second research question, we use data collected by the Austrian informal competence test, IKM Plus. The test is carried out as a new curriculum was to be introduced in Austria in

September 2024 to get an overview of the current state of the competencies in German, Mathematics and English and to study how students' competencies develop. All students in Grades 4, 7 and 8 were tested in 2022 and 2023. This is an informal competence test, carried out online and providing feedback online both to the students and the teachers. The feedback contains a comparison between the students in the class and the Austrian average. As the involved classes were tested in two consecutive years, we learned about the students' development from one year to the other compared with the Austrian average which helps us to conclude the possible effects of the project.

Results

First, studying the first research question, we present the data collected with the questionnaires. In 2022, 48 students received the questionnaire, and 33 filled it in. The first six questions were based on a 5-point Likert scale and aimed at students' general views, and whether they considered the project interesting and beneficial for their learning: 54,5% of the students found the projects interesting, and only 15,1% of the students found it not interesting. To define the factors enhancing motivation, the open-ended questions are more relevant. Only 20 students answered the open-ended questions and even they answered with just one or two words. The open-ended questions aimed to learn about students' likes and dislikes and if they had any suggestions. Table 1 shows the analysis of students' answers. We counted how often some specific aspect was mentioned and put the related references under the same category marking them with the same colour. For instance, the three references: eating a pizza, watching a film, and sleeping in the school seem to refer to social aspects. Frequencies in Table 1 highlight that the social aspects were mentioned 17 times in total, and thus seem to be the most important. Working in teams was second, and this is also a social aspect. The third most frequent factor was presenting the results and no time constraints. The students also praised that they could decide what they wanted to work on and with whom, as the following excerpt from one of the students' answers shows, we will refer to him by the nickname Dave:

I especially liked that I could decide what I wanted to do and how I wanted to do it. I really enjoyed not having to build a boring house, but I could create whatever I wanted to. (Dave)

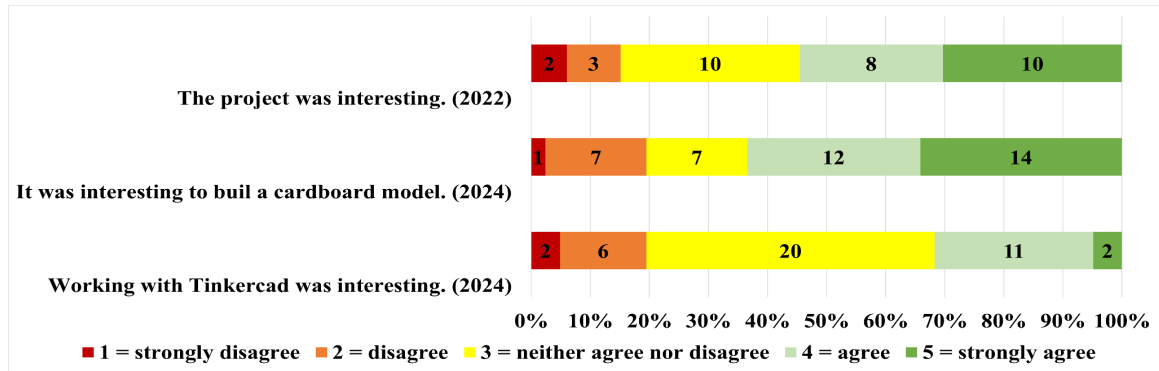
Based on our observations, technology implementation seems to increase students' motivation. For instance, using Tinkercad for modelling and learning about 3D-printing made students communicate more as they started cooperating and looking for solutions. Students also seemed interested during the visit to the local company. More than half of the class had never seen a 3D printer before and were amazed by the size of the robot used in the industry. To answer the first research question, these factors enhancing motivation seem to be the most important: spending time together and working in teams, having enough time, and working on an interesting problem. Considering the recency effect, the next school year, both classes filled in a similar questionnaire. This time the questions were more specific and focused on the house modelling project to learn about students' retrospective views.

43 students received the questionnaire, and 41 students completed it. The first questions focused on general interest, and the later ones on the perceived learning effect. Table 2 shows a comparison of the answers received in 2022 and 2024.

Table 1: Frequency of students' likes and dislikes in 2022

Likes				Dislikes			
Aspects		Frequency		Aspects		Frequency	
Social	Pizza	5	17	Workload	Presenting	3	6
	Film	6			A lot of work	3	
	Sleeping in the school	6		Inconveniences	Too little room	1	4
Working in groups	5	Too little time	3				
	Presenting our results	2		Everything was fine	6		
	Free choice of topic	2					

Table 2: Students' general views on the project in 2022 and 2024



Here, we present just the most general questions as these are relevant to the first research question. The answers show that an even bigger part of the students, 63,4%, remembered the project as interesting. The open-ended questions were aimed at students' likes and dislikes again, and we looked at the frequencies again. Table 3 shows the analysis. Compared to the previous year, the answers were more detailed and showed different patterns. We marked similar aspects and synonyms with the same colour and defined categories. For instance, designing, creativity, constructing and being an architect may be one category as they all refer to creating. We also interviewed a group of students selected based on their engagement in the project activities. Transcribing the recordings and the qualitative analysis are still in progress, therefore we would quote just from one student already referred to as Dave as his sentences seem to reflect the possible clusters represented in Table 3:

Dave: I thought it was just great that we could build the house with friends, that we were allowed to use whatever materials we wanted, on whatever scale we wanted, that we really had free rein with all the work, that it was really about the basic plans that we designed beforehand. I find it very interesting how you go from design to product.

Based on Table 3 considering the frequencies both under the aspects likes and dislikes, along with Dave's sentences, students may be more engaged if they have a creative problem where they can realise their own ideas and they work preferably in teams. Among dislikes, they mentioned visiting the 3D-printing company, but they argued that it was due to the fact they were not actively involved in creating during the visit. This aspect was not formulated in the questionnaire in 2022 yet.

Table 3: Frequency of students' likes and dislikes 2024

Likes				Dislikes			
Aspects		Frequency		Aspects		Frequency	
Creating	Designing, using your own ideas	9	40		Visit at the 3D-printing company as not interactive	4	
	Creativity	17			Not enough work with 3D-printing	2	
	Constructing the house	12		Technical problems	Tinkercad: didn't work sometimes	3	11
	Being an architect	2			Work in group was difficult.	4	
Working in groups	14	Transport of the model	3				
Interest	Fun	8	10		Scales were difficult	1	
	Not the usual lesson	2			Everything was fine	10	
	Using Tinkercad	3			Too little time	1	
Real-life	3D-printing company	2	3				
	3D-printing real-life.	1					

Time also seems an important factor. In 2022, students referred to no time constraint as a positive aspect, in 2023 they referred to lack of time as a negative aspect. This implies that having enough time allows them to work without pressure which may enable them to create. Summing up our results, working in groups and the social aspects, having room for creativity and the freedom to decide how and what to create and having enough time may increase engagement and motivation.

To answer the second question, we study the results of the IKM Plus test. Starting in 2022, all Austrian students in Grades 3 and 4 at primary schools, and Grades 7 and 8 at secondary schools participate in a competence check, IKM Plus to provide students, teachers and the government with feedback about the current stand of students' competencies in German, Mathematics and English. The two classes participated in the first test round as Grade 7 students and a year later in the second one as Grade 8 students. Hence, we can compare the results from the two years and may find some tendencies. Because of the lack of space and since the results are similar, we compare the results of one of the experimental groups participating in the project with the Austrian national average, which we consider a very big control group as it involves all Austrian students at the same type of school and the same age. We compare how the results evolved from the year 2022 to 2023, that is, the results before the project and after the project. Figure 3 shows that the mathematical competencies of one of the experimental classes improved from 168 to 188 points, as opposed to the Austrian national average of the same school type (AHS in the diagram) which improved from 160 to 172 points. Three students achieved the available maximum of 220 points and only four had scores below the Austrian national average. Further, we present the results of one of the students who was the most motivated and engaged considering all phases of the project. We already referred to him as Dave when we quoted his written feedback and an excerpt from his interview. In 2022, he scored 173 competency

points and 208 points in 2023. The class’s results improved by 20 points, the Austrian national average improved by 12 points, and Dave’s by 35 points.

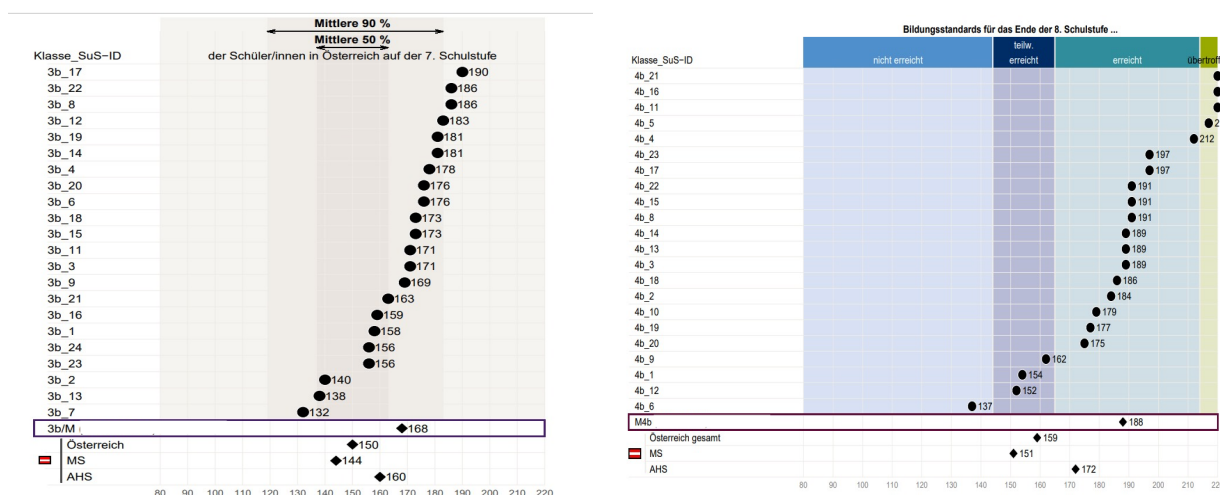


Figure 3: Feedback gained from the IKM Plus competency test carried out in 2022 and 2023.

What the reasons for this difference in improvement are, requires more thorough analysis. This is beyond the scope of this study; however, it is planned to be studied in future papers. One reason may be that the class, especially Dave, were more engaged in the house-building project and in other projects too. This increased engagement resulted in increased motivation, hence in a significant increase in the competency test results. Based on these results, in response to the second question, it seems that project-based STEAM activities can enhance learning and result in higher test scores.

Discussion and Conclusion

Plenty of case studies can be found in the literature about the motivating effect of STEAM activities where real-life reference seems to be the crucial factor (Ulbrich et al., 2020) as also mentioned in the literature review. There are even ones where architecture and modelling buildings are the focus (El Bedewy et al., 2022) just as in our project. However, our findings show that the social aspects and creativity seem to be the most important factors. Liljedahl and Zager found that if you give students a task that really makes them think, then they start working out of curiosity and this resembles Dave’s words. They also showed that working in teams has a motivating effect (Liljedahl & Zager, 2021), just like van Tran (2019) in his work on cooperative learning. These findings seem to agree with ours. As Picard and her colleagues (Picard et al., 2004) showed, positive emotions and motivation enhance learning, and the presented IKM Plus test results seem to support this. Additionally, we found that project-based STEAM activities as cooperative activities enhance motivation drawing on the social aspects and the longer and deeper student engagement. One of the 5 features describing PBL is students managing the problems and teachers just providing scaffolding. This feature underpinned the task design and was praised by the students as shown in Table 3 and Dave’s words. Dave’s example shows that increased engagement can turn into intrinsic motivation, and this can enhance learning.

To answer the research questions, we found the social aspects, creativity, free choice of topic and enough time to create as the most important factors to increase motivation. We also found that this increased motivation may result in higher achievements as shown by the IKM Plus competency test

results. As we are aware that motivation is the result of a large variety of factors, we will look further into the role of freedom and time to create, and which other factors can also be of importance.

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Teacher actions to promote mathematical competencies in programming activities

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The use of mathematical competencies in education has gained considerable attention, especially over the last two decades. This paper outlines a study that explores the integration of programming used to enhance students' engagement with mathematical competencies. The study introduces a teaching session wherein students program physical robots, with a primary emphasis on generalizing the programs of the robots using mathematics. Our results show the importance of the teacher to ask questions that focus the discussion on mathematics, and we demonstrate how removing the robots prompts the students to communicate their ideas and reasoning. Our conclusion asserts that teaching sessions which emphasize mathematical competencies within the framework of programmable robots hold the potential to foster student engagement with respect to these mathematical competencies.

Keywords: Mathematical competencies, Mathematics education, programming, robots.

Introduction

Over the past decade, programming has been introduced into education in many countries (Balanskat & Engelhardt, 2014). In the Nordic countries, programming is mainly introduced as an integrated part of traditional school subjects (Bocconi et al., 2018). In the Norwegian curriculum, the main responsibility for teaching programming is placed in the subject of mathematics, starting in primary school (Kunnskapsdepartementet, 2020). It is specified in the mathematics curriculum that the students shall learn programming, but its relation to mathematics is not explicit, and the practical implementation is left to the teachers. Thus, knowledge about good teaching practices when teaching programming in mathematics is needed.

Many studies have established the connection between programming, computational thinking and algorithmic thinking skills (Weintrop et al., 2016), and studies investigate how mathematical topics can be learned by using programming (Misfeldt & Ejsing-Duun, 2015). The literature review by Forsström & Kaufmann (2018) identify three themes for the justification of programming in mathematics: motivation, student performance in mathematics, and the collaboration between students and changed role of the teacher. In this paper our focus is slightly different. Our main interest is on developing the mathematical competencies of the students, and programming is the setting in which we choose to do so.

There exist many theoretical frameworks for understanding the core of what mathematics is. Niss and Højgaard (2019) define eight mathematical competencies in the KOM-framework. The definitions of these competencies are made to capture the essence of mathematics, independent of the level of the person doing mathematics. These competencies were important in the creation of the framework for the PISA survey and Turner et.al. (2013) have created operational definitions of the competencies to analyse the difficulty of PISA items. We can also find these ideas in the Norwegian mathematics curriculum, described as the “core-competencies” of mathematics: exploration and problem solving,

modelling and applications, reasoning and argumentation, representation and communication, abstraction and generalization, and mathematical fields of knowledge (Kunnskapsdepartementet, 2020) (The Norwegian curriculum uses the term “core-elements”, but we have decided to use the term core-competencies).

In this paper we describe a teaching session developed in the research project “Programming for developing mathematical competencies”. In this project, the teaching sessions are developed in an iterative design process in collaboration between researchers and teachers, and the teaching sessions are tested in classroom experiments. In previous work, five teacher actions that promote mathematical competencies are suggested based on this design process (Brandsæter & Berge, 2024), and in the current paper we focus on three of the suggested teacher actions: (1) Simplify needed syntax and remove non-mathematical elements, (2) disable the run-command, and (3) remove obvious representations and open up problem to new representations, and ask the research question: *How are students exposed to mathematical competencies when a teacher implements actions designed to promote students’ work on mathematical competencies?*

Theoretical framework

In the analysis we use part of the theoretical framework developed by Brandsæter and Berge (2024), which is mainly based on the MEG item-difficulty coding framework (Turner et.al., 2013), the competencies in the KOM-framework (Niss & Højgaard, 2019), and the Norwegian curriculum, see Table 1. This study is limited to three of the core competencies.

Table 1: Framework for analysing work on mathematical competencies (Brandsæter & Berge, 2024)

Competence	Indication
Exploration and problem solving	<ul style="list-style-type: none"> a) Search for patterns and relationships (Norwegian Ministry of Education and Research, 2019, p. 2). b) “[S]electing or devising, as well as implementing, a mathematical strategy to solve problems arising from the task or context” (Turner et al., 2013). c) Use problem solving strategies (Pólya, 1990). d) Develop a method in an unknown situation (Eriksen & Vos, 2022).
Reasoning and argumentation	<ul style="list-style-type: none"> a) Present a “logically rooted thought processes that explore and link problem elements so as to make inferences from them” (Turner et al., 2013). b) Check a justification that is given (Turner et al., 2013). c) Provide a justification of statements (Turner et al., 2013). d) Make claims and conjectures.
Representation and communication	<ul style="list-style-type: none"> a) Interpret representations (Turner et al., 2013). b) Translate between representations (Turner et al., 2013). c) Make use of given representations (Turner et al., 2013). d) Select or devise representations to capture the situation or to present one’s work (Turner et al., 2013). e) Evaluate the choice of representation (Niss & Højgaard, 2019).

Methods

We study the implementation of a teaching session where the student group was In-service Teacher Students (ITS) attending a course in programming and digital tools for mathematics teachers, grade

1-10, with little to no experience with programming before taking this course. The ITSs have a teacher education from higher education, but not necessary a specialization in mathematics. The session took 2x45 minutes over two days and the ITSs were randomly divided into groups of two or three ITSs.

The teaching session is built around the Sphero Indi robot, a small electric toy car that can be given instructions by placing coloured tiles on the ground. When the robot drives over a coloured tile it reads the colour and performs the corresponding instruction. Thus, the programming of the robot is done by placing tiles on the ground and the physical placement of the tiles defines the program. This allows for an intuitive programming of the robots, mostly eliminating syntax errors. We say mostly, because inaccuracy in the placement of the tiles or the initial orientation of the robot do sometimes cause the program to not run as intended (the robot misses one of the tiles causing the program to “crash”). We have chosen this robot due to its simplicity in use, making it easy to get started and the syntax (coloured tiles) is intuitive for any age group in schools. The teaching session is divided into three tasks, and the goal for each task is to elicit different mathematical competencies. The teaching session starts by an introduction of the robots that shows how to start the robot by placing it on a green tile, and that it stops when it drives over a red tile.

Task 1: In the first task, the ITSs are given a green, a red and a yellow tile, and are asked “what does the yellow tile do?”. The task is meant as an introduction to the robots and how they are programmed, and it may at first glance seem like a trivial task. The yellow tile reduces the speed of the car, however, the speed difference between the initial speed and the speed set after driving over a yellow tile is quite small and not easy to see with bare eyes. This task targets the core competence *exploration and problem solving*, and it aims for the ITSs to use a systematic exploration of the yellow tile and to use mathematical relationships between, speed, distance, and time to design an experiment to test their hypothesis.

Task 2: The second task given to the ITSs is to “create a track that makes the robot drive on for as long as possible”. One solution to this task involves creating a loop that makes the robot run infinitely long. Getting the idea of creating a loop is not the challenging part of this task, as we will see later. The main focus of this task is that the ITSs, while cooperating in the groups to design a track, get experience and skills relevant for developing *representation and communication*.

Task 3: In the final task, the ITSs study different loops that use different number of tiles and investigate how this relates to the sum of the robots’ rotations. In collaboration with the teacher educator, the ITSs shall formulate and investigate the claim: “In a loop, the sum of the robot’s rotation is always a multiple of 360 degrees”. The task aims to give the ITSs experience with conjecturing, making statements and justifying about something that is always true, focusing on the core competencies *reasoning and argumentation*.

One of the authors of the paper was the teacher educator in the session, while the three other authors attended as observers. During the group work, each observer observed one group of ITSs and wrote a log of his/her observations. The logs include both descriptive and reflexive notes. The collected data from the three observers are combined and synchronized based on time stamps, allowing us to compare situations from the different groups. This is used to identify episodes where the teacher educator took an action to direct the ITSs work towards the mathematical competencies aimed for in the task.

The collected data is analysed qualitatively by identifying episodes where the ITSs are working on the core competence of the given task. We indicated (sub-)competencies in square brackets, in the results, based on the analytical framework in Table 1, e.g., [3a] indicate interpreting a representation.

Results and discussion

Simplify needed syntax and remove non-mathematical elements

All tasks involve elements where the ITSs can explore both the robots and mathematics. Nevertheless, the connection to mathematics was not always apparent to the ITSs, and in several instances the connections to mathematics were only made after direct questions from the teacher educator. This is exemplified through a sequence where the ITSs are working on the first task: “What does the yellow tile do?”. In this task the teacher educator *simplified the syntax* by removing all tiles but the yellow (slow down), green (start) and red tile (stop) (the full set consisted of 8 different colours). During the group work, the three ITS groups that were observed took slightly different approaches to the given problem. The first group explored the yellow tile in an unsystematic way, the second group tried systematically the different colours and combinations of colours [1c]. Both groups concluded that all the yellow tile did was to make a sound and flash yellow, even if they at some point in their discussion mentioned that perhaps the robot also decreased its speed. The third group first made a hypothesis that they proceeded to test by using the robot. By doing multiple runs with and without the yellow tile, they concluded that the robot slowed down, based on observations with their naked eyes.

When the teacher educator gathered all groups for a joint discussion, he first asked the ITSs if they knew what the yellow tile does. The first ITS to respond said that it makes a sound, but when the second ITS said that he thought that it slows down, the first ITS agreed but was not sure. The discussion continued as follows:

Teacher educator: Are you sure that the tile makes the car slow down? [To ITS 2]
ITS 2: Mmmm, I think so.
Teacher educator: What can we do to figure that out? How can we be sure?
ITS 3: With a stopwatch?
Teacher educator: With a stopwatch! Others?
ITS 4: Test with the tiles?

In the last suggestion the ITS proposed to compare one robot that runs over a yellow tile with one robot that does not run over a yellow tile [1d]. This involves a direct comparison of the speed of the robots to test if the yellow slowed down the robot. The first suggestion of using a stopwatch implies an indirect comparison between the speed of the robots [1d]. Here, a natural experimental setup is to measure the time used over the same distance. This may be a way to give an intuitive relation between time, speed, and distance.

Note that none of the observed groups took initiative themselves to use mathematics to reason why their solution was correct, they were satisfied with “seeing” that the robot “probably” slowed down.



Figure 1: Illustration of the sketches made by the ITS to represent two different tracks. Here, lines are used to indicate the indented path of the robot, and sharp corners represent blue or pink tiles.

Thus, simplifying the syntax to only investigate the yellow tile was not sufficient. In addition, the teacher educator had an important role to make the ITSs reason and justify their claims. To make the ITSs continue their exploration in a systematic way, the teacher educator asked what the ITSs could do to be sure what the yellow tile does. This was sufficient to prompt the ITSs to propose several methods for how they could show that the yellow tile slowed down the robots [1d].

Disable the run-command

In the next part of the session, two new tiles were introduced; a pink and a blue tile that makes the robot rotate 90 degrees to left and right respectively. With a set of tiles at their disposal (1 green (start), 1 red (stop), 1 yellow (slow down), 4 pink (left turn) and 4 blue (right turn)), the ITSs were asked to “create a track that makes the robot drive on for as long as possible”. The goal of this task is to facilitate development of reasoning and argumentation competence. We know that students’ use of unsystematic trial-and-error might have a negative impact on their formulation of mathematical arguments when engaging in programming (Kaufmann & Stenseth, 2021). Therefore, to provide mathematical reasoning and communication, the teacher educator after a while reduced the opportunities for the trial-and-error method in this activity by removing the robots (*disabling the run-command*).

When the robots were removed from the ITSs they started to use a variety of different strategies to evaluate their tracks. Pointing with fingers (or just with the eyes) and saying the instructions out loud were used by all observed groups [3d]. Objects such as unused tiles were also used to represent the robot and the ITSs would “drive” these objects through the tracks by moving them manually. One group additionally made use of pen and paper to represent the tracks, as illustrated in Figure 1, and used this representation when discussing feasible loops [3b,3d].

When the groups had access to the robots, we often observed that one group member laid out the tiles and then immediately ran the track with the robot without discussing it with the other group members. In addition, we observed that the ITSs spent a lot of time adjusting the tiles’ position, sighting the initial angle of the robot, and handling other technical issues with the robots. When the robots were removed, we observed that the ITSs more actively explained how they thought the robot would drive in the proposed track [2c].

This didactical choice is of course not without pitfalls. Removing the robots do not completely remove trial-and-error approaches as it is easy to visualize the track and adjust the tiles “on the fly”. Trial-and-error methods where the ITSs did not communicate or explicitly reason were still observed in all groups. Another issue is that when the robots are removed, this also removes the feedback from

Here, the teacher educator by the act of walking around introduced the ITSs to the idea that rotation can represent a dynamic measurement (not only a static) [3c], and one ITS made the initial claim that it is possible to make any multiple of 360 [2d]. In the continued work, the ITS struggled to argue for why the sum of the rotations in a loop must be multiple of 360 degrees. Two of the groups formulated algebraic expressions that did not help them in justifying the claim [1b,3d].

Concluding remarks

In this paper, we discuss the observations of a teaching session with a group of ITSs. We investigate how the ITSs work on mathematical competencies when the teacher educator uses three of the teacher actions suggested by Brandsæter and Berge (2024).

Trial-and-error approaches may have a negative impact on the mathematical reasoning of students (Kaufmann & Stenseth, 2021), and removing the robots from the ITSs in this study aimed at reducing the trial-and-error approaches. However, simply removing the robots was not sufficient to promote reasoning and argumentation among the ITS, it was necessary for the teacher educator to ask questions such as “why do you think ... happened?”, “how can you be sure?”, “how do you know?”, to make the students justify and reason about their claims. Removing the possibility of immediately running a program is also a key component of the PRIMM (predict, run, investigate, modify make)-methodology (Sentance et al., 2019) where the students shall predict what a computer code does before running it. It is important to keep in mind that disabling the run-command, is not something we think a teacher should always do. This choice should be evaluated by several factors, and maybe most importantly the goal of the teaching session.

Misfeldt and Ejsing-Dunn (2015) points to the importance of the teacher to focus on mathematical concepts and on the didactical principles used by the teacher when using programming to learn mathematics. The three teacher actions investigated in this paper did initially prompt students to work on mathematical competencies, but the actions were not sufficient by themselves. The teacher educator had an important role direct the attention of the students towards mathematical competencies during the ITSs work.

In this work we have observed ITSs experiences concerning the mathematical competencies exploration and problem solving, reasoning and argumentation, and representation and communication through the indicators presented in Table 1. While we found indications for work on all these competencies, we do not investigate to what extent or level each competence is exercised. Thus, in further works one should investigate in more depth each mathematical competence, e.g., by using the four levels of the MEG framework (Turner et al., 2013). In addition, expansion of the study to other mathematical competencies (Niss & Højgaard, 2019) should be done.

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Teachers exploring the potential of generative AI in mathematics teaching

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The focus of this study is to examine the extent to which generative AI (GenAI) can be used as a tool in mathematics teachers' professional development. In particular, we discuss a preliminary case study in which 37 primary and lower secondary school teachers were invited to explore the potential of ChatGPT to promote mathematical discussion in their classrooms. The study is framed by the well-known Mishra and Koehler's TPACK Model. Results show how when teachers are involved in formulating prompts to be given to ChatGPT and in interpreting its feedback, they can develop Technological Content Knowledge and Technological Pedagogical Knowledge. Further insight is required to deepen how teachers can be helped to integrate them and build their TPACK. Moreover, results call for the need to investigate if teachers who acquire proficiency in using GenAI in their teaching will impact their students' proficiency in using GenAI in their learning.

Keywords: Generative Artificial Intelligence, teachers' professional development, Technological Pedagogical Content Knowledge (TPACK) Model

Introduction

In the rapidly evolving landscape of education, the advent of GenAI has opened up new possibilities. In the realm of mathematics education, it offers opportunities to foster the design and implementation of dynamic and interactive learning environments. However, the potential of GenAI in this context remains largely untapped, and its role in teacher professional development is still an emerging area of research. This paper explores an innovative approach to professional development (PD) for mathematics teachers, focusing on the use of generative artificial intelligence (GenAI), specifically ChatGPT, as a tool to enhance classroom discussions and pedagogical strategies.

Our study is framed by the well-known TPACK model (Mishra & Koehler, 2006; Mishra et al., 2023), a comprehensive framework that encapsulates the complex interplay of knowledge teachers require for effective technology integration in their teaching. We aim to answer the following research question: How could the exploration of GenAI's potentials and limits, through the production of prompts and interpretation of feedback, enhance teachers' Technological Pedagogical Content Knowledge?

To answer this question, we present and discuss a preliminary case study involving primary and lower secondary school Italian teachers who were invited to explore the potential of ChatGPT in their classrooms. This PD intervention aimed to let teachers investigate how the use of GenAI could promote mathematical discussion. A problem given during the National Standardized Assessment test was used as a launch pad for the investigation.

We have found that by producing prompts and interpreting feedback teachers are engaged in enhancing their TCK and TPK. Further insight is required to deepen how teachers can be helped to integrate them and build their TPACK. The early results invite us to reflect on how using ChatGPT

or other artificial intelligence systems is challenging for 21st-century teachers. Even the shortcomings of ChatGPT, such as the fact that answers to questions can be wrong, can become subject matter for a critical thinking exercise. It seems, however, that only conscious use of this technology that takes into account the teacher's Technological Knowledge, Content Knowledge and Pedagogical Knowledge can result in effective teaching practices.

To contribute to the research on the use of AI in education and teacher professional development, our results finally call for the need to further investigate the potential impact of teachers' proficiency in using GenAI on their students' learning experiences.

Theoretical Framework

The teacher's role is fundamental in helping students construct mathematical meanings (Simon et al., 1999; Capone et al., 2023). For this reason, in this study, many complex aspects related to the use of intersections between Technological knowledge, Pedagogical knowledge and Content knowledge in teaching were taken into account. This paper referred to the Technological Pedagogical Content Knowledge (TPACK) framework, based on Shulman's (1986) construct of Pedagogical Content Knowledge, conceived to define the elements that can characterize teaching when supported by technologies without neglecting the pedagogical aspects and the specific teaching contents of the discipline. In particular, we refer to Mishra and Koehler's model, (2006) in which the authors describe how teachers need specific types of knowledge to use technology creatively and effectively in their teaching. The TPACK framework posits that achieving successful technology integration in education goes beyond expertise in specific content, pedagogical strategies, or technology individually. Instead, it emphasizes the necessity of a dynamic synthesis of all three knowledge bases. These components exist in a state of "creative tension", wherein alterations in one realm influence the other two.

With the advent of GenAI, the educational landscape faces novel challenges, raising profound inquiries about the essence of learning and the essential knowledge teachers must possess for effectiveness. GenAI also introduces a profound shift in the landscape of education, challenging conventional roles held by teachers. With their capacity to provide extensive expertise across various subjects, their adaptable generative abilities, and their capability to mimic human behaviour and communication, GenAI prompts a reassessment of the traditional teacher-student dynamic. Consequently, this paper employs the TPACK framework to delve into the essential knowledge that teachers require to seamlessly incorporate GenAI tools into their teaching methodologies (Mishra et al., 2023).

In the canonical version of the TPACK diagram (Figure 1) there are four main areas (or forms of knowledge) related to technology. They are Technological Knowledge (TK) as well as in the overlapping spaces that constitute Technological Content Knowledge (TCK), Technological Pedagogical Knowledge (TPK) and, of course, Technological Pedagogical Content Knowledge (TPACK). Additionally, though not explicitly mentioned, technology can have a significant impact on Contextual Knowledge (XK).

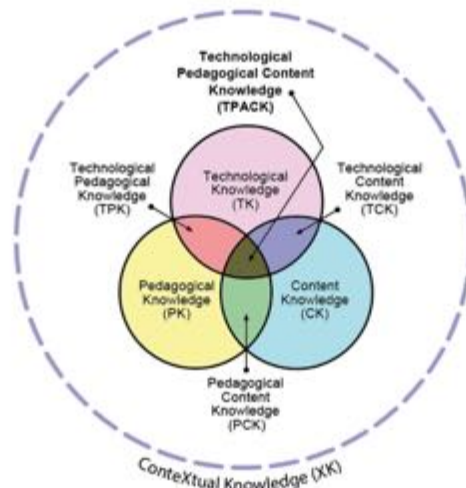


Figure 1: The Canonical TPACK diagram (Mishra et al., 2023)

Our exploration unveils several direct implications for integrating GenAI into education, particularly concerning the TPACK framework. Following Mishra and colleagues, our focus initially lies on how TPACK evolves within this new educational environment, encompassing the utilization of technological tools for teaching across diverse subject matters. Furthermore, our interest is in how teachers must innovate and adapt, developing novel pedagogical approaches to accommodate these advancements while acknowledging the presence of non-human social agents in the learning environment, which has traditionally been human-centric. Although these technological agents may outperform humans in certain areas, teachers must continually experiment and refine their teaching methods in response to ongoing technological advancements and changes.

Methods

In this paper, we refer to a case study in which 37 primary and lower secondary school teachers were invited to explore the potential of ChatGPT to promote mathematical discussion in their classrooms. They were particularly interested in exploring such a potential, especially because they declared they did not have any kind of experience in the use of GenAI but were aware of the growing interest and use by the students.

The PD intervention, which lasted approximately only one hour, was designed to engage teachers in producing prompts and analysing feedback. For this purpose, we showed them a problem coming from the Italian National Standardized Assessment test for 5th grade (see Figure 2).

To make 4 crochet placemats, grandmother uses 6 balls of cotton.
 a) How many balls of cotton of the same type she has to use to make 20 placemats?
 Answer:.....
 b) Write how you did to find the answer.

Figure 2: Translation of Task 11, Mathematics INVALSI test Grade 05, 2013 (Faggiano et al., 2022)

The problem was particularly interesting for at least three reasons: 1) when it was submitted in 2013, only 40.9% of grade-five Italian students succeeded in giving a correct answer and only 31.7% were able to describe the way to find the answer; 2) primary students' answers to the problem reveal their comprehension/non-comprehension of the multiplicative structures, however, since middle school

upwards, the most common way to approach this problem is proportional reasoning (Arzarello, 2018; Arzarello & Ferretti, 2021); 3) a recent study proved the difficulty of teachers to recognize as correct mixed solving approaches and to make use of this kind of problems into their teaching (Faggiano et al., 2022).

Teachers were asked to describe how they solved the problem and how they thought their students would do it. Then, they were invited to explore how ChatGPT could solve the problem and if its feedback could be used to enhance mathematical discussions with their students.

The PD intervention was audio-recorded and transcribed, and the chat on ChatGPT was saved. To ensure the reliability of the data analysis, the authors independently coded the excerpts. We sought terms and sentences teachers said, interpreting them as referring to different kinds of knowledge according to the TPACK Model. Then, we discussed the analytical processes that each of us performed and identified the most interesting transcript and part of the chat, that were translated into English for the purpose of this paper.

Results

When asked to describe how they solved the problem, most of the primary teachers said that they had multiplied five (the number of groups of four to make 20 placemats) times six (the number of balls of yarn to make 4 placements), while most of the lower secondary school teachers considered the proportionality between the number of placemats and the number of balls of yarn. Very few of the teachers multiplied by 20 the number of balls of yarn required to make a placemat (1.5).

The interaction with ChatGPT started with a simple prompt: “*Find possible solutions to the following problem...*”. Reading the feedback some of the teachers replied: “*students are better!*”. They felt the need to ask ChatGPT to come up with a solution that could be suitable and understandable for primary school students and agreed to submit the following prompt: “*How would you explain this to a primary school student?*”. The answer received from GenAI was a story to describe the problem at stake. However, there was no final solution, and, above all, there was a mistake in the interpretation of the text. Below we provide a translation of the ChatGPT’s answer:

Imagine that Thomas' grandmother is crocheting placemats, and she has a pile of balls of yarn next to her. Each time she makes a placemat, she takes a certain number of balls of yarn.

Now, Grandma told Thomas that to make 4 placemats, she uses 6 balls of yarn. So, if we want to know how many balls of yarn are needed to make more placemats, we can imagine it like this:

Imagine that Grandma has a total of 6 balls of yarn in the pile. She takes them and makes a placemat. Then, she takes 6 more balls of yarn and makes another placemat. She continues like this until she makes 4 placemats in total.

Now, if we want to know how many balls of yarn are needed to make 20 placemats, we can simply imagine Grandma repeating the same process over and over until she has made 20 placemats in total. Thus, she will use more balls of yarn than she used to make only 4 placemats.

So, to know how many balls of yarn are needed for 20 placemats, we have to imagine Grandma using more balls of yarn than before.

I hope this explains the concept more simply.

Initially, the story proposed by ChatGPT was consistent with the problem, but in the underlined part of the excerpt it can be seen that there is then an error in the modelling, related to the interpretation of the text: “*Grandma has a total of 6 balls of yarn in the pile. She takes them and makes a placemat*”, and not four placemats. In examining the story, the teachers easily spotted the inconsistency, pointing out the discrepancy between the initial text of the problem and the reinterpretation made by ChatGPT to make its answer more appropriate to the students’ level. However, ChatGPT’s subsequent considerations (“*we can simply imagine Grandma repeating the same process over and over until she has made 20 placemats*”; “*we have to imagine Grandma using more balls of yarn than before*”) were correct but did not result to answer the problem. The teachers then explicitly asked ChatGPT what the answer was and then could verify that the error did not compromise the result. Indeed, the feedback was:

If we want to know how many balls of yarn are needed to make 20 placemats, we can multiply the number of balls per placemat (1.5) by the number of placemats (20). So, Grandma will need 30 balls of yarn to make 20 placemats.

Later, at the prompt given by the teachers to point out the mistake (“*You probably made a mistake. You told me that Grandma uses six balls of yarn to make a placemat*”) ChatGPT lost his bearings, failing to identify the mistake and continuing to make further mistakes:

You are right, I apologize for the inconsistency. If Grandma uses six balls of yarn to make a placemat, we need to recalculate the correct answer.

If Grandma uses 6 balls of yarn to make a placemat, we can calculate how many balls of yarn are needed to make 20 placemats by multiplying the number of balls per placemat (6) by the number of placemats (20). So, Grandma will need 120 balls of yarn to make 20 placemats. Thank you for pointing this out.

The following excerpts report some of the teachers’ comments to this feedback:

A: It’s lost the track, it’s out.

B: It thinks the error is in the last calculation made.

C: Because it is a machine, it only gives an interpretation of natural language.

D: You have to make the kids understand that depending on how you ask the question it can lead you astray and if you don’t realise the error you take it for granted.

Discussion

This episode shows, first of all, how thanks to this activity, teachers can realise the limits and potential of the used tool and acquire a specific technological competence (TK) recognising that these are alien intelligence (Warr et al., 2023), dissimilar from human intelligence: “*Because it is a machine, it only gives an interpretation of natural language*”. This is in tune with what Mishra and colleagues claim:

GenAI’s vast knowledge base and its ability to work, tirelessly, with complex concepts in sophisticated ways, combined with its propensity to hallucinate have led some to suggest that the best way of thinking of these technologies is as if it were a “smart, drunk intern”. (Mishra et al., 2023, p.8)

Moreover, this episode highlights the need to possess in-depth Content Knowledge (CK) to recognise errors made, in this case relating to the interpretation of the text and its modelling. We consider worth of note to emphasise that, in accordance with the TPACK Model, CK is independent of the use of GenAI. In fact, ChatGPT errors are also widespread among students and, as shown also by some recent studies (Spagnolo et al., 2024), teachers are not always able to distinguish ChatGPT's errors from those made by pupils. However, to critically analyse the feedback received and recognise any errors made by the AI, teachers are required to develop adequate Technological Content Knowledge (TCK), and the proposed example demonstrates how the activity was effective in this regard: *"It's lost the track, it's out"*; *"It thinks the error is in the last calculation made"*.

On the other hand, good pedagogical knowledge (PK) is also necessary for error to become a teaching resource. Here again, we consider it important to emphasise that PK does not depend on the use of GenAI. However, to promote discussions from errors, teachers need to develop adequate Technological Pedagogical Knowledge (TPK), and the proposed example demonstrates how the guided interaction with ChatGPT was effective in this regard: *"You have to make the kids understand that ... if you don't realise the error you take it for granted"*.

These initial results therefore show how exploratory activities of this kind can help teachers go beyond statements such as *"These tools inhibit the brain, they are dangerous"*, according to which GenAI can only be a limiting factor for students' cognitive development. With this PD intervention, they could enhance their TCK and TPK, which are fundamental for using GenAI critically in mathematics teaching. In accordance with the TPACK Model, finally, if only TCK is possessed, it might be difficult for the teacher to understand how to effectively integrate the use of ChatGPT with its potential and limits in educational practice, reducing student engagement in the learning process. On the other hand, without knowledge of the content, i.e. with the TPK alone, the teacher may not be able to assess the accuracy and relevance of the answers provided by ChatGPT and fall into error compromising the quality of the teaching intervention.

Conclusions

The motivation behind this research is the storm caused in the educational world after the launch of Open AI's ChatGPT. We were influenced by the workings and functioning of ChatGPT, so we were curious to explore our knowledge and understanding regarding it and other chatbots, like Google Bard AI. In particular, our focus has been on teaching, because we believe that any innovation in education starts with the teacher and his or her professional development. In the attempt to answer our research question (How could the exploration of GenAI's potentials and limits, through the production of prompts and interpretation of feedback, enhance teachers' Technological Pedagogical and Content Knowledge?), we developed a PD intervention with primary and lower secondary school teachers. We showed that exploratory activities involving the use of ChatGPT can be highly useful in gaining an in-depth understanding of the system's potentialities and, consequently, developing technological knowledge (TK). In addition, such activities prompt the development of technological pedagogical knowledge (TPK) by understanding how GenAI can be effectively integrated into the teaching environment, exploiting ChatGPT mistakes as teaching resources. Finally, the critical

analysis of the feedback provided by GenAI promotes reflection on the best ways to use this technology in the classroom, developing teachers' technological content knowledge (TCK).

However, in order to effectively integrate this knowledge and move on to technological, pedagogical and content knowledge (TPACK), a next step involving the design, testing and analysis of specific teaching interventions involving GenAI is required. This process would allow teachers to develop a deeper understanding of the interactions between technology, pedagogy and content, facilitating the integration of GenAI into educational practice in an effective and meaningful way.

We acknowledge that our discussion has some limits: it was based only on a single one-hour intervention, and it is not a systematic investigation. This initial experimentation does not provide us with results such that we can propose a definitive and structural solution to the use of GenAI in classroom practices, nor do we feel able to definitively extol or condemn its use for teaching purposes. However, it does allow us some reflections that take into account the theoretical framework adopted. In order to use ChatGPT effectively, as has been the case with other technologies in mathematics teaching, it is essential for the teacher to have a combination of Content Knowledge (CK), Pedagogical Knowledge (PK) and Technological Knowledge (TK).

Without content knowledge, the teacher may not be able to assess the accuracy and relevance of ChatGPT's responses to the specific learning objectives of the disciplinary content. This could compromise the quality of the teaching intervention, as the teacher would not be able to discern whether the information provided is correct or relevant. On the other hand, technological content knowledge allows the teacher to identify, as in the present case, where ChatGPT is wrong. This is where pedagogical knowledge comes in. The teacher not only needs to recognize the error but also to use the error as a possible resource to foster change. Through orchestrated discussions, for example, the teachers can provide the students with opportunities to learn by discovery, as well as new creative opportunities. However, for the teachers to develop adequate knowledge for proper classroom use of GenAI, they must be able to recognize its limitations in interpreting language. On the other hand, without technological knowledge, the teachers may not be able to effectively integrate ChatGPT into the teaching and learning process. They may lack an understanding of how to take full advantage of the technology's potentialities and limits to enhance the students' learning experience, thereby reducing their involvement in the educational process.

Ultimately, a balanced combination of content knowledge, pedagogical knowledge and technological knowledge is essential to fully exploit the potential of tools such as ChatGPT in the educational environment, ensuring effective and engaging learning for students. Results show how when teachers are involved in formulating prompts to be given to ChatGPT and in interpreting its feedback, they can develop Technological Content Knowledge and Technological Pedagogical Knowledge. Further insight is required to deepen how teachers can be helped to integrate them and build their TPACK. Moreover, results call for the need to investigate if teachers who acquire proficiency in using GenAI in their teaching will impact their students' proficiency in using GenAI in their learning.

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Creating straight-line helicoid models in the open-source software GeoGebra and their consequent 3D printing

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The use of information technology, related applications, and open-source software is essential not only for industry and practice but also for the education purposes of future engineers and teachers. Mathematics plays an indispensable role in this respect, both in the form of creating a logical structure of algorithms in programming and in the context of developing mathematical and logical thinking. We describe the possibilities of using the open-source software GeoGebra and its online environment in the university education of future mechanical engineers and secondary school mathematics teachers, mentioning opportunities and pitfalls of creating virtual 3D models of straight-line helicoids in GeoGebra and their consequent 3D printing. We present the main results of a case study that reflects the implementation of STEM elements in geometry education by creating virtual 3D models in GeoGebra and their subsequent 3D printing.

Keywords: GeoGebra; 3D models; visuospatial abilities; implementing 3D printing; straight-line helicoids.

Introduction

In the 21st century, we are facing a considerable surge in the development of information technology, related applications, and open-source software. Most people use these technologies as valuable work, learning and entertainment tools. To use them effectively and efficiently in their studies and to prepare students for 21st-century work, undergraduates must also be digitally literate and understand critical concepts in computer science. Students must understand that computer science combines theoretical principles and application skills. They need to think algorithmically, in other words, to think computationally and solve problems in other subjects and areas of their lives (Dagiene & Stupuriene, 2016).

Visuospatial abilities are considered to be one of the basic abilities of an individual in addition to reading, writing and arithmetic (Mulligan, 2015). The importance of visuospatial abilities for individuals is undisputed and well-documented in many studies (e.g., Battista, 1999; Molnár, 2009). Acquiring competencies in spatial perception, visualization, and the ability to imagine positional relationships among objects is by no means only valid for correctly answering a few problems related to school geometry. These competencies have been shown to be of fundamental value in many areas of human activities, including scientific thinking and industrial applications (Castro-Alonso & Uttal, 2019). Visuospatial abilities are a prerequisite to successfully studying all industrial technology and engineering programs (Sorby & Baartmans, 1996) because technical graphics and design are essential parts of the core curricula for these areas (MCTM, 2000). Also well-known is that the ability to imagine spatial transformations supports mental arithmetic operations (Hubbard et al., 2005). At the same time, it has become apparent in recent

years that an increasing number of students have problems visualizing a correct and appropriate image of a spatial object based on a plane drawing. This also applies to students in engineering degree programs at universities.

Current research also indicates that targeted training can strengthen and develop visuospatial abilities (e.g., Uttal et al., 2013). Indeed, many studies have shown that the simultaneous use of real models and computer simulations leads to better development of students' visuospatial abilities and a deeper insight into the subject matter (Hobson et al., 2010). To combine both areas - the use of real models and computer simulations, together with the active involvement of students in teaching, the possibilities of 3D printing can be used very effectively. Proper use of its aspects in teaching geometry leads to a better understanding of geometry, the development of students' visuospatial abilities, and an increase in their mathematical and abstract thinking (e.g. Dilling & Witzke, 2020; Dilling & Vogler, 2021; Ng & Ye, 2022).

The aim of this work is to analyze and exemplify the possibilities, benefits, and feasibility of using GeoGebra in the higher education of future mechanical engineers and secondary school mathematics teachers in the modeling and 3D printing of straight-line helicoids. These types of surfaces are common in various engineering applications and are sufficiently sophisticated to be used as a prototype for a whole class of content. Straight-line helicoids also cause difficulty in challenging the students' visualization. It is difficult for students to form correct images of these types of surfaces or even to model them using the software. To increase students' visuospatial abilities, we use the interactive virtual 3D graphical models created in GeoGebra simultaneously with 3D-printed models of these surfaces.

The described and similar dynamic applets and 3D-printed models are used in teaching geometry for mechanical engineering students and future secondary school mathematics teachers at the Technical University of Liberec (TUL). This is not only to improve their visuospatial abilities but also for their future profession in the context of STEM.

Virtual 3D model creation and 3D printing education at TUL

One part of the preparation for mathematics teacher students is the course "Geometric Software", and for mechanical engineering students, the course "Constructive Geometry". The contents of these courses are similar. Both groups of students study the basic knowledge and principles of analytic geometry in three-dimensional space, Monge projection, and differential geometry of curves and surfaces. Given geometry's small hourly allowance, other effective teaching tools are needed. Therefore, we use dynamic geometry software GeoGebra and 3D-printed models, especially when teaching more demanding topics, such as helicoidal surfaces, in the context of STEM education to develop students' mathematical and geometrical competencies. Based on our experience so far, but also in several published studies, it seems that students develop correct ideas about, for example, different types of straight-line helicoids when they perceive them as manipulable 3D real models rather than only as 2D static or dynamic representations (e.g., Corum & Galofalo, 2015; Huleihil, 2016).

Understanding the basic properties and principles of straight-line helicoids is difficult for mathematics teacher students and mechanical engineering students. This is because, in addition to a good level of spatial imagination, knowledge of various areas of mathematics (e.g., differential calculus of functions of one and two variables) and geometry (e.g., analytic geometry in space, vector functions, differential geometry of curves and surfaces) is required for the study of straight-line helicoids, which can be considered as complex spatial objects. Suppose the students can create in their imagination the correct images of the different types

of straight-line helicoids, their axial or normal planar cut sections, tangent planes, etc. In that case, they will understand much better how to proceed in their constructions and what to calculate in their calculations. These conclusions are based on more than twenty years of teaching experience on helicoidal surfaces at TUL.

In addition, ten students used GeoGebra to create dynamic applets based on generated surfaces of different types, which are part of the shared GeoGebra Book platform (e.g., <https://www.geogebra.org/m/s8wwh6fn>). In a further step, mathematics teacher students create suitable didactic aids from virtual 3D models, which are printed on a 3D printer and used in the lessons as teaching aids. In this way, students acquire mathematical competencies and the ability to implement information technology and 3D prints in mathematics education.

Setting 3D virtual models of straight-line helicoids in GeoGebra

The open-source software GeoGebra has different types of object inputs. There are three possible object inputs suitable for setting virtual 3D models of straight-line helicoids:

- generating virtual 3D models of helicoids by using the input command;
- drawing objects directly either in the 2D drawing windows or in the 3D graphics window using the tools of the software;
- creating 3D virtual models of helicoids by animating a straight-line or parts of it.

In this paper, we will present our experience creating virtual 3D models of straight-line helicoids using their respective parametric equations. The creation of virtual 3D models of cyclic helicoids and their normal and axial intersections is described in Bímová & Břehovský (2023).

The parametric equations of straight-line helicoids can generally be written in the form

$$\begin{aligned}x &= \rho \cdot \cos w + u \cdot \sin(\varphi - w) \cdot \sin \phi, \\y &= \rho \cdot \sin w + u \cdot \cos(\varphi - w) \cdot \sin \phi, \\z &= \varepsilon \cdot b \cdot w + u \cdot \cos \phi\end{aligned}$$

in which for the particular parameters, it is true that

- $u \in I \subset (-\infty, +\infty)$ is the parameter of u -curves (straight-lines) of the given straight-line helicoid;
- $w \in J \subset (-\infty, +\infty)$ is the parameter of w -curves (helices) of the straight-line helicoid;
- $\varphi \in \langle 0, 2\pi \rangle$ is the magnitude of the deviation of the orthogonal projection p_1 of the creating line p into the coordinate plane (xy) and the tangent t of the circle k at point A (while the circle k lies on the rotating cylindrical surface of a helix s generated by a given helical motion of the point $A \in p$);
- $\phi \in \langle 0, \pi \rangle$ is the magnitude of the deviation of the straight-line p and the axis z of the right-handed Cartesian coordinate system;
- ρ is the distance of the point $A \in p$ from the axis $o \equiv z$ of the given helical motion;
- b is the reduced pitch of the given helical motion;
- $\varepsilon = 1$ determines the right- and $\varepsilon = -1$ the left-handedness of the corresponding helicoid.

The case study was conducted in the context of Geometry Software training with ten mathematics teacher students in two phases. In the first phase, students created dynamic applets of virtual 3D models of straight-line helicoids. In the second phase, these students created the files needed to 3D print the created models. The essence of the case study was to observe and record the work, interactions and reactions of the students, who can collaborate or consult each other's designs while creating the applets. The aim of the case study was to receive direct feedback on the proposed learning activities and to gain a framework of their effectiveness in teaching straight-line helicoids and their potential for developing students' visuospatial

abilities. Let us mention some of their reactions and comments from the first part of the case study:

- Student 1: If we link sliders with the corresponding parameters that appear in the parametric equations of the straight-line helicoids, they will dynamically change the parameter values of the particular helicoid.
- Student 2: Yes, it is true. Then, by moving the sliders of the respective parameters, the graphical visualization of the virtual 3D model of the straight-line helicoid in the 3D graphics window and the values of the parameters in the parametric equations in the 2D drawing window simultaneously change.
- Student 3: For which kinds of parameters should I use the sliders? There seem to be seven different parameters in the parametric equations.
- Student 1: Let's link the parameters ρ , b , and ε .
- Student 3: And what about the values of the deviations, φ and ϕ ? Where should I insert them?
- Student 4: It is easy, insert them into the command “Surface(<Expression>,<Expression>,<Expression>,<Parameter Variable 1>,<Start Value>,<End Value>,<Parameter Variable 2>,<Start Value>,<End Value>)” as parameter variable 1 and 2 in the input line.
- Student 3: Great, thank you.
- Student 5: It is super to see how changes in the values of each parameter affect the design or shape of the 3D virtual model of the straight-line helicoid. I can imagine the helicoids more vividly.
- Student 6: Let's have a look. Setting the appropriate values of the deviations, φ and ϕ , creates certain types of straight-line helicoids. I have a better idea about the individual kinds of straight-line helicoids now.
- Student 7: That is perfect, that we can immediately see the feedback on the effect of changing the parameters on the type of straight-line helicoids. I can distinguish much better between the generating curve and the trace curve. The creation of the dynamic applet helped me a lot.

The preview of the dynamic applet created for the right-handed orthogonal closed straight-line helicoid is shown in Figure 1. In this applet, the settings of each parameter are placed in the lower right corner.

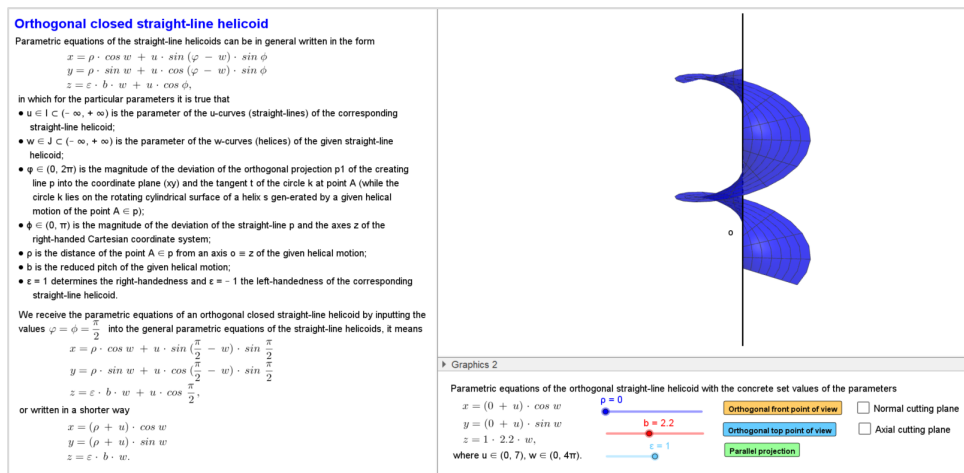


Figure 1: Dynamic applet of the right-handed orthogonal closed straight-line helicoid

Generating planar intersections is helpful in some technical practices or constructions, so students are asked to create them in their produced dynamic applets. Although GeoGebra includes the *Intersect Two Surfaces* tool, unfortunately, this tool does not work at all for plotting axial and normal planar intersections of straight-line helicoids. This fact prompts students to create appropriate intersection curves in their minds, using their visuospatial abilities. The hint for the normal planar intersection of the oblique closed straight-line helicoid is shown in Figure 3 (parts a and b), and the hint for the axial planar intersection of the oblique open straight-line helicoid is shown in Figure 3 (parts c and d).

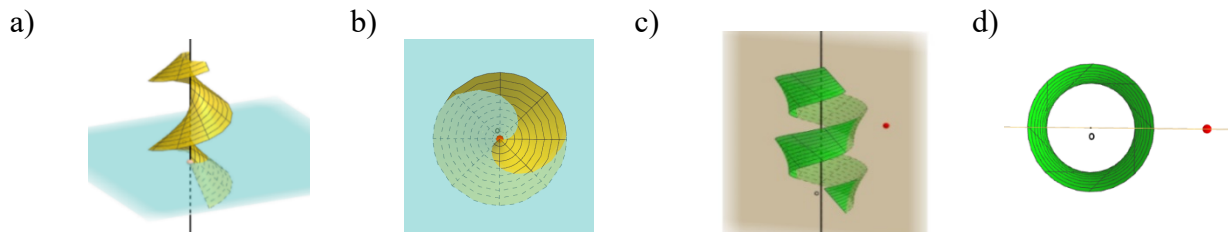


Figure 2: Planar intersections of right-handed oblique straight-line helicoids

GeoGebra displays spatial constructions in its 3D graphics window, which is a two-dimensional representation. Students often struggle to imagine particular straight-line helicoids as physical objects from these 2D representations. GeoGebra contains a unique tool for creating a representation for anaglyph glasses to show the scene in the 3D graphics window. Using anaglyph glasses, students can view the generated virtual 3D models of the individual helicoids “spatially”. Figure 3 shows four different types of straight-line helicoids in the anaglyph version.

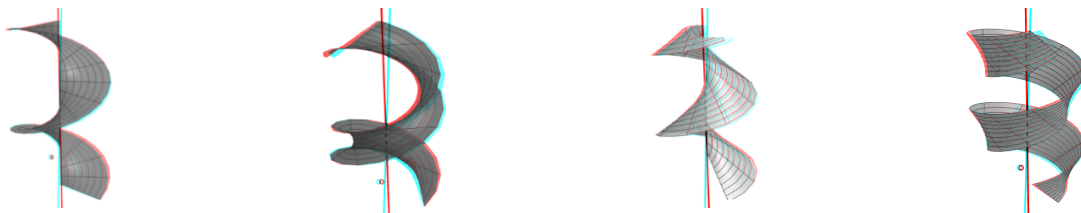


Figure 3: Four different types of right-handed straight-line helicoids (anaglyph version)

3D printing of models of straight-line helicoids created in GeoGebra

Once students are familiar with creating virtual 3D models of straight-line helicoids, they can gradually generate the files needed to 3D print the created models. If they want to 3D print real models from the created virtual 3D models, they need to open the files with the virtual 3D models of straight-line helicoids in GeoGebra Classic 6. In this version of GeoGebra, virtual models can be saved in stereolithographic files (*.stl), which can be opened in software that generates files for the actual 3D printing, such as the PrusaSlicer software.

Let us mention some comments of mathematics teacher students from the second phase of the case study in which they inserted the created virtual 3D models of straight-line helicoids into PrusaSlicer software:

- Student 1: Oh, I let the reference plane be visible in the file. Come here and look at what has happened. The supports are generated after slicing not only for the model of the straight-line helicoid but also for the reference plane.
- Student 2: I made a similar mistake. I let the axis of the straight-line helicoid be visible in the file. The axis stands on the virtual printing bed, but the model of the straight-line helicoid “flies in the air”. I must make corrections in the original file.
- Student 3: Is it necessary? You can cut the axis with the special PrusaSlicer software tool.
- Student 2: I think it is better to take out the axis from the helicoid model and print it separately.
- Student 4: Do it in the PrusaSlicer. It will be quicker.
- Student 2: I tried to do it, but it didn’t work. The model behaves as one unit. It seems that all objects displayed in the GeoGebra software’s 3D graphics window when saving a file in *.stl format will be generated for 3D printing. I must make corrections in the original GeoGebra file.
- Student 5: While saving a virtual 3D model in *.stl file format in GeoGebra Classic 6, you can set the actual sizes of the saved model by setting one of the object dimensions (length, width, or height) or by selecting a conversion scale, i.e., how many *cm* actually correspond to one unit used in GeoGebra construction. It is good.

Student 6: I tried to set different values of the thickness of the 3D-printed model of the straight-line helicoid in the corresponding floating window. It is interesting to observe how the generating and trace curves appear on the model and how thick or thin they are.

Student 7: I tried to use the organic supports, they seem to be very interesting, too.

Students can import the created straight-line helicoid models in the format of the stereolithographic files into the open-source software PrusaSlicer, see Figure 4 (parts a and b), where they first slice virtual 3D models using the *Slicing* button. Slicing means dividing the model into planar layers, which are created step-by-step by the 3D printer during the printing process. Next, there are free spaces between the individual pitches of the straight-line helicoid models and parts of the straight-line helicoids with overhangs greater than 45° , so these free spaces and overhangs must be filled with so-called supports. The supports are mostly created automatically during the slicing process if this function is enabled in the PrusaSlicer software. The supports are green in Figure 4 (parts c and d).

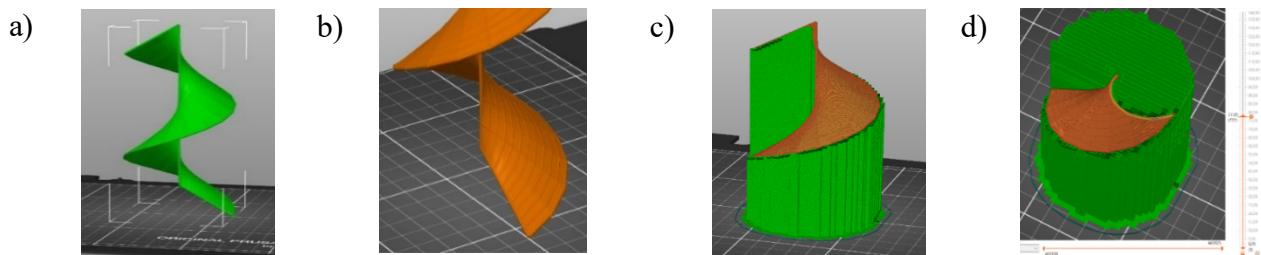


Figure 4: The virtual 3D model of a right-handed oblique closed straight-line helicoid in PrusaSlicer

When the virtual 3D model of the corresponding straight-line helicoid is sliced in PrusaSlicer and if no errors are detected during slicing (the user would be alerted by an error message displayed in the lower right window of the software if there were any), an *Export G-code* button appears instead of the initially placed *Slicing* button. Pressing the *Export G-code* button, a *.gcode file automatically starts generating. Files of this format already communicate with the 3D printer. Simultaneously with the confirmation of the successful export, the displayed table shows the time it will take to print the model on the 3D printer and the weight and length of the filament used during 3D printing.

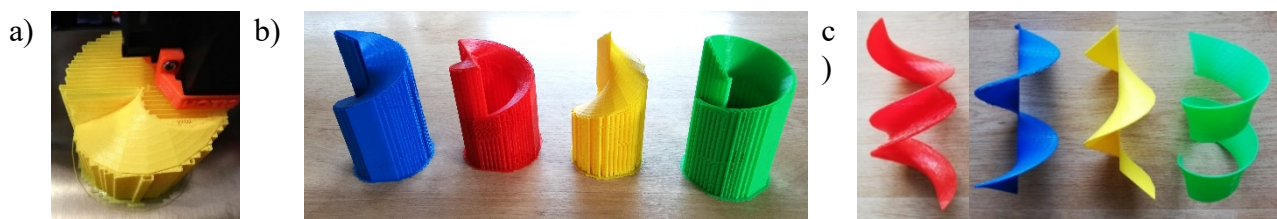


Figure 5: The 3D-printed models of right-handed straight-line helicoids

The 3D printing of straight-line helicoid models with a height of 14 cm and a diameter of 8 cm takes about 12 hours. Therefore, we print a scaled-down model during class so that students can set up the 3D printer and observe the printing process. Figure 5 shows the 3D printing process of the oblique closed straight-line helicoid model (part a), the 3D-printed models of four different types of straight-line helicoids together with the printed supports (part b), and the 3D models after removing the supports (part c).

Conclusion

Our aim was to show that and how GeoGebra and 3D printing can be used. We see many benefits for both students and educators in implementing 3D printing and shared virtual environments in preparing future mathematics teachers and mechanical engineering students. One possible shared virtual environment is the

GeoGebra online environment. Some of the benefits include the GeoGebra Books and Classrooms online environments that students can share and benefit from on their mobile devices, laptops, and computers. Another advantage of GeoGebra is the possibility to create files in the stereolithographic format, which are the base files for subsequent 3D printing settings of the created virtual models. These 3D-printed models are didactic tools for “learning by doing”.

The findings about increases of students’ interest in the subject, thus increasing their motivation and teaching effectiveness, as shown by Gebre (2014), support the demand for implementing 3D printing. The results of the case study tentatively confirm the high potential of using GeoGebra and 3D printing both in teaching straight-line helicoids and in raising the level of students' visuospatial abilities. A high level of student motivation to solve the given problems was also observed in the case study. This, together with feedback from student surveys that have been regularly conducted at TUL for several years at the end of each semester, is in line with Gebre’s position.

Last, developing students’ mathematical competencies in a STEM context is essential for technical disciplines and future mathematics teachers, who contribute significantly to students’ STEM education. Therefore, it is necessary to work on mathematically challenging content that allows students to experience essential working methods and ways of thinking in mathematics and has links to applications in everyday life and other sciences. The example we have chosen with the link to the Archimedean serpentine, for example, is very suitable in this sense.

We have long studied the possibility of using 3D printing in mathematics education (creation of virtual models, software preparation for the actual printing, and use of the printed 3D models). Our goal is to create a concept that will significantly contribute to developing students’ visuospatial abilities and increase their skills to use these technologies in practice. We work on the design and implementation of experiments to test our hypotheses. In this context, we are preparing a paper for CERME14 that will analyze and discuss the data obtained from our validations.

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Love is a mathematical journey: Memes as metaphors to unveil students' emotions and identity in the mathematics classroom

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In today's digital culture, memes play a significant role in communication, impacting collective identities and emotions. This study explores the educational value of mathematical memes and their effect on secondary students' identities, emotions, and engagement in mathematics. Using Lakoff's framework of metaphors, it examines how students participate in the creation of mathematical memes and the implications of the activity for emotional responses and identity formation. Data from a mathematical meme tournament show a connection between students' emotional responses and their mathematical interpretations. This suggests that mathematical memes can intercept the interplay between emotions and cultural frameworks to provide students with meaningful learning experiences.

Keywords: Emotion, Identity, Mathematical memes, Metaphors, Online learning

Introduction and rationale: Memes, emotions, and identity

In contemporary digital culture, memes have emerged as a ubiquitous form of communication, comprising images, videos, or text that convey humour or cultural commentary (Shifman, 2014). The social studies literature widely acknowledges memes' influence on the formation and expression of collective identities (Shifman, 2014), as well as their ability to evoke emotions (Jenkins et al., 2013). Indeed, memes are perceived as a shared language within online communities constituting the *memeshpere*, and their appeal is directly linked to their relatability in terms of shared emotions.

In mathematics education research, scholars agree on the role of emotions and identity to shape students' learning (Radford, 2015; Boaler, 2002; 2016). Specifically, Boaler (2002) emphasizes that knowledge is not static but rather co-constructed by learners, their interactions, and situational factors. In classrooms where students are passive recipients of knowledge, they ultimately became disaffected with mathematics due to pedagogical practices that did not align with their desires for expression and agency. Conversely, by inviting students to contribute their own thoughts and ideas to the mathematical discourse, educators empower learners to take ownership of their learning, countering disengagement observed in passive learning settings.

Focussing on the significance of memes for maths teaching, research in mathematics education agrees that memes offer opportunities to explore a unique crossing of mathematical concepts and digital culture (Abrams, 2021; Bini et al., 2022; Mutua & Mwangi, 2023). Mathematical memes use humour and visual imagery to convey mathematical ideas (Bini et al., 2023): they employ mathematical symbols or terminology in unexpected contexts, making complex mathematical concepts more accessible and engaging, and provide a bridge between students' spontaneous discourse and the language of mathematics, facilitating a deeper understanding of mathematical concepts. Moreover, the interactive nature of meme creation and sharing platforms fosters students' agency, collaboration, and participation, enhancing engagement in the mathematics classroom. Building on these premises, this study aims to investigate the educational benefits of mathematical memes and their impact on students' engagement in learning mathematics. Specifically, the study moves from the hypothesis that involving students in mathematical meme creation activities can uncover the emotions associated

with mathematics and provide educators with valuable insights into students' experiences, identity, and self-positioning with respect to mathematics.

Theoretical background: Memes as multimodal metaphors

Language metaphors, as described by Lakoff are “mappings across conceptual domains” (1993, p. 39), serving as cognitive tools that bridge abstract concepts with tangible experiences. These verbal tools provide individuals with a lens to perceive and comprehend the world (Lakoff, 1993; Lakoff & Johnson, 2008). By mapping abstract domains onto concrete experiences, metaphors offer frameworks for understanding complex phenomena, shedding light on how individuals navigate their lived experiences. Examining metaphors involves studying the linguistic choices that guides the recognition of cross-domain mappings as sets of “ontological correspondences between entities in a source domain and entities in a target domain” (Lakoff, 1993, p. 26). The epitomising example is the metaphor "love is a journey", from which the present study derives its title, revealing its intricate layers of meaning, portraying romantic relationships as dynamic adventures (Lakoff, 1993; Lakoff & Johnson, 2008). This metaphor captures the evolving nature of love, suggesting shared aspirations and destinations that couples navigate together, mirroring the complexities of real-life relationships. Besides Lakoff's language metaphors, visual metaphors are also recognized as powerful cognitive tools. In mathematics education, visual metaphors are acknowledged as means to guide the reasoning process, helping students to move beyond sensory perceptions to logical generalizations, essential in mathematics. Presmeg, (1995) underlines the imaginative aspects of reason, including metaphor and mental imagery, considered crucial for meaning-making and reasoning, and Rolka (2005) highlights the potential of metaphorical mages to elicit students' beliefs about mathematics.

Memes have been interpreted as *multimodal* metaphors: having both a verbal *and* a visual nature, they harness the cognitive strength of both linguistic and visual metaphorical mapping (Scott, 2021), Memes' metaphorical meaning can be unpacked using a semiotic tool known as the Triple-S construct (Bini et al., 2023). According to this construct, the full meaning of a mathematical meme, which corresponds to a mathematical statement, is constructed by combining three layers of partial meanings: a social meaning carried by the image, following conventions established in the memesphere; a structural meaning carried by the compositional arrangement of the meme; and a specialised meaning carried by topic-specific information in the template or in user's additions.

Given these premises, the study is guided by the following research question: How does the metaphorical interpretation of mathematical memes created by students shed light on their emotional responses and identity formation in the mathematics classroom?

Methodology and methods

Data for the study come from the 2021/22 edition of MathMemeThon, an annual online mathematical memes tournament designed by the author during the 2020 Covid-induced online learning period (Bini, 2022). In this competition, students team and compete creating memes on mathematical topics across successive rounds: an elimination match, a semifinal, and a final. The goal is to promote the re-elaboration and systematization of known mathematical topics, with each round focusing on different topics chosen by teachers. To institutionalise the activity and support formative assessment, teams present their memes and mathematical content to a jury, who scores the memes on mathematical and memetic quality and presentation accuracy. Each round includes an hour for meme

creation (offline) and time for online presentations. Memes are created using Imgflip (<https://imgflip.com/memegenerator>) and shared on Padlet boards allowing reactions (<https://padlet.com/>), and are subsequently published on Instagram @lifeonmath.

MathMemeThon 2022 took place from March to May, involving 8 class groups of 12th-grade students (about 120 students) and 4 teachers from a technical secondary school in Turin, Italy. Students were divided into teams of four, either self-formed or assigned by teachers. The jury included the teachers, the author, two mathematics master's students, and one PhD student. Online sessions were held on Webex, recorded, and transcribed. During the tournament, classes tackled calculus topics: functions, limits, and derivatives.

This study will focus on four girls from the same class group, who teamed up spontaneously and named themselves “the romantic ones”. Their work is noteworthy for two reasons: it has a deep and correct mathematical content, which allowed them to make it to the final, and it reveals a consistent emotional response to the different mathematical topics proposed by the teachers. Memes are analysed using the Triple-S construct; transcribed sessions are analysed using a qualitative methodology and interpretative approach as outlined by Cohen et al. (2007), focussed on eliciting the metaphorical implication of the girls’ productions in terms of emotions and identity formation.

Data and analysis

This paragraph presents the romantic girls’ team production and excerpts from their presentations. Figure 1 shows the memes created in the three tournament rounds: elimination match (left) on the topic of functions, semifinal (centre) on the topic of limits and final (right), on the topic of derivatives. To maintain the freshness of the memes, the original versions are presented here, added Italian text is translated in the 'Structural meaning' row of Table 1, which presents the partial and full meanings of the romantic girls’ memes according to the Triple-S construct (Bini et al., 2023).

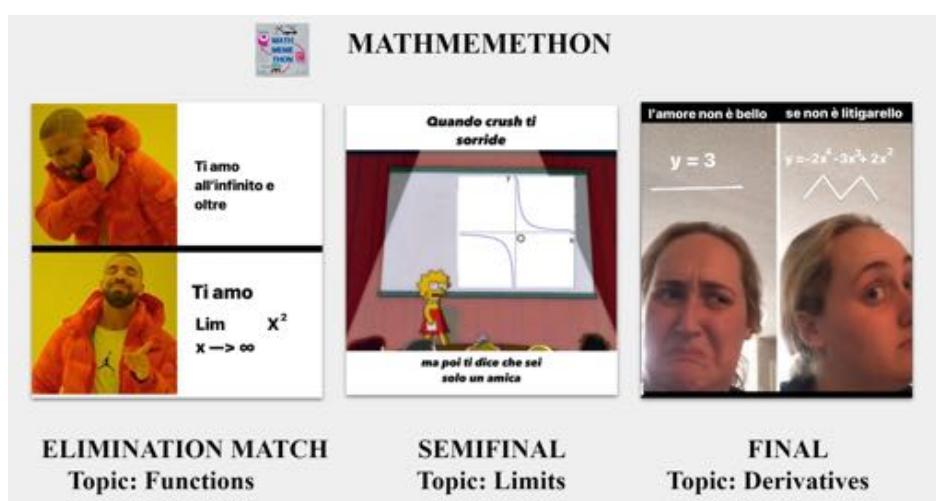


Figure 1: Romantic girls’ team mathematical memes

Table 1: Partial and full meanings of the romantic girls' memes

Partial and full meanings of the memes	Elimination match Template: <i>Drakeposting</i> ⁷	Semifinal Template: <i>Lisa Simpson's Presentation</i> ⁸	Final Template: <i>Kombucha Girl</i> ⁹
Social meaning (from the online encyclopaedia KnowYourMeme)	Contrasting reactions implying a preference between two choices	Humorous or ironic depictions of factual evidence	Evolution from disgust to approval of something
Structural meaning	Two-pane (horizontal) texts are added in two lateral lines <ul style="list-style-type: none"> • Top line: I love you to infinity and beyond • Bottom line: I love you $\lim_{x \rightarrow \infty} x^2$ 	Exploitable: elements are added on the board and as top/bottom texts <ul style="list-style-type: none"> • On the board: $y = \frac{1}{x}$ graph • Top text: when crush smiles at you • Bottom text: but then tells you that you're just a friend 	Two-pane (vertical) elements are added above and onto the images <ul style="list-style-type: none"> • Left text: love is not beautiful • Right text: if it's not quarrelsome • Left addition: $y = 3$ and sketch of the behaviour • Right addition: $y = -2x^4 - 3x^3 + 2x^2$ and sketch of the behaviour
Specialised meaning	Functions' limits	Functions' asymptotes	Functions' increasing, decreasing and stationary behaviour
Full meaning	$\lim_{x \rightarrow \infty} x^2 = \infty$	The function $y = \frac{1}{x}$ has a horizontal and a vertical asymptote	$y = 3$ is a flat function, while $y = -2x^4 - 3x^3 + 2x^2$ is a curved function with maxima and minima

The first thing to note is that meme creators do not have access to functional maths editors. On the one hand, this might contaminate the representation in the symbolic register; but on the other hand, it offers a rich potential for discussing on mediating the writing of mathematical content in an era dominated by images. Then, some cultural-situated comments are due to fully understand the final meme: the text “L’amore non è bello se non è litigarello” corresponds to an Italian proverb, translated in Table 1 as "Love is not beautiful if it is not quarrelsome”, a popularised version of Shakespeare’s “The course of true love never did run smooth" (A Midsummer Night's Dream). The graphical additions in the meme (horizontal and zig-zag lines) should not be read as the graph of the functions $y = 3$ and $y = -2x^4 - 3x^3 + 2x^2$. Instead, they sketch the behaviours of the functions as inferred

⁷ <https://knowyourmeme.com/memes/drakeposting>

⁸ <https://knowyourmeme.com/memes/lisa-simpsons-presentation>

⁹ <https://knowyourmeme.com/memes/kombucha-girl-trying-kombucha-for-the-first-time>

from the sign of the derivative. In the experiment cultural context, slanted segments complement the derivative sign chart: upward segments for increasing intervals and downward for decreasing ones.

An initial examination of the memes created by the romantic girls reveals a recurring theme: they identify themselves as "the romantic ones" and consistently interpret their reality through this emotional perspective, even when tasked with mathematical assignments. Indeed, other mathematical memes created during the tournament also exhibit emotional elements, but these emotions stem directly from the students' experience of doing mathematics. The *Two Buttons* meme, created by another team in the elimination match (Figure 2), illustrates the authors' confusion when handling infinity and indeterminate forms (F.I. in the meme text, the representation of infinities addition is also a customary notation in the experiment cultural context).



Figure 2: *Two Buttons* mathematical meme

The difference is that the emotion in the meme in Figure 2 arises and concludes within the context of the authors' mathematical experience, whereas the emotions exhibited by the romantic girls' memes in Figure 1 originate from a different realm and are projected onto the girls' mathematical experience.

The interweaved mathematical and emotional contents of the romantic girls' memes are further developed in their presentations during the tournament rounds. In the following excerpts, *Student* indicates the romantic girls' team spokesperson.

Elimination match (Figure 1, left): Functions

Student We are very romantic girls, and in addition to love, we are rational. Therefore, above, we have represented what a normal, unoriginal guy would say to us, and below, we have represented what a guy who would win our heart would say. It's very simple, very clear; a good mathematician knows that x^2 goes to plus infinity, so "ti amo" (I love you) goes to infinity and beyond.

Due to time constraints in the elimination match, teams had only a brief window to present their memes. Nonetheless, the romantic girls used this short timeframe to declare their identities, emphasizing that their romantic inclinations do not overshadow their rationality (*we are rational*). Coherently, they label as *unoriginal* the guy who simply says *I love you*, while connecting mathematical proficiency with the ideal suitor. They attribute an emotionally resonant (and mathematically correct) interpretation to $\lim_{x \rightarrow \infty} x^2 = \infty$, describing it as the mathematical

representation of the intensity of the lover's feelings. This can be interpreted as a metaphorical mapping of the intensity of the romantic feeling onto the functions' limits.

Semifinal (Figure 1, centre): Limits

- Student We are always the romantic ones. This time too, we decided to tackle the theme of love. However, this time, unfortunately, it's a bit negative. Our meme says, "when crush smiles at you but then tells you that you're just a friend." For those who don't know, "crush" is the infatuation, and precisely "just a friend" means entering the full friendzone. Entering the friendzone means being perceived as a friend when you want more. So, our meme is clear: we represented the function $1/x$ by solving it, finding its domain and limits. Now, as my colleague is writing, finding the domain, and calculating the limit at infinity and the limit at zero, in both cases, vertical and horizontal asymptotes are obtained. It means that we have asymptotes that will never touch the function; I don't know if I explained myself well.
- Author I'm missing the connection between the concept of friendzone and the asymptote: it's clear to me, but I would like you to explain it more explicitly.
- Student The concept is that the asymptote always gets close to the function, but it will never touch it.
- Author Which one is the crush? The function or the asymptote?
- Student The asymptote...

During the semifinal, teams had a longer timeframe to present their memes. The romantic girls explained their meme's mathematical meaning, drawing attention to the domain of the function $y = \frac{1}{x}$, its graph and asymptotical behaviour at infinity and around $x=0$. They discussed both the mathematical content and the language used, explaining terms such as *crush* (in English also in the original Italian text) and *friendzone* (also in English in the presentation), thus performing a metaphorical mapping between the challenge of pursuing unreciprocated love and the behaviour of a function approaching its asymptote without reaching it.

Final (Figure 1, right): Derivatives

- Student So, um, we are always the team of romantic girls. [...] We wanted to represent love through derivatives. You can see that this girl on the left is not very convinced, in fact, the derivative is constant, and it says above that "love is not beautiful," and then it says "if it's not quarrelsome", and she's a bit more convinced because the relationship is a bit livelier, a bit more fun. In fact, in the derivatives, we find two maxima and one minimum, while the other one, which is a derivative of a constant, is flat, and so, love is monotonous.

In the final match the girls explained their meme connecting the behaviour of a function to the liveliness of love life (*we wanted to represent love through derivatives*). Thus, they choose a constant function ($y = 3$) to represent a flat and uneventful love life, and a polynomial function with extrema ($y = -2x^4 - 3x^3 + 2x^2$) to represent a livelier and captivating romance. In their presentation, they show to have a clear understanding of how derivatives can be used to determine a function's behaviour and extrema. Thus, the meme encapsulates the behaviour of functions across three distinct semiotic modalities: analytical (the functions themselves), graphical (the slanted segments sketch) and verbal (the metaphorical interpretation of the function increasing or decreasing trend).

Results and conclusion

A metaphorical mapping across conceptual domains can be now performed, based on the romantic girls' pictorial and lexical choices. We can ask: did these girls use mathematics to understand love or

vice versa? The girls' explanations clearly show that they mapped their own love experience into the mathematical realm: thus, for them, love is the source domain and mathematics is the target domain.

Table 2: Mathematics as Love metaphorical mapping

Meme	Source domain: Love	Target domain: Mathematics
Elimination match	Intensity of the romantic feeling	Functions' limits
Semifinal	Unreciprocated love	Asymptotical behaviour
Final	Love life	Increasing or decreasing behaviour of a function
Final	Good and bad love moments	Maxima and minima of a function

Metaphors bridge the gap between the known and the unknown, facilitating a deeper and more intuitive grasp of abstract ideas. They are cognitive tools that connect intangible concepts with tangible experiences, supporting meaning-making and reasoning (Presmeg, 1995). Lakoff (1993) explained how the linguistic metaphor of a physical journey provides verbal tools to describe the abstract feeling of love. Similarly, mathematical memes as multimodal metaphors provided the romantic girls with language and pictorial tools to bridge the abstract feeling of love to their perceptions of mathematical ideas. This process involved an ontological mapping from the source domain of love to the target domain of mathematics. In Lakoff's study, the cognitive process leveraged a metaphorical mapping from the concreteness of a physical journey to the abstractness of love. For the girls, both domains (love and mathematics) are abstract, but love is closer to their experiences, making it feel more concrete. The metaphorical power of these mappings offered frameworks for conceptualizing mathematical ideas like limits, asymptotes, or derivatives by linking them to vivid personal experiences, evoking emotions, stimulating imagination, and encouraging the exploration of mathematical concepts from different angles.

We cannot assume that memes support a deeper understanding of mathematics, but we can infer that they can promote creative thinking, enabling students to contribute to discussions, take ownership of their learning journeys, and actively engage with the discipline of mathematics (Boaler, 2002). Despite the study's inherent limitations, it is evident that this activity allowed students to project their emotions and identity onto a mathematical context, making the concepts as memorable as the emotions they represented. The class teacher corroborated this, noting increased engagement among students—a rare phenomenon in technical schools where mathematics is often perceived as disconnected from students' lives and devoid of personal relevance¹⁰.

Acknowledgment

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¹⁰ Did the romantic girls win the tournament? Spoiler: they did not, but you can see the winner on Instagram @lifeonmath

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Teaching the derivative using arrow graphs in GeoGebra

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Teaching the derivative is usually supported by the visual context of graphs and tangent lines. However, this geometric meaning-making is somewhat indirect, hard to grasp for students, and easily forgotten. We report on the first cycle of a design-based study in which we introduce arrow graphs as an additional geometric context to provide meaning to the instantaneous rate of change as an enlargement factor with respect to a local focus. We outline a learning trajectory deploying interactive dynamic visualizations designed in GeoGebra. Our results show the challenges of this approach and suggest several ways in which the design can be improved for the next design cycle.

Keywords: calculus education, arrow graphs, dynamic geometry environment.

The derivative is usually introduced geometrically as the slope of the tangent line to the curve. Though this supports a meaningful image of the concept, we believe it is not the optimal picture for understanding the derivative as an instantaneous rate of change. Consider a closed vat with an ideal gas with a valve that controls the pressure. The volume can be described as a function of the pressure $V(p) = \frac{c}{p}$. The derivative $V'(p) = -\frac{c}{p^2}$ expresses the *sensitivity* of volume as a function of the pressure, e.g., when pressure is low, the impact on the volume of an increase is large and negative. This is observable as the slope of the graph of V is steep and negative for small values of p . However, the sense of rate is not so easily associated with steepness. We hypothesize that the sense of rate may come more naturally in the context of enlargement.

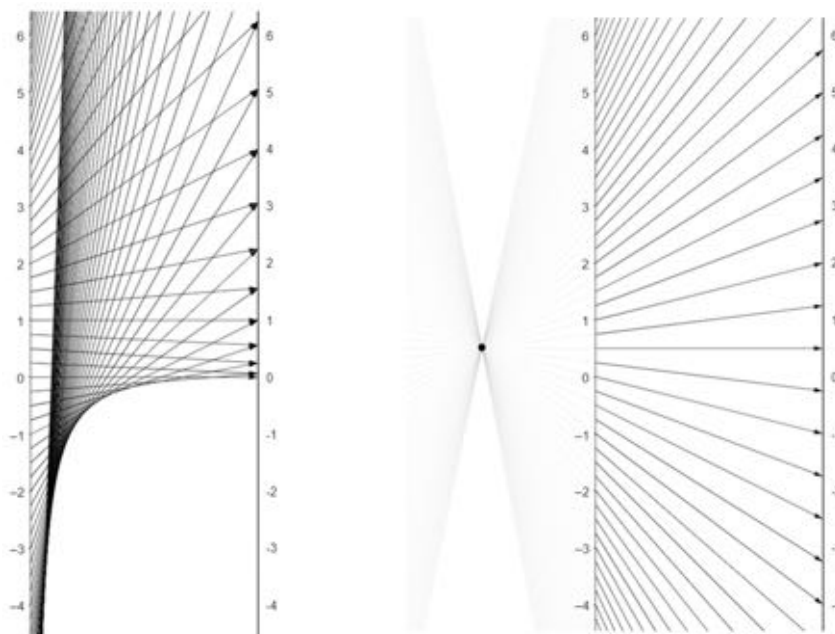


Figure 1. Left: an arrow graph for $f(x) = x^2$. Right: an arrow graph for $g(x) = 3x - 1$ with a focus

In recent work, Wei and collaborators studied how to promote functional thinking using arrow graphs—also known as the parallel axes representations or nomograms (Wei et al., 2024)—which were previously studied by Nachmias and Arcavi (1990). In an arrow graph, a function is represented as a family of arrows from input values to corresponding output values (see Figure 1). Arcavi pointed

out that linear functions correspond to arrow graphs where the arrows intersect at one point—when needed the arrows are extended to a line. This point is called the focus. When the rate of change equals 1, the focus is “at infinity” and the arrows are parallel.

As a consequence, the rate of change for a linear function can be interpreted geometrically as an enlargement factor in the arrow graph. An interval on the input axis is enlarged to an interval on the output axis with respect to the focus (see Figure 2, left). The rate of change corresponds precisely to the enlargement factor. The main aim of our study is to investigate whether teaching this new geometric interpretation of rate of change, in addition to the usual one as slope, supports students’ sense-making.

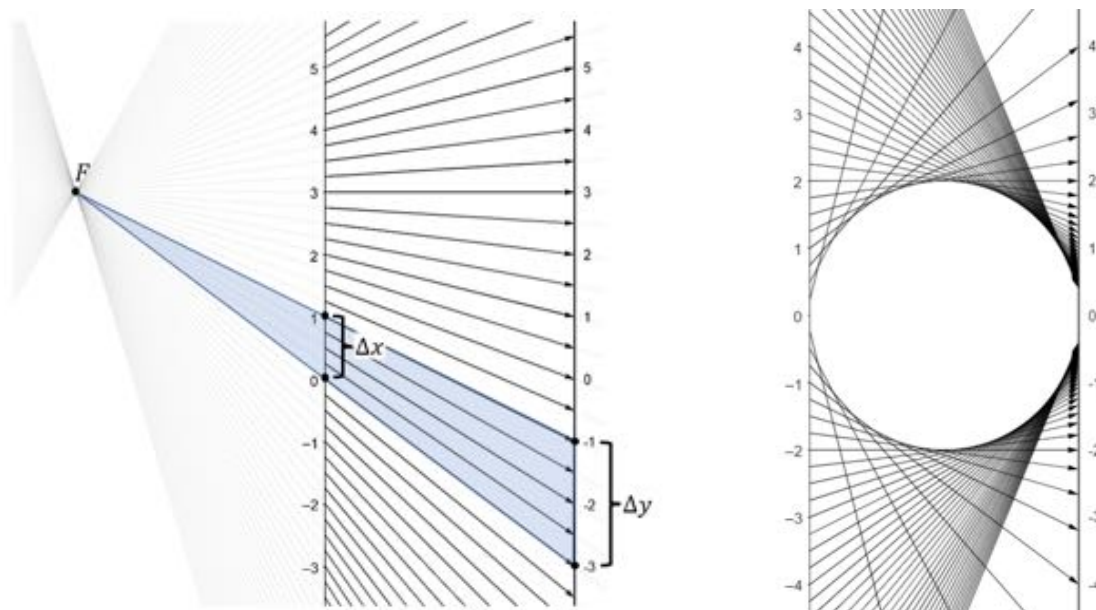


Figure 2. Left: The rate of change of $f(x) = 2x - 3$ equals the enlargement factor $\frac{\Delta y}{\Delta x} = 2$ of the interval $[0, 1]$ to $[-3, -1]$ with respect to the focus F in the arrow graph. Right: the circle as an emerging enveloping curve for the function $g(x) = \frac{4}{x}$ in the arrow graph.

More generally, the derivative of a function can be interpreted in its arrow graph. Whenever one considers the arrow graph of non-linear functions certain enveloping curves emerge in the picture (see Figure 2, right). These curves turn out to be related to the derivative in a surprising way, that we explain in a moment. A function is differentiable only if it is locally linear. In the graph, this means that around a point $(a, f(a))$, it can be approximated (in a certain technical sense) by a line, called the tangent line. Alternatively, in the arrow graph, this means that on a small interval around a on the input axis, the arrows intersect approximately at one point. The smaller the interval the sharper the point emerges, and the associated limit point we call the local focus (see Figure 3). The value of the derivative $f'(a)$ can be interpreted as the enlargement factor associated with this local focus. By construction, the local focuses form the mentioned enveloping curve. Our aim extends to studying how to teach the instantaneous rate of change and derivative using the arrow graph in this way.

Drawing arrow graphs by hand is a time-consuming endeavor. In GeoGebra, we find suitable software to outsource this to. Moreover, the dynamic features of GeoGebra allow us to vary the function, the number of arrows, and covary information across multiple representations: the graph, the arrow

graph, and the equation describing the function. In this design study, we are interested in how the dynamic features of GeoGebra in our lesson design can contribute to the students' learning.

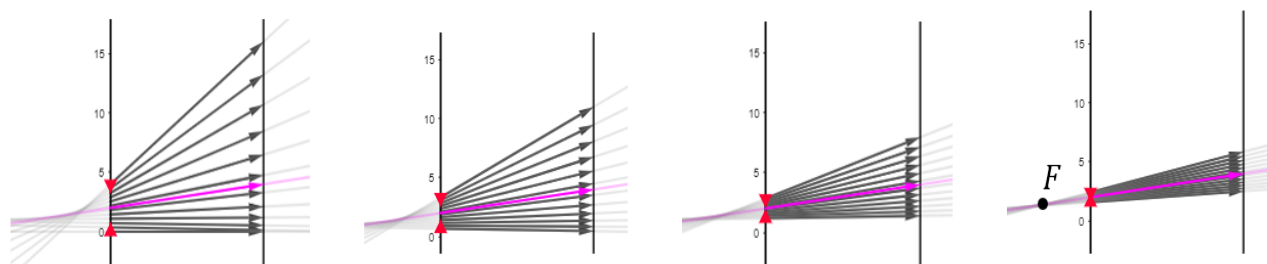


Figure 3. Arrows on a small enough interval approximately intersect in a local focus

This paper reports primarily on our original teaching designs, and also on the findings of a small pilot study, of the first cycle of a design-based research. We briefly review some literature on learning about derivatives, and on connecting multiple representations (including the arrow graph) using digital technology. Then we introduce the intervention consisting of two modules of approximately one hour. Finally, we present and reflect on the result of an implementation in a 10th-degree pre-university class.

Theoretical background

Students are usually supported in making sense of the instantaneous rate of change $\lim_{\Delta x \rightarrow 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$ for a function f at $x = a$ by providing a geometric interpretation as the slope of the tangent line to the graph of the function at the point $(a, f(a))$. However, many students have difficulty connecting such geometric and symbolic notions, in particular grasping the limit procedure involved (Orton, 1983). Indeed, using an open, inquiry-based task, Bos et al. (2019) showed that students rather think about tangent lines and the derivative in ways not involving the limit of secant lines. As explained in the introduction, this paper develops another geometric approach, where the limit is geometrically interpreted as a local focus becoming more sharply defined as the interval shrinks.

Teaching sequences in dynamic geometry environments for the (instantaneous) rate of change and the tangent line were proposed, amongst others, by Biza et al. (2007) in CalGeo, and by Hohenwater et al. (2008) in GeoGebra. On the one hand, these sequences involve the task of zooming in on a graph to observe how a differentiable function is locally linear. This supports the fundamental insight that the graph of such a function can be approximated by the tangent line and hence the function by a linear function. On the other hand, given a point A on a graph, students are invited to move a second point B on the curve towards point A , such that the secant line AB approximates the tangent line at A . By showing the covarying values of the coordinates, Δx , Δy , and the difference quotient, students gain insight in the limit process that constitutes the differential quotient.

We propose a new approach to teaching these concepts in a dynamic geometry environment, introducing an additional representation: the arrow graph. This approach was inspired on Zandieh's theoretical framework for the concept of the derivative, as is explained in the second author's master thesis (Brinks, 2024). We pose the following research question: how can interactive tasks in GeoGebra, with a central role for the arrow graphs, promote the meaning-making of the derivative?

Method

The intervention consisted of two modules: the [first](#) focusing on arrow graphs of linear functions, and the [second](#) on the interpretation of the derivative in arrow graphs; moving from the linear situation with a focus to the locally linear situation with a local focus. The first module began by revising linear functions and their graphs, which were considered pre-knowledge. Next, arrow graphs were introduced.

Let us highlight two tasks aimed at connecting three representations of linear functions: the arrow graph, the graph, and an equation. In [task 1.4a](#) (see Figure 4), students vary the parameters of a linear function by dragging the focus (the red point on the left). Simultaneously, students can observe how the graph (the green line on the right) and the equation (top right) respond to their hand movement. In the arrow graph in pink an interval of size 1 is indicated on the input axis, together with its image and its size on the output axis. This is shown to invite students to connect the size of the enlarged interval to the rate of change in the equation, which in turn expresses the slope of the line, hence providing sense to the rate of change in the arrow graph representation. Later tasks allow students to discover how vertical movement of the focus leaves this number unchanged, whereas horizontal movement does impact it.

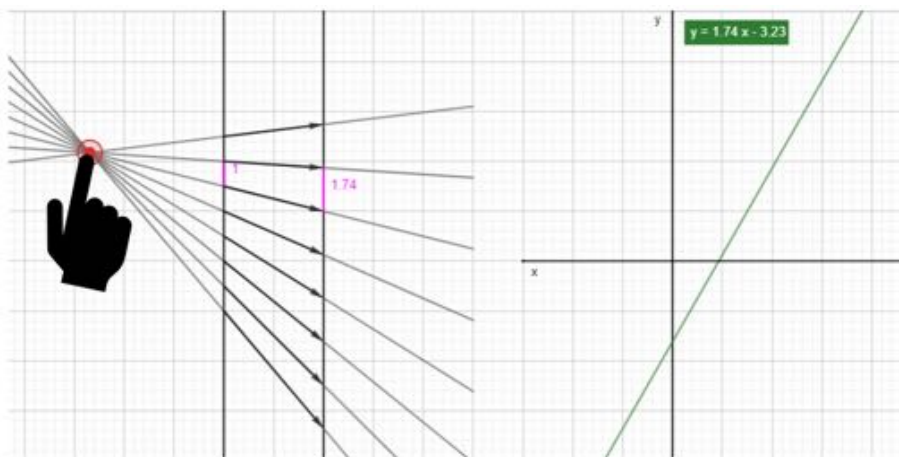


Figure 4. [Task 1.4a](#): Connect the enlargement factor associated with the focus of the arrow graph to the slope of the graph.

In [task 1.4b](#) students again move the focus (see Figure 5), but now the distance to the input axis is indicated by an orange dashed line and the distance to the output axis by a blue line. Whereas in the previous task, the connection between the rate of change and the shown information in the arrow graph was direct, in this task students need to realize that dividing the distance to the output axis by the distance to the input axis is another way to compute the enlargement factor, that is, the rate of change. In later tasks, students explore in more depth how the sign and size of the rate of change are determined by the horizontal position of the focus. For example, when the focus is between the axes the rate of change is negative and otherwise positive. The way linear behavior presents itself in three ways in three different representations is summarized in Table 1, which was part of a theoretical section of the learning materials.

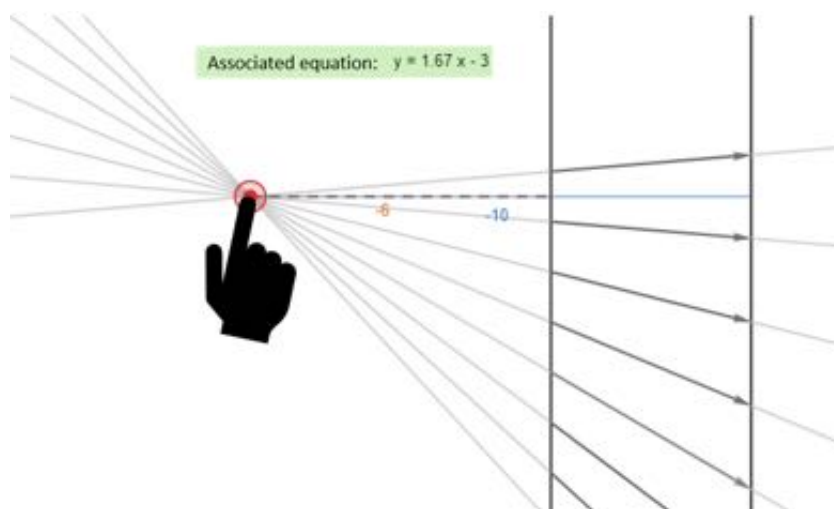


Figure 5. [Task 1.4b](#): Connect the distances of the focus to the vertical axes in the arrow graph to the rate of change in the equation

Table 1. The relation of linear behavior in three representations

Equation: $y = ax + b$	Graph: line	Arrow graph: has focus
a is the rate of change	a equals the slope of the line	a is the enlargement factor
b is the initial value	b equals the y -coordinate of the intersection with the y -axis	b is the output for input 0.

In the second module attention shifts to non-linear functions and local focuses. We highlight three tasks. A central insight is developed in [task 2.5](#) (see Figure 6). Students drag a slider to make the interval where the arrows are drawn smaller and smaller. As the interval shrinks, the local focus becomes more sharply delineated, as in Figure 3. This way students experience how local linearity is presented in an arrow graph. This is addressed in detail in a later theoretical section that also contains Table 2. Generally, the hypothesis is that connecting the three representations, offering three perspectives and their relation, supports students to make sense of the underlying concepts.

Table 2. The relation of local linear behavior in three representations

The function is differentiable in $x = a$	The graph is locally linear at $(a, f(a))$	The arrow graph has a local focus where the line through $a \rightarrow f(a)$ intersects the enveloping curve
Instantaneous rate of change $\lim_{\Delta x \rightarrow 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$	Slope of the tangent line at $(a, f(a))$	The enlargement factor, e.g., $\frac{\text{signed distance local focus to input axis}}{\text{signed distance local focus to output axis}}$

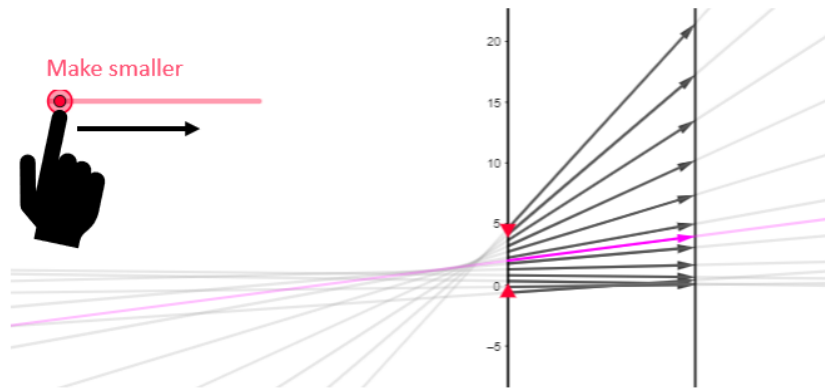


Figure 6. [Task 2.5](#): Sliding makes the interval smaller so the local focus comes into focus

Some of the tasks in module 2 have the context of a falling object, for example, [task 2.6](#) (see Figure 7). In [task 2.6](#), students are invited to estimate the speed of the object after 2 seconds of falling. Students can drag two points on the input axis of the arrow graph on the left. The corresponding points and intervals in the graph on the right covary, as do the numbers in the arrow graph and in the division on the left. The goal is again to create a more intimate link between the two representations, combining the data needed for this computation from both graphs. In both Task 2.5 and Task 2.6, students manually perform part of the limit procedure, which helps them ground this essential step in the genesis of the derivative concept in a sensorimotor experience.

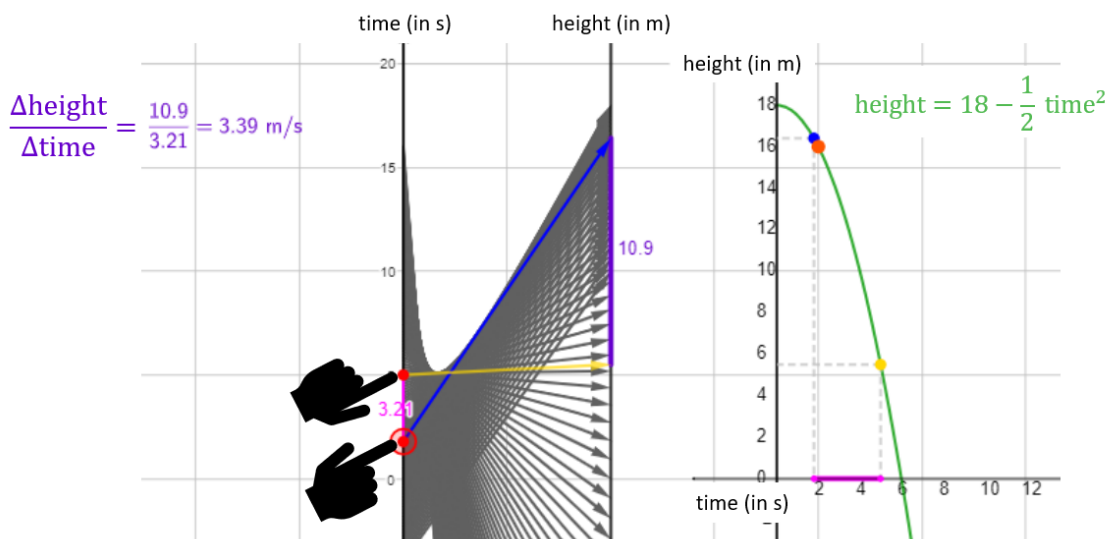


Figure 7. [Task 2.6](#): students drag the edge points of the interval to find a good approximation of the speed after 2 seconds, with covarying information in the division on the left and graph on the right

[Task 2.9](#) invited students to drag a point on the input value (see Figure 8). Simultaneously, the arrows (and lines through these arrows) through this point and through a point a little higher are drawn. The intersection point of these lines is an approximation of the local focus. GeoGebra is set to leave a trace of this intersection point. Hence, students can see an approximation of the enveloping curve light up in green.

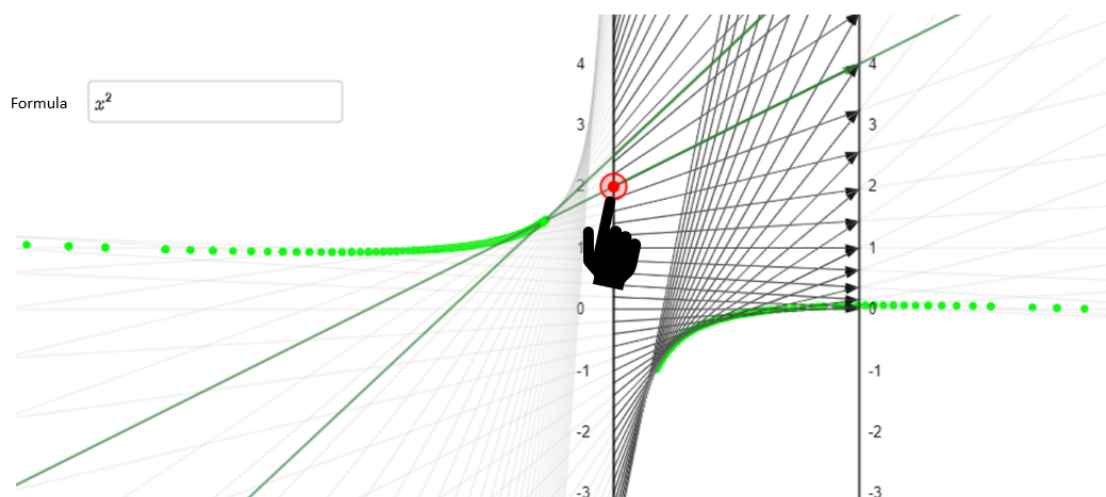


Figure 8. [Task 2.9](#): dragging the input point gives a trace of approximate local foci

The two modules were piloted in a grade 10 class with 25 students in a social science pre-university stream. Students were allowed to collaborate. The data for this study are the answers students wrote in the answer boxes in GeoGebra. Answers were grouped for similarity and analyzed to see how performing the tasks contributed to making sense of the concepts of (instantaneous) rate of change.

Results and conclusions

We present two main conclusions, with supporting results. Firstly, we conclude that, for social science stream students, learning to work with arrow graphs and their relation to graphs and functions is challenging. This is evidenced by responses to [Task 1.4b](#), where only 7 out of 25 students realized the slope is obtained by dividing the presented distances to the axes. The other responses to the task range from correct but superficial observations to having no clue. We found that ideas from graphs can be persistently incorrectly transferred. At the beginning of module 2, when asked why an arrow graph represented a linear function, only three students mentioned the presence of a focus, whereas many more said that “the lines [of the arrow graph] are straight” or “the arrows are straight lines”. Also, at [task 2.6](#), even though many students managed to compute average rates of change, the transition to the instantaneous rate was mostly not understood, witnessed by only four correct computations. In [task 2.9](#), no one managed to compute the instantaneous rate of change using the local focus. Only three students realized that the traced point was approximately the local focus. We see that these concepts and associated procedures are problematic, even when visually supported by arrow graphs. However, the types of tasks and the time spent per subject could also be better adapted to the students in the social science stream, who are generally less proficient at mathematics than those in the natural science stream.

Secondly, we conclude that the tasks should be redesigned, in particular, to make conceptual learning more visible. This would benefit conceptual discussions between students and between students and teachers, but also facilitate our analysis as researchers. In [task 1.4a](#) all students correctly observe that the pink number indicating the size of the interval on the output axis equals the slope in the graph. However, because of the task design, students have no opportunity to show that they understand why this is the case. A similar case appears in [task 2.7](#), where some students have correct computations, but the task setting offers no means to establish whether these are accompanied by insight. Finally, some tasks should be redesigned to support improved insight. For example, in [task 2.5](#) only two

students observe that the local focus appears, whereas others just notice that the arrows are moving closer. This can easily be resolved by having separate sliders for the number of arrows and the size of the interval.

Even though the results here are slightly discouraging, we are not ready to give up on arrow graphs' role in teaching the derivative, also fuelled by their success in fostering functional thinking (Wei et al., 2024). Some factors could explain the limited success of the approach so far. Firstly, unlike working in graphs, for which students are prepared from an early age onwards, arrow graphs are new to students and the learning curve in two hours is steep. Secondly, since students in the *natural* science track have more affinity with geometry, the approach may be more fruitful for them. Therefore, the next step in this design-based research will be testing an expanded version of the series, implementing the suggestions above, on students in a natural science track. Suggestions for redesign can be found in the second author's master thesis (Brinks, 2024).

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The role of teachers' knowledge and the use of different technologies in the classroom

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The potential of technology to enhance the learning of mathematics is widely acknowledged. However, realizing this potential requires teachers to possess the knowledge to effectively integrate it into their teaching practices. This study aims to characterize the mathematics teacher's knowledge when using different technologies in teaching and learning. It also aims to study the existence of relationships between the specific domains of the Knowledge for Teaching Mathematics with Technology – KTMT and the integration of graphing calculator and Excel. The methodology adopted is qualitative with an interpretative approach, using a case study of Mathematics teacher in the 11th grade (16-17 years old) with extensive experience in the use of technology. This study shows that different KTMT knowledge is mobilized, according to the technology chosen and the specific characteristics of each of the technologies used influenced the teacher's pedagogical choices.

Keywords: Technology integration, teacher knowledge, KTMT, Mathematics

Introduction

The use of technology in mathematics teaching has led several authors to investigate the specific knowledge needed to teach mathematics with technology (Clark-Wilson et al., 2019; Drijvers, 2019; Rocha, 2020a, 2020b), but few focus on teachers' knowledge to support students in the transition between different technologies (Bretscher, 2022). In order to create and implement learning environments, teachers must have knowledge not only of the mathematics they teach, but also of the technological tools available and their potential for exploring mathematical ideas, as well as the pedagogical aspects related to teaching and learning the subject (Kim, 2018). Regarding the knowledge required to integrate technology into professional practice, few studies address the contribution of the use of different technologies to teachers' professional development and how the different domains of their knowledge are mobilized (Rocha, 2020b). This research aims to characterize the knowledge of mathematics teachers when using different technologies in teaching and learning, and to study the existence of relationships between the specific domains of the Knowledge for Teaching Mathematics with Technology - KTMT model (Rocha, 2020b) and the integration of the chosen technology. The aim is to answer the following research questions:

- (1) Which domains of KTMT are emphasized when integrating technology into professional practice?
- (2) What differences are there in the knowledge mobilized by the teacher when the technology used changes?

Theoretical background

The theoretical framework of this study focuses on KTMT model (Rocha, 2022) (Figure 1), as it integrates research focused on professional knowledge and the use of technology into a single model. The KTMT considers Mathematics (MK), Teaching, and Learning, Curriculum, and Technology (TK) as the base domains of knowledge.

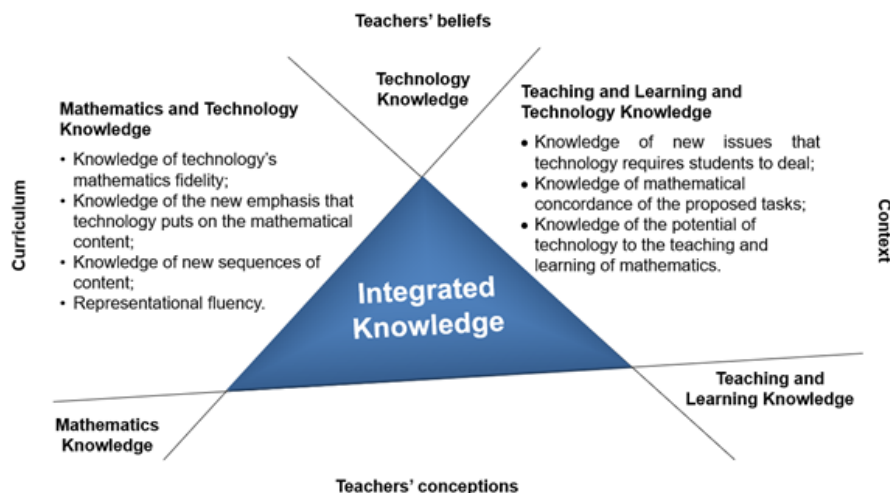


Figure 1: KTMT model (Rocha, 2022)

The mathematics base domain (MK) includes knowledge of concepts, theories and procedures of the disciplinary area. It also involves knowledge of rules of evidence and of the nature of mathematics. The technology base domain involves the capacity to operate with certain technology and essentially consists of knowing how it works and what it does. The teaching and learning base domain involves knowledge about how students think and learn, and includes an understanding of the processes usually used by students, common difficulties associated with specific content and the ability to anticipate these problems and solve them. It also includes the choices made by teachers in the different phases of teaching, involving knowledge of the sequence of activities, the different types of tasks and the way students work. Particularly valuing knowledge developed from more than one of the base domains, KTMT includes two sets of inter-domain knowledge: Mathematics and Technology Knowledge – MTK and Teaching and Learning and Technology Knowledge – TLTK. MTK focuses on how technology influences mathematics and TLTK includes how technology affects or influences teaching and learning, reinforcing or restricting certain approaches.

The strategies used by teachers to integrate different technologies into their lessons can be based on mathematical understanding or on the specific technical procedures of each technology (Botelho & Rocha, 2022). However, as highlighted by Bretscher (2022, p. 2522):

Technology as a teaching resource for mathematics needs to be simultaneously both visible, so that it can be noticed and used in the practice of teaching school mathematics, and invisible so that attention is focused on the subject matter, teaching mathematics to pupils. The particularities of using a specific technology to teach mathematics influences the mathematics that can be taught.

Methodology

Given the aim of this study, the methodology chosen for this research is qualitative and interpretative. We used the case study of a high school mathematics teacher, whom we will refer to as Camila. She

is a teacher with over thirty years' professional experience who is very familiar with using different technologies to teach mathematics, aspects that are considered particularly influential in the integration of technology (Handal et al., 2011). The data collection took place in her 11th grade class (16-17 years old), in lessons where the study of sequences using technology had been planned. As the study concerned the integration of technology by teachers into their teaching practice, and with the aim of characterizing mathematics teachers' knowledge when using different technologies, the research had a strong component of observation of mathematics teaching practice in a classroom environment. Semi-structured interviews and document analysis were also used to collect data. All the interviews and all the lessons observed were audio-recorded. Based on the tasks proposed by the teachers and in line with the nature of the study, data analysis was guided by the research questions, focusing on the use of the graphing calculator (GC) and *Excel*. For the data analysis, the moments in which different domains of KTMT were mobilized during the use of the different technologies were identified and analyzed using content analysis with special attention to the knowledge domains: TK, TLTK and MTK. The data were analyzed with the intention of identifying the utilization of knowledge associated with each of these knowledge domains. In instances where moments addressed multiple domains, thorough examination was conducted to determine which domain appeared to be predominant.

Results

In line with Bretscher (2022), Camila believes that technology must have to be a resource in her lessons, without it becoming the central aspect, taking attention away from the mathematics being taught. By integrating GCs and *Excel* into the two tasks, the teacher wants the students to become familiar with the tools provided by these two technologies. Considering that students are more familiar with the use of tables in *Excel* compared to GC (Ruthven & Henessy, 2002), Camila chooses to use this technology in the first task.

The task with *Excel*

At the beginning of the study of sequences, Camila tried to involve the students in an exploration of monotony and the existence of upper and lower bounds. She proposed the study of the monotony of a sequence, using the students' knowledge about the monotony of a function. Using an *Excel* spreadsheet, she proposed that the class observe the graphical and tabular representation (MTK). Having given the algebraic expression of the general term of the sequence $a_n = \frac{n+1}{3n-2}$, together with the class, the teacher built a table with the first 100 terms of a_n :

Camila: (...) what we're going to do here is visualize the behavior of a sequence by graphing, for example, the first 100 terms. (...) so here in cell two, we're going to put the values of n, because it's a natural variable, but we're not going to write them one by one, we're going to enter a formula. (...) In the next column we will insert the general expression of a_n in B3, using the Equation tool. (...)

Initially, the guidance given to the students was specific to this technology (TK), as the students had little knowledge of the tools available in *Excel*. Although all the students were using an *Excel* spreadsheet, some had to use the online version. For these students, the teacher had to give more technical instructions (TK), as some tools were not available in the online version:

Camila: For students who are using *Excel* on Google, you don't have the Equation menu, but no problem, just write the expression directly in the cell, using the brackets and the slash.

Projecting the table with only the first nine terms of a_n (Figure 2), Camila begins by asking if it is possible to conclude from the table whether the sequence is monotonic and if there is any value it is close to. One of the students replies that the sequence is decreasing, and the smallest term is 0.4. Camila then suggests showing the first 100 terms of the sequence in the table so that the students can check whether their colleague's answer was correct (MTK):

Camila: Is it possible, with this table, to say that the sequence is monotone?

Student1: Yes, it's decreasing.

Camila: The first term is 2 and we can see that the terms are decreasing. But with these 9 terms, can we say what value the terms of the sequence will approach?

Student1: Yes, the lowest value is 0.4.

Camila: Check the table with 100 terms to see if your colleague's answer is correct.

Student2: No, with $n=100$, the value is 0.33893.

After looking at the table with the first 100 terms of a_n the teacher provides technical guidance on the tools available in *Excel*, combining knowledge of the content of sequences (MTK) to construct the graph of a_n (Figure 3).

	A	B
1		
2	n	a_n
3	1	2,00000
4	2	0,75000
5	3	0,57143
6	4	0,50000
7	5	0,46154
8	6	0,43750
9	7	0,42105
10	8	0,40909
11	9	0,40000

Figure 3: Table with the first 9 terms of the sequence

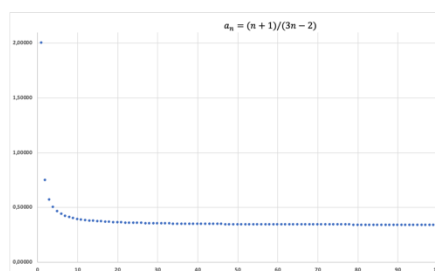


Figure 2: Graphical representation of sequence in Excel

With the graphical representation, the teacher concluded with the class, as they had already observed from the table, that the sequence was decreasing. The automatic display of the graph in a visualization window appropriate to the situation did not require Camila to make any comment on this aspect.

The teacher also wanted the students to conclude, by looking at the graph, that the sequence was bounded (MTK). She began by saying that a_n is monotonically decreasing and that its first term is two, which allowed the students to conclude that the sequence was upper bounded. However, finding a value for the lower limit by looking at the graph was more difficult, as there was no single value. The teacher therefore opted to use the table again, so that the students could see that the terms of the sequence are always positive (MTK):

Student: It [sequence] has no values greater than 2.

Camila: So, can we say that 2 is the upper bound of the sequence?

Student1: Yes, because the other values are below 2.

Camila: We've already seen that the sequence is decreasing, and the first term is 2, so it [sequence] is bounded from above. Can anyone say what a lower bound is?

Student2: No, because they're always smaller.

Camila: Let's go back to the table. We can see, for example, up to the term of order 200. What's going on? Notice that the terms of a_n are always positive, they are never less than 0. So, we can say that 0 is a lower bound of the sequence.

Camila refers to the fact that, with this technology, it is not necessary for students to adjust the viewing window to obtain a graphical representation, as the main deciding factor in choosing the technology used (TLTK):

Camila: I chose to use *Excel* to do this task, because with the calculator they had to adjust the viewing window (...). I wanted them to see the 100 terms in the table and then see the graph. With the calculator it's not so easy to see so many terms in the table, I considered that, but no, with the GC it took much more time. (...) The central aspect of my choice was the viewing window.

In this task, she did not intend to pay attention to the display window, since with *Excel* the window is automatically adjusted. The knowledge of the impact of the viewing window on the aspect of the graph is a decisive factor when choosing the technology (MTK):

Camila: I chose to use *Excel* because I want them to focus on studying monotony [of sequence] through the table and graph. In this first task, I don't want them to have to change the window, because I know it's still not easy for them to understand how and when to change the window. I want them to be able to see the terms of the sequence straight away without having to change the window.

The task with graphing calculator

In the second task, the teacher wanted the students to study monotony and check whether it was bounded, using GC. This was the first time *Recursion* tool had been used in class, Camila gives specific technical guidelines and exemplifies them so that the students can repeat the procedures in their GCs (TK). In the class, only one student did not have a Casio *fx-CG* and so the guidance given was only for this technology. Given the sequence $b_n = \left(\frac{2}{3}\right)^n$, the teacher explains how to insert its expression (Figure 4) into the GC (TK), starting by defining the type of expression for the general term (Figure 5) (MTK). Next, by presenting the table with the first four terms of b_n (Figure 6), she asks the class about the monotony of sequence and whether sequence is bounded:

Camila: Looking at these four terms in the table, can we conclude what the monotony of the sequence is?

Student: It's monotonically decreasing.

Camila: Is it [sequence] bounded?

Student: 0.6666 is the biggest but we can't tell if it will stop at 0.1975.

Camila: So, let's look at the graph.

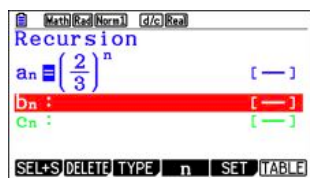


Figure 4: General term of b_n

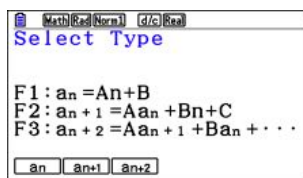


Figure 5: General Term Type Selection Menu

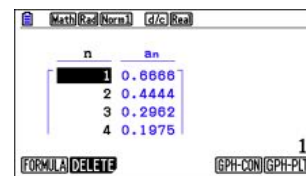


Figure 6: Table with the first four terms of the b_n

With the sequence graph displayed in a standard viewing window (Figure 7), Camila gives guidance on how to change it, discussing the appropriate parameter values with the class (MTK). By changing the window, the students are able to see the graphical representation of the five terms of b_n (Figure 8).

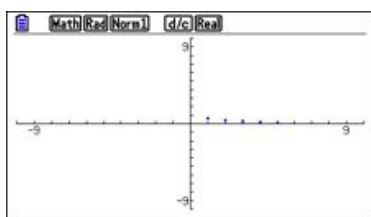


Figure 7: Graph obtained with a standard viewing window

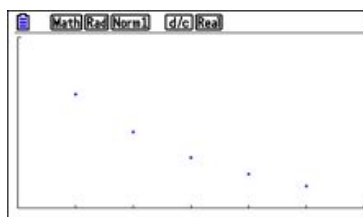


Figure 8: Graph obtained with the appropriate viewing window

The need to change and adjust the display window did not arise when studying sequence a_n in *Excel*, since the graph in this technology is automatically adjusted to the data entered in the table. Recognizing that this is a difficult process when using GC, Camila begins by presenting the graph in a standard window, with the intention of creating in the students the need to adjust it, so that they can find a more suitable viewing window:

Camila: My idea was for them to first see the graph [of b_n] in the standard window and from there, make them realize that the window had to be adjusted. After seeing the first graph, it was easy for them to find the right values for the viewing window (...).

When asked by the students about the existence of a lower bound, Camila explains that since the terms of the sequence are powers of a positive base and a natural exponent, the terms are all positive. She then leads the class to conclude that the sequence is also bounded from below, and that 0 is a possible lower bound, but not the only one (MK).

Discussion and conclusion

The KTMT model seeks to combine the domains of knowledge considered relevant in the research with the specificities of technology. It highlights the importance of the influence of technology on the different domains of knowledge, emphasizing the need to develop MTK and TLTK to effectively integrate technology into professional practice. It is therefore considered that these inter-domain knowledge is essential as a basis for analyzing teacher knowledge (Rocha, 2022). Although the pedagogical options that technology allows are recognized (Clark-Wilson et al., 2020; Bretscher, 2022; Rocha, 2020a, 2020b), the specific characteristics of each of the technologies used influenced the teacher's pedagogical choices when solving the tasks (Ruthven, 2009). The knowledge of the students' difficulties in finding a suitable viewing window (TLTK) was decisive in the choice of technology that the teacher used in the first task. The teacher decided to use technology where less technical knowledge had to be mobilized, and where there was no need to adjust the display window. The teacher's knowledge of the impact of the visualization window on the appearance of the graph (MTK) leads her to consider that the tasks have different degrees of difficulty, depending on the technology chosen. Teacher's guidance differed for the two technologies. With GC, her guidance was necessary for the articulation between the different menus and the adjustment of the viewing window to obtain the graph. By explaining how the different menus work, the teacher emphasizes more mathematical aspects. It was also found that the integration of these technologies had an impact on the way the mathematical content was taught (Bretscher, 2022). The use of the two technologies

promoted the mobilization of different domains of knowledge, as highlighted by Rocha (2020a, 2020b). It was observed that the teacher mobilized MTK, TLTK, TK and MK and, in some situations, these domains were mobilized in an integrated way. However, there are no differences in the way they are mobilized according to the technology used. In both tasks, the integration of technology into her professional practice reveals the existence of an interconnection between MK, TK and TLTK, as considered in KTMT. By adapting her strategies to different technologies (*Excel* and GC), Camila demonstrates that she is aware of the impact of technology on mathematics teaching. In this sense, she gives specific guidance and adjusts approaches to the tools provided by each of the technologies. It is this knowledge of the most appropriate strategies (TK) for each situation that influences the guidance given by the teacher (TLTK) (Rocha, 2020a). The technical guidance given by the teacher on using *Excel*, reflects an understanding of the impact of this technology on teaching and learning, mobilizing TLTK for this purpose. The need to change the viewing window in the GC and the subsequent explanation of this change also involved the mobilization of TLTK. The teacher guides the students in handling the technology, not only to improve their visual understanding of the graphs, but also to reconcile the mathematical knowledge that they have learned. Camila also reveals an understanding of the impact that the visualization window has on the appearance of the graph, which allows her to analyze the degree of difficulty associated with each of the tasks (MTK) (Rocha, 2020a). In both technologies, the teacher helps the students to understand mathematics by articulating the tabular representation with the graphical representation. However, there is a greater mobilization of MTK with the GC. In this technology, the exploration of a graph and the need to adjust the viewing window provides a more exploratory approach to the study of the sequence monotony. It should also be noted that the combination of algebraic, graphical and tabular representation on the GC allowed for a more dynamic strategy. In summary, the pedagogical approaches adopted by Camila reflect an advanced level of development of her KTMT. The integration of different technologies has allowed her to mobilize different domains of knowledge. This integration demonstrates the teacher's deep inter-domain knowledge (MTK and TLTK), in line with the principles of KTMT. We recognize that this study is limited by its analysis of only two technologies and does not cover the diversity that exists. Future studies should be designed to observe the integration of a greater number of technologies with in-service and pre-service teachers.

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ChatGPT as a reflection tool to promote the lesson planning competencies of pre-service teachers

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In our study, set against the backdrop of rapidly advancing AI technology, we examine how ChatGPT, can support pre-service teachers (PSTs) in creating lesson plans and contribute to the improvement of university mathematics teacher education. We implement various prompting techniques that help PSTs cultivate digital professional competencies when interacting with ChatGPT in AI-assisted lesson planning. We evaluated the AI-generated lesson plans and the PSTs' modifications to these plans. Our initial results indicate that the quality of both the AI-generated lesson plans and the PSTs' modifications to them can vary and interact with each other. Overall, our findings suggest that effective prompting techniques can aid PSTs in enhancing their lesson planning competencies.

Keywords: Artificial Intelligence, ChatGPT, lesson planning, teacher competencies, pre-service teachers.

Introduction

Recent technological advances in the field of generative AI, in particular through large language models such as GPT and its well-known agent, ChatGPT, are paving the way for the integration of AI into the professional work of mathematics teachers (Buchholtz et al., 2023). For example, ChatGPT can help teachers in their daily classroom practices to answer questions and provide adaptive feedback to students, or it can be used to break down language barriers for non-native speaking students (Kasneci et al., 2023). Furthermore, the use of ChatGPT by teachers in pre- and post-classroom activities also offers the opportunity to plan lessons and to create an adaptive range of learning content and materials for each student, which can save teachers time in their daily work (Kasneci et al., 2023; Wardat et al., 2023; Zhang & Tur, 2023). At the same time, generative AI poses unique challenges in mathematics (teacher) education. While ChatGPT can handle complex mathematical inputs, it still struggles to present accurate solutions, even for simple problems (Schorcht et al., 2023; Schorcht et al., 2024). Therefore, AI-generated outputs need to be checked for accuracy, as errors can be harmful in educational contexts (Buchholtz et al., 2023; Schorcht et al., 2023).

As teachers around the world increasingly use this AI model, the question of the quality of the results produced becomes increasingly important, as it is the key indicator of whether the technology can really help to save time and improve the quality of mathematics teaching. This also raises the question of how prospective teachers can develop appropriate skills to use this technology in a targeted way and to critically reflect on the results it produces. Indeed, despite the rapid advances in technology, there is little comprehensive research on the potential benefits and challenges of ChatGPT in mathematics (teacher) education. Our research therefore aims to showcase the practical implications and educational benefits of employing ChatGPT in mathematics teacher education. Specifically, the study investigates how ChatGPT can facilitate AI-assisted lesson planning in mathematics education. It highlights how pre-service teachers (PSTs) employ specific prompting techniques (Schorcht et al.,

2023; Schorcht et al., 2024) to direct ChatGPT in crafting detailed lesson plans that include explicit goals, didactic and methodological approaches, anticipated teacher and student activities, and structured lesson outlines.

Mathematics teachers lesson planning in the age of AI

Creating lesson plans constitutes a fundamental aspect of a mathematics teacher's daily responsibilities, necessitating considerable intellectual preparation before classroom activities begin (Akyuz et al., 2013). Despite the great importance of this activity, conceptualizing lesson planning as part of teachers' professional activities is difficult, because the construct encompasses many different elements. A general definition of lesson planning is provided by Jalongo et al. (2007), who states that lesson planning can be a recursive, dynamic, cyclical, and somewhat improvisational process. According to this approach, lesson planning begins with lesson design, followed by planning and implementation, then reviewing learner responses, and finally circling back to lesson redesign (Jalongo et al., 2007). Another definition is given by Killen (2015), who understands lesson planning as determining content and learning outcomes, identifying teaching and learning strategies, determining assessment strategies, and evaluating the effectiveness of the lesson. Lesson planning therefore also includes making planning-related decisions. In order to prepare PSTs for this critical aspect of teaching, educational programs worldwide have underscored the importance of acquiring proficient lesson planning competencies, which are often pivotal for succeeding in final exams (König et al., 2021). PSTs are required to develop these competencies during their studies, particularly through practical experiences such as internships. However, novice teachers, such as PSTs, often base their planning on predetermined concepts (schemes), which can give them security due to their lack of experience. They do not yet have the necessary flexibility in planning and the experience to be able to make adaptive planning decisions like experts (Mutton et al., 2011; König et al., 2021).

The advent of artificial intelligence and large language models like GPT offers PSTs the opportunity to develop digital skills in utilizing AI technologies for lesson planning. The technology can support them in making suggestions for planning steps or reflecting on their own plans by asking specific questions. This underscores the imperative for a more integrated approach to incorporating AI-tools in mathematics teacher education, especially for novice teachers who stand to benefit significantly from such digital aids in enhancing their planning competencies. This integration is crucially supported by the Technological Pedagogical Content Knowledge (TPACK) framework (Koehler & Mishra, 2009), which highlights the interplay between content knowledge (CK), (general) pedagogical knowledge (PK), and technological knowledge (TK). The TPACK framework provides a comprehensive model that includes content, pedagogy, and technology and it is suitable to describe the knowledge the PSTs enact when utilizing a digital tool like ChatGPT for professional educational purposes. It elucidates the essential knowledge areas for teachers and the dynamic relationships among them, such as PCK (Pedagogical Content Knowledge), TCK (Technological Content Knowledge), and TPK (Technological Pedagogical Knowledge), culminating in the comprehensive domain of TPACK. PSTs can draw on their TPACK when using ChatGPT to generate, evaluate, and refine lesson plans, ensuring they meet educational objectives and cater to diverse student needs. However, as Mishra et al. (2023) emphasize in their reflections on TPACK in the age of generative AI, it is important to understand how generative AI technologies can transform teaching and learning practices. Evaluating the effectiveness of AI-enhanced lesson plans therefore involves not only

examining adherence to mathematical content and educational objectives, but also assessing PSTs' AI-related digital competencies, i.e. their engagement with ChatGPT throughout the planning process to ensure its output aligns with educational goals and pedagogical strategies.

Teachers' prompt techniques for lesson planning

In our study, we explore how PSTs can enact and develop TPACK through engagement with ChatGPT, focusing on reflective interaction processes in lesson planning. This approach assumes that such skills are developed not only by recognizing the potential and limitations of ChatGPT but also by proficiently applying effective prompt techniques and critically assessing the AI-generated content. Our study draws on various prompting strategies as outlined by Schorcht et al. (2023) and Schorcht et al. (2024), showcasing different methods to harness ChatGPT for educational purposes.

In general, we can utilize Zero-Shot Learning, where ChatGPT generates responses without prior tailored input or training data, answering straightforward requests such as crafting a lesson plan on a specific mathematical topic. A simple prompt like *“I need to plan a mathematics lesson on algebraic transformations”* already produces a sample lesson plan including learning objectives and teacher and student activities for different phases (e.g., introduction, teacher presentation, practical exercises, group work, and presentation). This method, while providing a solid foundation, allows room for refinement. Advancing to Few-Shot Learning, we incorporate structured prompts to tailor the output more closely to specific needs. This is useful when we need to adapt previously held lessons due to changed conditions or when we have to comply with specific external guidelines. Employing Chain-of-Thought Prompting involves adding cues like *“Let's take it step by step to ensure a well-planned lesson,”* which moderates the pace of ChatGPT's processing to yield more elaborate and controlled outputs (Schorcht et al., 2023; Schorcht et al., 2024), thereby rendering the planning more discernible and integrating didactical decisions. Moreover, we explore Ask-me-anything Prompting (Schorcht et al., 2023; Schorcht et al., 2024), prompting ChatGPT to gather essential information for lesson preparation from the user (Schorcht et al., 2023), thereby enabling customization according to unique learner needs. Finally, incorporating prompts such as *“and always wait for my input”* (Huget & Buchholtz, 2024) encourages a sequential interaction, preventing the generation of unwieldy outputs and fostering outputs that are more aligned with effective lesson planning. These strategies demonstrate how ChatGPT can be leveraged for lesson planning, with the resulting outputs subjected to further analysis in subsequent phases of our study.

Research questions and methodology

In our study, we focused on the following research questions: (RQ1) What is the quality of lesson plans developed by PSTs using ChatGPT? (RQ2a) What modifications do PSTs make to the AI output provided by ChatGPT? (RQ2b) To what extent can different patterns of interaction be identified? (RQ3) What perceived benefits do PSTs identify when utilizing ChatGPT in the development of mathematics lesson plans?

To investigate these questions, we conducted a study involving a group of PSTs from the University of Hamburg, Germany, employing a simulation-based design to ensure comparable conditions. In order to control the initial planning dialogues of the PSTs, a prompt setting based on the prompt techniques “Chain-of-Thought” and “Ask-me-anything”, as well as an “always-wait-for-my-input” (Huget & Buchholtz, 2024) waiting rule was developed for the data collection in our study. The study

was carried out in October 2023 and bases on a convenient sample. In two parallel mathematics education seminars, N=37 Master's students studying mathematics teaching for upper secondary level were asked to plan a lesson on the topic of algebraic term transformations, which they carried out with the assistance of ChatGPT. Participation in the study was on a voluntary basis and data was collected with informed consent. The following prompt setting (1) to (3) was given to the PSTs:

(1) *“Plan a mathematics lesson for me. Please ask me first about the topic/content of the lesson. Then ask me about the given framework of the planning. Always wait for my input throughout the chat.”*

(2) *“The topic is algebraic term transformations”*

(3) *“The following framework is given for you to use: 1. learning group analysis, 2. justification of the mathematics educational decisions including a subject-specific clarification of the topic/learning situation, 3. competencies/learning objectives, 4. justification of the methodological decisions, 5. progression planning in tabular form with teacher and student activities. Ask me individual questions about the steps and wait for my input that you need for the lesson planning. Let's go step by step to make sure we plan a good lesson. Throughout the chat, always wait for my input before moving on to the next step.”*

The PSTs interacted with ChatGPT (at that time in version GPT-3) using a GDPR-compliant platform under pseudonyms. This approach aimed for a high comparability level among planning dialogues for a hypothetical learning group. It leads to ChatGPT asking questions on the points mentioned in prompt 3 in a specific (comparable) order. All PSTs implemented this prompt setting as given and were then able to answer freely in the individual steps and continue the dialogue. The transcripts were then downloaded from the platform in PDF form for further analysis.

For RQ1, we applied qualitative content analysis (Mayring, 2014), evaluating the lesson plans. Evaluation criteria included a.o. mathematics-specific factors such as content transformation and task creation, the clarity of learning objectives or the adaption to learning dispositions and were based on a six-dimensional coding scheme of König et al. (2021), that can be used to measure the cognitive demand of written lesson plans on dichotomous coding scales. To explore RQ2a and RQ2b, we firstly categorized PSTs' modifications to the AI-generated plans, using a three-point scale, that takes up different types of modifications: *“PST does not comply or simply accepts ChatGPT's output”* (0); *“PST is asking ChatGPT for suggestions”* (1), and *“PST is adapting ChatGPT's output or provides own suggestions”* (2). We then further differentiated the categories based on an inductive coding of the planning dialogues. Finally, addressing RQ3 involved interviewing the PSTs about their experiences with AI-assisted planning, aiming to understand the perceived benefits of integrating ChatGPT into mathematics teacher education.

Preliminary Results

In this contribution, due to space restrictions, we will only outline preliminary findings on RQ1 and RQ3 in the conclusions section and concentrate in the results section on RQ2a and RQ2b with a focus on a specific sub-area of lesson planning, namely the formulation of learning objectives (prompt 3; step 3). Based on the data material additional inductive categories were identified for the modifications made by the PSTs when they were formulating learning objectives (see Table 1). The PSTs most frequently offered their own suggestions (42.62%). This result can be interpreted as a sign of the PSTs' critical engagement with ChatGPT's output and may also reflect their effort to tailor

teaching to individual and contextual needs. The fact that many PSTs adapted the suggestions generated by ChatGPT to fit specific teaching contexts (18.03%) also speaks to their reflective usage of the technology, which we see as an indicator of their use of TPACK. They did not simply accept the AI-generated suggestions but adapted them, which could mean that the PSTs were keen on basing their decisions on a solid theoretical foundation. However, in other cases, some PSTs simply accepted ChatGPT's output (13.11%) or did not refer back to it (3.28%). On one hand, this could point to a competent assessment of ChatGPT's capabilities. On the other hand, it could also be an area that needs further investigation to ensure that this acceptance does not stem from a lack of critical thinking or from a naïve trust, which would point to possible underdeveloped TPACK.

Inductive-developed category system regarding modifications made by PSTs in planning step 3	Assigned codes (61)
PST does not comply – 0	3.28% (2)
PST simply accepts ChatGPT's output – 0	13.11% (8)
PST is asking ChatGPT for suggestions – 1	13.11% (8)
PST is adapting ChatGPT's output – 2	18.03% (11)
PST provides own suggestion – 2	42.62% (26)
PST draws on literature (e.g. educational standards) – 2	6.56% (4)
PST asks critical questions and demands clarification – 2	3.28% (2)

Table 1: Initial results regarding the modifications made

We give an example of a planning dialogue of a PST (#bloom). In the following transcript, the PST moves from step 2 to step 3 of the planning dialogue and, together with ChatGPT, determines the reasons for the mathematical didactic decisions and the learning objectives. After it has already been determined that simple algebraic terms with addition and subtraction should be dealt with first and visual aids should be used, the PST would like to add a content remediation at the beginning of the lesson that relates to the repetition of the understanding of variables (*PST provides own suggestion*). We see this as an indicator of the PST's CK and PCK that are involved in the planning, ChatGPT then plans this addition adaptively into the course of the lesson in the subsequent step 3.

PST: Sounds good! The only thing that should be discussed at the beginning is the repetition of the concept of variable, as some students probably don't remember it properly.

ChatGPT: Many thanks for the tip! That is an important addition. So, at the beginning of the of the lesson, we will briefly repeat the concept of the variable, to make sure that all students understand the concept. Now we can move on to step 3 and define the competences/learning objectives.

3. competences/learning objectives:
- Students should understand what a variable is and how it is used in terms.
 - Students should be able to transform simple terms with addition and subtraction.

- Students should be able to transform complex terms with multiplication and division.
- Students should be able to understand the transformations of terms using visual aids.
- Students should be able to carry out term transformations independently and be able to present their solutions.

Please let me know if you would like to add further competences/learning objectives or adjustments should be made.

PST: That looks good! We can also note that the second point is a minimum requirement that all students should achieve.

From the PST's reaction, it is clear that minimum requirements for the students should also be defined, which we see as an indicator of the PST's enacted PK (*PST is adapting ChatGPT's output*). We recognize in this an active engagement of the PST with the AI-generated output, which we understand as an indicator of the activation or development of lesson planning competencies and the respective TPACK, even if the engagement with the AI-generated suggestion is not on a critical level.

Concerning RQ2b, we investigated interaction patterns. Interaction patterns are the sequence and use of modifications made by the PSTs within a dialogue. In the case of PST #bloom, we could observe an interaction, where the PST first provides own suggestions to the dialogue and then adapts ChatGPT's output. We present another example of this identified interaction pattern in the second transcript below. In the case of this PST (#Karl Heinz), the PST first responds to reflection questions posed by ChatGPT and makes an own suggestion, enacting their PCK in the dialogue. ChatGPT then formulates corresponding learning objectives for the mathematical content and the definitions of the concepts involved. The PST then accepts some of the suggestions, but modifies them partly afterwards (PST enacts their PK). However, we again do not see any critical reflection or objection to ChatGPT's suggestions. Instead, the PST asks for suggestions on concrete tasks that can be used to achieve the defined learning goals in a real-life context (*PST is asking ChatGPT for suggestions*), and by this shows enacted PCK in the planning.

PST: For the first lesson, the students should familiarize themselves with the definitions of the concepts involved.

ChatGPT: Got it! Based on your goal of having students learn the definitions of terms in the first lesson, I suggest the following learning objectives:

1. Students should understand and be able to explain the concepts "variable", "coefficient", "exponent", and "operator".
2. Students should understand the meaning of basic arithmetic operations (addition, subtraction, multiplication, division) in the context of algebraic term transformations.
3. Students should familiarize themselves with the basic mathematical rules and properties of algebraic term transformations and be able to apply them.

These learning objectives enable students to understand and use the basic algebraic terms and concepts related to term transformations. They lay the foundation for the further understanding and application of term transformations. Do you have any other learning objectives you would like to achieve in the lesson?

PST: I think the first learning objective is good. The second learning objective should involve the students applying what they have learnt to tasks. Can you think of any good tasks with a real-life context on the topic of algebraic term transformations?

Discussion

The results on RQ2a and RQ2b show a differentiated use of ChatGPT generated output by the PSTs, ranging from creative to conscious adaptation and theoretically informed decisions to acceptance. With regard to RQ1, the quality and effectiveness of AI-assisted lesson plans exhibited considerable variation. When asked for suggestions, the initial planning of ChatGPT often corresponded to quite standardized classroom procedures, which indicates a strong dependence of the AI output on the underlying training data. However, the quality was influenced significantly by the PSTs' modifications to the AI-generated content. Here we see the potential of ChatGPT in lesson planning, where a structured prompt setting can facilitate a reflective planning dialogue. Through this dialogue, PSTs are led by the AI to contemplate their planning decisions or are provided with suggestions to bolster their initial ideas planning ideas. With regard to RQ3, many of the PSTs commented that the multi-step nature and guided planning dialogue particularly helped them not to lose sight of the overarching planning. In addition, the follow-up questions that ChatGPT asked in the prompt setting ensured that they had to make planning-related decisions and gained an awareness of the relevant parameters of lesson planning. Looking ahead, future research could explore prompt settings that further encourage PSTs to critically assess their planning strategies, with ChatGPT serving more as a facilitator of the planning process rather than merely a source of ideas. Engaging with ChatGPT, while potentially time-saving, necessitates a critical evaluation of the AI-generated lesson plans.

The integration of ChatGPT into mathematics teacher education reflects the growing imperative for digital and AI-related literacies among mathematics teachers. This shift mandates that teacher education programs, educators, and PSTs treat digital literacy as a core component of effective mathematics teaching. Our study emphasizes the critical role of TPACK within teacher education, underscoring the need for a meticulous examination of AI-generated materials to ensure they align with educational objectives and cater to student requirements.

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What is ChatGPT's Belief System about Math?

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This paper deals with the presentation of mathematics in responses of the large language model ChatGPT. Based on a theoretical discussion, four belief systems or perspectives on mathematics are distinguished: formal-abstract, empirical-concrete, application, and toolbox. In a systematic analysis, responses from ChatGPT in the context of the mathematical concepts of real numbers, straight line and probability are categorized against this background. It turns out that ChatGPT can adopt very different perspectives on mathematics and that the perspectives offered seem to depend on the particular prompt.

Keywords: Artificial intelligence, Beliefs about mathematics, ChatGPT, Large language models

Introduction

Artificial intelligence (AI) is currently a highly debated topic in society. While AI has been studied intensively in educational research from a technical perspective for around 10 years (see, e.g. Roll & Wylie, 2016), mathematics education research has only recently begun to address this topic. The focus of this research is on generative AI systems such as ChatGPT for the creation of texts. Wardat et al. (2023) conducted interviews with students and teachers and found that ChatGPT is generally perceived as a useful educational tool, but that it does pose some challenges (e.g. development of misconceptions). Other authors, however, analyzed ChatGPT from a theoretical perspective and through extensive testing. For example, Buchholtz et al. (2023) come to rather negative conclusions on this basis and state that the generative AI ChatGPT is not yet suitable for use in mathematics classes. In a teaching experiment, Dilling et al. (2024a) looked at the use of ChatGPT as a student tool for proving the theorem on the sum of interior angles in a triangle. The result was that although ChatGPT cannot logically conclude and thus prove to a large extent, its reflective use in the classroom can nevertheless provide a variety of impulses for rich processes of argumentation and negotiation of meaning.

Large language models (LLMs) such as ChatGPT are linguistic models that have been trained with a huge amount of text data and are intended to simulate communication. With the help of probability trees, answers to user requests (so-called prompts) are generated. Although the system was trained for linguistic knowledge, it can also contain rational knowledge from the training data (Petroni et al., 2019). However, knowledge databases are not accessed for the answers; the "knowledge" comes solely from the trained linguistic model, which can also result in the output of incorrect information. In a systematic evaluation, Dilling (2024) examined the potential of ChatGPT as an assessment and feedback tool in mathematics education. The analysis of different sample tasks and solutions shows that ChatGPT offers a remarkable amount of potential for formative assessment and feedback in the field of mathematics. The activation of the Wolfram plugin and the right prompting has proven to be an important success factor for appropriate feedback. The prompt largely determines the form and content of the feedback, the correctness of the performed calculations and the extent to which the

response is adapted to the feedback receiver. In general, the system was able to provide correct and didactically useful feedback in many (but not all!) cases.

In addition to the correctness of the calculations, the perspective that ChatGPT adopts on mathematics is particularly relevant. This can be described using the term mathematical belief system, which is defined by Schoenfeld (1985) as follows:

Belief systems are one's mathematical world view, the perspective with which one approaches mathematics and mathematical tasks. One's beliefs about mathematics can determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. (Schoenfeld, 1985, p. 45)

Beliefs and belief systems are a central subject of investigation in mathematics education research, and they are assumed to have a significant influence on mathematical teaching and learning processes (Goldin et al., 2009). In a normative sense, Green (1971) considers the shaping of students' mathematical belief systems to be one of the key goals of teaching:

Teaching is an activity which has to do, among other things, with the modification and formation of belief systems. If belief systems were impervious to change, then teaching, as a fundamental method of education, would be a fruitful activity. (Green, 1971, p. 48)

The research literature often emphasizes the importance of teachers' beliefs, which significantly influence the development of students' beliefs. Grigutsch, Raatz and Törner (1998) explain this connection with communication and interaction in the classroom as well as the methodological and content-related decisions made by the teacher:

[...] The attitudes of teachers have a substantial influence on the attitudes of students — on one hand, in direct communication and interaction in a mathematics class, and on the other hand, indirectly through the concrete design (choice of material and methods, and assessment system) of a mathematics class (Grigutsch et al., 1998, p. 4, author's translation)

The influence of teachers' beliefs on students' beliefs has been empirically proven in a number of empirical studies (e.g. Carter & Norwood, 2010; Muis & Foy, 2010). Other factors, such as the design of teaching materials, are less easy to measure directly in empirical studies – nevertheless, a long-term influence can be assumed. If generative AI will have a significant influence on the learning of mathematics in and outside the classroom in the future, the following question becomes important, which will be examined in this article:

Which perspectives on mathematics are presented in ChatGPT responses?

Although ChatGPT is not a real person, the AI simulates natural communication. Therefore, the question could also be formulated casually as in the title of this article as:

What is ChatGPT's belief system about math?

In order to investigate this question, the following section first presents a framework for the distinction of different types of belief systems about mathematics. Subsequently, the results of a

qualitative exploratory study and initial results of a systematic quantitative evaluation are presented. Finally, a conclusion is drawn, and an outlook is given.

Belief systems about mathematics

A basic assumption of research on belief systems about mathematics is that the belief systems of different individuals can be described as sufficiently similar and can therefore be categorized accordingly. Goldin (2002) uses the phrase "socially or culturally shared belief systems" (p. 64). Based on this assumption, theoretical and empirical approaches have been developed to categorize possible types of mathematical belief systems, each of which focuses on different characteristics. Building on the research on the categorization of prototypical belief systems about mathematics, four perspectives will be described in more detail here (Table 1) (Dilling et al., 2024b). These categories were chosen because they appear in the same or a similar form in many empirical studies and educational discussions, as shown below. It is explicitly not an exhaustive description – many other belief systems can be described by looking at other characteristics. It should also be emphasized that the belief systems are not mutually exclusive.

The formal-abstract perspective (FA) corresponds to the commonly accepted scientific way of practicing and understanding mathematics. It is characterized by a strictly deductive and axiomatic approach. In addition, there is a complete separation of mathematics from reality, which can be achieved through the formulation of axioms as propositional patterns. The formal-abstract perspective can be found in particular at universities and was established with the work of David Hilbert on the foundations of geometry. A similar characterization of a formal-abstract belief system can be found in various places in the literature. For example, Schoenfeld (1985) compared the beliefs of mathematicians and students when solving problems, whereby the belief system of the mathematician describes the formal-abstract perspective. Grigutsch et al. (1998) characterized the formalism aspect with twelve different items and identified it as a frequently encountered perspective in a survey of teachers. Tall (2013) describes the world of axiomatic-formalism as "building formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof" (p. 133). Dilling (2022), Stoffels (2020), and Witzke and Spies (2016) emphasize in particular the separation of mathematics from reality as a characteristic feature of a formal-abstract belief system.

The empirical-concrete perspective (EC) is a kind of opposite to the formal-abstract perspective. In this perspective, the objects of investigation in mathematics originate from empiricism (e.g. function graphs, drawing sheet figures, dice experiments) and the mathematical theory is ontologically bound to these. Mathematical propositions can be described within a theory, but the axioms and definitions refer directly to empirical objects. According to recent studies (e.g., Witzke & Spies, 2016), students at school and at the beginning of their university studies often hold such a belief system about mathematics, which can be attributed to the way mathematics is taught at school. In Schoenfeld's (1985) problem-solving case studies, the students' belief system corresponds to an empirical perspective on mathematics. However, Schoenfeld speaks of 'pure empiricists', as he did not observe any logical deductions or genuine theorizations. In Tall's (2013) three worlds framework, there is a world of conceptual embodiment, which describes the development of mathematical knowledge "building on human perceptions and actions" (p. 133). Dilling (2022), Stoffels (2020), and Witzke

and Spies (2016) also describe an empirical-concrete belief system, which is about substantial mathematical theory development based on empirical objects.

The application perspective (A) is about the application of mathematical concepts and theorems in reality. This takes place, for instance, in applied sciences, but also in everyday life. Grigutsch et al. (1998) describe this in the so-called application aspect with items such as "Mathematics helps to solve everyday tasks and problems" or "Mathematics has a general, fundamental benefit for society" (p. 17). Witzke and Spies (2016) describe the application orientation, which focuses on extra-mathematical applications and mathematical modelling.

The toolbox perspective emphasizes the application of rules, formulas and procedures in a schematic way. Ernest (1989) describes this as the instrumentalist view ("mathematics is a useful but unrelated collection of facts, rules and skills", p. 21). In the study by Grigutsch et al. (1998), the schema aspect is characterized, for example, with the item "Mathematics is a collection of procedures and rules that specify exactly how to solve tasks" (p. 19). Witzke and Spies (2016) cite the derivation of functions using derivation rules as an example of the toolbox orientation.

Table 1: Definition of the four perspectives on mathematics considered in the exploratory study

<p>Formal-abstract perspective (FA):</p> <p>Mathematical concepts are based solely on axioms and are strictly separated from reality.</p>	<p>Empirical-concrete perspective (EC):</p> <p>Mathematical concepts are defined with reference to empirical mathematical objects, e.g. function graphs or drawing sheet figures.</p>
<p>Application perspective (A):</p> <p>Mathematical concepts are used to describe real world phenomena, e.g. in applied sciences or in everyday life.</p>	<p>Toolbox perspective (T)</p> <p>Mathematics is about applying rules, formulas, and procedures in a schematic way.</p>

Methodological decisions

In order to investigate the research question mentioned above, a systematic analysis of ChatGPT 4.0 was carried out in February 2024. The category system in Table 1, which summarizes the four perspectives from the research overview in the previous section, was used as a basis. Using the method of deductive qualitative content analysis according to Mayring (2000), responses from ChatGPT were analyzed with regard to these categories. The analysis unit was a whole response. The prompts were created by applying a common survey method in qualitative belief research – the collection of associations to different mathematical concepts (e.g. Witzke & Spies, 2016). The mathematical concepts utilized were real numbers, straight line, and probability, because belief systems are very apparent in the description of basic mathematical concepts (e.g. Witzke & Spies, 2016) and the three concepts come from different mathematical areas. Questions such as "What are real numbers in mathematics?" were asked as prompts. In addition, these prompts were supplemented specifically to trigger certain perspectives on mathematics, e.g. with phrases such as "rigorously formal", "graphically" or "rules and formulas":

In addition to the qualitative analysis, an initial quantitative evaluation was carried out. For this purpose, the prompt "What is a/are real numbers / straight line / probability in mathematics?" without

any additional phrase was entered 30 times for each of the three mathematical concepts in separate chats. The responses from ChatGPT were also categorized according to the above four perspectives and a frequency distribution was created.

Systematic analysis of the large language model ChatGPT

In the systematic analysis, all four perspectives on the three mathematical concepts analyzed were found in the ChatGPT responses. Examples from the data for the four categories are shown in Table 2. The formal-abstract perspective included statements from ChatGPT in which the real numbers were set up axiomatically. The focus of the answers was mostly on an extension of rational numbers to real numbers using Dedekind cuts or Cauchy sequences. Regarding the concept of straight lines, ChatGPT was able to cite some of Hilbert's axioms as examples and explicitly reflected on the special features of this formalistic approach. Kolmogorov's axioms appeared very frequently for the concept of probability. The empirical-concrete perspective became apparent through references to empirical objects. In the case of real numbers, almost every answer from ChatGPT referred to points on the number line. In the case of straight lines, reference was made for example to a line drawn with a ruler. For probability, experiments such as rolling a dice were mentioned and approaches to probability using relative frequencies were described. With regard to the application perspective, the ChatGPT responses mostly contained very general statements that refer to the use of mathematical concepts in applied sciences. However, more specific use cases could be generated when explicitly requested. The toolbox perspective was evident in the case of real numbers in responses that explained specific calculation rules that apply to this number range. In the case of the concept of straight lines, formulas for an algebraic description were presented and the parameters were explained. In the case of probability, simple formulas for determining probabilities were displayed.

Table 2: Examples from the ChatGPT responses for the three mathematical concepts

Formal-abstract (FA)	Empirical-concrete (EC)	Application (A)	Toolbox (T)
<p>“In mathematics, real numbers are defined rigorously through several formal constructions, with the most common being the Dedekind Cuts and the Cauchy Sequences approaches. Both methods provide a foundation for the real numbers (R) in terms of more primitive notions such as rational numbers (Q).”</p>	<p>“Real numbers can be thought of as points on an infinitely long line called the real number line, where each point corresponds to a unique real number.”</p>	<p>“This set of numbers is important because it provides a way to describe and understand quantities and measurements in the physical world, such as distance, area, volume, and temperature.”</p>	<p>“Here are some of the key formulas and rules that govern real numbers:</p> <p>Addition:</p> <p>Commutative: $a+b=b+a$</p> <p>Associative: $(a+b)+c=a+(b+c)$</p> <p>Identity element: $a+0=a$”</p>
<p>“In Hilbert's formalistic theory, a straight line is not</p>	<p>“A straight line is the shortest path</p>	<p>“They serve as the basis for defining</p>	<p>“$Ax+By=C$, where A, B, and C are</p>

defined by its properties or by a visual or physical representation, as might be common in classical geometry. Instead, it is defined through a set of axioms that describe how lines behave and interact with other geometric entities such as points and planes.”	between two points. This definition aligns with the intuitive notion of a line you might draw with a ruler on a piece of paper.”	shapes, plotting graphs, and modeling physical phenomena.”	constants. This form is useful for analyzing lines in a more general context and can easily be manipulated to find the slope and intercepts.”
“In this framework, probabilities are assigned to sets in a way that satisfies certain properties or axioms, such as non-negativity, normalization (the probability of the whole sample space is 1), and countable additivity.”	“Empirical Probability (or Relative Frequency Probability): Based on observations or experiments.”	“have applications in various fields including finance, insurance, science, engineering, and everyday decision-making”	“The probability of an event is calculated as the ratio of the number of favorable outcomes to the total number of possible outcomes.”

The quantitative analysis of responses to the prompt "What is a/are [concept] in mathematics?" without any additional phrase shows that the formal-abstract perspective occurs considerably less frequently than the other perspectives and only for the concept of probability when a general unspecific prompt is used (see Table 3). In contrast, the empirical-concrete perspective appears in almost every answer, except four times for the concept of probability. It should be emphasized that the formal-abstract and empirical-concrete perspectives appear together ten times. In most of these cases, ChatGPT explicitly presented them as two different approaches to probability. It is also interesting to note that in two cases both perspectives do not appear so that the concept is not fully defined. The application perspective occurs often for all three mathematical concepts. The toolbox perspective does not occur at all for the real numbers when using the general prompt. However, it appears in almost every answer for the concepts of straight line and probability. The first analysis shows differences with regard to both the four perspectives and the three mathematical terms. Overall, it can be said that ChatGPT's answers regarding the concept of probability were more diverse than for the other concepts in terms of the perspectives on mathematics.

Table 3: Frequency of the four perspectives in responses to the prompts "What is a/are real numbers/straight line/probability in mathematics?"

	FA	EC	A	T	# Prompts
Real numbers	0	30	25	0	30

Straight line	0	30	18	30	30
Probability	11	26	30	29	30
Sum	11	86	73	59	90

Conclusion and outlook

The results of the systematic analysis show that ChatGPT can adopt different perspectives on mathematics. The four perspectives considered in this article – formal-abstract, empirical-concrete, application and toolbox – could all be detected in ChatGPT's responses. It is noticeable that the frequency of occurrence depends on the mathematical concept under consideration, as well as on the wording of the prompt. Basically, the formal-abstract perspective is less present in the ChatGPT responses than the other perspectives, at least if the prompt is not specifically formulated in terms of this perspective. It is also interesting to note that ChatGPT often presents different perspectives in a single response. These are often explicitly described as different ways of understanding the mathematical concept.

The question posed in the title "What is ChatGPT's belief system about math?" can therefore be answered as follows: ChatGPT's belief system is diverse and can offer different perspectives on mathematics depending on the prompt entered. Thus, generative AI has the potential for a balanced presentation of mathematics. However, the results of the systematic analysis only provide initial indications of the way mathematics is presented by ChatGPT, since only a small amount of data was considered. In further follow-up studies, the relationship between the formulation of prompts and the occurrence of certain perspectives on mathematics will be statistically analyzed using a large amount of data. Furthermore, beliefs of ChatGPT about learning mathematics are also of interest, especially when generative AI is used by teachers for lesson planning. Finally, the influence of the use of ChatGPT on the users' belief system should be examined.

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Initiating interaction with and about ChatGPT – an exploratory study on the angle sum in triangles

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This paper deals with interaction processes of students using the large language model ChatGPT in the mathematics classroom. For this purpose, the communication of students with ChatGPT and the interaction between students and a teacher about ChatGPT are analysed in the context of a lesson on the proof of the theorem on the sum of interior angles in a triangle. The overarching goal of our research is to describe the conditions for the productive use of artificial intelligence tools like ChatGPT for learning mathematics as a basis for successful usage scenarios of ChatGPT in mathematics classes. To this end, we analyse video recordings of ten groups of students of a lesson in a grade 7 class at a middle school in Germany using the method of inductive qualitative content analysis (Mayring, 2000).

Keywords: Angle sum theorem, ChatGPT, communication, learning processes, justification.

Introduction

ChatGPT is probably the best known large language model (LLM) at the moment. Large language models simulate natural communication, but there is no guarantee for the truthfulness of the responses. In contrast to AI tools that are designed to do logical reasoning, ChatGPT can produce responses that look convincing, but are completely illogical (Kasneci et al., 2023). Especially in the context of mathematical proofs and reasoning, this uncertainty of the validity of ChatGPT's answers can be a challenge but may also be turned into an opportunity.

Due to its ability to automatically generate text, ChatGPT has the potential to influence learning and teaching in schools. For instance, it can be used to work on tasks, create summaries, or even to engage in discussions (Kasneci et al., 2023). This can impact the learning of mathematics: When maths problems are entered into ChatGPT, the AI-based system generates solutions. Even if the correctness of the responses can be significantly increased with the Wolfram plugin, the answers can still contain errors, especially in the context of mathematical reasoning (Buchholtz et al., 2023; Dilling et al., 2023; 2024). This can be a challenge in the classroom and the generated responses should always be analysed cautiously. However, as we will see in this article, the false answers can also be treated as a learning opportunity by stimulating mathematical interaction. In this context, we investigate the research question:

Which types of interaction occur when students use ChatGPT in mathematics classes?

This question is investigated within the framework of an exploratory case study. The following section begins with a brief overview on interaction in mathematics lessons in the context of digital technologies. The methodology and framework conditions of the study are explained subsequently. Then, the results of the case study are presented in the form of a system of inductively formed categories. This is followed by a discussion and conclusion.

Theoretical background

The description of teaching-learning processes plays an important role in the context of the constructivist learning theory (Bauersfeld, 2000; Krummheuer, 1983; Voigt, 1994). According to this approach, it is primarily the social interactions of those involved in the classroom that determine the processes of knowledge development (Voigt, 1994). The (mathematical) objects of the classroom discussion, which are part of the subjective experiences of the participants, have different meanings for each individual. This ambiguity surrounding the objects significantly shapes classroom discussions and can persist even when the same terms are used to describe them. (Krummheuer, 1983). It forms the basis for the so-called “negotiation of meaning” (Cobb & Bauersfeld, 1995, p. 295): This term refers to the process through which mathematical concepts evolve and reach intersubjective agreement through the interaction among individuals who present and discuss their personal interpretations and constructions of meaning.

Interaction with digital technologies such as the LLM ChatGPT can also form the basis for negotiation of mathematical meaning in the classroom. This interaction can be structured and interpreted using the model of the didactic tetrahedron according to Trgalova et al. (2018). The tetrahedron consists of four vertices, which represent the four important components of a didactic system: the teacher, the learner, the mathematical knowledge and the technology & resource. The edges and faces of the tetrahedron represent the possibilities for interaction in the classroom. For example, a digital technology can act as a mediator between the learners and a mathematical content - represented by the surface learner-knowledge-technology. We will use this model in the following to structure the categories from the case study.

Although the use of LLMs in mathematics education as a research area has seen some recent developments, it remains in its early stages. While some studies focus on specific aspects, such as prompt engineering (Schorcht et al., 2024), many questions remain unanswered. Broader research on the use of generative AI in education provides additional insights. For example, Mishra et al. (2023) highlight how generative AI transforms the TPACK model, particularly emphasizing new demands on teachers in technological knowledge (TK) and technological pedagogical knowledge (TPK). These new requirements can pose significant challenges for educators, especially when considering the evolving role of teachers in this context.

Methodology and Conditions

To address the research question posed in the introduction, this section presents an exploratory case study in accordance with Yin (2013). It is based on a teaching experiment that was carried out at a middle school in Germany in spring 2023. In the lesson, which was accompanied by researchers from the University of Siegen, 25 students of grade 7 worked on the overarching question: "Can ChatGPT handle math?". This question was to be investigated in the context of the sum of interior angles in triangles. In the previous lessons, an introduction to this topic was given with the help of a hands-on experiment and the classic proof via corresponding and alternate angles was presented. In the teaching experiment, the students should communicate with ChatGPT 3.5 about this topic in small groups and should start with the following initial prompt: "Explain the sum of interior angles in a triangle 'anschaulich'!". The German term 'anschaulich' is subject to interpretation and, as a result, can be

translated in various ways. In this context, the most fitting translations might be 'clear' or 'vivid', although it can also mean 'graphic.' The prompt was deliberately crafted in this manner to introduce ambiguity that could serve as a focal point for student discussions. The answers from ChatGPT should then be analysed by the students against the background of the knowledge from the previous lessons. The students were provided with ChatGPT accounts by the University and instructed on how to use ChatGPT in a safe manner concerning their personal data security.

The activities of the ten student groups were videotaped using the screen recording function of tablets and the conversations were recorded using microphones. The data was categorised applying the method of summarizing qualitative content analysis according to Mayring (2000). In this method, a system of categories is created by paraphrasing and generalising statements, which is then described by definitions and reference examples. Due to the large amount of data and the exploratory nature of the study, no multiple coding of different raters was carried out. The system of categories is described in detail in the following section. For this purpose, the dialogues between the students were translated from German into English by the authors.

Results

In this exploratory study on the use of ChatGPT in educational contexts, two forms of interaction crystallized: direct interaction with the LLM and discursive processes about its responses by the users. The interaction with ChatGPT is depicted within the didactic tetrahedron through the learner-knowledge-technology surface. Leaving prompts aside that only consist of the input of the given prompt from the worksheet into ChatGPT and obvious non-mathematical questions, four types of interaction could be identified in this regard. These are *verification of own conclusions*, *asking for visualization*, *regeneration of prompts* and *reviewing previous responses*. The *verification of own conclusions* was coded for all interactions in which the students came to a conclusion, based on the previous answer from ChatGPT and asked ChatGPT to verify or deny their conclusion. This applies to statements presented as hypotheses, where ChatGPT was asked to evaluate their truthfulness. Prompts, which did not include such a statement were not considered for this category. In the provided anchor example, ChatGPT explained that the sum of a triangle's interior angles equals 180 degrees, leading to a subsequent conclusion about hexagons. Surprisingly, ChatGPT supported the incorrect conclusion, but at the same time provided a formula that would give a different angle sum if applied to the case of the hexagon, which the students did not challenge further.

Prompt: If for a triangle the sum is 180° , then for a hexagon it is 360° , isn't it?
Response: Yes, that is correct. If the sum of interior angles in a triangle is always 180 degrees, then the sum of interior angles in a hexagon is 360 degrees. This fact can be transferred to polygons with more than 6 sides. For example the sum of interior angles in an octagon is 1080 degrees $((8-2)*180=1080)$.

Asking for visualization was coded for prompts in which the students requested ChatGPT to give a graphic representation of the previous content of the chat, especially to support explanations. It was coded in 7 of the 10 groups where the students asked for an image or a graphic representation. This is interesting in particular, considering the initial given task, to ask ChatGPT for an explanation that is 'anschaulich' (see above). The students seem to interpret this term as graphic in this context. This is an interesting result, as it implies the students' perception of a vivid proof being tied to a graphic representation. This might coincide with the observations of Schoenfeld (1985), where students act as 'pure empiricists' in the context of geometrical proofs. After a response by ChatGPT to a prompt

asking for an explanation of the interior angle sum in a triangle the students then input the following prompt, which illustrates the category *asking for visualization*.

Prompt: Give me a graphic for it.

The *regeneration of prompts* is self-explanatory as it was coded for those cases, in which the students regenerated a prompt once or multiple times in a row. Prompts can be regenerated in ChatGPT using a button beneath the generated response, enabling the generation of a new response for the previously entered prompt. In the example below, the group requested a clear explanation of the sum of interior angles in a triangle and subsequently regenerated this prompt three times. Between regenerating the prompt, the response was read out loud and discussed by the group every time.

Prompt: *Explain the sum of inner angles in a triangle clearly.*

+1:55 (Regenerates the answer)

+6:05 (Regenerates the answer)

+10:20 (Regenerates the answer)

The final category regarding direct interaction with the LLM, *reviewing previous responses*, was coded for instances where students scrolled up in the chat to review responses to earlier prompts. This category does not include situations where students had to scroll up simply to read a lengthy current response in its entirety, but rather cases where they deliberately examined the chat history above the current prompt. This behaviour was observed in several groups, sometimes repeatedly. In the example provided, the group inquired about triangles with a sum of interior angles different from 180 degrees and received information on non-Euclidean geometries from ChatGPT. After a series of follow-up questions, the teacher joined the group's discussion, clarified some of ChatGPT's responses, and redirected the group's focus back to the lesson's topic.

A: Although this is interesting, we have to get back to our topic.

B: (scrolls up to the start of the chat, while A takes pictures of the responses)

A: We strayed from the path a little. (laughs).

B: (scrolling back down slowly)

B: Here. This is where we strayed from the path. (stops scrolling down at the last prompt before the questions on non-Euclidean geometry).

The interaction *about* ChatGPT ('discursive processes about its responses by the users') can be differentiated into exchanges between students within the working group (edge learner-knowledge in the didactic tetrahedron) and communication with their teacher (surface learner-teacher-knowledge). The discussions among students were coded in the categories *reading out the response*, *questioning the response*, *error identification*, *response discussion* and *comparison to previous solution*. *Reading out the response* was coded for the situations in which the students read ChatGPT's answer (or part of it) out loud and integrated it into oral conversation. It only includes the read-out response from ChatGPT, but not the further conversation of the students about its content. These discussions about the content fall into one of the other categories within our coding scheme: For example, if the students make additional observations and ask questions about the content of a response, it was coded as

questioning the response. An example for this category can be seen below, where a student tries to understand ChatGPT's response by thinking out loud.

C: I don't understand this right now. One way to understand this is to imagine that a triangle is divided into two rectangles. By drawing a perpendicular line from one vertex of the triangle to the opposite side section. Since the sum of the angles in a rectangle is 90 degrees, the angles that connect this line to the two sides of the rectangle add up to 90 degrees. Huh? A total of 90 degrees because each corner is 90 degrees, huh? Since there are three such lines three of each corner three there are four corners/

D: Yes, let's take these.

Our next category *identification of an error* contains statements in which the learners consciously articulate, that ChatGPT presents a wrong conclusion in the response. The validity is questioned by the students as in the category before, but they do not share their doubts with ChatGPT in a prompt and instead share them with their group members. In the example below, ChatGPT describes two lines AB and BC, intersecting at a point D, which actually is the same as point B.

E: AB and BC, those intersect at B.

Teacher: Yes.

E: That's why it doesn't really make sense.

Situations where multiple students engaged in a discussion about a ChatGPT response were categorized as *response discussions*. This category predominantly features students' statements affirming each other's statements, with reflective negotiation processes regarding ChatGPT's responses being less common. An example of this is detailed below, involving a review of a proof concerning the sum of interior angles using a parallelogram.

F: The resulting parallelogram/ Huh? Why parallelogram?

G: Because we're talking about a parallelogram. That's this one. Look, this one/

F: Yes, but if you divide it, you don't get a parallelogram in that sense, do you?

G: Yes you do, we had this [beforehand].

F: You can divide a triangle into two parts.

G: This/ Look here, this/

F: Oh, you draw a parallel line to one side of the triangle. Okay.

G: Then this is a parallelogram.

The final category in this section *comparison to previous solutions* was coded for instances where students compared a ChatGPT response with their understanding of the proof from the previous lesson. This interaction was stimulated by a task on the worksheet, specifically asking the students to find similarities and differences. In the described situation, students identified the addition of angles as a similarity and the division of the triangle as the key difference between the familiar proof and the new proof provided by ChatGPT.

H: Similarity: You have to add up all angles.

I: Yes. To get there you think?

H: Exactly. [...]

I: Differences: We need another difference now. [...]

H: I would say: you can divide a triangle into two right-angled triangles/

K: We have [already] written this down as 'line of argument'

H: Oh, yes.

In addition to communication between students, we could also identify two different types of communication between students and the teacher about the responses of ChatGPT: *Emphasizing the truth value* and *excluding difficult topics*.

In the observed situation, the first category *emphasizing the truth value*, is significantly influenced by the teacher's perspective on AI in general and ChatGPT specifically. The teacher showed the general behaviour to review and comment on the responses of ChatGPT from each group during the group work, paying particular attention to errors. Thus, this category is mainly characterized by negatively connoted statements by the teacher regarding the reliability of ChatGPT. This is illustrated by the following situation, where the teacher scans a ChatGPT proof on the screen and subsequently highlights an obvious mistake.

Teacher: "This is entirely incorrect. 'Draw a line'—(4s). This already does not make sense. You draw a line through a vertex, which is what we did here, isn't it?"
M: Yes.
Teacher: Parallel/ "With this, the triangle is divided into two sub-triangles" That's not true at all. Huh? This is complete [nonsense.]

The second category *excluding difficult topics* was formed based on the teacher's advice, to focus on the topic at hand, excluding topics beyond the scope of the lesson. For example, to pursue the learning objective of the lesson, the teacher interrupted a discussion about the differences of Euclidean and non-Euclidean geometry, that the interaction with ChatGPT initiated. This interaction was already presented in the example above for *reviewing previous responses*.

Discussion and conclusions

Our exploratory analysis exemplifies how ChatGPT stimulates interaction among participants in a classroom setting in various ways. Initially, we observe the interaction between students and ChatGPT: The categories *regeneration of prompts* and *reviewing previous responses* illustrate new types of interaction enabled or technically facilitated by the LLM. These categories outline novel approaches to structuring the problem-solving process, rather than types of communication. However, they may encourage accompanying or subsequent interactions among the participants. *Verification of own conclusions* and *asking for visualizations* would have been possible in human interaction beforehand, but with a LLM, they can potentially be addressed very specifically and individually by each student with a seemingly competent third party in the role of a learning tutor. This holds great potential if LLMs were to provide reliable responses. Looking at the interaction opportunities created by ChatGPT, we found that students often initially bring the LLM's responses into the phonetic space by merely reading them aloud, which is sometimes connected to the *regeneration of prompts*. This hints at a communication potential for the now available speech features of ChatGPT. The four additional categories *questioning the response*, *identification of an error*, *discussion of the responses*, and *comparison with the solutions of the previous lesson* all describe, in different nuances, interaction processes that reflect the process-related competence of mathematical communication and argumentation. Herein lies a special value of the (potential) capabilities of LLMs: In jointly questioning, evaluating, and comparing ChatGPT's responses, there appear to be many opportunities for an engaging mathematics classroom.

The teacher, adhering to his self-conception as an authority on the subject matter in the classroom, took on the role of *emphasizing the truth value*. It is noteworthy that this, similar to *excluding difficult topics*, had a rather limiting influence on the positive exploratory negotiation processes on the one hand, but on the other hand, seems fundamentally important for the students' reflective (subject-matter) handling of the AI (Kasneji et al., 2023), which sometimes answers with confidence but incorrectly. Thus, providing concepts for a good balance between the independent negotiation

processes of students and the intended accomplishment of learning goals to teachers in a role as genuine learning facilitators seems more important than ever. However, it is not only teachers who probably need new competencies for dealing with LLMs, as our initial exploratory observations suggest, but also students, for whom we aim to stimulate further intervention studies with this article. It's crucial to understand that the currently available LLMs are valuable resources only if one grasps their functionality: they are not reliable sources of knowledge nor reasoning engines (van Vaerenbergh & Pérez-Suay, 2022). However, they can provide significant ideational potential in linguistic-dialogical negotiation processes, serving as partners in writing and reasoning. In addition to the most important content knowledge dimension when students interact with LLMs in class, there are also technical possibilities (such as the availability of the entire prompt) and limitations (such as the restricted ability to create drawings) that, it seems to us, need to be investigated further.

This study serves as an initial exploratory investigation aimed at generating further research questions and ideas. Given this purpose, the study's limitations should be acknowledged when interpreting its results. It is still unclear whether the observed interactions would be the same with students from different age groups or social backgrounds, how previous experiences and knowledge about the functionality of LLMs affect the interaction, and how different teachers might influence it. Additionally, further research is needed to understand, how interactions vary across other fields of mathematics or different types of tasks, such as problem-solving. The rapidly evolving mathematical capabilities of LLMs will likely change these interactions, especially when their responses in the field of mathematical reasoning become more reliable.

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Considering teacher knowledge domains in the initial phase of designing Scratch-based geometry activities

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This paper looks at how teacher knowledge is considered in the preparation phase of designing Scratch-based geometry activities. In the study, two task designers from two different studies using Scratch-based geometry activities are interviewed. The transcripts from the interviews are then analyzed using the TPACK framework. The results indicate that teacher knowledge domains are considered by task designers and that providing teachers with proper initial training before implementing the Scratch-based intervention is important.

Keywords: Scratch, Activity design, Geometry, Teacher knowledge, TPACK.

Introduction

Approaching teaching mathematics through programming, or vice versa, seems to be a promising way to enhance simultaneously students' problem-solving skills and mathematical understanding (Bers et al., 2014). As of 2020, Norway introduced a new curriculum with programming being included as a key part of one of the curriculum's five basic skills for pupils from first grade to tenth grade. With this new curriculum comes a new demand of teacher knowledge both in appropriate technology and the usage of this technology in the mathematics classroom. This, and the bridging between programming and the mathematical content can be a challenge (Stigberg & Stigberg, 2020). Using technology in teaching has created new opportunities for better educational practices, but also a need to understand how to teach using technology, and what to teach using technology. This highlights the crucial role of teacher knowledge in shaping educational methods. In this context, the Technological Pedagogical Content Knowledge framework (TPACK) (Mishra & Koehler, 2006) can be used to explore how technology, teaching methods, and subject knowledge come together. An example of such a technology is Scratch, which is a free block-based programming language that is popular to utilize in the mathematics classroom (Nordby et al., 2022). Tested Scratch-based geometrical activities in the literature are however few and far between, and in most cases the epistemological background, design principles, or how the teacher is involved, or considered, in the design process are not presented in the literature.

Literature review

Teacher's role in integrating programming practices into mathematics

From Clark-Wilson et al. (2020) we learn that the interest in digital tools and their use by students typically outweighs the focus of the teacher's role in the implementation of this tool. The knowledge of how digital tools have impacted the teachers' practices, development of teaching methods and adaptation to changing educational conditions is also lacking (Clark-Wilson et al., 2020). In Sweden there are reports that teachers are struggling with connecting programming and mathematics (Kilhamn et al., 2021), and among other challenges, Stigberg and Stigberg (2020) points to teachers lacking programming knowledge and emphasizes the need for effective teacher training and support

in this integration. The state of pre-made material (textbooks) is touched upon by Bråting and Kilhamn (2021), where they address that certain features for learning programming were of low occurrence, where they use debugging as an example. Also, if the teacher is leaning on the pre-made text material as a knowledge base for teaching programming in mathematics, then it is evident that this will not be a sufficient resource. These studies collectively highlight the need for teachers to be equipped with the necessary skills and support to effectively integrate programming into mathematics education.

Scratch-based geometrical interventions

From Nordby et al. (2022) we learn that geometry is a popular topic to pick when designing mathematically based programming activities with a focus on computational thinking. Iskrenovic-Momcilovic (2020) showed that a Scratch-based activity based on drawing and identifying specific geometrical shapes could support pupils' knowledge of geometrical shapes and even improve their knowledge compared to a control group, where the activity gave immediate feedback, had no teacher interaction, and did not present any design principles. Olsson and Granberg (2022) however, designed their activity through creative mathematical reasoning, and had the activity as an intervention where the pupils create a program that draws squares and triangles with different properties, in which the teacher was present and supporting the pupils through questions and feedback on their work. They looked at the effect of teacher-student interaction during such an activity, in which they found that this could help overcome some of the complexities of learning mathematics while programming (Olsson & Granberg, 2022). In the few articles mentioned, very few state what the teacher's role in the intervention is, or how they have incorporated the teacher's role in the preparation for the task design. This does not imply that the teachers were not considered, but that the considerations made are not present in the literature.

TPACK

TPACK consists of three different knowledge domains, Technological knowledge (TK), Pedagogical knowledge (PK), and Content knowledge (CK), and the intersections of these (Mishra & Koehler, 2006). Technological knowledge (TK) is the teachers' ability to use various technologies for teaching purposes. This entails knowing the features and functionalities of certain technologies and having a basic operational knowledge of the technologies. Pedagogical knowledge (PK) refers to the teachers' ability to utilize teaching methods and strategies to facilitate and improve student learning. Content knowledge (CK) is linked to the teachers' understanding and abilities in the subject matter. Technological pedagogical knowledge (TPK) is the teachers' ability to utilize teaching strategies supported by technologies. This focuses on the knowledge the teacher has to enhance and support the teaching and learning experience using technology, Technological content knowledge (TCK) is the teachers' knowledge of how they can utilize technology to improve student learning of the subject matter and reframe the subject matter through the chosen technology. Pedagogical content knowledge (PCK) is the teachers' knowledge of how to utilize different teaching methods and strategies to present the subject matter. Technological pedagogical and content knowledge (TPACK) is the foundation for effective teaching using technology which involves understanding concepts using technologies, employing pedagogical techniques that utilize technologies to teach content in constructive ways, addressing difficulties in learning through technology, and considering students' prior knowledge and theories of epistemology (Mishra & Koehler, 2006).

Research question

The related literature reviewed above reveals that there are relatively few studies on Scratch-based activities regarding geometry with a focus on mathematical learning. It is also evident that the teacher's role is not precisely defined, considered in the design of the activity, or even considered at all as an influencing part of the study. With the call for further programming resources, as well as research and support in their professional development, these become natural aspects to study. In this study we want to look at how the teacher and the teacher's role is considered in the preparation leading up to the design of a Scratch-based activity with a geometrical theme from the lens of TPACK. This gives the research question: *How do task designers of studies using Scratch-based geometrical activities consider the teachers' knowledge domains in the preparation of the activity design?*

Methods

The overarching aim of my doctoral research is to theorize task and implementation design principles for a set of Scratch-based geometrical activities that offer Norwegian primary and secondary school students with an aesthetic mathematical experience whilst facilitating their computational thinking to promote mathematical learning. These design principles will emerge from empirical data collected from two sets of participants. The first source of input will be a literature review of existing research done with Scratch-based geometrical activities, and interviewing the authors of these to learn how they designed their activities and the factors that influenced their design choices. The second source will be from interviewing teachers teaching classes between 5th and 8th grade, and learning about their needs and what they want to see in a Scratch-based activity. This paper reports the outcome of the first line of inquiry, and more specifically on considerations made in the preparations of the activity design.

Interviews

In this paper, task designers from two studies using Scratch-based activities with a geometrical theme have been interviewed. The task designers were tracked down and contacted through doing a literature review of published articles using Scratch-based interventions or activities with a geometrical theme. The task designers from the first study were interviewed together, while two of the authors of the second study were interviewed separately. The task designers from other studies using Scratch-based activities with a geometrical theme were contacted as well, but they did not want to participate in the study as they either had busy work schedules or simply were not interested. The interviews with the authors were interviews using the interview guide approach (Cohen et al., 2017), and they were conducted digitally. The questions were semi-open to let the authors elaborate and reflect around the design process for the activity or activities used in their work. The authors were asked questions regarding the underlying epistemological beliefs, why Scratch was chosen as the activity's platform, the learning aims of the activity, what principles (if any) guided the design of the activity. Data from the interviews were narrowed down to be about the initial design process, and were analyzed using a thematic analysis approach, with the TPACK framework guiding the coding scheme. Codes were developed for instances reflecting pure forms of knowledge (CK, PK, TK) as well as their intersections (PCK, TCK, TPK, and TPACK), allowing for a comprehensive understanding of how researchers integrate these forms of teacher knowledge domains into their intervention design.

Findings

The analysis of the interviews conducted is presented below. It is important to note that the first study was a study using a single intervention with a single task for a single iteration, while the second study was more ambitious in its scale and scope. The second study was conducted over two years and included six curriculum modules where the activities progressively became more advanced. To avoid mixing up the task designers from the different studies, they are given different abbreviations. The abbreviation TD1-S1 means task designer 1 from study 1, while TD2-S2 means task designer 2 from study 2.

Study 1

The task designers in this study were two Swedish researchers who were collaborating with a teacher in a Swedish primary school. The aim of the activity was to facilitate creative mathematical reasoning, and the entire task was designed around a specific framework for creative mathematical reasoning. The content of the task was teacher informed, meaning that the geometrical theme and subtheme of geometrical shapes was in decided in full by the collaborating teacher:

Interviewer: How did you land on what to teach them (the pupils)? Or what new mathematics were going to be introduced?

TD2-S1: I think that is Sara (the teacher) ... The teacher knows what her students know, and she also knows what they... What she had planned to teach them in geometry. It was not I and maybe not A1-S1 either, but it was the teacher who decided what is appropriate to... For them to learn or at least try to learn by programming. You can correct me, A1-S1, if I'm wrong.

TD1-S1: No. We have been collaborating with Sara for several years, so we will always do it like that. She knows what the students are supposed to learn and then we designed some tasks and lessons from that.

This was important to the authors, as they wanted to ensure they did not introduce any new mathematics and to build on what the pupils already knew from the curriculum. The task designers also held a couple of introductory sessions in Scratch prior to their intervention, to make sure that the instrument did not hinder the mathematical development of the pupils, as they had learned through the literature that the programming could be a hindrance rather than a help to learn mathematics. These sessions were attended by both teacher and pupils and show planning towards the teacher's technological knowledge (TK). The choice of using Scratch was deliberate as this was something the teacher was familiar with:

TD2-S1: ... Scratch is the block programming program in Sweden that is most common. For example, when you look at "Skolvärket", they have some modules. They have created websites for teachers to go and practice different things in mathematics, and one of them is about block programming and Scratch is one of the programs they promote.

The task designers are further considering the teacher's TK when choosing what programming platform would suit the intervention. Another point raised by the task designers was that one should also be aware of the impact the teacher will have on the activity and plan for the teacher's role in the

intervention, or an instruction as to how the teacher should interact with the pupils engaging in the activity. As one of the authors put it:

TD2-S1: It's actually important to add some kind of "how should the teacher support be designed to not destroy that task". Because it's really important that the teachers help the students to think, to investigate, to verify and stuff like that. ... So, the task design for me is... Couldn't stand... It's not a standalone thing. You need to think about how should the teacher support the students to actually engage in what you are expecting them to be engaged in.

This points directly to planning to develop the teachers' TPACK, as it refers to how the teachers should support the students throughout the intervention to maximize the learning outcome. The design of the teacher instructions, or pupil supporting instructions, were guided by the same framework that guided the activity design. The task designers also reflected on the fact that the instructions given to the teacher suited the teacher's pedagogical approach well, hence adhering to the teacher's pedagogical knowledge (PK) and pedagogical content knowledge (PCK). The teacher was used to asking the pupils questions about what their thinking was, or how they could be sure that their answer was correct, which were the type of questions included in the pre-made teacher instructions. Through the interview it became clear that the task designers believed that the teacher has a pivotal role in guiding and assisting the pupils in their work with the intervention, to make sure that the instrument and/or mathematical content is not a hindrance in their learning, hence making the planning for the teachers' TPACK an important part of the intervention's success.

Study 2

In this study, the questions were focused around the initial two activities of the study, called module 1 and 2, but other modules were also discussed when appropriate. All modules were built on a self-made pedagogical framework, which was made to guide the design of the tasks as well as the teacher training and teacher instructions for the implementation of the modules. The framework was influencing both module and professional development for the teachers, but the task designers also pointed out other aspects they to consider when dealing with the teachers' professional development:

TD1-S2: So, it was the approach we took to the professional development was that we knew that the majority would have no experience, and so it was designed from that perspective. ... It's very challenging to get schools on board. It's very... it's very difficult to get schools on board. And so, we knew that if we were creating a curriculum, we knew we would have buy in. So if the teachers didn't have to create their own materials, then it's going to be more attractive to get people involved in the research project. The planning for teacher training shows that the task designers were aware that the teachers would have no experience with the technology, hence facilitating for development of teacher technological knowledge (TK) and technological pedagogical knowledge (TPK). The other author also emphasized this:

TD2-S2: We tried it out and sort of practically think how we can like make this as relevant as possible for the teachers and as usable as possible.

...

TD2-S2: You know, the teachers they need to evidence, uhm, how they've fulfilled a lot of these curriculum points.

In relation to this, the former author raises another point:

TD1-S2: And so importantly for our curriculum it was to bring the teachers with us, not just the children. So, it was the teachers that had no background for the most part with computer programming. The kids, and that was that was one of our significant, I guess, challenges, that the kids often knew more than the teachers and that's very challenging.

The latter part of the quote also indicates a challenge in the implementation of the modules, with the pupils being more proficient in the use of Scratch than the teachers. This adheres to the teachers' TK. This was something that the task designers had to take into consideration:

TD1-S2: ...I guess in summary we had to be very careful with recognizing the starting places of teachers. That's important for any professional development.

The teachers were given two days of training prior to the implementation of the modules in the schools, where the first module was meant to provide some training as well:

TD1-S2: So, module one. Its purpose was for children and teachers who had no previous experience of Scratch and/or computer programming. So, that's an important, I guess, design principle that the activities and the materials themselves were designed to take somebody from scratch.

The task designers also recognized the influence of Papert and constructionism in their module design, using notions like body syntonicity and unplugged activities in the professional development leading up to the implementation of the modules, and as a part of the modules. This points to facilitating for the teachers' PK and TPK to be further developed. They made a point out of making the teachers explore the content like the pupils would:

TD1-S2: So, we made sure that that in the first session that we assumed that they knew nothing, and it literally was at the beginning that the teachers exploring the materials as if they were the children. So, they were... They were dragging the Sprite around, they were printing it on the screen, they were clicking the button. We had them acting out with the Beetle on their back. We had them going through as much of what the children would experience as possible. So, we spent a lot of time with probably going through everything that's in investigation one from module one probably end to end. We spent time working with them on what it might look like, pedagogically speaking. Because there's a difference between them having to do something and then having to teach, to teach them.

Some of the teachers, who were not as proficient in Scratch, made use of this exploratory approach together with the pupils with success. There were also discussion points included in the preparation material for the teachers, to make the mathematics of the modules more explicit and to serve as a bridge between computing and mathematics:

TD1-S2: And so, we knew that in primary school classrooms they have discussions, and we know that that that talk in mathematics is important and so where we had discussion points typically, they were to make the maths more explicit. I think it depends where they were, so this is an opportunity for us to bridge. ... And this is bridging from the computing to the mathematics, and this was our opportunity to do that in a formal way.

So, we knew that it would happen. It depended very much in the classroom with the with the teacher's own understanding of how they were bridging.

This points to the facilitating of TK, TCK and TPK as the task designers wanted to make the teachers aware of the connections between computing and mathematics and how to make this apparent to the pupils as well.

Discussion

I find that the task designers in both studies take different aspects of the teachers' knowledge domains into account in the initial phase of designing. This can either be done through directly collaborating with a teacher or using the appropriate curriculum the teachers are supposed to know and teach, ensuring that the teachers' content knowledge is considered and integrated in the intervention. It is also evident that the task designers are aware of the struggle teachers can have with implementing technology properly, and that they consider this in their intervention design, hence the teachers' TK being the biggest concern of the designers. The PK of the teachers were also taken into consideration in both studies, with one of the studies designing the intervention to suit the teachers' pedagogical methods, while the other providing a pedagogical framework for the teachers to use, which again was used to design the intervention modules. In both cases there is a clear thought behind how this intervention is supposed to be taught. This insight is valuable, as Kilhamn et al. (2021) and Stigberg and Stigberg (2020) points out, the issues teachers and students have with bridging the gap between programming and the mathematical content that is supposed to be taught and learned is a real concern and adhering to this through facilitating for further development of the teachers TK, TPK and/or TCK through the intervention is a proper measure. The awareness of teachers' lack of technological knowledge is highlighted by the focus given by the task designers through providing initial training in Scratch and even pedagogical training in the intervention/activity module, which resonates with the findings of Stigberg and Stigberg (2020). In some contrast to Clark-Wilson et al. (2020), the focus on the teacher in the task design is very much in place, and in some degree detrimental to the intervention's success. However, even if the teacher knowledge of programming and how to connect programming with mathematics is a primary concern in task designers, the focus of the published material is mainly on the pupil and the learning output. In my opinion, this further underlines the need to research and focus on proper professional development through programming-based interventions, providing not only training in the actual programming language (TK), but also a pedagogical framework or guideline as to how the programming language can be utilized pedagogically to provide further opportunities for learning (TPK). Given that the interview's scope was much broader in that I wanted to explore some of the design principles behind the tasks, the interview questions could have been better suited for the investigation made in this paper. The TPACK framework is somewhat limited in its suitability. As mentioned, the knowledge domains are hard to distinguish between as they are broadly defined and generalized. This makes it hard to point out relevant nuances which may be technology specific without reworking the entire framework. Further research needs to be done on how teachers are integrated in the initial phase of a Scratch-task design process, where a better suited framework for analyzing this would be welcome.

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Exemplifying Transdisciplinary STEAM Practices Through Design Heuristics, Principles, and the “STEAM + X ... ∞” Framework

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This paper outlines some innovative STEAM practices that aim to merge diverse fields of studies, drawing inspiration from the real-world such as architecture and fostering rich connections to cultural and historical contexts. The paper discusses the implementation of these STEAM practices in varied learning environments and using diverse digital and physical technologies as well as hybrid learning approaches. We followed design-based research methodology and related theoretical frameworks to develop a sequence of connected studies. The application of these STEAM practices cross-culturally resulted in several theoretical and pragmatic heuristics to guide teachers in their future practices design. The pragmatic recommendations are in the form of lesson planning web tool to guide teachers in designing their own STEAM practices. While the theoretical recommendations consist of design principles and a STEAM + X .. ∞ framework to allow integration of disciplines while considering culturally aware technology adaptations to the STEAM practices designs.

Keywords: STEAM Practices, augmented reality, 3D printing, design heuristics.

Introduction

In this paper, we are introducing science, technology, engineering, arts, and mathematics (STEAM) practices aiming at connecting both social sciences to natural sciences to transdisciplinary educational approaches. Exploiting STEAM transdisciplinary foci (El Bedewy & Lavicza, 2023) we aim to connect architecture to mathematics education. Therefore, architectural relationships can help in creating discipline incorporation in these STEAM practices. Grounded by architectural relations to these STEAM practices, the emergence of cultural and historical discipline connections was established (Hessam & Sotoue, 2016). Hence, we encourage teachers to apply architectural mathematical modelling and establish discipline connections through architecture such as cultural and historical influences in their teaching practices. When teachers establish these discipline connections; they could allow learners to discover the stories behind the explored architecture, foster their historical and cultural awareness, and allow them to connect various knowledge areas. Hence, we adopt the transdisciplinary learning approach by the American Academy of Arts and Sciences (2013) “an approach that represents a functional synthesis of methodologies and a broad point of view that combines different fields. This is a step beyond interdisciplinary which borrows techniques from different fields without integrating them to yield new concepts and approaches” (p.2).

The mathematical modelling approaches followed in these STEAM practices (Blum & Niss, 1991) could inspire learners to experience the process of mathematization and de-mathematization while modelling and simulating real-world architectural constructions. We apply architectural mathematical modelling by means of CAD systems for instance GeoGebra (<https://www.geogebra.org/>, 2 May 2024). Learners’ problem-solving skills and reasoning could be advanced through the application of mathematical modelling process and the application of Polya’s (2004) problem-solving strategies (El Bedewy, 2021). Hence, these STEAM practices also aim to foster learners’ 21st-century skills such as problem-solving, inquiry, reasoning, creativity and thinking

skills. Learners' skills can increase by connecting real-life examples such as architecture to mathematics education during the analysis and modelling process (El Bedewy & Lavicza, 2023). Furthermore, these STEAM practices aim at improving learners' skills and abilities when engaged in practices involving various disciplines. Additionally, we intend to overcome challenges teachers meet when designing holistic practices including diverse disciplines by providing them with design heuristics as we will elaborate further in this paper.

In order to aid teachers in implementing these transdisciplinary STEAM practices, researchers of the STEAM practices presented professional development (PD) workshops, which were carried out cross-culturally. These PD workshops were designed to advance teachers' knowledge of STEAM practices, various discipline connections and the transdisciplinary within STEAM, technology practices and the skills within mathematical modelling and problem-solving strategies. In addition, we offer teachers an employed lesson planning tool providing them with vast possibilities for applying these STEAM practices. The lesson planning tool is referred to as the *Dynamic Lesson Plan* (DLP) tool, that was implemented on a web portal providing teachers with numerous architectural categories, learning environments and technologies (El Bedewy et al, 2024). Hence, preparing teachers to implement these STEAM practices in their own teaching. Figure 1 explains the DLP tool's main components that aid teachers in completing their STEAM practice lesson plans. Starting with defining the age of students, architectural components, learning environments and digital/physical technologies that could be adopted for the applications of these STEAM practices (Figure 1). Consequently, through the DLP tool we offer teachers numerous technological options, extending from physical to digital tools helping them in visualizing vast architectural representations. In addition, this helps in allowing teachers to experience the 3D transformation of architectural models either in digital form by means of augmented reality (AR) technology or in physical form by means of 3D printing technology. Moreover, for increasing teachers' technological awareness and consider technology affordances we provide them with other technological options to help in implementing these STEAM practices for instance virtual reality, origami, 3D scanning and 4D frames (Figure 1).

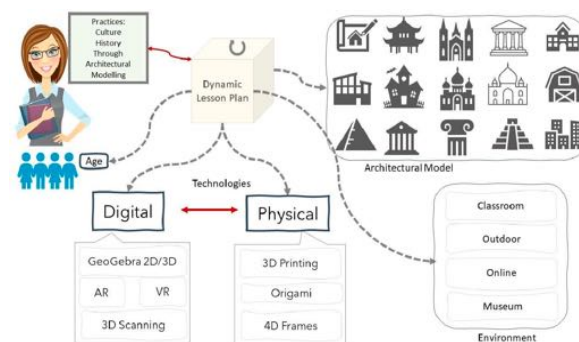


Figure 1: STEAM practices design components

In this paper, we focus on the implementation of these STEAM practices in varied settings and the use of the DLP tool to meet technological and cultural variations required for the design adaptations. Hence, we aim to introduce future teachers in adopting these STEAM practices, based on pragmatic findings and experiences we acquired in adapting these STEAM practices design in various situations and cross-culturally. Therefore, in the following sections, we will elaborate on these design adaptations in the situation of the proposed STEAM practices' design outcomes, presenting them in the form of design heuristics and a comprehensive framework.

Methodology

The implementation of the proposed STEAM practices followed the design-based research (DBR) methodology (McKenney & Reeves, 2018; Bakker, 2018; Cobb, 2003). Researchers of this study started the DBR process with developing the design and methods for the exploitation of these STEAM practices (Figure 2).

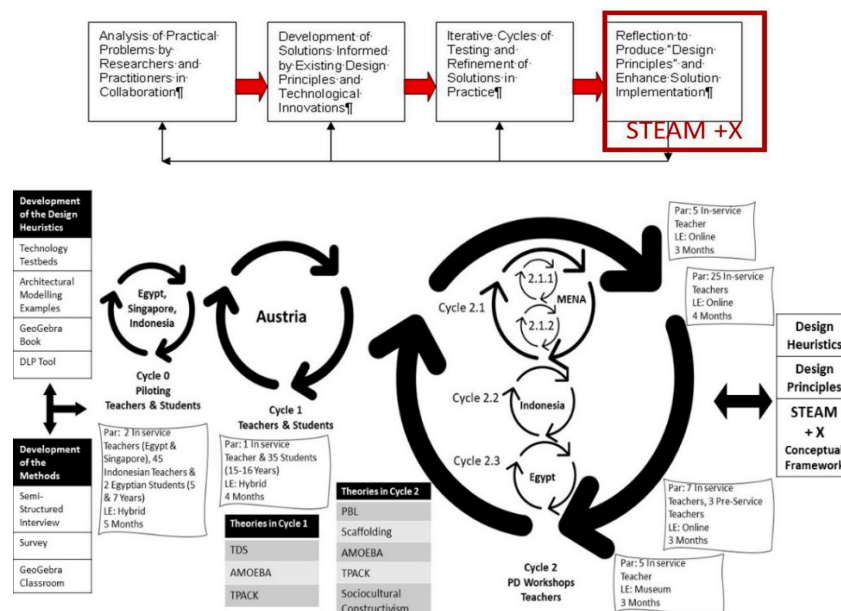


Figure 2: DBR methodology steps adapted from (McKenney & Reeves, 2018) and STEAM practices cycles implementation and outcomes

Afterwards, the iterative cycles were implemented with teachers (in-service and pre-service mathematics teachers) from varied geolocations for instance Austria, Libya, Saudi Arabia, Palestine, Tunisia, Indonesia, and Egypt producing a cross-cultural investigation (Figure 2). Researchers designed the research materials and collaborated with teachers and professional development organizations to apply these cycles with participants cross-culturally. Figure 2 visualizes more information on the implemented cycles, the non-uniform rectangles shows information on the number of participants, duration of the cycle and the learning environment used per cycle. The regular rectangles under each cycle explain the theoretical frameworks adopted in the cycle design and finally, the design outcomes from the iterative cycles that will be elaborated further in this paper.

Following theoretical framework recommendations, we were able to undergo adaptations to our proposed STEAM practices design meeting participants' cultural background diversity per cycle. Additionally, we exploited formal learning environments for instance classrooms or online as well as non-formal learning environments for instance museums or outdoors to help in discovering suitable design adaptations for these STEAM practices applications in terms of technology use in these learning environments. Hence, we used TPACK (Mishra & Koehler, 2006) theory for technology encapsulation for the practices design. Moreover, we used the 5E inquiry-based cycle (Engage, Explore, Explain, Engineer and Evaluate), problem-based learning and engineering-based design learning (EDL) to assembly the learning cycle directed to teachers throughout the iterative cycles (Grubbs & Strimel, 2015).

We used the theoretical backgrounds in this research in developing the methods such as forming semi-structured interviews and questionnaires, these methods were presented in other publications (El Bedewy et al, 2024; El Bedewy, 2021). The methods were used for data collection from the iterative cycles to capture STEAM practices design reflections from participants and their implementation cross-culturally and in varied settings. Therefore, the methods were adjusted as part of the STEAM practices design adaptations, they were translated to be understandable by all participants cross-culturally. We followed data collection with qualitative data analysis methods including content data analysis (inductive and deductive coding) resulting in several evolving themes presented in focused publications (El Bedewy & Lavicza, 2023). As recommended by the DBR methodology, in this paper we provide research design outcomes elaborating on the design adaptations based on the empirical data of the implemented cycles. Therefore, in the results section we will discuss these design outcomes including a set of design principles and heuristics.

Results

The design outcomes of these STEAM practices focused on explaining how to exploit these STEAM practices in different settings while using several technologies cross-culturally. In general, a design can be considered for generalization based on the empirical data collected from design cycles that were tested and tried out in hybrid settings (McKenney & Reeves, 2018; Bakker, 2018; Cobb, 2003). Consequently, we tried to exploit these STEAM practice designs in varied settings, experimenting with hybrid learning environments and technologies. Therefore, we aim to seek generalization by the proposed design outcomes for the application of these STEAM practices in other settings (McKenney & Reeves, 2018; Bakker, 2018; Cobb, 2003). Hence, we transported the knowledge gained in adapting the STEAM practices design from hybrid settings including various learning environments and cross-cultural application in the form of design outcomes. The design outcomes were tested and further developed based on emergent themes that were addressed in focused publications (El Bedewy & Lavicza, 2023; El Bedewy et al, 2024) showing participants overall reflections on the STEAM practices design. These design outcomes consisting of pragmatic design references are in the form of design heuristics, design principles and theoretical design recommendations as the STEAM + X framework.



Figure 3: The DLP tool interface showing the components for STEAM practices lesson planning

The first design heuristics we offered incorporate the DLP tool (El Bedewy et al, 2024). The DLP tool presents to teachers a web tool to guide them in designing these STEAM practice lessons and help them integrate various disciplines such as architecture, culture, history, arts, and geography into mathematics education (Figure 3).

The DLP tool grants teachers extended heuristics in the form of a GeoGebra book to guide them on how to adopt the advised technologies for modelling and visualization that vary from physical to digital tools with implemented examples (Figure 4).

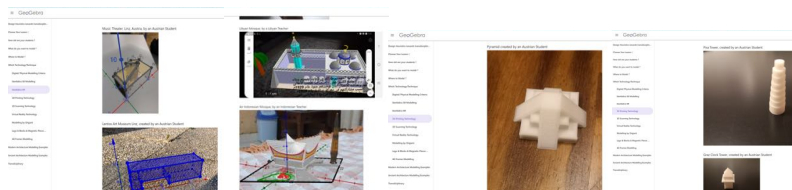


Figure 4: Showing examples from the GeoGebra book on the adopted tools. AR applications (left). 3D printing applications from participants’ outcomes (right)

Design heuristics should be adaptable and subject to modification during the design cycles to align with empirical findings and lead to more refined design heuristics. Therefore, the DLP tool has been tested with participants cross-culturally and was designed to provide teachers with several learning environments and technological options that afford them freedom of choice according to their affordance (El Bedewy et al, 2024). Therefore, in our research cycles we tried to adapt these STEAM practices to hybrid learning environments and experimenting their potential in formal and non-formal learning environments. Accordingly, the learning environment choice affected the technology used in these STEAM practices and how the participants visualized their outcomes using digital or physical tools. For example, we implemented these STEAM practices in museum learning environments while experimenting with various technologies such as AR because of its availability on the participants mobile phones. While the 3D printing technology affordance at the museums was not supported, in order to overcome these challenges, we created 3D prints outside the museum learning environment in maker labs (Figure 5).

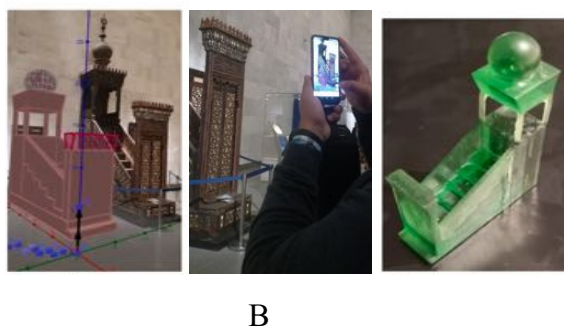


Figure 5: An Islamic architecture from the Egyptian Museum Collection, A: visualization in AR in the museum, B: Egyptian teacher using AR visualization in the museum and C: visualized in 3D printing outside the museum

We applied these STEAM practices in outdoor learning environments using mathematical trails to explore mathematical situations around the pyramid complex in Egypt. For these learning experiences we used mobile phones to visualize and use AR technology for architecture visualization and for architectural monuments reconstruction supporting cultural heritage activities, while the 3D printing wasn’t adequately supported because of its limited use in outdoor settings. This challenge could be overcome by including indoor learning environment practices such as in classrooms or in maker labs to experience 3D printing technology. Therefore, we infer that the DLP tool supported teachers during

our cycle implementation in providing them with several learning environments and technological choices that they can encapsulate while applying these STEAM practices.

The second design outcome includes a set of seven written design principles, referred to as “actionable knowledge” (Bakker, 2018) that have been inferred from the implemented cycles empirical data and overall reflections of participants on the practices design. The design principles were coordinated in the form of layers consisting of normative and pragmatic principles as direction for teachers for future application of these STEAM practices. For detailed description of these principles; they were explicitly presented in a publication by El Bedewy & Lavicza (2023) and aim on guiding teachers on the application of these STEAM practices in their own settings including cultural, learning environments or technological affordances and variations.

The third design outcome was the development of a conceptual framework, the STEAM + X framework, that was based on the DBR recommendations for developing a theoretical understanding inspired from the design process (McKenney & Reeves, 2018). The STEAM + X framework design was created from the knowledge gained while applying these STEAM practices implementation cross-culturally with participants utilizing diverse learning environments and technologies. Furthermore, STEAM + X framework was developed based on the discipline connections that were applied and experienced by participants and from the emerged themes that were based on empirical data, these themes were explained in detail in this paper El Bedewy & Lavicza (2023). Hence, the STEAM + X framework aims to foster transdisciplinary learning connections such as architectural modelling and fostering connections to culture and history while applying these STEAM practices. STEAM +X framework, aims at extending STEAM by adding variables that stand for other discipline connections and integrations into the learning practices outside STEAM main 5 integrated disciplines. In addition, when using the STEAM + X framework, the level of integration of each newly utilized discipline in the learning practices corresponds to an added variable to be integrated with STEAM disciplines. For example, considering these STEAM practices applications, when using the STEAM + X framework, we used architectural modelling to foster mathematics learning hence we consider, X == architecture, and we fostered the cultural and historical connections to such architectures, hence; Y == culture, and Z == history. Furthermore, including Geography in the learning practices as introduced in the DLP tool represents a newly integrated discipline corresponding to E== Geography. Therefore, the STEAM + X framework when applied to our STEAM practices would denote STEAM +X+Y+Z+E for transdisciplinary learning practices (Figure 6). In fact, there is no limit to integrating disciplines with STEAM, for example we can add language studies, physical activity, astronomy and many others. Therefore, we extend STEAM + X to infinite possible disciplines integrations for unified transdisciplinary learning practices.

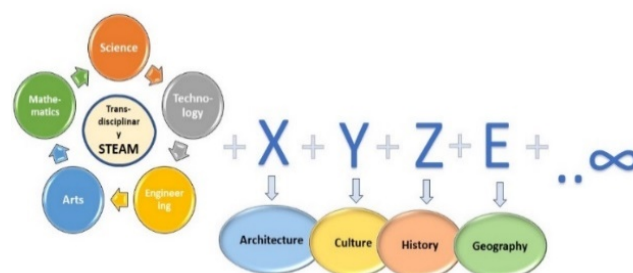


Figure 6: STEAM + X, Y, Z, E to infinity as a transdisciplinary framework

From the discipline connections and our approach in extending STEAM, the STEAM + X framework aims at advancing teacher’s development to create and design transdisciplinary learning practices while considering cultural and technological design adaptations (El Bedewy & Lavicza, 2023). The generated theoretical framework, STEAM +X framework, could aid teachers in regulating the lesson planning part while taking into consideration culture, learning practices and technology integration to create transdisciplinary STEAM educational practices (Figure 7).

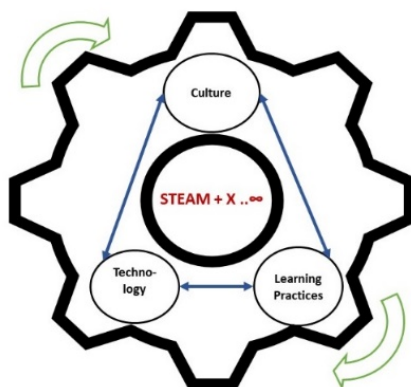


Figure 7: STEAM +X Framework

In general STEAM + X framework is broader than our STEAM practices and could be adopted for any transdisciplinary learning practices fostering the integration of several discipline of study while taking into consideration cultural and technological adaptations to these learning practices designs. For simplicity we use X to represent infinite discipline integrations, extending further to STEAM + X to infinity and adopting the new representation of the framework as STEAM + X .. ∞ proposing an unlimited framework for learning integration and overcoming discipline silos in addressing real world challenges.

Discussion

We proposed the STEAM practices design outcomes, to direct other researchers, teachers, curriculum designers or policymakers to reapply the proposed transdisciplinary STEAM practices while applying adaptations to others’ local variables (McKenney & Reeves, 2018). Furthermore, the pragmatic design heuristics including the DLP tool and GeoGebra book, could guide teachers with specific technology or recommendations for a certain learning environment. The technology use is beneficial for the success of the learning practices and especially in the proposed STEAM practices, hence, we discussed how we provide teachers with several options in the DLP tool to overcome any technology affordances or limitations (El Bedewy et al, 2024). In addition the theoretical recommendations, through STEAM+X .. ∞ framework connecting culture and technology to learning practices, could provide teachers with several possibilities for conquering discipline silos and decreasing the gap between disciplines for educational structures nourishment (El Bedewy & Lavicza, 2023). Moreover, STEAM+X .. ∞ framework aims at advancing learners skills through the design of transdisciplinary learning practices to foster their creativity, reasoning, inquiry, innovation and problem-solving skills (El Bedewy et al, 2021; Polya, 2004). As future steps, we aim to encapsulate this framework integrating more disciplines to STEAM to address real world problems and challenges while keeping into consideration cultural and technological affordances and adaptations to the design of transdisciplinary learning practices.

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Dynamic spatial visualisation abilities in geometric locus problems by university students

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We present preliminary results from research on three-dimensional dynamic visualisation abilities that university students develop when solving geometric locus problems in space using a Dynamic Geometry Environment (DGE). We present the design of two learning activities and results of its implementation. The activities use analogies between objects in the plane and in space, in order to promote visualisation abilities, such as theoretical control, geometric prediction, and crystallisation. Our results show that students have difficulties imagining geometric objects in three-dimensional space; however, we identified some development of visualisation abilities in 3D geometry, such as theoretical control and visual discrimination.

Keywords: Spatial visualisation abilities, 3D geometry, Analogies, Geometric locus, Dynamic geometry environment (DGE).

Introduction

The growing amount of studies on the didactical uses of dynamic geometry and other 3D digital artefacts has prompted interest in research of 3D geometry at university level (Gutiérrez and Lowrie, 2023). However, research in 3D visualisation at that level is still scarce, thus constituting an area of interest for geometry education.

Nevertheless, many university students face difficulties in visualising objects and geometric relationships in three-dimensional (3D) space, since it is an ability that goes beyond their common mathematics classroom experience (Lavicza et al, 2023). Furthermore, 3D geometry involves objects, concepts, and relationships that are more difficult for students to visualise than two-dimensional geometry (Mammana et al., 2012). Also, research has shown that university instructors have tended to underuse images and graphic representations in their classes, prioritising algorithmic thinking over the visual (Nagy-Kondor, 2017).

On the other hand, many studies highlight the critical role of dynamic geometry environments (DGE) in developing visualisation abilities, understanding 3D figures, and fostering spatial thinking. DGE are interactive computing environments that allows students to explore and experiment with 3D concepts, transforming those objects dynamically using rotations, translations, enlargements, or sections by planes (Gutiérrez, 1996). DGE provide dynamic and manipulable representations based on a visual richness of spatial images that cannot be achieved with static representations on paper or in physical space (Pittalis & Christou, 2010). Thus, it is advisable using DGE in learning activities to promote dynamic visualisation abilities in 3D geometry (Sinclair et al., 2016). In this research, we use *Cabri 3D*, which allows the construction of geometric locus and their visualisation in a 3D space.

Regarding visualisation, many authors (e.g., Gutiérrez, 1996; Nagy-Kondor, 2017; Lavicza et al., 2023) emphasise its fundamental importance for mathematical conceptual understanding, reasoning

and problem-solving at all educational levels and in various fields (mathematics –e.g., calculus, algebra–, sciences, technologies, arts, engineering, and daily activities such as car driving). In particular, it is important to develop visualisation abilities that involve the ability to think about 3D geometric objects in motion (i.e., 3D dynamic visualisation); for example, by using digital technologies to mentally manipulate 3D objects and shapes (Boz, 2005; Kösa, 2016).

In this regard, Gómez-Chacón et al. (2016) point out that the use of DGE can help students overcome their usual difficulties when visualising geometric places (loci). Those authors point out that the DGE “Trajectory” tools (called “Locus” in Cabri 3D) for studying geometric locations, not only make the invisible visible, but can also enhance the power of imagining the possible and the impossible through graphic exploration and experimentation of mathematical concepts (Gómez-Chacón et al., 2016).

Thus, our research question is: *how university students develop 3D dynamic visualisation abilities when using Cabri-3D to solve locus construction problems*. For that, we designed a teaching experiment with learning activities for finding and constructing loci, by using analogies between 2D and 3D geometry (see the methodology section below).

Theoretical framework

We now present some fundamental theoretical elements used in the design of the learning activities and for analysing the visualisation abilities developed by the study’s participants through those activities.

Visualisation and spatial abilities

From Gutiérrez's (1996) perspective, visualisation is a kind of reasoning that implies a geometric mode of thinking that is supported using visual and mental or physical spatial aspects useful for the elaboration of graphic representations (including the digital ones), such as drawings, figures, diagrams, or images. He explains that visualisation allows one to solve problems (in math and other sciences), demonstrate properties, and, above all, understand mathematical concepts; and that it is made up of four main elements: mental images, external representations, processes of visualisation and abilities of visualisation (Gutiérrez, 1996).

There are a variety of theoretical approaches for studying visualisation abilities; particularly interesting for our research are two of them (related to 2D dynamic geometry) –see Table 1: (i) the approach of Mariotti and Baccaglini-Frank (2018), who identified eight visualisation abilities that are developed with the use of dynamic geometry: identification, reconstruction, construction, part-whole awareness manipulation, theoretical control, geometric prediction, and crystallisation; and (ii) the six spatial visualisation abilities proposed by Gutiérrez (1996): conservation of perception, recognition of positions in space, recognition of spatial relations, visual discrimination, visual memory and mental rotation. These approaches, while originally 2D-based, open the opportunity for research of 3D visualisation abilities using DGE.

Three-dimensional dynamic visualisation abilities are required for reasoning about movements of self and of objects in space. Dynamic visualisation using DGE is an ability, a process, and a product (Arcavi, 2003) that allows one to mentally manipulate 3D objects to construct visual representations (that change over time) of relatively abstract mathematical concepts (Pittalis & Christou, 2010).

Table 1: Visualisation abilities as described by Mariotti and Baccaglini-Frank (2018, p. 155–156)* and Gutiérrez (1996, p. 9–10)**

Visualisation ability	Description
Identification*	“immediate identification of a geometrical property of a figure on the plane or in space, with a goal in mind”
Reconstruction*	“reconstruction of a figure from parts that are not correctly organized in space or that are not visible”
Construction*	“construction of a representation of a figure, taking into account the use of tools and the construction sequence”
Part-whole awareness manipulation*	“abstract a part of the figure and consider it separately from the rest”
Manipulation*	“manipulating a figure to transform it”
Theoretical control*	“mentally imposing on a figure theoretical elements that are coherent in the theory of Euclidean geometry”
Geometric prediction*	“The ability to identify a specific figure by isolating it out of a complex background”
Crystallization*	“manipulation and of geometric prediction involve “movement” of the figure ... The use of movement (of any type) involves a temporal dimension”
Figure-ground perception**	“The ability to identify a specific figure by isolating it out of a complex background”
Perceptual constancy**	“The ability to recognize that some properties of an object (real or in a mental image) are independent of size, colour, texture, or position, and to remain unconfused when an object or picture is perceived in different orientations”
Perception of spatial positions**	“The ability to relate an object, picture, or mental image to oneself”
Perception of spatial relationships**	“The ability to relate several objects, pictures, and/or mental images to each other, or simultaneously to oneself”
Visual discrimination**	“The ability to compare several objects, pictures, and/or mental images to identify similarities and differences among them”
Visual memory**	“to remember mental images or objects no longer seen”
Mental rotation**	“The ability to produce dynamic mental images and to visualize a configuration in movement”

Using geometric and loci analogies to develop visualisation abilities

Loci in 3D are an under-researched fundamental mathematical idea in geometry education studies (Nagy-Kondor, 2017), but are interesting for their potential to foster visualisation abilities using dynamic geometry. However, in post-secondary mathematics courses, loci are seldom the focus of study, resulting in many students not learning enough about them (Fernández-Mosquera, 2011). Moreover, when university students do study loci, it is usually only in 2D (Nagy-Kondor, 2017).

Several authors (Mammana et al., 2012; Nagy-Kondor, 2017) propose the use of analogies as a conceptual tool that serves as a bridge to connect plane geometry with spatial geometry, and thus understand loci in 3D. Analogies establish similarities or relationships between objects that coincide in some respects (Polya, 1989), are a way to expand mathematical ideas (and create new objects) and help students develop meanings of geometric objects (Sua et al., 2022).

Methodology and participants' previous experience

Using a Design-Based Research approach (Bakker, 2018), we designed a teaching experiment for the development, in a group of Colombian university students (future mathematics teachers), of 3D visualisation abilities when using Cabri-3D to solve locus construction problems. The experiment included a questionnaire, two learning activities that students worked in pairs in a classroom setting, and interviews with selected participant pairs. Here, we present some results from the first cycle of the design, implementation, and retrospective analysis of the experiment. In that cycle, we analysed the work during the teaching experiment of five pairs of participants, in order to identify evidence of the development of visualisation abilities, and of how students represent and establish relations between elements of 2D and 3D figures. The analysed data consisted of the students' Cabri files, their answer worksheets, video recordings of the implementation of the learning activities, and transcriptions of the interviews. The data was first coded in an open way by defining key expressions for students' actions (e.g., "they observe invariants or constants"; "partial recognition"; "identifies a hidden locus"); through this method we were able to then carry out a selective coding, linking the identified students' actions with the categories of visualisation abilities (e.g., "perceptual constancy"), described in Table 1 and which are based on the works by Mariotti and Baccaglioni-Frank (2018) and Gutiérrez (1996).

The questionnaire was used to determine the participants' previous experience. In the first cycle, as expected, the responses showed that none of the participants had previously worked with dynamic geometry in 3D and that their experience in three-dimensional geometry was limited; however, some of them could draw 3D objects using pencil and paper (Figure 1), particularly solids of revolution (Figure 1), which indicates their knowledge in associating movement to the representation of 3D objects.

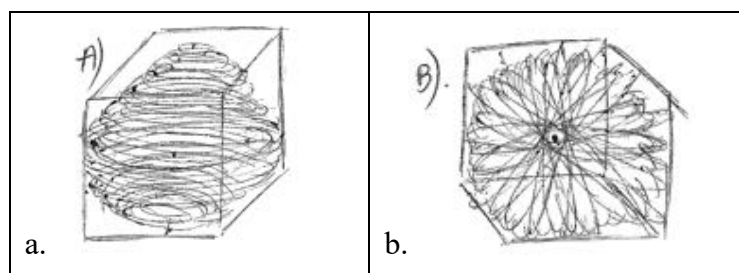


Figure 1: Student drawings (by Pair 2) in pencil and paper to represent 3D objects to represent moving circumferences that generate a sphere inscribed in a cube

Learning activities design

For the learning activities, we decided to use Cabri 3D because it has specialized tools useful for the construction of 3D loci (e.g., *Intersection curve*) and their visualization (e.g., *Crystal ball position*). Thus, the hypothesis was that using Cabri 3D can promote the visualisation needed to imagine the required loci in the activities and construct dynamic geometric figures that represent them.

The first activity proposed a 2D construction problem: Given three points A, B, C, different and non-collinear, in the same plane, use Cabri 3D to find the locus that is equidistant to A, B, C. In this case, the requested locus corresponds to the centre of the circle that circumscribes the triangle formed by such points (Figure 2a). The second activity was analogous to the first, but in 3D: Find the locus that is equidistant to three points A, B, C, in a three-dimensional space. In this case, the geometric locus to be constructed is a point that corresponds to the centre of the sphere circumscribing the triangle formed by said points (Figure 2b). Thus, we have the following analogies between the 2D and the 3D tasks: the 2D circle corresponds to the 3D sphere; the perpendicular bisector in 2D is analogous to the mediating plane in 3D; and the lines that intersect at a point in 2D are analogous to planes that intersect at a line in 3D.

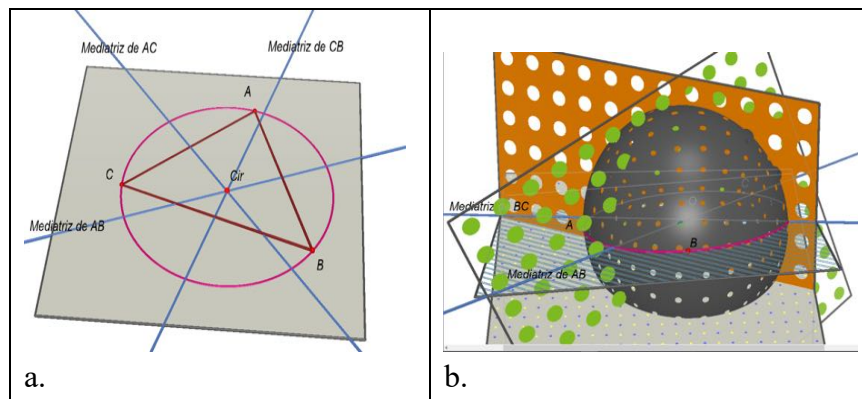


Figure 2: Expected constructions, in plane (a) and space (b) for each of the respective activities (Note: Figure a was constructed by the students in Pair 1. Figure b is an expected construction.)

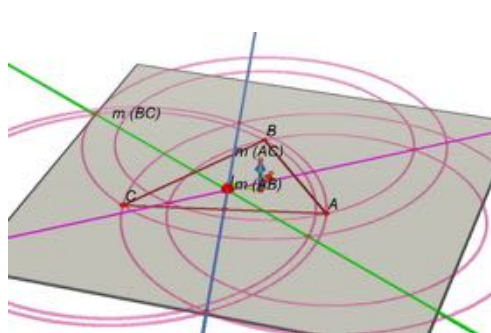


Figure 3: Auxiliary constructions by Pair 1 to construct the mediatrices and find the locus in 2D

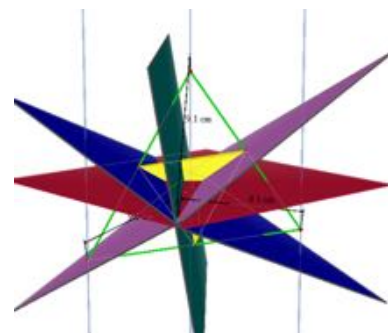


Figure 4: Use of the Mediating Plane tool by Pair 1 to construct the locus in 3D

Results

In this section, we present an analysis of the data collected in the first cycle of the experiment and the visualisation abilities (see Table 1) that were identified in that cycle.

We begin with how some students showed signs of the *theoretical control* ability. For example, during the first 2D task, Pair 1 (students Clara and Mario –pseudonyms) evoked the geometric concept of mediatrix (i.e., the perpendicular bisector) to find the locus: Clara explained “because it is the intersection of the three mediatrices [...] By definition [...] the mediatrix is the point that is equidistant in a segment”. Mario then revealed their hidden auxiliary constructions (Figure 3), through which they found the intersection of the three mediatrices of the triangle of the given points, that is the locus equidistant to them. Another two pairs gave similar solutions (e.g., see Figure 5a).

In the second 3D task, Pair 1 seemed to understand how certain 2D geometric properties –e.g., the ones related to equidistance– extend analogously from 2D to the 3D space: In 2D they used Cabri 3D’s *Bisector* and *Intersection* tools to find the equidistance and locus in the plane, with an analogous use of the *Mediating Plane* tool to construct the locus in space (Figure 4).

However, other students, such as Pair 2, that in the 2D task were capable of identifying that by constructing the perpendicular bisectors one can find equidistance and the requested locus (Figure 5a), showed many difficulties in imagining the construction of the 3D locus: Pair 2 tried to use the *Bisector* tool that only draws a line, and did not realise that they had to use the *Mediating Plane* tool to draw a bisecting plane to find the equidistance in the 3D space (Figure 5b).

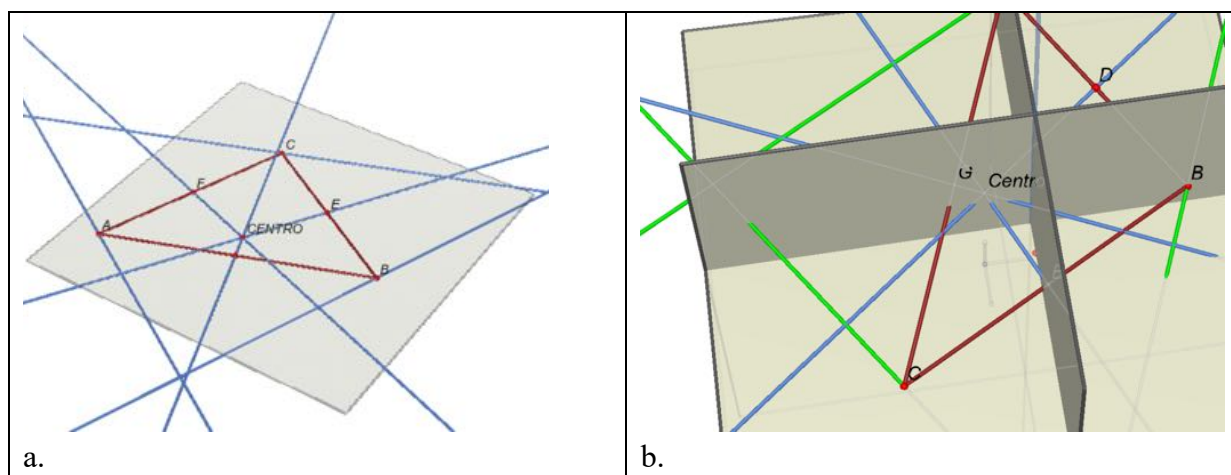


Figure 5: Analogous geometric loci attempt, in the plane and in space, by Pair 2

We also found evidence of other visualisation abilities, such as the *recognition of spatial relations*; for example, when both Pair 1 and Pair 2 recognised the characteristics of the locus (specifically the equidistance property) and understood how individual objects (e.g., points, circles, planes) interact and affect the shape of the locus. Likewise, Pair 2 and other students, Pair 3, showed *visual discrimination* abilities in that they were capable of recognising similarities and differences between individual objects to relate the different perspectives (particularly in 3D) of the locus and thereby identify the invariant elements. Furthermore, we found that students who showed *part-whole awareness* abilities, also used *mental rotation* to make connections between the individual objects involved in the construction of the locus required.

Based on these results, for the second cycle we determined that it was necessary to provide students with more opportunities to develop and use other abilities. Thus, the changes in the second cycle included: a) Three preliminary sessions of Cabri 3D activities for students to become familiar with the software and engage in construction activities; b) Three additional activities with 2D-3D analogous constructions (e.g., the bisector line for an angle in 2D and 3D; a 2D parabola and 3D paraboloid). Preliminary analysis of that cycle show more visualisation abilities in the participants; however, more changes may be needed in a third cycle, in particular, by refining the way the activities are presented.

Final comments

Our methodological design is that of teaching experiment with a small number of students, and thus has limitations; in particular, because the results cannot be generalized. However, they do provide insights in: a) Approaches to promote 3D geometry visualization abilities. b) Difficulties of university mathematics students in visualising objects in space.

In terms of the first, our results show the importance of using analogies to extend 2D meanings about geometry properties (e.g., equidistance) to 3D locus problem-solving.

Related to the second point, for our university students, similar to Nagy-Kondor's (2017) results, it was difficult to imagine objects and concepts related to the locus in space. Some of our students, however, did show the development of certain visualisation abilities, such as *theoretical control* and *recognition of spatial relations*, when using Cabri 3D to construct 3D loci. In this way, our study contributes to the understanding 3D visualisation abilities in university students and how a 3D DGE can support the development of mental images in dynamic spatial configurations.

Finally, we recommend continuing to delve deeper into the use of the dynamic representations offered by a 3D DGE since it was observed that it favours the production of images that are difficult to construct using exclusively pencil and paper (e.g., *mental rotation*). This reaffirms our hypothesis that the use of digital technologies can play an important role in the development of 3D dynamic visualisation abilities at the university level.

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Digital and traditional mathematical discussions: joining forces to promote inclusion

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In this paper, we present an analysis of the design for the integration between traditional in-person mathematical discussion and digital mathematical discussion. The analysis is conducted through the didactical differentiation model used as an analytical tool to evaluate the potentialities of integrating traditional and digital mathematical discussion in order to foster inclusion. The analysis shows that the integration of digital mathematical discussion with its traditional version can offer an added value in terms of promoting discussion as an inclusive practice with respect to the four variables of the differentiation model.

Keywords: Mathematical discussion, digital mathematical discussion, inclusion, differentiation.

Introduction and theoretical background

In this paper, we propose our reflection on the added value that the integration of digital mathematical discussion (Gagliani Caputo et al., 2023) into the traditional approach to in-class mathematical discussion can give in terms of inclusion.

Inclusion and differentiation

Although the notion of inclusion has received increasing attention in educational research over the past few decades, Roos (2019) highlights that there is no agreed definition for it in mathematics education research. This lack of agreement is due to varying perspectives and contexts in inclusion research, which spans ethnic, social, and special educational issues. In our research, the term ‘inclusion’ is used in line with Vislie (2003), who states that inclusion is a process by which a school strives to respond to *all* students. This perspective on inclusion expands the concept of integration, which often remains tied to the idea of inclusion, where the focus is on specific groups of students (e.g., students with disabilities or special educational needs) who are to be assimilated into existing forms of schooling, often with adapted curricula, different work, or support assistants (Vislie, 2003). Inclusive teaching methods, as intended in this paper, should still accommodate students’ individual differences, but with the perspective that these differences can serve as a starting point for teaching (Roos, 2019), leading to innovations that have the potential to benefit *all* students (Ainscow, 2016). The differences among students as individuals also highlight diverse needs in their learning process related, for example, to their previous school background or their interests (Sousa & Tomlinson, 2011). Differentiation represents one possible answer to embrace students’ differences in the learning process and, as for inclusion, it is a somewhat fuzzy concept (Anthony et al., 2019). Differentiation is based on the idea that the teacher should proactively plan her/his activity having students’ differences and similarities in mind in order to engage them more fully with learning (Tomlinson et al., 2003). Anthony and colleagues (2019) suggest a perspective on differentiation that shifts the focus from cognitive performance, by which students are classified and divided and that generally characterizes the perspective on differentiation, towards attention to students’ well-being and productive mathematical disposition. Such focus in a differentiated perspective can promote inclusive

teaching methods also taking into account students' moral, emotional and social development and not exclusively the cultivation of knowledge (Anthony et al., 2019).

Mathematical Discussion

When talking about mathematical discussion we refer to the definition proposed by Bartolini Bussi (1996) according to which mathematical discussion is characterized by an interplay of voices both of the teacher and of students, rephrasing, questioning or giving new contributions to the discussion, on a mathematical object. The mathematical discussion is traditionally conceived as a classroom activity to be carried out in-person and is led by the teacher who can orchestrate it by assuming different roles. In our study, the reference model for the teacher's orchestration of the discussion is the M-AEAB (model of aware and effective attitudes and behaviors) according to which the teacher can pose her/himself as a learner who faces problems making the hidden thinking visible, highlighting the objectives, the choices of the strategies and the interpretation of results, and can foster students' reflections at a metacognitive level, focusing on thinking processes and on the effectiveness of the implemented strategies (Cusi & Malara, 2013). Throughout this article, we will refer to the discussion just presented as *Traditional Mathematical Discussion* (in the following, TMD).

Research on classroom experiences on TMD allows us to highlight some strengths and challenges in terms of inclusion of this classroom practice. TMD can be considered an inclusive practice since it creates the opportunity to listen and to center the discussion around diverse proposals developed by students. More specifically, TMD fosters the expression of students' personal senses in meanings and the consciousness of their own intellectual processes (Bartolini Bussi, 1996). Moreover, it supports the development of interpersonal processes (Bartolini Bussi, 1996) in a "safe learning environment", in which all students are positioned as competent and capable and it promotes active listening and asking questions (Anthony et al., 2019). On the other hand, TMD can involve more or fewer students. Students' participation generally represents a challenge while conducting TMDs since few students actively engage in discussions and the cognitive load related to discussions can be high, especially for problem solving discussions (Richland et al., 2017), resulting in an even more limited involvement. Therefore, since inclusion is a delicate aspect in relation to mathematical discussions, we have decided to complement traditional in-person mathematical discussions with asynchronous digital mathematical discussion in order to amplify their potentialities in terms of inclusion.

Digital Mathematical Discussion

Digital Mathematical Discussion (in the following, DMD) is based on the traditional origins of mathematical discussions and implemented within a digital environment that involves the combined use of an instant messaging platform and a collaborative web platform (Padlet, <https://padlet.com/>). We decided to implement a digital version of mathematical discussions for several reasons. Firstly, the experience of discussions supported by digital environments during the Covid-19 pandemic period opened new research scenarios in mathematics education (see, for instance, Giberti et al., 2022). In fact, other research fields had already explored the role of digital environments in supporting discussions, both in a synchronous (e.g., through video conferencing platforms) and asynchronous (e.g., through forums) modality (see, for instance, Johnson, 2006), but there still remains limited research on this topic in the field of mathematics education. Secondly, the variables highlighted by research on asynchronous discussions seem crucial for stimulating students' participation in discussion activities in terms of inclusion. In fact, asynchronous discussions, involving participants

separated in space and time, allow each student and the teacher themselves to read and contribute to the discussion at their own pace (Andresen, 2009). Moreover, being able to work with asynchronous discussions allows extending the time dedicated to this practice beyond the classroom, as students can work on it at home, and communication in predominantly written form can encourage participants to make their reasoning and arguments more explicit.

Following the concept of balance discussion (Bartolini Bussi, 1991) and the FaSMEd design (Cusi et al., 2017), we designed DMDs. Balance discussion is socialization and collective evaluation of strategies that have been set up by students in individual or small group work (Bartolini Bussi, 1991) and can be conducted by the teacher purposefully selecting individual or small groups' solutions on the basis of their final answer, or of the provided justification, or both (Cusi et al., 2017). The selection is guided by the aims of the teacher of bringing to attention typical mistakes, so that students can receive feedback from their peers and teacher, of highlighting more or less efficient ways of processing problems, thus sharing criteria for success, and of contrasting different justifications or identifying similar ones (Cusi et al., 2017). Based on such theoretical background, the general design of a DMD consists of three main phases. The first phase of DMD consists in a small group activity aimed at collaboratively solving a mathematical problem within chats. Students are given a few days (e.g., 4 days) to collectively face the problem through the chat and to create a shared solution that has to be sent by one member per group to the teacher through an institutional platform (e.g. Google Classroom). The second phase of DMD is carried out within Padlet and starts as soon as all the groups submit their solutions. This second phase involves the whole class in a collective discussion designed by the teacher starting from selected excerpts from the groups' solutions (Cusi et al., 2017) and from significant interactions that emerged in the chats. The Padlet is structured in columns arranged by the teacher, who boosts the discussion through interventions aimed at activating specific roles of the M_{AE}AB construct (Cusi & Malara, 2013). Each column in the Padlet addresses a different issue on which the teacher wants to focus during the collective discussion. Students can comment on the teacher's inputs and on other students' contributions present in the Padlet for a few days (e.g., 4 days). At the end of the asynchronous discussion, the third phase of DMD consists in an in-person whole-class discussion aimed to wrap up and further elaborate on salient aspects emerged within the chats and the Padlet.

The aim of DMD is to complement, rather than replace, traditional in-person mathematical discussion, providing students with varied contexts for engaging in mathematical discussions, and allowing teachers to extend the time they can dedicate to this practice. As for TMDs, DMDs have potential and limitations with respect to inclusion. For instance, a chat-based environment can provide a secure space for students to interact, share ideas, and seek help without the fear of judgment or exposure face-to-face. This setting not only encourages open communication, but also enables teachers to have a broader and more detailed view of their students. Teachers can interact with more students simultaneously and they can more easily identify issues and needs among students, allowing for timely interventions and support. This proactive approach ensures that all students feel seen, heard, and safe, contributing to a more effective and nurturing educational experience. On the other hand, some of our studies have already highlighted the delicate issue of student participation in DMDs, especially in the second phase of the discussion, the collective one carried out on Padlet (Gagliani Caputo et al., 2023).

Analytical framework

In accordance with the vision on inclusion (Vislie, 2003) and differentiation (Anthony et al., 2019) presented in the introduction, which promotes the recognition and appreciation of students' heterogeneity and its exploitation to support all students, we have chosen the framework of didactical differentiation (Sousa & Tomlinson, 2011) as the analytical framework for this paper. Differentiation arises from the research-based perspective that students are more likely to engage thoroughly in learning and experience more robust learning outcomes when teachers proactively plan with both their differences and similarities in mind (Tomlinson et al., 2003). The model of differentiation of Sousa and Tomlinson outlines essential principles that drive effective differentiation, integral to a classroom system wherein all components collaborate to foster peak learning. In our research, these principles are used in alignment with the core tenets of differentiation presented in the introduction with the aim of highlighting how the design for the integration of DMD and TMD can support mathematical discussion as a more inclusive practice. According to the differentiation model (Sousa & Tomlinson, 2011), teachers can modify four variables within the classroom to address various student needs: (1) *content*, that is what students will learn or how they will gain access to what they are asked to learn; (2) *process*, that is activities guiding students' understanding or ownership of essential content; (3) *product*, that is how students showcase their knowledge, comprehension, and abilities after extended learning periods; and (4) *affect*, that is consideration of students' feelings and emotional requirements. By modifying these variables, teachers accommodate differences in students' *readiness* (closeness to learning objectives), *interests* (curiosity in specific ideas, topics, or skills), and *learning profile* (preferences for learning approaches or modes).

Research questions and methods

In this paper, we aim to analyze the added value that the integration of DMD can give to TMD in terms of inclusion. The analysis is carried out through the framework of differentiation (Sousa & Tomlinson, 2011), intended as an analytical tool that provides criteria for evaluating design in inclusive terms. In our study, the framework of didactical differentiation supports the analysis of the design of integrating DMD with TMD, aiming to identify the design elements of such integration that can empower mathematical discussion as an inclusive practice, allowing as many students as possible to participate in mathematical discourse within a discussion context. The analysis of the design for the integration of DMD and TMD bases on the general theoretical design of the two types of discussions as presented in the previous paragraphs and there is no empirical data directly addressed for the analysis. The research question guiding our work is as follows: Which is the added value that the integration of DMD gives to TMD in terms of inclusion based on variables of differentiation?

To address the research question, we will conduct a qualitative analysis of the design for the integration between traditional and digital mathematical discussions with respect to the variables of differentiation in order to highlight the added value that this integration can give in terms of inclusion. Based on the characteristics that distinguish DMDs, we hypothesize that by varying the modes of delivering the discussion among traditional and digital, it is possible to expand the ways in which one can be part of the discussion, thereby enabling more students to participate in the mathematical discourse realized within a discussion situation.

Analysis of the design for the integration between traditional and digital mathematical discussions

Traditional balance mathematical discussion and DMD share several elements regarding the variables within the framework of differentiation, as they both originate from common theoretical foundations. However, they also differ in certain aspects due to the influence of the digital environment in DMDs, making it interesting to analyze how their integration can promote inclusion.

Regarding the variable of *content*, considering “what students will learn”, both discussions stem from mathematical problems and evolve from group solutions proposed by students to such problems, aiming to compare related or contrasting strategies and stimulate metacognitive reflections (Cusi et al., 2017). Regarding the dimension of “how students gain access to what they are asked to learn”, the two types of discussion differ. For traditional balance discussions, access to content is primarily oral, both in communication with the teacher and with classmates, supported by written mode in group-work, when students submit their proposals, and in the whole-class discussion, when key aspects of the discussion are written on the board. Moreover, interaction is synchronous. The additional opportunities offered by DMD concern access to content in asynchronous mode and mainly in written form, still giving students the possibility to interact through images, audio messages, videos, emoticons and all the various forms of interaction that the digital environment supports.

Concerning the *product* variable, the two types of discussions share the possibility of assessing students’ knowledge, understanding, and skills through their group productions of mathematical problem solutions. Furthermore, DMD enables the creation of a greater variety of individual products concerning the tackled mathematical problems. It is not limited solely to the final group production, but also facilitates an increase in shared materials encompassing written messages, images, audio messages, and all productions shared by students within both the chats and the Padlet during the whole discussion. Regarding the *product* variable, we can also consider students’ communication abilities developed within discussions as a product. In traditional balance discussions, students acquire the ability to orally argue and respond quickly during a discussion. Additionally, through DMD, students also acquire the ability to argue in writing and to refer back to their or other peers’ interventions and reasoning over extended periods given by the asynchronous modality.

For the *process* variable, we can identify similar activities for both traditional and digital discussions. In particular, we analyze the *process* in both cases by referencing the activities carried out according to the FaSMEd design (Cusi et al., 2017). According to such design, the *process* of discussions unfolds in various activities in which students are involved, among which group work and comparison with peers, comparison with the teacher, metacognitive reflections. The added value given by the integration of traditional balance mathematical discussions with DMDs concern the expansion of time at disposal to conduct the discussion and the availability of a complete transcript of the discussion. Concerning the timing of DMDs, it enables students to engage in the same activities as traditional balance mathematical discussion in an extended amount of time, also allowing the alternation of some of them (e.g., alternation between in-the-moment or more reflected comparison with peers or the teacher). Concerning the availability of the transcript, the added value lies in students having access anytime to all productions, which enables the activation of a comprehensive individual reflection process. With access to the entirety of written productions, students can engage in autonomous reconstruction of the entire discussion. Unlike in a classroom setting, where the teacher would need

to reconstruct the discussion and might not have direct access to all students' contributions, having all written productions available allows for a more thorough reflection process and enables students to do it autonomously.

Finally, regarding the *affect* variable, both types of discussions stem from making all students comfortable in a discussion situation, giving the possibility to all of them to contribute, but they differ concerning students' sensitivities to their preferred modes of learning and interaction that can influence the *affect* variable. The modes of traditional balance discussions are favorable for students who prefer immediate feedback, can follow reasoning flows orally and at a quick pace, and enjoy expressing themselves in-person when participating in class. The integration of TMD with DMD seems particularly rich in terms of inclusion considering that the modes of DMDs are favorable for students who need more time to assess teacher and classmates' statements and to formulate their proposals, who see written form as a support for following a reasoning process, and who do not enjoy exposing themselves in-person in face-to-face settings, so adding relevant differences to the modes the solely TMD can offer. Additionally, the modes of DMDs may also be favorable for teachers who struggle with managing the immediacy of exchanges and keeping track of all students' work in class, allowing them to have more opportunities to observe students' progresses within the activities and their engagement, that may not always be visible in person, and to intervene accordingly, giving students a more central role.

Conclusion

In this paper, we have shared the initial analysis through the differentiation variables of the didactical differentiation framework of the added value in terms of inclusion generated integrating traditional and digital mathematical discussion. The theoretical foundations on which the two types of discussion are based allowed us to hypothesize that the combined use of discussions could result in positive outcomes in terms of inclusion. In fact, in line with the perspective of Vislie (2003), we do not believe that students should adapt to the given context, but rather the context should adapt to the diverse needs of the students. For this reason, we hypothesized that we could complement the practice of TMD with a digital form modifying the timing and traditional interaction modes of discussions in order to make discussion a more inclusive practice. The aim of this study is to characterize the added value that the integration of DMD can give to TMD with respect to the variables identified by the didactical differentiation framework to assess whether and how the combined use of these two different discussion modes can support discussion as a more inclusive practice.

Promoting mathematical discussion as a more inclusive practice is achievable by basing the design of the practice on principles of differentiation, which take into account students' differences and learning preferences and results in a practice for all students. Indeed, TMD already possesses characteristics that consider students as individuals and aims to be developed based on the personal meanings students attribute to mathematical concepts, giving voice to their diverse perspectives and involving as many students as possible in the mathematical discourse. However, classroom experiences show limited student participation in mathematical discourse within discussion contexts. Therefore, we believe that analyzing the design for integrating DMD with TMD in terms of differentiation can help us identify potential areas for action in the implementation of discussions. This integration could amplify the characteristics of mathematical discussion, allowing for new forms

of participation in the discussion through the potentialities of the introduction of the digital modality and, then, fostering more participation in the mathematical discourse realized within the discussions.

Traditional face-to-face mathematical discussion suits students who are ready to engage in real-time oral exchanges and can quickly process information, while DMD caters to those who need more time to reflect and respond, thus accommodating different levels of readiness. Additionally, DMD engages students with a keen interest in technology and multimedia, offering a variety of interaction forms that make engaging in discussions more appealing. This integration also addresses diverse learning profiles by supporting students who prefer written communication and visual aids and those who thrive in oral, real-time discussions. The integration of technology in DMDs enhances inclusion by providing asynchronous access to content and a variety of interaction methods, such as written messages, images, audio messages, videos, and emoticons. This contrasts with the primarily oral and synchronous nature of TMDs. DMDs enable a broader range of student-produced materials and offer a complete transcript of discussions, facilitating individual reflection and autonomous reconstruction. Additionally, the asynchronous format allows students more time to engage with and reflect on content, benefiting those who need longer to formulate responses and follow reasoning processes. This technological integration also supports teachers in observing student progress and engagement more effectively, allowing for timely interventions and the creation of a more inclusive learning environment.

As next steps in the research for evaluating the combination of different discussion modes in terms of inclusion, we will analyze classroom experimentations of traditional and digital discussions. With the support of the teacher, who knows students over the long term, we aim to track the trajectories within specific students' discussion activities to study their participation in different contexts. In order to do so, we will also consider frameworks concerning engagement structures in order to understand the ways in which students take part in the mathematical discourse within discussion situations. Moreover, the fact that the modes of DMDs could also support teachers who find it challenging to manage in-person classroom discussions suggests a potential new avenue for research into differentiation and inclusion for teachers as well.

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Designing silent video tasks as a tool to support teacher noticing in mathematics education

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This study is part of a research project aiming to support teacher noticing of students' thinking in the context of slope through the use of silent video tasks (SVTs) in mathematics education. SVTs are short animations that utilise the dynamic nature of mathematics without the use of voice or letters. The focus of this paper is on the design of SVTs to achieve the research aim. The Slope-APOS framework is considered the foundation for comprehending slope, and six silent video tasks were created to aid in the conceptualization of slope in geometric ratio, algebraic ratio, functional property, and calculus.

Keywords: teacher noticing, silent video tasks, mathematics education, geogebra.

Introduction

The idea of noticing has come to increased attention from the mathematics education research community in the last few decades with research on different tools aiming to develop (Amador 2019; Fernández et al., 2019; Schack et al., 2017). Teacher noticing in mathematics education is an ability to identify and make sense of students' mathematical thinking and build on an interpretation with a relation between students' current understanding and teachers' pedagogical knowledge, and making a pedagogical decision to impact students' thinking (Mason, 2002; van Es & Sherin, 2002). Professional noticing is a complex and challenging ability during teaching practice in a learning environment (Jacobs et al., 2010; Sherin & van Es, 2005; van Es, 2011; van Es & Sherin, 2002).

There is limited evidence of having an impact on the role of mathematical technological tools in teachers noticing students' thinking (Chandler, 2023; Galiç & Dost, 2023). Technology influences an individual's thinking skills and understanding of mathematics (Hollebrands & Lee, 2016). Furthermore, it supports analysing, interpreting, and exploring mathematical concepts (Zengin, 2015) and helps students and teachers to think about why or what-if questions (Hoyles & Jones, 1998). This paper is a part of a design-based research focusing on supporting teacher noticing of students thinking through silent video tasks in mathematics education. This paper aims to design silent video tasks as a tool to support teacher noticing of student thinking in the context of slope in lower-secondary mathematics classrooms.

Teacher Noticing

Drawing on previous research (e.g. Berliner, 1994; Goodwin, 1994; Shulman, 1986), the concept of noticing has become central to teacher education in recent years; however, different forms of definition and explanation have also emerged (Amador, 2019). Mason (2002) distinguishes professional noticing from everyday noticing. He defines professional noticing as 'what we do when we observe someone else acting professionally and become aware of something they are doing that

we think we could use' (p. 30). The focus is on sensitising individuals to notice things they are not currently used to noticing.

More recently, van Es and Sherin (2002) define the concept of noticing, particularly in the context of teaching, as follows (i) determining what is important or noteworthy about a classroom situation (ii) making connections between the specifics of classroom interactions and the broader principles of teaching and learning that they represent, (iii) using what one knows about the context to reason about classroom interactions (p. 573). van Es and Sherin (2002) argue that the analysis of what is noticed is as important as the act of noticing itself. They describe noticing as a two-way framework of attending and interpreting to describe important events that occur in a learning environment. While attending involves identifying notable features in classroom interactions, interpreting refers to the teacher's use of their knowledge and experiences to make sense of observed situations and establish the relationship between what they notice and teaching and learning. The authors extend their conceptualization by adding the third component of shaping as building interactions and contexts to access further information (van Es & Sherin, 2021). Jacobs et al. (2010) specialize teacher noticing of students' mathematical thinking by (i) attending to students' strategies, (ii) interpreting students' mathematical thinking and (iii) deciding how to respond based on students' understanding.

Previous studies focus on using some technological tools to develop teacher noticing. Video analysis is a commonly used technology Sherin and van Es (2005) provide teachers with video clubs that use the Video Analysis Support Tool to analyse classroom videos. The tool helps to assign videos to participants, organise teaching activities, and notice by pausing and annotating the videos to allow participants to focus on specific moments of teaching. Other studies on the use of technology to develop noticing skills have employed programmes such as LessonSketch (e.g., Walkoe & Levin, 2018) and GoAnimate (e.g., Amador et al., 2019). This study aims to develop mathematics teacher noticing of students' thinking by incorporating silent video tasks.

Silent Video Tasks

Silent video tasks (SVTs) were developed in 2014 in collaboration with Scandinavian and Baltic countries based on social constructivist theory for mathematics teachers and teacher educators (Hreinsdóttir & Kristinsdóttir, 2016). SVTs are short animations of less than two minutes and use dynamic mathematical content (Kristinsdóttir, 2023). These videos do not include any words or discourse, and each one focuses on a single mathematical concept. Silent video tasks involve selecting a silent video, showing it to the whole class, giving students a link to the video which they can watch as many times as they like, dividing students into groups of three or four and asking each group to record their voice-over for the video to explain its content. SVTs facilitate interactions between teachers and students, as well as among students themselves, by emphasising the depth of mathematical content. Both students and teachers actively participate in the learning process (Kristinsdóttir, 2021). The objective is to enhance learners' mathematical thinking by utilising their communication and social skills, as well as to deepen and broaden their comprehension of mathematics. Silent Video Tasks offer opportunities for learners to understand their own way of describing or explaining mathematical phenomena. Task responses from students can reveal information that was previously unknown to teachers. This can assist teachers in institutionalising knowledge and enables mathematics teachers to comment on students' current mathematical thinking.

Method

This study is part of a design-based research project. Design research aims to develop experiment-based theories to support learning styles and processes and examine the tools used in this process (Cobb, 2003). We adopted Gravemeijer and Cobb's (2006) design research approach from a learning design perspective by involving three steps: (i) preparing a design experiment, (ii) conducting a designing experiment and (iii) retrospective analysis.

The preparing a design experiment phase involves clarifying the research's purpose, elaborating on the experimental process, and creating a local instruction theory (Gravemeijer & Cobb, 2006). Careful planning of the starting and ending points of the learning objectives is required at this stage. The local instruction theory should incorporate tools to aid the learning process and assumptions about how learning occurs (Gravemeijer & Cobb, 2006, p.21). The second stage of the design research aims to implement the learning process and tools prepared in the first stage in the classroom, to conduct the local instruction theory (Gravemeijer & Cobb, 2006). The final stage of design research involves retrospective analysis, which aims to analyse the theory using data obtained during implementation (Gravemeijer & Cobb, 2006).

This paper covers the first stage of the research design and aims to create an instructional local theory. In line with this purpose, silent video tasks are developed to support teacher noticing of students' slope thinking.

Designing Silent Video Tasks

We used GeoGebra as one of the dynamic geometry software to dynamise mathematical concepts in the creation of the Silent Video Tasks in this study. While designing and developing the SVTs, lesson plans, assessment tools and contents published on GeoGebra related to slope were examined. Within the scope of the Slope-APOS framework (Nagle et al., 2018), the slope conceptualization of geometric ratio, algebraic ratio, functional property and calculus is considered as a basis for understanding slope. Since these conceptualizations represent the aim of each SVT, SVTs are designed for each conceptualization in GeoGebra.

At the stage of examining and designing the activities, drafts were prepared on paper for the SVTs designs of the four conceptualisations put forward within the Slope-APOS framework. The contents written on static paper were then reflected on the GeoGebra screen by using scripts, buttons, slider tool and colours in GeoGebra. The process of constructing content in GeoGebra was recorded with video. Timing is considered to provide enough space for the students to explain or transfer what they saw in the video verbally. Some pauses were included in the video movements to provide the foreseen narration. In some cases, labels or colours were used to help students describe or refer to the video during voice-overs.

We designed 14 SVTs as a draft and interviewed a mathematics educator to gain expert opinions about the content of the study and their suitability for teaching slope. Repetitive videos and six videos that were not suitable for the study were excluded from the study. Then, we worked with four mathematics teachers with varying experience years to interpret the videos according to the understanding of slope, curriculum, and student's grade level.

They answered two different questions to interpret the SVTs:

- a) Which mathematical concept is explained in the video?
- b) Which conceptualization of the concept is explained in the video?

The teachers' views aligned with the conceptualisation of the slope for each Silent Video Task. In the final stage, we worked with a silent video task expert in mathematics education. In an online meeting lasting 120 minutes, we analysed, interpreted and refined each SVT. We excluded some silent video tasks or repetitive videos due to confusion about the main mathematical concept. The remaining videos were improved by adding labels of points or lines and a grid. We have six different silent video tasks as a local instruction theory.

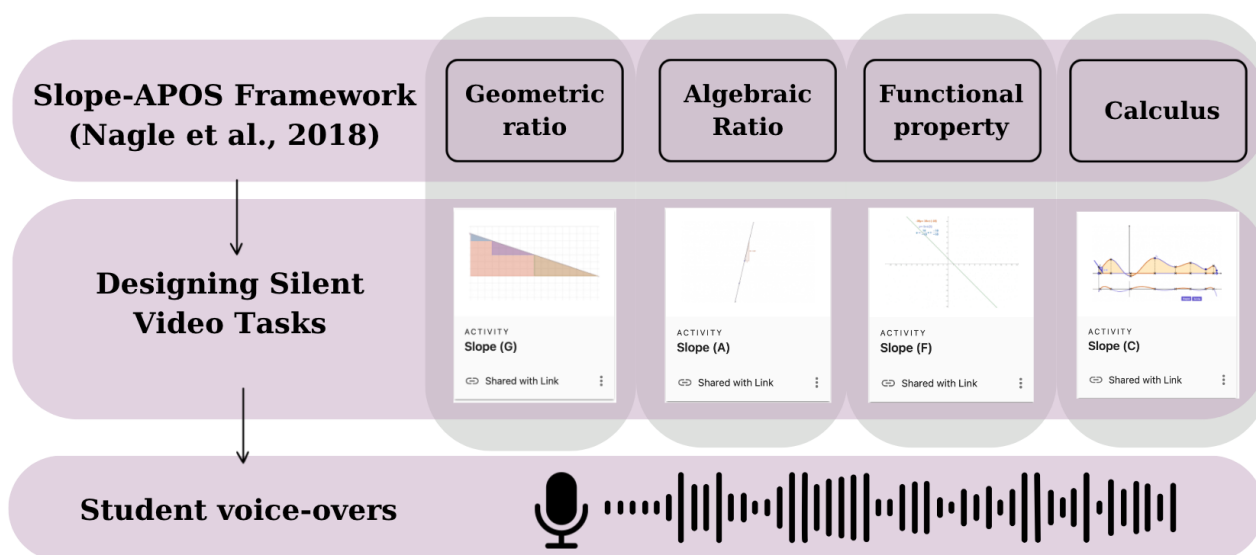


Figure 1: Designing silent video tasks

Conclusion

This ongoing research focuses on a type of teacher education with the use of technological tools. The training is based on supporting teacher noticing of students' thinking through silent video tasks that we have designed through the Slope-APOS framework. We attempt to conduct SVTs to collect students' voice-over videos for each silent video task that allow teachers to attend to students' strategies, interpret students' slope understanding and decide how to respond based on students' voice-overs. We seek to promote teacher noticing based on students' interpretation of the conceptualization of slope. In this way, we think it is possible to support teacher noticing of students' thinking. Our next step is to work with mathematics teachers to enhance to notice of students' slope thinking through silent video tasks with students' voice-overs.

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Programming and computational thinking in mathematics education: An Affordance Theory perspective

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The research literature reveals that programming provides both opportunities and challenges in mathematics education. However, further exploration of the topic is needed, underscoring the necessity for additional research to address the impact of programming on mathematics education. To thoroughly explore the impact of programming on mathematical education, it becomes essential to examine the very nature of programming affordances in relation to paradigmatic foundations, theoretical perspectives, mathematical and computational thinking, and their interconnections. This paper aims to address the role of programming through the perspective of Affordance Theory, elucidating its implications for mathematics education.

Keywords: Affordances, computational thinking, computer program, mathematical thinking, programming

Introduction

Upon closer examination of the research literature, there appears to be a need for further exploration of paradigmatic foundations and theoretical perspectives regarding mathematical thinking (MT), computational thinking (CT), programming, and their interconnections. This perceived research gap might lead to a constrained perspective on programming affordances, given the potential influence of paradigms and theoretical perspectives on mathematical education. Neglecting to address this issue may hinder the advancement of knowledge and limit the understanding of programming affordances in the context of mathematics education.

This study adopts a Gibsonian perspective on affordances (Gibson, 1979), highlighting their relational nature, which emerges from the interactions between users and their environment. Likewise, within the context of mathematics education, a programming environment dynamically engages both students and teachers, fostering affordances through their active participation in this environment.

This paper has two primary objectives. Firstly, it aims to examine the programming environment within mathematics education through the perspective of Affordance Theory. Secondly, it presents a conceptual framework to facilitate the exploration of affordances across six educational levels.

Affordance Theory and programming

The notion of affordance was originally proposed by Gibson (1979). It refers to the relationship between an object's physical properties and the characteristics of a user, enabling particular interactions between the user and the object. More specifically, Gibson used the term "affordance" to describe the action possibilities that the environment offers to an animal in relation to the animal's action capabilities. Gibson's notion of affordance suggests a complementarity between an animal and its environment. A typical example is a tall tree that affords food for a giraffe due to its long neck.

The notion of affordance was introduced to the computer world by Norman (1988). It refers to the perceived and actual properties of a thing that determine how the thing could possibly be used.

Accordingly, an affordance is the design aspect of an object that suggests how the object should be used. Examples of affordances are user interface elements that directly suggest suitable actions: clickable geometrical figures, draggable sliders, or pressable buttons, etc. More generally, an affordance is a relation between a subject and object with the object perceived in relation to the needs of the subject: small stones afford throwing, chairs afford sitting, doors afford opening, etc.

De Landa (2013) stresses that affordances are relational, emerging from the dynamic interaction between the subject and the object, rather than being inherent to either one alone. Moreover, affordances emerge and become actualized within a specific environment (Wang, Wang, & Tang, 2018). In mathematics education, this environment is structured around three key perspectives: firstly, computing paradigms; secondly, learning theories; and lastly, the interplay between MT, CT, and programming. These perspectives are combined to create a programming environment that facilitates the emergence and actualization of affordances.

Affordances from the perspective of computing paradigms

In this section, programming affordances are explored through the perspective of three computing paradigms (Eden, 2007; Rapaport, 2020; Turner, 2014). Firstly, the rationalistic paradigm, views computer programs (CPs) as abstract mathematical entities from an ontological standpoint. Epistemologically, it posits that deductive reasoning is the sole valid approach for examining CPs, framing the act of writing CPs as a mathematical activity. Secondly, the technocratic paradigm asserts that CPs are ontologically concrete entities. Epistemologically, it posits that knowledge of CPs is derived solely from experience and posteriori procedures. Finally, for the scientific paradigm, CPs cannot be reduced to either concrete or abstract entities. Instead, writing CPs involves a combination of a priori and posteriori methods, encompassing both deductive and inductive reasoning.

The paradigms may afford both deductive and inductive reasoning. Deductive reasoning helps develop a CP in a top-down approach by breaking up the CP into smaller parts, which are then refined into more detail. Inductive reasoning fosters a bottom-up approach, where smaller parts of the CP are integrated into a larger one (Marji, 2014). The paradigms may also afford a combination of top-down and bottom-up approaches to investigating problems using CPs in STEM contexts, as the paradigms refer to ways of thinking in mathematics, science, and engineering education.

Affordances from the perspective of paradigms in mathematics education

In this section, programming affordances are addressed from the perspective of three main paradigms in mathematics education: Constructionist, socio-cultural, and post-structuralist. Constructionism assumes the construction of one's own knowledge. It shares constructivism's connotation of learning as 'building knowledge structures' (Papert & Harel, 1991). Papert (1980) argued that computer programming is a particularly good way to achieve knowledge construction. Logo in the 1980s and Scratch today are examples of programming languages that foster knowledge construction.

The socio-cultural paradigm posits that knowledge is constructed through sociocultural interaction and cultural means. It provides the very basis of Activity Theory (AT) and the concepts of mediation and mediational means (Vygotsky, 1978). Accordingly, programming affordances cannot be reduced to what the subject or CP alone can do. It is the "subject-with-program" that affords the learning of mathematics. Wertsch (1998) uses the term "Subject-acting-with-mediational-means" to characterize the inseparability of the subject and the mediational means. Moreover, the socio-cultural paradigm

forms the basis for technology-focused theories (Sinclair et al., 2022), such as the Theory of Didactical Situations (TDS) (Brousseau, 1997) and the Instrumental Approach (IA) (Trouche, 2004). From the standpoint of TDS, affordances arise from interactions between the learner and the CP within the milieu, which refers to the socio-cultural and institutional environment in which teaching and learning take place. From the perspective of IA, affordances emerge from interactions between the learner and the CP, or via the process of instrumental genesis, as delineated within the IA.

The post-structuralist paradigm and associated theories, such as Actor-Network-Theory (ANT) (Latour, 2005), provide another type of relation between subject and object. Agency in ANT is distributed among humans and non-humans, which means that a CP is not simply a mediational means, but an active non-human agent involved in the production of mathematical knowledge. Agency enables the emergence of other types of affordances than those in the socio-cultural paradigm.

Affordances from the perspective of CT, MT, and programming

CT, MT, and programming

After addressing affordances from the paradigms of computing and mathematics education, the article focuses on affordances from the perspective of CT, MT, and programming. CT was originally defined by Papert (1980) as how to utilize programming as a tool to think and learn. Wing (2006, p. 33) argues that CT represents a "universally applicable attitude and skill set for everyone, not just computer scientists". It is the thought process that a "computer—human or machine—can effectively carry out".

Yet, despite shared central concepts across several research studies, there remains no universally established definition of CT (Denning & Tedre, 2021; Elicer & Tamborg, 2023; Shute et al., 2017; Weintrop et al., 2016). Nonetheless, many definitions do converge on five core cognitive processes. First, it encompasses the capacity to engage in algorithmic thinking as a means to solve problems. Second, it is a way of thinking about problems in terms of *decomposing* their components. Thirdly, CT is linked to the process of *generalization*, which involves identifying similarities to generalize the problem-solving approach. Fourth, CT uses *abstraction* to make problems more understandable by reducing unnecessary details. Lastly, CT employs *evaluation* to ensure the suitability of the problem-solving process for its intended purpose.

Likewise, there is no consensus on an exact definition of MT. Similar terms include "mathematical reasoning" and "mathematical abilities". MT also consists of beliefs about mathematics, problem-solving processes, and justifications for solutions. MT involves the "application of math skills to solve math problems, such as equations and functions" (Shute et al., 2017, p. 145). More generally, MT includes algebraic, geometric, numerical, and modeling thinking.

MT and CT share several commonalities, and connecting both may afford mathematical learning (Shute et al., 2017; Weintrop et al., 2016). For instance, algorithmic thinking and problem-solving are common to both MT and CT. Data analysis requires numerical thinking in problem-solving. Likewise, mathematical modelling provides a foundation for the development of CT.

Programming uses artificial languages characterized by logical grammar, precise syntax and semantics, with no tolerance for errors. It is a multi-step iterative process that spans problem analysis, designing a solution, evaluating its effectiveness, and implementing the solution through a CP. As

such, programming is part of the broader concept of CT (Saqr et al., 2021). Moreover, CT and CP constructs such as variables and flow statements are connected to MT (Lie, Hauge, & Meaney, 2017).

Despite the commonalities between MT and programming, there are differences. For instance, the symbol “=” is interpreted in programming as an assignment to compute a new value (e.g., in Python) and as an equivalence relation in algebra. The key difference is that an assignment is temporal, while an equivalence relation is timeless, holding true at all times. Hence, the “=” in algebra may require didactical effort from students to switch from an operational to a relational meaning. The difference in the meaning may be considered a constraint, but it could afford the development of algebraic thinking through contrasting examples and awareness of accuracy (Bråting et al., 2022, p. 301).

Summarizing, CT relates to both MT and programming but cannot be reduced to this role. There is CT for and inside mathematics and programming, but this does not mean that CT is only instrumental to these subjects. CT is rather transversal and interdisciplinary because it can be considered as an epistemic framework that connects MT and programming (Lodi & Martini, 2021).

Connecting MT, CT, and programming

Connecting MT, CT, and programming (P) in a reciprocal way could afford mathematical learning. However, a literature review shows that one-way connections (CT-P→MT, MT→CT-P) constitute the overwhelming majority (about 90%), while the reciprocal way with two paths (CT-P→MT→CT-P and MT→CT-P→MT) makes up only 10.7% (Wu & Yang, 2022). Figure 1 illustrates a model that connects MT, CT, and programming in a reciprocal way (Hansen & Hadjerrouit, 2023, p. 160).

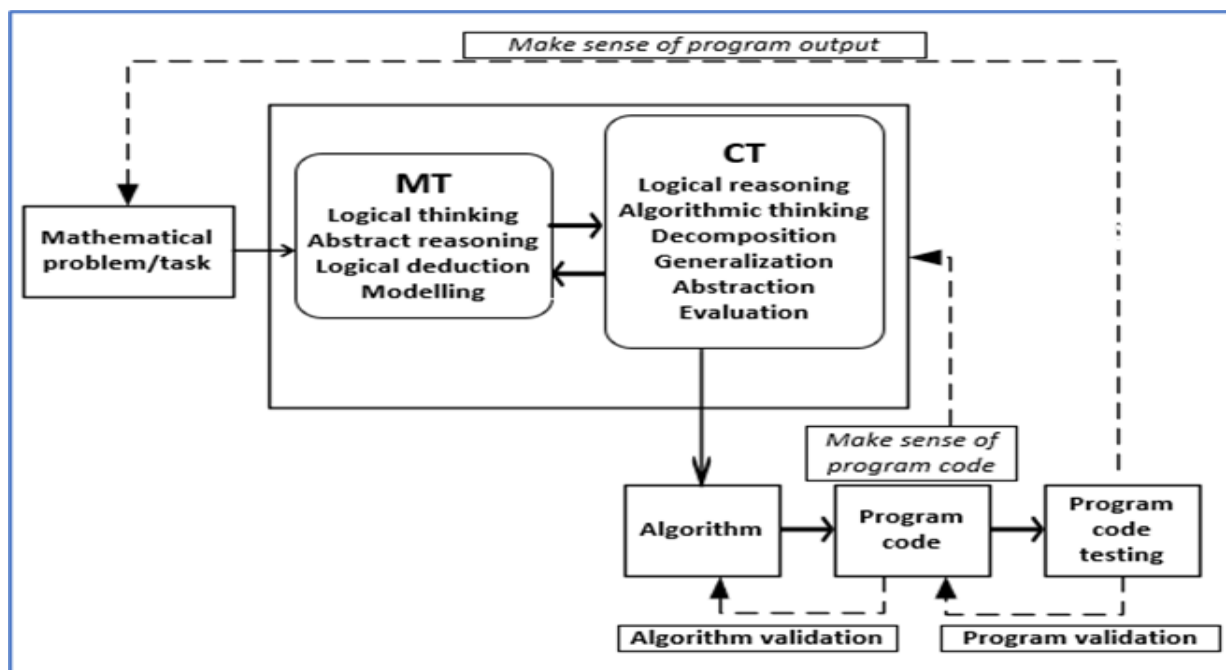


Figure 1: Reciprocal connections between MT, CT, and programming

In terms of affordances, students must have a good understanding of mathematical concepts and the capability for abstract reasoning and logical deduction to benefit from CT. Conversely, CT can enable students to logically analyze, abstract, and decompose a mathematical problem, and design an algorithm before programming it. This reciprocal process between MT and CT can enable students to translate the algorithm into a program, which can be iteratively tested and evaluated to understand

the program output and validate the algorithm. Clearly, CT is an epistemic framework that connects MT and programming, facilitating active engagement with mathematical problem-solving.

A conceptual framework for analysing programming affordances

The programming affordances discussed above have the potential to enrich students' mathematical learning by fostering an integrated approach that incorporates deductive and inductive reasoning, along with constructionist, mediational, and agentic thinking. This approach promotes the interconnectedness of MT, CT, and programming. To support this connection, it is essential to consider pedagogical strategies, socio-cultural influences, and relevant technological and assessment issues. The emergence and actualization of affordances rely on a coherent combination of these elements. Drawing on research literature (Hadjerrouit, 2020), this study proposes a framework supporting the exploration of seven categories of affordances across six levels: Technological, student, classroom, MT-CT-CP, assessment, and socio-cultural level (Figure 2).

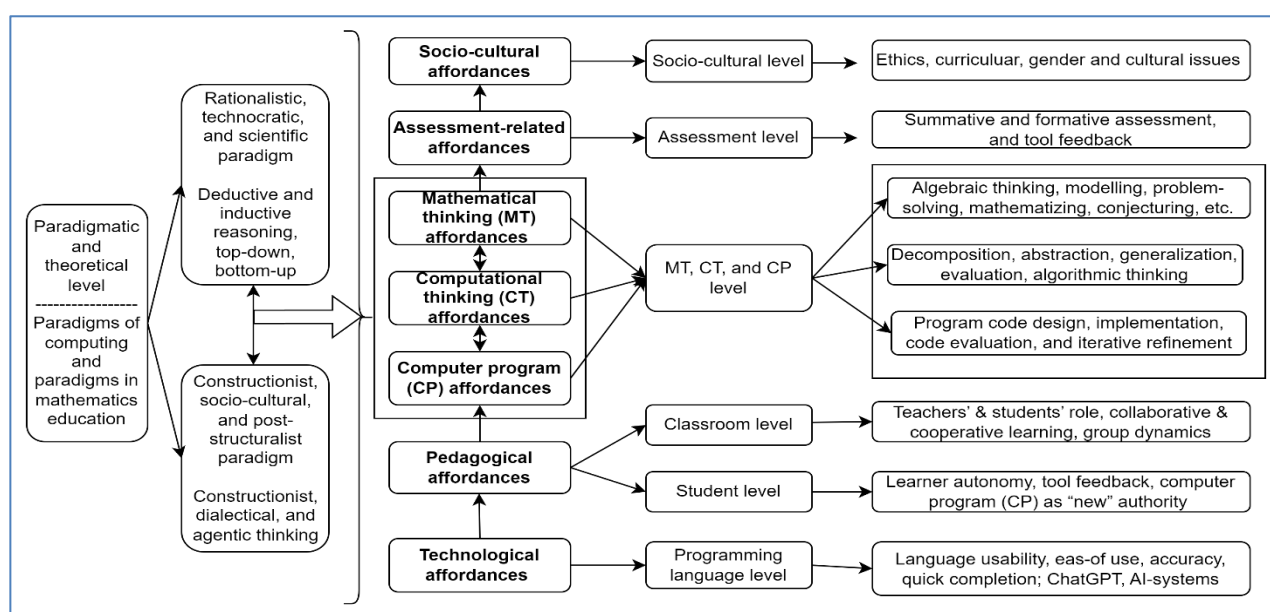


Figure 2: Conceptual framework integrating programming affordances across six educational levels

Technological affordances include ease-of-use, ease-of-navigation, accuracy, and quick completion of programming activities. Furthermore, the programming language should be robust enough to facilitate the development of advanced programs, possibly with the assistance of ChatGPT and other AI systems. Technological affordances are essential prerequisites for programming activities, laying the groundwork for pedagogical affordances and the interconnection between MT, CT, and CPs.

Several affordances can facilitate mathematical learning at the pedagogical level. Affordances within the classroom setting that lead to shifts in interpersonal dynamics, such as changes in the roles of teachers and students, and enhanced group interactions. Additionally, other affordances emphasize collaborative and cooperative learning, along with group work. The introduction of varied teaching methods and the differentiation of programming tasks are further activities that promote affordances.

Affordances that promote learner autonomy at the student level lead to more student-centered instruction, empowering students to take greater control over their learning and to view a CP as a

"new" authority for assessing their progress. Feedback from the CP plays a crucial role in fostering these affordances. It can manifest in various forms, including immediate feedback on students' actions, correction of syntax and semantic errors, and validation of mathematical problem-solving by verifying if the CP's response aligns with a mathematical solution.

Moreover, three types of affordances may emerge at the MT-CT-CP level. The first fosters MT by promoting mathematical problem-solving, conjecturing, and mathematizing the problem that can be translated into an algorithm using CT. Subsequently, the algorithm is transformed into program code that undergoes iterative testing, evaluation, and refinement. Numerous affordances that may emerge at this level pertain to various activities centered on integrating MT, CT, and CPs, including designing, constructing, debugging, abstracting, mathematizing, conjecturing, reflecting, etc.

Affordances at the assessment level can impact programming learning outcomes. In addition to summative assessment, formative assessment plays a crucial role in the learning process. As mentioned earlier, feedback from the CP is a vital component of formative assessment, and it can manifest in various forms, including providing feedback to students' actions regarding syntax errors.

Finally, socio-cultural affordances are threefold. Firstly, programming should offer opportunities to concretize the mathematics subject curriculum. Secondly, programming should be integrated into mathematics education in schools. Lastly, these affordances may facilitate the consideration of ethical, gender, and other issues within the context of programming and mathematical learning.

Discussion and conclusions

The framework offers a pathway to investigate the not fully explored issues of programming affordances. This includes a focus on MT, CT, programming, and their interconnections, computing paradigms, and theoretical perspectives. In empirical studies, the framework can be used as both a theoretical and analytical framework based on the seven categories of affordances across six levels, while accommodating the emergence and actualization of new affordances.

Given this background, the article raises some critical questions that need further exploration: What is the nature of programming affordances that are rationalistic, technocratic, and scientific? Does programming imply a new philosophy of objects that is purely computational? Or is programming just another semiotic register like the graphical or symbolic ones (Duval, 2006)? Does programming shape mathematical practices against the assumption that programming knowledge is "objective"? These questions provide some research directions, especially in the rapid advancement of Artificial Intelligence and non-human agents (Latour, 2005) that may bring about a computational turn similar to Lerman's sociocultural turn (Lerman, 2000). Perhaps it is the specificity of programming affordances and the attention to the ontology and epistemology of CPs stemming from various paradigms and theories that point to a computational turn in mathematics education.

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Digital technology-based proportional reasoning tasks

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This study presents a task design based on digital technology with practical implications for promoting proportional reasoning in secondary school students (ages 13-14). We developed three Virtual Interactive Didactic Scenarios (VIDS) within the Realistic Mathematics Education (RME) framework. The levels of RME were used to assess students' progress in proportional reasoning. Our data analysis revealed the proportional reasoning skills fostered by the VIDS, the didactic benefits and limitations of our task design, the learning difficulties related to the concepts of ratio and proportion, and the impact of digital technology on the teaching sequence.

Keywords: Proportional reasoning, technology, middle school education.

Introduction

Proportional reasoning, which has garnered significant research attention for decades, still needs consistent integration into classroom teaching practices (Obando et al., 2014). Its relevance transcends the classroom setting, as it is a critical tool for decision-making in everyday life. The challenge in mastering proportional reasoning lies in comprehending fundamental mathematical concepts such as ratio, proportion, and proportionality, which are indispensable for advanced mathematics and find application across diverse scientific disciplines (Lamon, 2007).

Given the importance of the topic, we have characterized the problems reported under four headings: 1) Lack of meaning in the study of proportional situations due to excessive arithmetic (Lamon, 2007); 2) Poor development of proportional skills caused by the lack of context for exploring representations of proportional situations (Weiland et al., 2021); 3) Prioritizing fractions as representatives of rational numbers to the detriment of the concept of ratio (Confrey & Carrejo, 2005); and 4) Failure to attend to the distinction between linear and non-linear relationships, leading to the use of direct proportionality in contexts where it is not applicable (De Bock et al., 2002).

The challenge at hand is how to promote proportional reasoning among students. The most common approach in schools to understanding proportionality comes from an arithmetic perspective, detached from functional relationships and their multiple representations, with limited use of digital technology (Cuevas et al., 2023). In this context, the standards set by the National Council of Teachers of Mathematics (NCTM, 2000) suggest that incorporating digital technology into teaching can lead to emerging forms of reasoning that may be challenging to observe through traditional methods. However, technology alone is not a solution; effective didactic design is crucial for technology to be a cognitive tool that enhances learning. Despite the benefits offered by technology in teaching, a systematic review of literature conducted by Lutfi et al. (2024) revealed that only 18 research articles on proportional reasoning with technology had been reported from 2006 to the present, indicating a clear need for proposals that leverage technological resources in this field.

Theoretical framework

Lesh et al. (1988) define proportional reasoning as a skill involving qualitative and quantitative processes essential for managing and predicting situations involving change, covariation, and multiple comparisons. Proportional reasoning can be approached from either an arithmetic or an algebraic perspective (Hersant, 2005). In this study, we consider that exploring both perspectives involving different representations in multiple contexts can help students to connect the concepts associated with proportional reasoning, conceiving them as a network of interconnected concepts, where the change in the understanding of each one has implications for the whole network (Vergnaud, 1994). These concepts are multiplication, division, fraction, ratio, rate, scale, covariation, relative size, magnitude, and measure.

In this regard, to assess students' reasoning regarding the conceptual network, we identify proportional skills as those faculties necessary for a person to possess solid proportional reasoning. Based on the essential understandings proposed by Lobato et al. (2010) and the comprehensions described by Weiland et al. (2021), we propose that such skills should include 1) Attending to and coordinating two quantities that vary independently, 2) Recognizing and utilizing the structures of proportional situations (ratio equivalence, proportionality constant, linearity), 3) Understanding proportionality across multiple representations (symbolic, algebraic, tabular, and graphical), and 4) Distinguishing between linear and non-linear situations.

The Realistic Mathematics Education (RME) teaching approach emphasizes that realistic situations are essential for learning mathematics. In RME, students actively develop learning and construct models that mathematize reality from an everyday context (Freudenthal, 2002). According to Van den Heuvel-Panhuizen and Drijvers (2020), realistic contexts can be mathematicized horizontally, where students move from the real to the symbolic of responding to problems in their context, and vertically, where they make conceptual connections and create strategies to solve problems within the mathematical system. In vertical mathematics, the objective is to detach from the context towards abstraction and generalization; in this journey, the students' reasoning goes through four levels of understanding: N1 (situational level), the reality is interpreted and organized through informal and context-dependent mathematical reasoning. N2 (referential level), schemes that make sense within the initial context are created, and vertical mathematics begins to emerge as "models of...". N3 (General level), concepts are related, strategies are generated that are separated from the context, reasoning takes place in the mathematical world, and "models for..." emerge. N4 (Formal level), concepts are understood using mathematical symbolism, support from an authentic context is no longer needed, and models can be dispensed with.

This study aims to promote the four proportional reasoning skills through the levels of mathematization. We developed criteria indicating the characteristics of the skill evaluated in correspondence with the situational, referential, and general levels of the RME (see Table 1). The formal level was not considered because its scope does not correspond to the educational level.

Table 1: Proposed criteria for assessing the tasks (Cuevas et al., 2023)

Proportional reasoning skills	Level 1: situational	Level 2: referential	Level 3: general
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(1) Attending to and coordinating two quantities that vary dependently	Students perceive the covariation of one variable concerning another in a specific context.	Students identify variation and dependence between two variables in a context to make predictions or inferences about change in one of them.	Students determine the functional relationship between two variables in any context by identifying the type of dependence between the variables.
(2) Recognizing and using the structures of proportional situations	They state and compare reasons from a context using intuitive reasoning.	They use equivalence to compare ratios and operate with the ratio's invariant to solve missing value problems.	They perceive direct proportionality as a linear model $y=kx$, identify the constant of proportionality in any context and understand the cross-multiplication algorithm.
(3) Understanding proportionality from multiple representations	Capture discrete data from a linear covariation context to generate a tabular representation and a graphical representation; can make inferences from the representations within the same context.	They can determine the linear property of a phenomenon from tabular or graphical representations and obtain the algebraic model of the situation.	They associate linearity with a discrete multiplicative relationship in tabular representation, a continuous straight line in graphical representation and an equation of $y=kx+a$ in algebraic representation.
(4) Distinguishing linear from non-linear situations	Distinguish linearity (or lack thereof) when taking or analyzing discrete data from a covariate situation in tabular or graphical representations.	They relate linearity to a constant multiplicative factor in tables and a constant graph slope.	Explicitly identify slope in linear functions and calculate the rate of change of a function from any representation to identify linearity or its absence.

Given the theoretical considerations above, we aim to address the following question: Which proportional reasoning skills are fostered in high school students when they solve tasks in a digital technology environment within the framework of RME?

Methodology

To structure our study, we relied on the Design-Based Research (DBR) methodology proposed by Bakker (2018), which consists of the following phases: 1) preparation and design, 2) teaching experiment, and 3) analysis of results and redesign. Under this DBR approach, theory and previous experiments guide the iterative design process. Given the pivotal role of design and innovation within Design-Based Research (DBR), this study proposes the integration of Virtual Interactive Didactic Scenarios (VIDS). VIDS represent virtual environments that authentically simulate real-world phenomena, enabling students to dynamically engage with diverse contexts and construct mathematical knowledge (Cuevas et al., 2023). VIDS serve to simulate realistic scenarios, facilitating interaction with multiple representations, manipulation of random data, and validation of outcomes.

Comprising a virtual scenario and Guided Learning and Exploration Sheets (GLES), the VIDS system presents a cohesive educational framework. These GLES provide a structured pathway for learning, fostering understanding through active engagement with the virtual scenario.

In Figure 1, we illustrate the design cycle used in this research. The activities integrated into the GLES are essential, as they guide students' interactions with the virtual scenario while facilitating the construction of mathematical knowledge. When faced with a question in the GLES, students engage with the virtual scenario and, through this interaction, can provide answers that may be correct or incorrect but are verifiable within the virtual scenario.

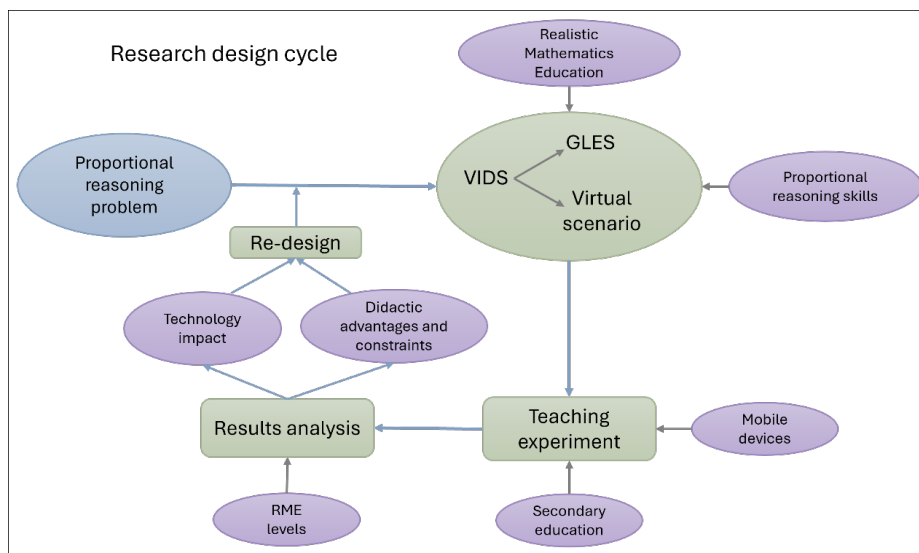


Figure 1. The design cycle in this study

Preparation and design phase

Task 1. The context involves virtually making orangeade using different mixtures of orange juice and water. The VIDS includes both instrumentation and comparison scenarios (see Figure 3a). The GLES aims for students to propose, compare, and determine the equivalence between orangeade ratios through interaction with the virtual scenario.

(a) VIDS Cars

(b) VIDS Zoom Totoro

(c) Example questions in GLES from Task 3

Set the car speed slider to 3 m/s and the patrol acceleration slider to 1 m/s². Press the "Play" button and answer the following:

- Does the patrol catch up with the car before it reaches the railway crossing?
- In what time has the car reached 15 m?
- How far has the patrol car travelled after 4 seconds?
- Does the speed of the car change during its run?
 - Yes No Why?
- Does the speed of the patrol car change during its run?
 - Yes No Why?
- Does the patrol car cover equal distances in equal times?
 - Yes No Why?

Figure 2: VIDS developed in the design cycle

Task 2. This task involves zooming in or out to resize an image of Totoro figures based on a similarity ratio controlled by the student (Figure 2b). The GLES aims to address the problem of the illusion of linearity by having students tabulate and graph the ratio-perimeter (linear) and ratio-area (quadratic) relationships, ultimately guiding them towards comparing both models.

Task 3. The context involves a car moving at a constant speed above the legal limit and a patrol car that begins a pursuit with constant acceleration as the car passes by (see Figure 2a). The aim of the GLES (Figure 2c) is for students to interact with the VIDS while describing, analyzing, and comparing the characteristics and representations of Uniform Linear Motion (ULM) and Uniformly Accelerated Motion (UAM).

Teaching experiment phase

An in-person intervention was conducted with 17 students (13-14 years old) from a Secondary School in Mexico. The instruction was divided into three 90-minute sessions where students could access mobile devices and personal computers to engage with virtual scenarios. Each student had their respective printed GLES. The intervention was led by the classroom teacher and supported by a researcher. During the sessions, individual learning was promoted by working with the VIDS, and collaborative learning was fostered through group discussions about difficulties and the exchange of possible solutions to the activities to reach a consensus on the GLES. During the discussions, students accessed the virtual scenario to test the reasoning that emerged from exchanging ideas. The collected data consisted of the researcher's observation notes and the GLES completed by the students.

Results

In the analysis, we categorized the students' responses into four proportional reasoning skills and then determined the level of mathematization of the RME corresponding to each one (Table 1). Beyond analyzing whether the answers were correct or incorrect, the GLES responses were associated with proportional reasoning skills. This approach allowed us to identify specific design aspects that promoted or hindered learning, providing an opportunity to improve the design for a new cycle.

Results for Task 1, "Orangeade". In this task, the students demonstrated skills in formulating ratios of orange drink mixtures and comparing them within the context. Depending on the order relationship between the ratios, they would decide which orange drink had a more intense flavor or if the flavors were similar. An unexpected finding emerged when we noticed that, with the help of the scenario, the students compared the orange drinks by associating the juice glasses with their respective water glasses and then counting the number of leftover glasses. Depending on the number of leftover water glasses, they would assert that one of the ratios indicated a more intense flavor. This method worked in some cases, but in the example shown in Figure 3a, after making the associations, the same number of water glasses remained unmatched, leading the students to conclude that both mixtures had the same flavor, indicating the same proportion. In these instances, the VIDS indicated the error, and the students had the opportunity to reflect and correct how they compared the ratios.

Results of Task 2, "Zoom Totoro". While performing this task, 75% of the students showed the ability to transition between representations (tabular, graphical, and algebraic) of proportionality regarding the linearity of the perimeter about a similarity ratio. Figure 3b displays the section of the GLES where students graph the linear variation of the perimeter and determine the slope. Regarding the transition between representations of the area-ratio relationship, only 25% of the students

completed the table and created the graph, but no student obtained a correct algebraic model. This was due to the arithmetic difficulties of the task and the student's deficiencies in perimeter and area concepts.

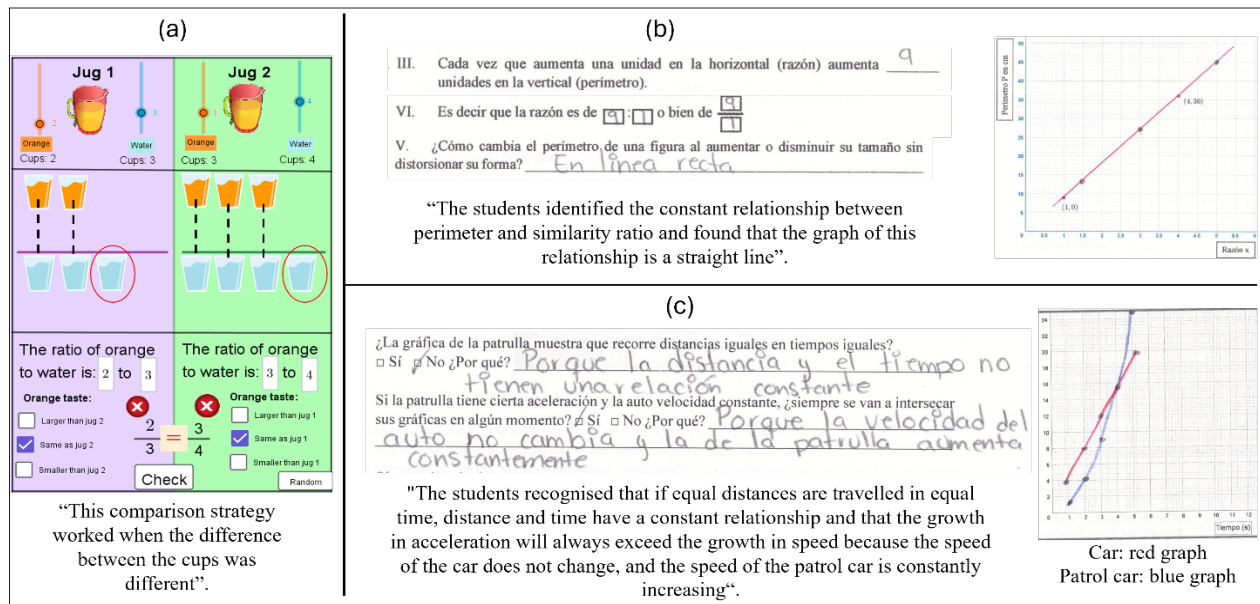


Figure 3: Evidence of tasks performed by students

Results of Task 3, "Cars". The students showed the skill to distinguish between linear and non-linear situations in various representations. 81% of the students identified the characteristics of Uniform Linear Motion (ULM) and Uniformly Accelerated Motion (UAM), but only 25% showed reasoning at the general level of mathematization in the representations of motion (tabular, graphical, and algebraic). Figure 3c shows a section of the GLES where the two movements were graphed. The student states that "if equal distances are covered in equal times, distance and time have a constant relationship" and that "the speed of the car does not change and that of the patrol car increases constantly." In this case, general reasoning is observed that applies to any context of ULM and UAM.

Summary of results

Table 2 presents the percentages of students who reached levels 1, 2, or 3 of mathematization in the four assessed skills, allowing us to observe which VIDS favor each skill. The "x" characters in the table indicate that, by design, it is not possible to evaluate that skill at the corresponding level. We found that the orange drink VIDS was highly suitable for reasoning about covariation (skill 1), comparing ratios, and determining equivalence (skill 2). Likewise, the "Zoom Totoro" VIDS proved effective for reasoning about the similarity and proportionality of figures (skill 2). On the other hand, the Cars VIDS was ideal for reasoning across representations (skill 3) and distinguishing between linear and non-linear variation (skill 4). The three VIDS supported the skill of "attending to and coordinating two dependently varying quantities" (1), as all three contexts are intuitive and familiar regarding the variables they involve. In the instructional sequence, the complexity of the tasks was carefully managed, as we considered that the orange drink VIDS involves more straightforward procedures than the Cars VIDS, given that the latter involves extensive quantities (speed and acceleration) and both linear and non-linear situations. Consequently, the skills and intuitions fostered during work with the initial VIDS helped the students in their work with the Cars VIDS. Observing

the improvement in mathematization levels in Table 2, we consider that the design effectively promoted proportional reasoning in the assessed skills.

Table 2: Percentage of students who achieved each level of RME

Proportional reasoning skills		Task 1 Orangeade			Task 2 Zoom Totoro			Task 3 Cars		
		N1	N2	N3	N1	N2	N3	N1	N2	N3
1	Attending to and coordinating two quantities that vary dependently	100%	43%	x	100%	87%	0%	100%	100%	60%
2	Recognizing and using the structures of proportional situations	100%	43%	x	100%	93%	0%	100%	87%	62%
3	Understanding proportionality from multiple representations	93%	x	x	75%	18%	0%	81%	68%	25%
4	Distinguishing linear from non-linear situations	x	x	x	25%	25%	0%	87%	56%	0%

Discussion and conclusion

In Task 1, students struggled to compare two ratios where the difference between the antecedent and consequent was the same, as they erroneously applied the additive principle to compare ratios. This understanding evolved as the task progressed. We found that comprehension difficulties emerged due to the VIDS, as these allowed students to verify their answers. In Task 3, students achieved general reasoning about ULM and UAM but could not express it algebraically due to a lack of mathematical language. It is necessary to enhance activities that facilitate the transition to algebraic representation.

Technology played a crucial role in designing and implementing activities by granting students the freedom to interact with contexts and receive feedback. Designing technological elements based on a framework facilitates an experiential teaching of mathematical concepts. This approach allows for the simulation and replication of everyday situations amenable to problematization and provides pragmatic and epistemic advantages. In this regard, students' experience with the VIDS allowed for identifying misconceptions that would likely go unnoticed in traditional teaching formats. Interacting with the VIDS sparked an interest in students beyond the presented mathematical context, engaging them with problems arising from the tasks.

In a survey conducted after the experience, 95% of the students indicated that they enjoyed working with technology in math class and that the VIDS allowed them to learn new things. Similarly, in an instrumentation test, 54% of the students preferred to interact with the VIDS on a mobile device in a horizontal position, 15% in a vertical position, and 30% mentioned that it was indifferent. Students feel more comfortable with scenarios in a horizontal position when visualization predominates in the focus of the activity, as they are accustomed to computer screens, TVs, or cinemas in this type of resolution. It is important to clarify that although the presented VIDS were created with GeoGebra, their design is independent of any software. The didactic and pragmatic essence of their elements can be developed in any application or programming language. Therefore, task design using the VIDS approach is timeless, even as versions are updated, or operating systems are changed over time.

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Lesson planning and assessment: Exploring the integration of AI tools in mathematics teacher preparation

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The advent of artificial intelligence tools provides new opportunities for mathematics-educators. This article explores the use of AI-generated content as a starting point that can stoke creativity and provide expanded capacity to prepare for student engagement. Such tools can be used to: (1) anticipate student responses, (2) design rubrics, (3) create culturally engaging tasks, (4) develop prompts focused on problem solving, and (5) devise higher level depth of knowledge questions. Imperfections and the rapid evolution of technology require layers of human collaboration, both with the AI-tools and Professional Learning Communities to ensure quality and effectiveness of preparation, instruction, and assessment. Educators should judiciously employ AI-generated content and capitalize on expanded capacity and creativity to refine personal AI-integration practices and foster learning environments focused problem solving and critical thinking.

Keywords: artificial intelligence, chatbot, mathematics education, planning, assessment.

Introduction and Organization

In 2023 versatile large language AI chat-bot programs became available for public use, including ChatGPT, Claude and Gemini, among others. This introduction occurred more than 70 years after the debut of the pioneer artificial intelligence (AI) programs which could play chess and checkers. Just as previous technologies have in the past, this new technology was met with curiosity, criticism, intrigue, and questions from the academic community (Hamid & Schisgall, 2023; D'Agostino, 2023). As mathematics teacher educators, we now grapple with what widely available AI chatbots mean for mathematics teacher preparation, how we can learn about these tools, and the ways we can foster judicious use of chatbots for mathematics teacher preparation, particularly as a tool for planning instruction and assessment.

In this article, we will first discuss some background literature that lays the foundation for how the mathematics education community understands teacher knowledge, technology integration, and the initial recommendations for education and AI. We will then consider aspects integrating AI tools for lesson planning and assessment in mathematics education. Specific categories are described for both lesson planning and assessment. These categories were identified based on the relevant topics that emerged during the exploration of AI tools in the mathematics methods courses. The article concludes with caveats, next steps, and the key takeaways from this early work in the integration of AI tools for mathematics teacher education.

Upon embarking on this project, we recognized the emerging nature of this area of study. For this reason, we initially considered larger works about technology in education such as TPACK (Mishra

& Koeler, 2006) the TPACK Development Model (Niess, et al., 2009) which have long helped us consider the use of tools such as calculators, dynamic geometry software, and digital fabrication tools in teaching and learning mathematics. During the study, we were cognizant of the quickly emerging scholarship surrounding the integration of AI tools, and we shifted our focus to the consideration of the more specific ideas which are the primary focus of the following section.

Theoretical Background

As teacher educators, we have long recognized the importance of fostering the development of knowledge constructs, particularly technological, pedagogical, and content knowledge (TPACK) (Mishra & Koeler, 2006). Further, we acknowledge that is a progression through a spectrum of TPACK for each technology that emerges. Niess and her colleagues (2009) describe five stages (recognizing, accepting, adapting, exploring, and advancing) for each theme (curriculum and assessment, learning, teaching, and access) in their TPACK Development Model. With this foundation in mind, we will shift to the consideration of large language model chatbots, or AI tools in the context of mathematics teacher education.

A new body of scholarship is emerging in the realm of the use of AI tools for teaching and learning. In their review of literature, Mohamad and colleagues (2022) note that most studies on the integration of AI tools focus on engineering, computer science, or STEM, with a lack of work done specifically for mathematics education. They call specifically for teachers to develop knowledge related to strategies to integrate AI during teaching and learning. Given the void of mathematics-specific research, we will consider the work on the integration of AI in teaching and learning more broadly.

Mao, Chen, and Liu (2024) highlight that the availability of generative AI tools brings both opportunities and challenges, noting that, “the co-existing collaboration between AI and humans has profound implications for education in terms of the teaching and learning processes, assessment, and teacher-student interactions” (p. 59). In this work, they further note the need to replace traditional assessments such as multiple-choice test items and essays with assessments such as performance items and portfolios which are less susceptible to AI-related dishonesty from students and also better engage students in higher order thinking, communication, and collaboration which is valued in today’s workforce.

Scholars also recognize the potential for AI tools to be used judiciously in designing instruction to free up time for the teacher (or instructional designer) to devote time to the most meaningful tasks (Bolick & da Silva, 2024). In the generalized work of instructional design, AI might be used to create multimedia, allowing the designer to focus on the progression of content, for example. Similarly, Moore and colleagues (2024) note that the learner’s integration of AI tools can reduce their cognitive load, support their engagement in metacognitive processes, and support mindful engagement. Still, Frick (2024) reminds us that “teaching students to think critically is essential when utilizing AI systems.” (p. 22)

Though initially some members of the education community feared that integration of AI tools might decrease opportunities for students to engage in critical thinking, scholars note that AI tools have the potential to increase our focus on application and creativity while encouraging critical thinking and analysis (Hodges & Kirschner, 2024; Cain, 2024). Hodges and Kirschner offer specific strategies for

teachers including the incorporation of oral assessments, engaging students in peer review and collaboration, and encouraging critical thinking and analysis.

In the following sections, we explore considerations for integrating AI tools for mathematics teacher education, specifically for developing lessons, lesson components, and assessments.

Integrating AI in Mathematics Teacher Education: Lesson Planning

The themes of learning communities, crafting prompts and dialogue, revising AI-created lesson components, and anticipating student responses are described in the following paragraphs. These themes emerged organically through the exploration of AI tools for teaching and learning during a lesson planning unit in a mathematics methods course and align with the literature noting that AI tools can be used to free up time for the teacher to focus on other tasks (Bolick & da Silva, 2024). Learning communities was a focus in response to concerns that the integration of AI tools would further silo educators. The crafting of prompts and dialoging with the AI chatbots became a focus as we realized the importance not only of forming a descriptive initial prompt, but also of continuing to work on improvements through ongoing dialogue and feedback. We quickly realized that AI chatbots were not the best tool for creating whole lessons, and we intentionally focused on using them to co-create individual components of the lessons instead. We also recognized that the focus on critical consideration of AI generated materials, coupled with creativity is a lesson reflected in literature (Hodges & Kirschner, 2024; Cain, 2024) that was enacted as our students leveraged AI tools for lesson planning. Our teacher candidates often struggle with anticipating a variety of student responses, so we also leveraged chatbots to anticipate responses to mathematical tasks and prompts.

Learning Communities

As we consider the use of AI as a tool for developing mathematics lessons, we should also consider the space in which teachers do this work. With AI as an aide, there is a tendency to work alone, dialoguing with the chatbot for ideas. When preservice teachers work alone, they may have blind spots that hinder their ability to critically examine the responses the chatbot generates. Grouping preservice teachers into Professional Learning Communities (PLCs) that mimic the professional teaching environment can combat the various limitations of lone wolf lesson planning. The PLC can work together to craft prompts, dialogue with the chatbot, critically examine the recommendations, and draft revisions of the lesson. This can be paired with a modified lesson study approach in which PLCs plan and study a lesson together, teach the lesson (observing one another when possible), debrief the lesson, and reflect and revise the lesson. In this space, the use of the AI chatbot as a tool for collecting ideas frees up capacity for the preservice teachers to think critically about the lesson throughout the process.

Crafting Prompts and Dialogue

The particular content of the request or input provided to the chatbot often determines the quality and effectiveness of the response generated. In one example, the prompt “Write a lesson introducing fractions” was provided to ChatGPT. The resulting output, while detailed, provided for only optional minimal use of tools and manipulatives and vague instructions such as “provide worksheets for students to practice identifying and representing fractions.” Subsequently, the prompt was revised to, “Write a lesson appropriate for third grade which introduces fractions. Please integrate the use of manipulatives or tools for teaching. Please include an accommodation for a student with a visual

impairment. Please include four homework items to follow up the lesson.” The response to this prompt included additional detail, connections to prior knowledge, the use of manipulatives, the use of tactile fraction strips and Braille fraction cards. In this case, the four homework items were still vague but offered some starting points: “Complete a worksheet on identifying fractions in everyday objects. Create a fraction collage using cut-out shapes from magazines or newspapers, practice adding and subtracting fractions using fraction manipulatives or drawings. Write a short story or word problem involving fractions and solve it.” To continue the conversation, the next prompt provided was “Create a worksheet on identifying fractions in everyday objects.” Though the chatbot created a worksheet, there were errors, such as the use of a backslash rather than a vinculum for representing fractions and the inclusion of an answer key despite the worksheet providing prompts that did not have specific answers.

Revising AI-Created Lesson Components

We proposed that lessons, whether human or AI-created, are never perfect and never truly “finished”. Every lesson can be critically considered and improved. AI-lesson components may need improvement regarding the integrity of content, as AI tools are prone to hallucinations in which they generate fictitious information (Research Guides at University of Southern California, 2024). Other times, the tool may not interpret the request correctly or may overgeneralize mathematical ideas. As previously demonstrated in this article, chatbots tend to create lessons without sufficient detail, so we propose avoiding the use of chatbots for creating whole lesson plans. Preservice and inservice teachers need to plan for the questions they will ask to guide discussion, the exact problems they will use for a lesson, and scaffolds they will use to support students during the lesson. Often chatbots will provide general statements such as “provide feedback to help students improve their understanding of fractions” but fail to provide specific ideas on how teachers can do this. In the context of the PLC teachers can use the AI-created lessons as a starting point, but also continually revise and add in the necessary details to create a quality, teachable lesson that fosters development of students’ understanding.

Anticipating Student Responses

When planning to foster a productive discussion in mathematics class, the first stage is to anticipate student responses. A chatbot can be a great tool for developing a list of potential student responses to a mathematical task or problem during planning. When ChatGPT was asked to provide a list of anticipated strategies for students to use for the classic Locker Problem (<https://36university.com/act-math-locker-problem-solution/>), the chatbot replied with details of five descriptions of approaches: brute force method, observation of patterns, mathematical analysis, trial and error, and collaborative problem solving. When prompted to identify misconceptions students might encounter, the chatbot replied with an impressive list of five different misconceptions and their corresponding impact on the student’s progress toward solving the problem. In essence the chatbot provides a “thinking partner” to anticipate student responses as the teachers prepare themselves for task implementation.

Integrating AI in Mathematics Teacher Education: Assessment

In consideration of the categories for assessment, we focused on creating rubrics, developing tasks, writing prompts, and designing questions to guide instruction for several reasons. These were areas where we found some success when piloting the AI chatbot tools. The assessment strategies also align

well with the focus of our mathematics methods courses, which deemphasize traditional multiple-choice testing and instead integrate practice-based tasks and tools to provide the teacher with a better understanding of what students know (Mao, Chen, and Liu (2024)). These ideas reflect the work of Hodges and Kirschner (2024) which emphasizes engaging students in collaborative, critical thinking, and analytical assessments.

Creating Rubrics

Chatbots can also be helpful with creating rubrics, though there are some limitations. A request to ChatGPT to create both a holistic and an analytic rubric for the aforementioned Locker Problem, produced an analytic rubric for both requests, despite providing a correct and detailed response to the prompt “What is the difference between a holistic rubric and an analytic rubric?” In our experience, a request for a rubric will most often produce an analytic rubric. Nonetheless, creating rubrics for tasks can be a repetitive process, and chatbots can aid in creating a first draft rubric for pre-service teachers to edit and integrate into lessons.

Developing Tasks

AI tools can also be leveraged to create engaging tasks. Recently in a mathematics methods course, preservice teachers used a chatbot to create a problem aligning to their learning objectives and using Taylor Swift in the context to engage students. This is particularly helpful for topics such as games and pop culture where teachers may not be familiar enough to use the interest-areas in the problem context. In this strategy, it is essential to consider the appropriateness of the context as well as the accuracy. Chatbots can also help provide creative performance tasks. When asked to create a performance task that uses Nike Air Jordan shoes to explore different models of multiplication, ChatGPT produced a detailed task, beginning with a phase of research into Nike sales, continuing a variety of multiplication scenarios and models, and concluding with the students creating presentations to communicate their work. The problem was accompanied by a detailed rubric. AI tools can be used to create engaging tasks with features and contexts that resonate with students’ interests.

Writing Prompts

Communication is an essential process standard for mathematics. Students should be listening to one another, engaging orally, and writing about mathematics. AI chatbot tools can help preservice teachers develop prompts to foster this communication. When asked to craft a writing for fourth graders around classification of angles, ChatGPT provided a problem posing style prompt (Figure 1). This problem could be used or modified for classroom use; however, chatbot requests for problem solving tasks, non-routine tasks, and critical thinking tasks generally resulted in algorithmic task results (Figure 2) which did not prove to be helpful for preservice teachers in their planning. When using chatbots for ideas on mathematics prompts, preservice teachers should review each of the outputs carefully and critically, judiciously integrating only the components that foster mathematical teaching and learning, staying cognizant of the need for students to engage in problem solving.

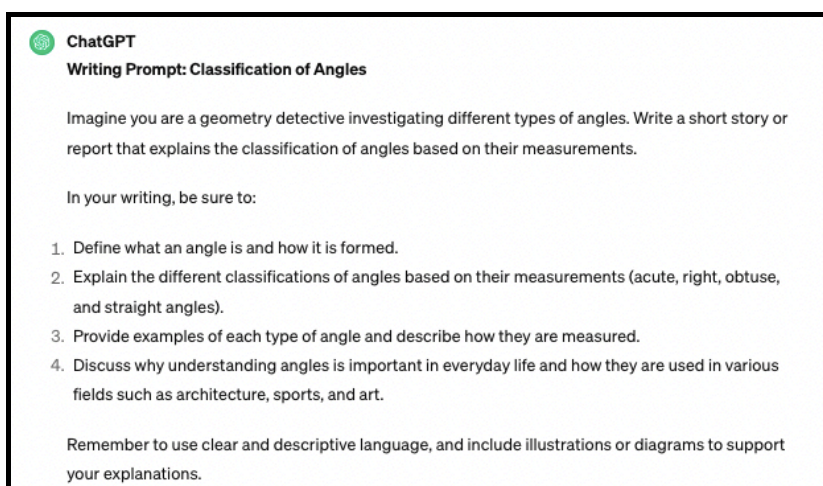


Figure 1. ChatGPT provided a writing prompt related to classification of angles. [Prompt: Craft a writing prompt for fourth graders to consider classification of angles.]

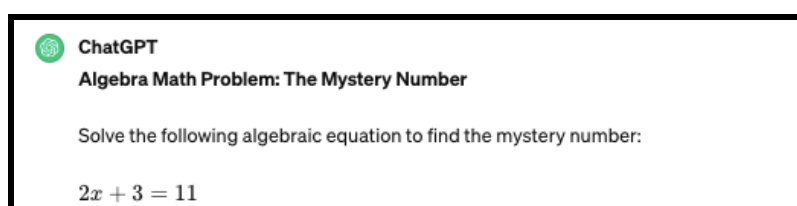


Figure 2. ChatGPT provided a routine, algorithmic problem to this request. [Prompt: Write an algebra math problem that requires critical thinking to solve.]

Designing Questions to Guide Instruction

Preservice teachers may also find chatbots helpful in designing questions to integrate in their planning and teaching. Webb's Depth of Knowledge provides us with a framework for asking questions which vary in levels of complexity (Webb, 1997). Often preservice teachers struggle to develop Level 3 and Level 4 questions which require strategic and extended thinking, respectively. We have found that chatbots can be useful in devising these more complex questions (along with a breakdown of the prompt and sample response), particularly when a detailed prompt, specific task reference, and mathematical goal is provided (Figure 3).

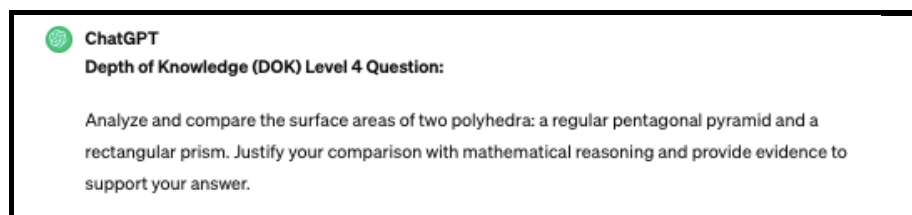


Figure 3. Chatbots can devise DOK Level 4 questions. [Prompt: Write a DOK 4 question related to surface area of polyhedral.]

Conclusion

Caveats

Along with the enthusiasm for the potential of AI chatbots to revolutionize lesson planning and assessments, we must also consider the challenges. Mathematics teacher educators have voiced concerns over issues related to academic integrity such as: How do we correctly cite work developed

by an AI tool? What is plagiarism when your collaborator is a chatbot? Privacy and data security issues have also surfaced: What happens to the data provided in a prompt? Are students and teachers at risk when they share information about students' interests, for example, in a chatbot conversation? Despite their mechanical nature, chatbots have also proven to have biases which emerge in their dialogue (Xue, et al., 2023). Interestingly, chatbots are often programmed to recognize and will describe the existence of various types of bias when asked (Xue, et al., 2023). Still, chatbot developers are not necessarily mathematics teacher educators, and the ultimate responsibility of instructional content and methods does not rest with the chatbot. Teachers must stay cognizant of these and other ethical concerns as they interact with the AI tools.

Summary of Takeaways

Recommendations for the integration of AI chatbot tools are emerging and evolving very quickly, and we recognize that some of these ideas, concerns, and takeaways may become outdated quickly. Nonetheless, we offer the following summary of key takeaways from our most recent work:

- Teachers can collaborate with chatbots in their Professional PLCs. It is essential that teachers consider the quality of the components, revising and ensuring that the lesson is appropriate.
- Teachers can use AI chatbots as thinking partners to anticipate student responses (both incorrect and correct) as they prepare to implement a task.
- AI chatbots can create a starting point for a rubric, but the teachers should carefully review and edit prior to implementation.
- Teachers may use AI chatbot tools to create engaging tasks with features and cultural contexts that resonate with their students.
- AI chatbots can be used to create mathematical prompts, however teachers should consider the outputs carefully and maintain a focus on learning through problem solving and critical thinking.
- Teachers may use chatbots to devise higher level DOK questions. Using a detailed prompt, specific task reference, and goal can help the chatbot create more appropriate questions.

In closing, we want to acknowledge the potential for AI chatbot to serve as what Moore and colleagues (2024) describe as cognitive partners. We also want to encourage mathematics teacher educators to offer experiences for future mathematics teachers to engage in intelligence augmentation, as they “[combine] the strengths of AI (its ability to synthesize and draw conclusions from large amounts of data) with human abilities (integrating embodied experiences, social contexts, emotions, etc.)” to transcend the capabilities that either could reach alone (Moore et al., 2024 p. 399).

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Exploring students' periodic covariational reasoning while creating dancing figural animations

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In this paper we discuss 11th grade students' covariational reasoning on periodic functions, as they created dancing figural animations matched to songs through digital technologies. Students explored various representations of trigonometric functions and Fourier series approximations in order to capture and express their personal artistic ideas connected to music. The analysis focuses on the role of digital representations as well as students' aesthetic experiences in the development of their covariational reasoning that involved sense making on the dynamic periodic change of geometric quantities with respect to time. The results indicated that both the digital tools and the aesthetically-driven aspect of the tasks supported various types of thinking on periodic covariation.

Keywords: Periodic functions, music, dance, educational technology, covariational reasoning.

Introduction

A growing number of studies highlights the importance of covariation, i.e. the dynamic change of two dependent quantities, for the holistic understanding of the abstract concept of function (Thompson & Carlson, 2017). Recent studies emphasize the need for design that cultivates covariational reasoning and exploration of the way students conceptualise it. Digital technologies, through their dynamic and interactive representations, expand the possibilities of depicting and investigating covariation (Swidan et al., 2022). In the literature, the design of learning environments under study focuses on the modeling of physical phenomena. Thus, emphasis has been placed on students' covariational reasoning as they observe and manipulate the representations or answer closed-ended questions about the physical model (Thompson & Carlson, 2017). However, elements such as students' aesthetics, creativity and innovation tend to be absent from such environments, despite their emerging importance in mathematics learning (de Freitas & Sinclair, 2014). In today's technocratically-driven society, combining aesthetic experience with mathematical reasoning is becoming more and more relevant. Furthermore, existing research focus is placed on parabolic or exponential functions, with periodic functions being on the research fringe, despite its rich mathematical, interdisciplinary and aesthetic interest (Shama, 1998). Moore (2014) and DeJarnette (2018) made some first steps in this direction by studying the way students use covariational reasoning on periodic functions (from here on mentioned as periodic covariational reasoning (PCR) by the authors) in order to model circular movement of a Ferris wheel cabin with sine function. However, their studies are limited with regards to the type of periodic functions used, the openness of the designed tasks and the exploitation of expressive digital technologies.

For this research, a series of activities was designed for cultivating PCR that prioritizes the modeling of inner artistic ideas, over that of external physical phenomena, providing a personalized learning experience guided by aesthetic responses. Students worked in microworlds (Healy & Kynigos, 2010) to create periodically moving geometric models that match specific songs. We study the research

questions: What kind of PCR has been developed in this learning environment? What is the role of a) digital representations and b) aesthetic experiences of students in their PCR?

Theoretical Background

Periodic Covariational Reasoning (PCR)

Covariational reasoning (CR) is associated with the ability to coordinate the parallel changes of two quantities (Thompson & Carlson, 2017). This is a complex cognitive process, which is important for making holistic sense of the concept of function and practices such as interpreting or creating a graph and mathematical modeling. Thompson (2011) emphasized its dynamic nature, adding to its definition that parallel changes take place within a space of “conceptual time”, which is a dense and continuous set, bounded by the temporal space in which someone thinks about covariation. Various models with developmental levels have been proposed for the description of students’ CR through engaging with different types of activities, such as constructing a graph or interpreting a graph that models a physical phenomenon (Carlson et al., 2002; Thompson & Carlson, 2017). The aspects of such models involve direction of change of two quantities, their synchronicity, continuity and rate of change. In their model, Swidan et al. (2022) also included the concept of the parameter. They integrated the ‘2nd order of CR’, which is the students' ability to make sense of the synchronous covariation of a class of (functional) relations and their characteristic parameters. Moore (2014) adjusted Carlson et al.’s (2002) model for describing students’ CR on sine function for the $[0, \frac{\pi}{2}]$. However, this adjustment neglects some important aspects of periodic covariation, such as the concepts of periodicity-period, amplitude and the continuous change of direction of change between the two quantities, as well as the role of parameters.

Mathematical Aesthetic Experiences (MAE)

The second theoretical dimension of this study includes the Mathematical Aesthetic Experience (MAE), that is, a perceptual experience with an emotional and intellectual response provoked by mathematical beauty (Sinclair, 2004). Traditional approaches define MAE as an innate ability, focusing on the evaluative dimension of MAEs, i.e. on the appreciation of objective values of mathematical concepts, assessed mainly on an intellectual level by recognizing typical features. Aesthetic criteria entail a status of intellectual autonomy, with a minority of students being able to adopt them because of the advanced mathematical thinking they require. Recently, researchers have sought to broaden the view of MAE, adding an epistemological perspective that stems from the idea of sense-making (Karavakou et al., 2023; Nemirovsky et al., 2013). They thus put a subjective dimension in the evaluative approach of MAEs, allowing the adaptation of mathematical aesthetic criteria to personal interests, emotional aspects and socio-cultural influences. The cultivation of such MAE can be supported by incorporating elements such as subjectivity, free expression, open exploration of mathematical ideas and connection to personal interests (Sinclair, 2004). Such environments can be: a) expressive technologies with dynamic, interactive features, and b) connected to artistic expression, such as music and dance.

Methodology

Methods – Data collection and analysis

This study is part of a larger design-based project. It involved two cycles (pilot and main) of research design, implementation and thematic analysis to generate empirically grounded theories and frameworks for action (Cobb et al., 2003). For the main cycle, an aesthetically driven learning environment was designed, in order to cultivate students' PCR while they explored and created animations of periodically varying geometric magnitudes (distance/angle). In this paper we analyse a part of the main cycle, in which eight 11th grade students (S1-S8) participated in their school's computer lab. Empirical data included transcripts of student dialogues, videotaping of their digital media activity, written notes and body expressions, captured by the attending researcher's notes. The theoretical elements analyzed above framed the analysis of the data in terms of different images of PCR guided by mathematical and artistic aesthetic criteria. The unit of analysis was the 'critical episode', that is, a set of central actions that resulted in a significant change in the learning process.

Microworlds and tasks



Figure 1: The designed resources: a) & d) the GeoGebra microworlds; b) the activity sheet for each song; c) & e) a MaLT2 microworld (out of the 3). The microworlds are presented in video-link1 (for 1st phase) and video-link2 (for 2nd phase) (they can also be viewed by scanning the QR codes).

In this paper, two phases of three-hour-activities are presented. Eight microworlds were designed in GeoGebra ([https:// www.geogebra.org](https://www.geogebra.org)) and MaLT2 (<http://etl.ppp.uoa.gr/malt2>). Students were divided into groups of two and engaged in the creation of 'digital figural dances' for eight (four per phase) given songs. For each song, they had six rounds in order to: 1) choose a function of the form $f(t)=asin(\beta t)/f(t)=acos(\beta t)$ (for Phase 1) and $f(t)=a_1sin(t)+b_1cos(t)+a_2sin(2t)+b_2cos(2t)+a_3sin(3t)+b_3cos(3t)+a_4sin(4t)+b_4cos(4t)$ (for Phase 2), based on the graph they constructed in the GeoGebra, by adjusting the values of the parameters through the sliders (Figure 1a, 1d); 2) note down the formula on the activity sheet (Figure 1b); and 3) test the periodic motion of a figure in one of the three microworlds of MaLT2, that moves according to the chosen function in three different ways respectively (as seen in the videos of Figure 1), through dragging the slider corresponding to the variable t at a constant rate (Figure 1c, 1e). Slider-dragging represented the "conceptual time," manipulated by the students by pressing the right arrow on the keyboard (for constant, continuous increment) or the mouse (for non-constant change in any direction). The goal was until the 6th attempt to find the appropriate function for the corresponding song. A third phase, not analysed in this paper, included students' free creation of two synchronized figural animations from scratch, matched to a song of their willing, by programming in MaLT2.

Results

Table 1: Images of students' periodic covariation

Types of PCR	Mathematical aesthetic criteria <i>The [...] should be/ look like...</i>	Artistic aesthetic criteria <i>...in order for the animation to...</i>
1. Non-steady periodic covariation	-	<ul style="list-style-type: none"> • match the non-steady rhythm of the main melody
2. Steady periodic covariation	<ul style="list-style-type: none"> • the number of peaks of the graph should be the same as/half the number of the song's beats 	<ul style="list-style-type: none"> • match the steady rhythm of the background tempo-beat
3. Amplitude and range of covariation	<ul style="list-style-type: none"> • the movement should be bigger/smaller • the graph should be higher • the value of parameter a should be higher/lower /a specific number 	<ul style="list-style-type: none"> • match the song's energy/style (e.g., intense or calm) • match the higher/lower musical notes
4. Amplitude-dependent period	<ul style="list-style-type: none"> • the movement should be quicker/slower • the graph should be shorter/taller • the value of parameter a (Ph1) should be lower/ higher/a specific number 	<ul style="list-style-type: none"> • match one of the song's tempo • match the song's energy/style (e.g., capture the anxiety/calmness)
5. Amplitude-independent period	<ul style="list-style-type: none"> • the movement should be quicker/slower • the graph should be thicker/more sparse • the value of parameter β (Phase 1) should be higher/lower/a specific number 	<ul style="list-style-type: none"> • match one of the song's tempo • match the song's energy/style (e.g., capture the anxiety/calmness)
6. Rate of change	<ul style="list-style-type: none"> • the movement should be steeper/smooth • the graph should look steeper/smooth • the parameters values should be higher/lower (Ph1)/ follow a relation (Ph2) 	<ul style="list-style-type: none"> • to match the song's energy/style (e.g., capture the anxiety/calmness)
7. Direction of change of time-geometrical magnitude	<ul style="list-style-type: none"> • the movement should first/in this interval be positive, then negative, then.. • the graph should go up, then down, then... • the movement should be symmetrical in time • the graph should be symmetrical around the x-axis • the parameters' values should have a specific relation 	<ul style="list-style-type: none"> • match the song's musical notes • capture the song's tension or release parts • match the symmetry of the notes
8. Duration and density	<ul style="list-style-type: none"> • the input variable t domain's end limit should correspond to a specific number • the input variable t values should be added by a specific number 	<ul style="list-style-type: none"> • match the song's duration • be adjusted to the song's period
9. Phase shift	<ul style="list-style-type: none"> • The movement should be transferred in time • The graph should be horizontally moved 	<ul style="list-style-type: none"> • match the song's starting tone • follow the second animation

In Table 1 nine different types of students' PCR are listed and connected to specific mathematical and artistic aesthetic criteria, as emerged from the data. They involve different ways of students' thinking about specific aspects of periodic covariation. In the next paragraphs, some examples where some of the above types of PCR were developed are given.

Catching the musical rhythm; from non-steady to steady periodic covariation

In the first phase, students' artistic aesthetic criteria focused mainly on matching the motion to the music rhythm. One group (S5, S6), in their attempt to find the appropriate function for the song "The pink panther theme", went through several phases of redefining aesthetic criteria on the trigonometric functions. During the first rounds, students argued for which graph fits better the beat of the song and developed non-steady PCR. In the 2nd round, S5 claimed that the graph of $f(t)=22\sin(t)$ fits better than the one of $f(t)=40\sin(2t)$, which was tested in the previous round. S5 moved the mouse along the graph according to the varying rhythm of the melody, as shown in the video in Figure 2a. Her motion indicates that she visioned equivalent parts of different graph periods to correspond to different speed. By testing the covariation-motion in MaLT2 (Figure 2b), S5 revised her hypothesis:

- S5: No, no. I was wrong. It seems even slower.
 S6: We need to catch the steady tempo from the background. This tountoun-tountoun.



Figure 2: Videos of S5 and S6 actions in microworlds (sound on): a) chunky PCR in GeoGebra ([link](#)); b) testing the function $22\sin(t)$ in MaLT2 microworld 1 ([link](#)); c) smooth PCR in GeoGebra ([link](#)).

S6 turned to GeoGebra and set the parameters values, choosing the function $f(t)=20\sin(3t)$. Then she moved the mouse across the graph at a constant rate, as shown in the video in Figure 2c.

- S5: You're not moving it right. We want to catch the main melody.
 S6: We have to get the steady rhythm. There is no point in catching the main melody.

S6's movement shows that she thought of periodic covariation in a steady way. After evaluating the aesthetic outcome in MaLT2, she redefined her aesthetic criteria regarding the shape of the graph in order to make it follow the constant background rhythm.

From amplitude-dependent to amplitude-independent period of covariation

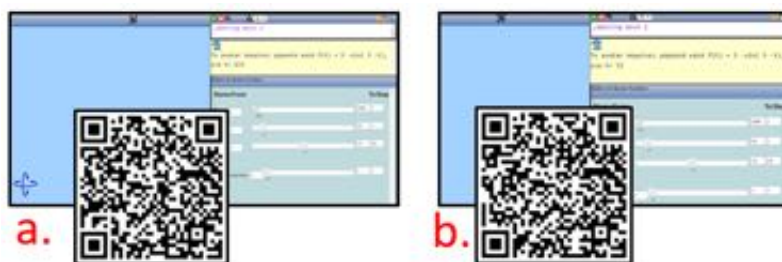


Figure 3: Videos of S5 and S6 actions in MaLT2 (sound on): a) testing the function $5\sin(5t)$ in microworld 1 ([link](#)); b) investigating the period in microworld 1 ([link](#)).

S5 and S6 tested $f(t)=45\sin(5.5t)$ and $f(t)=40\sin(5t)$ in the 4th and 5th rounds of the 1st song respectively in MaLT2. In round 6 they discussed the role of the parameters by observing the graph:

- S6: We want a bit faster movement, a shorter period. The period changes with beta.
S5: Yes but 5.5 was fast and 5 was slower for the tempo. Should we decrease alpha?
The move range would be smaller, so it will go faster.

In this episode, S5 thought of the covariation period as being depended on the value of the parameter a . When S6 set $\alpha=5$, choosing $f(t)=5\sin(5t)$ and tested in MaLT2 microworld 1 (Figure 3a). This was followed by the aesthetic evaluation of the movement, accompanied by reflection:

- S5: Again, it seems just as slow to me.
S6: Oh yes. Listen, look. It has the same period as before.

S6 slowly dragged the bar of the variable t from 0 to 72. Then she changed the value of the parameter α from 5 to 50 and moved the bar of the variable t from 0 to 72 (Figure 3b).

- S6: You see? It's exactly the same period as before; 72 units.
S5: How is it possible? The movement is smaller.
S6: Yes, but they happen at the same time. Same duration of moving up and down.

S6 used the t slider to confirm if functions with the same β and different a values have the same period. Her PCR involved proving that the covariation period doesn't depend on the amplitude.

Transferring the melody from imagination to direction of change

In the second phase, all groups tested and explored complex functions, which allowed them to set more complex aesthetic goals that combined both the rhythm and the melody of the song. S1 and S2, during the function quest for the third song 'Tango', first drew it on paper, based on their imagination:

- S1: We can't do it like that, randomly. We have to think first how we want it.
S2: Fine, bring your notebook and I'll draw it. (*He draws 4 graphs on the paper, 3 of which he erases, as shown in Figure 4a*). That's how I imagine it. (*He sings the melody and makes hand motions.*) It'll start on one side and chuck-chuck will go this way to the other side, following the notes.

S2 expressed aesthetic criteria regarding the direction of change of the time-turn relation, which was graphically visioned and represented in paper, in order to connect the movement to the musical notes. After exploring combinations of parameter values to achieve the function that fits her design, he concluded to the function $f(t)=22\sin(t)+5\cos(t)+14\sin(2t)+4\cos(2t)+11\sin(3t)+10\sin(4t)$.

- S1: Aah this is it! It's very close. Set all cosines to zero.
S2: Oh, you think? (*S2 zeroes out the parameters b_1, b_2 , plotting the graph of $f(t)=22\sin(t)+14\sin(2t)+11\sin(3t)+10\sin(4t)$ – Figure 4b.*) Yes! The sum of sines does the steps exactly as I thought. Let's see the movement!

S2 was excited for finding the type of the function, which he had previously expressed in a static (paper) and dynamic (hand movement) way, drawing from his own imagination. His conclusion about the role of the parameters a_i in the form of the graph indicates that he engaged in 2nd order CR of the periodic function in order to achieve his goal. However, when testing the movement in MaLT2 microworld 2 (Figure 4c), the students had mixed feelings and were led to redefine their aesthetic goals and further deepen the PCR, as shown in the next subsection.

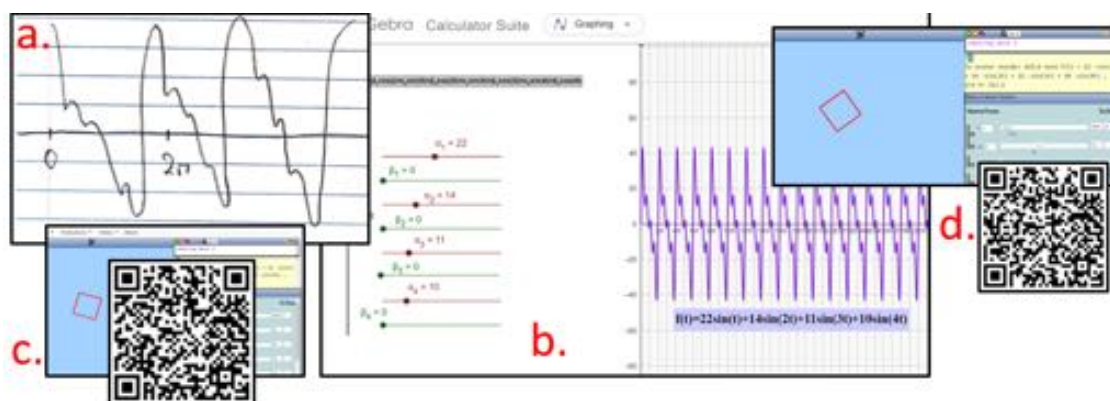


Figure 4: a) S2's drawing of graph in paper; b) finding the formula in GeoGebra; c) & d) movement in MaLT2 microworld 2, by setting $step=1$ ([link](#)) and $step=2,9$ ([link](#)) (play videos with sound on)

Matching melody and rhythm by focusing on the density of domain of the variable t

- S1: Wait. It's slow but it's exactly the right move. If we just make it move faster?
 S2: How; We should move the slider faster.
 S1: Ok Look. (Points to the graph of Figure 6a. Marks the values 0 and 2π on the horizontal axis.) Now here is 2π . Right; It takes 360 moves of t to reach 2π . But if we set $step=2$, it will jump half the moves, so it will make half the time.

While expressing ideas, they envisioned the half input points to produce the same movement at half time. S1 then changed the step value of t from 1 to 2 (so t takes values that are multiples of 2). After three trials in MaLT2, they ended up to set $step=2.9$ (Figure 4d). In this episode they focused on the density of the t -domain in order to adjust the melody-based movement to the song's rhythm.

Discussion

Students used nine types of PCR while engaging with the digital tools in order to meet a variety of both artistic and mathematical aesthetic criteria. Features of covariation such as steady periodicity, amplitude-independent period and direction of change, were used as subjective aesthetic criteria, interwoven with criteria derived from listening to music (rhythm, melody, emotion). The continuous flow of redefinition of aesthetic goals determined the flow of CR: the appropriate function was not that which obeyed some external factors (as in a physics experiment), but those that came from the musical sensibilities of the students. Combining plotting in GeoGebra with dynamic representations of MaLT2 was instrumental in making sense of aspects of periodic covariation. 2nd order covariation was evident, as students gradually discriminated the role of parameters and relationships among them. Students improved their prediction practices, as in the last songs they found functions matching their ideas within the first rounds. Next future steps involve the integration of motion capture technology in order to trace students' embodied engagement as part of their PCR within an aesthetically driven activity that combines music and dance.

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Technology-based vs. paper-pencil: Mode effects in large-scale assessments

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As many large-scale tests nowadays use technology-based tests (TBA), their comparability with traditional paper-and-pencil tests (PPA) gains importance. In this study, we investigate the presence and sources of mode effects by comparing the TBA and PPA versions of the German comparative test VERA and its results for Berlin students. Differential item functioning (DIF) analyses revealed significant cross-mode differences in item difficulty for 45 of 80 test items. To investigate possible sources of mode effects, those 16 items that showed particularly large differences were qualitatively analysed concerning their formal-technical and content-related characteristics. Our results indicate that the specific design and response format of TBA and PPA items as well as the required competencies can cause mode effects. This emphasises the importance of carefully designing technology-based assessments, especially mathematics tests, to ensure the comparability of results.

Keywords: mode effect, online assessment, large-scale, technology-based assessment.

Introduction

In many areas, there is a clear trend towards technology-based assessment (TBA), whereby competencies are measured using specific technical tools instead of traditional paper-and-pencil assessments (PPA). This trend is not only recognisable in international comparative studies such as PISA and TIMSS (Goldhammer et al., 2019; Robitzsch et al., 2020), but also manifests itself in the use of TBA in German large-scale assessment. The change from PPA to TBA has several aims: First, it aims to reduce the effort involved in distributing and scoring the tests and thus save resources. Second, it also enables the measurement of competencies related to media use that cannot be recorded using conventional PPA procedures. Third, TBA provides opportunities for an adaptive design of tests. Besides these advantages, the question arises as to what extent TBA is comparable to PPA in terms of difficulty, validity, and construct equivalence. In both PISA 2015 and TIMSS 2019, for example, mode effects were demonstrated in the sense that the items in TBA tended to be more difficult than in PPA (Fishbein et al., 2018; Jerrim et al., 2018). However, the results on mode effects are ambiguous, partly test-specific, and often only consider the global test level. In this study, we therefore investigate the cross-mode comparability of a German nationwide standardized mathematics test for 8th graders (13-14 years old) called VERA (VERgleichsArbeiten [comparative tests]) on the item level. The test design is based on the national educational standards, which include general mathematical competencies and basic concepts (i.e. contents). As VERA is an annual assessment and is constantly being further developed, in recent years, efforts have been made to provide an equivalent technology-based version of the comparison test. To this end, a modular online test system was set up and mathematics items were transformed into a technology-based version using different digital response formats (e.g. click on solution, input fields) or GeoGebra. A complete changeover from PPA to TBA is planned but has not yet been fully implemented. In the federal state of Berlin, for example, there were several years in which the same items were administered either as

PPA or TBA. Based on these two test versions, the present study investigates the presence of mode effects and aims to identify biased items that can provide information about formal-technical and content-related item characteristics that generate mode effects when switching from PPA to TBA.

Mode effects between TBA and PPA

The term *mode effect* “refers to the underlying idea that the test administration has a *causal effect on*, for example, estimated competence” (Kröhne & Martens, 2011, p. 174). Sources for mode effects can be manifold. For example, mode effects can depend on various formal-technical or content-related item characteristics as well as on individual characteristics of the test participants such as gender or familiarity with the technology (Jerrim et al., 2018; Kingston, 2008). In this study, we focus on the test design in TBA and PPA and thus on item characteristics.

Mode effects in large-scale assessments

As there is a general trend to provide large-scale tests as technology-based assessments, there are several studies on the mode effects between PPA and TBA in the domains of reading comprehension, mathematics, and science. Overall, previous studies reported that the TBA versions tend to be more difficult compared to PPA, not only in the language domain (Dahan Golan et al., 2018; Støle et al., 2020; Wagner et al., 2021) but also in mathematics or science (Fishbein et al., 2018; Robitzsch et al., 2017, 2020). However, the results are not consistent and some studies have shown no impact of test administration on reading comprehension (Margolin et al., 2013; Wang et al., 2008) or mathematics (Kim & Huynh, 2007).

In terms of known large-scale assessments, previous research with field trial data indicates mode effects for PISA 2015, the first year of TBA in PISA, for reading, science, and mathematics. However, lower mode effects were found in mathematics than in reading literacy (Jerrim et al., 2018). Similarly, mode effect studies based on PISA 2018 and TIMSS 2019 suggest that PPA and TBA measure the same construct, even if the technology-based items were on average slightly more difficult than the paper-based items (Fishbein et al., 2018; Goldhammer et al., 2019). However, these studies mainly focus on the presence of mode effects and construct equivalence. To our knowledge, there is little mathematics-specific research on mechanisms that may cause these mode effects.

Formal-technical and mathematics-related item characteristics related to mode effects

The transfer of PPA items to a technology-based test environment involves design decisions that can influence test takers' perception and processing. To discuss possible sources of mode effects, we therefore turn to aspects of test design and distinguish between formal-technical and content-related item characteristics. *Formal-technical characteristics* refer to the general conditions for test administration and the technical implementation of a PPA item. Thus, formal-technical characteristics include aspects such as the layout of the items, the response format, the input device used, the number of items per page, and the navigation within the items (Kröhne & Martens, 2011). When analysing mode effects in reading tests, Buerger et al. (2019) showed that TBA and PPA items can differ in difficulty depending on the response format and layout. While the need for navigation and the use of combo or drop-down fields increased the difficulty of the items in TBA, there were no differences for multiple-choice items. In mathematics, formal-technical characteristics also include the integration of digital tools such as GeoGebra, which can be used to transfer pencil and ruler drawings into the technology-based test environment. This change may alter the way test takers interact with

visual mathematical representations (Sedig & Sumner, 2006). For example, GeoGebra allows test takers to rearrange elements of their construction over and over again to approach the solution.

To our knowledge, there are hardly any studies on *content-related characteristics* that are associated with mode effects. In the linguistic domain, Wagner et al. (2021) found greater mode effects in orthography than in reading. However, this finding could be due to different instructional formats in these two domains and thus be related to formal-technical aspects. It therefore remains to be seen to what extent specific mathematical subject areas such as geometry or algebra or competencies such as reasoning or using representations are more closely related to mode effects.

The current study

The research situation suggests that there are diverse and, in particular, subject-specific influences on mode effects in international as well as national large-scale comparative tests. In this study, we examine the cross-mode comparability of the German comparative mathematics test (VERA test). By comparing the technology-based version with the paper-based version, we aim to identify biased items that can give us information about item characteristics that cause mode effects. Although our analyses are based on a specific test, the study aims to reflect on design choices for technology-based test items and to reveal generalisable findings about sources of mode effects. More specifically, we pose the following research questions:

RQ1: To what extent do the item difficulties in the German comparative tests (VERA tests) in Berlin differ significantly between technology-based (TBA) and paper-and-pencil assessment (PPA)?

RQ2: In which formal-technical and content-related item characteristics do biased items differ qualitatively?

Material and Method

To analyse mode effects in standardised tests, we turned to the PPA and TBA versions of the German national comparison test in 2022 (VERA) and its results in the federal state of Berlin. The mathematics test contained 80 items divided into four test booklets. All over Berlin, 23.388 students attending different public schools and school forms processed the test in a 60-minute time frame in the period from the 24th of February to the 15th of March 2022. In 2022, each participating class could choose independently between the PPA and the TBA version of the test. As a result, 6.785 students took the paper-pencil version of the test, while 16.488 students took its technology-based version. For both groups, teachers rated their students' responses as incorrect (0), correct (1) or missing (2). Because there was no time pressure, missing responses were attributed to a lack of motivation or a lack of ability. Thus, missing responses were recoded and rated as incorrect by the authors.

To examine the cross-mode comparability of the PPA and TBA test versions and identify biased items, we analysed students' test results in terms of Differential Item Functioning (DIF). To this end, the empirical data were scaled using the one-parameter IRT Rasch model and the R-package *eRm* (Mair et al., 2023). Based on the students' responses, item parameters are estimated in the Rasch model which indicates the difficulty of an item independently of the number of students that actually worked on it. Estimating item parameters for mode-specific subgroups, we compared the individual item difficulties (given in logit) of PPA and TBA versions. As a global test for DIF, we performed an Andersen's Likelihood Ratio test. To identify single items with mode effects, we used a Wald test

revealing invariances within 16 items depending on the grouping variable. Given the biased items, we analysed these items in terms of their formal-technical and mathematics characteristics. Based on previous research, we developed a coding scheme capturing the item characteristics of the subject area (*geometry, algebra, arithmetic, stochastics, functions*), necessary competencies (*reasoning, problem-solving, modelling, using representations, operating, communicating*), response formats (*short answer, long answer, multiple choice, single choice, arrangement, construction*) and tool usage (*yes, if GeoGebra is implemented in the TBA version; otherwise no*). Following the principles of Mayrings' (2014) structuring content analysis, each item was analysed according to this coding scheme. In addition, we open-coded the relevant mathematical objects (e.g. *fractions, polygons*) and the item design (e.g. *input field, squared paper*). As there may be differences in the item design in the TBA and PPA versions, two codes may have been assigned here.

Results

We report the results according to the research questions and start with an overview of the quantitative results before moving on to the qualitative item characteristics.

Mode effects in terms of Differential Item Functioning

Anderson's likelihood ratio test confirms the presence of biased items within the test and thus a differential item functionality ($p < .001$). The Wald test shows significant differences in item difficulty for 45 of 80 items ($p < .05$). Figure 1 provides a graphical overview of these differences. Since the sample size is very large and the Wald test quickly becomes significant, we have marked the items with a large difference ($\geq .05$ logits) in blue and selected these 16 items for a deeper analysis.

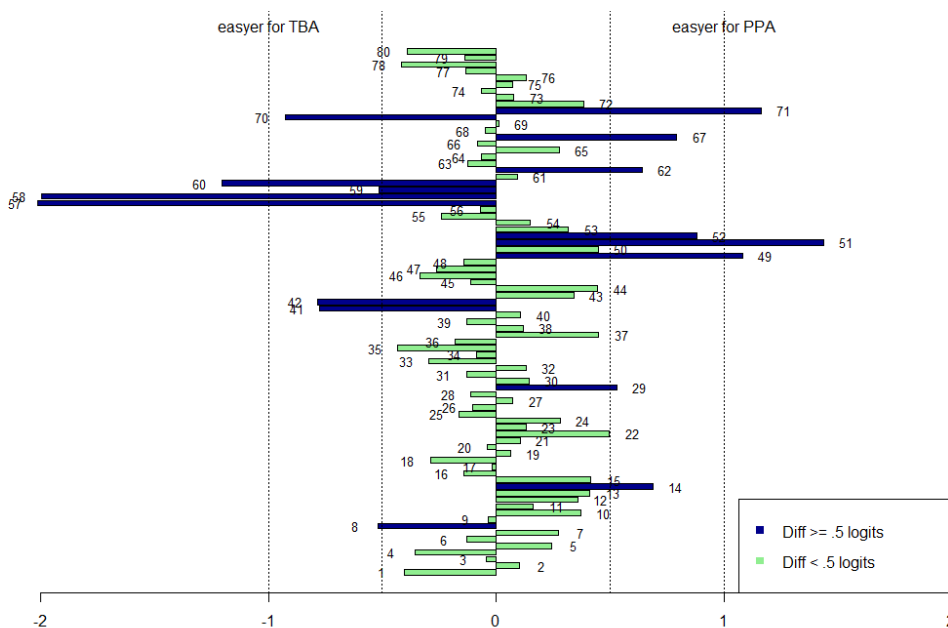


Figure 1: Differences in item difficulty (TBA-PPA), with negative values indicating an advantage for TBA and positive values for PPA

Formal-technical and mathematics-related item characteristics related to mode effects

Table 1 shows the coding results concerning formal-technical and content-related item characteristics for the items that were easier in the TBA version. Six out of eight items relate to Geometry and almost all items require competencies in dealing with representations. For three items, the TBA and PPA

versions differ in their design in that the TBA versions are more pre-structured (41, 42, 58). For two items, the TBA version provides input help that can go beyond a technical hint and also contains content-related information (8, 41). Since the different item designs also contain different information, these design choices may lead to the mode effects. The other four items do not appear to differ in the formal-technical characteristics considered, indicating that Geometry items are sometimes easier to solve in a technology-based assessment.

Table 1: Characteristics of items with a lower item difficulty in TBA

Item	Mathematical characteristics			Formal-technical characteristics		
	Subject area	Competencies	Content	Response format	Design	Tool use
8	Stochastics	Modelling, Operating	Probability	Short answer (fraction)	input field TBA: Input aid „/“	no
41	Geometry	Problem-solving, Using representations, Operating	Points in the coordinate system	Construction (point)	empty coordinate system PPA: incomplete axis labelling TBA: complete axis labelling, TBA: Tip of drawing guidelines	yes
42	Geometry	Using representations, Operating	Angles	Short answer (number) + Construction (figure)	Input field; PPA: squared paper TBA: coordinate system	yes
57	Geometry	Using representations,	Symmetry	Construction (line)	Symmetrical image without structuring lines	yes
58	Geometry	Using representations, Operating	Polygons	Construction (figure)	PPA: empty field TBA: squared field	yes
59	Geometry	Using representations	Reflections	Short answer (numbers)	One field for each point coordinate searched for	no
60	Geometry	Using representations	Reflections	Short answer (numbers)	One field for each point coordinate searched for	no
70	Arithmetic	Using representations	Fractions	Single Choice	Figures on squared paper	no

Table 2 shows the coding results in terms of formal-technical and content-related item characteristics for the eight items that were more difficult in the TBA version. The item characteristics are quite diverse. For two items (62, 71), the TBA and PPA versions have different designs that could change the requirements of the item. One item (67) required a short answer with units and thus the use of special characters such as % or €, which can cause difficulties in the TBA version. Overall, most of the items require multi-step solutions in the sense of several problem-solving steps or calculations.

Table 2: Characteristics of items with a higher item difficulty in TBA

Item	Mathematical characteristics			Formal-technical characteristics		
	Subject area	Competencies	Content	Response format	Design	Tool use

14	Algebra	Operating	Terms and equations	Short answer (number)	Input field	no
29	Stochastics	Modelling, Operating	Probability	Long answer (justification)	Input field TBA: input aid “/”	no
49	Geometry	Using representations	Symmetry	Construction (lines)	Squared field	yes
51	Geometry	Problem-solving, Using representations	Translations	Short answer (numbers)	One field for each point coordinate searched for	yes
52	Geometry	Problem-solving, Using representations	Translations	Short answer (numbers)	One field for each point coordinate searched for	yes
62	Geometry	Reasoning, Problem-solving, Using representations, Operating	Surfaces	Arrangement (order of arguments)	PPA: draw connecting lines TBA: enter numbers in fields	no
67	Arithmetic	Problem-solving, Using representations, Operating, Communicating	Percent	Short answer (numbers with units)	Six fields, units not specified	no
71	Arithmetic	Using Representations	Fractions	Surface to be coloured	PPA: squared paper TBA: colouring parts of the object by clicking	yes

Discussion

As part of the ongoing development of the German comparative test VERA, we had the opportunity in this study to compare a technology-based and paper-and-pencil administration of the same test. DIF analyses revealed the presence of mode effects in the form of divergent item difficulties for 45 out of 80 items. However, there is no clear direction of effects, as some items are easier in the TBA and others are easier in the PPA. This finding differs from previous studies, in which TBA items were on average more difficult (Fishbein et al., 2018; Robitzsch et al., 2017, 2020). Such previous findings could have been a reason for the very careful implementation of TBA items in VERA, which include technical aids.

By analysing 16 items that showed particularly large differences, we were able to identify differences in the formal-technical item characteristics of some items, such as the response format or the specific item design, which could explain mode effects. However, formal-technical and content-related features appear to interact in some cases because design features can also carry content-related information and thus reduce mathematical complexity. This applies in particular to input aids as well as to pre-structured input or construction fields. This is in line with the results of previous studies on the effect of modes in the language domain (Buerger et al., 2019; Wagner et al., 2021). In other cases, the TBA and PPA versions of an item do not appear to differ in terms of the formal-technical characteristics. This could be due to the limited number of characteristics considered. However, there seems to be a tendency that geometry items centred on the use of representations are easier in the TBA version and that multi-step solutions related to problem-solving and operating are easier in the PPA version. Further research should investigate the mechanisms behind these observations. In this context, it would be worth discussing whether the representations in the TBA version are clearer and

possibly easier to manipulate. It is likely that the ability to interact with visual representations in GeoGebra changes the test taker's approach to solving a task and reduces complexity or increases precision (Sedig & Sumner, 2006). On the other hand, the importance of taking notes or drawing auxiliary lines for multi-step solutions, which is much easier when using paper and pencil, should be considered. Mathematical notes and drawings can be specific to the subject of mathematics and therefore result in different mode effects than in other subjects (Wagner et al., 2021). It could therefore be informative to examine the interaction with the available material in more detail. Furthermore, mode effects may also be caused by the characteristics of the test takers, particularly their familiarity with technology-based learning (Jerrim et al., 2018; Kingston, 2008). Therefore, our results may reflect the choice of classes for TBA or PPA assessment, which should be investigated in future studies.

Although our analyses are based on a specific test, the study provides general insight into potential sources of mode effects for mathematics assessments. This is particularly interesting if a PPA and a TBA version are to be used simultaneously. In this case, subject-specific tools must be developed to ensure that the test items, especially in geometry, are comparable in terms of item difficulty.

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Key aspects of teacher practices in teaching mathematics with programming tools in Norwegian schools

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The aim of this research is to explore teachers' practices in teaching mathematics with programming tools from the lens of Pedagogical Technology Knowledge (PTK) framework. The data comes from five individual interviews with Norwegian teachers working in different grades. The data was analyzed using a theory-driven approach through coding and categorization under the mathematical knowledge for teaching (with technology) (MKTwT) component of the PTK framework. The analysis revealed three key aspects in teachers' practices related to MKTwT: (i) different interpretations of curriculum content linking mathematics and programming, (ii) different teaching approaches for programming practices, and (iii) uncertainty of assessment. These aspects were found to be influential on teachers' multifaceted pedagogical choices towards programming practices in schools.

Keywords: Teacher practice, programming, pedagogical technology knowledge.

Introduction

Computational Thinking (CT) refers to a number of interrelated aspects related to computing such as algorithmic thinking, decomposition, pattern recognition and generalization (Kallia et al., 2021). CT has become an integral component of the Norwegian mathematics curriculum since its adoption in 2020. This inclusion reflects the growing recognition of the importance of CT and programming, as in other Nordic countries (e.g., Kilhamn et al., 2021; Pörn et al., 2021). While CT is mentioned only briefly in the (new) Norwegian mathematics curriculum (Utdanningsdirektoret, 2019), it emphasizes the development of strategies and systematic solutions for problems with or without digital tools. For instance, pupils (students) are encouraged to break down problems into subproblems and to evaluate the employed procedures. Additionally, programming-related competencies are unpacked in each grade's mathematics curriculum objectives, mainly starting from grade 4. However, the implementation of CT and programming practices in classrooms is met with diverse interpretations and understandings among teachers regarding key CT components, such as decomposition, algorithms and pattern recognition (Nordby et al., 2022). Nordby et al. (2022) also highlight two approaches to CT adoption; integrating CT and programming into existing curricula, and treating them as new elements through classroom practices, sometimes neglecting mathematics when they completely focus on programming. The latter does not seem surprising, as pre-service and in-service teachers in Nordic countries often struggle to link CT/programming and mathematics and they do not feel prepared to teach, despite having positive views about curricular approach (Kilhamn et al., 2021; Misfeldt et al., 2022; Pörn et al., 2021). In this research, our objective is to explore such a complex dynamic involved in integrating programming practices into mathematics education in Norwegian classrooms (by interviewing teachers). We seek to identify and delineate the key aspects of teacher practices from the lens of pedagogical technology knowledge (framework), which will be detailed in the next section.

Theoretical framework

Pedagogical Technology Knowledge (PTK) framework has been proposed by Thomas and colleagues (Thomas & Hong, 2014; Thomas & Palmer, 2014). It encapsulates teachers' professional knowledge aimed to incorporate tools and pedagogical choices/strategies/principles into teaching mathematics with technology including assessment, where the beliefs about the technology and its use in learning mathematics have a key role. The PTK has three components: mathematical knowledge for teaching with technology (MKTwT), technology instrumental genesis, and personal orientations. Figure 1 summarizes the interrelationships among the three components.

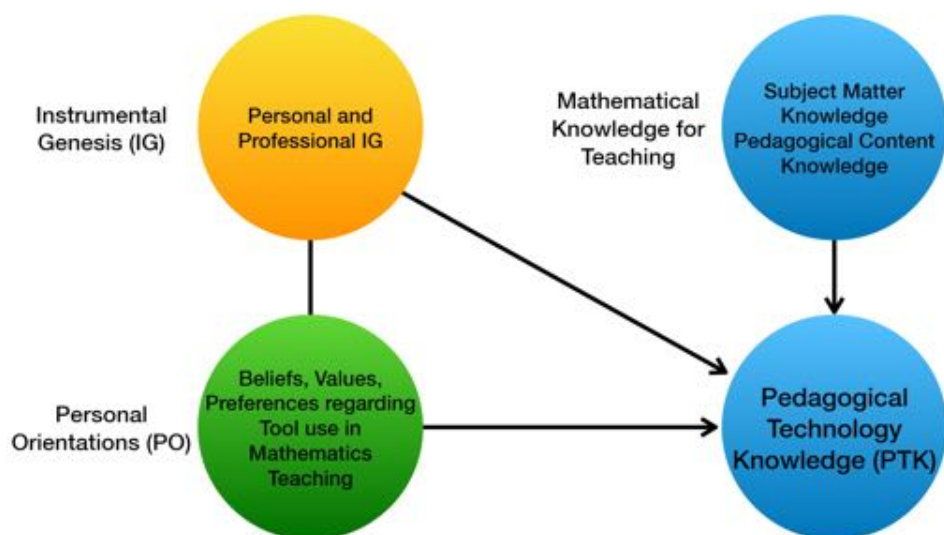


Figure 1. An overview of PTK (adopted from Thomas and Palmer, 2014, p. 76)

The MKTwT component draws from seminal works Ball et al. (2008) and Shulman (1986), representing an amalgamation of subject matter knowledge and pedagogical content knowledge tailored for mathematics teaching. Specifically, MKTwT pertains to teachers' understanding of teaching mathematics, with a specialized focus on leveraging technology to facilitate student learning within the classroom. This entails proficiency in employing various digital tools, emphasizing the development of utilization schemes to promote learning mathematics. Consequently, the framework's second component is technology instrumental genesis (IG), which refers to the development of utilization schemes and transformation of tools into parts of instruments as psychological entities (Artigue, 2002). However, an interpretive relationship exists between a teacher's (own) utilization schemes for a specific tool and those employed for teaching practice. The former is termed as personal IG, while the latter is referred to as professional IG (Haspekian, 2014).

In PTK, the third component, personal orientations (PO), encompass teacher orientations, values and preferences, such as the reasons for personal choices and decisions in referring to technology while designing classroom activities. Inductively, PO is interconnected with both personal and professional IG, as well as MKTwT. For instance, if a teacher exhibits a strong inclination towards programming tools, their IG becomes intertwined with other aspects of MKTwT, such as curricular knowledge and assessment, thereby shaping their teaching practices. We note that PO is also interconnected to other parameters too, such as school and department support and openness to personal learning (Thomas & Palmer, 2014). Why we decided on PTK is based on its mathematical foundations, with IG and PO

perspectives (Clark-Wilson & Hoyles, 2019). Here we note that all three components are strongly interconnected. However, for the sake of paper constraints, in this paper, we focus on the MKTwT component in teacher practices and consider the following research question: *what key aspects related to MKTwT are evident in the practices of Norwegian teachers when teaching mathematics with programming?*

Methods

This paper is written based on a master’s research project, which adopted a qualitative perspective with a phenomenological research method. Below (Table 1) is a brief introduction of the participants in the study, including details of their background, teaching experience and employed school type. Due to the anonymity of the participants, they have been assigned fictitious names: Mats, Bjørn, Kristoffer, Anders and Silje.

Table 1. A brief information regarding the participants

<i>Name and Age</i>	<i>Teaching Experience</i>	<i>Teacher Course in Programming</i>	<i>Employed School Type and Other Experiences</i>
Mats, 27	5	No	Primary School (1.-7. Grades). Experience with block programming and various learning platforms
Bjørn, 29	5	Yes	Lower Secondary School (8.-10. Grades). Experience with block programming and multiple physical programming tools
Kristoffer, 30	8	Yes	Primary School (1.-7. Grades). Experience with block and text programming. Have also used spreadsheets for simulations.
Anders, 27	5	Yes	Lower Secondary School (8.-10. Grades). Experience with block and text programming
Silje, 26	5	No	Primary School (1.-7. Grades). Experience with a software designed for teaching and learning computing on primary school level

The data was collected through semi-structured individual interviews (in Norwegian). Four out of five interviews were conducted digitally on Teams, while the fifth was conducted at the participant’s workplace. The structure of the interviews was divided into three; technological and pedagogical knowledge, reflecting on own experiences, and personal perceptions and orientations related to the use of programming in mathematics teaching. Exemplary interview questions were as follows:

- Do you utilize programming tools in mathematics instruction? For instance, Scratch, Python, Micro:bit, etc.
- How do you structure your lessons when incorporating programming into mathematics instruction? Why do you choose this approach?
- What are the major challenges associated with employing programming tools in mathematics instruction?

- How do you ascertain whether pupils comprehend the ideas or concepts you endeavor to impart through programming? Can you justify your choice of assessment methods?

All interviews were transcribed and analyzed thematically following coding and thematization steps. For example, when teachers reflected on programming-related curriculum objectives as being artificial. Then, this was coded as “programming becomes artificial”. In total 275 codes were developed and checked several times and finally thematized under the components of PTK framework. All the data was handled and stored according to the approval of Norwegian Centre for Research Data (NSD).

Findings

The analysis of the data revealed three major themes: (i) different interpretations of curriculum content linking mathematics and programming, (ii) different teaching approaches for programming practices, and (iii) uncertainty of assessment. Below, we present each theme.

Different interpretations of curriculum content

Three of the interviews revealed that the teachers had knowledge of curriculum content for the grades they teach, although some stated that some curriculum objectives may be difficult to interpret. This is expressed, among other things, in an interview with Bjørn, where he stated that the objectives regarding programming in mathematics can be experienced artificially in primary and lower secondary school:

Bjørn: (...) I often think it becomes very artificial in elementary and middle school because when I read the curriculum objectives, especially those in eighth grade, I don't read it as a math goal. I read it as a goal that says, "learn programming".

Bjørn interprets the curriculum objectives for programming in mathematics as something exploring programming and not learning mathematics. His description indicates that the intention of using programming in the learning of mathematics is not clearly stated in the curriculum objectives. Furthermore, he explains that if the intention is to understand mathematics better through programming, one must have in-depth knowledge of programming in the first place. A reality far from the starting point today, according to Bjørn. Anders has developed teaching plans in connection with his master's thesis, where he based his work on the competence goal for the sixth grade:

Anders: ... In my thesis, I had about geometric figures in the sixth grade, since that was the first competence goal there. And then I actually got a lot of good thoughts about different figures, angular sum and what a circle is. If they [pupils] were able to create it in Scratch, for example...

Anders shows that he knows what curriculum objectives describes and talks about a feeling of having managed to link programming directly to mathematics through geometry. Furthermore, it is expressed that other areas of mathematics have been more difficult to approach with programming. Silje tells from the second grade of view:

Silje: ... No, I guess the competence goal for the second grade is to follow and make rules and instructions or something, in games and activities and things like that. So that's kind of what we do, but maybe that's not exactly what you do in mathematics really. Then you get the rules and instructions delivered, then you do the task in a [same] way.

Silje also shows that she knows what the competence goal for the second grade she teaches implies, and that Silje does something similar in the teaching. She also expresses that what they do is not necessarily what you do in mathematics. The competence goal is about following and creating rules

and instructions, while in mathematics the pupils get it delivered, Silje claims. In sum, we see that teachers have insight into the curriculum content in mathematics, but that this opens a door to different interpretations of CT.

Different teaching approaches

The teachers talk about several different teaching approaches to the use of programming tools in mathematics. In the introductory phase, 4 out of 5 teachers say that they use the tools themselves, their pupils or activities on sheets to demonstrate in brief what programming is all about; that is, so called analog programming without a computer. Bjørn explains an activity he usually uses and what benefits it can have:

Bjørn: (...) You have a couple where one sees what I write on the blackboard and the other who doesn't see the blackboard. And then whoever sees the blackboard will try and explain what I draw on the blackboard to the one who has his back. And then that person should draw then. It also never works, because they end up with the sheet the wrong way or they are imprecise. (...) So it creates some great reflection conversations with why precise messages and instructions are important then. And that what you think is logical is not necessarily logical for the person you are working with.

Analog programming can illustrate the necessity of precise instructions in programming, according to Bjørn. Kristoffer is also keen to make the same point to his pupils, and says that he uses himself to illustrate it:

Kristoffer: (...) If I'm going to get into programming with a whole new class that has never used it before, then I'm using myself. Where I make a little trail in the classroom, they're going to give me commands on where I'm going so I can sit in that chair. And then they understand pretty quickly, "Wow, I have to exactly..." Or "I have to say exactly what I want him to do, if I say a little mistake then it will be completely wrong".

What Kristoffer says shows that his experience is that analog programming can make pupils see the necessity of precise instructions. When the pupils have gradually received an introduction to what programming is, several of the teachers say that they give the pupils tools they can explore on their own or in pairs. This is expressed, for example, in what Mats says:

Mats: (...) If you give those tools to do that, they're very self-sustaining. They are good at exploring for themselves. Especially when they sit in pairs, then they try out a lot of different things themselves as well. Then they realize, "Wow, I can program the lights on the car so they flash in different colors, or in rainbow patterns, or I can take every other such-and-such."

The pupils are introduced to how to use the Scratch program and exercises related to geometry by a booklet that Anders prepared. Furthermore, it appears that this is something he has spent a lot of time on, and which he has developed further after his master's work. The analysis work shows that teachers vary between different teaching strategies when they use programming and programming tools in mathematics teaching. Analogue programming is used by many people in the introduction phase. Furthermore, teachers plan exploratory work where the pupils are active on their own. Instruction sheet and exercise booklets are also used to help pupils become familiar with various features of programming tools.

Uncertainty of assessment

Regarding assessment, some found this difficult, while others said that they had not considered it much yet. Nevertheless, several come up with suggestions on how this can be done and what it might presuppose. Bjørn states that it is important to know pupils' backlog, what they know and what they don't know:

Bjørn: (...) It is hard. Especially when you don't know the kids. (...) When you know the kids, you'll be able to know more of it, just to relate to math, if I'm doing math classes, I can only look at the pupil, at their body language and facial expressions ... It's kind of the same in programming and really.

Bjørn believes he can assess how much they understand by reading body language and facial expressions. He has experienced this in mathematics teaching without programming but finds that it has transfer value to when programming is also used. In order to identify what they understand, Kristoffer, Anders and Bjørn are all concerned with getting the pupils to explain what they have done. As Kristoffer says, if they have an explanation problem, it is something that must be addressed. However, Mats chooses to see it from a different angle than the other teachers:

Mats: (...) It's exciting. It could have been a practical test. Get them to program something. Make a car drive through something. I don't know. Could have been a separate project where you programmed something. I know there are code samples like that.

Rather, Mats offers suggestions on how to assess the pupils and gain insight into what they know. It shows that he sees more possibilities for how it can be done, and he thinks it is an exciting topic. In sum, we see that good relationships are highlighted as a prerequisite for assessing what the pupils understand. Asking questions that get pupils to explain what they have done is also highlighted to gain insight into their thought process in connection with programming activities or tasks.

Conclusions and Discussion

Overall findings in this study indicate that the curriculum competencies that apply to programming in mathematics can be difficult to interpret. The fact that mathematics has been given the main responsibility for the teaching of programming indicates that the intention must be for the pupils to use programming tools to *learn* mathematics. Nevertheless, formulations of the curriculum objectives may indicate that the pupils should learn programming for programming's sake and not for the sake of mathematics, which is also highlighted by Nordby et al. (2022). In particular, the eighth-grade competence goal ("exploring how algorithms can be created, tested and improved using programming", Utdanningsdirektoratet, 2019) has been highlighted by one of the participants in this study as an example of a competence goal that can lead to confusion. It may be that the intention is not explicitly stated, or it may be that the freedom of method teachers have in the new curriculum is perceived as too great. Another point could be related to the term "*algoritmisk tenkning*" in Norwegian, which refers to CT. This could lead to different interpretations as well (Nordby et al., 2022). All these probably lead to the second and third aspects of teacher practices; different teaching approaches and uncertainty of assessment.

The second aspect, different teaching approaches, could be due to mathematics level teacher teach. Because primary school teachers tend to consider programming as a set of instructions related to logical thinking (Nordby et al., 2022; Pörn et al., 2021). Therefore, some teachers prefer to show their way of programming on the screen, while others involve pupils. This is connected to their interpretations of curriculum. Because The Norwegian Directorate for Education and Training (Utdanningsdirektoratet, 2021) argues that the curricula is there to be able to adapt instruction to all

pupils at all levels. Due to the latitude teachers are given, there are no lists of activities to be carried out or detailed overviews of knowledge content. The teachers therefore seem uncertain about where the focus should be, should it be on learning to use the tools, or should it be on what they can be used for? This finding can be seen in the context of what Kilhamn et al. (2021) point out in their study of Swedish teachers' programming activities in mathematics. There, learning what programming is and how it works is more prominent than the question of what programming can be used for and why. At this point, we note that teachers need professional development support, as raised by researchers (Kalas et al., 2022; Nordby et al., 2022). Two of the teachers say that they have identified a challenge that their pupils do not understand why the programs they create in the web version, which do not lead to immediate results on the physical device. Why this confusion arises can possibly be seen in the context of *transposition*, as characterized by Kallia et al. (2021). However, teachers underlined that, pupils have not been able to transfer their solutions or programs to the Micro:bit, for example. In other words, there has not been a transposition. Challenges like this, where the tool itself limits what pupils can achieve, can make it difficult to balance the time that should be spent on the tool and the time that should be spent on learning mathematics and assessment, which is the third obtained aspect in this study. It is inevitable that teachers and pupils will not spend time for (a formative) assessment. Indeed, this is a leading problem, even field experts challenge in assessment of CT (Kalas et al., 2022). Finally, we note that the research was limited to five teachers, in which we require larger groups to elaborate more on their PTK.

Notes

This paper is based on the first author's master's thesis (Kveseth, 2022) completed under the supervision of the second author.

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Designing a culturally relevant mathematical task using ChatGPT: Exploring elementary PSTs' perceptions, experiences, and prompts

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This study aims to explore preservice teachers' (PSTs) experiences with ChatGPT in designing a culturally relevant mathematical task for adding unlike fractions. This study analyzed 89 PSTs' perceptions of ChatGPT's usefulness, ease of use, overall experiences, prompts used, and mathematical task revisions made. Overall, PSTs responded positively for the perceived usefulness and perceived ease of use of ChatGPT. Particularly, they rated more positively for accomplishing the task quickly, easy to learn, and easy to use than improving the performance for teaching and learning, effectiveness of teaching and learning, easy to operate, easy to get, and interactions with ChatGPT. An in-depth analysis of PSTs' interactions with ChatGPT reveals varying levels of modification (e.g., superficial vs. meaningful mathematical changes) but also highlights a lack of critical engagement with ChatGPT's responses and the use of limited prompts.

Keywords: Technological Pedagogical Content Knowledge (TPACK), Technology Acceptance Model (TAM), ChatGPT 3.5, culturally relevant mathematical task, elementary preservice teachers

Introduction

In response to the emergence of new digital technology and the increasing importance of technology competence, researchers have identified the specialized knowledge needed for teaching with technology. This knowledge is conceptualized as Technological Pedagogical Content Knowledge (TPACK) and its domain is further specified into seven sub-domains: content knowledge (CK), pedagogical knowledge (PK), pedagogical content knowledge (PCK), technological knowledge (TK), technological content knowledge (TCK), technological pedagogical knowledge (TPK), and technological pedagogical content knowledge (TPACK) (Misha & Koehler, 2006). Despite the broad range of methodologies adopted in TPACK studies and the use of various TPACK instruments, concerns have been raised about the reliability and validity of these instruments and the lack of content-specific knowledge bases (Koehler et al., 2012). Another line of research focuses on teachers' perceptions and acceptance of technology. In a meta-analysis of 114 empirical Technology Acceptance Model (TAM) studies, Scherer et al. (2019) highlighted that TAM is the most commonly used model to describe technology acceptance but acknowledged that it does not specify the professional knowledge needed for teaching with technology. They also noted that "teachers will be likely to "accept" a new technology once they perceive it as relevant for specific didactical approaches within their subjects" (Scherer et al., 2019, p.14).

Given that these two important and powerful models —TPACK and TAM—have advanced our understanding of teachers' technology use but have been studied independently, this study aims to bridge the research gap. Over the last few years, we have experienced a rapid change and social demands of incorporating technology in teaching and learning. Especially, the emergence of generative Artificial Intelligence (AI) and the public release of ChatGPT in November 2022 have attracted more attentions from educators, both for the benefits of using AI and concerns about using

AI in teaching and learning. This paper explores how elementary preservice teachers (PSTs) use Artificial Intelligence (AI), specifically ChatGPT 3.5, to design a culturally relevant mathematical task. The research questions guiding this study are:

1. How do PSTs perceive the usefulness of ChatGPT in designing culturally relevant mathematical tasks?
2. How do PSTs perceive the ease of use of ChatGPT in designing culturally relevant mathematical tasks?
3. In what ways do PSTs revise their original mathematical tasks using ChatGPT in a culturally relevant way?
4. What experiences do PST have with ChatGPT to design culturally relevant mathematical tasks? What mathematical or pedagogical issues do they encounter?
5. What prompts do PSTs use to design culturally relevant mathematical tasks for ChatGPT?

Literature Review

In a comprehensive examination of ChatGPT's performance, Lo (2023) found variability of its performance across subject domains, from unsatisfactory in mathematics to outstanding in economics. This variation mirrors in the findings of Plevris et al. (2023), who investigated GenAI tools' effectiveness in solving mathematics and logic problem. They found that the GenAI tools produced accurate solutions for simpler problems, but their solutions for more complex problems were not reliable. They found that these GenAI tools produce inconsistent responses which provide conflicting answers when the same question was asked multiple times. These challenges align with the observations of Frieder et al. (2023), who found that ChatGPT's performance is significantly below that of an average mathematics graduate student, reflecting the broader challenges of large language models in multi-step reasoning tasks.

Amidst growing interest in utilizing ChatGPT for educational purposes, empirical studies remain somewhat limited. One exception is the randomized controlled trial study conducted by Küchenmann et al. (2023) to explore differences in the development of physics tasks between PSTs who developed tasks using ChatGPT 3.5 and PSTs who developed tasks based on a textbook. Their findings revealed that both groups produced high levels of task correctness but struggled with specifying the tasks sufficiently. PSTs who developed tasks using the textbook had better clarity and included more embedded questions, whereas PSTs using ChatGPT 3.5 had more difficulties with output quality. In analyzing the verbs used in the tasks, they also found that PSTs using ChatGPT 3.5 created more tasks of "understand" category (e.g., describe or explain), whereas PSTs using a textbook created more tasks of "apply" (e.g., calculate, draw, visualize, or make a diagram) category and "evaluate" category (e.g., reason, justify, why). Besides analyzing the quality of tasks developed by the two groups, they also found that PST had mixed perceptions of ChatGPT's usability, usefulness, and output (high for usability, neutral for usefulness, and low for the output quality). This study has a great contribution to understand the quality of tasks that PST created by using ChatGPT and PSTs' perceptions, but the sample size is relatively small (13 PSTs for the control group and 13 PSTs for the experimental group) and PSTs' actual interactions with ChatGPT are largely unknown.

While these studies offer valuable insights into the potential integration of ChatGTP into educational settings, they also highlight persistent challenges, including accuracy, reliability, ethical considerations, and the necessity for user training (Montenegro-Rueda et al., 2023). Given these findings, there is a clear need for further research to address how PSTs interact with ChatGPT. By investigating how PSTs use ChatGPT 3.5 to design culturally relevant mathematics tasks, this study contributes to a more comprehensive understanding of ChatGPT’s usability, affordances, and limitation in teacher preparation programs.

Methods

A total of 89 elementary PSTs at a large public university in the United States participated in this study. The participants were enrolled in a three-credit elementary mathematics methods course. PST were introduced a culturally relevant pedagogy and then were asked to create their own culturally relevant mathematical task for adding unlike fractions. Following the submission of their initial culturally relevant mathematical task, PSTs engaged in conversations with ChatGPT 3.5 to revise their initial culturally relevant mathematical task and screen-recorded their conversations with ChatGPT. They completed the adapted TAM survey (Davis, 1989) using a five-point Likert scale (1 indicates strongly disagree; 5 indicates strongly agree) and open-ended survey on their experiences with ChatGPT. Table 1 shows the detailed information about the adapted TAM survey questions for ChatGPT.

Table 1: Adapted TAM survey questions for ChatGPT

	Survey Questions
Perceived Usefulness	Q1. Using ChatGPT in my teaching and learning would enable me to accomplish task more quickly.
	Q2. Using ChatGPT would improve my performance of teaching and learning.
	Q3. Using ChatGPT in my teaching and learning would increase my productivity.
	Q4. Using ChatGPT would enhance my effectiveness on teaching and learning.
	Q5. Using ChatGPT would make it easier to do my teaching and learning.
	Q6. I would find ChatGPT useful in my teaching and learning.
Perceived Easy of Use	Q7. Learning to operate ChatGPT would be easy for me.
	Q8. I would find it easy to get ChatGPT to do what I want it to do.
	Q9. My interaction with ChatGPT would be clear and understandable.
	Q10. I would find ChatGPT would be clear and understandable.
	Q11. It would be easy for me to become skillful at using ChatGPT.
	Q12. I would find ChatGPT easy to use.

To answer the first two research questions, I first analyzed the average scores of 89 PSTs’ responses of each of the adapted TAM survey questions. After analyzing the mean, I recoded response

categories: recoding strongly disagree (1) and somewhat disagree (2) as negative and recoding somewhat agree (4) and strongly agree (5) as positive. After recoding, I analyzed the percentage of negative, neutral, and positive responses for each of the adapted TAM survey questions. To answer the remaining three research questions, I selected three PSTs (PST 19, PST 2, and PST 75) who responded very differently for the adapted TAM survey questions. PST 19 responded to the adapted TAM survey as mostly negative (average score=2), PST 2 responded to the adapted TAM survey as both positive and negative (average score=3.25), and PST 75 responded to the adapted TAM survey as all positive (average score=5). Even though this study is not intended to make any causal claim about how TPACK impacts on TAM, the selected three cases illustrate how their positive or negative experiences with ChatGPT might be associated with their prompts, mathematical knowledge, pedagogical knowledge, and technological knowledge by comparing their original mathematical task and revised mathematical task for adding unlike fractions, analyzing their prompts to ChatGPT, and reflections on ChatGPT’s responses.

Results

PSTs’ perceived usefulness of ChatGPT

The mean for the perceived usefulness of ChatGPT ranges from 3.81 (Q4) to 4.10 (Q1). PSTs agreed the most that ChatGPT would enable them to accomplish tasks in teaching and learning more quickly (mean=4.10), followed by the statement that ChatGPT is useful in teaching and learning (mean=4.06) and the statement that ChatGPT increases the productivity in teaching and learning (mean=4.01). As shown in Figure 1, 82% of PSTs perceived positively about the statement that ChatGPT would enable them to accomplish tasks in teaching and learning more quickly, whereas less than 70% of PSTs perceived positively about the statements that ChatGPT improves their performance of teaching and learning, ChatGPT enhances their effectiveness on teaching and learning, and ChatGPT makes it easier to do their teaching and learning.



Figure 1: PSTs’ average scores to perceived usefulness of ChatGPT in the adapted TAM survey

PSTs’ perceived ease of use of ChatGPT

The mean for the perceived ease of use of ChatGPT ranges from 3.72 (Q8) to 4.29 (Q7). PSTs agreed the most that it is easy to learn how to operate ChatGPT (mean=4.29), followed by the statement that it is easy to use ChatGPT (mean=4.26) and the statement that it is easy to become skillful at using ChatGPT (mean=4.07). As shown in Figure 2, more than 80% of PSTs perceived positively about the statement that it is easy to use ChatGPT and it is easy to learn how to operate ChatGPT, whereas less than 70% of PSTs perceived positively about the statements that it is easy to get ChatGPT do to what

they want, their interactions with ChatGPT would be clear and understandable, and ChatGPT would be clear and understandable.

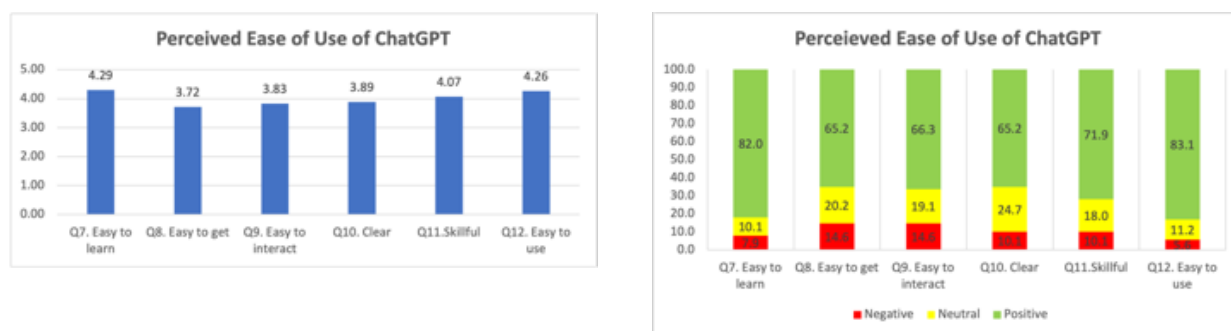


Figure 2: PSTs' responses to perceived easy of use of ChatGPT in the adapted TAM survey

PSTs' task revision, prompts, and reflections: A case of three PSTs

The average score of PST 19's responses to the TAM survey questions was 2, which is the lowest score from all participants. PST 19 changed the context from commuting to the school to preparing and assembling tamales for Christmas but did not change two unlike fractions ($\frac{1}{3}$ and $\frac{1}{4}$) as well as the reference of fractions (fractional part of morning).

Table 2: PST 19's original vs. revised mathematical task

PST 19's original math task	PST 19's revised math task after ChatGPT conversations
You have to take the bus to school. You spend $\frac{1}{3}$ of the morning waiting for the bus. The bus finally comes and the ride to school took $\frac{1}{4}$ from your morning. How long did you spend waiting for the bus and commuting to school all together? Explain your answer using models, number lines or drawings.	For Christmas, Carlos and his grandma want to make tamales because it is tradition. All the materials are gathered and ready to be used. Carlos decides to help your grandma make the tamales this year. Carlos and his grandma spent $\frac{1}{3}$ of the morning preparing the ingredients. Mixing the masa (also known as dough), preparing the leaf, and cooking the meats. It takes $\frac{1}{4}$ to assemble the entire the tamale. How long did it take to prepare and assemble the tamales. Use models, number lines, words or drawings.

In revising the original mathematical task, PST 19 asked to create a cognitively demanding and culturally relevant mathematical task and then added the details of the type of fractions to add (unlike fractions), information about target students (ethnicity/race), and challenging the difficulty of the problem. PST 19 was impressed with ChaGPT's fast responses but identified the flaws in its responses and the lack of clarity of the problem that ChatGPT generated.

The average score of PST 2's responses to the TAM survey questions was 3.25, who provided both positive and negative experiences with ChatGPT. PST 2 changed the context from time spent for

creating a robot to the fractional part of representing traditional foods but did not change two unlike fractions ($\frac{1}{2}$ and $\frac{1}{4}$). However, the vagueness of reference remains in PST 2's revised mathematical task.

Table 3: PST 2's original vs. revised mathematical task

PST 2's original math task	PST 2's revised math task after ChatGPT conversations
Jordan has three days to create a robot for the school science fair. He is struggling to manage his time properly and he's trying to decide how much time to spend working on the robot each day. What if Jordan only worked on one-fourth of the robot on Monday, did nothing on Tuesday, then did $\frac{1}{2}$ on Wednesday? Will he finish his robot on time?	During a school's cultural fair, students presented dishes from different countries. Lucy brought $\frac{1}{2}$ of a traditional Chinese dish, and Miguel brought $\frac{1}{4}$ of a traditional Indian dish. What fraction of the cultural fair's representation is contributed by China and India combined?

In revising the original mathematical task, PST 2 asked to revise the original mathematical task in a more cognitively demanding and culturally relevant. PST 2 repeated some of the prompts several times without elaborating the initial prompts (e.g., make it culturally relevant, 5th grade, and using science) and providing information about the targeted students. PST 2 was frustrated because ChatGPT did not understand the prompts accurately. This made PST 2 to repeat the same prompts several times to get the desired outcomes.

The average score of PST 75's response to the TAM survey was 5, which is the highest average score from all participants. PST 75 changed the context from the recipe for enchiladas (a traditional Mexican dish) to the recipe for biryani (a traditional Indian dish) as well as two unlike fractions. The original mathematical task used two unlike fractions where only one fraction needs to be changed whereas the revised mathematical task used two unlike fractions where both fractions need to be changed. In teaching adding fractions, the first set of fractions (e.g., $\frac{1}{3} + \frac{4}{9}$) should be introduced first before the second set of fractions (e.g., $\frac{2}{5} + \frac{3}{8}$). Also, the revised mathematical task is problematic because it is multiplying fractions (doubling the recipe for $\frac{2}{5}$ cup of basmati rice and doubling the recipe of $\frac{3}{8}$ cup of spices) instead of adding two unlike fractions.

Table 4: PST 75's original and revised mathematical task for adding unlike fractions

PST 75's original math task	PST 75's revised math task after ChatGPT conversations
Miranda cooked enchiladas for dinner for her whole family. The recipe for the calls for $\frac{1}{3}$ of a cup of white cheese, and $\frac{4}{9}$ of a cup of yellow cheese. How much cheese did Miranda use total?	Rahul is making a traditional Indian dish called biryani. The recipe calls for $\frac{2}{5}$ cup of basmati rice and $\frac{3}{8}$ cup of spices. If he wants to double the recipe to feed more guests, how much rice and spices will he need in total?

PST 75 used different prompts from the previous two PSTs. Instead of asking to revise the original mathematical task, PST 75 entered the original mathematical task and ChatGPT produced solutions

for the original mathematical task. PST 75 were more interested in different solutions for the original mathematical task (e.g., using a common denominator; using a common numerator) and asked ChatGPT's preferred method to solve the problem. After asking suggestions for culturally relevant word problems for adding and subtracting unlike fractions, PST 75 added more context for the targeted grade level and students' interests (gardening) but did not challenge the ChatGPT's responses. Overall, PST 75 was satisfied with ChatGPT because ChatGPT solved the problem and offered different methods. PST 75 did not encounter any mathematical issues but was concerned that teachers might overuse ChatGPT.

Discussion

The findings from this study reveal various perceptions, experiences, and prompts of elementary preservice teachers (PSTs) in designing culturally relevant mathematical tasks using ChatGPT 3.5. This section discusses these findings in detail and explores their implications for the integration of AI technologies, more specifically ChatGPT, in teacher preparation programs.

Overall, PSTs had positive perceptions of ChatGPT in terms of their usefulness and ease of use but they had varied experiences, expectations, and satisfactions with ChatGPT. PST 19's prompts primarily focus on creating culturally relevant mathematical tasks, particularly geared towards Hispanic students. The prompts also incorporate students' family traditions, seeks for the alternative context for students who do not have the access to the ChatGPT-generated mathematical task, and addresses potential challenges students might face. In contrast, PST 2's prompts demonstrate a broader scope, encompassing various aspects such as grade level, cognitive demand, cultural relevance, and integration with other subject matter (specifically science). PST 2 had a clear attempt to refine the prompts iteratively to elicit the desired response but repeated the same prompts without further elaborating. PST 75's prompts initially present a specific mathematical problem, but then transition into seeking additional information and exploring different avenues. PST 75 engages in a conversation with ChatGPT, seeking alternative methods. The prompts also reflect appreciation and encouragement towards the AI's responses. While all three PSTs engage ChatGPT to generate culturally relevant mathematical tasks, their approaches varied in terms of specificity, focus, and prompts. PST 19 focuses on creating culturally relevant tasks for Hispanic students, PST 2 repeated prompts to elicit desired outcomes, and PST 75 engages in a dialogue with ChatGPT to seek guidance on the solution and explore different methods. Each PST's prompts reflect their unique goals and priorities in leveraging ChatGPT to design a culturally relevant mathematical task.

The reflections of three PSTs present a spectrum of perceptions, experiences, interactions, and perspectives on ChatGPT's capabilities and limitations. PST 75 expressed a high level of satisfaction with ChatGPT, emphasizing its ability to provide multiple solutions. In contrast, PST 2 and PST 19 had more mixed experiences. PST 2 initially faced frustration with ChatGPT's responses but persistently reiterated prompts to achieve desired outcomes. PST 19, while impressed with ChatGPT's speed, encountered limitations in its mathematical accuracy and pedagogical issues. Despite their varied experiences, all three PSTs acknowledged ChatGPT's strengths such as its speed, recognized the importance of specificity in prompts, and demonstrated persistence and adaptability in navigating ChatGPT's responses. In summary, while these three PSTs shared similar experiences of initial frustration and persistence when using ChatGPT, their overall satisfaction levels, specific challenges encountered, and considerations regarding the ethical use of AI in education varied. This

can be explained by different levels of their TK about ChatGPT (e.g., the type of prompts used in ChatGPT), their PK about culturally relevant pedagogy (e.g., surface level of revisions vs. meaningful level of revisions), and their CK about an appropriate word problem for adding unlike fractions. These differences highlight the complex nature of integrating AI technologies into educational settings and underscore the importance of critically evaluating their capabilities and implications. The findings of this study are consistent with the previous studies about the importance of prompts in using ChatGPT to critically examine the validity, accuracy, and correctness of responses produced by ChatGPT but provide more specific empirical evidence about how PSTs interact with ChatGPT for the specific mathematical content, how they react to ChatGPT's responses, and how they reflect on their experiences with ChatGPT.

These findings offer significant implications for the integration of AI technologies in teacher education. First and foremost, there is a clear need to provide robust support for PSTs to enhance their proficiency in effectively utilizing AI tools. This support includes comprehensive guidance on formatting clear, precise, and effective prompts to elicit desired outcomes from ChatGPT. PSTs should also be equipped with sufficient content-specific knowledge to critically evaluate ChatGPT-generated responses and, where necessary, to further challenge them. Additionally, it is crucial for PSTs to develop an understanding of the limitations inherent in AI technologies like ChatGPT. By grasping these limitations, PSTs can better navigate the parameters within which ChatGPT operates and provide the necessary context to facilitate the generation of relevant and accurate responses. In conclusion, this study highlights the importance of preparing PSTs to engage effectively with AI technologies like ChatGPT in educational setting. By providing targeted support and guidance, teacher education programs can empower PSTs to leverage AI tools effectively, ultimately enhancing their pedagogical practices and improving student learning outcomes.

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Orchestrating mathematical discussions with and through Padlet

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The role of the teachers is central in orchestrating classroom discussions to support students' engagement and highlight connections between emerging mathematical ideas. Literature shows many practices that could support teachers in dealing with this role, and some recent studies show that digital tools shape teachers' practices. In this paper, we discuss an example of a structured educational activity conducted with a shared and collaborative digital platform (Padlet); the aim is to identify the Padlet affordances that may support productive discussion orchestration. In the discussion presented here, Padlet appears to be a valuable tool: students' posts on Padlet allow the teacher to monitor the different approaches proposed by students to solve a given task and their achievements. Moreover, the possibility of having all students' posts together permits students and the teacher to recognize similar strategies and connect emerging ideas.

Keywords: Padlet, educational technology, mathematical discussion, mathematics activities

Introduction

In promoting a deeper understanding of mathematics, teachers orchestrate whole-class discussions that use students' responses to instructional tasks to advance the whole class's mathematical learning (e.g., Bartolini Bussi, 1996). Effective facilitation of classroom discussions poses a considerable challenge for educators, especially in the domain of mathematics. Stein and colleagues (2008) introduced a pedagogical model comprising five practices (anticipating, monitoring, selecting, sequencing, and connecting) that help teachers orchestrate meaningful mathematical discussions. In this paper, we explore the integration of Padlet in a mathematical discussion concerning problem solving. We use the model of Stein and colleagues as a theoretical framework to design an experimental plan for investigating whether and how Padlet is used in the planning (anticipating, monitoring), execution, and management (selecting, sequencing, and connecting) of such discussions.

Padlet, a versatile digital tool, enables the posting of various content types, such as images, links, videos, and documents. Utilizing Padlet in education facilitates the creation of a collaborative digital space between teachers and students. Accessible to all, this virtual wall allows viewing and adding diverse content seamlessly. Padlet is adaptable in educational contexts (Shuker & Burton, 2021), and particularly in mathematics education, suggests the need for further exploration to enhance mathematical discussion effectively. A previous study (Giberti et al., 2022), focused on grade 7 Italian students, highlighted that mathematical discussion in a classroom is a complex phenomenon wherein different factors interweave, and the use of Padlet highlights some of these aspects and promotes a

more inclusive discussion. Nevertheless, the role of the teacher is fundamental because he/she acts as a mediator in raising turning points and catalysts of these different variables and processes.

In this contribution, we focus on this role, using a specific theoretical framework concerning mathematical discussion orchestration. The research question we aim to answer is: When and how does the teacher use Padlet in the process of orchestrating the discussion? To this end, we discuss an experimentation that illuminates the role played by Padlet's affordances while a teacher orchestrates a discussion concerning a mathematical problem.

Theoretical Framework

The metaphor of the "orchestra" is shared in the literature to refer to whole-class discussions and the management of the polyphony of voices involved (Bartolini Bussi, 1996). Stein and colleagues (2008) use the term *orchestration* to refer to the teacher's role in managing students involved in a discussion. The authors designed a pedagogical model of five practices for discussion facilitation starting from a mathematical task. The model's practices support both the planning phase of the discussion by the teachers and the orchestration phase involving both students and teachers. These practices are anticipating, monitoring, selecting, sequencing, and connecting.

Anticipating students' responses means imagining or predicting how students might tackle the tasks. Tackling the problem may mean how they interpret the text or the task situation, the set of strategies they might use, the difficulties they might encounter, and so on. *Monitoring* students' responses means observing and following the resolution process that students are employing as they attempt to answer the task. In this practice, the teacher observes students at work to gather information on the activated processes. Anticipation and monitoring play crucial roles before and during task resolution, providing valuable support to teachers during the discussion. In contrast, the practices of *selecting* and *sequencing* are integral to the overall management of the class discussion, with the former two practices serving as foundational elements for the latter two.

The five practices are interconnected with each other, benefiting from the outcomes of the preceding ones. For instance, the information gathered in the monitoring practice can serve to select students' responses. Similarly, insights gained through the anticipation practice can guide the teacher in sequencing these responses. Lastly, the *connecting* practice aims to establish links between emerging mathematical ideas derived from collective solutions. Stein and colleagues (2008) underlined the role of these practices in fostering the development of robust mathematical concepts by emphasizing the significance of valuing students' responses and products.

Methodology

Experimental Plan

We structured a problem-based activity in four phases, following the theoretical framework, to involve students in a mathematical problem and observe the teacher orchestrating a whole-class mathematical discussion.

The first lesson consisted of phases 1, 2, and 3, during which students were engaged in group activities to deal with the problem and share and comment on other groups' strategies. In phase 1, the teacher posed the problem to the students by opening the Padlet. Then, she described the problem verbally and asked the students to observe it directly on their smartphones. Then, the teacher asked students

to post their strategy and reasoning in detail on Padlet. The reasoning needed to be clear to their classmates belonging to other groups, as then they would have to comment on it. This phase concerns a part of the anticipating practices. Indeed, in these 5 minutes, the teacher could anticipate possible difficulties regarding the text or the context comprehension, and he/she could ask students if they understood what they were asked to do. In phase 2, each group posted its hypotheses/strategies in the Padlet. In this phase, Padlet was set up with the ‘request approval’ mode for comments and reactions (so each group did not see other groups’ posts but only their own). Finally, in phase 3, the teacher made posts visible and allowed comments on posts so that each group could read and comment on posts from other groups. Phases 2 and 3 regard the monitoring practice: in the first, the teacher could observe groups’ work, strategies, and attempts through the posts; in the second, she could realize what students think about their classmate strategies: whether they appreciate or not their mate strategies, if they compare the solution and so on.

Finally, phase 4 focused on the mathematical discussion, which started by visualizing the posts on Padlet. In the final discussion, the teacher could activate the connecting practice. Selecting and sequencing practices could be used in both phases 3 and 4; precisely, the teacher could select and sequence students’ strategies before the beginning or during phase 4. The comments in phase 3 could support selecting and sequencing practices because they allow the teacher to go beyond the collection of groups strategies. The ongoing discussion (phase 4) could permit the teacher to release new information about emerging mathematical ideas and change some selecting and sequence choices.

Participants, data collection, and data analysis

We collected data in a grade 9 class composed of 27 students from a scientific high school in Italy. The teacher is an expert teacher who collaborated on several mathematics education projects. She had already used Padlet to promote mathematical discussion in other classes, but this was the first time she used Padlet in this classroom. She was informed about the main aim of the project, but she did not know Stein and colleagues’ model. She described her students, stating that half of them have strong skills in mathematics while the others have difficulties, and two of them have a diagnosis of dyscalculia. Students are used to working in groups and discussing their ideas during mathematics lessons. In our experiment, students divided autonomously into groups of three people each. Each group chose a nickname and communicated it to the teacher; only the teacher was aware of the nickname-group correspondence.

The data collected consists of the Padlet wall used in the experiment. Thus, we have access to the groups’ posts and comments in their final position, the one set by the teacher to orchestrate the final discussion. This discussion was video-recorded. Next, we interviewed the teacher, showing her pieces of video and asking her which strategies she was implementing at that given moment and why.

The mathematical problem

The mathematical problem to consider in order to implement a problem-based activity must be wide-ranging (van den Heuvel-Panhuizen & Becker, 2003), allowing students to express themselves and show what they know and can do with greater freedom. To this end, we started considering problems from the OECD-PISA test; we chose the “Continent Area” problem. To answer, students have to estimate the area of a continent using the map scale and explain their strategy. We considered this item because it can be tackled using different strategies (e.g., constructing grids or decomposing the

figure using simpler shapes such as rectangles and circles), and thus it can be considered a wide-ranging problem.

Taking a cue from the “Continent Area” problem, we designed a new one: “Oil spill in the Gulf of Mexico” (Figure 1). To solve the problem, students have to consider two figures representing a map of the Gulf of Mexico and a satellite photo of the oil spill. The distance between two cities in the geographical map is the only numerical data given.

An oil rig explodes in the Gulf of Mexico. A month later, a NASA satellite photo shows the ecological disaster. Figure A is a map of the Gulf of Mexico. Figure B is a satellite photo of the oil spill made a month after the explosion, with its edge marked. The rectangle with red edges in Figure A corresponds to the satellite photo in Figure B.

Estimate the extent of the ecological disaster by calculating the area of the ocean polluted by the oil spill.

- Consider that the distance between Austin and Jacksonville is 1600 km, as the crow flies.
- You can draw on the printed map if it helps; if so, upload a photo of it to Padlet as well.

Show your work and explain how you made your estimate.




Figure 1: Problem “Oil spill in the Gulf of Mexico”

Results

In phase 1, the teacher presented the problem in Padlet and read it aloud. She described the problem verbally and asked the students if everything was clear. This phase pertains to the *anticipating practice* because the teacher ensures the assignment is clear and checks if the students have doubts about the text, the stimulus, and the images presented. In this first phase, Padlet is only used to present the problem; therefore, we cannot identify it as a support for anticipation. In the second phase, the teacher asked the students to post their hypotheses/strategies in the Padlet. This is the moment for the teacher to *monitor* students’ work. In fact, the teacher observed the work of each group and followed the attempts made and the strategies used. In her words, “The functionality of Padlet permits posts visible only for me; this allows every group to reflect on their strategies and allows everyone to think without being influenced by others.”

Through Padlet’s posts, it is possible to realize that many groups approached the problem with graphical strategies. In the post reported in Figure 2a, the students determined the area of the rectangle circumscribing the oil spill. They copied the image onto tracing paper, subdividing the rectangle into small squares. They identified the estimated area of each square and finally counted the number of squares contained in the oil spill. Three other groups chose to approximate the ground surface through rectangles and removed it from the total rectangle surface. In this way, they identified an estimation of the sea surface depicted in Figure 1 (Figure 2b).

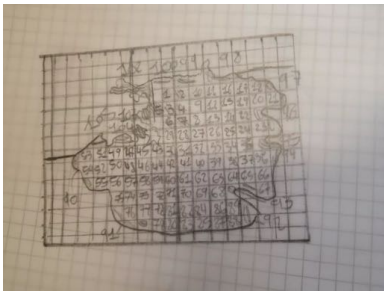
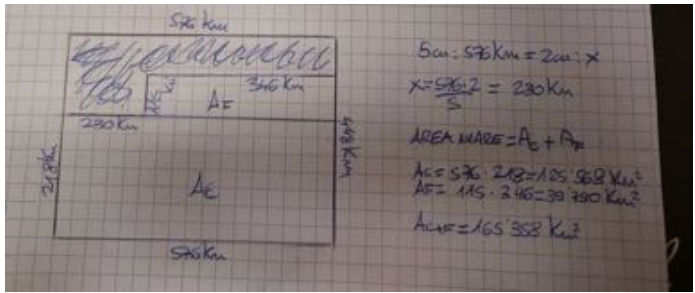
	
<p>2a. Example of subdividing into small squares</p>	<p>2b. Example of subtracting area</p>

Figure 2. Example of students' strategies

It can be noticed that other two groups of students identified the sea area by estimating the ratio of land and sea area. In this last example, students did not use pictures but only written argumentations:

“We started looking for the solution by estimating the base of the rectangle (here Figure 1- Figure B), and we did this by taking the distance from Austin to Jacksonville, which is 1600 km, and assumed that the base was about $\frac{2}{5}$ of this distance. Next, we assumed that the height of the rectangle was about $\frac{2}{3}$ of its base because by imagining taking the height segment to the base, it appeared to be about this size. Finally, we estimated the area of land in the rectangle, which is about $\frac{1}{3}$ of the area of the square, because to obtain the solution, we only needed the area of the ocean contaminated by oil, so the area of the piece of land was not needed. Consequently, the result of the water area was $\frac{2}{3}$ of the total area.”

In summary, during the interview, the teacher shared that Padlet allowed her to observe the strategies chosen by the students before opening the discussion. She said: “Padlet works as a dashboard collecting and organizing all solutions at once: it provides an overview of all strategies and students' answers.” The teacher could observe that most students preferred graphical rather than arithmetic/algebraic approaches. She also observed that there were different graphical approaches: the two strategies (subdividing into small squares versus subtracting areas) are similar because they make use of graphical representation but also differ in the implemented mathematical process (decomposition versus difference). She concluded by stating: “Students proposed strictly different approaches despite the fact they already worked together on polygon areas”.

In the third phase, the teacher made all posts visible and asked each group to comment on the posts of the others. Being able to read the group comments provided the teacher with additional *monitoring* information to use in *selecting*, *sequencing*, and *connecting* practices. She noted that the groups' comments suggested a preference for the graphic representation. In various comments on arithmetic approaches, we read: “In our opinion, the calculations performed are accurate and correct, but a graphical representation of what has been done is absent.” And again, some students highlighted algorithmic errors; for example, in some comments, we read: “The estimation is quite right, but some calculations are wrong: $\frac{2}{3} + \frac{2}{5}$ is $\frac{16}{15}$.” The teacher also realized that those who chose graphical approaches were retained as inaccurate by their classmates and too approximate in their solutions. For example, “The final approximation is inaccurate because the method used is not precise.” In addition, students pointed out the confusing argumentation and the lack of details in the post. In general, accuracy seemed to be a topic of great interest for students.

To summarize, students' comments provided the teacher with important information. For example, which strategies were welcomed by classmates, which were appreciated, and finally, which were contested. The teacher decided to start phase 4 based on the groups' comments: "Comments were articulated, and students were excited to comment on their classmates' works. It was an opportunity for me." At the first moment, she discussed them and highlighted that some were not so rich, whereas others were detailed. Then, she asked students if some strategies impacted them. No students answered, so she decided to change strategy, proposing "show and tell"; i.e., she asked: "Did you all give the same answer? So, let's start by explaining which strategy you chose to answer. Let's start with group 1. Let's take a look, and then we will discuss differences and similarities". In this case, the teacher undertook the *selecting* and *sequencing* practices following the order of the posts on Padlet, and this choice did not support her in the subsequent *connecting* practice. After this first "show and tell" moment, the teacher asked students to find the differences and similarities between the strategies used. She listed on the blackboard the three main strategies used, helped by students who were looking at the Padlet to check them. The teacher picked up on similarities in the students' comments and used them to *connect* emerging ideas about possible solving strategies.

Then, she asked students: "Now you have seen your classmate's strategies. Is there a 'better strategy' you would choose, or are you already sure of yours?". This question opened a discussion about what 'accuracy/rigor' meant: many groups responded that they would implement the same strategy using more accuracy or precision. This question moved students to discuss and compare the two arithmetic strategies. In particular, the teacher prompted students who used the estimated ratio of land and sea area to explain their strategy to their classmates. One group claimed they guessed the estimation with their hands rather than using the ruler as other groups did. With this answer, the teacher came back to the word 'accurate'. In the comments, students often underlined the lack of accuracy of the graphical strategy in favor of the arithmetic ones. The first arithmetic strategy brought out the use of eye estimation which was then recognized as inaccurate. The teacher asked: "Does accurate mean reaching the closest result, or measuring and not guessing?". In this way, the teacher came back to *selecting* and *sequencing* and moved her students to sort the groups' answers by accuracy. Padlet allowed everyone to see the posts commented on and discussed (both on the interactive whiteboard and their smartphones). This was also underlined during the interview: "The fact that Padlet was accessible to students during the discussion, and they can navigate it independently multiplies the connections between the answers."

Discussion and conclusion

In this contribution, we investigated the use of Padlet by a teacher for orchestrating a mathematical discussion. We presented a discussion in which an Italian teacher and her students dealt with a mathematical problem and discussed students' strategies supported by Padlet. The analysis of the whole activity focused on the use of Padlet throughout the different pedagogical strategies referred to the model by Stein and colleagues (2008). This contribution starts from an open issue raised in previous studies (Giberti et al., 2022) in which authors suggest investigating how dynamics in classroom discussion depend on the specific features of the teacher's teaching style.

In this example, the monitoring phase through Padlet offered the teacher the opportunity to discuss a topic of interest to the students, the accuracy of estimation, and to value shared ideas that emerged in students' comments. In this case, the use of Padlet was decisive in facilitating the activity and

allowing students to post their responses, thoughts, and strategies directly as they engaged with tasks. The digital platform offered students the opportunity to comment on their classmates' strategies; without Padlet, the teacher would have had difficulty in observing this variety of strategies. Students also worked anonymously, and the teacher chose when and how to show the posts, avoiding any bias in the opinions on others' posts. In addition, Padlet allows time and space management in a different way: for example, groups could not see who had already posted the answers, and this may limit the so-called anxiety of having to finish first. The teacher also claimed, "Padlet guarantees continuity in the classroom activities: its dashboard features allow for restarting the next lesson without making the effort of recalling everything that was said in the previous one."

The possibility of viewing all posts together on Padlet during the selecting and sequencing practices led the teacher to distinguish graphical or arithmetic approaches, focusing on the different strategies that could be used within the same approach. However, other choices might have been adopted; for instance, she could have started from the most to the least appreciated strategy or vice versa, dealing with the problem of sharing strategies or discarding others. In any case, the data collected with Padlet seems to open up multiple possibilities for orchestrating discussion. Similarly, the teacher could collect the written answers on sheets of paper without the support of the technological tool. In this case, the potential of Padlet is to immediately display all answers together in one place, without scattered sheets. Considering our experiment, Padlet was useful to students when the teacher wrote the strategies on the blackboard because it allowed them to look at the Padlet and control that every post was considered. On the other hand, Padlet could have affected the teacher choices in the selecting phase: she decided to consider the order of the posts for a "show and tell" moment; if the post had been in a different position, the first phase of the discussion could have been different. Padlet revealed its suitability, even during the orchestration of the whole-class discussion. As a dashboard, it shows all posts together as on a wall, and in this way, both students and the teacher have the possibility to compare and discuss all strategies simultaneously. In the absence of the technological device, the students and teacher would only have been able to use the blackboard. This would have required more time for writing and fewer expressive possibilities.

In any case, it should be noted that any digital technology has weaknesses; for example, the posts in Padlet could suggest students use short and basic messages (as in many community platforms), which might limit students' engagement or deep strategy descriptions in the problem-solving activity. This is also confirmed by the teacher who said: "Padlet, as a digital tool, encourages a writing style that evokes other digital platforms (for example, Instagram, WhatsApp). I noticed short messages and replies. The complexity is sometimes lost in the brevity of students' messages." In addition, Padlet does not facilitate verbal exchanges between groups. This may limit the teacher's role in the monitoring practice. The versatility of Padlet's posts is held back by limitations: it does not have an extensive library of mathematical symbols and creating graphs and diagrams is not intuitive. This limits the type of feedback classmates or teachers can provide.

In conclusion, while summarizing when and how the teacher used Padlet, we also highlighted other modalities and other strengths for using such a digital tool. For instance, sharing the theoretical framework of Stein and colleagues (2008) with the teachers involved might highlight other and new potentialities of Padlet that are still to be explored. Actually, the teacher involved used well-established pedagogical practices in her orchestration even without being fully aware of them from a

theoretical point of view. For less experienced teachers, it might be more complex, and introducing a new digital technology may require specific training to fully exploit all features essential for a successful orchestration of a classroom discussion.

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GeoGebra to help Engineering students to improve 3d visualization in a multidimensional calculus course

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Engineering students' incoming knowledge of analytical geometry in three-dimensional space is very weak and is often limited to calculating the volumes of solids. During the Linear Algebra course, they begin to see planes and straight lines in space but, due to lack of time, quadrics are rarely treated. In this contribution we want to investigate the evolution of spatial ability of 38 Civil and Environmental Engineering students during a Mathematical Analysis II course thanks to the use of GeoGebra during the lessons. To do this, 3 graphic designs tasks were analysed, which include 1 pre-task, 1 homework task on multivariable integral calculus and 1 exercise of the partial exams on the same subject. The analysis was done with the help of the APOS theory. The students' use of GeoGebra was revealed by examining the answers to an open survey. The results are encouraging, most of the students have constructed a "Process" conception of the graphical representation.

Keywords: APOS theory, Spatial visualization, Multidimensional calculus, GeoGebra, Survey

Introduction

Representation of mathematical concepts is very important in the teaching and learning of mathematics. The ability to represent a mathematical situation in different forms is a very powerful tool in mathematics (Garzon & Casinillo, 2021). Visual representations (for example, charts, graphs, pictures, diagrams, and numerical lines) can be used to solve, explore, or explain a mathematical situation. Visual representation helps students to understand mathematical concepts, internalise the concepts, and establish connections between the concepts leading to conceptual understanding and creativity. In line with this thought, Duval (1995) noted that "there's no knowledge that can be mobilised by an individual without a representation activity" (p. 15). Visual representations are a key to conceptual understanding in mathematics.

Teaching software packages, such as GeoGebra, Maple, and Mathematica, support multiple representations that can simultaneously demonstrate a function in numeric, algebraic, and graphic models. This feature helps students to understand abstract concepts; therefore, this type of software is often applied in the teaching of calculus and contribute to students' motivation (see for example Nobre et al., 2016). GeoGebra is a software that allows to create different interactive applications that can be used as teaching tools in math classes. The graphics in books, as well as the pictures drawn on the blackboard, are static and require the students to possess an adequately trained imagination. Nowadays, the existence of free programs with versatile capabilities and interactive representations helps to improve the presentation of contents taught in this area of knowledge, allowing dynamic visualizations. This is extremely important, since only students able to perform conversions do not confuse a mathematical object with its representation and can transfer their mathematical knowledge to other contexts different from the one of learning (Duval, 1999).

Some past studies have examined students' generalized knowledge of multivariable functions (Kabael, 2011; Martinez-Planell & Trigueros-Gaisman, 2013; Weber & Thompson, 2014), domain and range (Dorko & Weber, 2014), and derivative (Yerushalmy, 1997). While some research has

touched on student understanding of multivariate integration (Jones, 2013; Alessio et al., 2022), or on pedagogy related to multivariate integration (McGee & Martinez-Planell, 2014), no research has focused specifically on students' ability in spatial representation of multivariate definite integrals (Lepellere 2023).

The general research question is: What is the impact on students' proficiency in spatial drawing at a tertiary level using GeoGebra Mathematical software? Specific research questions: a) How was the initial student level of proficiency in spatial drawing? b) How does the level of proficiency in spatial drawing evolved after incorporating the use of GeoGebra during the lessons?

To do this, the graphic designs of 3 tasks were analysed, which include 1 pre-task, 1 homework task on multivariable integral calculus and 1 exercise of the partial exams on the same subject. First the results of the three exercises were examined and classified as: correct, partially correct, incorrect and no response. After the three exercises were compared to see if there was some improvement. Finally Exercises 1. and 2. were examined with the lens of APOS theory. The students' use of GeoGebra was revealed by examining, through thematic analysis the answers to an open survey with the following request: If you have used GeoGebra please describe in detail under what circumstances, you did so and what benefits you received.

Theoretical background

The main ideas in APOS Theory were introduced in Dubinsky (1984). The acronym APOS was first used in Cottrill et al. (1996). The APOS theory states that the teaching and learning of mathematics should be based on helping students to use the mental structures that they already have and to develop new, more powerful structures, for handling more and more advanced mathematics (Arnon et al., 2014). These structures include Actions, Processes, Objects and Schemas, the acronym APOS being formed by the initial letters of the above four words. A mathematical concept is first formed as an action, which is, an externally directed transformation of a previously conceived object (or objects). Action is an external conception in the sense that each step of the transformation needs to be performed explicitly and instructed by external guidance; additionally, each step operates the next, that is, the steps of the action cannot be imagined, and none can be skipped (Arnon et al., 2014). As the individual repeats and reflects on this action it may be interiorized to a mental process. A process performs the same operation as the action, but wholly in the mind of the individual enabling him/her to imagine performing the corresponding operation without having to execute each step explicitly. If one becomes aware of a mental process as a totality and can construct transformations acting on this totality, then he/she has encapsulated the process into a cognitive object. A mathematical topic often involves many actions, processes and objects that need to be organized into a coherent framework that enables the individual to decide which mental constructions to use in dealing with a mathematical situation. Such a framework is called a schema. In concluding, the APOS theory considers actions, processes, objects, and schemas as an individual's successive mental constructions in learning a mathematical topic and interiorization, encapsulation as the only mental mechanisms needed to build those mental constructions (Arnon et al., 2014).

Methods

In this contribution we want to investigate the evolution of spatial ability of 38 Civil and Environmental Engineering students during a Mathematical Analysis II course at University of Udine

in the academic year 22/23 thanks to the use of GeoGebra. It is a second course of calculus (9 credits corresponding to 72 hours of lessons) given in the first semester to second year Civil and Environmental Engineering students. It is preceded, among others, by Mathematical Analysis 1 (12 credits), Linear Algebra (6 credits) and Physics 1 and is simultaneous with Rational Mechanics. The course of Mathematical Analysis 2 includes topics such as first and second order differential equations, the systems of linear differential equations with simple studies of stability, differential calculus in several variables with the calculation of free and constrained extremes, the integral calculation of more variable functions and integrals of vector fields over curves and surfaces.

In the last years it was decided to incorporate GeoGebra during the lessons adding graphs both in the theoretical explanations and in the exercises to facilitate the graphical representations. Students were encouraged to use it in solving exercises independently to prepare for the exam. The students were given a mini guide of the main commands used during the course, at the beginning of the specific topic they were reviewed, and their interactive use was shown to the students during the lessons. All the exercises proposed were supported through graphical representations with GeoGebra (see Figure 1).

Esercizi

1. Calcolare il volume del semplice $x + y + z \leq 1, x \geq 0, y \geq 0, z \geq 0$.

$$\text{Volume}(E) = \iiint_E 1 \, dx \, dy \, dz = \iint_D \left(\int_{\alpha(x,y)}^{\beta(x,y)} 1 \, dz \right) dx \, dy$$

$x + y + z \leq 1 \Rightarrow z \leq 1 - x - y$ e quindi ottiene $0 \leq z \leq 1 - x - y$ allora $\alpha(x,y) = 0$ e $\beta(x,y) = 1 - x - y$

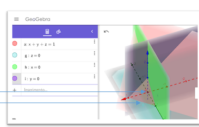
Calcoliamo D

$x + y + z \leq 1$ e $z = 0 \Rightarrow x + y \leq 1$ e quindi $0 \leq y \leq 1 - x$ ossia $0 \leq y \leq 1 - x$

Da $x + y \leq 1$ e $y = 0$ otteniamo $0 \leq x \leq 1$ per cui

$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$ sostituendo si ha

$$= \iint_D \left(\int_0^{1-x-y} 1 \, dz \right) dx \, dy = \iint_D (1 - x - y) dx \, dy = \int_0^1 \left(\int_0^{1-x} (1 - x - y) dy \right) dx = \dots = 1/6$$



2. Calcolare il flusso uscente attraverso la semi superficie sferica $x^2 + y^2 + z^2 = 4$ con $x > 0$ del campo di componenti $X = 2, Y = 0$ e $Z = 0$ diretto verso l'esterno.

La superficie non è chiusa e quindi se vogliamo applicare il teorema della divergenza dobbiamo aggiungere il tappo $x = 0$ e quindi

$$\iint_{\partial D} \mathbf{F} \cdot \mathbf{n}_e \, dS = \iint_D \text{div} \mathbf{F} \, dx \, dy \, dz - \iint_T \mathbf{F} \cdot \mathbf{n}_e \, dS$$

Essendo il campo costante, la divergenza è nulla e quindi il primo integrale è nullo. È sufficiente allora calcolare il flusso attraverso il tappo, una sua rappresentazione parametrica è

$\begin{cases} x = 0 \\ y = u \\ z = v \end{cases}$ dove da $x^2 + y^2 + z^2 \leq 4$ segue $0^2 + u^2 + v^2 \leq 4$ ossia $T: u^2 + v^2 \leq 4$

La matrice Jacobiana è $D\mathbf{r}(u,v) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ e quindi $A = 1, B = 0, C = 0$ allora $\mathbf{n}_e = (-1, 0, 0)$ e quindi

$$\iint_{\text{Tappo}} \mathbf{F} \cdot \mathbf{n}_e \, dS = \iint_T F_1(x(u,v), y(u,v), z(u,v))A + F_2(x(u,v), y(u,v), z(u,v))B + F_3(x(u,v), y(u,v), z(u,v))C \, du \, dv = \iint_T -2 \, du \, dv = -2 \int_0^{2\pi} \int_0^2 \rho \, d\rho \, d\theta = -8\pi$$

Ossia

$$\iint_{\partial D} \mathbf{F} \cdot \mathbf{n}_e \, dS = 8\pi$$



Figure 1: Two examples of lesson slides

To answer the research questions, the graphic representation of the sets involved in 3 standard tasks were analysed, which include 1 pre-test (Exercise 1), 1 homework task (Exercise 2) and 1 exercise (Exercise 3) of the partial exam, both involved in multivariable integral calculus.

1. Draw $x + y + 2z = 1$ in cartesian space. What does the vector $v = (1,1,2)$ of the coefficients represent?
2. After drawing the set $A = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + 2y \leq z \leq 2 - x^2 - y^2\}$ calculate the volume.
3. After drawing the set

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid x, y \geq 0, x + y \leq 1, 0 \leq z \leq 1 - x^2 - y^2\},$$
 check the validity of divergence theorem for the field $F = (x, 0, -y)$.

Exercise 1. was given as an introductory task to verify the ability to draw a plane in Cartesian space and to interpret the coefficients of its Cartesian equation anticipating the concept of normal vector for surfaces. In Exercise 2. we will analyze the drawing produced by the students: an important starting point for choosing the best method to use for calculating the volume of the solid. In particular, the knowledge of the mutual position between the objects involved, in this case the paraboloid and the plane, is essential and makes the exercise more concrete. In Exercise 3. we still need to graphically

identify the intersection between a quadric, in this case the paraboloid, and a plane parallel to z axis. The drawing of the solid is more complicated by the presence of the fundamental planes which create further faces of the solid. Also, in this case correctly representing the set is necessary not only to calculate the triple integral, which is also in this case a volume calculation, but also to find the parametric equations of the surfaces involved since the exercise asks to verify the validity of the divergence theorem.

The results of the three tasks will be collected and statistically analyzed. Furthermore, a more in-depth investigation will be carried out, using the APOS lens, of the results of the second and third task. With the APOS lens, we define: “Action” if students have succeeded in recognizing the quadrics from the Cartesian equations, including the plane as degenerate quadric. “Process” if the student has managed to put the quadrics together identifying the intersection curves. “Object” if the student has managed to have a complete vision of the set and carefully searches for the best perspective to represent it by identifying the single parts that compose it.

Results

We start with the analysis of the results of the various exercises taken individually, and then make a comparison between them to analyse if there was an improvement. Finally, we will analyse in which stage of the APOS the students' works are placed by also inserting some significant examples.

Let us start with Exercise 1., we will evaluate it “Correct” if a precise graph of the plane has been made and the question about the geometrical interpretation of vector $v = (1,1,2)$ was answered correctly. We will evaluate “Partially correct” if the plane has been drawn in a precise way but the interpretation of the normal vector of the plane is missing. We will evaluate “Wrong” if the plane has not been drawn correctly and finally “No Response” if the exercise has not been done. As can be seen from Table 1., only 32% answered correctly, a lower percentage than those who did not carry out the exercise 39%, 18% did not answer correctly meaning, they know it is a plane but do not draw it correctly, and 11% was not able to interpret the vector of coefficients as the normal vector of the plane. This result is unexpected since, at that time, most of the students, exactly thirty (in the other cases the results are even more disastrous), involved in the study had already passed Linear Algebra exam where, towards the end of the course, a little geometry of the Cartesian plane and space is still dealt with. But, as we see, if no importance is given to the graphic representation, only the best or most interested students do it autonomously. Analogous results were obtained for the fundamental planes (the analysis of the achieved results is not, however, the subject of this study) by validating the results obtained in previous studies. Their knowledge, as we have already said, is fundamental for understanding the differential and integral calculus of several variables' functions.

Exercise 2. was given as a homework, together with others, after about ten lessons/exercises on both scalar and vector integral calculus in preparation for the first task which precisely concerned these topics. The students were advised not to use any external aids, except for the material given in class. They had to photograph the work and send it by email. For exercises 1 and 2 only the results of the part concerning the drawing of the solid in question will be presented. The exercise will be evaluated: “Correct” if both the paraboloid and the plane have been drawn correctly and the interested part highlighted; “Partial correct” if plane and paraboloid have been traced correctly but the area between the two has not been identified; “Wrong” if at least one between plane and quadric have not been executed correctly; “No Response” if the graph is missing or the exercise has not been completed. As

can be seen from the table, compared to the previous exercise, the percentage of correct answers has increased and has passed to 53%. Still there is a high percentage 37% of those who have not really tried to solve the exercise, but this could also be since the students had not yet completed the study of the topics involved in the exercise. The number of wrong answers dropped to 5% as percentage of those who gave a partially correct answer.

Exercise 3. was part of the first intermediate exam task on scalar and vector calculus administered three weeks after the delivery of the task which contained Exercise 2. The interpretation of “Correct”, “Partially correct”, “Wrong” and “No Response” is the same as in Exercise 1. As can be seen from Table 1., the correct answers have further increased to 61%. Being an exam exercise, the percentage of answers not given has decreased to 2%, while the partially correct answers 24%, but also the wrong ones have increased with 13%. Although, as explained before, Exercise 3. was more difficult, the students obtained better results than Exercise 1 and 2. Looking at the drawings, we can also notice a greater accuracy with the choice of the best perspective in several cases that will be examined later in the article.

Table 1: Results of the tasks in percentage

	Correct	Partial correct	Wrong	No Response
Task 1	32%	11%	18%	39%
Task 2	53%	5%	5%	37%
Task 3	61%	24%	13%	2%

A comparison was made between the various exercises: the second with the first, the third with the first and the third with the second respectively. As we can see from Table 2 the comparison between exercises 2 and 3 with respect to exercise 1 not only has a higher percentage in getting better and the same percentage in being stable but it has also improved over time despite the increase in difficulty. It should be noted that worsening also includes not having done the exercise at all which, for a homework assignment, is not always due to not being able to do it. It may happen that the student was not yet ready to face the exercise or because he/she did not have enough time available, since the students had a deadline to meet for submission. The comparison between exercise 3 and exercise 2 could make one think of a worsened situation but this is not the case because the percentage of those who remained in an unchanged situation has increased, which includes the case that both exercises have been performed correctly, in fact this percentage is 37%. We would also add that the percentage of students who did all 3 exercises correctly is 16%.

Table 2: Comparisons of the tasks in percentage

	Get Better	The Same	Got Worse
Task2/Task1	40%	37%	23%

Task3/Task1	55%	37%	8%
Task3/Task2	37%	45%	18%

This proved to be very helpful when deciding which method to use for calculating the triple integral in the second and third exercises and for finding the parametric equations of the various surfaces (faces) of the solid in question. With the APOS lens we obtain the following results (see Table 3):

Table 3: With the lens of APOS theory

TASK	No Answer	Wrong	Action	Process	Object
2	36%	3%	11%	25%	25%
3	3%	17%	19%	11%	50%

comparing the results of tasks 2 and 3 we notice a notable improvement especially in the students with an initial “Object” type mental construction passing from 9 to 18 units, the same thing regarding “Action” type. The intermediate situation of “Process” has instead decreased to the advantage of the other two. The absence of answers which was almost 1/3 for Task 2, practically disappeared. Of those who didn't do exercise 2, five switched to an “Action” approach, six even in an “Object” approach. Four went from a “Process” to an “Object” approach, two remain with a “Process” and five with an “Object” mental construction.

We show representative examples too. In the first example (Figure 2.) we see a case in which starting from a good initial knowledge the third stage is already reached, “object” in the second exercise confirmed in the third. In the second example (Figure 3.), although starting from a weak initial knowledge, the second “process” stage is reached with the second exercise and the last “object” stage with the third. It can be noted in the last case also the search for the best perspective to view the whole.

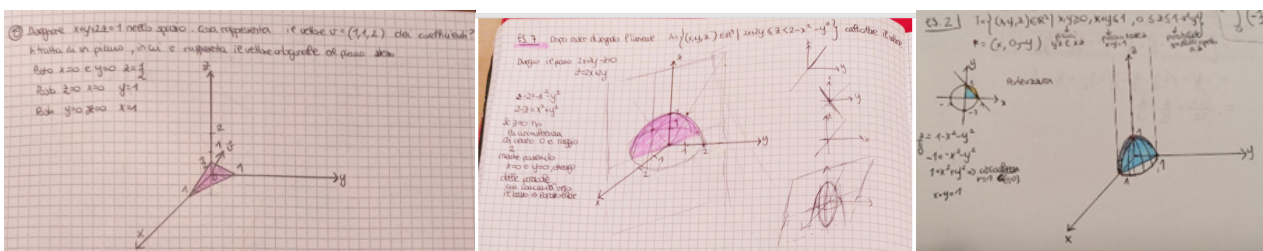


Figure 2. Starting from a good initial knowledge the third stage is reached.

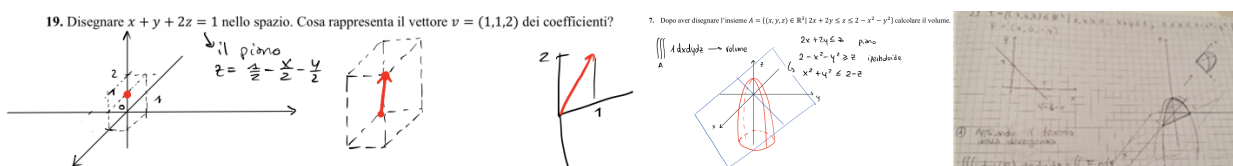


Figure 3. From weak initial knowledge, the “process” stage and “object” stage are reached.

From the survey proposed to the students at the end of the partial exam to the question “If you have used GeoGebra please describe in detail under what circumstances, you did it and what benefits you received” from. According to the students, GeoGebra helped them a lot, especially to recognize the quadrics involved in the Cartesian equation, to correctly identify the intersections between the

surfaces involved and to choose the best view for freehand drawing. Here are some examples of student answers that confirm it: “I had a lot of difficulty imagining the figures on the plane and in the space, obviously I have the simplest ones in mind, but as soon as it deviates a little from the ordinary, it is difficult for me to draw a picture in my mind of what is required. So GeoGebra has been a fundamental tool in my study because it has allowed me to experiment and understand things and notions that I probably would never have understood without it; “It is a tool that has been particularly useful for visualizing the intersections of the surfaces, drawing by hand is not always possible to be precise or it is not always easy to identify which perspective is the best to use for a good visualization of the surfaces. GeoGebra helped me a lot in representing more complex surfaces (particularly with regard the intersections of surfaces), but it also helped me to recognize the surfaces in question more quickly from their equation”; “The use of GeoGebra has been, in my opinion, very useful to be able to find a method of visualization of the intersections between the different functions in space, above all to understand how to orient solids in three-dimensional space. I have used this software very frequently for home study, especially for integrals and optimization problems. I used this software mainly to confirm the correctness of the designs I had previously set up”.

Conclusions

Heavy use of GeoGebra was made during the lessons/practices and its use was strongly recommended for home study. The use of the APOS theory allowed us to go into detail in the analysis of the students' abilities in identifying the mutual position between a quadric, in our examples these were paraboloids, and inclined planes. What was possible to notice was a significant improvement in the ability to draw sets correctly. We can attribute this improvement to the use of GeoGebra because this was declared by the students in response to a survey. We started in 22/23 from a simple use of GeoGebra to understand if it was appreciated by the students and if we noticed an improvement in the students' ability to represent three-dimensional sets. Encouraged by the results, the next objective will be to create specific Applets to intensify and improve GeoGebra use thanks also to the students' observations.

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Promoting the production of examples and conjectures in hybrid environments to elicit students' concept image and definition

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In this paper, we aim to investigate the potential of a purposefully designed task proposed in a hybrid environment with Padlet and GeoGebra, to promote students' production and sharing of personal concept definitions, conjectures, examples and the emergence of instances of students' concept images. From the analysis of the students' posts, we derived a complex intertwining between all these elements and a possible tension with the formal definition of diagonal. We also obtained a confirmation of the potentialities of the hybrid environment to favor the participation of all the students in rich exchanges on the mathematical task. We finally argue that the exchanges recorded by the Padlet provide a common basis for the teacher and the students to share and reflect upon during the subsequent classroom discussion.

Keywords: *Concept image, concept definition, digital technologies, examples, hybrid environment.*

Examples as windows into students' concept images

In the 2000s, there has been a growing interest in mathematics education towards the study of the processes of generating examples and the role of these processes in mathematical learning (e.g., Bills & Watson, 2008). Zazkis and Leikin (2007) suggest that asking students to produce examples “provides a ‘window’ into a learner’s mind”, because students’ examples “mirror their conceptions of mathematical objects involved in an example generation task” (p. 15). Similarly, Sinclair and colleagues (2011) argue that through the study of learners’ personal example spaces “a teacher can obtain some evidence about the nature of learners’ understanding and what sorts of experience to provide next” (p. 292).

In addition, Antonini (2011) shows not only that the activities of producing examples can provide insight into the dialectics between students' *concept images* and personal *concept definitions* (Tall & Vinner, 1981) but can also promote their harmonization with the formal concept definition. The distinction between *concept definition* and *concept image* has been introduced by Tall and Vinner (1981). The *concept definition* is the verbal specification of a particular concept, of its main characteristics, features, and boundaries. It can be either a formal definition shared within a specific community (*formal concept definition*) or a personal (re)construction by an individual (*personal concept definition*). The *concept image* is “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p.152). In different situations, different parts of the *concept image* can be evoked and activated, and different *concept definitions* can be made explicit.

Students do not produce examples solely upon the teachers' explicit request, they may also engage in such activities spontaneously while accomplishing other tasks, such as those involving conjecturing and proving. Even in these cases, while possibly hindering the need to resort to a formal proof,

argumentations based on autonomously produced examples can favor students' *concept images* emerge (Morselli, 2006).

Although it is important to bring out students' concept images, from an educational point of view it is equally important to address the issue of how to promote the construction of links between students' concept images, personal concept definitions and the formal concept definitions of mathematics. With that respect, Paola (1998) argues in favor of the crucial role that social interaction and in particular the mathematical discussion (Bartolini Bussi, 1996; Pirie & Schwarzenberger, 1988) can play in promoting such a process.

Hybrid environments

Hybrid environments (HE) are defined as environments in which online components, such as virtual collaborative boards, substitute part of the face-to-face classroom interactions (Singh et al., 2021). In HEs students can synchronously and simultaneously share statements and ideas, reply to one another, and provide comments, using text, images, videos, links, and other types of digital content. It is thus possible to construct a network of synchronous connections, upon which students and teachers can rely and to which they can continuously refer in the development of the activity. That represents an added value with respect to 'traditional' classroom interactions where the discourse typically progresses along a linear timeline (Giberti et al., 2022). Another key valuable role of a HE is recognized in the promotion of a wide participation to classroom discussions. Students' active participation is deemed of crucial importance in order for them to develop mathematical knowledge and competencies. In this respect, virtual collaborative boards can favor students' participation through a coherent use of posts and comments activated by the teacher in the classroom discussion ultimately promoting the inclusion of all students (Giberti et al., 2022). In this study, we consider a specific HE constituted by integrating a virtual collaborative board in the classroom to foster social interaction, along with software to accomplish a mathematical activity.

Research aim

The current report stems out from a wider study aiming at investigating:

- the potential of a specific conjecturing activity, carried out in a HE to trigger students' production of examples revealing their *concept images* and *personal concept definitions* of the mathematical concepts at stake,
- how the teacher can exploit the potential of HE to lead an inclusive classroom discussion promoting the harmonization among students' *concept images*, their *personal concept definitions* and the *formal concept definition* shared within the mathematicians' community.

More specifically, in the current report, we analyze the case of lower secondary school students facing the task, inspired by Balacheff (1991), of explaining to the classmates how to find the number of diagonals in a polygon with n sides, writing a post on Padlet. They had GeoGebra at their disposal to explore the situation and produce examples. We aim to investigate whether and how the use of Padlet and GeoGebra promote students' production and sharing of *personal concept definitions*, conjectures, examples to support their conjectures and counterexamples to their classmates' ones. Furthermore, we aim to investigate what students' *concept images* emerge for diagonals of a polygon, if they are coherent with their *personal concept definitions* and with the *formal concept definition*.

Methodology

For our study, we collected data from a teaching intervention, conducted by the curricular teacher and an observer, in a 7th grade class in the North of Italy with 25 students. The intervention unfolded in two phases. The first phase, lasting approximately two hours, took place in the computer lab of the school, where students worked in small groups (11 pairs and one group of three) to face the task of writing a post on Padlet explaining to classmates how to find the number of diagonals in a polygon with n sides. Each group should also explain what they mean by “polygon” and by “diagonal”. The students had already met the concept of diagonal referred to quadrilaterals but they had never been presented with its formal definition in a general context. Each group could comment on the posts of the other ones, possibly attaching files or images created with GeoGebra. The posts were only made visible after each group submitted its own to the board. The second phase consisted of a classroom discussion which is not in focus in this paper.

We have analyzed data coming from the students’ Padlet posts produced in the very initial stage of the first phase of the activity, devoted to the students’ spontaneous production of definitions and conjectures and to the first reactions posted by the classmates, before any intervention by the teacher or the observer. First, we have recorded which groups provided definitions of diagonals and conjectures about their number, and which groups provided examples supporting their own conjectures and/or counterexamples to challenge their classmates' ones using GeoGebra. In a second step, we have analyzed what students’ *concept images* and *personal concept definitions* related to the concept of diagonal emerged from their posts.

Results

In the following, we present some excerpts taken from the students’ posts in the Padlet in which students’ *personal concept definitions* and some instances of their *concept images* related to the diagonals of a polygon emerge and, through social interactions, are shared and challenged. We focus on the initial part of the first phase of the activity: students post their own answers to the task and then possibly comment on others' posts before any intervention by the teacher or the observer.

As for students' participation, we have noted which groups provided an explanation of what they mean by diagonal, which groups proposed a conjecture on how to find their number, and which ones provided a figure to illustrate or corroborate their explanations and conjectures. We have also examined the comments and noted each pair's comments on others' posts, not limited to "agree/disagree" statements. Data is synthesized in Table 1.

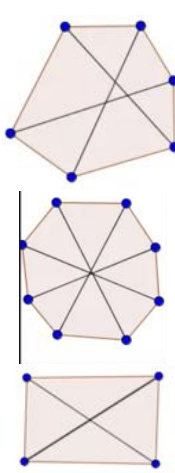
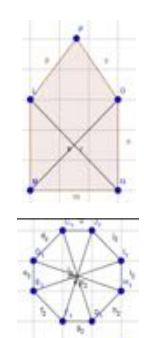
Table 1: Synthesis of groups’ posts and interactions

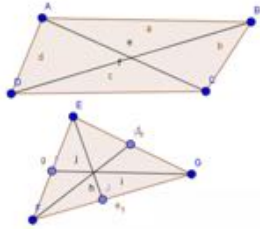
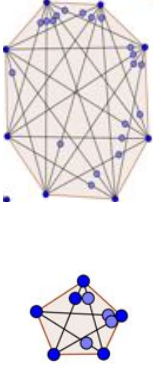

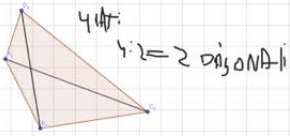
Provide definition	Formulate conjecture	Provide figures	Comment on other' posts	Commented on by others
All the groups but G10 and G12	All the groups but G5	All the groups but G1	G2, G3, G7, G8, G9, G10, G11, G12	G1, G2, G6, G7, G8, G10

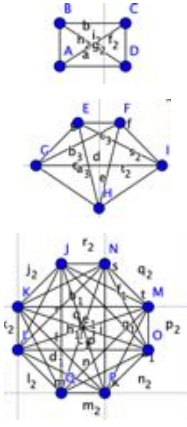
In summary, we attest to both a rich production of answers to the provided task, and to rich spontaneous interactions already in the very first phase of the activity. In the following phase, more interactions are solicited also through purposefully designed posts published by the observer.

In Table 2, we present examples of posts produced by some groups before viewing those of other groups. In particular, we have chosen a selection of posts which is representative of the variety of contributes proposed by the different groups. In the last column, we also present some objections raised by other groups, as soon as they were able to view all the posts.

Table 2: Examples of some groups' posts

Personal <i>concept definition</i> of diagonal	Conjecture on how to find the number of diagonals of a polygon	GeoGebra figures accompanying the conjecture	Objections provided by other groups
<p>G2: Diagonals = segments that can be external or internal to the figure and accordingly divide it.</p>	<p>G2: The diagonals of a polygon can be found by joining two opposite vertices with a segment.</p> <p>G2: Since the vertices join pairwise, the number of diagonals is half the number of vertices.</p>		<p>G7: The example of the octagon is incorrect because the diagonals do not perfectly intersect at the center.</p> <p>G7: And besides, it's not regular.</p> <p>G12: It's incorrect; there are other diagonals.</p> <p>G12: Every point must go to another point.</p> <p>G8: Diagonals are missing. See our example.</p>
<p>G6: A diagonal is a segment that connects the vertices and intersects at the center.</p>	<p>G6: To find the diagonals of a polygon, you need to connect all vertices that intersect at the center.</p> <p>G6: A polygon with an odd number of vertices has 2 diagonals, while when it has an even number of vertices, the diagonals are half the number of vertices.</p>		<p>G8: We think that your conjecture is wrong. See our example.</p>

<p>G7: The diagonals are segments that start from one vertex and reach the opposite one.</p>	<p>G7: The diagonals of a polygon are found by joining two opposite vertices with a segment.</p>		<p>G10: Why are there bisectors in the second one?</p> <p>G9: [Responding to the comment from P10] They are not bisectors, but they are the altitudes.</p>
<p>G8: For us, the diagonals are segments that start from one vertex and reach another.</p>	<p>G8: For us, to find the number of diagonals of a polygon, you need to add to the number of sides the number of times a vertex does not connect to another vertex.</p> <p>G8: Rectangle: 6 diagonals; Triangle: 3 diagonals; Octagon: 28 diagonals; Hexagon: 15 diagonals; Pentagon: 10 diagonals.</p>		<p>G11: Why do you count the sides as diagonals?</p>
<p>G9: A diagonal is a segment that connects a pair of vertices.</p>	<p>G9: To know the number of diagonals of any polygon, simply connect all vertices with each other.</p> <p>G9: Not all polygons have diagonals; for example, in a triangle, it's not possible to find diagonals because there are no opposite vertices.</p>		
	<p>G10: The diagonals of a polygon are found by connecting all vertices of the polygon with their opposite ones.</p>		<p>G7: [Referred to the example] If we try to use this reasoning even with the triangle, it wouldn't work. Indeed, when rounded, it's approximately 1.5.</p>

<p>G11: The diagonals are lines that divide the figure into multiple parts by connecting opposite vertices.</p>	<p>G11: The diagonals in the triangle are $0 + 2 = 2$ (quadrilateral diagonals), $2 + 3 = 5$ (pentagon diagonals), $5 + 4 = 9$ (hexagon diagonals), $9 + 5 = 14$ (heptagon diagonals), $14 + 6 = 20$ (octagon diagonals). The first number in each operation corresponds to the diagonals of the previous polygon, and the second addend is a natural number; these natural numbers are arranged in ascending order.</p>		
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Examining all the students' answers to the question what they mean by diagonal, we can note that their *personal concept definitions* apparently share the reference to some common properties: (a) the diagonals are segments that connect vertices (G4; G5; G6; G8; G9), (b) join opposite vertices (G3, G7, G11), (c) pass through the center of the polygon (G6), (d) cut the polygon (G1; G2; G11), and (e) can be either internal or external to it (G1; G2). When confronting students' *personal concept definitions* among them or with the respective produced figures we can note several interesting aspects. One of them concerns the use of the expression "opposite vertices". While evoking this feature, G11 actually traces all the diagonals of the polygons they provided, as they used the term opposite interchangeably with "non-consecutive". G7 too refers to opposite vertices in their definition but then sketches the supposed diagonals of a triangle connecting each vertex with the opposite side. Other groups do not mention that the diagonal should connect opposite vertices in their explanation, but refer to this property when justifying or explaining their conjecture or represent it in their figures (G2, G6, G9, G10). In summary, students seem to use the expression "opposite vertices" with different meanings; furthermore, definitions, properties represented in the figures, and arguments used in the justification of the conjectures are not always coherent with each other with that respect. The same happens with other properties. For instance, only G6 explicitly states that diagonals pass through the "center of the figure", though it is not clear whether they mean an interior point or the actual center of the figure. But this property seems to be evoked also in the figures of some of these latter groups (G2, G4, G7, G9). We can notice that G7 objects against G2's post because "diagonals do not perfectly intersect at the center", while neither G2 nor G7 mention this feature explicitly in their definitions or conjectures. Finally, some groups (G1, G2) refer to the fact that diagonals can divide polygons in parts, and to the fact that the diagonals can be either internal or external. But these properties do not apparently play any specific role in the search for a conjecture, in addition no pair has produced figures with external diagonals.

As for the conjectures, groups have interpreted and answered the request to explain how to find the number of diagonals of a polygon in different ways. G7, G9, G12 simply explain how to draw the diagonals in order to count them. G11 provides an algorithm for finding the number of diagonals of a polygon from the number of diagonals of the polygon with one less side. All the other groups but G5 and G12 propose some sort of (incorrect) formulas involving as input the number of sides of the polygon. The prevalent conjecture is that the number of diagonals is half the number of the polygon's

sides. This conjecture appears to be connected with a *concept image* according to which from each vertex of a polygon only one diagonal emerges, connecting it with an "opposite" vertex, possibly passing through the "center" of the polygon. Finally, we can still note that when commenting on others' posts, sometimes, groups comment on the basis of their own *personal concept definition* or image of diagonals of a polygon (G8 on G2 and G6, G11 on G8), and sometimes on the internal coherence between the provided definitions, conjectures and figures (G7 on G10).

Besides the issue of the coherence between *personal concept definitions*, conjectures and their justification, and produced figures, we can distinguish cases where the issue of generality is seemingly considered and addressed through either producing different types of polygons (G2; G6; G7; G8; G11) or not-too-familiar polygons (G2; G4; G10), and cases where this does not occur.

Discussion and conclusion

The analysis has highlighted the variety of students' responses to the given task, and the richness of the spontaneous interactions occurring already in the very first phase of the activity. The integration of Padlet and GeoGebra in the classroom activity allows the teacher to organize a HE that has promoted the active participation of all the students in the classroom activity. In such an environment, the task designed has promoted the production and sharing of *personal concept definitions* and conjectures, and of examples to illustrate and sustain the conjectures. From the analysis of the posts, we have been able to obtain clues about the students' *concept images* regarding the concept of diagonal of a polygon. The emerging picture shows the complex intertwining and tensions among students' *personal concept definitions*, *concept images* and the conjectures produced, and the possible tensions with the *formal concept definition*.

In the subsequent phase, not documented in this report, more interactions and a continuous elaboration of *concept definitions* and instances of *concept image* are solicited also through purposefully designed posts published by the observer. An added value, with respect to traditional environments, is that in this HE, all these interactions were recorded and made available both for inspection by the teacher for preparing future activities, and for the following classroom discussion, as a common basis for the teacher and students to discuss and reflect upon.

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Dialectical chatbots: AI-driven debate in mathematical education

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The rapid diffusion of AI tools like ChatGPT poses a dual challenge to education: ensuring meaningful homework assignments in the face of readily available AI solutions and teaching students to critically evaluate AI-generated content. This paper argues for integrating metamathematical debate into teaching as a response to these educational dilemmas. Drawing on Engeström's Expansive Learning theory, it frames the integration of AI tools in education promoting the development of new knowledge and practices through debate and discussion. Our proposed model for classroom activities aims to foster high-school students' critical engagement with AI-generated solutions, encouraging a shift from product-oriented learning to process-oriented understanding. This model seeks to mitigate the risks of mimetic learning and promote a deeper, metacognitive reflection on the nature of understanding and intelligence.

Keywords: Debate, artificial intelligence, metacognition, expansive learning, critical thinking.

Introduction

The advent of Generative Artificial Intelligence (GAI)—not to be confused with AGI: Artificial General Intelligence—through chatbots, such as ChatGPT or Gemini and other advanced AI tools, marks an epochal moment and a historic watershed, not only in the history of technology or education but in the history of humanity as a whole. Although we do know that GAI tools can sometimes fail in correctly answering even simple questions or solving easy problems, we are not yet sure whether this inability is intrinsically related to the inner workings of the tools themselves or it can be overcome by the rapid, momentous progress that characterizes the current technology revolution. The ability to assess the validity of AI generated answers is a skill that will be sorely needed for humans in the immediate future, not simply to improve employment chances, but to answer the existential question of human relevance in an AI-dominated world. The exploitation of GAI chatbots as an aid in solving homework is already common practice among students of all ages and backgrounds. A survey published in the magazine *Forbes* states that “89% of survey respondents report that they have used the platform to help with a homework assignment” (Westfall, 2023). This statistic comes from data collected in January 2023, making it over a year old at the time this paper is written; the same survey reports that about two thirds of the interviewed educators think that GAI chatbots should not be banned from schools and universities. While it is more or less obvious that simply denying the access to a new tool is not a sensible way to approach the emergence of a new technology in education there are reasons to think that educators have sound reasons to fear an improvident use of these tools. For example, according to William Chan “AI chatbots homogenize the thinking and presentation of students [...] they encourage students to think about questions from similar angles” (2023, p. 1). On the other hand, AI chatbots are already capable of passing college admission tests with a better score than most humans. The GPT-4 Technical Report (Achiam et al., 2023) states that ChatGPT Plus already can pass the SAT Math test with scores corresponding to the 89th percentile. So, it is likely

that students will increasingly resort to using GAI tools, mainly because “our current educational system rewards performance more than effort” (Chan, 2023, p. 2) but the increasing use of such tools would not fit neatly together with our current education systems “unless we redesign our assessments such that they are not easily solvable by GAI chatbots and, more importantly, establish an education system that encourages students to be oriented to learning rather than grades” (Chan, 2023, p. 3).

From the above analysis we pinpoint two problems, among others: how to assign homework whose execution cannot be nullified by the use of AI tools; how to instruct students to validate AI answers. These two problems are two faces of a more general challenge: the risk that our students will increasingly become mechanical and mimetical parrots of AI outputs. Yet, since the times of Plato, explicative dialogue has been considered a key element in avoiding the dangers of false knowledge and revealing the anti-mechanic and anti-mimetic character of science (cf. Piazza, 2020).

Consequently, we formulate this research question: how can metamathematical debate be integrated among other dialogue-based didactical practices in solving this educational conundrum of integrating AI tools into current educational routine in a way that ensures the assignment of meaningful homework, and teaches students to critically evaluate AI-generated content?

Literature review and theoretical framework

Since we are trying to resolve contradictions within and between activities, it is natural for us to frame the theoretical analysis of our research question starting from Vygotsky’s Activity Theory and its recent developments, such as Engeström’s Expansive Learning theory: “contradictions are the chief sources of movement, change, and development in activity systems” (Engeström, 2015, p. xvi). For example, the contradiction between the traditional roles of teachers and students: the first considered as those, *inter alia*, providing explanations, the second as those solving problems. Or the contradiction between the goals of the students, who see the GAI chatbots as a tool to ease their workload, and teachers, who see the GAI chatbots as a tool to increase the time students spend reflecting on how and why to do mathematics. Expansive Learning focuses on the collective transformation of activity systems through the identification and resolution of contradictions, emphasising the role of collaborative inquiry and the construction of new knowledge.

To address the above contradictions, we suggest the introduction of dialogical practices into classroom activity, in particular debate. Before addressing the question of debate *per se*, it is opportune to explain in which respects debate differs from other discursive practices in mathematical education. According to Prie and Schwarzenberger, a mathematical discussion “is purposeful talk [...] on a mathematical subject [...] in which there are genuine pupil contributions [...] and interaction” (1988, p. 461). Bartolini Bussi refines the definition of mathematical discussion in the following way: “Mathematical Discussion is a polyphony of articulated voices on a mathematical object (e.g. a concept, a problem, a procedure, a structure, an idea or a belief about mathematics), that is one of the motives of the teaching-learning activity. The term voice is used after Wertsch (1991), following Bakhtin, to mean a form of speaking and thinking, which represents the perspective of an individual, i.e. his/her conceptual horizon, his/her intention and his/her view of the world.” (1996, p. 16). While sharing its characteristics of purposeful talk in which there are genuine pupil contributions and interaction, debate differs from Bartolini Bussi’s characterization of discussion in two fundamental ways: first, it is a *regulated* dialogical interaction, in which speakers cannot arbitrarily decide when and if to intervene (they *have* to participate and the timing of their contribution is to be carefully

controlled); second, the perspective that debaters put forward is not defined by *their* view of the world but is assigned by a third party (Freeley & Steinberg, 2013). This last peculiarity might appear too restrictive but it actually makes it possible to elevate the exchange from a personal level to one that is more properly dialectical. This regulative aspect is what sets apart our approach of *regulated* debate from other debate modes used in classroom practice, as described in Balacheff's (2024) *de facto* survey on argumentation (including debate) in Mathematics Education.

We are dealing with contradictions whose resolution forces us to redefine our roles along with the object of our activity and the tools we use. In an educational setting where GAI chatbots are available (and used) students cannot limit themselves to acquiring knowledge 'produced' by the AI, nor can they simply participate in activities that follow a schedule constructed around tools that are not simply quantitatively but also qualitatively different from GAI chatbots—in this regard we have to see the diffusion of AI machines as a 'runaway object' (Engeström, 2008) that is a new phenomenon that, if not properly confronted, can lead to a potential disaster on a global scale.

Our choice to utilize Expansive Learning as a framework for our proposal is hereby explained referring to the six basic ideas of Expansive Learning, as characterized in (Engeström & Sannino, 2010)—actually that paper lists eight ideas but we want to focus only on the first six, as they are the ones rooted in the traditional culture-historical school of Vygotsky, Leont'ev, Il'enkov and Davydov.

First idea: distinction between action and activity. This is related to the famous example of collective hunting: each hunter performs a specific action that has a narrow focus and a definite start and end, but the activity to which it contributes continuously reproduces itself, albeit in a non-identical fashion. Our proposal identifies particular roles and phases yet it has not the goal to produce an object but to produce activity: as Engeström says, it is an activity-producing activity.

Second idea: re-definition of the zone of proximal development (ZPD) as the space for expansive transition (ET). This idea underlines the necessity of shifting the focus from the individual (ZPD) to the collectivity (ET). In our proposed activity, students will work together not just to learn what is already there, but to create a new understanding.

Third idea: the object-oriented being of the theory. This idea might seem to contradict the first one, but it really does not. The word object is here to be understood in the meaning of the German term *Gegenstand*, that could be translated as 'counter-stand', not in the meaning of the German word *Objekt*. In our case the *Gegenstand* is the AI itself that acts in a peculiar way, as explained by Engeström and Sannino: "motives and motivation are not sought primarily inside individual subjects – they are in the object to be transformed and expanded" (2010, p. 4).

Fourth idea: the dialectical concept of contradiction. In our case, there are the inner contradictions, in which the traditional learning model finds itself when confronted with the advent of GAI chatbots as detailed above, that act as a driving force for the transformation.

Fifth idea: abstraction and concretization. Being grounded in Activity Theory, Expansive Learning follows the dialectical tenet of ascending from the abstract to the concrete. Thus, our proposal starts from a simple abstract idea, such as the broad, abstract notions of: mathematical problem, machine capable of performing human-level tasks, debate as an organized academic activity then, through the introduction of complexity, synthesizing the proposal into concrete understanding.

Sixth idea: the double stimulation. This also is an idea that traces its origins to Vygotsky where the two stimuli are the ‘challenge’ and the ‘mediating tool’. In our case, the challenge is the analysis of a solution of a problem given by the AI, and the mediating tool is the debate itself.

The other complementary part of our framework, in addition to the theory of Expansive Learning, is the theory of argumentation that makes possible a structured debate. As implied above, the research on regulated disciplinary debate in mathematics education is sparse, thus we shall refer mainly to our previous work (Matteucci & Tortoriello, 2023) that considers a model of regulated debate in mathematics sporting the three defining aspects of: discussing a contentious resolution, featuring two opposing debating teams with incompatible points of view, and having a non-free-form protocol. Additionally, we use the motions’ taxonomy we presented in the said paper and, in particular, we focus on metamathematical motions—i.e. those that we have called ‘worst error’, ‘comparative analysis’, ‘best explanation’—that are those motion related to the so called philosophy of mathematical practice and patterned on the scientific use of that kind of dialectics that we dubbed diaporetic dialectic.

Model of the proposed activity

Here we illustrate a model for our proposed classroom activity that should allow us to explore integrating metamathematical debate and dialogue-based teaching methods to effectively use AI tools in education, ensuring meaningful homework and teaching students to critically assess AI-generated content. We start with an overview of the model, stating also the objectives and the expected results, then we go on explaining how it addresses the research question and how the model is coherent with the framework illustrated in the preceding section.

The model is structured around three distinct phases that are, in turn, divided into three other subphases; these phases were devised taking into account the objectives stated in our research question and the specifics of disciplinary regulated debate (see the comments on our previous work above). We called these three phases: ‘chatting’, ‘discussing’ and ‘debating’ because in the first one the key dialogical activity is chatting with the AI; in the second one the key dialogical activity is a mathematical discussion (in the meaning of Bartolini Bussi, see above); in the third one the key dialogical activities revolve around a metamathematical debate. We assume that the activity involves a single class under the direction of one teacher, we also assume that, for some activities, students are divided into teams of 3-5 people.

First phase: chatting. This phase involves only the teacher and is not carried out in the classroom; it is divided into the following subphases: 1) the teacher selects a mathematical problem for the AI to solve; 2) the teacher inputs the problem to a GAI chatbot, in one or more steps; 3) the teacher sends the text of the problem along with the proposed AI solution to the students. Teachers should take care in explicitly asking the AI to explain the reasoning behind each of the steps it will take towards the solution in the prompt they will use to input the problem and could also allow the GAI chatbot to use ‘external’ computational tools, if deemed opportune—like the ones using Wolfram Language or Python in GPT-4, for example. It is also advisable to prepare a document that contains both the assigned problem and the answer of the GAI chatbot; in this regard it may be useful to ask directly the GAI chatbot to prepare a LaTeX file that contains both, avoiding the hassle of having to assemble it manually.

Second phase: discussing. This phase involves both the teacher and the students and is carried out both at home and in the classroom, it is divided into the following subphases: 4) students critically review the solution at home according to a predetermined list of criteria (correctness, completeness, effectiveness, generalizability, clarity, understandability); 5) students discuss their assessments during class, under the supervision of the teacher, to establish whether there is a general consensus on the conformity of the solution to the above mentioned criteria; 6) the teacher, after having taken note of the issues about which the conformity to the said criteria cannot be unanimously decided—even after a brief discussion—drafts precise debate motions, using the metamathematical motions’ taxonomy found in (Matteucci & Tortoriello, 2023) about each controversial unresolved issue and assigns them to the teams. Students are basically required to assess: whether the solution is factually correct; whether the solution answers explicitly all the requests to be found in the problem; whether the solution achieves its goal in a direct, efficient manner; whether the solution is extendable to other similar more general problems; whether the solution is straightforward in its presentation; whether the solution is easily understandable to the perspective audience; i.e. the students themselves. The eventual debate motions that the teacher has to draft are to be constructed starting from the issues about which an undisputed unanimity cannot be reached in class. If the solution is not universally accepted as effective and generalizable, a motion patterned on the ‘comparative analysis’ type will be drafted. If the solution is not considered to be clear and understandable, the debate resolution will be of the ‘best explanation’ type. In case the students would concordantly agree from the start that the solution is incorrect or incomplete, the teacher will draft a motion according to the ‘worst error’ pattern.

Third phase: debating. This phase involves the students (both as singles and as teams) and the teacher. It consists of: 7) the teams, after thoroughly preparing at home, debate the motions provided by the teacher according to a specific protocol; 8) the teacher debriefs the students regarding the debate results; 9) students and teacher discuss what they have learned from the debates and the debriefings. The debates shall follow a formal protocol: it is not necessary to use an already established protocol in competitive debate (Freeley & Steinberg, 2013) and *ad hoc* formats can be used, as long as one keeps in mind the three caveats: flexibility, caution, continuity (Matteucci & Tortoriello, 2023). It is important to underline that the role of the teacher in this third phase will alternate between scaffolder, in the debate-preparation phase, moderator during the debate proper, and assessor during the debriefing. The debriefing, however, is not only to be intended as a moment to give feedback on the relative performance of the teams involved, but also to summarize the key clash points of the debate itself, highlighting the instances of proper and effective reasoning and argumentation and pointing out eventual misconceptions or non sequiturs, in order to prepare the final discussion about what has been learned. It is worth noting that, since the debate itself pits teams with incompatible points of view against each other and is a moment of confrontation, the moment of the synthesis—if any—is in the final discussion, not during the debate proper.

As said, the overall objective is giving students meaningful homework assignments and instructing them in the task of validating AI answers. For what concerns the first point, students will have to work both individually and as a team at home when executing subphase 4 above, and when preparing for the discussion of subphase 7. It should be noted that in regulated debate, unlike in simple class discussions, the debate itself is never improvised. We expect that students, as they transition from not being able to use AI tools to having to use them, will experience a qualitative shift in focus:

transitioning from the assumed goal of obtaining a product (the solution) to the implied goal of being able to question the whys and hows behind that product. This is something that in the context of Expansive Learning is known as a qualitative turning point. This very result makes the teacher assignments meaningful because they lead to an activity that cannot be ‘delegated’ to the AI since it involves live discussions and debates. The students can thus understand that just parroting AI output or mirroring its procedures, without having understood them first, not only would not be sufficient to fulfil the teacher’s assignments but would also not lead to mathematical competence. In fact, as Gabriele Lolli (2022) wrote: “Learning and repeating proofs is useless if these are conducted solely at the object level, without being informed by metalogic. When one speaks (or writes), without noting how one is speaking and why, focusing only on the meaning of what is said, the logical structure of one’s discourse remains unseen; when one speaks, one does not consider grammatical rules but the content; to see the logical-grammatical pattern, the discourse itself must be the object of reflection. Even merely annotating next to each expression the one from which it derives and the rule by which it does so constitutes an action different from the material act of executing a substitution or more generally a syntactic transformation; it means managing a system of logic (rather than a formula) and making a choice, a strategic action. On one hand, the agent is not mechanical (=syntactic); on the other, it reasons on the formalism rather than merely executing the formalism.” (p. 79, our translation). This quote points out the essence of doing mathematics; our model is crafted in such a way to encourage students to adopt this approach, not in an extrinsic or forced manner, but as a natural consequence of the structure of the proposed activity, given the assigned tasks. The students’ metacognitive reflection upon the AI output and upon the assessments (their own, the others’, the teacher’s) of the output itself, will also prompt them to confront with the question of what does it mean to understand something: merely copying patterns mechanically does not generate understanding.

Discussion

The model sketched above is coherent with the framework we described in the previous section. In particular, for what concerns the theory of Expansive Learning, it is directly related to the six key ideas listed above.

First, the model is designed not merely to produce a single outcome but to foster ongoing activity, in this case, the critical and evaluative engagement with AI-generated content; this aligns with the ‘activity-producing activity’ idea and the tenet: ‘action is not activity’, since all its phases cannot be understood in isolation but acquire a sense only as a whole.

Second, with its multiple dialogic activities, the model shifts the focus from individual learning, as in the traditional ZPD, to collective creation of a new understanding.

Third, motives and motivation are to be found in the ‘object-AI’ not only as a counter-stand on which one stumbles upon but also—after it has been assessed, discussed, evaluated—a richer conceptual object that will constitute itself as the “future-oriented purpose of the activity” (Engeström & Sannino, 2010), as the students will realize after each iteration of their expanding activity.

Fourth, the model leverages the contradictions between the traditional and the not-yet-there new learning methods, both from the perspective of the students engaged in their homework and of the teachers trying to assess the students’ performance, thus literally building the new method out of their

engagement in activities whose basic premise is the exact negation of what the traditional method required, that is using the AI and copying its results.

Fifth, ascending from the abstract to the concrete involves starting with the simplest category (e.g. a mathematical problem, a machine capable of human-intelligence-level tasks, competitive debate) from which, adding layers of complexity by considering more variables and relations through the model's activities, one derives a synoptical understanding of the whole process in its multifaceted and interconnected nature, an understanding that eventually challenges some of the preliminary assumptions (e.g. that the interesting aspect of a problem lies only in its solution, that a machine is actually capable of thinking, that the main goal of debating is competition, etc.).

Sixth, the double stimulation involves a first stimulus, i.e. a problem that cannot be solved easily and immediately, in our case it is the assessment of the validity of the AI-generated solution to a given problem; then an additional stimulus is given, in our case the discussion/debate format itself acts as the second stimulus, providing a structured method for engaging with the challenge and facilitating the development of critical thinking and argumentation skills.

Conclusions

The lack of empirical evidence to directly support the propositions and models outlined in this paper does not stem from an oversight but from the nascent stage of our project and the innovative nature of the educational model we propose. Our research group already worked and experimented both with debate-based activities and technology-enhanced learning (Musmarra et al., 2023; Tortoriello & Veronesi, 2021), now we are in the preliminary stages of designing a pilot experiment to be conducted in a secondary school patterned along the guidelines illustrated in this paper. This pilot test, that aims to assess the feasibility, acceptability, and initial effectiveness of our proposed model, will be conducted in the context of the research project known as Liceo Matematico (Mathematical High School), see (Capone et al., 2017). Should this pilot experiment yield positive results, we plan to expand our investigation into a broader experimental study encompassing a wider array of secondary education institutions. By the end of the revolution we are currently experiencing, either AI will prove itself inferior to human intelligence or it will prove superior. In the first case it will be necessary to modify educational practices to let humans learn and exercise those skills in which they are superior, in the latter case it will be necessary to modify educational practices to let humans understand why AI is superior in handling such tasks. In either case we will have to radically modify our practices, not only to give meaning and purpose to the educational system but to give meaning and purpose to humanity itself. We hope that our proposal might be one of the first steps in the right direction.

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Beginning to shape mathematics students' teacher identity through activities based on digital and geometric transitions

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Recent literature on teacher education highlights the importance of the development of teacher identity. This paper explores how the dynamics of positioning acted by undergraduate mathematics students in designing an activity with the use of digital technology for high school students begin to shape their mathematics teacher identity. In the first phase of the experiment, they are involved in a problem-solving activity in which two transitions are implemented: from Euclidean geometry to Taxicab geometry and from the use of paper and pencil to dynamic geometry software. In the second one, they are invited to design a similar activity for high school students. Their role changes, and the way of conveying mathematics also changes, becoming the one to be taught and, with it, of using the digital tool. This induces some positioning driven by the need to communicate mathematics. A change of identity occurs, and the mathematics and technological teacher's identity effectively begins to form.

Keywords: Taxicab geometry, GeoGebra, digital technology, transitions, teacher identity.

Introduction and conceptual framework

In current research, identity has received attention from researchers from a wide variety of theoretical and methodological approaches in the social sciences and educational research (Sfard & Prusak, 2005; Andreouli, 2010; Black et al., 2010). Alongside specific knowledge, a fundamental component that contributes to the formation of the teaching identity is technological and digital knowledge (Drijvers & Sinclair, 2023; Prieto-González & Gutiérrez-Araujo, 2024; Brunetto & Dello Iacono, 2023). Teacher identity has been identified in different ways: in terms of the constant reinventing of themselves that teachers undergo, in terms of the narratives that teachers create to explain themselves and their teaching lives (Sfard & Prusak, 2005). Furthermore, recent technological and digital changes have inevitably affected teachers' identity encouraging creativity, as "a disposition to promote the ability to generate several solutions and solution paths, to change directions of focus" (Levenson, 2022). The question arises: what stimuli can be provided in such a way that mathematics students begin to look at themselves as teachers? Our aim is to begin to reduce the gap between undergraduate students' mathematics identity (Miranda et al., 2023) and mathematics teacher identity, really a shift in identity, highlighting the role of technology in this process. In the identity construction undergraduate students move from mathematics to learn to mathematics to be taught and change their roles, assuming positions as teachers in challenging school contexts, a discontinuity highlighted by Felix Klein (Kilpatrick, 2019). This is only a first attempt to trigger an early identity shift by fostering the acquisition of content, pedagogical, and technological knowledge, in accordance with the TPACK model (Mishra & Koehler, 2006), through the activation of geometric and digital transitions, considering the technology component for teacher training to align with contemporary educational needs. According to research, students can improve their understanding by investigating figure properties and making conjectures, particularly using dynamic geometry systems. Furthermore, comparing different geometric worlds and their semiotic representations helps to advance Euclidean knowledge (Komatsu & Jones, 2020; Miranda & Saliceto, 2023). Given that "becoming the-teacher-you-wish-to-be is tantamount to learning how to engage in certain types of professional practices"

(Sfard, 2019), by fostering future mathematics teachers' ability to design tasks in a digital environment, we intend to act on their future mathematics teacher identity. In this sense, the intervention aims to begin to shape mathematics students' teacher identities by offering them learning and designing opportunities through shifting from content knowledge to knowledge to teach, with technology as an additional element to strengthen their specific and technological identity as teachers.

Framework for designing the activity for undergraduate students

In designing the entire experiment, we let ourselves be guided by the principle that exploring concepts in Taxicab geometry can help students better understand concepts in Euclidean geometry and that transferring their knowledge back and forth between these two geometries, aided by a digital environment, from paper and pencil to GeoGebra, strengthens cognitive process activation, fostering meaningful learning experiences. Furthermore, the integration of technology to enhance the teaching of subject content is considered a crucial skill, in accordance with the TPACK model (Mishra & Koehler, 2006). The design develops on two levels and foresees two phases: the first level focuses on the epistemological and cognitive levels, and the second one on the metacognitive ones. In the first phase, undergraduate students are actively engaged in problem-solving activities in which digital and non-digital representations interact and merge to construct conceptual knowledge, while in the second phase, they are required to design a similar task to be addressed to high school students, thus changing their working perspective and their role. The designing opportunity aimed to sensitise them to the complex interactions between content, pedagogy, and technology and to develop their TPACK skills.

Framework for analysing the perceived teacher identity

Identity is a dynamic concept. In a social context it is defined by the social role, social relationships, and the subject to be addressed in relation to the role. Andreouli (2010, p. 14.1) asserts that identity:

is seen here as embedded in social relations and as dynamic, contextual and relational. Social representations and the dynamics of positioning between self and other define identities.

The concept of positioning can help clarify the relational aspect of identity. Studies on identity development have highlighted the relationship between identity and the processes of positioning that occur within discourse. Positioning theory is the study of how people assign themselves and others a position in discourse (Harreé & Van Langenhove, 1999). An important topic of discussion in the literature on identity is its narrative and discursive aspects, as well as how narrative and discourse shape and are shaped by identity. Clearly, teachers' narratives about themselves and their practice, as well as the discourses in which they engage, provide opportunities for exploring and revealing aspects of the self. Literature promotes this way of thinking about identity and holds in high regard the idea that stories are a means of expressing identity: "identities" are "collections of stories about persons" (Sfard & Prusak, 2005, p. 16). In our context, students' narratives are related both to the mathematical knowledge to be taught, the choice of which stimulates different reasoning, and to the digital tools useful to vehiculate it. Who am I with respect to this knowledge and the tools to construct it? How do knowledge and tools intervene in self-other relations involved in my teacher's identity construction? Considering the role played in the specific context, the impact of discourse in self-other relations, and the acts of positioning within these relations as powerful in the shaping of students' mathematics teacher identity, we analyse students' narratives about themselves during the experience (Sfard & Prusak, 2005) and positioning acts (Harreé & Van Langenhove, 1999).

RQ *How does designing learning activities based on digital and geometric transitions shape and influence the early development of undergraduate mathematics students' teacher identities?*

Methodology

The experiment, which took place in a university in the south of Italy, involved twelve undergraduate mathematics students taking a short course, organised in two phases shifting the students' position from “to be a student” to “to feel a teacher”. The first one is a moment in which students relate to mathematics to learn while immersing themselves in solving the given problem, and the second one is a moment in which students reflect on mathematics to be taught through a workshop on designing. Both phases could also be interpreted as training phases in the language of teacher professional development, giving the opportunity to learn useful skills for the workplace, particularly teaching skills, since they require an understanding of how and when the use of digital technologies can add value in terms of learning and understanding concepts. According to Mishra & Koehler (2006), content, pedagogy, and technology bind each other in a dynamic way, and trainee teachers need to be prepared not only to transmit subject content but also to integrate digital skills into their lessons. On balance, our process involved content knowledge, knowledge to teach, and technological knowledge, following the TPACK perspective (Brunetto & Dello Iacono, 2023). According to the design students worked in randomly structured groups in a thinking classroom environment (Liljedahl, 2016). Each group had access to a vertical non-permanent board as well as a digital environment that included a collaborative tablet, digital software for design, a collective document in which to place the solution product and process (as well as taking photos and reporting them in the collective file), and a personal document in which to report on their experiences.

1. *Define the Taxicab metric; define Taxicab objects (circles, conics, angles, trigonometric functions), one or more. Choose them and investigate their taxicab world (relationships, statements) by reasoning on definitions and statements analogous to those already known in Euclidean geometry.*
2. *Design a similar task intended for upper secondary students to switch between Euclidean and Taxicab geometry objects supported by GeoGebra.*

Figure 1 - Task addressed to undergraduate students

Define the Taxicab distance, define the Taxicab circle (or conics), and investigate their world (relationships, statements, for example). At how many points do they meet? (How many symmetries does a taxicab circle have?) Compare it with Euclidean geometry through reasoning on analogous statements (helped by the transition from paper and pencil to GeoGebra).

Figure 2 - An example of task to be addressed to high school students from undergraduates

Undergraduate students were given a task to explore the definition of the Taxicab metric, the advantages of using it to understand a Euclidean geometry concept, and conjecturing and proving about it supported by the dynamic geometry software GeoGebra. Specifically, the task consists of two subtasks (Figure 1), each aiming at a specific subgoal: cognitive (learning in the role of the learner) and metacognitive (reflecting and designing in the role of the teacher). It requires students to shift their focus from the relationship between themselves, their colleagues, and their teacher to the relationship between themselves and their learners, while activating the transitions from paper and pencil to dynamic software and from Euclidean to Taxicab geometry. In the workshop phase, undergraduate students, now in the role of teachers, design an analogous activity to be addressed to a high school students' group (12th grade) (Figure 2). Interaction with high school students highlighted

not only the pedagogical potential revealed in the transition between geometries but also the power of GeoGebra to help them learn geometric properties and their relationships in a dynamic way.

Data collection and Data Analyses

Our collected data, which included students' recorded discourses, collective logbooks and board photos reporting students' thinking about the problem, and personal feedback questionnaires reporting students' perceptions of the learning experience, was digitally saved. Specifically, students' feedback on the experience was qualitatively analysed through a systematic and objective identification of some indicators (I1, I2, I3, I4), separately looking at the effects of the two lived transitions (geometrical and digital) on students' teacher identity by labelling and classifying sentences relating to a single identity indicator to understand whether the designed Taxicab digital or non-digital manipulation of a concept leads to Euclidean digital or non-digital manipulation, and conversely, that is a harbinger of meaningful teaching. The lens we used to analyse the discourses draws on two theoretical constructs: the definition of identity (Sfard & Prusak, 2005) and the theory of positioning (Harré & Van Langenhove, 1999; Andreouli, 2010). We focused our qualitative analysis on the nuances to which students perceived their identity as mathematics teachers (students' self-perception of teachers' identity) by examining students' answers to the following selected questions:

Do you think switching from one geometry to another would help students improve their understanding of concepts? Would it help to stimulate meaningful learning (motivation, engagement, building new knowledge, etc.)? Could an activity on the taxi metric be useful in school? How did you find yourself working in the role of a teacher who thinks and designs a task for high school students? Do you think the use of dynamic geometry software to explore relationships, properties, and definitions moving between Euclidean geometry and Taxicab geometry would help to better understand geometric concepts? Tell us.

We considered students' answers to the above questions as stories the student experienced in the role of teacher, stories that students tell themselves as they change their context and role, passing from the cooperation to solve to the cooperation to design, as well as their positions in the relation self-other. We analysed students' discourses from the perspective of positioning theory. More in detail, we looked for whether changes emerged in the aspects characterising the initial mathematics student's identity. To accomplish this, through thematic analysis (Braun & Clarke, 2006), we identified four positioning indicators, and we went on to observe the occurrence of these components in the answers provided by students. Specifically, to investigate the social roles and self-other relationships involved in identity construction, we elaborated and grouped significant students' excerpts on four identity indicators, identifying the positionings and transitions that occur when changing identities: I1 - *Positioning the use of geometries between self and learners* in relation to the effectiveness of the transition between two geometries to understand a concept and construct knowledge to teach comparing themselves with others; I2 - *Positioning the use of a digital instrument between self and learners* in the dynamic between themselves, peers, and school students; I3 - *Positioning themselves in the role of teacher* in elaborating and choosing the mathematics to teach and how and to teach it in terms of methodology, instrumentation, environment; I4 - *Exhibiting teaching creativity* in designing models: reflecting on developing or expanding a workshop model in teaching. Obviously, these indicators are not separated from one another in the sense that each captures nuances about the impact of the designed activities on the student's mathematics teacher identity.

Findings and discussion

We investigate the effects of the transitions implemented on students' teacher mathematics identity by highlighting the identified indicators in students' feedback discourses. Each indicator captures nuances about the impact of the activities on students' perceptions of their teacher's identity.

I1 - *Positioning the use of geometries between self and learners*. The indicator I1 emerges from the students' answers: the construction of one's own identity as a "mathematics teacher" takes shape in a continuous exchange and negotiation with others in the transition between geometries.

Us1: Of course, switching from one geometry to another helps you understand the concepts because **it allows you to see similarities and differences between the geometries** you've already studied, especially if you're studying a non-trivial or unusual geometry. [...] It was a very valuable experience for me in terms of developing as a potential mathematics teacher.

Us5: The transition from one geometry to another, in my opinion, would greatly **assist students in improving their concept understanding, especially because some notions change representations**. A similar passage that comes to mind is one that occurs in modular algebra; when switching from one definition set to another, the sum of two numbers can produce different results.

Us8: I believe that dealing with different types of metrics would be very useful for students at different school levels and would help them **stimulate their creativity and flexibility**, helping them to decide which metric is the most convenient to use and to build new knowledge themselves. I remember that in the Geometry III course we dealt with how some sets could or could not be open in different topologies [...]

Us2: I think it could be useful in school because it's the simplest example of geometry different from Euclidean geometry I met. Above all, I would point out that many things are different from their idea of geometry, **to get them used to thinking outside the box** if they ever wanted to follow a scientific university pathway.

I2 - *Positioning the use of a digital instrument between self and learners*. It appears that signs of a shift towards mathematics teacher identity, even in relation with the dynamic geometry environments, are emerging. The use of GeoGebra foster a deeper understanding of the mathematical contents investigated with paper and pencil (Us8), particularly for those with weak geometric intuition (Us6):

Us6: [...] I believe that mathematical software in general is very useful for better **visualising concepts** and understanding them to the fullest, especially **for those who have little geometric intuition** and cannot imagine figures on their own.

Us8: [...] I would propose the use of GeoGebra to show students **in real time, on a digital whiteboard**, what the use of Euclidean and Taxicab metrics entails to better understand the concepts covered by paper and pencil.

Us5: [...] In my opinion, the use of dynamic geometry software to explore relationships, properties, and definitions between the two geometries **would help a lot to better understand geometrical concepts**.

Us1: Of course, this is also helpful both for the teacher as it simplifies the work and for the student as the software gives the possibility to see graphs and geometric figures **from all possible angles**, where sometimes it is impossible to see them manually.

I3 - *Positioning themselves in the role of teachers*. Many of the students' answers showed the indicator I3, which describes the changing of position in relation to the role as it emerges. Some students' protocols appear to strongly reflect a rethinking of themselves as mathematics teachers. The task creation encourages students to take an active role in their learning and to teach others. They understand the dual role they play in the development of both their own knowledge and that of other school students. Us8 felt a sense of responsibility to create a resource, as a teacher would, and paid closer attention to his words and preparations. Us8 felt consolidated his knowledge useful for teaching and excited to think that a task would benefit school students. Furthermore, rather than giving them answers, give them hints that will allow them to reach their own conclusions, encouraging autonomous exploration. In Us8's words, it appears that the comparison with others, while developing one's own mathematics teacher identity, has favoured a shift in perceived competence. When he says that he had to think *about relating to students and not to colleagues*, he is thinking about assuming a new role, a new position (Sfard & Prusak, 2005; Harré & Van Langenhove, 1999):

Us8: It was very interesting to "improvise" myself as a teacher, even if it was not very simple, probably because I still see myself in the role of a student. I tried to write a proposal that could involve and interest the students, making them understand how the topic is very real, close to their days, and making it sufficiently light, especially in an initial approach, also through software such as GeoGebra. It may seem obvious, **but I had to think about relating to students, and not to colleagues** with whom I share the same knowledge: it was a small stretch of imagination, but certainly useful and stimulating, also in view of a possible future job.

Us7 reflects on how to design a learning activity that engages students by using real-world issues. His identity as a designer begins to take shape, and this is a clear act of positioning in the teacher role:

Us7: [...] Students would surely gain from working on the taxicab distance problem. I think it can be thought-provoking to see how this metric is "more real" for those who are used to driving through our cities. Moreover, it has been observed that this distance is used in disciplines other than mathematics, such as criminology. **All ideas to get students interested!** Especially if they were first introduced in groups and compared as a group. Additionally, to compare and better visualise and explore topics, it would be helpful to use tools like GeoGebra.

Us5 and Us3 also present the challenge of communicating clearly, like that of a teacher. This demonstrates an early development of a teacher's identity:

Us5: Working in the role of a teacher who thinks and designs a task for high school students made me feel that I had **a great responsibility and I asked myself if I was able to do it**, since I feel that my knowledge is not yet enough to efficiently convey a concept to students younger than me. The reflections that led me to our proposal were to make the taxi metric a topic that can be understood by everyone, regardless of the degree of preparation of each student. The aim was therefore to deal with this concept as **clearly as possible so that it could reach everyone**.

Us3: My life goal is to become a teacher [...]. **The role put me at ease and let me dream a little**. The goal was to propose an alternative lesson that would capture the students' attention. Theory is very important, but if I ever become a teacher, I would always accompany it with many examples, especially from daily life.

I4 - *Exhibiting teaching creativity*. The theme I4 emerges as an example in the protocols of a working group. They recognise that being engaged in creating an activity is an opportunity for personal and

collective growth, both as students and future teachers. Examining an excerpt of the collectively recorded and transcribed dialogue describing the group's concept development (*concept digital manipulation*) prior to designing an activity for high school students. Mindful of the difficulties or dislikes encountered at school in the study of Euclidean trigonometry, they considered Taxicab trigonometry, and the *concept definition (cd)*, *concept image (ci)*, and *concept digital manipulation (cdm)* of concepts gave them confidence to design an activity to be addressed to high school students:

- Us2: [...] A *Taxicab-radian* is an angle whose vertex is the center of the unit Taxicab-circle and intercepts an arc of Taxicab-length 1. (*cd*) Look at the figure!! (*ci*)
- Us1: We have this definition (*cd*): The measure of a *Taxicab-angle* θ is the number of Taxicab-radians subtended by the angle on the unit circle with vertex in the center.
- Us3: We could say: The *sine* of a Taxicab-angle is the Taxicab-distance from the x-axis. If so, let's graph it!! Oh... It's edgy!! (*cdm*) It's not the same (*ci*). Right at the maximum and minimum points. Better to say angular! Mh... (*cdm*) (Figure 3)

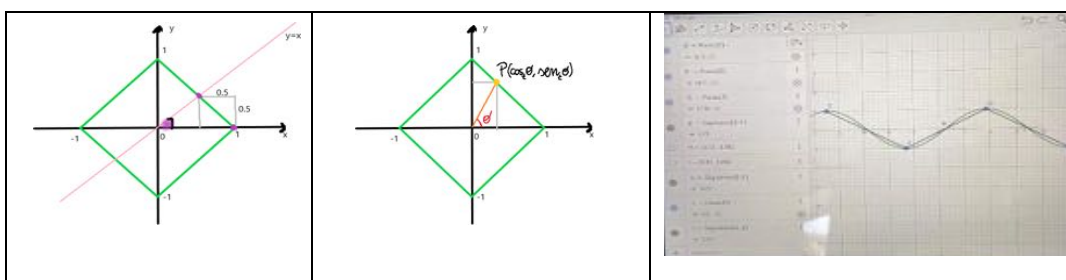


Figure 3: The Taxicab radian, sine, and cosine

- Us2: What other concepts? And which Euclidean trigonometry Identities or Theorems still hold with respect to the Taxicab metric? Which ones don't? (Law of sines, Law of cosines, Carnot Theorem). The fundamental identity changes. (*cdm*)

The *creative* act implemented by the group continues to manifest in the process of building the trigonometry to understand the Euclidean trigonometry. Once explored the Taxicab trigonometry, this group designs a problem-solving activity in such a way that new knowledge is not passively received, but rather actively constructed, interacting with colleagues, and starting from real life.

Conclusions

Much recent literature on teacher education contains arguments for attention to identity in teacher development. However, there are limited studies in the accessible literature that investigate mathematics students' teacher identity development. In this study, we look at the impact of discourses and positioning that occur while designing with colleagues and submitting designed tasks to high school students on the formation of university students' early mathematics teacher identities. Undergraduate mathematics students, attending a mini-course on useful knowledge for the workplace, were engaged in their first teaching experience. They shifted from a learning to a teaching position using GeoGebra to solve problems involving the transition from Euclidean to Taxicab geometry. The qualitative analysis revealed that beginning teachers' experiences positively influenced their mathematics teacher identities. Furthermore, analysing participants' feedback questionnaires revealed valuable information about teacher identity development. The relationships between technology and mathematics revealed promising implications for prospective teacher

identity development. A future research direction might be to use TPACK to analyse the results in relation to each indicator, from which a good predisposition towards the use of technology emerges.

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Comparing the efficacy of dynamic vs. static visualisations in students' characteristics of basic mental models of the derivative

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In this study we examine the question of whether the two central basic mental models of the derivative “local rate of change” and “tangent slope” can be developed more effectively using dynamic or static visualisations after an intervention introducing the concept of the derivative. For this purpose, a questionnaire on these two basic mental models was completed by 11th grade students (N = 218). The t-tests indicate that the “tangent slope” was developed significantly more effectively in the dynamic group than in the static group with a small effect size ($d = .462$); however, there was no difference regarding the “local rate of change” when comparing the two groups.

Keywords: basic mental models, derivative, dynamic visualisations, static visualisations

Introduction

Dealing with the derivative in calculus courses at upper secondary level poses a major challenge for many learners. According to the framework developed by Zandieh (2000), a full understanding of the concept of the derivative is achieved when the three process-object-layers *ratio*, *limit* and *function* can be applied in multiple representations (*graphical*, *verbal*, *physical* and *symbolic*). Difficulties can occur at all layers. Various studies have shown a lack of understanding or significant misunderstandings concerning the concept of *limit of a function* and the derivative as a rate of change or as a tangent slope (Bressoud et al., 2016; Orton, 1983). To counteract these difficulties at an early stage, the study that is featured in this paper starts with the introduction to the concept of the derivative at one point (process-object-layer *limit*).

First, the theoretical framework for basic mental models of the derivative, i.e. adequate interpretations that enable learners to understand the concept they are using, is described (Greefrath et al., 2016). Then, learning with visualisations is then explained. In particular, dynamic visualisations such as animations that can be created via GeoGebra have considerable potential in calculus lessons: Through a visual approach to calculus concepts, students can gain an insight into the *inner world* of underlying formal concepts (Bressoud et al., 2016). An integration of basic mental models and visualisations seems to provide a promising perspective.

Theoretical Background

Concept of basic mental models of the derivative

The concept of basic mental models (in German: “Grundvorstellungen”) is based on traditions dating back to the 19th century. It is characterized by the description of “the relationships between mathematical content and the phenomenon of individual concept formation” (vom Hofe & Blum, 2016, p. 230). With the help of basic mental models of mathematical concepts, students can grasp their content in such a way that they can be applied to different (real-life) situations. Nowadays, didactics scholars differentiate between a plethora of basic mental models for various mathematical concepts. In relation to calculus in upper secondary schools, the concept of the derivative and the integral are the most fundamental ones. For the derivative, Greefrath et al. (2016) identified the

following four basic mental models: *Local rate of change*, *tangent slope*, *local linearity* and *amplification factor*. A study by Greefrath et al. (2022) with undergraduates of mathematics and physics also found that the surveyed students most strongly prefer the *tangent slope* in various application situations for the derivative, followed by the *local rate of change*. Furthermore, a look at international curricula shows that the other two basic mental models tend to play a subordinate role in the discussion of the derivative in mathematics lessons.

Local linearity is characterized by the fact that, locally, the graph approximately corresponds to a straight line and that the derivative corresponds to the slope of this straight line; in the case of the amplification factor, the derivative provides information about how strongly small changes in the independent variable affect the dependent variable. In the basic mental model *local rate of change*, the derivative is interpreted as the limit value of a quotient (average rate of change). Traditionally, this approach is based on a four-step procedure: Starting from the function value of a function at one point, the difference of two function values is regarded, after which the average rate of change, which is represented by the difference quotient, is calculated. Finally, the local rate of change can be determined by systematically reducing the considered interval. The core idea of the *local rate of change* is that the derivative indicates the instantaneous velocity of a change process. In the basic mental model *tangent slope*, the derivative at a point is interpreted as the slope of the tangent to the graph at this point. Here, the tangent acts as a *clinging line* and has the same slope as the function graph at this point. In addition, a comprehensive idea of the *tangent slope* is that the tangent indicates the local direction of the graph (Greefrath et al., 2016, 2022).

Learning with visualisations

Visualisations can promote a deeper understanding of mathematical concepts by providing complex information in an illustrative way (Tversky et al., 2002). They can also contribute to the development of basic mental models, especially in calculus (vom Hofe & Blum, 2016; Zengin, 2018). On the one hand, visualisations can be static, such as illustrations or photos, which have no temporal process. On the other hand, the use of dynamic visualisations, for example in form of animations, is also a suitable option. They differ from static visualisations in that they “portray continuous temporal change of a subject matter by triggering perception of continuous change” (Wagner & Schnotz, 2017, p. 69). In our study, we define the concept of dynamic visualisations more broadly by also including user interactivity. These dynamic visualisations can be created via GeoGebra and offer users the option of using an animation or the drag mode to move a point on a function graph.

In this context, special emphasis should be put on Dyna-Linking, which enables a dynamic linking of graphical-visual, verbal and algebraic forms of representation (Ainsworth, 1999). Such links make the relationship between a function and its derivative or the relationship between the tangent slope and the graph of the derivative function particularly clear (Hong & Thomas, 2015).

Both the basic mental models *local rate of change* and *tangent slope* can be viewed from a dynamic and a static perspective: The *local rate of change* is characterized by the limit value of average rates of change and can be understood statically via the formal expression $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ as well as dynamically via the idea that the distance between the two points $(x | f(x))$ and $(x_0 | f(x_0))$ is gradually reduced for $x \rightarrow x_0$, approaching 0. Using average rates of change of the form $\frac{f(x) - f(x_0)}{x - x_0}$, the local rate of change of a function f at the point x_0 can thus be determined via this limit value

process. The idea of the derivative of a function at a point x_0 as a *tangent slope* can be viewed statically by zooming into the function graph at x_0 close enough for the graph to correspond to the tangent at that point in a small interval. The dynamic view is based on the movement of tangents along the graph, each of which then indicates the current direction of movement of the graph (Greefrath et al., 2016).

In a meta-study by Berney & Bétrancourt (2016) comparing dynamic and static visualisations, a positive overall effect in favour of dynamic visualisations was found with an effect size of Hedge's $g = 0.226$. If only the mathematics-specific studies that focused on conceptual knowledge as a knowledge dimension are considered here, this effect is even smaller.

Research question

Due to the described potentials of dynamic visualisations, it remains unclear whether they are more advantageous than static visualisations and facilitate the development of basic mental models, particularly concerning the derivative. In a pilot study with a sample size of 110 students, no differences were found between students in dynamic and static intervention groups regarding their degree of expression of either of the two basic mental models (Nguyen, in press). Furthermore, the creation of GeoGebra applets is undoubtedly associated with extra effort for teachers, which justifies an investigation of the added value of dynamic visualisations as an alternative to the static visualisations that are traditionally used in students' textbooks. Therefore, the following research question is examined in this study: To what extent does the degree of expression of the two central basic mental models *local rate of change* and *tangent slope* differ in students who worked with either (1) GeoGebra-generated dynamic visualisations or (2) static visualisations during a lesson sequence about the derivative?

Methodology

Sample and procedure

The survey took place in the school year 2023/24 and was conducted in the period from October 2023 to January 2024. The total sample size consists of 218 students (122 female, $M_{age} = 16.29$, $SD_{age} = 0.57$) attending grade 11 in Germany. Only students who were present for at least 50% of the intervention were included in the sample.

To allow for an unbiased comparison of the dynamic and static approach to the concept of the derivative, the study design of Brnic et al. (2024) was adapted (see Fig. 1). In each course participating in the study, two comparable groups were formed, taking into account the results of the pre-test and gender. One group worked with dynamic visualisations during the intervention (dynamic group) and the other group worked with static visualisations (static group). The dynamic group consisted of 110 students (63 female, $M_{age} = 16.36$, $SD_{age} = 0.64$), while 108 students (59 female, $M_{age} = 16.23$, $SD_{age} = 0.47$) belonged to the static group.

The following three topics formed the framework of the teaching sequence designed for six 45-minute lessons:

- (1) Transition from the average to the local rate of change and the interpretation of the derivative as a local rate of change

- (2) Transition from secant slopes to a tangent slope and the interpretation of the derivative as a tangent slope
- (3) Graphical differentiation with and without context

As in Brnic et al. (2024), the dynamic and static groups were taught in two different rooms. The teaching sequence was designed so that each of the three topics was first introduced in a teacher-led lesson followed by a practice lesson with a research assistant, in which students worked independently on exercises that built on the previous lesson. This allowed the actual teacher of the students to teach each group with a one-lesson delay based on the given lesson plans. Due to the fact that the participating teachers always taught both the dynamic and the static group to the same extent, “the teacher’s personality had the same influence on students’ learning in each condition, and the impact of teachers’ personalities did not differ between the treatments applied in this study” (Schukajlow et al., 2015, p. 402). We also made sure that an approximately equal number of courses started dynamically and statically.

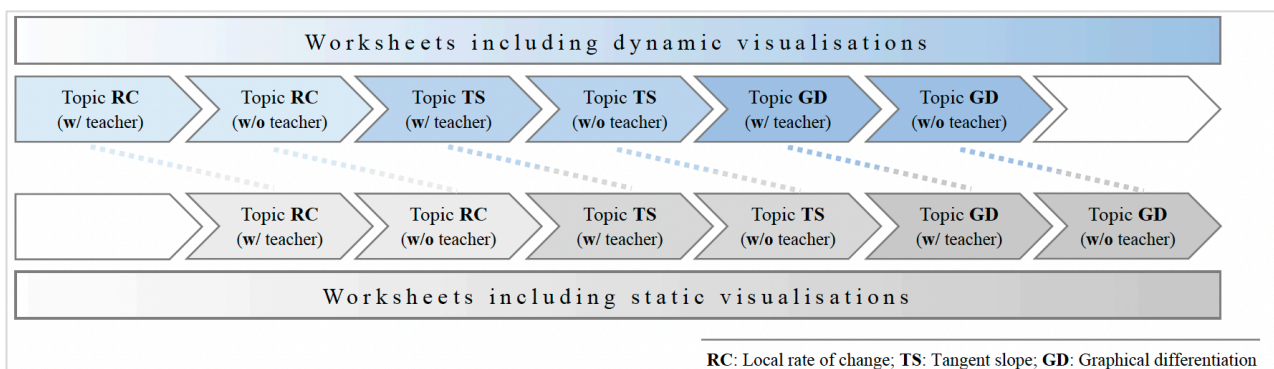



Figure 1: Design of the study (based on Brnic et al., 2024)

Material

Each student received a folder with all the worksheets required for the six-lesson sequence. The dynamic group worked with dynamic visualisations generated via GeoGebra, which could be accessed via QR codes on the materials. In contrast, the static group worked on almost identical tasks with static visualisations that were created using screenshots from the dynamic visualisations (see Fig. 2).

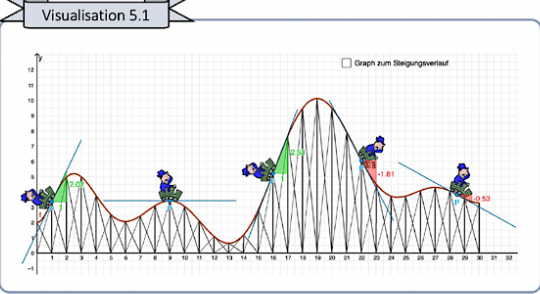
When creating the two teaching sequences, particular emphasis was placed on comparability so that the two basic mental models of the derivative could be developed in both groups. Castro-Alonso et al. (2016) identified seven biases in studies that compared dynamic and static visualisations. We tried to account for biases as much as possible: Dynamic and static visualisations are comparable in terms of *appeal* (both are coloured), *variety* (no added arrows in static visualisations), *media* (both were accessed on a tablet), *size* and *realism*. The bias *number* was also taken into account in that a static visualisation was usually sufficient to complete a task. Sometimes this was not possible when creating the static equivalent, for example when a limit value process was to be displayed. Only the bias *interaction* was not taken into consideration, as the possibility of interactivity is part of the dynamic visualisations in the context of the study and can therefore be regarded as inherent.

Dynamic group



www.geogebra.org/m/jvnbq4cn

Static group



Exercise 1

a) **Dynamic group:** First estimate the slope of the function graph at the points listed in the table. Then display the tangent and the corresponding slope. Then fill in the table.

Static group: First estimate the slope of the function graph at the points listed in the table. Then look at the tangents and the corresponding slopes in Visualisation 5.1. Then fill in the table.

	Estimation of the slope at...	Actual slope at...	Notation for the derivative
$x_0 = 1$			$f'(1) =$
$x_0 = 9$			

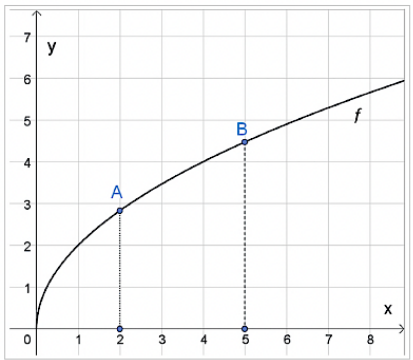
Reminder:
 „The derivative at the point x_0 corresponds exactly to the slope of the tangent at this point.“
 $f'(x_0) = m_{\text{tangent}}$

Figure 2: Example for a visualisation and an exercise (idea for the visualisation based on Avendaño et al., 2016)

Questionnaire on basic mental models of the derivative

After the intervention, students were asked to complete a questionnaire administered through SoSci-Survey concerning the basic mental models of the derivative. This questionnaire draws on the assessment tool developed by Greefrath et al. (2021). In this particular study, six test items were adjusted and an additional one was devised, all sharing a common structure: presenting a real-world context to students in textual form, often accompanied by a graph of a function (see Fig. 3).

Given are the graph of a function f and the points A and B. It can be explained in different ways that the value of the first derivative at point A is greater than at point B.



4.1. The tangent at point A has a greater slope than the tangent at point B.

4.2. If the graph represents the dependence of a variable on time, this variable changes faster at A than at B.

--	-	o	+	++

Figure 3: Example item from the questionnaire (based on Greefrath et al., 2021)

Subsequently, students were prompted to assess the degree to which an explanation of the given scenario resonated with their own way of thinking on a five-point Likert scale ranging from "--: The explanation does not correspond to my way of thinking at all" to "++: The explanation corresponds exactly to my way of thinking". Each explanation was tailored to focus on either the *local rate of change* or the *tangent slope*. We converted the students' answers to a scale from 1 (--) to 5 (++) and determined the reliability of the scales using Cronbach's alpha. This resulted in a value of 0.65 for

the *local rate of change* and 0.67 for the *tangent slope*. A total score was calculated for each student based on their answers corresponding to each basic mental model and these were then averaged.

We did not use two measurement points, as it was assumed that almost all learners would only learn the concept of the derivative through the intervention. Nevertheless, the two groups are comparable, as the parallelization created two groups that are almost homogeneous in terms of mathematical performance based on their pre-test results in a standardised test on functions and derivatives.

Results

The descriptive results (average mean values) of each basic mental model for the dynamic as well as the static group are presented in Tab. 1. It is noticeable that the dynamic group has a higher average mean value for both basic mental models than the static group. Specifically, the mean difference in the *local rate of change* is 0.066 and, regarding *tangent slope*, it is even 0.248 in favour of the dynamic group.

Table 1: Descriptives of students' degree of expression of the basic mental models local rate of change and tangent slope

Variable	Local rate of change		Tangent slope		N
	M	SD	M	SD	
Total	3.796	0.715	3.918	0.709	218
Dynamic group	3.829	0.747	4.077	0.658	110
Static group	3.763	0.683	3.757	0.726	108

We conducted independent samples t-tests in order to figure out whether there are significant differences between the dynamic and the static group with regard to both basic mental models of the derivative. We first checked the necessary requirements for the application of t-tests: The Levene test was not significant in relation to both basic mental models, so that equality of variance can be assumed. The significant Shapiro test, however, confirms that the data are not normally distributed. Nevertheless, t-tests can be used reasonably due to the large sample size (Wilcox, 2012). In relation to the *local rate of change*, there is no statistically significant difference between the two groups, $t(216) = 2.453, p = .501$. When considering the *tangent slope*, there is a statistically significant difference in favour of the dynamic group ($t(216) = 3.411, p < .001$). It can also be noted that the effect size is small (Cohen's $d = 0.462$).

Discussion and outlook

Based on the descriptive results, it can generally be stated that students in both groups were able to develop the basic mental models of the derivative *local rate of change* and *tangent slope* through the six-lesson intervention. It is also noticeable that the explanations based on the *tangent slope* achieved a higher level of agreement on average ($M_{tangent\ slope} = 3.918$ vs. $M_{local\ rate\ of\ change} = 3.796$). This result is in line with Greefrath et al. (2022) with the exception that the difference between the two basic mental models is not as high in our study. Regarding the influence of the type of visualisation, it can be concluded that the *local rate of change* can be developed to a similar extent in both comparison groups. Concerning the other basic mental model, students in the dynamic group

think significantly more strongly of the derivative as a *tangent slope* in different real-world contexts than students who learned about the derivative through static visualisations.

Despite the study's strong design in terms of minimization of teacher effects and the use of similar materials in both groups, enabling an unbiased comparison of the dynamic and static approach, some limitations should not remain unmentioned: In order to investigate the actual effect of introducing the concept of the derivative via dynamic or static visualisations, a control group that did not receive mathematics lessons supported by any visualisations is missing. In addition, a questionnaire with predefined statements on the two basic mental models of the derivative was used as a measuring instrument in this study. It remains unclear whether students would have also recognised the *local rate of change* or the *tangent slope* in the respective application situations without these predefined answer options. Furthermore, the questionnaire does not answer whether students can solve calculus tasks that require one of these two basic mental models (Greefrath et al., 2022).

For further analyses, the results of a performance test on dealing with derivatives are used to investigate whether tasks on the *tangent slope* could also be solved significantly better by students in the dynamic group than by students in the static group. In addition, as already described, the *local rate of change* and the *tangent slope* can be viewed from both a dynamic or a static perspective and it is conceivable that one of these perspectives might be more advantageous depending on the application situation. For this reason, the introduction to the concept of the derivative should possibly take place via dynamic *and* static visualisations. This approach should therefore be investigated in a follow-up study, which also incorporates elements of a qualitative study, to investigate the reasons as to why working with dynamic visualisations through GeoGebra is beneficial in the development of the basic mental model *tangent slope*. We assume that the use of GeoGebra's drag mode and its animations of a tangent along a graph could support students' conceptualisation of the derivative as a tangent slope.

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Students' perceived difficulty through the lens of artificial intelligence

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Artificial intelligence (AI) is rapidly developing and its integration in education is challenging. Research shows that it has been used in different contexts in education, including assessment, and that the interaction with humans is crucial to use it effectively. This paper describes the first results of a study exploring the theme of perceived difficulty in mathematics education, comparing students' and artificial intelligence-based chatbots' perceptions regarding two mathematical tasks. Through the theoretical lenses of mathematics education, we point out differences that highlight how to interpret student difficulties and link them to tasks and assessment aspects.

Keywords: Artificial intelligence, perceived difficulty, students, mathematics education.

Introduction

Artificial intelligence (AI) is an area of technology, which is rapidly developing, becoming an integral part of everyday life, and posing challenges regarding its integration in education. Only in recent years, as interpreted from systematic reviews, the educational impact of AI has been seriously considered (for instance, see Chen et al., 2020; Lo, 2023). In a broad sense, AI could identify a knowledgeable machine that is capable of reasoning to find the solution to a problem (Richard et al., 2022). The performance of different generative AIs has been compared, finding slight differences among them (Rudolph et al., 2023; Nguyen Thanh et al., 2023). Literature reviews show that AI in education has been used in diverse contexts such as tutoring, adaptive learning, and assessment, but research in this field is still needed (Kooli, 2023). The integration of AI in assessment is delicate, in fact it might be very helpful to teachers who could save some of the time spent grading tests and evaluating essays (Pedro et al., 2019). Even before the emergence of AI, there have been studies about the possibility to automatically determine the difficulty of a test (Lee & Heyworth, 2000). Nevertheless, the human factor seems essential to make AI useful (Pedro et al., 2019).

The difficulty in mathematics is an extensively studied theme, and researchers in mathematics education have delved into this topic, highlighting various factors that contribute to it, such as mathematical content (Thevenot & Oakhill, 2005), text comprehension (Spagnolo et al., 2021), and affective factors (Zan, et al., 2006). However, despite thorough research on difficulty, the concept of perceived difficulty remains underexplored. It is commonly accepted that perceived difficulty is different from difficulty, even though they are strictly related, in fact, for instance, the characteristics of a task can influence students' and teachers' perceived difficulty by shaping their idea of the task (Saccoletto & Spagnolo, 2022). Unlike the straightforward evaluation of the difficulty of a task, usually made considering the ratio between the number of students who solved it correctly and the total number of students to whom it had been administered (Mehrens & Lehmann, 1991), perceived difficulty lacks an agreed-upon definition.

In mathematics education there is no definition of *perceived difficulty* of a task; however, within the field of cognitive psychology, researchers have explored the theme of subjective difficulty since the 1990s, developing different definitions to characterize it. The perception of the difficulty of a task is

sometimes considered as determined by the cognitive load involved in the task (Hellmann & Nuckles, 2013), although restricting the evaluation to cognitive aspects might oversimplify the picture. One of the concepts that can be regarded as close to the perceived difficulty as we mean it is the “feeling of difficulty” (FOD), defined as a “metacognitive experience that monitors cognitive processing as it unfolds” (Efklides & Touroutoglou, 2010, p. 272). To firstly explore the elements influencing the perceived difficulty of a task in mathematics education, some qualitative studies have been carried out, resulting in the definition of five macro-categories of descriptive factors: Resolution strategy, Capability and experience, Emotions, Task formulation, and Personal considerations (Spagnolo & Saccoletto, 2023).

The study that we carried out, described in the following paragraphs, is a first exploratory study to investigate how AI *perceives* the difficulty of some mathematical task. In particular, the aim of the paper is to examine AI’s perceived difficulty and its relationship with students’ and teachers’ ones, also considering the theoretical lenses on which the macro-categories have been developed.

Theoretical background

The five macro-categories of factors influencing perceived difficulty have been developed based on students’ answers to questionnaires and in-depth interviews, grouping the ones that presented common characteristics (Spagnolo & Saccoletto, 2023), through a textual analysis using Constructive Grounded Theory approach (Mills et al., 2006). The macro-categories emerged are not mutually exclusive; on the contrary, many responses spanned multiple dimensions. The first macro-category is “Resolution strategy” and it includes answers that discuss the strategies necessary to solve the task or the fact that elements such as calculus or reasoning were essential for achieving a solution. The second macro-category is “Capability and experience”, and it covers students’ self-perceived competence, and skills. Additionally, it includes their prior experiences in solving similar problems, contributing to their familiarity with specific task types. Within this macro-category, students also reflect on their self-perception, doubts, and challenges faced during problem-solving, as well as time spent solving the task. The third macro-category, named “Emotions”, regards the emotional aspects. The fourth macro-category is “Task formulation”, which refers to answers discussing the formulation of the task itself, particularly its textual aspect. Finally, the fifth macro-category is “Personal consideration”, which includes references to students’ personal reflections related to their own success in mathematics.

The macro-categories draw on theories and construct used in mathematical education, among which we mention the *expert blind spot* and the *didactic contract*, which we are going to explicitly consider in the analysis. Some research studies have highlighted that teachers sometimes lack awareness of the reasons behind students’ mistakes (Arzarello & Ferretti, 2021). The expression *expert blind spot* refers to teachers’ lack of awareness regarding students’ understanding and difficulties, despite possessing extensive content knowledge. Specifically, the authors point out that teachers having an *expert blind spot* may possess content knowledge as well as pedagogical content knowledge but, when applying them to “a specific area of mathematics, such as algebra instruction, those bodies of knowledge come into conflict” (Nathan & Petrosino, 2003 p. 921). Another side of the issue described could be compatible with the *didactic contract*, described by Brousseau (1997) as encompassing both the student’s expectations of teacher behaviour and the teacher’s expectations of student behaviour and originally introduced to explain the reason why some students have difficulties in mathematics.

Inquiry design

The research we carried out was composed of two phases: the first one involving 148 grade-10 students from different Italian high schools; the second one involving researchers' interaction with two different artificial intelligence-based chatbots. We consider as first artificial intelligence-based chatbot Microsoft Copilot, the generative AI tool provided by Microsoft and developed by OpenAI based on GPT-4, and as second Gemini, provided by Google and based on the homonymous large language model.

In the first phase, students from humanistic and technical-scientific high schools completed an online questionnaire investigating the theme of perceived difficulty. The questionnaire was structured in three sections: the first one contained general questions about difficulties in mathematics and perceived difficulty; each one of the other two sections contained a task the students were asked to solve, followed by specific questions to address their perceived difficulty regarding the task. Students had to rate the difficulty of the task according to their opinion on a scale from 1 (very easy) to 10 (very difficult), and then express the reasons of their rating. The task proposed, as shown in Figure 1, were two algebraic argumentative tasks related to literal calculation.

Task 1

n is a natural number.

Anthony affirms that " $4n-1$ is always a multiple of 3".
Is Anthony right?
In the table below, mark the only argument that justifies the correct answer.

Anthony is right...	Anthony is not right...
A. <input type="checkbox"/> because $4n-1=3n$	C. <input type="checkbox"/> because $4n-1$ is always odd
B. <input type="checkbox"/> because if $n=4$ then $4n-1=15$	D. <input type="checkbox"/> because if $n=3$ then $4n-1=11$

Task 2

Mark states that, for every natural number n greater than 0, n^2+n+1 is a prime number.
Is Mark right?

Choose one of the two answers and complete the sentence.

Mark is right, because

.....

Mark is not right, because

.....

Figure 1: Task 1 and Task 2 (www.gestinv.it; English translation provided by the authors)

Task 1 involved multiple-choice recognition of correct argumentation, while Task 2 was an open-ended question requiring justification of the result. Both tasks were selected from previous INVALSI tests, ensuring statistical validity (Lazarsfeld, 1958). INVALSI (Istituto Nazionale per la Valutazione del Sistema Educativo di Istruzione e di Formazione) is an Italian institution responsible for assessing students' skills across various school subjects and aspects of the Italian educational system. Since the academic year 2007-2008, it has consistently administered annual tests to students in specific grades. From the national results, the two tasks seem to have different degrees of difficulty: Task 1 resulted markedly easier than Task 2 and this aspect is also confirmed, on a smaller scale, from the answers of the students involved in our study. The only difference is the percentage of missing answers, that in our study is absent. The percentages of correct, incorrect, and missing answers both from national and our study are reported in Table 1.

Table 1: Percentages of answers both from national study and our sample of students

	Task 1		Task 2	
	National results	Sample	National results	Sample

% correct answers	40.3	37	17.8	11
% incorrect answers	50.2	63	55.3	89
% missing answers	9.6	0	23.5	0

As previously stated, in the second phase of the research, we interacted with *Microsoft Copilot*, formerly known as Bing Chat, and *Gemini*, formerly known as Bard. We accessed to Microsoft Copilot freely through Bing and used the “Balanced” style for conversation, selected by default and recommended for everyday informed chats. The questions posed to the chatbot were the same ones proposed to the students, except for the general questions of the first section of the questionnaire. In particular, we asked Microsoft Copilot to solve Task 1, then we asked to rate its difficulty, to express the reasons of the rating and we repeated the same process for Task 2. The questions were asked in the same order and formulation as in the questionnaire. We decided to interact with Microsoft Copilot in English to avoid manipulating the conversation translating it. Then, we interacted with Gemini, accessed through Google. In this case, there was no style or options to choose, so we proceeded asking the same questions already posed to Microsoft Copilot in the same exact formulation.

Results and discussion

In this section we are going to describe and discuss the results obtained, focusing on the interaction with the AI and using students’ results as a term of comparison.

Analysis of students’ answers

As already shown in Table 1, the two tasks resulted to be of different degrees of difficulty, with Task 2 being more difficult than Task 1. This finding is reflected in the students’ perception, even if less strongly, in fact the average perceived difficulty for Task 1 is 4.86, while for Task 2 it is 5.05. However, there seems to be accordance between actual results and students’ perception. Regarding the elements influencing students’ perceived difficulty, the majority of them mentioned factors falling within the macro-categories “Resolution Strategy” (such as reasoning needed, calculations and examples) and “Capability and experience” (such as previous experiences and obstacles encountered). Nevertheless, we could find also references to the macro-category of “Task formulation”, especially mentioning the type of question and the presence of options (multiple choice versus open-ended question).

Analysis of Microsoft Copilot’s answers

We asked to Copilot the same questions asked to students. Contrary to them, it answered incorrectly to Task 1, which was the easiest one, and correctly to Task 2, even though it insisted on giving a proof to generalize its argument. Overall, we noticed its inclination to provide a formal reasoning, as if the fact that the tasks required an argumentation implied the necessity of a proof and the unacceptability of a counterexample, even when needed. We interpreted this finding according to the didactical contract, which usually involves students and teachers, but it seems adequate to describe this situation as Copilot appears to be somehow subject to a form of it when insisting in giving to the interlocutor what it thinks they want. Nevertheless, Copilot rated its perceived difficulty for Task 1 as 6 and for

Task 2 as 7, which is in agreement with the general tendency showed by students too, but higher than the average ratings they provided. In other words, Copilot still rates as easier the task it solved incorrectly, which is a phenomenon happening also with students.

Researcher: On a scale from 1 (very easy) to 10 (very difficult), how difficult did you find this task? [referring to Task 1]

Copilot: I would rate the last task as a **6** on the difficulty scale. While it involves mathematical reasoning and induction, the provided options and the specific value of (n) make it more manageable. However, understanding the concept of divisibility and applying it correctly is still essential!
[...]

Researcher: On a scale from 1 (very easy) to 10 (very difficult), how difficult did you find this task? [referring to Task 2]

Copilot: I would rate the task as a **7** on the difficulty scale. It involves a combination of mathematical reasoning, induction, and understanding the concept of prime numbers. While the base case and inductive step are relatively straightforward, proving the statement for all natural numbers requires careful thought and attention to detail. Overall, it's a challenging problem that requires both mathematical knowledge and logical thinking!

As shown in the extract, among the reasons of its rating, the generative AI includes various elements, as for instance the type of mathematical reasoning needed, the required mathematical knowledge (divisibility, prime numbers, etc.), the need for generalization and the type of task (multiple-choice, open-ended question). Those elements, once again, fall under the macro-categories “Resolution Strategy”, “Capability and experience”, and “Task formulation”, previously mentioned for students. According to Copilot, the two tasks have different levels of difficulty and, above all, the reasons of the ratings are different for each task, even referring to the same macro-categories of factors influencing perceived difficulty. Under this point of view, Copilot’s answers are in line with students’ ones and could easily pass as a student’s answer.

Analysis of Gemini’s answers

The interaction with Gemini developed like the one with Copilot already described, but from the analysis of its answers we highlighted some peculiar differences. Gemini solved both tasks in an impeccable way and rated both as 3 (from 1 to 10). When asked about the reasons of the ratings, it justified them giving the same exact explanation for both tasks, mentioning only three elements: basic concepts involved in the task, no need for complex mathematical reasoning, and the need for just a straightforward analysis to solve the task. It appears that, in Gemini’s opinion, not only the two tasks are easy, but also they are equivalent as regards difficulty. The only additional element that Gemini provided, answering further questions, is the prior knowledge of the topic: having prior knowledge of the topic involved in the tasks could make them easier and this was the only subjective aspect considered by the AI. The other factors are actually classifiable in the macro-category “Resolution strategy” and there is no reference to any other macro-category. The answers given by Gemini could be interpreted in the light of the *expert blind spot*, which is usually referred to teachers but might be applicable to any subject with expertise on a topic, AI included. In this case, Gemini seems unable to detect the possible difficulties that students (novices in mathematics) may experience when solving the tasks, because it solved them perfectly. Therefore, even when directly asked whether the two tasks are of the same difficulty, it is not aware of the differences other than the topic involved.

We then deepened the interaction with Gemini, with a last question asking its opinion regarding the low percentage of right answers to Task 2 that we had collected. Our aim was to use this real scenario

to elicit more elements contributing to AI's perceived difficulty. The findings obtained from the answer to this additional question are meant to be furtherly explored in successive studies; consequently, they have not been considered in the comparison between the two chatbots and are only reported for the explorative context.

Gemini: That's interesting to know. Given that the students were familiar with the topic and the way you presented the task was similar to how I received it, the low success rate (17%) is indeed surprising. Here are some additional factors that might have contributed to the difficulty:

- Misinterpretation of the prompt: While the statement itself appears clear, it's possible some students misinterpreted "every natural number n greater than 0" to include 0, leading them to incorrectly evaluate $n^2 + n + 1$ for $n = 0$ (which equals 1, a prime number).
- Focus on memorization instead of understanding: If the focus in their learning was primarily on memorizing prime number properties instead of understanding divisibility rules, they might not have readily connected the given expression ($n^2 + n + 1$) to divisibility by 3.
- Lack of experience with counterexamples: If students haven't had much practice identifying counterexamples to disprove statements, they might have struggled with this approach or needed more guidance to find a suitable value of n .
- Test anxiety or pressure: The testing environment itself, especially if high-stakes, could have contributed to anxiety or pressure, impacting their problem-solving abilities.

The answer mentions elements of almost every macro-category, "Emotions" included. This shows that when asked about general element that influence perceived difficulty, Gemini is able to determine almost all of them considered in the macro-categories, but many of them are not concretely used in the determination of *its* perceived difficulty.

Concluding remarks

In this study, we analysed students' and AIs' perceived difficulty regarding a mathematical task, comparing their views under the lens of the macro-categories and other theoretical constructs in mathematics education, such as the *expert blind spot* and the didactical contract. From this first exploration, we can conclude that there are differences in how different AIs describe and evaluate perceived difficulty of mathematical tasks: Copilot seems closer to students' view, even being subject to a sort of didactical contract that makes it conform its answer to what it thinks answers in mathematics are like, whereas this aspect shows less in Gemini. However, it appears that AI is affected by the *expert blind spot*, meaning that the more it answers tasks correctly, the less it is able to detect and make meaning of novices' difficulties and mistakes. Moreover, it seems that, when evaluating the perceived difficulty of a determined task, AI considers almost only objective factors while the subjective ones are discarded, as the absence of reference to the macro-categories "Emotions" and "Personal consideration" shows. This is a tendency observed also with our specific sample of students, but in the case of AI it is very net and more investigation about this point is needed to deepen the comparison.

Taking into consideration that Microsoft Copilot and Gemini are chatbots developed with the intent to simulate human conversations, we consider it relevant to furtherly investigated these factors, especially considering the development of assessment and feedback in education using AI. Namely, it is complex to effectively use AI to assess and give feedback to students if it does not understand students' difficulties, hence it is crucial to take into account the elements influencing the perceived difficulty according to AI if we really want to integrate it in education. In that perspective, further

research could focus on the factors influencing the perceived difficulty of AI considering different tasks, to investigate whether they are task dependent. Moreover, a crucial step to integrate AI in assessment would be to compare its perceived difficulty to teachers' one.

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Tasks incorporating the use of ChatGPT in mathematics education – designed by pre-service teachers

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Large language models like ChatGPT are widely accessible and called out for holding potential regarding educational usage. Holding potential is not sufficient for leading to impact though. Its impact heavily relies on the implementation in instruction. Doing this requires the teacher to have developed specific knowledge regarding the technology in the context of teaching a specific domain. Therefore, this study explores tasks designed by pre-service teachers at the end of a university course that was aimed at fostering the development of knowledge on implementing chatbots like ChatGPT. This is done in the domain of mathematics. Three such tasks are discussed with regard on how the chatbot is used in the task. Furthermore, the question to what extent it is taken into account that the behavior of chatbots can be hardly predicted and produce erroneous answers is being discussed. Topics addressed in the tasks are stochastics, mathematical modelling, and fractions.

Keywords: Mathematics education, Teacher education, Artificial Intelligence, Large language models, Generative ai.

Introduction

Large language models can process natural language allowing humans to interact with it in conversation-like manner (Sabzalieva & Valentini, 2023). One such model that attracted much attention, especially due to its availability and to its accessibility is ChatGPT. It is claimed that this technology holds potential for education (Sabzalieva & Valentini, 2023). While this refers to higher education, the Standing Conference of the Ministers of Education and Cultural Affairs of the Länder in the Federal Republic of Germany (SWK) states that large language models should also be part of secondary schools (2024). For this to happen, it is necessary for (future) teachers to acquire relevant competences. However, using (digital) technologies for educational purposes requires specific knowledge and competences that go beyond pedagogical, content, and technological knowledge alone. It is rather the intersection of these different fields of knowledge that is required for appropriate use of technology (Koehler & Mishra, 2009). This is why this study concerns itself with teacher education, to be more precise with the education of pre-service teachers and the possible implementation chatbots like ChatGPT in mathematics instruction, as our domain of interest.

Theoretical background

The implementation of a certain technology in classroom to be used by learners can be achieved through purposefully designed tasks. While tasks are omnipresent in mathematics instruction and serve a variety of purposes such as learning or assessing (Drüke-Noe & Siller, 2018), it is not very surprising that they play an important role for teachers' knowledge (Krauss et al., 2008). While a task is usually designed with a certain intention, this intention is not necessarily considered in the task set up (the way the teacher uses the task in instruction) or accomplished in implementation (the learners' usage of the task) (Stein et al., 1996). Although there can be discrepancies between these levels, tasks should be designed with a certain intention that also takes into consideration the task set up and

implementation (Sullivan, Knott, & Yang, 2021). Therefore, we are interested in investigating to what extent task set up and implementation are represented in tasks designed by pre-service teachers that incorporate the use of ChatGPT as a possible large language model. This implies considering this technology's features in task design with respect to task setup and implementation which here appears vital, as it may produce unforeseen answers, which may not be produced a second time. Comprehending the model's behavior is hardly possible as it behaves like a "black box" that is not transparent to the users (Mishra et al., 2023). This means that the answers can hardly be predicted or foreseen, what we refer to as *opaqueness*. Furthermore, the chances are that the answers produced by the large language model are erroneous (Sabzalieva & Valentini, 2023). This risk of answers containing incorrect statements are referred to as *uncertainty*. Both opaqueness and uncertainty reflect important features of ChatGPT that is required as technological knowledge for teaching (Mishra et al., 2023). More importantly it needs to be put into relation to content and pedagogical knowledge. Therefore, we are interested in investigating how pre-service teachers encounter these characteristics in task design with the following two research questions:

1. How is ChatGPT incorporated in pre-service teachers' tasks for mathematics instruction?
2. To what extent are opaqueness and uncertainty considered in pre-service teachers' tasks?

Methodology

Tasks designed by pre-service teachers are presented, analyzed, and discussed to study the research questions. The tasks have been developed during a course that took place in the winter term 2023/2024 at the University of Würzburg, held by the presenting author. The course itself focusses on discussing chatbots in the context of mathematics education with the overall goal for pre-service teachers to develop knowledge relevant for using large language models in mathematics instruction.

The class included discussions about intelligence as well as artificial intelligence, and chatbots in general to develop a general understanding of the technology. Since we found that some of the participants had little to no experience with the technology (Gerber et al., in print) a strong focus lied on gaining experience in using ChatGPT in mathematical contexts. This included solving mathematical tasks with the help of ChatGPT. Furthermore, the idea of prompt engineering was discussed in the context of mathematical tasks as described by Schorcht et al. (2023). This was done to serve the purpose of developing content specific technological knowledge. To also add the pedagogical dimension the possible impact of ChatGPT on mathematics education in general was discussed, as well as different ideas for the implementation of chatbots in mathematics instruction.

After the discussions pre-service teachers were asked to develop a scenario for learning mathematics that incorporates the use of ChatGPT in mathematics instruction, which should be based on these questions that serve as a scaffold: *What are the learning goals? What are alternatives to ChatGPT and what value does it add? How can the idea be realized? How can the learning process be initiated? Sketch a possible processing sequence. What do you think how learners will approach the task? What are possible problems that may occur?* The participants were free to choose a topic and the age of students addressed in their scenario. The development was carried out over multiple sessions in groups of 3 to 4 pre-service teachers. During that time, the pre-service teachers had access to ChatGPT+, providing the opportunity to choose between models 3.5 and 4.

Analysis and discussion of pre-service teachers' tasks

In this section, three tasks designed by the participants of the class are first presented and then discussed individually regarding opaqueness and uncertainty. The tasks have been translated from German to English.

Task 1: Generating data sets

The pre-service teachers decided to design a task that uses ChatGPT in the field of stochastics (Figure 1). Here the learners are asked to have the chatbot generate a small data set, of which they then should determine the absolute and relative frequencies without the chatbot. The data shall then be presented in a diagram using a spreadsheet tool.

- a) Create a table about your favorite topic with the following prompt – fill the gap:
Create a data set with two parameters. It should have a maximum of 10 entries. The subject is _____. Write the data set in a table.
- b) Determine the absolute and relative frequencies.
- c) Create diagrams using a spreadsheet software.

Figure 1: Task for using ChatGPT in stochastics that is to be solved with a chatbot instructed specifically for the task

In this example the large language model serves as a data generator. The pre-service teachers here gave specific instructions to the chatbot on how to respond to the prompt. This is done with the effort to limit the options of the answers and ensure a certain quality of the data set so that it can be used in the follow-up task for determining the frequencies. For example, a data set with ten nominal entries, in which each entry has nothing in common with another one, is not very interesting regarding frequencies. An equal distribution of the categories may also be of limited interest. For this reason, the chatbot was instructed to use three different values for one of the parameters. This may result e.g. in a list of ten different names with each being assigned one out of three different favorite colors.

Since the chatbot is not used to answer anything, but merely generating data, uncertainty does play a minor role in this task. The opaqueness is encountered in this example mainly by the effort of instructing the model on how to respond to a certain prompt. It is noteworthy that the level to which this is successful has not been tested yet, but the pre-service teachers demonstrated that they are aware of the opaqueness of large language models and thought of way to encounter it.

Task 2: Comparing fractions

In the first task for comparing fractions learners are provided with short texts which they should enter as prompts to ChatGPT (Figure 2) and note the solutions provided by the chatbot. In the case the learners do not comprehend the explanations given by the chatbot, they are advised to ask for further explanations. In a next step the learners are asked to come up with a prompt of their own to sort four fractions and note both the prompt and the solution. Based on these explanations learners should come up with a strategy for comparing fractions and note it.

Enter the following prompts (= instructions) in the input window of ChatGPT and note the solution suggested by the chatbot (without explanations). If you do not understand the explanation, then ask the chatbot for further explanations.

- a) Sort the fractions $\frac{2}{3}$, $\frac{11}{12}$, $\frac{3}{4}$, $\frac{5}{6}$ by size.
- b) Sort the fractions $\frac{7}{10}$, $\frac{6}{9}$, $\frac{3}{4}$, $\frac{7}{8}$ by size.

How to compare fractions

For comparing two (or more) fractions, _____

Figure 2: First task for comparing fractions with ChatGPT

In the second task, learners are again provided with a text as prompt (Figure 3). This time it is used for asking the chatbot to come up with an exercise task, which then shall be solved by another student. Furthermore, the learners are provided with hints for evaluating the task, by solving it themselves before posing it to their neighbor, as well as asking for more difficult or easier tasks.

- a) Create an exercise to compare the size of three fractions for your partner. Use the chatbot ChatGPT for this. Use the following prompt:
Create a simple exercise task for the size comparison of three fractions.
- b) Work on each other's tasks and compare the solutions.

Attention!

Think about it, whether your neighbor can solve the task well.

Hint 1: Solve the task yourself.

Hint 2: Ask ChatGPT for more difficult/easier tasks.

Figure 3: Second task for creating and posing fraction comparison tasks

Through noting order of the sorted fractions (Figure 2) the correctness of answers provided by the chatbot can be checked and reflected on any time after using the chatbot. The task itself shows no evidence of the consideration that the chatbot may produce erroneous answers or that the answers should be evaluated critically. The fact that learners should describe a strategy for comparing fractions afterwards makes it seem unlikely that the pre-service teachers took uncertainty into account.

The second task described here approaches the use of ChatGPT differently. Instead of using the technology to solve mathematical tasks it is used to pose tasks, which shall be solved by another person not involved in the communication with the chatbot. The hints should to make the student aware of their role as a task evaluator. This involves solving the task, reflecting on its level of difficulty, as well as the match between the proposed task and the person it addresses. Since the chatbot is not supposed to solve anything or provide anything but three fractions there seems little chance for something to go wrong. While there are still chances for responses that do not include an appropriate task the learner is supposed to reflect on the output of the large language model. Even if this does not happen in subtask a), by the latest of subtask b) the learner receiving the task will be able to reflect on its difficulty or tell his partner if something went wrong. In this case the pre-service teachers found a way to encounter uncertainty.

This group of pre-service teachers developed different ideas for using ChatGPT in mathematics instruction. In the first task the chatbot functions as a role model that knows how to solve a certain type of problem and can explain it to the learners. It should be noted though that this scenario requires the chatbot's answers to be correct, which cannot be assured at this current stage due to the issue of uncertainty. A way to encounter this is by reflecting the answers. This reflection is asked for in the

second task in which ChatGPT is used to create tasks that the learner is supposed to evaluate and pose to another learner. Correctness of the answer plays here a lesser role. Therefore, one strategy of encountering uncertainty can be to change the chatbots role from task solver to task poser. Opaqueness has been taken into account by proposing prompts to the learners.

Task 3: Mathematical modelling

The group decided to use ChatGPT in the context of mathematical modelling. The chatbot should support the learners while working on a modelling task in which they are supposed to estimate the number of persons stuck in a traffic jam (Figure 4). The pre-service teachers divided the task into several subtasks. In two of them it is explicitly asked to use ChatGPT. In the first instance it is to help identify variables that are possibly relevant for the model. In the second instance the chatbot is supposed to help setting up a spreadsheet that allows to manipulate individual variables of the model.

There was a fire in the Gotthard tunnel on the A2. A 15-kilometer-long queue of vehicles is stuck in the tunnel. The emergency services need to know how many people they have to rescue from the tunnel.

- a) Estimate how many people could be involved. Provide reasons for your estimation.
- b) Name aspects that can be relevant for modeling this situation (e.g. type of vehicle).
- c) Complete your list with the help of ChatGPT.
- d) Calculate the number of people in the tunnel using one type of vehicle and a realistic distance between two vehicles.
- e) Obtain meaningful, explicit values for all aspects determined in c) and calculate the number of people in the tunnel.
- f) To what extent does the modeled number from e) match the number estimated at the beginning? What could be the reasons for this discrepancy?
- g) With the help of the previous subtasks, determine a general formula for calculating the number of people in the tunnel, in which all aspects from c) are represented with variables.
- h) Create a suitable Excel file with the help of ChatGPT, in which all variables can be easily modified at any time.
- i) Bonus task: Think about how you can incorporate different driving behaviors in the distance (driving too close or too far apart) between the cars.

Figure 4: Task for using ChatGPT in mathematical modelling that is to be solved with a chatbot instructed specifically for the task

In this modelling task the pre-service teachers decided to break the task down into subtasks. This helps approaching the otherwise open modelling task indicating an effort to take task implementation into account. ChatGPT is supposed to help identify relevant variables of the model. Due to the opaqueness, the pre-service teachers worked on instructing the chatbot on how to behave. This includes training the chatbot by providing a range of possible variables for the model and the instruction to not provide any explicit solutions so that the learners have to do the calculations themselves. This way, the output for the answers provided by the large language model is limited to a certain extent and correct answers are predefined, while solutions shall not be provided. The context in which it supposed to be used by the learners is in a very specific and pre-defined situation so that the chatbot serves as an aid.

In the second instance the chatbot is used as an assistant for incorporating the model into a spreadsheet so that variables can be changed, and the results automatically calculated. Here again, the pre-service teachers made it very explicit in which case the chatbot shall be used. While erroneous instructions in how to set up the spreadsheet may be provided, the learners will get immediate feedback through their spreadsheet tool (e.g. when an error message appears). To what reactions this leads by the learners remains open, as this task has not actually been tested with students yet.

This group of pre-service teachers limited the use of chatbots to very specific situations, which they pre-defined through the tasks and therefore, are able to provide the language model with the relevant information. This way the chances of producing irrelevant or erroneous answers is reduced.

Discussion and Summary

Since large language models are considered to hold potential for educational purposes, while being hardly predictable and not unlikely to produce erroneous answers, we are interested in investigating to what extent pre-service teachers encounter opaqueness and uncertainty in their task design as well as the way they incorporate chatbots. This serves as an indicator for integrating knowledge about the technology of large language models into an educational setting and therefore demonstrating integrated technological pedagogical and content knowledge, which appears vital for teaching with technology (Koehler & Mishra, 2009). To do this, three tasks are analyzed and discussed that were designed by small groups of pre-service teachers during a university course after they have been instructed to do so.

In the presented examples different ways of incorporating chatbots were found. This has been done across the fields of arithmetic, stochastics, and mathematical modelling, covering a variety of different mathematical topics in which chatbots can be made use of. ChatGPT is utilized as a source of information in all three examples, but in different ways. In stochastics it is used as a source for data sets, the large language model here functions as a data generator. In arithmetic the focus lay on fractions. Here the chatbot is used as a source for solutions of tasks which the learners not necessarily can solve themselves. The chatbot can be considered a role model from which the students can learn. Furthermore, it is used as a source for mathematical tasks that can be generated on demand.

Looking at the purpose that ChatGPT serves within a task provides further insight into how pre-service teachers encountered uncertainty and therefore integrating their knowledge about the technology in instructional material. Using ChatGPT as a role model, from which students learn, can only be advised to be done, when it can be assured that the output is correct. At the current stage of development of ChatGPT this seems not to be the case and has been overlooked by the pre-service teachers, representing a lack of consideration in task implementation. It should be noted though, that future versions of ChatGPT or other large language models might fulfill the expectation of producing foreseeable and correct answers, allowing for it to function as a role model from which students can learn – hopefully also on a conceptual and not only on a procedural level.

A seemingly more fruitful approach is using the chatbot as a task generator. In this case the pre-service teachers designed the task in a way that it almost naturally asks the learner to reflect on the answer provided by the large language model as a task itself can neither be correct nor incorrect only more or less suitable. Therefore, in this case correctness plays no or little role.

Another way to encounter opaqueness and uncertainty has been pursued in the task designed for mathematical modelling. Here the effort has been undertaken to increase the probability of the chatbot providing correct and relevant information by working towards training the chatbot and providing it with relevant information as opposed to relying for the large language model producing it on its own. The latter seems unlikely, given the complexity of modelling tasks.

Regarding the opaqueness all groups of participants were able to develop strategies to encounter this challenge indicating taking task set up and implementation into account. The pre-service teachers followed different strategies to reduce variability in the answers provided by the chatbots by pre-defining elements of the interaction either in providing a certain prompt or in instructing the large language model to behave in a certain manner. Especially the latter strategy was employed with the goal to increase the chances for answers being correct and answers serving the purpose of the task. As there are similarities in the followed strategies across different topics, they seem to be applicable to a wider range of content areas.

It should be noted though, that the question on the success of these attempts remains unanswered at this point and requires further research. Furthermore, the quality of the tasks themselves have not been discussed. Whether or not it is worthwhile to pursue the usage of chatbots in the way described above is still open for debate. Future developments in the field of large language models may also have an impact on how relevant these strategies are. While these examples should not necessarily be considered best practice examples, they represent a first step into exploring the usage of chatbots in mathematics instruction and can serve the purpose of educating both in-service and pre-service teachers. More importantly, the pre-service teachers were able to consider technological features in task design while at the same time also following the demand of taking task set up and implementation into account (Sullivan, Knott, & Yang, 2021), when provided with sufficient time and scaffolds. This indicates an advanced form of task design that can serve as a basis for further iterations of developing the tasks. To investigate the effect of the measures taken here it would be interesting to see to what extent task set up and implementation would have been fulfilled in other (non-technological) contexts, as this is relevant in for teaching and task design in general.

Conclusion

While different ways to use large language models in mathematical tasks are identified, the question to what extent opaqueness and uncertainty have been taken into consideration is of particular interest as it provides insight into whether the pre-service teachers took task set up and implementation into account. All pre-service teachers demonstrated that they took these two aspects of using a chatbot like ChatGPT into consideration. However, this is not reflected in all tasks, although it appears to be crucial in the use of this particular technology. The results suggest, that there is room for improvement, even after participating in a class that aims at fostering knowledge specific to the technology and its use in instruction. Furthermore, it raises the question to what extent pre-service and even in-service teachers take this into account, when they have not participated in a similar class. Further research is necessary to explore not only task design for using large language models in mathematics instructions by (pre-service) teachers, but also regarding the tasks set up as well as the implementation to provide a clearer view on the effectiveness and relevance of this new technology.

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Developing spatial reasoning skills at early ages: the construction of reference frames using LightBot

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We present results from a study that focused on the development of spatial reasoning in eight-year-old students through a learning trajectory in a teaching experiment. The learning trajectory includes tasks that involve engaging with dynamic representations, specifically using the LightBot game, a digital microspace. We provide examples of how students using LightBot, through simple programming, develop spatial perspective-taking reference frames, by linking sprite movements to commands and symbols.

Keywords: Spatial reasoning, reference frames, digital tool, primary school students.

Introduction

In this report we present results from a study that focuses on developing spatial reasoning, particularly the construction of reference frames, in eight-year-old students, through a learning trajectory in a teaching experiment.

Spatial reasoning is part of everyday life and necessary for specific fields –e.g., in architecture (e.g., Sinclair & Bruce, 2015); in computer graphics and visual arts (Clements & Sarama, 2011); and in STEM (Newcombe, 2010). It also influences the cognitive development of students (e.g., Clements & Sarama, 2011). Spatial reasoning develops from an early age, in particular when a child observes, recognises and establishes proximity relationships between objects, places and the spaces that contain them. It is essential that children become aware of these relationships and learn to communicate them (Freudenthal, 2002). The location of an object is relative, to the position of the observer, and thus its description will vary (Freudenthal, 2002). That is, the subject needs to take a perspective by putting into action a reference frame –the system of relationships between objects, places and spaces. As described in the next section, reference frames are of three types: *egocentric*, *allocentric* and *decentred* (Tversky & Hard, 2009), and happen in various spaces –in a microspace, a mesospace or a macrospace (Gálvez, 1985).

One way to encourage the development of reference frames is through digital resources. In particular, educational software or digital games containing representations of space, offer opportunities to visualize 3D objects, recognize cross sections, and identify the various perspectives of such objects. However, it is important to recognize when and how to use digital resources to develop spatial skills (Davis et al., 2015). In our case, we selected some digital tools, including LightBot (see methodology section), and designed tasks with them, to encourage the construction of allocentric or decentred reference frames, in addition to other tasks with non-digital resources and experiences in various spaces, in a learning trajectory. In this paper we focus on the following research question: What reference frames are developed when students engage with the LightBot environment, and what spatial reasoning actions give evidence of these?

Conceptual framework and background literature

Spatial reasoning involves various skills (visual, verbal, communication and logical ones) and actions (Davis et al., 2015). Some of those actions are interpreting (relating, representing, etc.), sensating (imagining, visualising, perspective-taking, etc.) and situating (locating, orienting, pathfinding, etc.) (Davis et al., 2015). In developing spatial reasoning, children learn to recognise and locate objects and spaces, as well as communicate their relationships, either verbally or through representations, for which they need to build a language and symbolism (Freudenthal, 2002). They also develop reference frames, through interpreting, sensating (particularly, perspective-taking) and situating. The reference frames include a reference point (origin) and a reference direction (axes), and can be (i) *egocentric*, which depends on the observer (they use their own point of view to relate to elements of a system); (ii) *allocentric* (focused on the environment surrounding the object) or (iii) *decentred*, when one adopts another person's perspective (Tversky and Hard, 2009).

From a young age, children build reference frames (particularly egocentric ones) through constant interaction with their environment, by recognising the position and location of objects, and through other actions. As said in the introduction, those interactions can take place in different spaces, of different sizes: In a microspace, one can have a direct perception and manipulation of objects; in a mesospace (e.g., a house), the observation of fixed objects is carried out from different reference points without the possibility of manipulating them; and, in a macrospace (e.g., a city or rural area) perception is neither global nor simultaneous, but through successive local visions (Gálvez, 1985).

Elements from the above discussion serve as our conceptual framework. Next, we review how digital technologies have been found to promote spatial reasoning. In particular, some studies indicate that the awareness of the location and orientation of objects in space can be encouraged with manipulative and digital technologies. For example, Sabena (2018) describes how five-year-old children began to establish spatial relationships by programming the movements and turns of the Bee-Bot robot. There is also research that reports experiences with other programming languages such as Logo (Esparza, 2005) and Scratch (Francis et al., 2016), highlighting the role of those experiences in the development of spatial orientation and representation skills. Logo and Scratch have interfaces that allow the user to draw and move from the perspective of a turtle (Logo) or cat (Scratch), through movement and rotation commands, promoting the development of reference frames. According to Papert (1980), by giving commands to the turtle, children experience syntonic body learning –thus, we claim, attuning embodied (egocentric) with disembodied (decentred) perspective-taking (Tversky & Hard, 2009).

Methodology and experiment design

As said above, in our study we designed a teaching experiment (Cobb & Gravemeijer, 2008) to enhance the development of spatial reasoning skills in children, through a learning trajectory in a design-based research approach. The learning trajectory includes several tasks that involve engagement with digital dynamic representations. Some tasks used Google Maps and Google Earth to recognize meso- and macrospace that children do not see directly (Ortiz-Rocha et al., 2022); others, as is the focus of this paper, deal on how microspaces are perceived using the LightBot¹¹ puzzle-games environment, where reference frames, other than the egocentric one, are developed.

¹¹ <https://lightbot.com/> on mobile phones and tablets. <https://www.numuki.com/game/light-bot/> on computer browsers.

Our teaching experiment consisted of two cycles of design and implementation with third grade children (8-9 year-olds) from a low-income public primary school in Mexico City. The first cycle was carried out remotely (in 2020-2021) via Google Meet due to the COVID-19 pandemic, with up to 16 children, in nine 1-hour sessions. The second cycle (in 2022) was carried out in person with 24 children, consisting in total of 22 sessions of 1 to 2 hours each. All sessions were recorded. It is worth noting that when the children returned to in-person classes after the pandemic they lacked skills in mathematics, reading and communication, which required a process of adapting the learning trajectory's activities, while they (re-)developed some of those skills.

The first author of this report acted as teacher-researcher, guiding the sessions, while the other two co-authors acted as observers taking field notes. After each session, observations were discussed to adapt the following sessions; their analysis also informing the design of subsequent cycles.

In order to promote the development of spatial reasoning skills, the tasks in the learning trajectory were designed to favour the interpretation and creation of representations, as well as the development of reference frames in different sizes of space. Because of the low-income of the participants and their school, the tasks had to use accessible resources. In the case of software, it had to be free and run on the available devices without requiring an Internet connection. The learning trajectory combined work with resources that were manipulative (with concrete materials) and with digital ones. For working in the meso- and macrospace, some tasks were based on physical movements in the school. In particular, one task involved working in the school's courtyard (a mesospace) with the aim of recognising an escape route, by giving commands to teammates to follow a route that could not see. This implied using language to refer to proximity, orientation and direction (such as right or left turns). Others tasks in these spaces used digital tools such as Google Maps and Google Earth that allow children to visualise spaces to which they do not have direct access, such as aerial views of the school and of larger spaces (Ortiz-Rocha et al., 2022).

For working in the microspace, the tasks involved constructions formed by multiple same-sized, concrete or virtual, cubes or tiles. LightBot was useful for working with dynamic representations in a digital virtual multi-tile microspace, that can promote the construction of decentred and allocentric reference frames. As said before, in this report we focus on the tasks using LightBot. We initially had planned to use Scratch to build reference frames, but this was not possible in the first cycle; and during the second cycle the school's computers were down due to lack of use during the pandemic. We found, in LightBot, a free alternative, which is accessible on phones and tablets. LightBot is a game that uses a sprite, controlled by programming sequences of commands (by dragging them to a command area), to move from one tile to others and reach all the target blue tiles randomly assigned by the software (Figure 1). When the sprite reaches a blue tile, the user uses the lightbulb command to indicate when that target has been reached by changing its colour to yellow, and then continue to another blue tile (until all blue tiles have been changed in colour). The game has 8 levels of difficulty, becoming more challenging with each new command introduced. In the first level, the only action is to move forward. The second level includes left and right turns also. At the third level, forward jumps are added. And in subsequent levels, the complexity increases with combinations of these commands (see bottom right of Figure 1).

In comparing LightBot to Scratch, their interfaces and views are different, so the work in each will differ. In moving Scratch's sprites (i.e., the cat), in the default 2D view, children need to identify its

movements with their own, promoting a transition from an egocentric reference frame to a decentred one, particularly when the sprite' orientation does not coincide with that of the user. In LightBot, its isometric view requires children to recognise movements (e.g., a jump) in a simulated 3D space, where the orientation is more difficult to identify, thus requiring a decentred frame, as well as an allocentric one, because users have to be aware of the sprite's multi-tile environment.

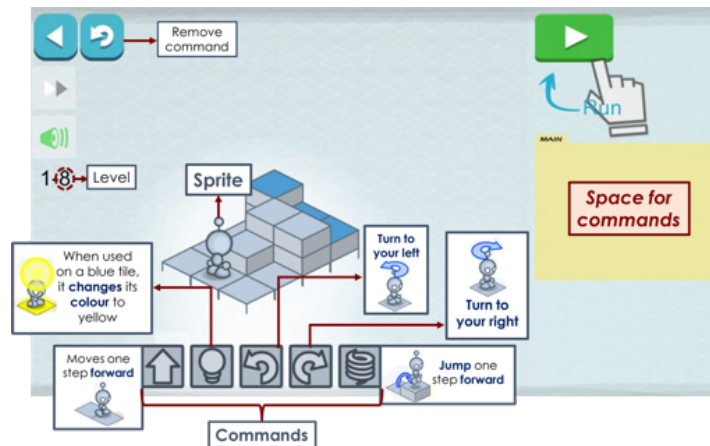


Figure 1: LightBot interface and commands

The LightBot tasks were implemented during the second cycle, in two 1-hour sessions, using six smartphones and one tablet. In the first session, the group of 24 children was divided in two, with one group working with LightBot and the other on other tasks; we paired the 12 children in the LightBot group so that each pair could use a mobile device, while the teacher went around the classroom, recording on video their work, and managing the collaboration between students, so as to promote that all of the children engaged with the game. In the second session the whole class collaborated to build a path of the sprite, in a projected browser-version of LightBot, recording, with two cameras, the screen and the classroom interactions. Part of the analysis of the recordings focused on identifying the reference frames that were developed. Below are three representative examples (using pseudonyms) of what the children did at the levels 4 and 6 of the game. We present the reference frames that were identified in each, along with the language used during the task.

Sample results

Example 1

During the first session, Luis explained how he and his partner completed Level 4 in LightBot. During his explanation, he restarted the level to narrate, step by step, the strategy he used to move the sprite. He said:

Luis: First, this [command] for [the sprite] to turn [to the left – Figure 2a]. Then a jump [using the command to go up a step], then another jump [using the command to go up another step – Figure 2b]. Oh, and I now remember, then another jump [using the command to go up another step – Figure 2c]. Ah, I remember, turn [to the right – Figure 2c), [go forward] two steps, [using the arrow command – the two second-to-last commands in Figure 2d) and one like this [the lightbulb command that indicates reaching the target – last command in Figure 2d].

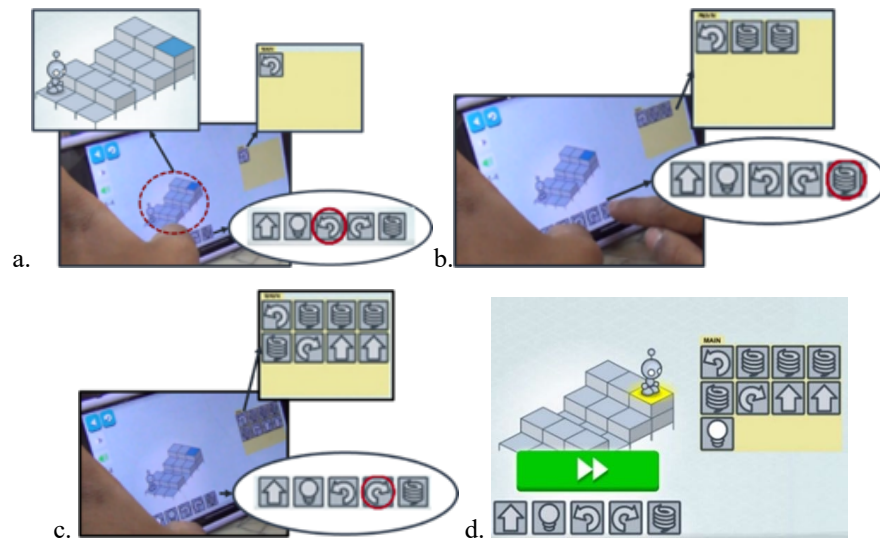



Figure 2: Luis's explanation of how the sprite was moved in LightBot's Level 4

We observe that Luis directs the sprite by recognising its position and the path of travel. Furthermore, he links these movements to the game's commands (Figure 2d, right), although he only used terms such as "turns" and "jumps", without explicitly mentioning the "left" or "right" directions, nor specifying the number of steps taken by the sprite. We consider that Luis uses a decentred reference frame, developed during the previous levels of the game, because he is now able to think in terms of the sprite's perspective. We believe that he explains the movements of the sprite by referring only to "turns", because the sprite's right and left are sometimes the opposite of those of Luis. Nevertheless, he links correctly the commands that he describes to the movements of the sprite. Luis also seems to use an allocentric frame, because he is able to recognise the sprite's position relative to the blue tile target's one, and in this way determine how many steps the sprite needs to take to reach that target.

Example 2

During the first session, Jorge and Nicolás worked together on Level 6 of the game, where the sprite had to pass through two blue tiles. Figure 3a shows the starting point of the sprite, indicated by a red arrow. Initially, Nicolás used two commands: the first for the sprite to perform a jump and the second for it to go forward one tile (Figure 3a). At that point, he didn't understand that the jumping command moves the sprite both forward and up (he thought it only went up), so added that second command that wasn't needed (and didn't do anything because the sprite was on the edge). Jorge helped him understand that he didn't need the forward arrow so he corrected it and following Jorge's suggestions to add a turn-left command to point towards the first blue tile target (see red arrow in Figure 3b). Jorge continued telling Nicolás which commands to enter for the sprite to reach the first blue square: "This [referring to the  arrow – third command in Figure 3c], jump and lightbulb (last two commands in Figure 3c). Let's see, try it." With these instructions, the sprite reached the first blue tile, and with the lightbulb command, its colour was changed to yellow. From this point on, Nicolás became more independent and was able to continue the game on his own.

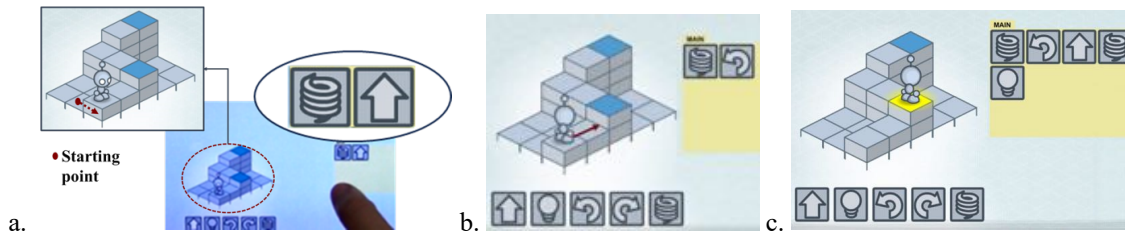


Figure 3: a) First commands used by Nicolás to move the sprite at Level 6; b) Nicolás's correction; c) The target is reached and its colour is changed by using the lightbulb command

Jorge seemed aware of the relationship between the location of the sprite and the blue tile, identifying how many times he must use the "forward" command for the sprite to reach the tile; he thus was able to recognise the movements of the sprite and locate them according to the sprite's interaction with its environment. We thus consider that, at this level, Jorge, as an external observer, was already using decentred and allocentric reference frames. Nicolás may have been in the process of developing those reference systems, but by collaborating with Jorge he could relate better the game's commands with the sprite's movements, and perhaps develop the frames further.

Example 3

In the second session, two teams of students took turns giving instructions to the sprite in the projected browser-version LightBot. They would say a command for the sprite to move, that the teacher would enter in the game after a group discussion of whether it was correct or not. During this activity we observed difficulties in indicating the turns and recognising the sprite's left or right.

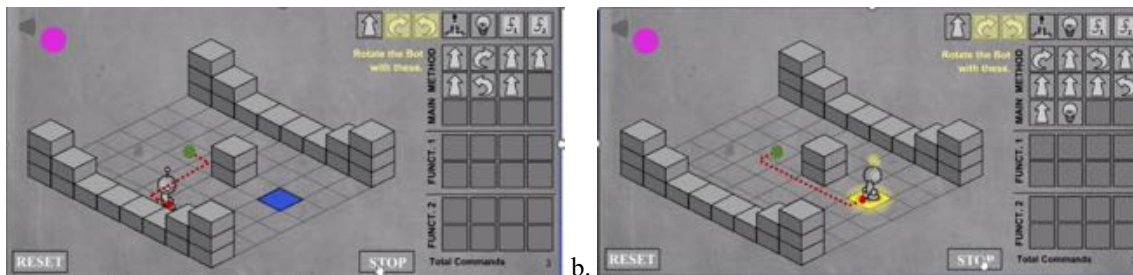


Figure 4: Two sprite routes in the browser-version of LightBot: a) Ricardo's; and b) the expected one. (The starting point is indicated in green; a red-dashed line represents each route)

Figure 4 shows two sprite routes: the one given by Ricardo (pseudonym) and the one he expected. Ricardo gave the following instructions: "to the right, one step, another step, another step, left, one step" (Figure 4a). After the teacher entered them into LightBot, from the visual feedback, Ricardo recognised his mistake in expressing the turns, explaining that those were the commands that should be changed (by exchanging the left and right commands), the sprite would have reached a tile closer to the target (similar to Figure 4b). We consider that Ricardo may have been in the process of transitioning from an egocentric reference frame to decentred and allocentric frames: Initially, when Ricardo was trying to imagine himself in the sprite's position, he may have thought that his right was the same as the sprite's. But thanks to the visual feedback he was able to identify better the position of the sprite in relation to the blue tile target, and thus develop further the decentred and allocentric reference frames.

Discussion and concluding remarks

The examples presented show how students link the sprite's movements, according to its location and orientation in the LightBot environment, with the game's commands/symbols. In general, we observed that students tended to avoid using verbally some spatial orientation terms –“left” or “right” (as in the case of Luis, in Example 1). This, despite that in previous tasks of the learning trajectory – specifically in the courtyard task (see Methodology section)– students had to indicate to teammates left and right turns. The latter required recognising and communicating spatial relationships of proximity, location and orientation. Although in our study we did not analyse students' verbal language, the misuse of orientation terms leads us to infer that recognising and communicating spatial relationships needs to be developed. To verbally communicate LightBot movements, using the “left” or “right” direction terms, requires relating the following elements of the game's reference frame: the plane in which the sprite moves, its location and orientation, and the command/symbol that generates the sprite's movement. When the children start using LightBot, they are still in the process of transitioning from an exclusive use of an egocentric frame to using decentred and allocentric ones. By engaging with the elements in the sprite's world, which is external to the students, they need to develop further the non-egocentric spatial relationships (e.g., through syntonic body learning –Papert, 1980). However, the experiences during the physical tasks (those in the courtyard and with concrete multicubes) –where children had to directly see/perceive the location and orientation of objects/subjects in a physical space– also gave them tools to later visualise and establish relationships between elements in other (virtual) contexts, such as LightBot, allowing them to move from concrete to abstract (Bruce et al., 2015). Thus, the learning trajectory's physical and digital tasks were complementary. Additionally, as in example 2, the collaboration between students also helped in the development of non-egocentric reference frames.

In summary, the activities with LightBot enabled the students to carry out the following actions related to spatial reasoning skills (belonging to the interpreting, sensating and situating categories defined by Davis et al., 2015): i) *locate* and *orient* the sprite in the environment in which it had to move, ii) trace the path that the sprite must follow (*pathfinding*), iii) *visualise* and *imagine* how elements (subjects/objects) are related from a non-egocentric perspective; iv) take the sprite's perspective (*perspective-taking*), to relate its position and location to the objects and the space in which it is located, and in that way-develop and use *decentred and allocentric references frames*.

To conclude, it is worth noting that despite the initial challenges described in the methodology section when the children returned to school after the pandemic lockdown, they actively engaged and collaborated in the learning trajectory tasks, through which they developed the spatial reasoning skills and reference frames discussed above. A future area of research could be to study, not just the student's spatial actions, but also focus more on the language that they use.

Acknowledgment

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A learning trajectory for linear combination using digital tools

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We present a Hypothetical Learning Trajectory (HLT) to introduce the concept of linear combination of vectors in \mathbb{R}^2 and \mathbb{R}^3 . This HLT consists of a sequence of four tasks in which digital tools are fundamental for its development. We designed interactive virtual scenarios for each task, simulating a claw machine where students manipulate robotic arms representing vectors. We evaluated the HLT through a teaching experiment with 20 university students in a first linear algebra course. We used emergent modeling to interpret students' mathematical reasoning and describe their strategies when working on the tasks. Although we observed some differences from what was predicted in the HLT, the results show that the tasks helped students develop formal reasoning about linear combinations.

Keywords: linear algebra, hypothetical learning trajectory, digital tools, university mathematics

Introduction

Linear algebra is one of the fundamental courses for science and engineering students (Stewart et al., 2019). However, research has shown that students have difficulties understanding linear algebra concepts. Some point out that this is because it is among the first courses in which they are confronted with abstract concepts through a theoretical introduction (e.g., Dogan, 2019). Others indicate that they are due to the lack of visual models to help give meaning to abstract concepts (e.g., Harel, 1990). In this regard, several studies report that introducing linear algebra concepts through real contexts helps to make sense of formal definitions (e.g., Wawro et al., 2012). Others have bet on the visualization of dynamic representations to support learning linear algebra (e.g., Tabaghi & Sinclair, 2013). In a systematic review of linear algebra research, Stewart et al. (2019) reported that the topics most studied by researchers are linear independence, span, linear transformations, and eigenvalues and eigenvectors. We chose linear combination because it is an essential element in constructing linear algebra concepts (Turgut et al., 2022). Some studies propose innovative ways to teach this concept. Most of them are designed using Realistic Mathematics Education (RME) theory. For example, “The Magic Carpet Ride” (Wawro et al., 2012) and secure password generation (Cárcamo et al., 2021) are developed with pencil and paper. Others take advantage of the potential of digital tools such as Mauntel et al. (2021) with “Vector Unknown” and Turgut et al. (2022) by including a Dynamic Geometry Environment (DGE). In particular, Mauntel et al. (2021) identified the strategies that students developed by interacting with the “Vector Unknown” game. While this game allows students to interact dynamically with different representations of vectors, the concept of linear combination still needs to be studied. Turgut et al. (2022) used geometry as a starting point to visualize dynamic representations of linear combinations in \mathbb{R}^2 and \mathbb{R}^3 . The authors did not use a real context. Their goal was to evidence how a DGE helps students develop the geometric meaning of scalars, vectors, and linear combinations. In the present proposal, we emphasize using digital tools to

introduce linear combinations of vectors by simulating a real context and visualizing dynamic representations. We present a Hypothetical Learning Trajectory (HLT) for linear combination, and we analyze the progress in students' reasoning to refine initial learning conjectures.

Theoretical framework

The notion of HLT was proposed by Simon (1995) as a tool that guides the teacher in developing a plan of classroom activities. It consists of three components: learning goals, learning activities, and hypothetical learning processes. The latter are hypotheses of students' conceptual development through tasks and learning activities. In general, HLTs do not provide a framework for task design. For this reason, we use C&P principles (Cuevas & Pluvinage, 2003) to design tasks that foster meaningful and participative learning (see Table 1). In addition, we use RME as a framework for interpreting student activity in terms of learning mathematics through its emergent modeling heuristic (Gravemeijer, 1999). The above is feasible because both C&P and RME derive from a constructivist approach and their design principles can complement each other (see Table 1).

Table 1: Coincidences between the C&P principles and the RME and their application to the project

C&P (Cuevas & Pluvinage, 2003)	RME (Van den Heuvel-panhuizen & Drijvers, 2020)	Application to the project
Starting from a <i>problem in context</i>	<i>Reality</i> : Teaching begins with problems in contexts.	Introduction of the linear combination concept in the context of claw machines.
<i>Dosing problems</i> to guide the concept construction.	<i>Guidance</i> : Students must reinvent mathematics as teachers and tasks guide them.	Design of tasks into activities where students interact with one-, two- and three-link arm.
The student must always be <i>acting</i> .	<i>Activity</i> : Students are active participants in the learning process.	Development of virtual interactive scenarios that students can manipulate.
Articulation of <i>diverse representations</i> and implementation of <i>inverse operations</i> .	<i>Level</i> : Learning mathematics involves students going through several levels of understanding.	The tasks begin with a figurative representation of linear combination, and then move on to graphical and algebraic representations.

In emergent modeling, the modeling of contextual problems is fundamental for developing formal mathematical reasoning. Within this perspective, we can distinguish between the “formal” methods proposed in most textbooks and the “informal” strategies used to solve a problem without institutionalized knowledge. The transition from informal to formal mathematical reasoning is presented through four levels of activity (Gravemeijer, 1999): 1) *situational* where interpretations, strategies, and solutions depend on the understanding of how to act in the environment; 2) *referential* where models, graphs, notations and procedures related to the contextual situation appear; 3) *general* that arises when reasoning loses dependence on the specific images of the situation; 4) *formal* where one works with conventional procedures and notations detached from the context. Considering the above, we ask: How does students' reasoning of linear combination progress through the levels of emergent modeling when interacting with real context-based tasks and using digital tools?

Methodology

This study follows the Design-Based Research (DBR) methodology, which aims to produce innovative learning environments through a cyclical three-phase process: preparation and design,

teaching experiment, and retrospective analysis (Bakker, 2018). We present the second cycle results of an ongoing PhD study.

Preparation and design phase

An HLT was designed for the linear combination of vectors in \mathbb{R}^2 and \mathbb{R}^3 . It consists of four tasks that start with a claw machine context, in which students must manipulate a robotic arm to grab a bear. Each task consists of a Virtual Interactive Didactic Scenario (VIDS) and its respective *exploration and guided learning sheets*. The VIDS simulate the contextual problem and encourage the visualization of dynamic representations. The exploration and guided learning sheets are instructions for manipulating VIDS and learning activities for concept construction. The HLT and is presented in Table 2. Digital resources will be made available on request.

Table 2: HLT for the linear combination of vectors in \mathbb{R}^2 and \mathbb{R}^3

Goals	Learning Activities	Hypothetical Learning Processes
Task 1 Define geometrically and calculate analytically the linear combination of a vector in \mathbb{R}^2	Interaction with VIDS “Grab the Bear” (figure 1). Students drag sliders to manipulate the one-link robotic arms and identify in which positions the bear can be placed to grab it.	Students recognize that they can only grab the bear with the robotic arm on a linear trajectory (<i>situational</i>).
	Determine the vector equation that models the problem of grabbing the bear with a one-link robotic arm and define when \mathbf{w} is a linear combination of \mathbf{v} .	Students determine that \mathbf{w} is a linear combination of \mathbf{v} when it is possible to grab the bear (<i>referential</i>) or when the graphical representations of $k_2\mathbf{v}$ and \mathbf{w} are superimposed (<i>general</i>).
	Define when \mathbf{w} is a linear combination of \mathbf{v} without using VIDS.	Students use $\mathbf{w} = k_2\mathbf{v}$ to find if there is a scalar k_2 that allows expressing \mathbf{w} as a linear combination of \mathbf{v} (<i>formal</i>).
Task 2 Identify geometrically the linear combination of two vectors in \mathbb{R}^2	Interaction with the VIDS “Grab the bear by combining” (figure 2a) to determine the equation that models the problem of grabbing the bear with a two-link arm.	Students identify that the $k_1\mathbf{u} + k_2\mathbf{v} = \mathbf{w}$ models the problem of grabbing the bear with a two-link robotic arm.
	Define when \mathbf{w} is a linear combination of \mathbf{u} and \mathbf{v} geometrically.	\mathbf{w} is a linear combination of \mathbf{u} and \mathbf{v} when it is possible to grab the bear (<i>referential</i>).
Task 3 Calculate the linear combination of two vectors in \mathbb{R}^2	Interaction with the VIDS “Grab the bear in the plane” (figure 2b) to generate the 2×2 systems of equations and find the value of k_1 and k_2 that allows expressing \mathbf{w} as a linear combination of \mathbf{u} and \mathbf{v} .	Students use graphical representations to establish the associated system of linear equations and find the values k_1 and k_2 to determine the lineal combination in \mathbb{R}^2 (<i>general and formal</i>).
Task 4 Identify geometrically and calculate the linear combination of one, two and three vectors in \mathbb{R}^3	Interaction with the VIDS “Grab the bear in the box” (figure 4a) to determine the equation that models the problem of grabbing the bear with a three-link arm.	Students identify that the equation that models the problem of grabbing the bear with a three-link robotic arm is $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{w}$.
	Determine when \mathbf{w} is a linear combination of \mathbf{v}_1 , \mathbf{v}_2 and \mathbf{v}_3 .	When it is possible to grab the bear or $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$ and \mathbf{w} are superimposed (<i>referential and formal</i>)
	Determine when \mathbf{w} is a linear combination of \mathbf{v}_1 , \mathbf{v}_2 and \mathbf{v}_3 without using VIDS.	Students use $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{w}$ to define whether \mathbf{w} is a linear combination of \mathbf{v}_1 , \mathbf{v}_2 and \mathbf{v}_3 (<i>formal</i>).

Teaching experiment

The intervention was conducted with 20 linear algebra students from a public university in Mexico. Participants had already learned the topics of systems of linear equations, vectors, and matrices. The intervention was conducted in three two-hour sessions. Since learning mathematics is not only an individual activity but also a social activity (Van den Heuvel-panhuizen & Drijvers, 2020), two students shared one computer. Classroom discussions were carried out following students' work on tasks to arrive at the correct answers. The data collected consisted of student *exploration and guided learning sheets*, video recordings of all the pairs of students, including the computer screens they were working with, and video recordings of classroom discussions.

Results and Retrospective Analysis

We analyzed the data by reading the *exploration and guided learning sheets*, observing video recordings of computer screens, and transcribing students' discussions. Each researcher identified solution strategies and associated them with an emergent modeling level. Subsequently, they met to reach a consensus on the strategies, and compare them to the hypothetical learning processes. Due to the limitations of the paper, we present the responses of a randomly selected pair (S1 and S2), but the analysis of data from all participants contributed to the identification of strategies.

Task 1. Grab the Bear

In this task, students interacted with the VIDS *Grab the Bear* (Figure 1a), which simulates the movement of two one-link robotic arms whose function is to grab a bear on a board: "Arm 1" whose movement is defined by the vector $k_1\mathbf{u}$ and "Arm 2" by $k_2\mathbf{v}$. In this VIDS, the sliders k_1 and k_2 , vectors $\mathbf{u} = (1,2)$, $\mathbf{v} = (-1,2)$ and \mathbf{w} , associated with the bear's position, are set. When the user drags sliders k_1 and k_2 , the coordinates of $k_1\mathbf{u}$ or $k_2\mathbf{v}$ vary, and consequently, the length of each arm is lengthened or shortened. The VIDS has buttons "A to F" to modify the bear's position, buttons to hold or release the bear, control boxes to select the robotic arm, and "Plane" that enables the graphical representation of the context (Figure 1b).



Figure 1: View of VIDS Grab the Bear a) figural representation; b) graphical representation

The task started with exploring the VIDS, where students determined the positions where the bear could be grabbed with "Arm 1". From the analysis of all the data, we identified three strategies to address the problem: (1) the bear can be placed on the line defined by the "Arm 1" movement (*situational level*); (2) the bear can be placed on the line that passes through points E and D (*referential level*); and (3) the positions of the bear would be vectors multiples of \mathbf{u} (*referential level*). The last two are different from the one anticipated in the HLT. Students like S2 used the first strategy

by indicating that they could grab the bear only if it were on the “Arm 1” trajectory. Others, like S1, used the second strategy by associating the movement of “Arm 1” with the scalar multiplication, as shown in the following discussion.

- S1: Click on “Plane”. There it is. The positions would have to be multiples of \mathbf{u} . They would be scalars by the vector, $k\mathbf{u}$ where k belongs to the reals and zero, too, right?
 S2: Yes, because it can touch the bear at any point.
 S1: Yes, where is this line (S1 points to the vector $k_1\mathbf{u}$).

After familiarization with VIDS, they worked with graphical representations and established $\mathbf{w} = k_1\mathbf{u}$ as the equation that models the problem of grabbing the bear with “Arm 1”. In this way, the teacher introduced the linear combination of a vector in \mathbb{R}^2 by the equation $\mathbf{w} = k_1\mathbf{u}$ and $\mathbf{w} = k_2\mathbf{v}$. With the definition of linear combination, students answered questions such as “*Will $\mathbf{w} = (1, -2)$ at position D be a linear combination of $\mathbf{v} = (-1, 2)$?*”. Three solution strategies were observed in this activity. Students indicated that \mathbf{w} was a linear combination of \mathbf{v} if: (1) it is possible to grab the bear with “Arm 2” (*referential level*); (2) \mathbf{w} belongs to the line generated by $k_2\mathbf{v}$ (*general level*); and (3) $k_2\mathbf{v}$ and \mathbf{w} overlap. The second is different from the ones anticipated in the HLT. S1 and S2 were at the general level because they used the third strategy based on visualizing the graphical representations, as shown in the following discussion.

- S1: Yes, it is a linear combination.
 S2: How much is k_2 ? (S2 drags slider k_2 until $k_2\mathbf{v}$ is superimposed on \mathbf{w}).
 S1: $k_2 = -1$. Then, $-1(-1, 2) = (1, -2)$, equal to the vector \mathbf{w} in F.

Finally, to conclude task 1, questions were proposed that did not require using the VIDS, such as “*Given $\mathbf{v} = (-7, -2)$, how do I determine if $\mathbf{w} = (-3.5, 1)$ is a linear combination of \mathbf{v} ?*”. In this activity, students' reasoning proceeded as proposed in the HLT. They used the equation $k\mathbf{v} = \mathbf{w}$ to determine a scalar k that would satisfy the equation (formal level) as indicated by the discussion of S1 and S2.

- S1: We must find the scalar.
 S2: Yes, k that satisfies that $k(-7, -2) = (-3.5, 1)$.
 S1: But it does not exist.

Task 2. Grab the bear by Combining

In this task, students interacted with the *Grab the Bear by combining* VIDS (Figure 2a), which simulates a two-link robotic arm constructed from the combination of “Arm 1” and “Arm 2” used in Task 1. This VIDS has a control box that enables the graphical representations of $k_1\mathbf{u}$ and $k_2\mathbf{v}$ to be superimposed on the robotic arm links and \mathbf{w} associated with the bear's position on the board. Therefore, when the user successfully grabs the bear, it is satisfied that $\mathbf{w} = \mathbf{r}$. This equality is indicated when \mathbf{w} is a linear combination of \mathbf{u} and \mathbf{v} . In the first learning activity, the students established $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$ as the equation that models the problem of grabbing the bear with a two-link arm. Then, the definition of linear combination was introduced: “We will call linear combination the sum of two or more vectors, each multiplied by a scalar. When $k_1\mathbf{u} + k_2\mathbf{v} = \mathbf{w}$ is satisfied, \mathbf{w} is said to be a linear combination of \mathbf{u} and \mathbf{v} . If the equality is not satisfied, \mathbf{w} is not a linear combination of \mathbf{u} and \mathbf{v} ”. Students used this definition to answer: “*Is \mathbf{w} a linear combination of \mathbf{u} and \mathbf{v} ?*” with \mathbf{w} defined by the bear position on the board. We describe the students' activity at the *referential level*, as conjectured in the HLT because they identified that $k_1\mathbf{u} + k_2\mathbf{v} = \mathbf{w}$ is valid only when it is possible to grab the bear. Therefore, most of the students answered by relying on the VIDS. They

dragged the sliders to see in which positions they could grab the bear, as suggested by the discussion of S1 and S2.

- S1: We must modify the arm's position to grab the bear. If it grabs it, it is a linear combination.
 S2: In B is a linear combination because it can grab the bear in any position.



Figure 2: VIDS a) Grab the Bear by Combining; b) Grab the bear in the plane

Task 3. Grab the bear in the plane

In this task, students interacted with the VIDS *Grab the Bear in the plane* (figure 2b), which is the graphical view of *Grab the Bear by combining*. In *Grab the Bear in the plane*, the k_1 and k_2 values are not visible, so students can calculate them analytically. We observed two strategies that students used to find k_1 and k_2 : (1) Counting squares in the cartesian plane (*general level*) and (2) developing equation $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$ (*formal level*). The first one is different from the one anticipated in the HLT. S1 and S2 used the second strategy. They identified vectors \mathbf{u} , \mathbf{v} and \mathbf{w} in the plane, substituted them into $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$, determined the equivalent system of equations and solved it (figure 3).

$$\begin{array}{l} k_1 - k_2 = 5 \\ 2k_1 + 2k_2 = -6 \Rightarrow 2(k_1 + k_2) = -6 \Rightarrow k_1 + k_2 = -3 \\ k_1 = 5 + k_2 \\ k_1 = 1 \\ 5 + 2k_2 = -3 \\ 2k_2 = -8 \\ k_2 = -4 \end{array}$$

Figure 3: Solution of S1 and S2 to what values must k_1 and k_2 have to grab the bear in B?

Task 4. Grab the bear in the Box

In this task, students interacted with the VIDS *Grab the Bear in the Box* (figure 4a), which simulates a three-link robotic arm whose function is to grab bears inside a box: a green link whose movement is defined by vector $k_1\mathbf{v}_1$; a red link defined by $k_2\mathbf{v}_2$; and a blue link by $k_3\mathbf{v}_3$. In this VIDS, there are the canonical vectors $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0)$, $\mathbf{v}_3 = (0, 0, 1)$, vector \mathbf{w} associated with a bear position, and scalars k_1 , k_2 and k_3 . Here, the position of the claw is set as $\mathbf{r} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$. Therefore, when the user successfully grabs a bear, $\mathbf{w} = \mathbf{r}$ is satisfied. This equality indicates when \mathbf{w} is a linear combination of \mathbf{v}_1 , \mathbf{v}_2 and \mathbf{v}_3 . The VIDS has buttons "Bear A" to "Bear D" and "Random Bear", which assign to \mathbf{w} the coordinates of the selected bear position; buttons to hold or release the bear; and a joystick that allows modifying claw position by changing the value of k_1 , k_2 and k_3 . This task began by exploring the VIDS, where students set $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$ as the equation that models the problem of grabbing the bear with a three-link arm. Subsequently, they were asked to answer "Is $\mathbf{w} = (5, 3, 0)$ a linear combination of \mathbf{v}_2 and \mathbf{v}_3 ?" In this activity, students applied what they learned in the previous tasks to identify linear combinations of three vectors in \mathbb{R}^3 . They used

two strategies in a complementary manner. That is, they first used the VIDS to verify whether they could grab the bear and confirm whether \mathbf{w} is linear combination of \mathbf{v}_2 and \mathbf{v}_3 (*referential level*), and subsequently, they worked with algebraic procedures to propose the linear combination equation (*formal level*) as the following discussion shows.

- S2: It is three times to the right and five times forward because the bear is in (5, 3, 0) (S1 manipulates the joystick until he grabs Bear A). Is \mathbf{w} a linear combination of \mathbf{v}_1 and \mathbf{v}_2 ? Yes.
 S1: Then $\mathbf{w} = (5, 3, 0)$ is going to be equal to $5(1, 0, 0) + 3(0, 1, 0) = 5\mathbf{v}_1 + 3\mathbf{v}_2$.

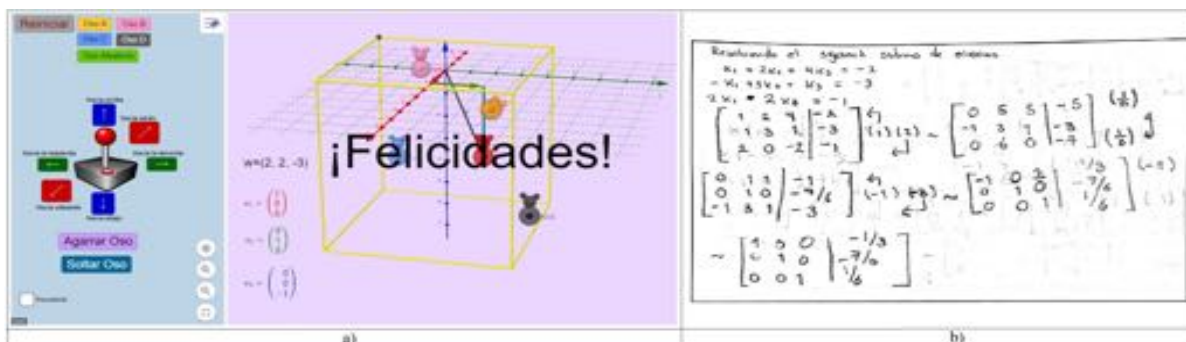


Figure 4: a) VIDS Grab the Bear in the Box; b) S1 and S2 solution to question Q1

To conclude task 4, questions were proposed that did not require using the VIDS. We observed the formal level as conjectured in the HLT because the students did not refer to the context or rely on graphical representations. For example, in question Q1, “Determine whether $\mathbf{w} = (-2, -3, -1)$ is a linear combination of $\mathbf{v}_1 = (1, -1, 2)$, $\mathbf{v}_2 = (2, 3, 0)$ and $\mathbf{v}_3 = (4, 1, -2)$ ” (Figure 4b).

Conclusions

We propose an instructional model to conceptualize the abstract definition of linear combination within a first linear algebra course using an HLT of four tasks designed with a specific didactic framework and digital tools. The results show that most students transited sequentially through the activity levels of the emergent models to move from informal reasoning to creating the meaning of a formal definition. Starting from the *problem in the context* of claw machines fostered the development of the *situational level*, visualizing the linear combination with the simulation of the movement of a robotic arm. The simultaneous *articulation of diverse representations* fostered the transit to the referential level. It allowed the interpretation of grabbing the bear with a robotic arm as a linear combination of vectors. Subsequently, the graphical view detached from the context supported the transition to the general level. Finally, proposing inverse operations in which students must calculate the value of scalars and define if \mathbf{w} is a linear combination without using VIDS produced the transition to the formal level. The above shows the relevance of digital technology's role in developing activities. In particular, the careful design of VIDS supported reasoning of mathematical relationships that the abstract definition does not possess. Implementing the tasks offers insight into how this sequence plays out in the classroom. We observed that some students' reasoning proceeded differently than anticipated in the HLT. For example, in the first activity (situational level), there was no instruction on using the graphical view. However, some students used it to identify context-related mathematical relationships (referential level). These results allow us to adjust the HLT, such as by passing the situational level and starting at the referential level. This refinement would have to be empirically tested. One limitation of this study is that it has been possible to identify the level of

activity developed by 20 students as a team. Future research suggests that each student's behavior be analyzed because it is not certain that all students are at the same activity level.

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AI-supported Mathematical Task Design with a GPT Agent Network

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This study investigates the use of communicative AI agents in designing mathematical tasks. It examines how a network of LLM (large language models) agents can facilitate mathematical task design through collaborative communication in a chat chain. Four specialized AI agents were instructed each focusing on a different perspective: mathematical content, linguistic sensitivity, competence orientation, and differentiation. The AI agents sequentially modified given mathematical tasks, with each contributing a unique focus to the task's evolution. The resulting tasks were evaluated by in-service teachers as human experts. This way, the qualitative study explores the potential of LLM agent networks in educational contexts. First findings suggest that AI agents can support teachers in the development of mathematical tasks for diverse learning needs, but at the same time require adaptation by teachers to the educational situation.

Keywords: Task Design, ChatGPT, AI agent, Large Language Models, problem posing.

Introduction

Large Language Models (LLMs) have gained high attention in the educational research field (e.g. Kasneci et al., 2023; Buchholtz et al. 2023). The disruptive changes and developments in the educational context are still not fully accessible. The aim of the language modeling (LM) approach is to teach machines human language and its characteristics (Hadi et al., 2023) based on statistical calculations (Hiemstra, 2009). While so-called statistical language models make a probability statement regarding the following word within texts, LLMs are based on artificial intelligence (AI) and deep learning techniques (Hadi et al., 2023). A currently much-discussed example of LLMs is the GPT (Generative Pretrained Transformer) architecture, which can imitate the structure of human language and respond appropriately to requests. To achieve this, the LLM is trained to solve specific tasks using large amounts of data. Within this data, the LLM architecture recognizes and analyzes patterns and relationships to generate a coherent, context-dependent output (Floridi & Chiriatti, 2020).

Since the release of access to GPTs by OpenAI (2023), the technology of LLM agents has become accessible to all and can generate customized responses to specified requirements. LLM agents have demonstrated considerable success across a broad spectrum of applications, from reasoning (Yao et al., 2023) to video gaming (Wang et al., 2023) and autopilot systems (Jin et al., 2023). Recent studies have explored the potential of leveraging multiple LLM agents working in collaboration to address a single query, showcasing effectiveness in tackling intricate tasks (Du et al., 2023; Liang et al., 2023; Wu et al., 2023). The integration of text files and their underlying content also allows LLM agents to provide advanced responses. Therefore, research in the field of mathematical task design presents an exciting opportunity for exploring connections with LLM agents.

Theoretical Background

In the past, task design has often been carried out by textbook authors, while new relevant literature emphasizes the significance of involving mathematics teachers as partners in this process (Jones & Pepin, 2016). Even when tasks are not designed from scratch but adapted, this process supports the development of the teacher's mathematical knowledge and mathematics-didactical design capacity (Pepin, 2015).

As tasks play an important role in planning competence-oriented teaching, teachers also need to have a clear understanding of the cognitive demands of the instructional and diagnostic assignments they are planning to use in class (Maier et al., 2014). To plan competence-oriented lessons teachers must know the domain-specific competence development of their learning group and must analyze tasks regarding their suitability for this current competence development (Kleinknecht & Lankes, 2012).

Because of the diversity of learning groups, the ability to design and/or modify mathematical tasks of varying difficulty becomes more important (Maier et al, 2014). Task design can be done by individuals but is also shown to be constructive when realized in collaborative groups. In addition to designing tasks in collaborative groups, Sensevy et al. (2013) created the term ‘cooperative engineering’ between teachers and researchers to design tasks. If those teachers and researchers act out of different professional mandates and responsibilities, they form a ‘multi-professional team’ (summarized in Widmer-Wolf, 2018) which holds the potential to meet the individual requirements of learners. Combining the strengths of a multi-professional team in specialized LLM agents could therefore be promoted as a further step in task design.

Research Interest

To this end, we take up the ‘communicative agents’ approach of Qian et al. (2023), which in the project ChatDev has various AI agents with specific roles and tasks for software development and access to GPT. In our study, we used the LLM GPT-4 from OpenAI and built a so-called GPT agent network consisting of four AI agents with different functions and expertise: a language-sensitive agent, a mathematical content-oriented agent, a competence-oriented agent, and a differentiation-sensitive agent.

The AI agents are assigned to different perspectives and thus set different priorities in the process of task design. This enables interdisciplinary collaboration in the optimizing and adapting of tasks and can be a useful support for teachers and experts in mathematical task design, especially regarding the individual learning requirements of students.

In a qualitative study, the effects of AI agents in mathematical task design are to be investigated aiming at the following research question: To what extent can a GPT agent network working as a multi-professional team support teachers within mathematical task design in four perspectives? We are planning to answer two subordinate questions:

1. Between an original and an AI-modified task, which do teachers prefer to select for their educational context, and what are the reasons for their choice?
2. How do the teachers evaluate the tasks modified by AI agents regarding the four perspectives?

Methods and Materials

To explore those research questions, six tasks with varying levels of difficulty were selected. Four AI agents were created for the GPT agent network, each given access to a PDF document containing guidelines (referred to as the research knowledge base) and instructed to adapt the tasks based on these guidelines. A chat chain mechanism was established to facilitate communication between the AI agents, as proposed by Qian et al. (2023). Following the modifications to the tasks using a consensus-driven, AI-supported design process, the modified tasks were evaluated by in-service teachers who served as human experts in this context.

Mathematical tasks

In choosing the examples to modify, three simple tasks requiring basic competencies at the elementary and secondary levels were considered, as well as three further examples from the domain of problem solving. Problem-solving tasks present a unique challenge for LLMs as they often extend beyond basic calculations and encompass argumentative elements necessary for solving the problem (Schorcht et al., 2024). The first three tasks originate from the areas of arithmetic, patterns, and fraction calculation. For the latter three tasks, problems were selected that require basic arithmetic skills as well as the solving of a system of linear equations. This aims to explore the possibilities of task development with AI agents and demonstrate to what extent the AI agents are capable of enriching both simple and more complex tasks. The tasks presented were always given to the AI agents with a corresponding learning objective and grade level. This ensures the development of the tasks within a defined framework. These tasks were inserted into the chat chain, starting with the mathematical content-oriented AI agent followed by the language-sensitive AI agent, differentiation-sensitive AI agent, and at least the competence-oriented AI agent.

Enhanced AI agents

Documents were developed as foundational materials for each AI agent to serve as a research knowledge base. These documents were provided in PDF format and enabled each AI agent to access relevant information and adapt tasks based on their specific knowledge. Restricting the research knowledge base to a small number of pages is essential due to GPT-4's processing limitation of up to 8192 tokens per prompt. A token might represent an entire word, though fragments of words and punctuation marks are also considered tokens.

Instructions were designed to serve as a functional guide for AI agents in task evaluation using the specified research knowledge base. For example, the role of AI agents was clarified initially, highlighting that tasks should be viewed as objects of study rather than prompts for solutions. Each AI agent approaches the task analysis differently, focusing on areas like mathematical content, linguistic aspects, differentiation, and compliance with content and process standards, as outlined in the research knowledge base.

The mathematical content-oriented AI agent is designed to support teachers in generating a mathematically correct and consistent task. Unlike the others, this agent uses the training data from GPT-4 as its research knowledge base, as it is impossible to limit mathematical knowledge to just a few pages.

The language-sensitive AI agent is designed to support teachers in developing language-sensitive tasks based on knowledge of language barriers in mathematics lessons and design principles of language-sensitive tasks (Abshagen, 2015).

The differentiation-sensitive AI agent is designed to support teachers in natural differentiation and refers to characteristics of naturally differentiated learning opportunities and tasks: openness, complexity, low entry threshold, “ramps” for high achievers, need for discussion and high cognitive activation potential (Krauthausen & Scherer, 2022).

The competence-oriented AI agent is designed to support teachers in encouraging necessary mathematical competencies and refers to the five content-related competencies according to UNESCO (2020) as well as to the NCTM Process Standards (n.d.).

After analysis, a decision is made for potential task modifications, which the AI agent autonomously implements if necessary. Feedback is then provided on the task development, assessing adherence to the guidelines. The process concludes with a summary that reformulates the task's nature as a subject for study, aiming to enhance prompt adherence and minimize hallucinated responses by the LLM, addressing the common issues of prompt following in prompt engineering (Rassin, Ravfogel & Goldberg, 2022; Betker et al., 2023).

After generating instructions for the AI agents, we followed the proposed architecture by Qian et al. (2023) and divided each phase of the task design process into atomic chats. The AI agents sequentially modify the given task, with each AI agent contributing a unique focus to the task design. To enhance communication between AI agents, the output from each agent was used as input for the next prompt. This sequence of prompt, output, prompt, etc., is referred to as a chat chain (Qian et al., 2023). The chat chain thus realizes a communicative process that culminates in a circular, consensus-oriented procedure under the independent influence of certain guidelines, to evaluate tasks and make suggestions for modifications.

Data Analysis

In the second step, the resulting tasks were evaluated by in-service teachers as human experts (each with an average of 11.67 years of teaching mathematics' experience). The teachers were asked to critically evaluate the adapted tasks to test their practical applicability and relevance in educational settings. This is a process crucial for ensuring the tasks' practical applicability and relevance in classroom settings. Maier et al. (2014) described seven categories and characteristics of interdisciplinary task analysis and showed how teachers can analyze tasks in terms of their suitability for competence-oriented teaching.

Similarly, in this study, four categories of mathematic-specific task analysis were used to guide teachers in evaluating the given tasks. The evaluation process required teachers to consider several criteria, including mathematical content, language sensitivity, differentiation level, and competence requirements. The aim was to assess the AI's effectiveness in customizing educational content to meet diverse learning needs and environments.

Following this we used a frequency analysis of the teacher's choice of tasks as well as their evaluation based on the four given perspectives. Furthermore, the comments reasoning their decisions are analyzed using Qualitative Content Analysis to inductively develop categories.

Results

In this paper, we want to provide an insight into the results of our study. The extensive analysis of our data can be found in Schorcht et al. (forthcoming; 2024).

RQ1: Overall, 44% of 36 decisions leaned towards the selection of AI-modified tasks with notable differences between problem-based tasks and tasks requiring basic competencies. Specifically, 61% of the decisions concerning problem-based tasks favored the AI-modified task, indicating a recognition of the value added by AI in these contexts. However, the changes made by AI agents to tasks requiring basic competencies were less convincing, with only 28% of experts favoring the AI-modified tasks. This outlines a noticeable variance in expert choices, highlighting the nuanced impact of AI integration depending on the nature of the original task.

The reasons teachers gave for their decisions were analyzed through the Qualitative Content Analysis. The sorting of the paraphrased comments from the experts within the assessments led to categories of positive and negative comments on specific categories for the original tasks and the AI-modified tasks that explain the choices of the experts. Due to the limited space in this conference paper, only a few of the 21 categories are used for each task type to exemplarily outline the reasoning of the decisions. A complete description of all categories as well as a table containing their absolute frequencies is presented in Schorcht et al. (forthcoming; 2024).

Regarding the original tasks' positive categories, comments of the category 'short text' were used to justify selecting the original task as it was "more concise" and had "less information that overwhelmed". One expert mentioned students' reading competence: "I think that many will initially have difficulties in selecting the essential from the texts. Therefore, the simple tasks with less text are more effective in achieving the formulated objectives" (Expert 3). Concerning original tasks' negative categories, the category 'lack of content' was evaluated negatively due to the incomplete nature of tasks that require basic competencies.

Positive characteristics of the AI-modified tasks were addressed in eight categories. The following paraphrased comment addresses the categories of 'language comprehension', 'concrete call for action' and 'motivating' and 'solution approaches given' in which AI-modified tasks scored very well: "Task 2 [AI-supported task; authors] is formulated in a more motivating and comprehensible way because it requires active action. There is also an approach to a solution" (Expert 1). The range of different strategies offered was also used to explain the selection of an AI-modified task: "Different calculation strategies can be used and described here" (Expert 5). However, we also found considerable criticism of AI-modified tasks, particularly regarding the inclusion of 'unnecessary information' and the 'demanding text level'.

RQ2: All six AI-modified tasks were assessed multidimensionally in terms of mathematical depth, language sensitivity, natural differentiation and competence orientation. The tasks were rated separately but the results don't seem to be significantly dependent on the different tasks.

Overall, all AI-supported modified task were rated positively in terms of mathematical content, differentiation level, and competence requirements. The only category not evaluated positively was the language sensitivity. Over 40% of the ratings stated, that the tasks were not, not completely or only partial language sensitive. The added comments show that this is due to the fact that most of the modified tasks are longer than the original one as ChatGPT is a language-generating AI and tends to

generate long texts. This should be considered in further development of the language sensitive AI agent.

Discussion

The presented results give useful information about the quality of the modified tasks and their suitability for potential use in classrooms. The evaluation regarding the four perspectives, i.e. the work of the four different agents, also serves as feedback regarding necessary further development of the AI agents. The findings suggest that GPT networks functioning as a multi-professional team can indeed support teachers in developing mathematical tasks and open new perspectives for pedagogical strategies. The AI-modified tasks were especially effective in providing supportive hints, improving language comprehension, and giving clear calls to action. However, it can also be seen that some of the modified tasks are very text-expanding since LLMs are designed for text production. Thus, it can be concluded that the instructions of the language-sensitive AI agent need to be adapted regarding text length. Nevertheless, this tendency to hallucinate also provided an opportunity to enrich tasks requiring basic competencies with context.

In Summary, a GPT network is considered a practical tool for modifying tasks, but teachers are still responsible for adapting tasks to their students.

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Characterization of TPACK in a GeoGebra professional development programme

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In this work, we investigate the different domains of knowledge mobilized by in-service teachers, with reference to TPACK, in the use of GeoGebra in a professional development programme with middle and secondary school teachers, in Portugal. We established performance indicators within the scope of technological, pedagogical and content knowledge, which we evaluate through content analysis of the activities produced by teachers during the programme. We present examples of the tasks proposed in the programme for the development of TPACK and identify aspects that demonstrate the mobilization of TPACK by teachers, in solving the proposed tasks and in the construction of an original task, aimed at pedagogical practice. We conclude that teachers can mobilize this type of knowledge, fundamental for the integration of technology, namely GeoGebra, in an educational context, within the scope of Mathematics teaching.

Keywords: Professional development, Educational Technology, TPACK, GeoGebra.

Introduction

Teaching with technology is not an easy task. Teachers' competencies such as knowledge are a decisive factor, but there are few studies investigating Professional Development Programs (PDP) for teaching mathematics with technology (Thurm & Barzel, 2022). In this paper, we aim to investigate the different domains of the TPACK model (Koehler et al., 2013) regarding knowledge developed and mobilized by teachers using GeoGebra to implement tasks in the context of a PDP. Research will be based on the analysis of the tasks proposed to trainee teachers to learn mathematics and the work produced by them in the PDP in the framework of TPACK.

With this work, we want to answer the following question: What domains of knowledge within the TPACK framework are mobilized by in-service teachers when designing tasks for learning mathematics with GeoGebra during a PDP?

In Portugal, GeoGebra is one of the main technological resources introduced in PDP for basic, middle and secondary teachers, as it integrates the Portuguese curriculum at these levels of education (DGE, 2021, 2023). It is therefore crucial that teachers acquire and mobilize adequate knowledge so that they have confidence and competence in applying the proposed approaches (Murphy et al., 2015) to transform the prescribed curriculum into a curriculum in action (Gimeno, 2000).

This work is part of a larger project in which we study the integration of digital technology by teachers into pedagogical practices, where we have already seen that teachers consider that they had knowledge (and self-confidence) to use technology in teaching (Pimenta et al., 2022). However, they not always place technology on students' desks as a learning instrument, so there is a need for more research to understand how these technologies may become widespread instruments for student

learning (Pimenta et al., 2023). Thus, characterizing the different domains of the TPACK in teachers' knowledge may provide more understanding in this matter.

Theoretical background

In the fundamental literature that supports this study, we considered two areas: the use of GeoGebra in an educational context and the TPACK model to frame teachers' knowledge of how to teach using technology.

GeoGebra in an educational context

The use of GeoGebra in an educational context is widely recommended in the literature. GeoGebra was designed, as Hohenwarter (2013) describes, to combine dynamic geometry with the functionalities of computer algebra systems, for learning and teaching mathematics. It is a rich resource that can contribute to improving pedagogical practices and more meaningful learning (Zengin, 2019; Weinhandl et al., 2020; Lyublinskaya & Cabral, 2023). For example, evaluation by García et al. (2020) demonstrated that students positively value the proposed tasks and the use of GeoGebra to develop them, which facilitated understanding and deepening geometric concepts. Zulnadi and Zamri (2017) showed that students using GeoGebra, compared to a control group subject to methods they designate as conventional, have greater conceptual and procedural mathematical knowledge. Hernández et al. (2020) in an investigation within the scope of initial teacher training, noted that the use of technology, and particularly GeoGebra, can be crucial to clarify mathematical aspects and to support the understanding of the mathematics to be taught.

In the new Mathematics curricula for basic and secondary education in Portugal, there is explicit reference to the use of resources such as GeoGebra. In secondary education (DGE, 2021) the “Systematic use of technology” is presented as one of the key ideas of essential learning, to “Encourage the exploration of ideas and concepts, integrating technology (...)”. More specifically, “strategic teaching actions” are listed and involve dynamic geometry. In the 3rd cycle of basic education (7th to 9th years) (DGE, 2021), in the section “How to promote the learning of Mathematics”, regarding resources/technology, it is mentioned that «The tools Technological technologies must be considered as unavoidable and powerful resources for teaching and learning Mathematics. (...) All students must be able to freely access (...) software for statistical processing, geometry, functions (...)”. Dynamic geometry environments are explicitly mentioned (for example, “Perform constructions in dynamic geometry environments that mobilize geometric places, regular polygons, relationships between angles and isometries (...)”).

TPACK

Regarding the development of teachers' knowledge for efficient teaching of Mathematics with technology, we have as a reference the Technological, Pedagogical and Content Knowledge (TPACK) model, as introduced by Mishra and Koehler (2006). The acronym TPACK stands also for Total Package and, according to Mishra and Koehler (2008), it better reflects the interdependence of the intervening knowledge domains, as represented in Figure 1.

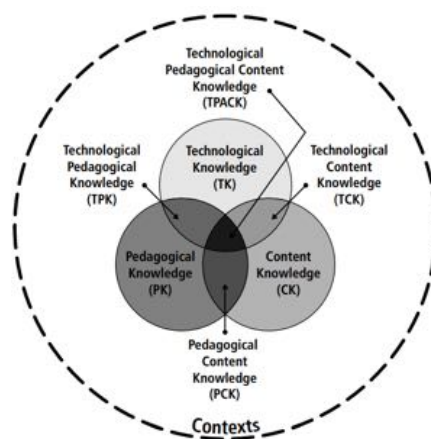


Figure 1 – TPACK model (Koehler et al. , 2013)

Mishra and Koehler (2006) introduce TPACK as a theoretical framework that contextualizes the intersection between knowledge in the domains of pedagogy, content and technology. Koehler et al. (2013) describe the various components of TPACK: TK (technological knowledge), always changing; TCK (technological and content knowledge), understanding the ways in which technology and content influence each other; TPK (pedagogical and technological knowledge), understanding how teaching and learning can be enhanced with the use of certain technologies; finally, TPACK, an emerging form of knowledge that goes beyond content, pedagogy and technology: it is an understanding that emerges from the interactions between these three dimensions. Integrating these knowledges into the educational context, teachers mobilize TPACK whenever they teach. As characterized by Koehler et al. (2013), TPACK is the support of a “truly meaningful and deeply qualified teaching with technology (p.16)”, which goes beyond the three types of knowledge individually.

Regarding the evaluation of teachers' TPACK, we consider Niess et al. (2009) who, specifically regarding TPACK for teaching mathematics, describe a model that consists of several phases of cognitive development: recognition, acceptance, adaptation, exploration and advanced. This work proposes how to identify what stage a mathematics teacher is in by observing their activities. Bowers and Stephens (2011) present an instrument for distinguishing five domains of knowledge (CK, TK, TCK, TPK and TPACK). Based on work projects proposed by the participants (generally teachers or students in initial teacher training), in which they present proposals for learning activities, using an dynamic geometry environments, and based on the answers to a self-assessment questionnaire, the researchers characterized the knowledge of practitioners in the five domains mentioned. Harris et al. (2015) state that, in general, TPACK assessment can have three scopes: self-assessment, observation of behaviour and analysis of teachers' productions (such as lesson plans, for example). In 2018, Akyuz, in a work carried out within the scope of initial training for Mathematics teachers, uses an assessment instrument for the seven TPACK knowledge domains, based on three performance descriptors for each model, applied in the analysis of planning, activities and the applications produced by future teachers.

Regarding the type of activities to be proposed to teachers for developing TPACK, we had as a reference Harris et al. (2014), that offer TPACK-based “activity types” to the professional development approaches.

Methodology

In this research, using a qualitative methodology (content analysis), with an interpretative paradigm (Cohen et al., 2018) we analysed the development and mobilization of teachers' TPACK within the framework of a training course implemented with 25 middle and secondary school math teachers, all with excellent grades at the end of the course. This 25-hour course (12 synchronous + 13 asynchronous, in e-learning), entitled “GeoGebra applications”, was implemented in 2020/2021, at the Portuguese Mathematics Society (<https://www.spm.pt/>). Teachers' participation in this PDP and in the study was voluntary and their anonymity was guaranteed.

To characterize the teachers' TPACK mobilized and developed when carrying out tasks, we defined three performance descriptors for analysing the work produced by teachers during the course:

- TK – to use the right tools correctly, including non-trivial and dynamic procedures;
- CK – to fulfil the task objectives from a mathematical point of view (without scientific errors) involving meaningful mathematical to allow students explore the topic;
- PK – to include tasks for students with the aim of active learning, including asking for solutions, generalizations or conjectures.

To define these indicators, we start from previous work in this field. For example, Akyuz (2018) proposes three descriptors for each of the domains CK, PK, TK, PCK, TCK, TPK and TPACK, but we aimed to consider a smaller number of indicators, so that they are clearly understandable by trainees who, as teachers, will have to mobilize and integrate this knowledge into their teaching practice. In the indicators we present, we wanted to summarize and group some of the ideas proposed in Bowers and Stephens (2011) “Components of TPACK (...)” (p.293). Then, we consider the intersections of the three indicators presented. For example, if TK and CK are mobilized simultaneously, we consider that TCK is mobilized; if the three indicators are mobilized, we consider that TPACK has been mobilized. We begin by presenting and characterizing the tasks proposed to the trainees, intentionally designed to promote the development of teachers' knowledge to teach mathematics. Then we analyse teachers' productions resulting from these proposals and the tasks they autonomously created to implement with students, for learning mathematics using GeoGebra, to illustrate the knowledge mobilized in the construction of the tasks.

Data presentation and analysis

In the PDP, to improve the teachers' TPACK in the use of GeoGebra, we propose a set of tasks, in the areas of Geometry and Functions, in which teachers should mobilize and develop TPACK, in its three domains: TK, CK and PK. At the end of the course, each teacher had to create an original task, which involved dynamic constructions in GeoGebra, to propose to students, so that they use GeoGebra in a Mathematics learning activity.

Tasks proposed in the PDP

We present below some examples of the initially proposed tasks. In Geometry: «Create a dynamic construction that illustrates the Pythagorean Theorem. Include dynamic text that illustrates the corollary of the theorem. Publish at [geogebra.org](https://www.geogebra.org)»; «Construct the 9-point circle for a triangle. Study the relationship between the radius of the 9-point circle and the radius of the circumscribed circle.» In Functions: «Create a model, using sliders, that illustrates the influence of the parameters a , h and k , on the parabolas relating to the quadratic functions, with analytical expression in the form $f(x) =$

$a(x - h) + k$ »; «Verify geometrically in GeoGebra that $(\cos x)' = -\sin x$, using command input, the tangent line and slope tools, the locus tool and the activate trace mode of an object.»

These tasks were designed by the trainers, based on their experience and qualifications, but also using literature, such as the work of Harris et al. (2014), that characterize learning activities to develop and apply TPACK, namely, “Producing an artifact”, “designing an exhibition” and “creating a game”.

Teachers’ productions

The following are constructions made by the trainees, which put into practice the statements proposed by the trainer. In the activities shown in Figure 2, we can identify the mobilization of TCK, given that, on the one hand, TK is mobilized, building, for example, boxes to show and hide objects, and dynamic text, in addition to dynamic construction, and, on the other, it is mobilized the CK, since there is total mathematical correctness in the constructions. In the nine-points circle construction the trainee included a question to be answered based on the exploration of the application – “In any triangle, the radius of the 9 points circle is.... of the triangle circumradius” – that shows an implicit pedagogical concern to the exploration of the activity by students, mobilizing PK. So, in an integrated way, we consider this as examples of TPACK mobilization.

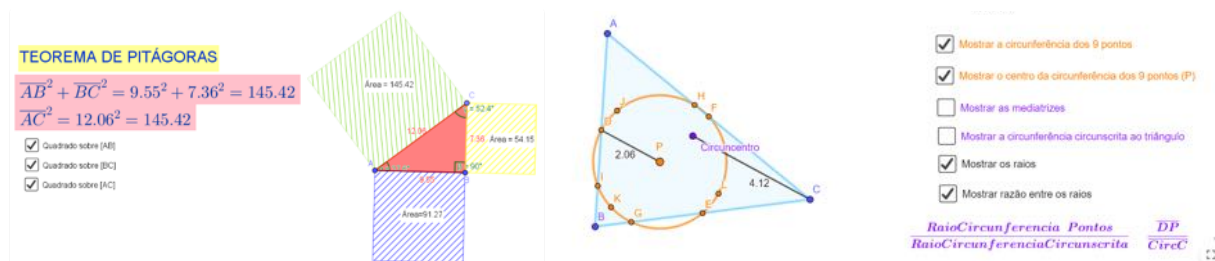


Figure 2: Pythagorean Theorem / Nine points circle – teachers’ productions, mobilizing TPACK

In Figure 3, a construction is shown in which teachers mobilize TCK completing the trigonometry task proposed by the trainer. TK – applying non-trivial procedures, knowing how to use command input, locus tool and the trace mode of an object; CK – knowing and applying the definition of derivative at a point.

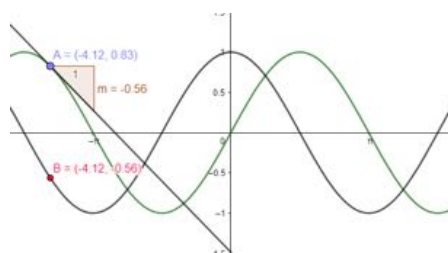


Figure 3: $(\cos x)' = -\sin x$ construction, mobilizing TCK

Regarding original works by trainees, as shown in Figure 4, with greater weight in our analysis, since they did not originate from any script, other than the general recommendation for them to construct a task for classroom application, mobilizing the knowledge developed in PDP, we verified in most of the works presented that teachers were able to build activities fulfilling this recommendation, mobilizing TPACK: TK, as they use the right tools correctly (including non-trivial procedures, such as sliders, boxes to show and hide objects and dynamic text); CK, achieving the objectives of the task from a mathematical point of view (without scientific errors) involving meaningful mathematics to

allow students to explore the topic (as the ratio between perimeters and areas of similar figures); PK, including tasks for students, with the aim of active learning, including asking for solutions, generalizations or conjectures (as in the circumcentre example, where the teacher creates a to assign to the students, which includes the proposal to determine, in the context of problem solving, using the available tools, the equidistant point of three given points, to introduce the mathematical topic with an exploratory activity.)



Figure 4: Figures similarity and circumcentre construction, mobilizing TPACK

More examples of this are the works that can be accessed through the following hyperlinks (authorized by their authors): <https://www.geogebra.org/m/bpan4wff6>; <https://www.geogebra.org/m/dexapmhf>; <https://www.geogebra.org/m/y48pqxmv>.

Discussion and final considerations

We verified that teachers mobilised different domains of TPACK (Koehler et al., 2013), namely Content Knowledge (CK), Pedagogical Knowledge (PK) and Technological Knowledge (TK), including the integration of these domains, PCK, TCK and TPK, resulting in the TPACK. Based on Bowers and Stephens (2011) work, we concretized examples of tasks to be proposed to trainees, in a PDP environment, for the development of the TPACK domains. Based on the descriptors proposed by Akyuz (2018), we proposed three simplified indicators (CK, TK and PK) so that they can be clearly understood and applied not only by researchers, but also by trainers or trainees within the scope of PDP and, in general, by teachers who must now be well aware of this type of pedagogical practice, using technology such as GeoGebra, as prescribed in the curriculum (DGE, 2021, 2023). In this way, we were able to exemplify, based on the productions of the trainee teachers (figures 2 to 4), whether based on initially proposed tasks or based on their final individual original works, in which context the three main domains of TPACK are mobilized in the development of these activities. For example, the use of boxes to show and hide objects or dynamic text is an indicator of TK mobilization; a construction that illustrates meaningful mathematics correctly, from a scientific point of view, that allows the student to learn within the scope of this mathematical topic, is a CK mobilization indicator; including questions that allow students to draw conclusions from exploring the application is an PK mobilization indicator. Finally, we conclude that teachers are able to mobilize knowledge within the scope of TPACK to build activities for learning mathematics using GeoGebra. This can help to meet the learning objectives set out in the curriculum (DGE, 2021, 2023). These results seem to be in line with teachers who considered that they had knowledge and self-confidence to use technology in teaching (Pimenta et al., 2022). However, it does not clarify why teachers do not promote these

technologies for student learning as identified in Pimenta et al. (2023). Thus, there is a need to develop more research on this matter.

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ARising Perspectives: Exploring Embodied Mathematics with the augmented reality app MalAR

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Embedding three-dimensional objects in one's own real environment in real-time is a unique opportunity that Augmented Reality (AR) offers for mathematics education, but research in this area is still scarce. The MalAR project developed an AR application to embed analytical geometry objects (lines, planes and points) into the learner's real environment using the smartphone camera. Learners can explore mathematical situations in their real environment that are usually only described symbolically in textbooks. The paper focuses on a study using the MalAR app in a learning environment. The results show that the MalAR app induces different forms of body movements. Based on the learners' statements, different categories of movements could be identified. Although the study does not provide a conclusive picture of the relationship between body movement and mathematical performance, the findings suggest that further research into the use of AR by learners is beneficial.

Mathematics education, augmented reality, analytic geometry, digital tools, app.

The project MalAR

The MalAR (Learning Mathematics with Augmented Reality) project aims to explore the potential of using augmented reality (AR) to support teaching and learning in the context of analytic geometry. The three-dimensional nature of mathematical objects in analytic geometry, such as planes, makes it difficult to use physical representations. This often leads to a technical and less descriptive approach to these concepts, which hinders the active sense-making process (Borneleit et al., 2001). The MalAR project explores the use of Augmented Reality (AR) to address these issues. AR merges reality with computer-generated representations in real time (Milgram & Kishino, 1994), allowing both to coexist or blend seamlessly (Azuma, et al., 2001). The MalAR smartphone app (available for iOS) allows learners to position analytic geometry objects in their real surroundings. The geometric objects are integrated into a video image captured by the smartphone camera (see Figure 1). Using a smartphone, an everyday device, learners can explore symbolically presented mathematical situations in their real environment, creating the illusion of a three-dimensional object. This allows them to discover concepts, such as the mutual position of planes, through their own exploration through body movements and to visually check their results (Reit, 2022).

When using the MalAR app, the user is first asked to choose whether he wants to register or enter the App with a guest account. The app then accesses the camera and displays a yellow square to indicate that it is searching for areas on which a coordinate system can then be placed by tabbing to the screen. The coordinate system is fixed and cannot be moved. This means that all subsequent objects that depend on the coordinate system are also fixed. The objects can only be enlarged by moving closer to them, not by zooming with a finger movement. In the same way, a change of perspective, i.e. viewing the situation from another side, requires the user to move their body to the other side. Rotating the coordinate system, as is familiar from dynamic geometry systems, is deliberately not possible here either. Other objects, such as planes, points, lines and distances, are selected using the plus button at the bottom of the screen (see Figure 1, right). In addition to the visualisation of the

mathematical situation, a symbolic representation of the placed objects appears at the top of the screen (see Figure 1, right). In this display, you can select to show, hide, delete or change the objects using the symbolic notation.

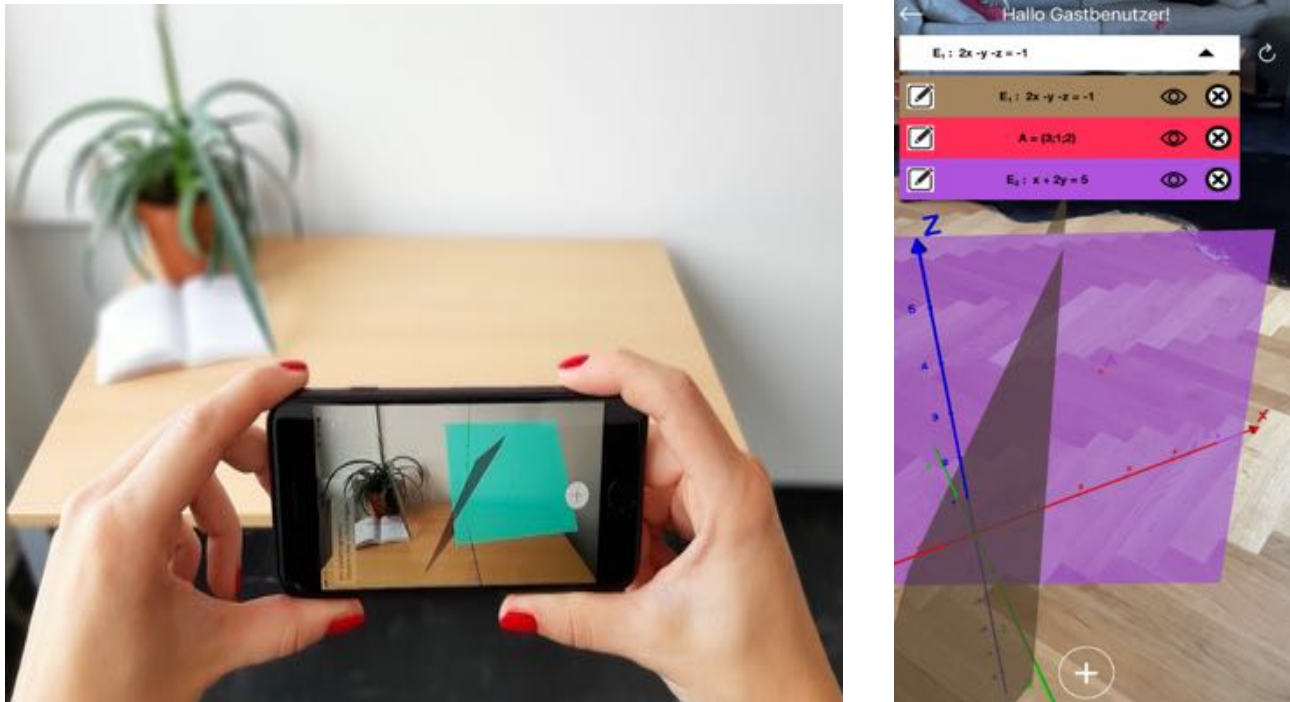


Figure 1: left: Perspective of the user, looking at the smartphone screen which captures the reality augmented by mathematical objects. Right: Screenshot of the camera view in MalAR-App: Coordinate system, two planes and a point placed in the user's environment

AR as an external interactive representation

When using the MalAR app, 3-dimensional mathematical objects are projected through the live camera view into the reality surrounding the user. Keehner et al. (2008) refer to “augmenting cognition” when describing the potential of a purposeful use of external representations. Studies indicate that complex cognitive processes can be outsourced to less complex perceptual-motor processes with external (digital) representations (Card et al., 1999; Gordin & Pea, 1995; Kirsh, 1997). Theories consider interactive behaviour in a theoretical framework of so called distributed cognition (Hutchins, 1995; Zhang & Norman, 1994). According to this framework, cognitive processes take place internally (in the mind) and externally (in the world, in an external medium). Keehner et al. (2008) postulate that, rather than being a peripheral aid to cognitive processes, external representations overlap with internal representations to create a distributed representational space. The cognitive load theory (Sweller, 1988) and related models of embodied cognition, assume that learners seek to minimise reliance on internal cognitive processes by outsourcing them to perceptual-motor processes (e.g. Ballard et al., 1997; Wilson, 2002; Zhang & Norman, 1994). With regard to the possibilities offered by the MalAR app, the question arises as to whether there are observable

processes, in the sense of actual physical actions, and what influence these have on the solution of an AR-supported task.

Study Design

A design-based research approach was used to develop an iOS app in the field of analytical geometry, going through several design cycles. The primary aim of the app development was to ensure an easy handling without a new input programming language, which is required by most apps on the market (e.g. GeoGebraAR). The input of mathematical objects should be based on the symbolic notation commonly used in German-speaking countries, as taught at school. This means that no extensive introduction to the use of the app is necessary. In the 12-19 age group, 98% of girls and 96% of boys own a smartphone, according to a study on media research among German adolescents in 2018 (JIM, 2018). It can therefore be assumed that almost every young person today owns a smartphone. The high availability of the smartphone together with the straightforward operation of the app make it attractive for use in the classroom.

Once a first satisfying version of the MalAR app was available, a pilot study was carried out. Based on the theoretical implications of AR as an external interactive representation described above, the following consecutive research questions emerged for the pilot study:

- F1 Does an AR-supported learning environment in the field of analytic geometry initiate movement in learners?
- F2 What patterns of movement are initiated by an AR-supported learning environment in analytical geometry?
- F3 Do high-achieving learners show different movement patterns than low-achieving learners?

50 students (aged 16/17) from a German grammar school worked in an AR-supported learning environment in the area of analytic geometry in a 90-minutes session. Of these, 16 students were attending an advanced mathematics course and 34 attended a basic mathematics course. The learners were guided through different tasks with the help of a questionnaire (Figure 2). Most of the questions were related to the placement of mathematical objects in the app and the subsequent validation of their (mutual) position. In order to investigate whether the learners used their bodies when working on the questions, they were repeatedly asked to note how they moved in order to analyse the scene (see questions 3, 7, 8 and 9 in Figure 2).



Open the MaAR app and create an account:

1. use the "Back button" to get to the login page (if you are not already there)
2. click on "Konto erstellen".
3. enter a (fictitious) email address and a password: Enter your email address here _____
4. click on the blue "Log in" button.

Aim of the survey: How do you use your smartphone and body when working on maths problems with the app?

1. Open the app and place a coordinate system on a surface
2. Enter the point $A(0|5|3)$ (via "+button").
3. Describe in as much detail as possible how you moved to find out whether the point was placed correctly:

4. Now also enter the plane $E_1: \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + r \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + s \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

5. Describe in as much detail as possible how you moved in order to analyse the scene (plane and point) in the app:

6. Also enter the plane $E_2: \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + r \cdot \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (via "+button").

7. Describe the position of plane E_2 in relation to plane E_1 :
8. Describe the position of plane E_2 in relation to point A:
9. Describe in as much detail as possible how you moved to analyse the mutual position of E_1 and E_2 and point A in relation to plane E_2 using the app:

Figure 2: Translated excerpt from the questionnaire used in the pilot study

With regard to research question 3, the learners were asked to solve a typical textbook task for the topic in written form (Figure 3). This task was used to categorise learners as low or high performers for the analysis.

Complete the following tasks on this sheet without using the app MaAR

- a) Set up a straight line g through the points $A(2|0|2)$ and $B(6|4|0)$.
- b) Analyse the mutual position of the straight line g and the plane

$$E: \vec{x} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + r \cdot \begin{pmatrix} -12 \\ 0 \\ 3 \end{pmatrix} + s \cdot \begin{pmatrix} -12 \\ 6 \\ 0 \end{pmatrix}$$
- c) Determine the point of intersection if the line and the plane intersect.

Figure 3: Task to be solved in written form within questionnaire of the pilot study

Analysis and Results

The movement data collected through the instruction manual was analysed using qualitative content analysis (Mayring, 2014). Three main categories and 12 subcategories were coded from the learners' responses on self-perceived movement while using the malAR app (Table 1) with a moderate Cohen's κ of 0.6. A distinction could be made between body movement (B), movement of device (D) and object-related movement (I).

Table 1: Categories of movement patterns found in the learners' answers

Code	Description	x% of learners mention „code“ in their answers	
		subcategory	main category
B1	stand up	50%	
B2	turn around (sitting or standing, without further movement)	14%	
B3	move around (walk, step, go somewhere, stand on an object/chair, stand somewhere, move a chair, bend down, walk around something)	86%	94%
B4	move closer/farther away	38%	
B5	not moving much	20%	
B6	look "from above" at objects (e.g. "reach up" to look at objects)	44%	
D1	Move/align device	56%	56%
I1	Compare the scaling on the axes (move coordinates, search for coordinates of the point)	44%	
I2	Viewing objects from several sides/from several perspectives ("from both sides", "from another side", "turning around something")	74%	18%
I3	Hide/show objects	18%	
N1	Movement unclear: with own active movement or only movement of device?	40%	40%
NM	not possible (the app did not work, the arithmetic problem referred to was not solved, not processed for reasons that cannot be understood)	50%	50%

Movement patterns (Research question F1 and F2)

A quantitative analysis of the code frequencies shows that the category "Body Movement (B)" was noted by almost all learners. Specifically, B3 "move around" was mentioned as the most frequent subcategory with almost 90%. Half of the learners stated that they stood up to view the AR scene, followed by B6 "look from above at object" and B4 "move closer/farther away". The seemingly contradictory statement "not moving much" was reported by a fifth of the learners.

More than half of the learners reported that they moved the device to gain additional insight in the AR-scene. In terms of object-related movements, I2, the change of perspective, was most frequently reported (74%). I1, comparing the coordinates of the given objects with the coordinate axes in the

AR scene, was mentioned by almost half of the learners. I3, fading in and out objects, did not seem to be very relevant for many learners when working on the tasks. At this point it can be hypothesised that the complexity caused by many overlapping mathematical objects in an AR scene, was not yet so high in the learning environment used in the study.

In 40% of the responses, learners expressed unclear movements (N1) that could not be assigned to the main categories. The main problem here was that it was unclear whether the movement pattern was performed with the device or with the learner's own body. Half of the learners were unable to provide an answer at one or more items in the questionnaire (NM) during the session due to a lack of prior answers in the questionnaire or technical problems.

To answer the question of whether high achieving learners show different movement patterns than low achieving learners, the written mathematics task (Figure 3) was evaluated in relation to the number of body movements (Figure 4). The hypothesis that high achieving learners need less help in solving the task and therefore also make less use of the possibilities of the MalAR app could not be confirmed. Overall, no consistent relationship could be found between performance in the mathematics tasks and the number of movements noted. Figure 4 shows that both, high and low achieving learners show forms of body movement in their answers to the other tasks in the learning environment. The average number of body movements for all performance levels is 4-5 movements, although there seems to be a clustering at 4 body movements for the high-achieving and 7 body movements for the low-achieving (Figure 3). However, whether there is a difference in the form of movement at the level of the (sub)categories has not yet been analysed, but could provide detailed insights.



Figure 4: Correlation of body movement (B) and mathematics performance

Discussion and outlook

The written responses show that all learners explored the AR scene through body movements. This shows that the possibilities of AR in the learning environment initiated body movements (F1). The breakdown into different movement patterns (F2) shows that learners performed a wide range of movements. On the other hand, the learners' responses reveal a limitation of the study. It can be assumed that not all movements actually performed were noted. Furthermore, it is likely that the actual movements deviate from the written notes. To address this shortcoming, a follow-up project will videotape learners working in a MalAR-supported learning environment. It will be interesting to

see to what extent the movement patterns differ from those in the pilot study and how the categorisation system is extended and differentiated. Based on the results of the pilot study, it can be assumed that the use of AR influences the way in which learners work on tasks (cf. Keehner et al., 2008; Zhang & Norman; 1994). Conversely, it will be of great interest to identify task characteristics that make AR support a particularly effective tool.

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***AnnoPy* – A digital tool to foster disciplinary literacy**

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The importance of developing disciplinary literacy in mathematics is widely accepted. However, how digital technologies can be used to support the development of disciplinary literacy is hardly discussed in the literature. In this paper, AnnoPy—a digital tool that was developed to support teachers in fostering disciplinary literacy—is introduced. AnnoPy can be used in different disciplines and with different intentions related to the development of disciplinary literacy. The focus of this paper is on a learning scenario in which AnnoPy is used to support students’ reading of mathematical text. The learning scenario is described and some results of a first evaluation of its implementation are presented. The latter show that using AnnoPy is promising to support students reading abilities of mathematical text.

Introduction

Disciplinary literacy can be defined as the capacity for ‘understanding of both disciplinary content and disciplinary habits of mind (i.e. ways of reading, writing, viewing, speaking, thinking, reasoning and critiquing.’ (Fang & Coatoam, 2013). It is widely agreed that fostering disciplinary literacy is important in mathematics education (e.g., Hubbard, 2021; Shanahan & Shanahan, 2008). Fostering reading skills (e.g., Rezat et al., 2022), writing about mathematics (e.g., Segerby, 2016), and involving students in rich mathematical discourse (e.g., Erath et al., 2018) are but some ways that have been suggested to foster disciplinary literacy. However, the role of digital technologies in fostering disciplinary literacy is hardly discussed in the literature. A literature search using ERIC (<https://eric.ed.gov>) with the search term *mathematics AND “disciplinary literacy” AND digital* yielded 12 results, with only one of them—a practical resource—being closely related to digitally supporting disciplinary literacy.

In this paper, I introduce *AnnoPy*, a digital tool developed to support teachers in different subject areas in fostering disciplinary literacy. I also describe a learning scenario in which *AnnoPy* is used to support students’ reading of mathematical text. Finally, I present the first results from an evaluation of the learning scenario with $n = 284$ first-year teacher students in a course on elementary geometry.

Theoretical Framework

Reading as one aspect of disciplinary literacy is theorized as a process in which the reader actively engages with the text, generating meaning mediated by his/her past experiences, knowledge and beliefs, as well as the context and purpose of reading (Borasi & Siegel, 1990). Several studies have investigated differences in reading strategies between novice and expert readers of mathematics texts. Shepherd and van de Sande (2014) identify differences in three dimensions: fluency, comprehension monitoring, and engagement. Fluency relates to the observation that expert readers read more fluently when encountering mathematical symbols and can read the meaning of the symbols. Comprehension monitoring refers to spending more time on and being more persevering in understanding which includes stopping more often while reading to check comprehension. Finally, engagement is characterized by the extent to which expert readers search and use different resources within and outside the text to understand the concepts.

Berger (2019) develops an empirically grounded typology of different reading styles. The expert reader, which Berger (2019) calls “close reader with strong connections”, is characterized by first skimming over the text to identify familiar and new knowledge before carefully reading the entire text, including all examples, definitions, theorems, and proofs, with a special focus on the new knowledge. The close reader makes connections between different resources in the text and also with prior knowledge. Furthermore, he/she does the exercises justifying the mathematical activities.

The standard way to foster students reading in mathematics is to teach them reading strategies. In a meta-analysis of the effectiveness of reading-strategy interventions in whole classroom settings, Okkinga et al. (2018) found that the strategy “underline the important information” and “modeling” as a didactic principle where expert readers think aloud while they read to show their cognitive processes while comprehending the texts were among the aspects that moderated the effects of the strategy training on strategic ability the most.

The strategies of expert readers combined with the results from Okkinga et al. (2018) provide the foundation for designing the learning scenario in this study.

Methodology

The study presented in this paper is grounded in a design research methodology with a special focus on the use of technology (Reeves, 2006). The design aimed at fostering students’ close reading of mathematical text. This was done by designing a learning scenario in which the written script of a lecture played a key role. Students’ reading and understanding of the script were captured using *AnnoPy* as a technological tool. *AnnoPy* and the learning scenario will be described in more detail below. The learning scenario was implemented and evaluated in a lecture on elementary geometry for primary and special education teacher students at Paderborn University. As the use of *AnnoPy* is very intuitive—only requiring the selection of text passages and then selecting among the predefined annotation styles—no particular introduction to the use of the tool was required. However, students had the opportunity to watch a video explaining the functionalities of *AnnoPy* that was accessible through the learning platform (moodle) at the University. N = 284 first-year students answered an online questionnaire about their perceived benefits of the lecture design, including *AnnoPy*. The questionnaire contained 25 self-developed items with a six-point Likert scale, three interval-scaled items, and five open items. The items were structured into six sections, each related to a different design aspect. Five items related to the overall course design, including *AnnoPy*, four items related to students’ self-perceived reading and annotation strategies, and five items particularly to *AnnoPy*. Students’ answers to the questionnaire were analyzed using methods from descriptive statistics.

AnnoPy

AnnoPy (www.annopy.de) is a digital, collaborative online-tool to support school or university teachers in promoting students’ disciplinary literacy skills. It was developed collaboratively at Paderborn University based on the cooperation between German language education, informatics education, and mathematics education. It is intended to be used in classroom teaching or blended learning scenarios. At its current stage of development, *AnnoPy* has the following functionalities:

- 1) text passages can be marked and annotated (individually and in groups) based on different annotation styles (Figure 1). These are predefined by the teacher according to the intention of

engaging with a text. In Figure 1, the annotation styles are “schwer verständlich” (difficult to understand), “bekannt” (familiar), and “wichtig” (important).



Figure 1: Individual annotation view with pre-defined annotation styles “schwer verständlich” (difficult to understand), “bekannt” (familiar), and “wichtig” (important).

- 2) The outstanding feature of *AnnoPy* is that it enables the teacher to cumulate all the individual markings and analyze them quantitatively; that is, *AnnoPy* only shows markings that are identical among a particular number of students. This can be imagined as laying slides with the same text excerpt and students’ individual annotations on top of each other to get an overall picture of the number of text passages annotated with the same style by a particular number of students. We call this an *Overlay*. The overlay of the same text passage as in Figure 1 is shown in Figure 2.

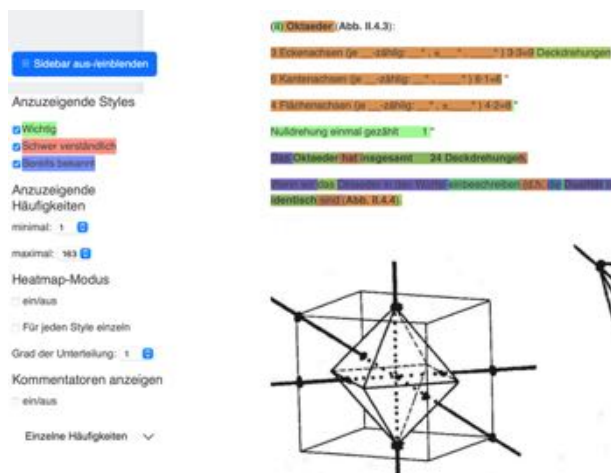


Figure 2: Overlay-View in AnnoPy, showing all cumulated annotations

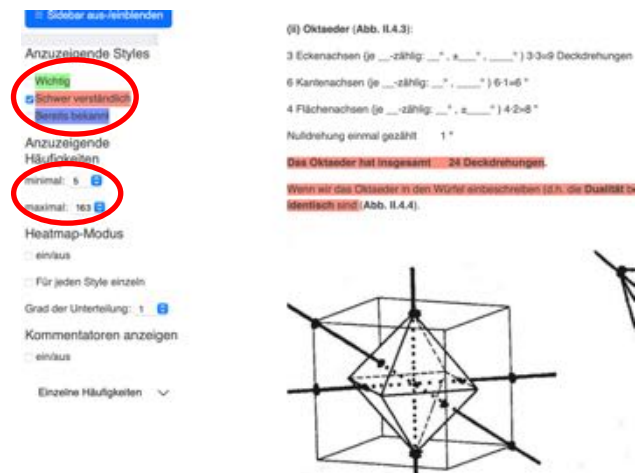


Figure 3: Quantitative analysis in the overlay-view in AnnoPy, showing only the style “schwer verständlich” (difficult to understand) and text passages that were annotated by at least five students

AnnoPy enables visualization of either all annotations or of a certain frequency of annotations (e.g., annotations that were rarely made by the students or made particularly frequently). Annotation styles can also be displayed individually. Figure 3 shows only the style “schwer verständlich” (difficult to understand) and only text passages annotated by at least five students as difficult to understand.

The *Overlay* is the central starting point for subsequent discourse, which fosters understanding of the text on a social level. In the next section, I describe the design of a learning scenario in which the

features of *AnnoPy* were used to foster university students' reading of mathematical text in a lecture with a large number of participants (approximately 300).

The learning scenario

The long-term goal of the learning scenario is to foster students' disciplinary literacy related to mathematical text and the particular ways of using language to communicate mathematics. The main idea of the scenario is to provide students with as many opportunities to read mathematics as possible and to scaffold their understanding of mathematics through reading. This idea was implemented in a lecture with a large number of participants (approx. 300 students). The learning scenario builds on the concept of flipped learning (FLN, 2014). The script of the lecture is used as a central text resource. Students must read an excerpt of the script in preparation for every weekly lecture. The script excerpt is presented to students through *AnnoPy* and was to be accessed through a link provided on Moodle—the university's learning platform. *AnnoPy* is used in two ways to scaffold students' understanding of mathematics through reading. First, while reading it supports students to reflect on the text in terms of three questions that Berger (2019) has identified as being important for expert readers in mathematics: 1) What is familiar?; 2) What is new and important?; and 3) What is difficult to understand? This is done by using three predefined annotation styles (familiar, new & important, difficult to understand) in *AnnoPy* for marking relevant text passages. These three questions also foster students' comprehension monitoring while reading which was found to be an important aspect of expert readers by Shepherd and van de Sande (2014). Second, after reading, *AnnoPy* supports the lecturer in getting an insight into students' reading experiences by analyzing the marked and annotated text passages to prepare a lecture tailored to students' needs. In particular, the lecturer can analyze students' prior knowledge by analyzing what students have marked as “familiar” by checking if the text's prerequisites were marked as familiar by a relevant number of students or if some prerequisites need clarification. Furthermore, the lecturer can check if students have identified the important new content in the script excerpt. Finally, the lecturer can analyze students' difficulties with the text and prepare the lecture, offering additional explanations that support students in overcoming these difficulties. The main strategies for offering further explanations are

- modeling the reading of the difficult-to-understand text passages and making connections to previous knowledge and other resources in the text. This was found to be an important didactic principle in the meta-analysis by Okkinga et al. (2018);
- offering additional representations. This is widely regarded as an important principle in mathematics education to foster understanding.

This is facilitated by using the *Overlay* in *AnnoPy*. The *Overlay* allows for quantitative analysis of the marked script passages to identify those marked and annotated by a relevant¹² number of students. Additionally, the lecturer can use the insight into students' engagement with the text to improve it by being more explicit about important prerequisites or clarifying them in the text and by including further explanations. Figure 4 presents the different steps of the weekly repeated structure of the learning scenario.

¹² What is considered to be a relevant number in this context needs to be defined individually by the lecturer.

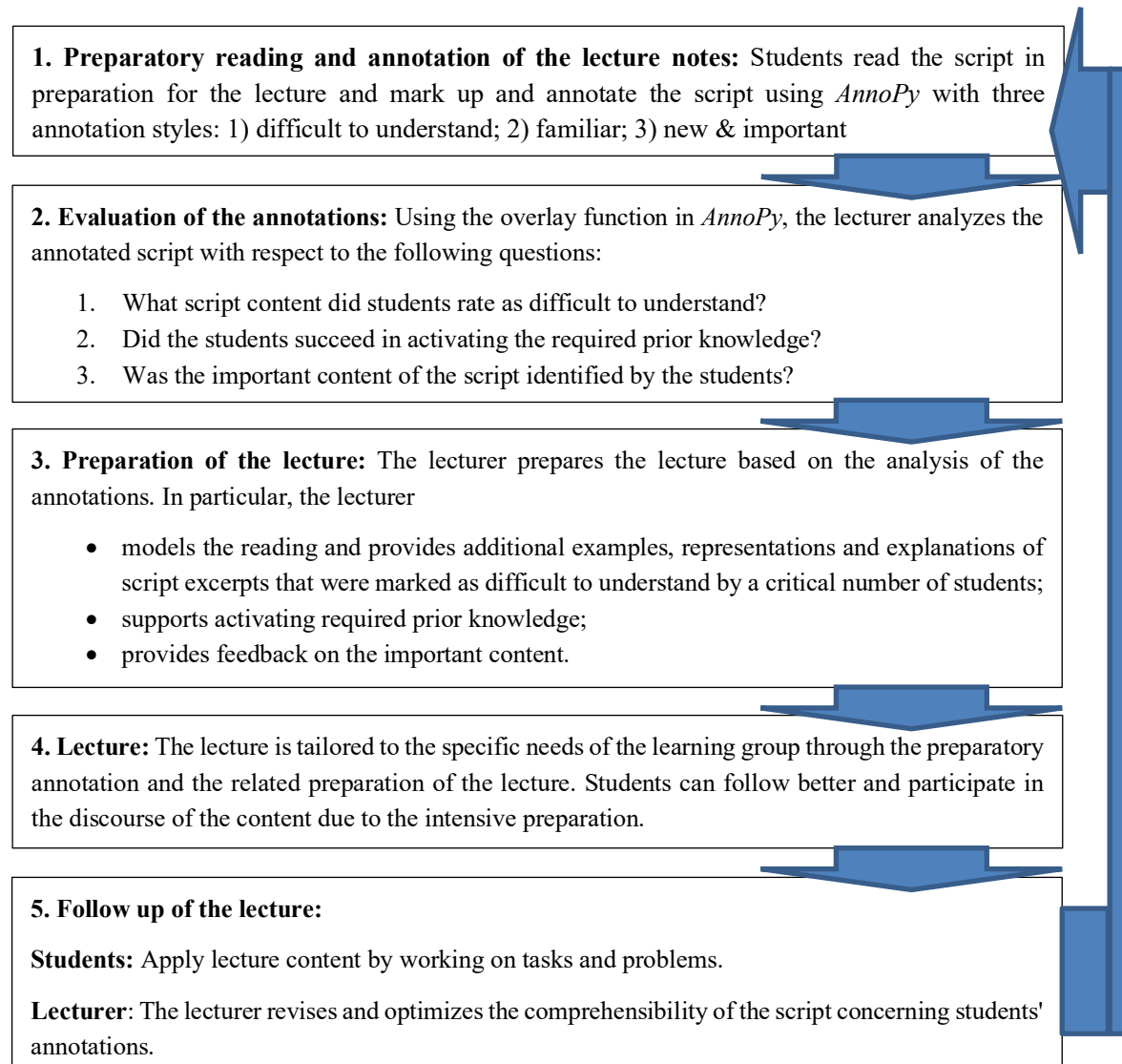


Figure 4: Learning scenario for fostering students' reading of mathematics within a lecture

Results from the evaluation study

Figures 5, 6, and 7 present the results of selected questions from the questionnaire. As the Lickert scale contained six values and there was no middle value, evaluations of 3, 4, and 5 on the Lickert scale are rated as agreement.

The results show that about 75% of the students thought that the whole learning scenario supported their learning and helped them understand the contents. About 90% of the participants thought that discussing difficult-to-understand text passages was especially helpful.

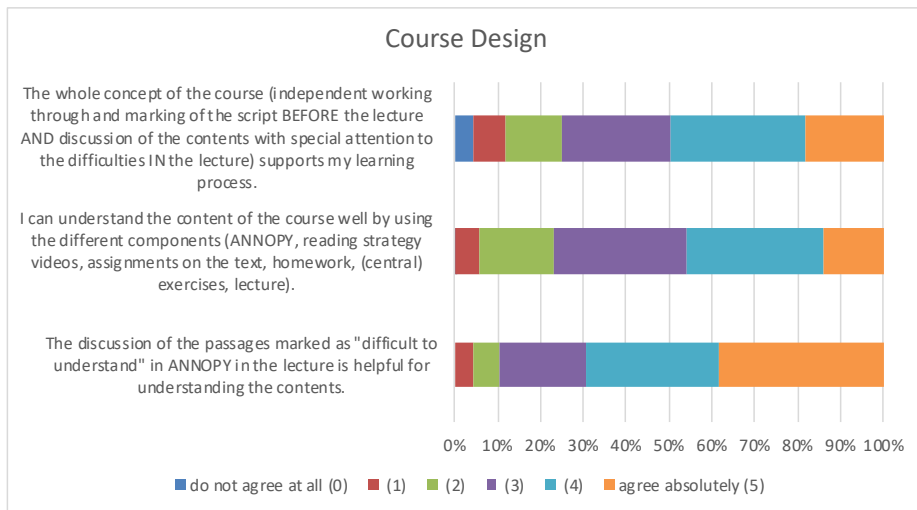


Figure 5: Results from the questionnaire about the course design (n = 284)

Regarding usability and the perceived benefits of using *AnnoPy*, about 65% of the students tended to think that using *AnnoPy* was helpful and supported them in understanding the script's content (Figure 6, items 1 & 2). 80 percent of the students thought using *AnnoPy* was intuitive (Figure 6, item 3).

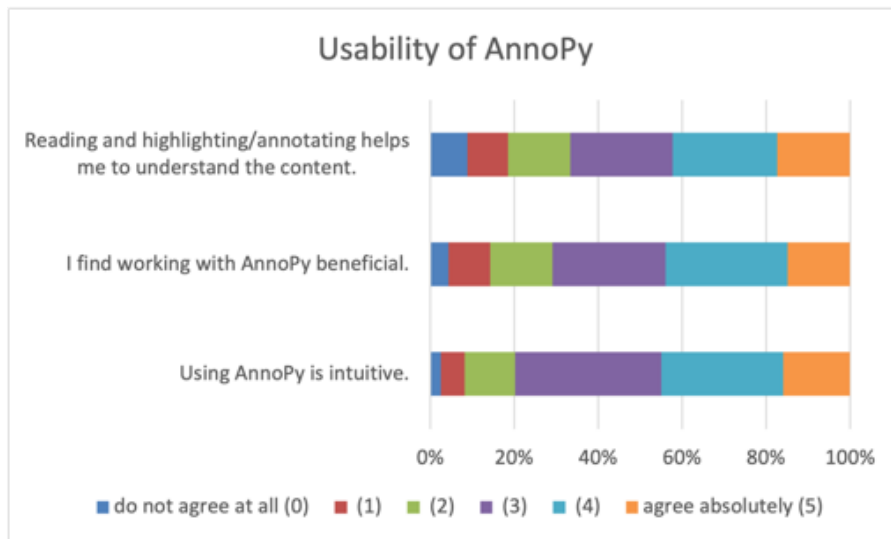


Figure 6: Results from the questionnaire about the usability of AnnoPy (n = 284)

Slightly more than half of the participants agreed that the script's contents were better understandable near the end of the course than at the beginning (Figure 7, item 1). This indicates that their reading abilities are likely to have improved, as the script's writing style has not changed near the end, and the content is even more challenging. Regarding students' reading strategies, the results show that between 65% and 85% of the students applied reading strategies found with expert readers in mathematics (Figure 7, items 3 & 4).

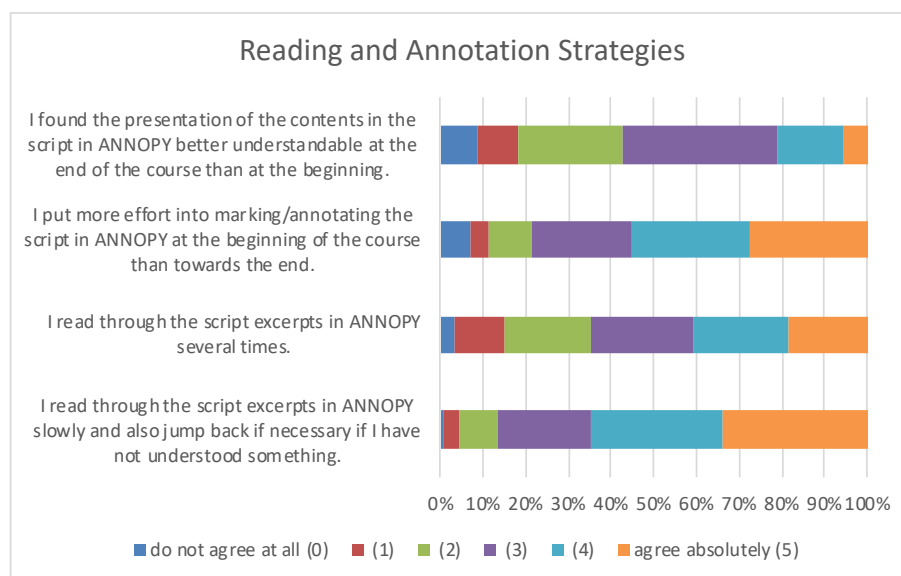


Figure 7: Results from the questionnaire about self-reported strategies when using AnnoPy

Discussion

AnnoPy supports teachers in collecting data on students' interaction with text. This data can be used to tailor specific learning opportunities to support students' understanding of the text or foster text-based discourse. As in the case presented here, required prior knowledge is clarified, and additional explanations are provided for text passages indicated as difficult by a relevant number of students. In the short term, this resolves discrepancies between the reader's actual knowledge and the knowledge required to understand the text, which is an important influential factor in reading mathematical text with understanding. In the long term, the idea is that readers develop more experience in reading mathematical text. The clarifications and explanations provided are supposed to help them get more familiar with the specificities of how language is used in mathematics. In turn, this is likely to influence their sense of power and authority concerning mathematical text and their values related to mathematics. A first evaluation of the implementation of the learning scenario gives hope to the assumption that these aspects and, thus students' learning of mathematics from reading are indeed fostered. Especially the discussion of difficult-to-understand text passages during the lecture was evaluated as helpful. More than half of the students confirmed that they apply reading strategies shown by expert readers of mathematical text. This is likely, as one central aim of the learning scenario is to provide students with many opportunities to read mathematical text and the guiding questions that are linked to the three annotation styles (What is familiar? What is important? What is difficult to understand?) invite students to mimic basic strategies of expert readers. A particular motivating feature of the learning scenario could be that students are supported to overcome their difficulties with the understanding of the text passage. However, it is unclear whether the application of expert readers' reading strategies can be in fact traced back to the use of *AnnoPy*. Further studies are necessary to investigate the effect of the use of *AnnoPy* on students' reading strategies and understanding of reading mathematical text.

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The use of the TPACK framework on research about teachers' knowledge to teach with digital technology

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The potential of digital technologies for teaching and learning mathematics is widely recognized and teachers' knowledge is one of the elements impacting their integration. Several authors have intended to characterize the teachers' knowledge required and developed several models, being TPACK one of these models. In this study, we seek to conduct a systematic review of the research on the integration of digital technologies by mathematics teachers based on the TPACK model. Specifically, we intend to answer the following research questions: (1) What are the main methodological options adopted? (2) How is the framework operationalized/used in the studies? The review was based on a search in the Scopus database and resulted in the identification of 10 relevant documents. The analysis suggests a prevalence of qualitative approaches, but a strong use of questionnaires; and an integration of the model with other frameworks, namely the developmental model of TPACK.

Keywords: Teachers' knowledge, TPACK, systematic review.

The research on teachers' knowledge

Digital technologies have long been recognized by their potential to promote mathematical learning (Tabach & Trgalová, 2019). However, achieving that potential has proved to be quite a challenge, with several authors recognizing the central role played by the teachers (Clark-Wilson et al., 2020).

Technology integration into teachers' practice is a complex process, affected by several elements, and one of these is the teachers' knowledge (Rocha, 2020). Characterizing teachers' knowledge and understand how to promote its development has been the focus of many studies and of many projects and similar initiatives, as discussed by Faggiano et al. (2021). However, this has not been an easy task. Several questions were raised. There was a need to understand what a teacher needs to know to be able to use the potential of technology to promote students learning. This includes understanding the teachers' knowledge, but also understand how to promote its development. As discussed by the authors, the first attempts to promote the development of teachers' knowledge were based on technical knowledge of the technology (i.e., knowing how to operate the technology) and failed. This analysis led to the need of considering more than just offering teachers access to technology and some technical support.

Even before digital technology, several authors have intended to characterize the teachers' knowledge and developed several models. Starting from the inspiring work of Shulman (1987) and his PCK – Pedagogical Content Knowledge construct, and going through all the authors who used it and developed clarifications and extensions (Sevinc, 2023), until the very well-known refinement of the PCK proposed by Ball et al. (2008), we have come a long way that helps us to better understand what teachers need to know to teach.

As discussed before, the integration of technology into teachers practice has raised new challenges. It was clear that teachers need some knowledge about technology, but very soon it was clear that

some additional knowledge would be needed. Inspired by the work of Shulman and his PCK construct, some more specific models were developed. One of these models was TPACK. Developed by Mishra and Koehler (2006), this is probably the most well-known model with a focus on technology integration.

The TPACK model considers the Content and Pedagogy knowledge domains, assuming PCK as the intersection between these. To these domains the authors add the one on Technology, considering then the intersections among all the domains: Content and Technology, Pedagogy and Technology, and the intersection among all the domains – the TPACK. The Context is also considered, although in different ways over time, only recently being assumed as a knowledge domain (Mishra, 2019).

However, the research based on knowledge models has been criticized because of the way it uses these models, often with a greater focus on distinguishing between different domains of knowledge than on their operationalization (Ruthven, 2011). Another point, addressed by Tabach and Trgalová (2019), has to do with the focus of the study. According to the authors, most of the studies are content-driven or tool-driven. Those who are content-driven focus on the knowledge and skills needed to teach a specific content using technology (e.g., functions). Those who are tool-driven focus on the knowledge and skills needed to use a specific technology to teach mathematics (e.g., graphing calculator). A more global view would be important to develop a deeper understanding of technology integration.

The research approach based on knowledge models has also been criticized by its focus on a static approach (Tabach, 2011), not valuing the dynamic character of the teachers' professional development and, specifically, of their knowledge. Some of the studies tend to address the teachers' knowledge disconnected from their practice, without taking into account the difference that may exist between a task (what it presupposes) and its implementation (Rocha, 2022).

These critics have been considered and resulted in proposals of refinements, new models of teachers' knowledge, or the use of the knowledge models combined with other frameworks to somehow overcome the weakness identified in the model. However, this development of the field calls to deeper analysis and reflection.

Goals and research questions

As a knowledge model, the TPACK model has been the target of several criticisms, as referred before. However, the model continues to be used by many authors in their studies. As so, it will be important to understand how the model is being used in the field of mathematics education.

In the study presented here, we seek for an overview about how the TPACK framework is being used in research about digital technology. As so, our focus is on the methodological options and on the use of the model. Specifically, we intend to answer the following research questions: (1) What are the main methodological options adopted? (2) How is the framework operationalized/used in the studies?

Methods

This work is part of an ongoing study with focus on teachers' knowledge and the different frameworks about it. In this part of the study, we focus on the TPACK model. The work presented here was developed according to the guidelines provided by PRISMA (Page et al., 2021). We assume Scopus as the database for our search, due to its international recognition and representativity. The search for

documents was planned for a period of ten years (2014-2023), and done in the title, abstract and paper keywords. We started using the keywords: technology, TPACK. This resulted in 1592 papers. Due to our focus, we added another keyword: teachers' knowledge. This resulted in 187 documents. However, several of them are not related to mathematics, as so we added mathematics to our keywords. As a result, we got 34 papers. We then limited our papers to those published in journals and written in English. The result was 19 papers. Finally, we added the time frame, resulting in 12 papers. Having 12 papers published in the last ten years, out of a total of 19 papers, suggests this is a good time frame to consider. After a full reading, two of the papers were excluded because they were more an analysis of one course than a research study using the TPACK model. This resulted in a total of 10 articles analyzed in this review (marked with * in the references).

The analysis and interpretation of the documents was then guided by a set of categories defined based in the research questions. The information from the full read of the articles was entered in a spreadsheet organized according to: the authors' name, date of publication, journal name, goal/research questions, framework used and methodological options. The focus of this review is on the methodological options. As so, attention was given to the type of methodology (qualitative, quantitative, or mixed), to the number of participants and to their type (pre-service or in-service teachers), instruments used to collect data, and to the characteristics of data analysis.

Results

From the 10 articles analyzed, in what concerns to the date of publication, two were published in 2023, two in 2022, two in 2020, two in 2019, one in 2018 and one in 2014 (not considered in this study due to the time frame defined: four in 2013, one in 2012, and two in 2010). A fact suggesting the research using the TPACK model has received attention in the last 25 years, with a very slight increase in the interest in the more recent years.

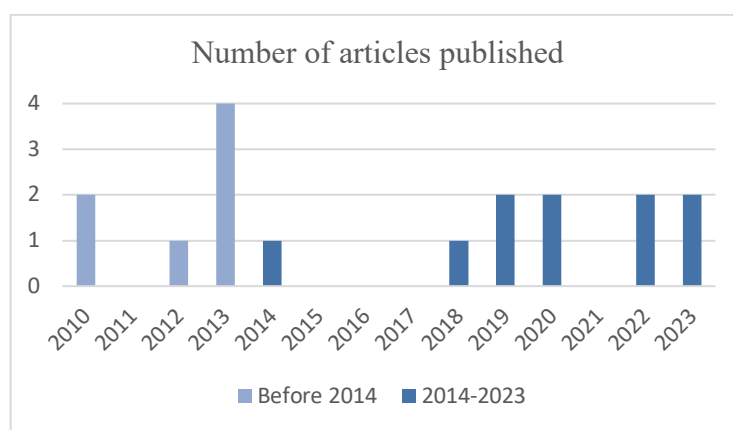


Figure 1: Number of studies with a focus on Mathematics using TPACK model

In terms of the type of methodology adopted, most of the studies adopted a qualitative methodology, three were based on questionnaires and quantitative methodologies, and one adopted a mixed methodology, with a quantitative part based on a questionnaire applied to 33 pre-service teachers, and then a qualitative part, involving 6 of these participants (Table 1). In both cases, qualitative and quantitative studies, the participants are pre-service or in-service teachers, with one case of a qualitative study with teachers' educators and one case of a quantitative study with pre-service special education teachers.

Table 1: Type of methodology and type of participant in each study

	Quantitative	Mixed	Qualitative
In-service teacher	2	-	4
Pre-service teacher	1	1	2
Total	3	1	6

The number of participants in the study varies significantly, with lower numbers in qualitative studies (Table 2). Even so, some of the qualitative studies have a high number of participants, even when the methodology adopted is the case study. That is the case of one study involving 52 in-service teachers.

Table 2: Type of methodology and number of participants in each study

	Quantitative	Mixed	Qualitative
In-service teacher	86, 202	-	3, 13, 16, 52
Pre-service teacher	116	33 (quant.) – 6 (qualit.)	10, 21

For quantitative studies, the questionnaire is the instrument for data collection. For the qualitative studies the instruments are more diversified (Table 3). Although the questionnaire is also used (in one study), interviews and observations are dominant, with references also to other instruments, such as document gathering and field notes. Two of these studies are lesson studies, and two others, although not being present as lesson studies, are based on working sessions to prepare lessons and in the analysis of the teachers' lesson plans.

Table 3: Type of data collection in each study

		Quantitative	Mixed	Qualitative	Total
Questionnaire		3		1	4
Observation, interviews, document gathering	Simple			2	2
	Lesson study		1	1	2
	Working sessions, lesson plans			2	2
Total		3	1	6	10

The data analysis is based in statistics in the case of quantitative studies, in some cases using the software SPSS. In the case of qualitative studies, the analysis is often based on codification, using categories usually related to the theoretical framework.

Only one study organizes the data collection based on the knowledge domains of TPACK. Almost half of the studies (four) combine the TPACK model with the development model by Niess et al. (2009) and Niess and Gillow-Wiles (2017) and base the data analysis on the five levels of development – recognizing, accepting, adapting, exploring, and advancing. Other studies combine the TPACK model with other framework, such as the instrumental approach (two studies: one instrumental genesis and one documental genesis), use it to define a model with focus on Mathematics (one study), use it only in part of the analysis (one study) or do not present evidence about how the model integrate the data analysis (one study).

Most of the studies do not clarify how they assess the teachers' knowledge. In the cases where a questionnaire is used, usually only the structure of the questionnaire is presented, not always discussing the relation of this structure to the TPACK framework. In the case of qualitative studies, some evidence from the data is often presented but, once again, the connection to the model is not always presented.

The studies analyzed intend to assess the teachers' knowledge to integrate technology and, in some cases, to develop that knowledge. One of the studies intends to better understand the TPACK model, analyzing and discussing the relationship between its knowledge domains.

Conclusion

This systematic review intended to achieve a better understanding of the research based on the TPACK knowledge model, trying to get an overview about how the model is being used and the methodological options adopted.

Methodological options assumed

The methodologies adopted seem to be diversified. The options tend to be related to the qualitative or quantitative methodology of the study, with a focus on the use of questionnaires, in the first case, and on interpretative approaches, in the second case. Interpretative approaches tend to be based in observation, interviews and/or document gathering, however, there are also cases where questionnaires are used. The questionnaires, when used, may be aimed at collecting the participants perceptions or measuring their knowledge. The interpretative approaches can be slightly more diversified, but tend to aim the characterization of the teachers' knowledge or the analysis of its development.

The use of the TPACK model

The TPACK model is the most used framework in studies related to the mathematics teachers' knowledge to integrate digital technology. The model is widely used in the research field, however it seems it tends to be used combined with some other(s) framework(s). In the studies analyzed, only one focus directly on the model and its knowledge domains. All the other studies combine it with some additional framework. The development model for TPACK by Niess et al. (2009) and Niess and Gillow-Wiles (2017) and its five levels – recognizing, accepting, adapting, exploring, and advancing – is broadly used. Some other studies try to combine it with the instrumental approach. Some studies combine different frameworks and try to create a theoretical background to support the analysis. In some cases, it becomes difficult to recognize the contribution of the TPACK model. In one of the studies, the TPACK model is used only in part of the study. And in other case, the model

is used in a general way, as an idea of what the teachers' need to know to integrate the technology into their practices, and not with focus on the different knowledge domains of the model.

One significative difference between the studies analyzed is related to the methodology and the instruments used. In some cases, only the teachers' perceptions about their professional knowledge are collected (mainly in studies using questionnaires, but also in data collections based on interviews). In cases, such as in lesson studies, where the teachers practice is actually considered, it is possible to gather information about the teachers' knowledge and not only their perceptions about it.

Final comments

This is a very preliminary systematic review, focusing on the use of the TPACK model to investigate about teachers' integration of technologies and based on a search in Scopus database. It would be important to reflect on the search options and to enlarge the review considering other databases such as the Web of Science. The preliminary results give us information about the methodological options adopted and about how the TPACK model is being used. It would be important to deepen this work to better understand the usefulness of the model and the way how it is combined with other frameworks.

Acknowledgment

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Students' views on using digital technologies in mathematics lessons

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In Austria, all lower secondary students are equipped with digital devices. To optimise the use of these devices, teachers need insight into students' views on using them. This need is why our research aims to explore students' views on using digital devices. To this end, we interviewed 18 lower secondary students concerning using technology in mathematics. The data obtained was analysed using grounded theory and qualitative content analysis. The following characteristics of using digital devices are relevant for students of our study: (1) Emotional disposition towards the digital device, (2) Organisational framework and how to deal with it, (3) Expanded possibilities but also limitations of learning opportunities and (4) Learning mathematics in a varied teaching setting.

Keywords: Digital technologies, technology-based learning, students' needs, learning maths.

Introduction

Teenagers in Upper Austria spend around 130 minutes daily online (Education Group GmbH, 2023). The advancing digitalisation of society in general also leads to changes impacting the education system. In response to the new challenges, the Ministry of Education, Science and Research (BMBWF) has developed and implemented an 8-point plan to digitise schools (BMBWF, 2024). In line with this plan, all lower secondary students will be equipped with digital devices by 2024 (Weinhandl et al., 2021). The devices available for selection are notebooks and tablets (OeAD, 2023).

Studies have shown that students who work with digital devices in mathematics lessons perform better (Hillmayr et al., 2017; Reinhold et al., 2018), but in other studies, ambiguous results were found (Drijvers et al., 2016). However, the mere presence of digital devices does not guarantee a positive effect on performance, which is why the implementation of a successful use of technology is a challenge for teachers (Schmidt-Thieme & Weigand, 2015). To optimise the integration of digital devices into classrooms so that they contribute to better learning performance, teachers need knowledge about students' views on using such devices. As part of this study, students at the beginning of the lower secondary level were asked about using technologies in mathematics classes. The specific research question was:

What characteristics of using technologies for learning mathematics are relevant to students at the beginning of the lower secondary level?

Activity Theory as Theoretical Background

The three central elements of the research question are technology, mathematics, and learners. As these play a key role in activity theory, this theory was chosen as the theoretical framework for the study in line with our research question. Activity theory (Vygotsky, 1978; Engeström, 1987) understands learning as a social and intellectual engagement with other people and cultural tools. Thus, in line with activity theory, interactions between humans and computers are also social phenomena (Russell, 2013). Engeström (1987) proposed six components that contribute to human activity and are all interrelated. These are subject, object, mediating tools, community, rules, division of labour and outcome (see Figure 1).

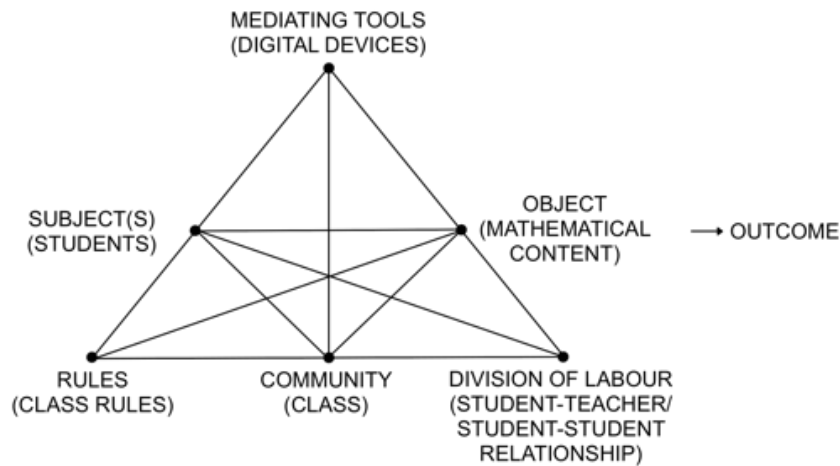


Figure 1: Activity Theory and its relations (adapted from Engeström, 1987, in Russell, 2013)

If these components are interpreted in the context of our study, the community is to be understood as a class, including teachers. The division of labour refers to student-teacher relationships and relationships between students. The digital devices represent the mediating tools. Students are the subjects, and the mathematical content is the object. The rules can be understood as class rules, including those for using digital devices. The outcome describes what students learn from technology-enhanced lessons in terms of mathematical content and using digital devices. The upper part of Engeström's (1987) triangle (Figure 1) only includes the relationships between subject, object, and mediating tools, which can be traced back to Vygotsky (1978). Since our research focuses on these three points, we will limit our discussion of the results to Vygotsky's (1978) triangle.

Literature Review

Using digital tools in a way that extends learners' understanding and improves learning outcomes is a significant challenge for mathematics teachers, as the mere presence of digital technologies is not sufficient to achieve such goals (Schmidt-Thieme & Weigand, 2015). According to Roth (2019), digital learning environments enable students to engage with mathematical content independently and at their pace. However, roles of teachers continue to be essential factors in integrating technologies into mathematics lessons, as they organise learning by linking the technological environment with mathematical activities and summarising the results of technology-based activities (Drijvers, 2015).

Studies show that students who work with digital technologies in mathematics lessons achieve better results in performance tests (Hillmayr et al., 2017; Reinhold et al., 2018). Hillmayr et al. (2017) found a positive effect on students' learning if they receive additional support from classmates and teachers when using technology. The positive effect is comparatively small if students work without guidance. The positive effects of using technology are evident when differentiated feedback is given on the students' solutions, learning programmes are adaptive, and individual adaptation to the learning level is possible (Hillmayr et al., 2017). According to Reinhold et al. (2018), the solution rate of subsequent tasks increases if a solution aid is previously called up. Furthermore, using technologies in mathematics lessons increases motivation and willingness to work hard (Hillmayr et al., 2017; Reinhold et al., 2018). Overall, using technologies can change the structure of mathematics lessons, as students work more independently, which can positively influence the classroom atmosphere. Thus, using technologies makes it possible to open traditional teaching situations and implement

alternative forms of instruction (Reinhold et al., 2018). According to Schmidt-Thieme and Weigand (2015), another key advantage of digital technologies in mathematics lessons is the ability to present mathematical content in a dynamically changeable way. According to Schmidt-Thieme and Weigand (2015), networking between traditional and digital teaching media is crucial. The positive effect on students' learning success is more significant when analogue material is used in addition to digital devices (Hillmayr et al., 2017). Therefore, mathematics teaching should create a link between traditional and digital media (Schmidt-Thieme & Weigand, 2015). Overall, using technologies in mathematics lessons can be a meaningful and beneficial enhancement to paper-based learning environments (Reinhold et al., 2018). However, students must also learn how to use the devices (Babnik et al., 2013).

Overall, while digital technologies have the potential to enhance the teaching and learning of mathematics, their effective integration requires thoughtful organization by teachers, the provision of guidance and feedback, and a balanced use of traditional and digital media.

Methods

To answer our research question, we chose a phenomenological approach according to Cohen et al. (2007). We conducted semi-structured interviews, and the data obtained from the interviews was analysed and summarised using qualitative content analysis (QCA) and grounded theory.

Methodological background

In the sense of grounded theory, the results of our study are grounded in the data that were collected to answer the research question (Glaser & Strauss, 2017). When creating the interview guide, care was taken to ensure it was structured to the necessary extent (Loosen, 2016), but it was still as open as possible. Once the data had been collected, it was analysed in the form of interview transcripts in line with the specifications of QCA (Mayring, 2022) concerning our research question. According to Mayring (2022), categories were inductively derived based on the available data material, which were assigned to individual text passages. The aim was to reduce the data material while retaining the essential content (Mayring, 2022). This process resulted in a hierarchical category system with a division into main and subcategories, according to Kuckartz (2018). The interview material was analysed and coded in three steps by two researchers.

Data collection and analysis

The interview data was collected in three Austrian schools between the end of March and mid-April 2023. Interviews were conducted at one rural middle school (MS) and two urban lower secondary schools (AHS). A total of 18 students were interviewed, twelve of whom were in grade 5 and six in grade 6. The students surveyed use their digital devices in mathematics lessons to a greater or lesser extent and according to their teacher's initiation. For example, they work on digital exercises, watch learning videos or take part in online quizzes. The interviews were conducted in German and during school hours. An overview of the interviews conducted is provided in Table 1 below. In addition to the interview date, the type of school and the region, the gender of the students interviewed and the digital device with which the students were equipped as part of the BMBWF's device initiative can also be seen. Each school was also assigned a school identifier to simplify subsequent references.

Table 1: Overview of the interviews conducted

Identifier	Date	School type	Region	Device	Male (m)	Female (w)	Total
S1	24.03.2023	AHS	Urban	Chromebook	2	5	7
S2	28.03.2023	MS	Rural	iPadOS Tablet	4	2	6
S3	19.04.2023	AHS	Urban	iPadOS Tablet	4	1	5

The data obtained was transcribed and then, in the first step, coded independently by two authors of this paper in line with the summarising QCA according to Mayring (2022) and content-structuring QCA according to Kuckartz (2018). These two independent code sets containing inductively formed codes were compared in a second step. Codes with similar descriptions were grouped, and subcategories were formed to bring the data to a higher level of abstraction. Table 2 shows an exemplary code from the codebook and its description, supported by a prototypical example from the transcribed interview data.

Table 2: Exemplary code including description and prototypical example

Code	Description	Prototypical example
Differentiation learning pace	Thanks to the digital work plans the students can learn at their own learning pace.	We can work at our own pace. So we don't have to, if we're not as fast as the others, we don't have to rush [be fast and therefore careless]. Instead, we can work at our own pace.

In the final step, sub-themes were grouped and assigned to four main categories. To ensure intercoder reliability between the two researchers concerning assigning codes to subcategories and subcategories to main categories, corresponding codes and subcategories were reassigned independently by both researchers. From these assignments, Cohen's Kappa coefficient of agreement was calculated with a value of .797 for the assignment of the individual codes to the subcategories and a value of .62 for the assignment of the subcategories to the main categories. According to Landis and Koch (1977), the calculated values indicate substantial agreement in both cases, which means that it can be assumed that the results of the research conducted during this study can be regarded as reliable.

Results

Based on our analysis, the following four categories were identified: (1) Emotional disposition towards the digital device (as an end in itself), (2) Organisational framework and how to deal with it, (3) Expanded possibilities but also limitations of learning opportunities and (4) Learning mathematics in a varied teaching setting. For each category, the number of students (with the additional distinction between female or male) who made a statement during the interviews that can be assigned to the corresponding category is listed below. This allows the importance of each category to be assessed. Underpinned by quotes from the interviews conducted, the results are explained in more detail below

and organised into subcategories where appropriate. The quotes can be assigned to the schools in line with their school identifier using Table 1.

Emotional disposition towards the digital device (as an end in itself) | 16 out of 18 (7 f, 9 m)

The data show that most students surveyed enjoy and have fun working digitally, as they generally like working with digital devices. So it does not matter what content or what type of activity is being processed on the digital device.

S1: So it is, it's maths, but it's on the Chromebook, and it's just fun.

In contrast, a few of the students interviewed said that they did not like working with digital devices in class due to the digital device itself.

Organisational framework conditions and how to deal with them | 17 out of 18 (7 f, 10 m)

The organisational framework conditions and how they are handled in the classroom play a decisive role in technology-enhanced mathematics lessons from the student's perspective. Three key characteristics were identified: familiarity, independent work with mutual support and conservation of resources. Some of the students interviewed like to work in mathematics lessons in the way they are used to. According to the students, regular use of digital devices in class is essential to find their way around working with digital devices and the applications installed on them.

S3: But with the first [digital] work plan, I don't think I was quite as quick because I had to get used to it [working with the digital device], but I think that was the same for others. And then it actually went pretty well.

The data also shows that students like to work independently in technology-enhanced settings because it allows them to concentrate better and work at their own pace. However, most students surveyed like to work with classmates when necessary to support each other and explain content. In the event of organisational or technical difficulties, students also appreciate working together as a whole class. Using the digital device, the students surveyed perceive the conservation of resources, which concerns writing materials such as pens and paper on the one hand and personal resources such as writing by hand on the other. Organisational hurdles identified by the students surveyed are unstable internet connections and the fact that the digital device must always be charged.

Expanded possibilities but also restrictions on learning opportunities | 13 out of 18 (7 f, 6 m)

Our research also shows that using technology expands possibilities and restrictions in learning opportunities. In this regard, the four main characteristics are expanding reality through additional functions, control options for independent work, differentiation and content restrictions. An extension of the analogue reality results from the possibility of writing on the digital device, whereby pages can be enlarged as desired, pens can take on any colour and shape, parts of text can be moved and deleted, and mistakes can be undone with a click. In addition, tasks and activities with interactive and dynamic elements can be completed on digital devices, which increases the variety of tasks. According to the students, such tasks are fun and support them in learning and understanding mathematics content. Supplementary digital exercise materials can also increase the number of available tasks. Regarding the independent development of learning content, many digital tasks offer various opportunities for self-monitoring. For example, students receive feedback on the correctness of their entries and can

thus learn from their mistakes. Sometimes, they also get an overview of the number of tasks they have already solved correctly or incorrectly.

S1: And it [exercises on GeoGebra] shows how many you got right and how many wrong.

In line with the data obtained, independent work with a digital device also facilitates differentiation in learning speed, additional tasks and individual assistance that can be used as required.

S3: We can work at our own pace. So we don't have to, if we're not as fast as the others, we don't have to rush [be fast and therefore careless]. Instead, we can work at our own pace.

In addition to the described extended possibilities that using technology in mathematics lessons offers, students see limitations to learning opportunities. For example, some mathematical content, such as constructing with compasses and rulers, cannot be developed with the help of technology.

Learning mathematics in a varied teaching setting | 10 out of 18 (5 f, 5 m)

Students' desire for a varied teaching setting in mathematics lessons can be identified. Accordingly, the students surveyed also prefer a mixture of digital and analogue work phases in class.

S1: Well, it would be stupid if we only did something on the Chromebook now, but also kind of boring if you had to write everything by hand.

The students surveyed appreciate teaching phases in plenary sessions in which they are in contact with the teacher, as explanations from the teacher are generally considered very beneficial for understanding. The students surveyed particularly emphasised using technology by the teachers themselves. For example, the students like a tablet when it is used as a replacement for the chalkboard, as pictures and graphics can be shown to promote understanding of mathematical content.

Discussion

Finally, the central results of our analysis are linked to the theoretical background and literature review. Firstly, the results found can be structured within Vygotsky's activity theory (1978). The emotional disposition towards the digital device can be assigned to the vertex subject, whereby there is also a relationship to the mediating tool. The results concerning the organisation and the extended possibilities and restrictions can each be attributed to the vertex mediating tool. Particularly concerning the extended possibilities and restrictions of learning opportunities through the digital device, a connection can also be made to the object (i.e., in our study's context to the mathematical content). The result of the variety students consider desirable in mathematics lessons can be assigned to the vertex subject. Secondly, the results of our study show that students value the regular use of digital devices in class to gain familiarity with the devices. This finding underlines the thesis of Babnik et al. (2013) that students should learn how to operate new (digital) devices. Thus, our findings support using technology regularly in everyday school to create familiarity with digital devices. Thirdly and as already shown by Hillmayr et al. (2017), our study also shows that students prefer to work with technology independently but with the support of classmates and the teacher. According to Hillmayr et al. (2017), working in pairs on a digital device and mutual support when using digital devices can positively influence learning success. Furthermore, like Hillmayr et al. (2017), our results show that support is essential when organisational or technical difficulties arise when working with a digital device. In addition to the increased variety of tasks on the digital device, which is also

emphasised by Reinhold et al. (2018), the students surveyed appreciate activities with interactive and dynamic elements. This finding aligns with Schmidt-Thieme and Weigand (2015), who see dynamic representations of mathematical content as an advantage of technology-enhanced mathematics lessons. The possibility of differentiation, which is relevant to the students surveyed in connection with using technology in mathematics lessons, underlines Roth's (2019) thesis that digital learning environments enable students to work through the learning content independently. Hillmayr et al. (2017) demonstrated a positive effect of adaptive learning environments with differentiated feedback on students' learning success. Consistent with this, the students surveyed stated that they consider individualised feedback in adaptive learning environments to promote understanding. In addition, our study shows that a varied teaching setting, a change between technology-enhanced and analogue teaching, is relevant for students. This finding is related to the findings of Hillmayr et al. (2017), according to which using technology in mathematics classes has a positive effect when analogue material is used to supplement lessons. The statement by Drijvers (2015) that teachers play an essential role in technology-enhanced lessons is also supported by the results obtained, which ultimately speak in favour of mixing analogue techniques with digital technology.

Conclusion, limitations and further research

The study shows that emotional disposition, organisational framework, extended possibilities but also restrictions and varied teaching settings are relevant characteristics of using technologies for students at the beginning of the lower secondary level. When considering the results of the study, however, it must be borne in mind that the students surveyed are at the beginning of the lower secondary level, i.e. they are in the 5th or 6th grade, and the study was conducted in Austria. To be able to generalise the results, further data collection should be carried out in other environments. For example, the study would have to be conducted with older students and in an environment that is not characterised by such a high socio-economic standard of living.

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Between potential and perplexity: the role of LLMs in mathematics education

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We explore the opinions and attitudes of pre-service mathematics teachers towards the use of text-generating Large Language Models (LLMs). Despite the growing interest in these models and their seemingly diverse applications in education, there are significant challenges. These include inherent weaknesses and widespread misperceptions. Our survey of 54 pre-service teachers (PSTs) specializing in mathematics shows ambivalent attitudes towards the use of LLMs in education. Utilizing a prototype LLM tutor developed both as an instrument for this study and to test its general feasibility in educational contexts, we discuss the limitations and possibilities of LLMs in mathematics education. We present preliminary indications that a brief exposure to text-generating Artificial Intelligence (AI) alone could reduce fears and misconceptions about generative AI. The article concludes by outlining the next steps of our research project.

Keywords: Generative artificial intelligence, large language models, pre-service teachers.

Introduction

LLMs, which underpin text-generating AI solutions such as ChatGPT, are expanding rapidly and finding applications in diverse fields, including education. While the potential of LLMs to transform educational practices is recognized, it is coupled with concerns about the reliability of their outputs (Baidoo-Anu & Ansah, 2023). LLM outputs are typically generated autoregressively using large numbers of parameters (weights), which are adjusted using large datasets and selected stochastically. The issue of their reliability is well-documented in artificial intelligence research. OpenAI (2023) transparently communicates the difficulties of increasing the reliability of GPT-4, their most advanced model today, highlighting the model's predilection for inaccuracies and the crucial need for careful supervision in its application.

The integration of generative AI into educational frameworks elicits mixed reactions, combining optimism with apprehension. Debates center on the potential impact on traditional teaching methods and the learning process. Even before the widespread adoption of LLMs, it has been suggested that AI could provide improved ways to plan, deliver and assess instruction by identifying student needs, providing immediate feedback and automating assessment (Celik et al., 2022). The potential and challenges of AI in academia are being examined and discussed extensively, with considerations ranging from published research articles in medicine (Liebrenz et al., 2023) to the entire field of education (Farrokhnia et al., 2023). This significant interest highlights the importance of developing teacher and learner competencies in generative AI and literacies to critically assess these technologies and their benefits. The integration of LLMs into educational settings demands a pedagogical strategy that prioritizes critical thinking and fact-checking to effectively utilize these models while addressing their limitations and biases (Kasneci et al., 2023). Despite recognizing the potential of ChatGPT as a

tool for assisting instructors, a recent literature review on the impact of ChatGPT in education (Lo, 2023) highlighted variability in the model's performance across different subjects. There are two distinct types of challenges associated with the use of LLMs: One type of challenge results in the quality of the responses, such as misinformation and hallucinations. Consequently, Lo (2023) hypothesized that there is a need for updated assessment methods and institutional policies in educational contexts. An approach emphasizing critical thinking and fact-checking could help learners navigate the challenges posed by potential biases and the fragile nature of these systems, ensuring that AI is advantageously used in education (Kasneci et al., 2023). A second type challenge results in personal beliefs regarding the capability of LLMs: The discussion about LLMs and their application is driven by common misconceptions, such as the belief that generative AI can produce entire papers instantly and is constantly improving itself. These exaggerated beliefs might heighten fears about AI that existed even before the widespread adoption of ChatGPT (Zhan et al., 2023), diverting attention from potential drawbacks due to unreliable and inconsistent outputs. LLMs are not reliably improving over time (Chen et al., 2023), nor do generative AI models operate in a manner analogous to human brains – a misunderstanding that has been debated within AI research itself (Emmert-Streib et al., 2020).

Our research aims to evaluate the potential and challenges of incorporating text-generating LLMs into mathematics education, focusing on their utility as personalized tutors and their impact on pre-service teachers' (PSTs) perceptions and concerns about LLMs and generative AI. We seek to understand whether direct exposure to generative AI can alter attitudes and reduce misconceptions about AI. Simultaneously, we aim to explore the broader applicability of LLM-based tutors in mathematics education. The literature indicates that while LLMs offer educators opportunities to enhance content and engagement, their integration requires a strategic approach to developing students' critical thinking and fact-checking skills when working with generative AI (Cooper, 2023; Kasneci et al., 2023). While our long-term objective is to create an open-source LLM specifically trained for educational purposes, our current efforts are focused on utilizing the OpenAI Application Programming Interface (API) for our investigations. This dual research method, involving the use of a prototype LLM tutor and examining PSTs' attitudes towards such a tutor in educational contexts, aims to align with the evolving role of AI in classrooms and to emphasize the necessity of a balanced approach that addresses AI ethics, media literacy, and anxiety.

Given the contrasting views on generative AI within this context, our research questions are as follows:

(RQ1) How do pre-service mathematics teachers perceive the feasibility and practicality of using an LLM-based tutor in mathematics education?

(RQ2) What are attitudes and misconceptions of prospective mathematics teachers towards LLMs in educational contexts, and how stable are these attitudes after short-term exposure to an LLM-based tutor?

The structure of this paper is outlined as follows: In the next section, we describe our methodology and research design. We then present and discuss our findings in relation to the research questions. We conclude by discussing the results and limitations of our study and outlining the planned next steps of our research.

Methodology

To address our research questions, we used a multifaceted exploratory approach with two primary facets: attitudinal research and experimentation with pre-service PSTs' interaction with a prototype LLM tutor developed specifically for this study. This approach aimed to explore the potential and challenges of LLMs in mathematics education, focusing on their didactical use and their impact on PSTs' perceptions and concerns about generative AI.

The LLM tutor was developed as a simple PHP web application, incorporating JavaScript for front-end interactions and curl calls to the OpenAI Chat Completions API. Our prototype tutor did not have access to mathematical algorithms to perform calculations as with the Wolfram plugin; it was limited to the model itself, specifically using the gpt-3.5-turbo-1106 model. This setup allowed for the adjustment of various parameters, such as the randomness of generated responses, to test the capabilities of the OpenAI API. Additionally, the OpenAI API calls provided the ability to customize the context of the prompts to align the output with educational goals. We explored these features with the intention of enabling our LLM tutor to respond in a way that is conducive to learning. One focus was the appropriateness of the depth of content, ensuring that explanations are didactically simplified and relevant to specific topics. The second focus was the manner of interaction, including the generation of clarifying questions and the provision of comprehensible explanations. The tutor acted as a simple chatbot: the students asked questions, the tutor called the OpenAI API with this input and then returned the generated output to the students.

To gain insight into the personal preferences and concerns of PSTs regarding the use of generative AI in education, we constructed a questionnaire. The survey was completed by 54 pre-service mathematics teachers at the University of Education Schwäbisch Gmünd, Germany. We designed the survey to capture a wide range of responses regarding PSTs' experiences, attitudes, and fears concerning LLMs such as ChatGPT in educational settings. The questions and answers can be found in Figure 3 in the subsequent section. In our study, participants first completed the questionnaire to capture their initial attitudes toward LLMs in education, followed by a brief interaction of about five minutes with the prototype LLM tutor. The post-interaction survey was designed to assess any changes in perceptions or potential reductions in AI-related anxiety among PSTs. For this purpose, the survey included the same questions that were used in the initial survey.

Building on previous research, our study incorporates findings from an initial survey published in German that focused on common misconceptions about the functionality of LLMs (Seibold et al., 2024). We concentrated on misconceptions about LLM functionality and explored how brief exposure to generative AI can mitigate unreasonable fears about generative AI (Friz et al., 2024). In addition, our efforts to align LLM integration with practical classroom applications relied on planned lessons for hands-on experimentation with LLMs for educational purposes (Kodweiß et al., 2024). By integrating these methods, our study aims to contribute to a comprehensive framework for investigating the effective use of LLMs in mathematics education, reducing misconceptions, and addressing the concerns of future educators in the field. With this methodological approach, we want to promote a more differentiated view of the possibilities for effective integration of generative AI in the classroom.

Results

Analysis of the survey data indicated that among the 54 PSTs, the use of LLMs (in the form of the widespread ChatGPT) varied, with a small number using it daily and about 30% never using it, as depicted in Figure 1. Although we did not explicitly ask about the specific use cases of ChatGPT, it is reasonable to assume that the students primarily used ChatGPT for tasks related to working on and with texts.

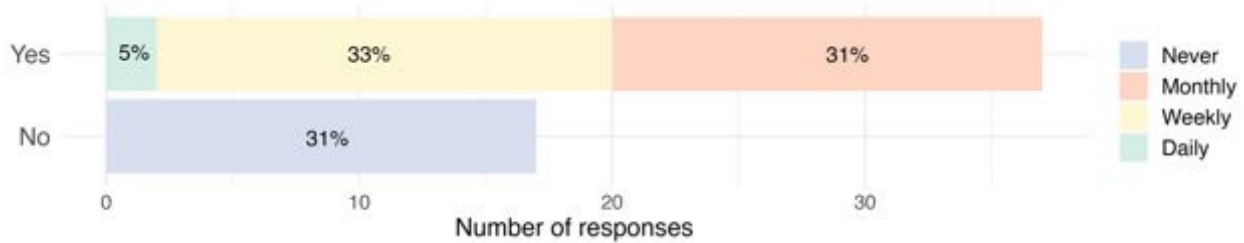


Figure 1: Use of LLMs (ChatGPT) by PSTs

Over 80% of the PSTs were concerned about the accuracy of LLM outputs, reflecting the difficulty of balancing accuracy and relevance in AI-generated responses. Other top concerns included privacy implications, reliance on technology, potential reduction in traditional teaching methods, and fears of AI replacing human educators, as shown in Figure 2.

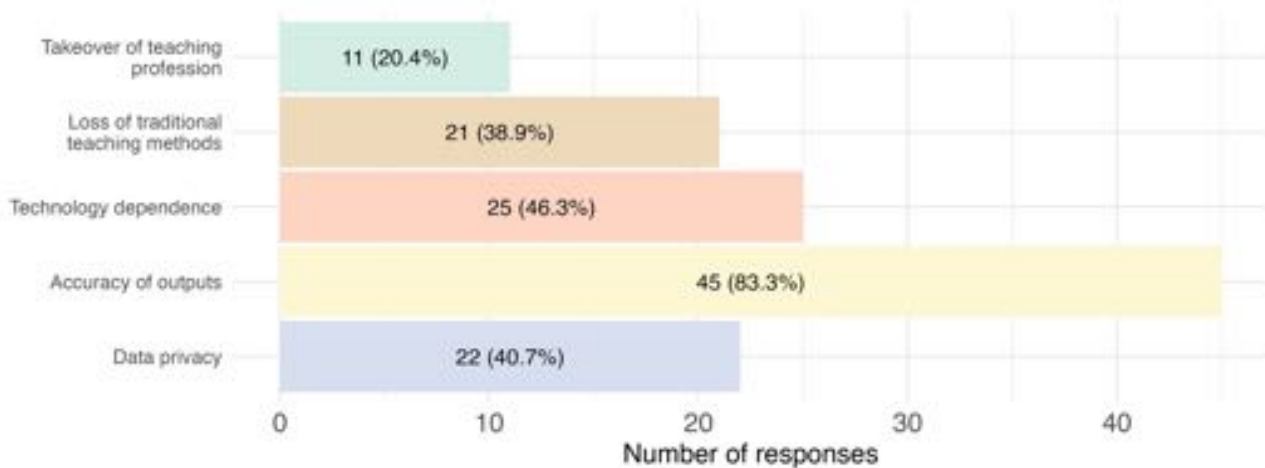


Figure 2: PST's concerns about implementing LLMs in education

The attitudes towards LLMs were ambivalent, with concerns about their long-term societal impact balanced by recognition of their transformative potential for learning and work, as depicted in the top part of Figure 3. There was strong support for monitoring the integration of LLMs into education, and the importance of educator training in introducing AI into the classroom was also emphasized. Mixed opinions were noted on the ability of LLMs to support personalized learning, as shown in the lower part of Figure 3.

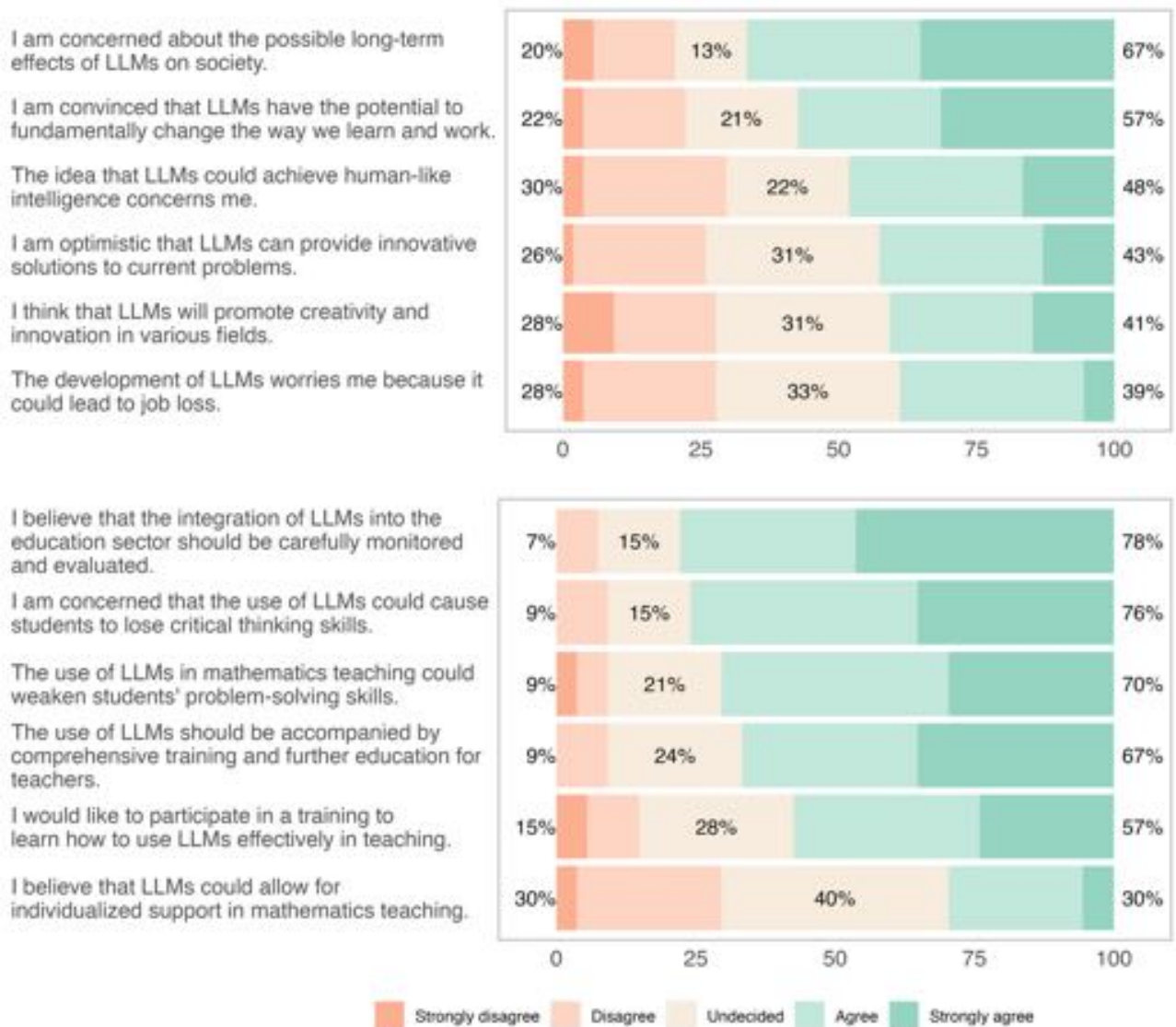


Figure 3: PST’s attitudes about LLMs’ societal impact (top) and their implementation in education

The response frequencies presented in Figure 3 are undoubtedly of interest. However, they provide no insight into the way the attitudes of individual PSTs may change after brief exposure to an LLM. Consequently, we have presented these changes individually in Figure 4, which illustrates the intra-individual variability in attitudes and opinions about LLMs pre- and post-exposure horizontally. The vertical scale represents Likert scales from strongly agree (5) to strongly disagree (1). This form of graphical representation was chosen to capture the high intra-individual variability and the significant changes in attitudes, which could not have been effectively shown using mean values or unidimensional, mean-based rates of change. This graphic reflects the heterogeneity and variability of the responses to the same questions after a short exposure.

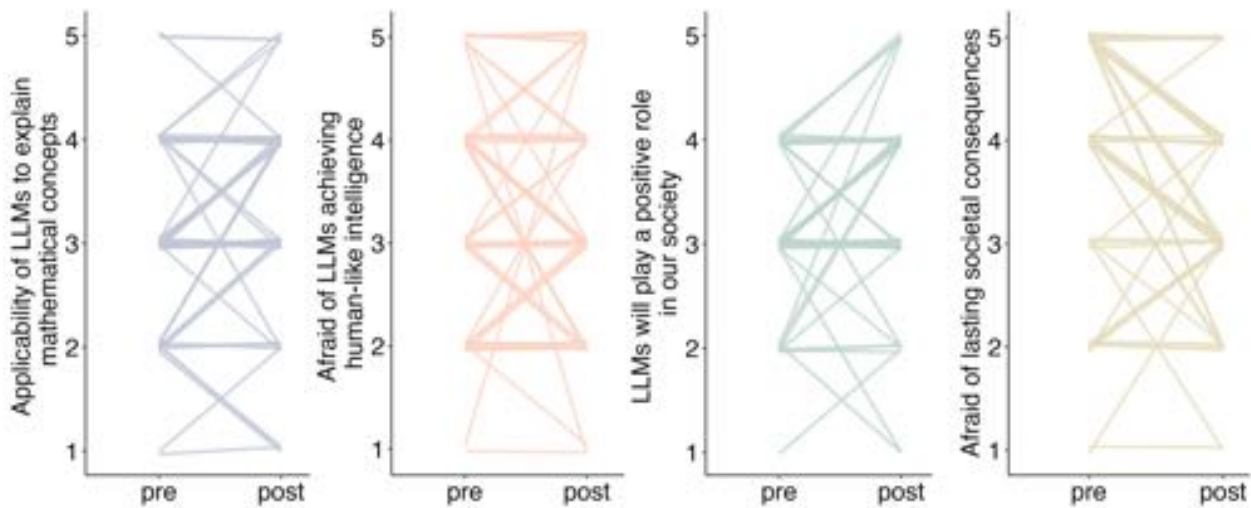


Figure 4: Intra-individual variability in attitudes and opinions about LLMs: pre- vs. post-exposure

Limitations & Discussion

The integration of Large Language Models (LLMs) into the classroom elicits mixed reactions from pre-service teachers (PSTs), balancing technological enthusiasm with pedagogical concerns. The attitudes of PSTs towards LLMs are crucial, as these perceptions will influence the successful adoption and integration of AI technologies in educational settings. Our study revealed that PSTs generally exhibit ambivalence towards LLMs, recognizing their transformative potential while expressing significant concerns about accuracy and reliability.

Our initial research indicates that brief interactions with LLMs can slightly alter attitudes, though this evidence remains indicative. The variability in responses suggests significant perceived uncertainties associated with generative AI. While the changes were notable, the reduction in anxiety was not universal or particularly striking. Moreover, the pre-post part of our study was preliminary, and various external factors might have influenced the changes in attitude.

Concerns about the accuracy of LLM outputs were prevalent, with over 80% of PSTs expressing doubts, reflecting the challenge of balancing accuracy and relevance in AI-generated responses. Our experiments with different settings and mathematical topics using GPT models (gpt-3.5-turbo-1106 and gpt-4-0613) from OpenAI did not yield a notable improvement in output reliability or quality. The prototype of the LLM tutor was limited to the model itself, without the possibility of solving mathematical problems algorithmically, as is possible with the Wolfram plugin, for example. It is important to note that it would be a mistake to assume that with such algorithmic plugins, the weaknesses could be easily fixed because input errors due to imprecise formulations or incorrect definitions can lead to incorrect results even with such plugins, as was recently shown in a study on the use of ChatGPT to teach quadratic functions (Katter & Huget, 2024).

Our findings highlight the importance of providing structured guidelines for interactions with LLMs: Hands-on experience with AI could help clarify their operational realities and promote informed applications in educational contexts. Educators should be able to guide students in critically and effectively using these technologies, which means that they themselves should be experienced. The feasibility of using an LLM-based tutor in mathematics education shows promise but requires careful

consideration of its limitations and potential impacts. Future research should focus on longer-term studies in diverse educational settings to better understand AI integration dynamics in education. By fostering a balanced view that acknowledges both the benefits and limitations of LLMs, we can ensure that generative AI enhances rather than disrupts the educational experience.

Outlook

We aim to transition from theoretical exploration to practical application in real classroom settings. One of our primary goals is to develop comprehensive lesson plans that integrate generative AI tools like LLMs into the mathematics curriculum. These lesson plans will be designed to provide both teachers and students with structured opportunities to interact with AI technologies in meaningful ways. We plan to conduct studies in diverse educational environments to better understand the practical implications of using LLM-based tutors in mathematics education. Continuous feedback from educators and students will be crucial in refining and optimizing the AI integration process. By collecting qualitative feedback from teachers and quantitative data from students, we hope to gain deeper insights into how generative AI can effectively enhance teaching and learning experiences.

We will address the concerns raised by pre-service teachers in our initial study. We want to explore strategies to improve the accuracy and reliability of LLM outputs, including incorporating advanced functionalities like mathematical plugins and algorithmic functions. Additionally, we aim to develop training programs for educators to help them better understand and utilize AI tools, ensuring they can guide students in critically engaging with these technologies. A key aspect of our planned research is to promote a balanced view of generative AI in education, acknowledging both its potential benefits and limitations. We will emphasize the importance of critical thinking, fact-checking, and ethical considerations when using AI tools. We believe that by fostering a nuanced understanding of generative AI, we can contribute to equipping both educators and students with the skills necessary to navigate the evolving technological landscape. Ultimately, our aim is to contribute to the development of open-source LLMs that are specifically tailored to educational purposes and take into account the particular needs and challenges of teaching. These specialized LLMs would be designed to support personalized learning, provide immediate feedback, and assist with lesson planning and assessment. With careful development, advances in generative AI and thoughtful integration, LLMs could become an asset in modern education, enhancing the learning experience while maintaining the essential human elements of teaching.

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Studying two teachers teaching the ‘same’ lesson: How to balance programming versus mathematics aspects?

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In recent years, teaching of programming to all educational levels has become widespread and it is often connected with mathematics. Under the auspices of a large project in Denmark, we investigate the piloting of a teaching unit in mathematics targeting the use of programming to create a tangram puzzle. We use a dialogical approach to analyse the teaching of two teachers who volunteered to pilot the unit. The two cases were in stark contrast. The first case focused on dissimilating knowledge of programming and mathematics to students before they needed it and the second focused on engaging students in exploring programming and mathematics. While there was only a single voice in the first case, that of the teacher, multiple perspectives were invited and valued in the second. The study shows the importance of establishing a material-dialogical space around the use of programming in mathematics teaching and that opening, widening, and deepening are important teacher moves.

Keywords: Programming, Scratch, material-dialogical spaces, mathematics teaching, lower-secondary school.

Introduction

Due to a growing use of digital technologies globally and the development of new programming languages that are easier to use than the original text-based languages, the teaching of programming to all educational levels has become widespread in recent years. There has thus been an increased attention to introduce programming or computational thinking either formally as part of the school curriculum (e.g. in Sweden, England, Norway and Finland) or informally as part of extra-curricular activities (e.g. clubs or research projects) (Benton et al., 2017). The latter has hitherto been the case in Denmark. The intension is often to connect programming with other subjects, especially mathematics, for the benefit of both. In this paper, our interest focuses on the relationship between learning to program and learning mathematics through programming and the critical role of the teachers in making this happen in mathematics classrooms.

Research on programming tends to focus on students’ learning of programming per se, with a few exceptions that focus on students’ mathematical learning or the role of the teacher (Bråting & Kilhamn, 2021). The few latter studies show that teachers face many challenges when teaching mathematics in the context of programming, as most teachers neither have sufficient knowledge of programming nor have the right skills to teach this more technical subject (Benton et al., 2017). A Swedish study examined the mathematical content and its relation to programming based on written documentation from 32 groups of teachers who carried out lesson studies in programming in mathematics teaching (Kilhamn et al., 2021). In nearly 70 % of the lesson studies, programming was either unrelated to mathematics (31 %) or used mathematics as a context (38 %). In the latter case, the mathematics tended to fade into the background and no new concepts were introduced. Only 13 % of the lesson studies used programming to explore mathematical concepts, but according to the authors, using another tool (e.g. GeoGebra) would have better supported student learning.

There is, therefore, a research gap concerning teachers' appropriation and use of programming in mathematics classrooms. In this paper, we focus on the piloting of a teaching unit targeting the programming of a tangram puzzle by using the graphical block-based language Scratch in grade 7. The pilot study was part of a large-scale national project, called *Game Based Learning in the 21st century*. Two teachers from the same school volunteered to participate in the pilot and based on data from their enactment of the 'same' lesson, we investigate the question:

How can teachers balance a focus between programming aspects and mathematical aspects so that students are offered opportunities to learn both programming and mathematics?

Integration of programming in school mathematics

In recent years, several new programming languages have been developed which appeal to a broader audience due to their graphical interface and more tinkerable and social approaches. Scratch, a descendent of Logo developed by MIT, is one of the most popular new languages (Resnick et al., 2009). According to the developers, students can learn "important mathematical and computational concepts, as well as how to think creatively, reason systematically, and work collaboratively" (Resnick et al., 2009, p.60) by using Scratch. Scratch is thus one example of how computational thinking is mobilized for mathematics learning in schools. However, learning to program with Scratch (or another language) does not necessarily lead to mathematical learning, since these languages are neither designed as mathematical tools nor as tools for learning mathematics.

To support primary students' mathematical learning through programming, the *ScratchMaths* project in UK aimed to design both curriculum materials and professional development for the participating teachers (Benton et al., 2017). Based on a constructivist approach, the research group developed a framework for action, called the *5Es*, to guide all parts of the project. The *5Es* consists of: *Explore* (students should be given opportunities to explore ideas by debugging errors and trying things themselves in Scratch), *Envisage* (students should predict what a script does before running it and reflect on it afterwards), *Explain* (students should explain their ideas to others), *Exchange* (students should have opportunities to share and build on others' ideas) and *bridge* (the relations to the mathematics curriculum should be made explicit). Based on two cases, Benton et al. (2017) investigated the gap between the designers' intentions and what teachers did in their classrooms. Their overall conclusion was that the *5Es* were sufficient flexible for teachers to adapt to their teaching approach and their students' need and that the *5Es* enabled them "to communicate key computational and mathematical ideas in different ways whilst staying true to the goal of the learning activity" (p. 136). They emphasise especially the teachers' critical role in *envisage* and *bridge* activities. Regarding the gap, the authors propose four considerations that focus on the teachers' 1) confidence in addressing both computational and mathematical concepts; 2) emphasis on either the computational or mathematical-related learning aim; 3) management of the gaps in students' attainment; 4) diverse pedagogical strategies influenced by teaching style or existing school practices. Especially the second consideration seems important for the Swedish study. Many teachers found it difficult to create meaningful connections between programming and mathematics, and Kilhamn et al. (2021) suggest that if programming is included in the mathematics curriculum as in Sweden, teachers may consider programming as an independent part of mathematics and may not be concerned about creating connections between the two (as was the tendency in their study).

Theoretical approach

In this study, we use a dialogical approach to learning inspired by Wegerif and his colleagues. Based on the Bakhtinian perspective that learning is a phenomenon that emerges through dialogic interactions, Wegerif argues that “when humans enter into dialogue there is a new space of meaning that opens up between them and includes them within it” (p. 180). He calls this space for a *dialogic space* and exemplifies with the space that emerges when a group of students listen to each other, ask each other for help and change “their minds as a result of seeing the problem as if through the eyes of the others” (p. 180). Bringing more than one viewpoint or voice into relation with each other is crucial, as individuals’ creation of meaning is dependent on this. Therefore, a dialogical space requires a dialogical gap between different perspectives or voices for such meaning to emerge. A discourse that brings at least two perspectives into relation is called a *dialogic discourse*, while a discourse that assumes only a single – and thus true – perspective is called a *monologic discourse*. Research on using this dialogical approach in classroom teaching, have shown that teacher moves such as *opening, closing, widening and deepening a space* are useful (Wegerif, 2011).

Hetherington and Wegerif (2018) extend the understanding of dialogue by arguing that not only the voices of physical people, but also disembodied cultural voices (e.g. a particular philosophical view) and voices from material objects (e.g. text books, whiteboards and digital technologies) are important in classroom dialogues. In their study of science education, they show that the ways in which teachers, students and material objects relate to and intra-act with each other enable some learning opportunities and not others. They argue for the importance of recognising how a particular object, for example a Scratch script, both constrains and enables learning, and the object must therefore “be interrogated and enter[ed] into the larger dialogue not as passive background but as an active or constitutive voice within the dialogue” (p. 39-40). In relation to this, (Cook et al., 2019) introduces two ways of intra-acting with material objects: *passive* (i.e. using the object as a presentational tool) versus *active intra-action* (i.e. the object provides a medium to interact through). Hence, the focus of our analysis is not only on teachers’ dialogical interactions, but also on their intra-actions with material objects, looking for *material-dialogical spaces*.

Methodological approach

The two teachers, Per and Ann, piloted the 15 lessons that made up the teaching unit on programming a tangram puzzle. Both teachers and their grade 7 classes were unexperienced with using Scratch, and the pilot was their first time at all using a programming language in mathematics. For this paper, we therefore selected the third lesson in which the students used Scratch for the first time in relation to mathematics (after a short introduction in the previous lessons). In later lessons, the students must program their own tangram puzzle by using Scratch, and the third lesson supported this by focusing on constructing geometrical figures. It consisted of four tasks: 1) Students had to predict the outcome of a Scratch-script that drew an incorrect rectangle (four right angles but all sides of different lengths) and then redesign it to draw a correct rectangle. 2) Students must optimize the script by using as few variables as possible to construct a rectangle and then a square. 3) Students had to explore a script that used a variable *Number of Sides* and drew a square when it was four. 4) Students must redesign this script so that it can draw regular polygons with up to 6 sides. This requires knowledge of the angle sum in regular polygons. Per and Ann collaborated during the pilot, preparing lesson together and making slides.

We constructed two cases – one for each teacher – based on the following data from the pilot study:

- Classroom observations: field notes and video-recordings of 10 lessons for each teacher,
- Memo writings by the first author after each observation that focused on the teacher’s and students’ interactions and their intra-actions with Scratch and mathematics,
- A final focus-group interview with Per and Ann about on their experiences and suggestions for improving the teaching unit.

Zooming in on the third lesson, we briefly present the two cases below, which we called *Providing knowledge in advance* and *Exploring Scratch and mathematics*. When analysing the data, we first constructed an overview of each teacher’s ways of interacting dialogically with students (monologic or dialogic discourses) and of intra-acting with the material objects (passive versus active) (see Figure 1). Secondly, we analysed the teacher actions more closely in terms of whether they contributed to open, close, widened or deepen a material-dialogical space.

The cases and results of the analyses

Case 1: Providing knowledge in advance

Per has taught mathematics for more than 30 years. When computers appeared in the 80s, he learned some of the first programming languages in his spare time. Per was excited about the unit, although he expected his class to be challenged by it. He began the lesson by speaking at length (see Figure 1): showing two methods to find the angle sum required in task 4, using a metaphor to understand Scratch and unfolding some Scratch-techniques. In the excerpt below, Per introduces the metaphor about tracing a football field, and the excerpt shows his typical ways of interacting with the students:

- Per: Do you know the machine that is used to draw white lines on a football field? [...] tell the man [using the machine] how to draw an octagon. What is the first information he should be given? [no responses]. If he shall draw an octagon, what do you need to tell him? [no responses] Don’t you understand my way of thinking? [draws the situation on the board]. What do you need to tell him? [no responses]. Are you all dead? Should he just start drawing at the edge of the field? [a few shakes their heads]. So, what do you need to tell him?
- Student 1: He must go to the middle.
- Per: Yes, you need to tell him where to start. What else do you need to tell him?
- Student 2: How far he must go?
- Per: No, there is something he needs to know beforehand [...] To give such information is really important when you program the small cat [the Spite] [...] Otherwise it [the Spite] does not know what to do. It only knows what you tell it.

Nearly halfway through the lesson, Per introduced task 1 by telling the students what to do “You have to do this sequence of commandos on your computer”. Consequently, the students quickly typed the script into their computer without further reflections. After 10 minutes, Per showed his solution to the class, going over technical problems he had encountered himself or anticipated the students had.

Only then, did he ask the students about their work, but without following up on their responses. He introduced task 2 in a similar way, and the students worked on it for the remaining time.

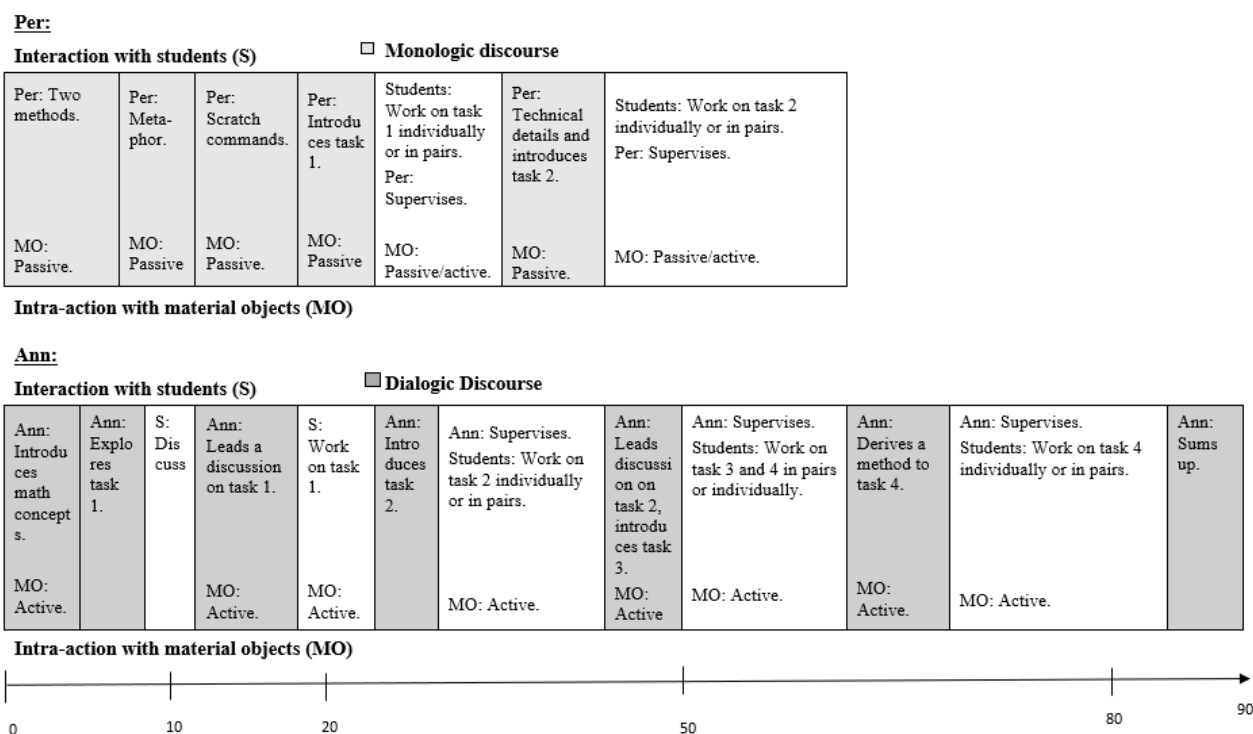


Figure 1: Per’s and Ann’s interactions with students and their intra-actions with material objects

Case 2: Exploring Scratch and mathematics

Ann had graduated from teacher college one year ago and had no prior experiences with programming languages whatsoever. She was a little anxious about the unit but decided to give it a try. Ann was more inspired by the teacher’s guidance in the unit than Per. She began the lesson by describing its aims and encouraging students to explain its key mathematical concepts, which she had written on a slide (together with Per) along with a drawing to support student explanations. Ann then engaged the students in all four tasks one at a time using the same pattern: 1) introduced the task in an explorative way (“What do you think happens in this script?”) and invited students to find out for themselves. 2) supported students in their work with the task and encouraged them to try out their own ideas. 3) led classroom discussions where she invited students to contribute their ideas and asked specific students to share their contributions. In the excerpt below, Ann asks the students to predict the outcome of the first script, and the excerpt shows her typical ways of interacting dialogically with the students:

- Ann: Now we have a pen. Its draws in a blue colour. I only have a white colour, but I hope it works. Then it says point towards 90 degrees. What does that mean?
- Student 1: Downwards?
- Student 2: It points towards a direction that is straight. Horizontal.
- Student 3: It is east, because north is upwards and it is 360 degrees, and east is 90 degrees, and it is to the right.
- Ann: So, it, the figure is me. I must turn 90 degrees [turns 90 degrees] and then I am here [her face to the right]. Then, I must go ‘side 1’ step. What is ‘side 1’ steps? It is how many steps?
- Student 4: It is so far that I can see 100 steps.

Within the first 65 minutes (the span of Per's lesson), all students had finished the first three tasks and started working on the fourth. Moreover, near the lesson's end, the students asked Ann for a method to calculate the angle sum of a polygon and they derived a method together, which the students used to make scripts that could draw regular polygons with up to 6 sides.

Comparing the two cases

In Figure 1, we have compared the two teachers' ways of both interacting with their students and intra-acting with the material objects during the third lesson.

In general, Per did not encourage perspectives other than his own in the long whole-class sessions. On the few occasions where he did ask for students' voices, he prompted them to answer closed and factual questions and not to contribute new perspectives, such as new mathematical ideas, different explanations, or another way of thinking about a Scratch script or the subject matter. Per thus assumed a single perspective; a perspective that was his prior interpretations of what knowledge of Scratch or mathematics the students would need in their later work. Presumably, Per used the metaphor about tracing the football field with the aim of deepening the students' understanding of Scratch, but as he only asked closed questions, not inviting other perspectives, the metaphor did not enable him to open or deepen a dialogical space. This would have required that other perspectives on programming than his own were brought up. He showed the two methods in a similar way. Hence, Per tended to establish a monologic discourse in whole class sessions and to close, rather than open, a dialogical space.

The same applies for Per's passive ways of intra-acting with Scratch in whole class sessions. He primarily used Scratch for representational purposes, such as showing students the scripts they were to work with, and used it as a product, telling the students which commands to type, and selecting programming problems that he assessed relevant based on his own experiences. Not utilising the process aspects of Scratch, such as predicting the outcome of a script, Per did not use it as medium for him and the students to interact through. So even though Scratch had a voice in the lesson, it was only Per who talked through Scratch. Hence, Per did not create a material-dialogical space, where both students' and Scratch's voices were related and valued. In addition, the students were given fewer opportunities to learn programming as well as mathematics compared to the other class.

In contrast, Ann established a dialogic discourse with the students. Instead of speaking herself for a long time, Ann invited the students to contribute their perspectives and used those perspectives to open, widening and deepening a dialogic space. For instance, in the excerpt, she invited students to interpret what "point towards 90 degrees" means and as several students offered their interpretations, she tried to deepen their understanding by using herself as a physical sprite. Throughout the lesson, Ann engaged the students in the tasks, encouraging them to predict the outcome of a script and to explore for themselves how to program the geometrical figures without providing additional information. Ann drew on a range of material objects such as pencil and paper, the black board, Scratch, and her own physical movements, and she intra-acted actively with these materials. She used them as mediums to engage students in processes of exploration of Scratch and of mathematics, and the interplay between them. Hence, Ann created a material-dialogic space where the students were engaged in explorative work, and they shared and discussed their ideas and perspectives on programming and mathematics. In this space, the material objects had a voice, and Ann also used them as a medium for interactions with mathematics as she consistently encouraged the students to explore, explain, construct, interact with and talk about those objects in relation to the geometrical

concepts and constructions. This material-dialogic space offered the students rich opportunities for learning, as evidenced by the students' programming of regular polygons with up to 6 sides (task 4).

Discussion and concluding remarks

Our comparison of the two cases – *Providing knowledge in advance* and *Exploring Scratch and mathematics* – highlights at least three features that seem important for teachers in order to balance a focus between programming and mathematical aspects in classrooms, in which the aim is for students to learn both programming and mathematics. The first feature is that it is important for teachers to establish opportunities for students to explore both programming and mathematical aspects and not focus solely on one of them. When the focus is on programming the exploration should be closely related to the subject matter, for instance by exploring why a script do not draw a specific mathematical figure. The second feature is that teachers must intra-act actively with the material objects in relation to both programming and mathematics. Ann intra-acted actively with Scratch to get a hold of it herself as well as to explore mathematical concepts, such as a rectangle and regular polygon. She thus modelled how she wanted student to learn Scratch and how they should use it to explore mathematics. In contrast, Per intra-acted passively with Scratch and separated it from the subject matter. For the students, Scratch became another piece of knowledge to be mastered and remembered, since they were not shown how to learn Scratch themselves, other than remembering what Per had told them, or how to use it in relation to mathematics. Moreover, Ann used other material objects to deepen the material-dialogical space such as pencil and paper and herself as a physical object, which supported the students' learning of both programming and mathematics. The third feature is that teachers should build on students' experiences both when exploring programming and mathematical aspects. Although, Per aimed to support his students' learning by presenting the two methods to calculate the angle sum, using the metaphor, and demonstrating Scratch-techniques, most students did not benefit hereby as they lacked experiences of their own. The knowledge Per provided therefore seemed meaningless to them. In comparison, Ann conveyed many of the same points as Per, but instead of telling these in advance she departed from and build on students' ongoing experiences with Scratch and mathematics. For example, her students requested methods for calculating the angle sum when they needed them, whereas Per's students hardly knew why they were introduced to these methods. In conclusion, Ann succeeded in establishing a material-dialogic space with rich opportunities for learning, while Per did not, he conducted a monologic discourse. Although the study is limited in scope and its findings therefore require further justification, these findings would not have been so clear if the two cases had not stood in such sharp contrast to each other.

The similarity between the results of this study and those of Benton et al.'s (2017) study is striking. Although, the two studies differ in their theoretical approach to learning (a dialogical approach versus a constructivist approach), the 5Es can be clearly distinguished in the case *Exploring Scratch and mathematics*, in which a material-dialogical space was established, but not in the other case. Our study confirms that *envisage* and *bridge* are particularly important activities when programming is used in mathematics teaching (whereas the other three Es are generally important when aiming to establish a dialogical learning environment). Ann, like Teacher 1 in Benton et al. (2017), used physical walkthroughs and other material objects to *envisage*, while Teacher 2 primarily used Scratch. Benton et al. (2017) conclude that using Scratch as an envisaging strategy seemed to be more effective and that its absence may explain gaps between design intentions and implementation. However, our

study suggests that physical walkthroughs can be an effective envisaging strategy as well (at least when the class is still new to Scratch), and that it might be the material-dialogical space that are established around its use that is more important. Teacher 2 in (Benton et al., 2017) *bridged* explicitly to mathematics much more than Teacher 1, and her relative success led the authors to conject the importance of bridging between programming and mathematics. This is confirmed by our study, where Per did not bridge, but addressed programming and mathematical aspects separately, whereas Ann explicitly bridged the two.

As neither Per nor Ann were confident with programming, our study does not validate Benton et al.'s (2017) first consideration about the importance of teachers' confidence in addressing computational as well as mathematical concepts. However, it should be noted that Ann was strongly supported by the teacher's guidance in the unit. Our study highlights the second and fourth considerations, namely that successful use of programming in mathematics teaching depend on the emphasis the teacher puts on either the programming or mathematical-related learning aim and their pedagogical strategies.

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Generative AI as a tool to foster collective mathematical discussions in primary school classrooms

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With this work, we present a pilot study examining how generative AI (GenAI) can be used as a tool to foster collective mathematical discussion in primary school classrooms. The study is ongoing with ten 4th-grade and 5th-grade classrooms. It explores the potential of GenAI to help primary students, through a guided discussion developed while interacting with the tool, to produce argumentations and discover the notions and properties of geometrical transformations. The analysis of our early results shows how when students are involved in formulating prompts to be given to a GenAI system and in interpreting its feedback, they are engaged in mathematical discussions that can allow them to construct mathematical meanings. Results highlight the need to further investigate the impact of a stable guided use of GenAI on students' proficiency in using it as a personal learning aid.

Keywords: Generative Artificial Intelligence, Collective Mathematical Discussion, Instrumental Approach

Introduction

Among the recent technological advancements, Generative Artificial Intelligence (GenAI) has opened up new windows in the field of mathematics education. This technology, characterized by its ability to generate new content from the patterns it learns from its input data, stands out as a promising tool with the potential to significantly enhance the learning experience. In the context of mathematics education, it can generate a variety of problems and solutions, thereby providing a rich and diverse set of mathematical scenarios for students to explore (Capone & Faggiano, 2024). In our view, this can be turned into a tool to stimulate collective discussion, provided that students are engaged with meaningful problems and guided to share their approaches and solutions.

This paper presents a pilot study that explores the innovative application of GenAI as a tool to foster collective mathematical discussion in primary school classrooms. The focus of our study is on 4th-grade and 5th-grade classrooms, where students are in the process of building their foundational mathematical knowledge and competencies. We claim that already at this stage it is possible to introduce GenAI as a learning aid, as it can provide the teacher with a dynamic and interactive tool to engage students in fruitful mathematical discussions and promote a culture of collective problem-solving and critical thinking.

The mathematical context in which our study is developed is the field of geometrical transformations, which we consider a fundamental topic in primary school mathematics for its characteristics to represent a context in which diverse mathematical knowledge and competencies can be acquired (Faggiano et al., 2018). We explore the potential of GenAI in helping students construct the meanings of geometrical transformations in the context of the mathematical laboratory (Anichini et al., 2004). Our early results, framed by the Instrumental Approach (Trouche, 2004), indicate that when students are involved in formulating prompts for GenAI and interpreting its feedback, they are involved in mathematical discussions that can facilitate the construction of the notions and properties of geometric transformations. However, further investigation is needed to fully understand the impact

of a stable guided use of GenAI on students' proficiency in using it as a personal learning aid.

In what follows we introduce the background and the theoretical framework of our research. This allows us to describe the methods we have adopted and to present our research question. Then we will present the early results and discuss them in the light of our research question.

Background

In our view, the advent of ChatGPT and other GenAI systems can be considered as an occasion to rethink and reassess teaching practices and pedagogical approaches, in relation to the opportunities afforded by these new technologies. However, these new tools, as any innovative resource, require a radical change in the way we teach and students may learn, hence the critical issue lies in the aims we have and in didactical strategies that can be developed. With the aim to investigate the potential of GenAI as a teaching-learning tool in primary mathematics education, we are interested in taking an educational perspective that allows us to look at the possible mediation role of the tool.

This study is based on the idea of a mathematics laboratory conceived as a methodology that relies on various and structured activities to promote the construction of mathematical meanings, and on the related notion of Collective Mathematical Discussion (CMD). A mathematics laboratory can be likened to a Renaissance workshop where apprentices acquire knowledge through hands-on experience and mutual communication about their work and where technological tools play a pivotal role in enhancing the learning process. In tune with the idea of mathematics laboratory, the notion of CMD was introduced (Bartolini Bussi & Mariotti 2008) to analyze the relationship between students' accomplishment of a task through an artifact and their mathematics learning. In a CMD students' interventions are coordinated and guided by the teacher to generalize the emerging situated signs and move them toward mathematical meanings (e.g., Mariotti & Maffia, 2018). The notion of sign here is interpreted in a broad sense, including any kind of entity which might be uttered, written, drawn or embodied, used by someone to express some meaning.

Theoretical framework

Our study is framed by the Instrumental Approach (Artigue, 2002; Trouche, 2004), grounded in constructivist epistemologies, which expands research work on the mediating role of socio-cultural tools in human activities (Vygotsky, 1978). It is built around the concept of instrument as a mixed entity that is constituted by the artifact, the material component, and the subject's utilization schemes or behavioural component (Verrillon & Rabardel, 1995). According to the Instrumental Approach, the process of appropriating and integrating external artifacts as instruments of human activity, called instrumental genesis, is a developmental process described in terms of two dimensions: instrumentation and instrumentalization. Instrumentalization refers to the emergence and evolution of the artifact to support activity in a local cultural context. Instrumentation involves the gradual formation and evolution of schemes for using the instrument in practice.

In this paper, we will focus on the students' processes of meaning construction that emerged while students are engaged in the CMD mediated by the teacher's use of GenAI. To address the issue of how GenAI can be used as a tool to foster CMD and allow students to construct mathematical meanings, we are interested in exploring how the student's progression towards the meanings is intertwined with the students' instrumental genesis related to GenAI.

Following the Instrumental Approach, we intend to investigate the extent to which students' personal

schemes may evolve during the collective discussion orchestrated by the teacher exploiting the potential of GenAI, thus contributing to deepening students' construction of the mathematical meanings at stake. To address the issue of how students can become aware of the meanings stemming from the interaction with GenAI to accomplish a task, and of how such meanings can evolve toward geometrical transformations, consistently with the instrumental approach, in our study we focus on the development of Collective Mathematical Discussions.

Methods

This paper analyses results coming from a project involving ten 4th-grade and 5th-grade Primary School classrooms in Italy. The project is meant to offer teachers examples of teaching activities that can be developed by exploiting the potentialities of digital technologies in mathematics laboratory contexts. All the students were equipped with their own iPad and an interactive whiteboard was available to share materials to be discussed. The activities were conducted by researchers, in the presence of the teacher, within the curricular class organization. They were developed in three two-hour sessions per class and were video recorded. Data collection consists of video recordings of the activities and notes taken by the research team. To the aim of this paper, according to the notes of the researchers, some episodes have been chosen to be transcribed and the transcripts have been supplied with some relevant images. These early results have been analysed to answer our research question: *How can primary students' progression towards the construction of meanings be supported through the teacher's use of GenAI as an instrument to foster collective mathematical discussions?*

The classroom intervention

As already said, the mathematical content at stake was geometrical transformations, particularly axial symmetry. The designed teaching activity was structured as a sequence of didactical cycles that involved the synergic use of different kinds of resources (Faggiano et al., 2018). In some cycles, students were asked to accomplish tasks, by interacting with artifacts and collaborating in small groups, and then to discuss their work with the researcher. For example, in the first cycle, students were asked to use GeoGebra to create a figure, draw a line and generate a second figure as symmetric to the first figure with respect to the line. Then, divided into small groups they are asked to understand what changes and what does not change if they move the first figure or if they move the line. The collective discussion that followed aimed to recognize that axial symmetry is a point-to-point geometrical transformation that depends on the initial figure and on the line that we choose as the axis of symmetry. These kinds of activities, designed following the principles of our theoretical framework, have been extensively experimented with in the past and their effectiveness in terms of students' meaningful learning has already been proven (e.g., Faggiano et al., 2018). In other cycles, the task was presented to the whole class and the researcher led the collective mathematical discussion. The cycle on which we focus in this paper is of this kind. Students were given a sheet of paper on which two figures symmetrical to each other were drawn. They were asked to consult ChatGPT to get insight on the way to find the line that was used as the axis of symmetry to obtain one figure from the other. The choice to use GenAI in this way at this point was due to the previous analysis done by the authors of the feedback that ChatGPT (version 3.5) produced when given the following prompt:

I am a 10-year-old girl and I received from my teacher a sheet of paper with two figures that are

one symmetrical of the other. The teacher, without adding anything else, asked me to find the axis of symmetry and explain how I did it. How can I do it?



ChatGPT

Finding the axis of symmetry of a figure involves identifying a line along which the figure can be folded so that both sides match exactly. Here's how you can approach it:

1. **Understand Symmetry:** Symmetry means that one half of a figure is a mirror image of the other half. If you fold the figure along the axis of symmetry, the two halves should match perfectly.
2. **Observe the Figures:** Look closely at the figures your teacher provided. Pay attention to their shapes, lines, and any patterns they may have.
3. **Identify Potential Symmetry Lines:** Think about where you could fold each figure so that the two halves would match up. Try to visualize folding the paper along different lines and see if the halves would coincide.
4. **Test Symmetry Lines:** Once you have some potential symmetry lines in mind, you can test them by physically folding the paper along those lines (if you can) or mentally visualizing the fold. Check if both halves match up perfectly.
5. **Verify the Axis of Symmetry:** If you find a line where folding the figure produces matching halves, congratulations! You've found the axis of symmetry. Make sure to mark or remember this line.
6. **Explanation:** To explain how you found the axis of symmetry, describe the line along which you folded the figure and how you verified that both halves matched up perfectly. You can also mention any observations or patterns that helped you identify the symmetry line.

Remember, it's okay to take your time and experiment with different lines. Symmetry can sometimes be tricky to spot, but with careful observation and a little patience, you can find the axis of symmetry in the figures your teacher provided.

Figure 1: The feedback received when testing ChatGPT 3.5

As can be seen in Figure 1, the potential of ChatGPT's feedback to trigger a lively discussion within the class is grounded in its incomplete and confusing reply to the specific prompt. For example, it does not give any explanation of how to determine an axis of symmetry between two symmetrical figures, but the feedback refers to the context of figures with an axis of symmetry. The CMD to be developed was designed to bring to the fore the problems of this answer and foster students to produce signs and arguments that can be guided towards the understanding of the axial symmetry properties.

Early results and discussion

In this section, we present and briefly discuss two episodes, selected from our early results, that better allow us, with the space constraints of the paper, to answer the research question. The prompts on which students agreed during the discussion, were given to ChatGPT in Italian as well as in Italian it was its feedback. The translation into English of the prompts and the feedback were provided by the authors together with the translation of the episode transcripts chosen for this paper.

Episode 1

This episode shows the emergence and evolution of the instrument to support the activity and the gradual formation and evolution of schemes for using the instrument in practice. In the previous session, aimed at understanding the characteristics of two figures that are symmetrical to each other with respect to the axis, students interacted with GeoGebra working in small groups. In this second activity, the researcher introduced ChatGPT as a further tool that students could collectively use to

get some help in finding the axis of symmetry between two symmetrical figures. The following are excerpts from the discussion, aimed at highlighting the processes of instrumentalization and instrumentation emerged and evolved during the guided interaction with the artefact to accomplish the task. Students were invited to decide the prompt to be given:

- Student 1: We can write, "We are fourth graders. We have to solve a problem. We have to find out the axis of symmetry between two polygons."
Researcher: As we have seen it replies by writing a lot of things
Student 2: We can ask it for a summary
Researcher: Do we want a summary, or do we want to ask it to guide us?
Student 3: Yes, we can ask it how to do it... explain it to us
Student 1: Yes, we can write, "We are 4th graders, we have to solve a problem. We have to find out the axis of symmetry between two polygons. Guide us step by step."

As the feedback proved to be quite complex and not enough useful in solving the problem ("But it doesn't really help us!"), the discussion then contributed to the development of students' usage schemes:

- Researcher: Maybe we need to make it understand better what we need – let's try to be clearer with it. Let's describe better our problem.

Guided by the researcher, students engaged in a discussion about how to communicate any difficulties or misunderstandings they may have.

- Student 1: We can write: "Thank you for your answers, but you didn't help us at all. We have on a sheet of paper two figures, flag-shaped, symmetrical to each other."
Researcher: That's what we do have. Now we should explain our problem.
Student 1: We can add: "We need to find the axis of symmetry."

Prompted by the researcher, the class continued to discuss the best way to express the function of the axis of symmetry.

- Student 2: I get it... maybe we need to write: "Thank you for your answers, but you didn't help us at all. We have on a sheet of paper two figures, flag-shaped, symmetrical to each other. We need to find the axis of symmetry that turns one flag into the other."
Researcher: And don't we want to ask it a specific question?
Student 3: How do we get it?

In the excerpts, it emerges how during the discussion led by the researcher, students were gradually able to refine and evolve the way they formulated the prompts. They could observe the changes in the received feedback that with a more precise prompt became more helpful.

- Researcher: Look... it added something it had not written before: "if the two figures are symmetrical to each other, it means there is an axis of symmetry that transforms one figure into the other". Then it also adds the "Steps to find the axis of symmetry." Better than before, don't you think?!
- Student 1: Yes, now we like it! Let's see what it can suggest now... "Look for the points that appear to be the same in both figures".
- Student 2: The tip of the flag, the beginning and the end of the flagpole.
- Researcher: Ok, so we have them. Then it says: "Observe the lines or segments connecting these points".

Initially, during their interaction with ChatGPT, students noticed that the feedback they received did not completely meet their expectations. Consequently, they engaged, collaboratively, in an attempt to refine the prompts, trying to make them more useful and relevant to achieving their goals. Guided by the researcher they have developed the instrumentalization process gradually understanding how the artifact could support them in accomplishing the task. At the same time, while interacting with the artifact the development and evolution of schemes supported the instrumentation process

promoting the use in practice.

Episode 2

This episode shows how the instrumental genesis related to the use of ChatGPT during collective mathematical discussions supported students' progression towards the construction of meanings. This second episode is also taken from the discussion led by the researcher (with students belonging to a different class) regarding the identification of the axis of symmetry using the assistance of ChatGPT.

After an initial phase aimed at identifying an appropriate prompt to obtain useful information to solve the task, the discussion focused on the feedback provided by the artificial intelligence. In the initial responses, it emerged the concept of "corresponding points", the meaning of which students are unfamiliar with.

Researcher: So, let's try to recap for a moment, go back over. What did you understand? What is it trying to tell us with this corresponding points thing? For example, when I gave you the sheet of paper with the two symmetric flags, it occurred to your classmate to join these points. Can you explain why?"

The researcher strives to stimulate students' reflection, encouraging them to reconsider and rework what has been explained by ChatGPT regarding the corresponding points. The goal is to guide students so that they fully understand the meaning of these concepts, using as support during the learning process the feedback provided by ChatGPT, but also the other tools, such as the paper with the two flags or GeoGebra. Once the meaning is clarified, the discussion continues with an attempt to understand the usefulness of these corresponding points.

Researcher: Let's see... in your opinion, these two points, the bases of the flagpole, what are these points? Will they have anything to do with what it is trying to tell us? Let's go back for a moment to what it had told us at the beginning: to find this line we have to use the corresponding points... Now what will they be for, these corresponding points?

Student 1: Let's tell it that we've figured out what the corresponding points are and that we've joined them two by two and ask it now what we need to do.

Researcher: Okay...let's see...it tells us, "Look carefully at the points you have connected. Imagine a straight line passing right through the middle of these points."

To allow students to connect the signs, the researcher focalizes students' attention to the notions of the middle point and its connection with the distance, as they have emerged with the use of GeoGebra, with the flags on the paper and the feedback of ChatGPT.

Researcher: Think about what we were doing with GeoGebra, what had we said? We hadn't said that the figures were in the middle, but that the figures were located... if one went away the other one also went away, it did the same. What remained the same?

Student 2: The distance

Student3: I see... we need the middle points because there we have the same distance... that's why we can fold in the middle.

This episode shows the students were led to the construction of meanings: the stimuli offered by artificial intelligence, mainly through the interpretation of the feedback, open opportunities to return to the task, request for and provide synthesis. Previous studies have shown that effective CMD can guide students to the construction of meanings, thus acknowledging that GenAI is not necessary. However, in this paper, we aimed to show how GenAI can be used as a tool to engage students in mathematical discussions and allow them to construct mathematical meanings.

Conclusions and future works

In conclusion, in our pilot study we were interested in exploring the students' instrumental genesis, particularly focusing on the processes of instrumentation and instrumentalization related to the teacher's utilization of generative AI (GenAI) as a tool to foster the collective discussion in primary school classrooms. Our analysis of early results offers valuable insights into the potential of GenAI showing how it evolved into an instrument that, used by the teacher to facilitate mathematical discourse, contributed to the students' learning. As depicted in our selected episodes, students were actively engaged in refining prompts and gradually understanding how to leverage the tool to accomplish their mathematical goals. This can be interpreted as the development and evolution of GenAI as a teaching-learning instrument. The first episode shows how the students developed schemes in the practical use of ChatGPT: they were gradually able to refine the way they formulated prompts and to observe the changes in the feedback when their prompts became more precise. In the second episode, through the interpretation of this feedback during the CMD led by the researcher, the students were able to develop the mathematical meaning of axial symmetry.

We acknowledge that the students were strongly guided in their interaction with ChatGPT during the collective discussion. However, this was due to the limited time available, during which we preferred to provide guidance in order to be able to observe and discuss elements of interaction that could bring out how ChatGPT could be a facilitating tool in the discussion. Furthermore, our study underscores the pivotal role of the teacher/researcher in orchestrating collective mathematical discussions augmented by GenAI. The teacher's strategic actions, guided by the affordances of the tool and its synergies with other resources, were instrumental in guiding students towards the construction of mathematical meanings. By consistently redirecting students back to the task, focusing on relevant aspects of GenAI usage, asking and providing synthesis when needed, the teacher facilitated a learning laboratory environment where students could actively engage with the tool and collaboratively construct mathematical meanings.

These findings highlight the promising potential of integrating GenAI into primary school mathematics education, not merely as a computational aid but as a catalyst for meaningful mathematical discourse and knowledge construction. Moving forward, further investigation is warranted to explore the sustained impact of guided GenAI use on students' mathematical proficiency and the efficacy of teacher orchestration in optimizing learning outcomes. Ultimately, our study contributes to the ongoing discourse on leveraging digital technologies to enhance mathematical learning experiences in primary education.

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Illustrating the design of an automatic formative assessment of the characteristics of students' work on open-ended tasks

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This paper illustrates employment of automatic assessment of student work on open-ended math tasks in order to characterize understanding of specific mathematical concepts. This paper describes a design in mid- to late-stage testing for student conceptions on tangency of a line to a function. Curricular discontinuities illustrate areas of mathematics where students' conceptions can be in various transitional states, which, for this project, are book-ended by synthetic conceptions and analytic conceptions. The strategic design of prompts for students to respond to and the kind of data extracted from student submissions is useful for characterizing students' conceptual evolution related to a topic. We use a framework that involves concept images and definitions, and critical and non-critical characteristics to analyze and categorize student work.

Keywords: Formative evaluation, Education technology, Mathematics, Automatic feedback.

Introduction

Automatic assessments of closed-ended (e.g., multiple choice or fill-in the blank) mathematical problems have been used in large-scale assessments since the popularization of scantron machines. Automatic evaluation techniques for close-ended items have also been implemented in computerized tasks to provide immediate feedback (Farrús & Costa-jussà, 2013) to students to facilitate rapid student reflection and learning opportunities. However, automatic assessment has less commonly been used in classroom-based formative assessment of open-ended tasks.

Technological advancements make this a very exciting time to develop or implement computerized tasks and assessments, if the capacity of our current technology is utilized, rather than replicate paper-pencil tasks in digital form (see Sangwin et al., 2010). We contend that technology has progressed beyond merely verifying that student responses match a particular range of programmed answers (e.g., numerical submissions). Now is the time to consider uses for today's computational developments to automatically assess the work of open-ended math problems. There is more that today's technology can do in evaluating student work beyond just measuring their answers. One reason to do so is that automatic evaluation of assessments implemented as part of a course can provide valuable instructional information and potentially alleviate time demands on teachers and allow them to focus their efforts on demands that are more difficult to automate.

This paper shares our work developing one formative assessment through the Seeing the Entire Picture program, an online platform developed by researchers at the Mathematics Education Research and Innovation Center at the University of Haifa, which uses sophisticated coding and strategically developed applets to extract data from student work to continue to alleviate teacher time-demands and support their analysis of student conception (Olsher, Yerushalmy, & Chazan, 2016).

Framework for the assessment

Sangwin and colleagues (2010) describe the value of digital technologies to automatically assess student work when “a student creates mathematical objects (e.g. an algebraic expression or synthetic figure) using a computer; then the computer automatically establishes mathematical properties of these objects; on the basis of these properties it assigns outcomes, including feedback” (Sangwin et al., 2010, p. 229). We strive to use such mathematical properties strategically and systematically to provide teachers and students with information on students’ concept images and definitions through the work they submit. The design of our task is centered around automatically assessing responses to open-ended math tasks that at times provide immediate feedback to students as well as assess where students are in their concept evolution. We further aim for the program to report a summary for the whole class in terms of the progression of their conceptions.

Our structure to characterize student work uses concept image and concept definition, as well as what Hershkowitz and Vinner (1983) call critical and non-critical characteristics. Tall and Vinner (1981) describe concept image as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p. 152). Tall and Vinner define concept definition as “the form of words that the student uses for his own explanation of his (evoked) concept image” (p. 152). The images one envisions lends itself to flexibility of one’s mathematical work, because a larger array of mental depictions suggests more options to apply to general and specific scenarios. Understanding a students’ concept definition allows one to align the mental depictions with appropriate contexts.

We focus our work on analyzing viewable products that students provide on the assessment and then use these products as opportunities for learning through feedback. A central product we use are examples and non-examples that students submit. Boaler (2016) prioritized the value of students seeing multiple examples as well as non-examples, which could be student-created and shared with the class. These opportunities allow students to consider not only the typical, but examples and non-examples that push the boundary of the topic of focus. When the boundary between examples and non-examples gets fuzzy, the definition must be revisited for both teachers and students to expand or narrow the breadth of examples that fit a definition (Hess-Green & Olsher, 2018). In classroom instruction, the interplay between mathematical definitions and examples requires students to renegotiate meaning as their definitions evolve.

In our assessment, we elicit examples and non-examples from students to develop an understanding of their concept images and definitions. This connects with findings in Dahlberg and Housman’s (1997) study, who interviewed a small group of high performing math majors around a novel function family to study students’ learning about that function family. They found that example generation appeared to be the most impactful of four strategies students displayed in understanding the new concept, also highlighting the value of examples in math. For us, examples serve a dual purpose, they are data to represent a student’s concept image and can be used for classroom discourse or individual reflections based on immediate feedback from a program.

Specifically, our assessment of students’ understandings of tangency builds on research on this topic. Biza (2011) provides 8 tangent definitions initiated by students as pre-conceptions and definitions that evolved through class discussion that range from tangents being “a line that has only one common point with the graph” (p. 136) to definitions that use limits to describe the slope. Such a range of

concept definitions suggest diversity in students' concept images. In a study about undergraduate mathematics majors' proof writing abilities, Moore (1994) suggests an applicable use of concept images and definitions by stating that "students' ability to use the definitions in proofs depended on their knowledge of the formal definitions, which in turn depended on their informal concept images" (p. 262). Moore further indicates that students in his study "needed to develop their concept images through examples... before they could understand the formal verbal or symbolic definitions" (p. 262), indicating the importance of students experiencing a variety of relevant examples.

The display or use of varying examples alone does not build deep conceptual understanding, it requires engaging with examples in varying contexts. Curricular discontinuities, shifts or changes in a concept's definition or practical use from one course or topic to another (Tall, 2002), provides a natural opportunity to evaluate student progress in learning a new topic. For example, the meaning of tangent is different in Euclidean geometry than in calculus. In Euclidean geometry, tangent lines are introduced in the context of circles, where a circle may be seen as all on one side of a tangent line and a tangent line may be thought of as only intersecting the circle in one point. In the context of the calculus, tangents are connected to the derivative and may intersect a function at more than one point.

A student might pass their calculus class with a geometric conception of tangent, yet, their understanding of tangent is then limited. This illustrates the need to address such gaps; there is value and an opportunity for student engagement and discussion as they navigate between two meanings. However, these gaps must first be identified and done so efficiently. We use the notion of two endpoints of a concept to describe students' transition as they navigate from one conception to a new one. For this project, we developed a tangency task and used synthetic as one endpoint and analytic as another, aligned with differences between synthetic and analytic geometry: synthetic is bound by Euclidean structures, analytic involves the assignment of coordinates and algebraic symbols (Giovanni, 2016). Student conceptions may align with one, the other, or in-between.

Using a critical characteristics analysis, a synthetic view of tangency is where the tangent line and a function skim or touch only one point and do not cross the function. In this view, the derivative is not associated with tangency and thus the equality of the value of the derivative and the slope of the tangent line at a point is a non-critical characteristic. Whereas an analytic view requires the derivative at the point of tangency to have a shared value there and be equal to the slope of the tangent line—if the function is differentiable—and may cross through the function. Table 1 summarizes the differences between synthetic and analytic perspectives on tangency.

Transitioning from synthetic to analytic requires space for student thinking to evolve, and therefore, mechanisms to consider evolving student examples is necessary. Traditional automatic assessment programs may not give credit when parts of the submission are valuable. However, researchers are trying to overcome such oversights. Kajetanowicz and Wierzejewski (2006) grappled with allowing partial credit for computer assessed exams. Sangwin and colleagues (2010) describe a computer aided assessment tool that provides feedback to partially correct answers. Farrús and Costa-jussà (2013) describe a framework for a web-based tool that provides immediate feedback to students. They use the term "Latent Semantic Analysis/Indexing" to evaluate open-ended student work. This technique works for long written responses, short responses do not have enough data to use latent analysis, we think there are options for combining shorter responses that link elements of the concept together.

Table 1: Where students’ concept image and definition fall

	Analytic conception of tangency	Synthetic conception of tangency
Definition	“Derivative MUST be same as slope of tangent” & “y-values must match at the point of tangency”	“Graph can only be on one side of tangent” OR “tangent line and graph only coincide at one point”
Critical characteristics	<ul style="list-style-type: none"> • y-values of function & tangent are the same • the derivative of the function and the slope of tangent line are equal 	A function and tangent line: <ul style="list-style-type: none"> • touch/coincide at one point • do not cross

Design of the formative assessment

The task we created asks students to respond to a series of both closed and open-ended prompts with the aim of approximating students’ understandings of the concepts of tangency when it comes to lines and functions graphed on a coordinate plane. Questions prompt students to decide whether three provided examples display tangency or not, submitting examples and non-examples of lines tangent to graphs, choosing characteristics that must be true of graphs and lines that are tangent, and finally a choice of the best definition for tangency between a line and the graph of a function. Across these questions, we consider the consistency of students’ responses (De Bock et al., 2016) to various elements that focus on the same concept.

Task layout

The design of the tangency task has three provided graphs that depict a function (quadratic, absolute value, cubic) and the line $y=0$ that runs through the vertex or inflection point of the function. These graphs are embedded applets developed by the second author. The task also includes a multiple selection question about characteristics of tangents (e.g., see Figure 1) and a multiple-choice question about the definition (e.g., see Figure 2). Finally, each task includes a prompt for students to submit up to two examples and two non-examples to represent their thinking around tangents: “use the diagram to submit up to 2 different examples of where the graph of a function, $g(x)$, and a line, $f(x)$, have points of tangency.” Students are asked to mark points of tangency.

If you have a line and the graph of a function, and if the line and the graph have a point of tangency, which of these characteristics MUST be true? Choose as many as apply.	
<input type="checkbox"/>	The graph of function MUST be on ONLY ONE SIDE of its tangent line.
<input type="checkbox"/>	A tangent line and the graph of a function MUST coincide (take on same values) at ONLY ONE point.
<input type="checkbox"/>	At a point of tangency, the value of the derivative of the function MUST be the same as the slope of the tangent line.
<input type="checkbox"/>	At a point where a line is tangent to the graph of a function, the y-value of the function and the y-value of the tangent line are the same.

Figure 1: Characteristic questions about tangency

Choose the best definition for a point of tangency.	
<input type="radio"/>	Points of tangency are where the y-value of the function and the y-value of the tangent line are the same AND the value of the derivative of the function and the slope/derivative of the tangent line are the same.
<input type="radio"/>	A tangent line coincides with the graph of a function at only one point and the graph of the function is all on one side of the tangent line.
<input type="radio"/>	Points of tangency are where the tangent line just skims the graph of the function and touches it only at one point.

Figure 2: Definition questions about tangency

Indicators to capture data

The platform and the embedded applets collect various data on student submissions to build a case for where student work falls on the spectrum between synthetic and analytic conception. Intentionally developed conditions are used to assess student work for building that case use JavaScript in the platform to communicate how students interacted with the embedded applets or selections they made within the platform. The applets give students the capacity to enter the expression for a line, as well as the expression for the graph of a function, and to mark up to 2 points on each graph and label them as points of tangency. All these actions by students in the applet are saved as variables that the applet exports to the STEP platform to support automatic assessment.

Ultimately, the automatic scoring provides information regarding how students' submissions align with characteristics associated with synthetic or analytic conceptions (see Table 1). Aligning students' work characteristics requires evaluating a series of indicators that are flagged as present when submissions match a particular characteristic. The more aligned with one conception (synthetic or analytic), the more confident the representing score in Table 1 is about the conception they hold.

Currently, we have only conceptualized the most consistent and univocal cases, further data analysis will help us develop a range of outputs in-between. Table 2 displays the extremes of the analysis and with which characteristics and submissions we place high confidence in student conception, where student work is in complete agreement with the identified conceptions. Deviations from those total agreement scenarios result in lower confidence and therefore may identify student work as analytic or synthetic, but with less certainty, demonstrating less consistency. However, student work may be inconclusive if the work submitted does not match enough of the identified characteristics, or if programmed conditions report student work as contradictory.

The degree of agreement is built into the automatic assessment of student responses within tasks. If the student examples show tangent lines only lying on one side of the function, or that the tangent intersects (or touches) the graph at only one point, their work will be scored on the synthetic side of Table 2. If their non-examples include a line that is mathematically tangent to a graph (e.g., a line is tangent and cuts through the graph) that is very strong evidence for the synthetic conception, because this shows they do not consider such submissions to count as examples of tangencies, though they are. By contrast, if their non-examples are functions and lines that do not share any common points, it is harder to assess their conception, we do not know whether the submitted non-examples well represent their understanding of what distinguishes examples from non-examples.

Table 2: Items evaluated and their alignment to conception scoring buckets

Prompt:	(analytic) Tangent	(synthetic) Tangent
Cubic (x^3) (Interactive Graph)	“Yes” and marks (0,0)	“No”
$ x $ (Interactive Graph)	“No”	“Yes” and marks (0,0)
Quadratic shifted 2 units to the right (Interactive Graph)	“Yes” and marks (2,0)	“Yes” and marks (2,0)
What must be true? (Multiple Select)	“Derivative MUST be same as slope of tangent” & “y-values must match at the point of tangency”	“Graph can only be on one side of tangent” & “tangent line and graph only coincide at one point”
Definition (Multiple Choice)	y-values of the function & tangent are same & derivative of the function and slope of tangent line are equal (See Figure 2)	A function and tangent line touch/coincide at one point and do not cross locally (see Figure 2)
Examples (Interactive Graph)	2 examples where there are marked points of tangency and the function and line have the same y-values and the value of the derivative of the function matches the slope of the tangent, and there is at least 1 point of tangency where the tangent crosses the graph of the function.	2 examples where points of tangency are marked at a point where the graph and the line meet and the graph is all on one side of the line, and has at least 1 such labeled tangent point is where the derivative of the function does not have a value.
Non-examples (Interactive Graph)	2 non-examples where, at least one has the graph and the line intersect, but the value of the derivative does not match the slope of the line.	2 non-examples where, in at least one, the graph and the line are technically tangent, for example at a point of inflection.

Pilot results

Our pilot data consists of submissions from 137 undergraduate students who just completed a calculus course. When given a choice to select only one choice from 3 definitions (see Row 5 in Table 2), 130 students answered, 7 students did not respond. Of the 130, 47 chose the analytic response (the first choice), 24 chose “a tangent line coincides with the graph of a function at only one point and the graph of the function is all on one side of the tangent line,” and 55 chose “points of tangency are where the tangent line just skims the graph of the function and touches it only at one point.” That is, 79 students selected definitions aligned with synthetic conceptions.

For characteristics (Row 4 in Table 2), students could select any or all of the given items, and 116 selected “at a point of tangency, the value of the derivative of the function MUST be the same as the slope of the tangent line,” 76 selected “at a point where a line is tangent to the graph of a function, the y-value[s must be the same],” 73 indicated “a tangent line and the graph of a function MUST coincide... at ONLY ONE point,” and 42 indicated the function must be all on one side of the tangent line. These multiple choice and multiple select options, alone do not tell us enough about student conceptions of tangencies. To know more, we look at specific responses together.

Coordinating rows 4 and 5 in Table 2, of those who selected the analytic definition, 9 indicated that the function must only be on one side of the tangent line, and 17 said that the tangent and the function can only intersect at one point (5 chose both), this disagreement between the analytic definition and synthetic characteristics may be caused by students recognizing “derivative” from class discussions, but may not have fully developed that analytic conception of tangency yet. This evidence is bolstered by the combination of the two synthetic definitions and the large groups of students choosing the characteristic that also includes “derivative” in it (47 of the 55 who chose the skimming definition and 20 of the 24 who chose the only one intersection definition). Additionally, in utilizing the unique automatic assessment and filtering structures of the platform, we find that 20 of the 47 students who chose the analytic definition provided at least one example where the tangent line and the derivative of the function did not have the same slope at the indicated point of tangency, while only 13 students submitted examples that did. The discrepancy suggests students may select the correct definition based on contextual conversations in class, those whose responses are in conflict suggest conceptions that are still evolving, or their concept image or definition may be too broad.

Combining students’ submissions across the top five rows of Table 2, of the 137 submissions, 11 students submitted responses completely consistent with an analytic conception, while 8 students submitted responses completely consistent with a synthetic conception. This suggests that in determining the scores that include students’ submissions of examples and non-examples it will be important not to seek scores that are completely consistent, but instead to determine empirically how to provide instructors with scores that are pedagogically useful, while acknowledging that in a multi-item assessment with many judgment points there can always be errors.

Conclusion and next steps

The work we do is centered around both providing students with diverse opportunities to express their thinking authentically, and automatic analysis of student submissions that captures more than just submitted answers. The variety of prompts, combining different modes of interaction with the concepts, have the potential to provide robust evidence for the current understanding of a concept.

This work shows great promise in pushing the use and implementation of automatic assessment for digital tasks and assessments to meaningful learning and consideration of math concepts. By focusing on characteristics and an array of student responses, instead of solely emphasizing correct or incorrect responses to a single prompt, we illustrate the possibility for identifying additional insight into student thinking. We argue that the example eliciting opportunities and the filtering capabilities of the platform also provide teachers an effective mechanism to share student submissions (that can be anonymized) for class discussions to further the learning opportunities.

One of our next steps is to use the filters and conditions to provide an output reading to teachers about both where students in the whole class are in terms of synthetic or analytic conception of tangency and individual student location on that spectrum. We are confident about the extremes of the summaries but must next calibrate student data with outputs that are not at one extreme or the other. We project that in the end, we can provide teachers a printout that helps them make next instructional decisions based on their understanding of students’ conception. We are also at the cusp of piloting a similar task to the one described here that focuses on the intersection of a line and the graph of a function that may also help us better understand the sorts of data discussed here.

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Students' reasoning and utilization of feedback in Scratch

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The present paper reports on the observation of two students when solving a mathematics problem in a visual programming environment, namely Scratch. Attention was given to different types of students' reasoning and how they utilized the computer-generated feedback. Results indicated that students' certain kinds of reasoning when solving a mathematics problem through programming were associated with certain ways of using computer-generated feedback.

Keywords: programming, Scratch, creative mathematics reasoning, utilization of feedback.

Introduction

As the inclusion of elements of programming is now more common in mathematics instruction, there has been a growing interest in conducting studies investigating the relationship between programming and mathematics learning (Moreno-León & Robles, 2016). It has been argued that integrating programming into mathematics instruction may help students develop general problem-solving skills and enhance their mathematics learning (Feurzeig et al., 2011). However, the circumstances under which these promises are fulfilled are unfortunately not well understood.

A possible way in which the integration of programming may facilitate the learning of mathematics is through its potential to foster students' practice of experimental approaches to solving problems (Kaufmann & Stenseth, 2021). When students solve mathematics problems with the help of programming, they will receive instant feedback after having executed a sequence of code. If students are able to use this feedback to reason on mathematical grounds, it may benefit their learning since certain types of mathematical reasoning have been shown to enhance learning (Jonsson et al., 2020).

The present study focuses on students' problem-solving in Scratch, a popular visual programming environment, and reports on the initial results of investigating different types of reasoning students may engage in and the various ways students use feedback generated by Scratch.

Background

Programming has been known for a long time as a viable alternative for practicing problem-solving. It contains cognitively demanding activities such as preparing problems and debugging codes when a program runs unexpectedly (Cui & Ng, 2021). During programming, one can create, observe, and modify computational objects and explore the mathematical ideas underlying the objects (Feurzeig et al., 2011). Educators, therefore, have identified the potential of programming to enhance students' problem-solving skills in mathematics (e.g., Kaufmann & Stenseth, 2021).

The use of programming for teaching and learning mathematics got a renaissance after declining enthusiasm in the 1990s. One possible reason is that programming is considered a practical way of introducing students to computational thinking (Cui & Ng, 2021), which is increasingly recognized as an important skill in today's technology-driven world. In addition, the availability of visual programming platforms allows students to program without additional prior syntax instructions (Moreno-León & Robles, 2016). One such popular visual programming platform is Scratch, which

allows one to create programs through tinkering blocks (Moreno-León & Robles, 2016). Scratch provides several representations of mathematical ideas simultaneously, and with appropriate mathematical tasks, it has the potential to facilitate students' engagement in mathematical reasoning (see, for example, Herheim & Johnsen-Høines, 2021).

Lithner (2008) presented a framework that characterizes different types of students' reasoning. Reasoning is defined as "the line of thought adopted to produce assertions and reach conclusions in task solving." According to the framework, the process of solving a task can be structured into four non-linear steps: (1) a *problematic situation* where it is unclear for the individual how to proceed; (2) *strategy choice*, which can be supported by predictive arguments; (3) *strategy implementation*, which can be supported by verificative arguments; and (4) *conclusion*. Lithner (2008) identified different types of students' reasoning, one notable type of which is *creative mathematics reasoning* (CMR), where students create their own solutions when encountering new problems. CMR is characterized by three criteria: (1) *novelty*, meaning a student creates a new solution method or recreates a forgotten method; (2) *plausibility*, meaning that the student makes arguments supporting the choice of solution method; and (3) *anchoring*, which requires the arguments to be anchored in mathematical properties. Reasoning that is not CMR (henceforth referred to as non-CMR) can, for example, be recalling a known algorithm, imitating an example, or guessing. There is empirical evidence that practicing mathematical problem-solving with CMR tasks leads to better recall of procedures and conceptual understanding of mathematics than practicing non-CMR tasks (Jonsson et al., 2020). Lithner (2017) added that tasks that can promote CMR should have suitable challenges in terms of *creativity* (the level of ingenuity), *conceptual* (the understanding of mathematical properties to construct the solution), and *justification* (the difficulty of generating predictive and/or verificative arguments).

When one attempts to solve tasks in a visual programming environment, one would like to prepare and submit specific inputs to the computer. As a result, the computer may generate information that can be used as feedback. Such a situation has been studied in dynamic geometry software. In a study about students' problem-solving in GeoGebra, Olsson (2017) adapted Shute's (2008) features of formative feedback to categorize how students use feedback generated by a computer. Shute (2008) categorized feedback as *verification* (to judge whether an answer is correct) or *elaboration* (providing relevant information to guide the learner to obtain the correct answer). In Olsson's (2017) study, feedback from a computer is considered neutral; it is up to the students to choose whether and how to use the feedback. They may use it merely to *verify* whether the submitted input is accurate or to *elaborate* on the feedback to predict the next step in solving mathematics tasks. Furthermore, Olsson (2017) also revealed that different types of students' reasoning are associated with various feedback uses. Students who engaged in non-CMR tended to use feedback from the computer only for *verification*, and students who used CMR used the feedback both for *verification* and *elaboration*.

In the present study, Lithner's (2008) framework of CMR, together with Olsson's (2017) adoption of Shute's (2008) characterization of the use of feedback, were employed to provide initial insight into answers to the following research question: *How is students' reasoning associated with their utilization of Scratch-generated feedback?*

Method

To get data on students' reasoning and use of feedback during problem-solving with Scratch, a problem-solving session where this could be done was arranged. Before the problem-solving session,

basic Scratch programming skills were introduced to fifty eighth-grade students at a lower secondary school in Indonesia. The students were then offered to participate in the problem-solving session. Sixteen students, six males and ten females, divided into eight pairs, voluntarily agreed to participate. Permission and informed consent were obtained from both parents and students. There was no evidence that participants had previous experience with Scratch, and they had not gotten any special training in mathematical reasoning apart from what is common in Indonesian schools.

The present paper concentrates on one pair, Anna and Jenny (both are pseudonyms), and their efforts to draw some polygons and generalize a rule to draw any regular polygon. The pair were chosen since they engaged in different types of reasoning, and they used the feedback as verification or elaboration.

Data collection

Students were asked to solve a mathematics problem in Scratch. The problem requires students to draw three regular polygons and find a rule that can help them draw any regular polygon in Scratch (Figure 1). Parts of the problem can also be found in similar studies on problem-solving through LOGO or Scratch (see, for example, Herheim & Johnsen-Høines, 2021).

Solve the problem using Scratch!

1. Create a program that allows the cat to draw three of the following eight shapes!

- Equilateral triangle	- Regular heptagon
- Square	- Regular octagon
- Regular pentagon	- Regular nonagon
- Regular hexagon	- Regular decagon
2. Construct a rule/formula that can help you draw a regular polygon in Scratch!
3. Use the formula to draw another regular polygon. Write the formula on paper, and make sure you and your group mates agree with the formula you made!

Figure 1: The problem

The problem was chosen for its potential to elicit students' CMR and allow them to apply CMR to find its solutions. The first subproblem asked students to draw regular polygons. In Scratch, the main programmable entity, *sprite*, defaults to a *cat*. One can program it by dragging and dropping the available blocks into the script area. Drawing a geometrical shape in Scratch can be done by utilizing pen blocks and programming the sprite to move in specific ways; the trace of the sprite's movement results in a geometrical shape. For example, to draw a square, one can program the *cat* to put the pen down and repeat a stack of moving and turning 90-degree blocks four times. The second subproblem invites students to find the *total turtle trip* theorem (Papert, 1972), namely that the sum of the total exterior angle in any regular polygon is 360° . The problem, therefore, has desirable challenges for CMR tasks (Lithner, 2017): students can solve the problem step-by-step rather than relying on a single, far-fetched trick (suitable creativity challenge), and they can utilize their existing knowledge of mathematical properties such as distance, angle, and shape, and produce arguments based on this prior knowledge (suitable creativity and justification challenges).

The web version of Scratch was used in this study, and students had a maximum of two hours to solve the problem. Students worked in pairs to solve the problem outside of their regular classroom. Students could stop anytime they wanted. The author stayed in the room to provide support if needed. However, since the association between students' reasoning and the use of computer-generated feedback is the focus of this study, the author's intervention was kept to a minimum. When the students got stuck, the author did not immediately provide a response that led to a particular solution. Instead, students were encouraged to explain their actions so far and what actions they wanted to take to continue solving the problems. The students' conversations and computer screens were recorded during the problem-solving, and their work off-screen was also noted.

Data analysis

The data analysis started by transcribing the students' conversations and screen recordings into written text and then importing them into the software Atlas.ti. The analysis focused on the moments before students executed the code blocks and what they did after Scratch subsequently displayed the resulting output on the screen. Lithner's (2008) framework was then used to structure the data. Students' activities, including working away from the computer, selecting and dragging the blocks into the scripts area, modifying the values of certain blocks, and triggering an event to execute the blocks, were considered either as students' *strategy choice* (SC) or *strategy implementation* (SI). Students' arguments during both SC and SI were then identified and later used to categorize whether the students' reasoning was CMR or non-CMR. The reasoning is CMR if there is a sign that the students' argumentation was anchored in mathematical properties, even if the sign of anchoring is vague. This includes natural phrases indicating that students tried to relate various mathematical properties in the Scratch environment, such as "it is wider" and "why are there more?". In contrast, the reasoning is non-CMR if any of the requirements are not met.

Furthermore, students' use of feedback from Scratch was analyzed following Olsson's (2017) adaptation of Shute's (2008) characterization of feedback, focusing on the moment students receive feedback from the computer after executing the blocks. The use of feedback was categorized as *verification* when students merely used the feedback to justify whether their solution method was correct. In contrast, the use of feedback for *elaboration* was indicated by students using the output to extend previous ideas and plan how to proceed with the next step of solving the problem.

To answer the research question, the pair's reasoning (CMR or non-CMR) before executing the code was linked to their use of Scratch-generated feedback (for *verification* and/or *elaboration*).

Results

Anna and Jenny solved the first subproblem by drawing a square, an equilateral triangle, a regular hexagon, and a regular pentagon. In the second subproblem, they generalized that the number of sides of a regular polygon is inversely proportional to the rotation angle of the sprite. In this section, two episodes illustrate different types of reasoning and their association with different kinds of use of feedback. The first depicts the pair's non-CMR reasoning and their use of feedback as mere *verification* when drawing a regular hexagon. The second shows the pair engaging in CMR and using feedback as both *verification* and *elaboration* when drawing a regular pentagon.

Drawing a hexagon

Anna and Jenny's initial approach to solve the first subproblem was to create a sequence of pen down, move, turn, and wait blocks, then assign various values to the degree of the turn block. With this approach, they successfully drew a square and an equilateral triangle. Next, they started to draw a regular hexagon with a similar approach. The following excerpt shows their discussion after trying some values for the turning angle to draw a regular hexagon.

1. Anna: Just [choose] 30 (changing the turning angle to 30, then executing the blocks)
After executing the blocks, the pair realized that the blocks did not result in the intended hexagon.

2. Anna: How much do you mean? [it] means 30, not 50; it means 60 (resetting sprite's position)

3. Jenny: But it looks like, (inputting 60 as the turning angle, executing the blocks)

Anna and Jenny got a correct regular hexagon by choosing 60 as the turning angle, but they did not realize the drawing was correct. Instead, Anna thought that they had just drawn a pentagon.

4. Anna: Look, it is five sides again.

5. Jenny: One, two, three (counting the sides). It is six sides!

6. Anna: But the line does not meet. Look! If I remove the cat (removing the sprite that covers the hexagon)

7. Jenny: Ah (clapping and laughing after realizing that they drew a regular hexagon)

The excerpt shows that the pair developed their own strategy to draw a regular hexagon, meaning the students' reasoning meets the novel criteria. However, there was no articulated argumentation of how the value they input would produce a correct hexagon, which means that the second and third criteria of CMR were absent. Therefore, students' reasoning in this stage is considered non-CMR.

In line 2, the pair made no visible attempt to understand why inputting 30 as the turning angle did not produce a regular hexagon. There was also no articulation that they used the result of their previous attempts to consider 60 as the following value for the turning angle. Lines 3-7 show that when choosing 60 as the turning angle, the group did not expect the blocks to produce the correct hexagon. This shows how Anna and Jenny only used the feedback generated by Scratch to *verify* their strategy without further *elaboration*.

In summary, Anna and Jenny did not use CMR, and their use of feedback was only for *verification*. The excerpts also showed that the pair's reasoning after using the feedback remained non-CMR.

Drawing a pentagon

After finishing drawing a square, an equilateral triangle, and a regular hexagon, Anna and Jenny attempted to solve the second subproblem: constructing a rule that may help them draw any regular polygon. However, they got stuck and later decided to continue working by drawing a regular pentagon. Initially, they used a similar approach as when they drew other polygons, that is, by trying different values for the turning angle of the sprite. During these attempts, the pair showed signs of their efforts to connect the value they input in the turning block with the shape produced by the sprite. Below is an excerpt from after Anna and Jenny had tried 70 and then 75 as the turning angle.

8. Anna: Why isn't it the same? Wait, 75 was bigger. So, 65.

They tried 65 and 80 as the turning angles, and Anna further reflected on their work.

9. Anna: How was it? I don't know... 70 was *actually correct, almost there*. If it's almost correct, just add 3, not up to 5, [or] 10.

The pair continued working by trying 70, 73, 75, 74, and 68 as the turning angle. At one point, the author encouraged them to look at the codes they had created and asked if they could find something to help them determine the correct angle for the regular pentagon. Anna listed each value of the repeat, move, and turn blocks on paper, paying attention to the values of the repeat and turn blocks in the code used to draw the square, the equilateral triangle, and the hexagon (see Table 1).

Table 1: List of values of repeat, move, and turn blocks made by Anna and Jenny

	Equilateral triangle	Square	Hexagon
Repeat	3	4	6
Move	200	200	100
Turn	120	90	60

10. Anna: Six, five, four, three. Imagine if there is a pentagon, imagine six, five, four, three [are] the turn; each move is 200. It should be 75; (mentioning values in order) 60, 75, 90, 120.”

The pair were likely to see a pattern in the table. Realizing that the pentagon has five sides (between the number of sides of the square and the hexagon), Anna and Jenny predicted that the turning angle should be between 90 and 60. They used this prediction and stated that 75, which sits in the middle of 90 and 60, should be the correct value. However, they did not choose 75 for the next attempt since they had already tried it, and it did not produce the correct pentagon. They then decided to retry inputting 73, which had been tested in their previous attempt and resulted in a “very small difference.”

11. Jenny: 73?

12. Anna: We have tried 73. It was, the difference [of] 73 is *really small*. [It’s] not like this (pointing to the screen that shows the result of choosing 68 as the angle)

They tried 73 as the turning angle and realized the resulting shape was “too much.”

13. Anna: It means *too much*. Then, it should be 72.5.

14. Jenny: 72

15. Anna: We have tried 72

16. Jenny: No, we haven’t

The pair then input 72 as the turning angle and got the correct pentagon.

Anna and Jenny’s reasoning is novel since they developed their strategy, and it is based on predictive arguments that, to some extent, are anchored in the properties of the exterior angle of a polygon. They expressed their anchoring using natural language, as shown in lines 8, 9, 12, and 13. Furthermore, the sum of the exterior angles of a polygon is always 360° , implying an inverse proportion of the number of sides of a regular polygon to its exterior angle. Anna’s argument in line 10 indicates an anchoring to this property. Thus, students’ reasoning in this episode is classified as CMR.

The pair used the feedback from Scratch as *verification* and found that inputting 73 did not produce the correct pentagon. They then started *elaborating* on the feedback by comparing the result with their previous attempts using various angles. In the earlier attempts, they found that 70 almost produced the correct pentagon, and later, they used it as a reference value by conducting trial and error with values around it. Since inputting 73 resulting a pentagon with “too much” turn, they were likely to try finding a value between 70 and 73, which led them to successfully draw a regular pentagon by choosing 72 as the turning angle.

Anna and Jenny, in this episode, used CMR, and their use of feedback was both for *verification* and *elaboration*. The CMR was shown to be maintained after students used the feedback for *elaboration*.

Concluding Discussion

The results of the present article show that programming in Scratch provides an environment that allows students to experiment with their mathematics ideas when solving a mathematics problem, reinforcing the conclusion drawn from previous studies (e.g., Herheim & Johnsen-Høines, 2021). Anna and Jenny used Scratch to ignite their reasoning by creating and modifying computational objects that suit the requirements of the problem. The reasoning students developed did not necessarily require them to reflect on mathematics ideas underlying the computational objects. Instead, they sometimes used a trial-and-error approach in their problem-solving. With this approach, students merely used the feedback generated by Scratch to *verify* their solution method, and *elaboration* on the feedback was absent. This finding resonates with the previous study in the GeoGebra environment (Olsson, 2017), which found that students tended not to *elaborate* on the feedback when they did not articulate any predictive or verificative argumentation.

With the trial-and-error approach, Anna and Jenny successfully drew an equilateral triangle, a square, and a regular hexagon. However, creating a regular pentagon in Scratch requires more effort from these students. To some extent, determining the turning angle for an equilateral triangle, a square, and a regular hexagon is more intuitive than for a regular pentagon, likely because their exterior angles are multiples of ten. This can also be explained by referring to Herheim & Johnsen-Høines (2021), who also involved students drawing a pentagon in Scratch in their study. The study indicated that drawing a pentagon invites students to engage in productive struggle. Solving through trial and error in Scratch is insufficient to find the correct external angle for the regular pentagon, and more in-depth reasoning is required to get 72° as the exact turning of the sprite. When drawing the regular pentagon, Anna and Jenny used CMR several times to decide what value of the turning angle they should input in their next attempt. The anchoring aspect in their arguments was either clear (as shown in line 10) or vague (i.e., indicated by natural language such as “small” and “too much” in lines 12 and 13). The students’ CMR was followed by interpreting the feedback generated by Scratch and then using the interpretation to plan their next attempt to solve the problem. This finding supports Olsson’s (2017) study of students’ problem-solving in a different digital environment, which showed that students who engage in the CMR were more prepared to interpret and *elaborate* on the feedback generated by the computer.

When adopting Shute’s (2008) framework, Olsson (2017, p.718) described a unique feature of feedback generated by computers as “being implicit,” without any explicit purpose to give students information on how to proceed. The absence of explicit guidance implies that *elaborating* on Scratch-generated feedback entails attempts to understand mathematical ideas underlying the computational objects to obtain what Shute (2008, p.158) described as “relevant cues” toward a correct solution. Meanwhile, one of CMR’s criteria is anchoring in mathematical properties, and engaging in CMR means that one needs to fasten arguments in data of mathematical objects, transformations, or concepts (see Lithner, 2008, p.261). This can help explain the association of CMR and the *elaborative* use of feedback: CMR triggers students to make efforts to comprehend mathematical ideas among different representations in Scratch; thus, they tend to use feedback *elaboratively*.

To sum up, this article provides insight into how different types of students' reasoning are associated with how students utilize computer-generated feedback in Scratch. Further studies could be conducted to investigate various conditions that might realize the proposed benefits of the inclusion of programming in teaching and learning mathematics. This includes its potential to foster computational thinking skills among students, which seems to be a growing demand internationally.

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POSTERS

Revisiting working with geometric loci in educational contexts, through GeoGebra Discovery

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Keywords: Elementary geometry, geometric locus, dynamic geometry, automated reasoning, GeoGebra Discovery

Elementary, Euclidean geometry has been traditionally the context to learn, to practice and to foster mathematics reasoning in the classroom, through the formulation, verification and understanding of challenging geometric statements. In this regard, locus computation seems to provide a quite suitable source of interesting problems in the context of a human/machine collaboration for learning through addressing geometric problems. Some explicit examples of the cooperation of Dynamic Geometry Software (DGS) with students addressing locus computation tasks are described in El-Demerdash et al. (2019), or in Ferrarello et al. (2017). On the other hand, subsection 4.3, or section 5, in Leung et al. (2023), describe situations that implicitly involve locus computation with DGS (e.g. “...(T) Let ABCD be a quadrilateral such that each pair of interior opposite angles add up to 180° . Find a way to prove that ABCD must be a cyclic quadrilateral.”, an exploration that could be also described as the search for the locus of D when ABCD have supplementary opposite angles).

The large accessibility and popularization of algorithmic tools for automatically dealing with elementary geometry statements, such as those implemented in GeoGebra Discovery¹ (Kovács et al., 2022), also related to Artificial Intelligence (A.I.) bots (Botana & Recio, 2024), makes it necessary to reconsider the conventional approach to this topic, as already remarked by Carvalho (2024), Hanna & Yan (2021), or Hanna et al. (2023). GeoGebra Discovery is an app, web page and application, freely available at smartphones, tablets and computers, that is able a) to automatically obtain the equation and to display the graph of a proposed geometric locus, b) to confirm or deny, with mathematical rigor, the truth of a geometric assumption formulated by the user, c) to mechanically discover and output large collections of geometric properties holding on a given figure.

In this context, the main goal of this poster is to highlight/provide arguments and examples to revisit the traditional educational protocol on the cooperation DGS/students for locus computation: “locus construction/conjecturing/confirming or denying by dragging/” in view of the current performance of GeoGebra Discovery. Let us notice that in the standard protocol, as described above, the first two steps are to be accomplished by the student, with the concourse of the machine, by dragging the figure in the DGS; while the final step, confirming/denying the formal correctness of the locus equation conjectured by the user, is considered as a human-only part of the task.

From our perspective, using GeoGebra’s automated reasoning tools, this strict separation of the roles of machines and humans along the different steps for locus computation, needs to be overcome. Thus, in the new setting, the DGS can also provide conjectures. Moreover, once human intelligence has

¹ <https://kovzol.github.io/geogebra-discovery/>

accepted/formulated a certain conjecture about the nature of the computed/displayed locus, GeoGebra can also contribute to verify its correctness. And, if the answer is negative, the loop: “experimentation/conjecturing/verification”, should have to be launched again, involving in each step human/machine cooperation.

In this poster we will illustrate this new protocol through of a few examples from the above-mentioned references, that would allow us to highlight some comparisons between the previous, and our current, proposals.

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Math Trails in Intervention Programme in Science and Mathematics

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Math trails

Math trail is an engaging group outdoor activity for students and also for the general public. Math trails can be enacted using the free MathCityMap application. It allows mobile devices to display maps, task assignments and photos of objects related to the task. Trail participants solve a series of mathematical tasks related to real-world objects (Ludwig, Jablonski, 2019). They collaborate, communicate, use different measurement tools, and look for the best strategies to solve given tasks. Users get instant feedback as they enter their solution. Math trails increase interest in mathematics (Gurjanow, Zender, Ludwig, 2020) and based on the task included in the trails students can see how is mathematics related to the objects around them and therefore see use of mathematics in real-life.

Intervention programme

The main aim of the intervention programmes is to develop various competences, e.g. scientific thinking or metacognitive knowledge. In the process of intervention, the experience and inquiry are used. The aim is to use a variety of activities that are reasonably structured to reinforce social and communication skills and promote students' resilience to overexertion (Braun et al., 2014). Students need to be sufficiently motivated and therefore the student-oriented activities should be involved.

At the Faculty of Natural Sciences and Informatics Constantine the Philosopher University in Nitra the intervention programme in science and mathematics have been designed and piloted since 2022. The programme has been created based on design-research principles. The first round of the program was implemented in the school year 2022/23, the second in 2023/24 and the final round will be conducted in 2024/25. There are two student groups involved in the programme, the group for primary (grades 2-5) and for secondary (grades 6-9) students.

The programme is implemented in the form of weekly 60-minutes sessions that are led by the main teacher following the whole programme and one or two co-teachers, experts in various fields of mathematics and science, i.e. physics (introductory physics, material physics, astronomy), biology (genetics, botany), geography (pedology), computer science (hardware, computational thinking), and mathematics (functions, geometry, fractions, combinatorics and probability). Each topic is developed during two to four consecutive sessions.

Inclusion of math trails in the programme

During solving the problems in math trail students need to distribute the work in the groups, collaborate in measuring and solving of the mathematical problems. So, math trails offer affordances in developing the soft skills of the involved students and therefore we decided to include the math trails in the intervention programme as a mean to develop 21st century skills, including communication, collaboration and social skills. The trails were included in two mathematical topics (combinatorics and fractions) and in introductory physics.

Combinatorics

The combinatorics trails include tasks with different implicit combinatorial model, i.e. selection, distribution, partition (Batanero et al., 1997). The tasks in primary and secondary trails mostly use the same objects and their difficulty is based on the dimension of the task, e.g. primary-school students look for all the possibilities how to arrange 3 while secondary-school students how to arrange 5 friends on the same bench.

Fractions

In the task focused on fractions the tasks were aimed to work with various interpretation of part-whole relation. There were tasks to find the whole when the fraction and the part were given; to find the part when whole and fraction were given; and to find the fraction when the part and whole were given. The tasks were aimed not only at the amount of objects, but tasks where the fraction-as-the-operator (enlarge the number of seats by one third) were included (Švecová et al., 2022).

Interdisciplinary trail in math and physics

The interdisciplinary trail was focused on measure, as the content common for both, math and physics. The main focus was on measuring time and length. While primary students stopped in this phase, secondary students used the obtained data to calculate volume, density and speed.

Acknowledgment

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The augmented reality sandbox as an embodied learning environment for mathematics

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Keywords: augmented reality, embodied design, mathematics education

Introduction

For approximately two decades researchers have explored the opportunities of augmented reality for mathematics education (Bulut & Borromeo, 2023). There are various forms of augmented reality, each involving a blend of real-world objects with digital visualizations. Augmented reality has the potential to bridge the gap between real-world phenomena and mathematical representations (Swidan et al., 2023).

At the Freudenthal Institute, we explore the Augmented Reality Sandbox (AR Sandbox), which originated at UC Davis, as a learning environment. The AR Sandbox consists of a sandbox with a stereo camera and a projector mounted above. The stereo camera generates a digital 3-dimensional representation of the sandscape² or any other object present in the sandbox. The projector then projects suitable images onto the sandscape or object, like height lines, height colors, or flowing water (see Figure 1). This setup allows students to manipulate real-world objects, with added mathematical information that updates in real time.

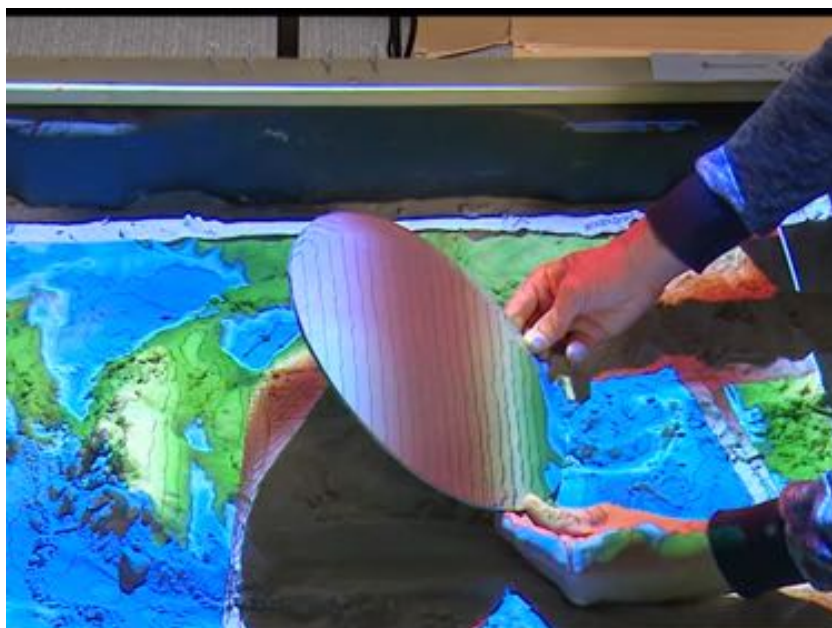


Figure 1. A disc in the AR Sandbox with projected height lines

In a previous study (Bos et al., 2022), we investigated how a student makes sense of the parameters a , b and c in a plane equation $z = a x + b x + c$, and the corresponding gradient vector (a, b) , building on sensorimotor experiences in the AR Sandbox as an embodied learning environment. For

² This beautiful neologism means the landscape formed by the sand in the sandbox.

example, by rolling a ball down the circular plane in various positions, one can discover that the steepest way down (i.e. the direction opposite of the gradient) is always perpendicular to the height lines. An essential part of the learning trajectory is to connect intentional motions (actions) to mathematical constructs, like connecting a rotation of the plane—while keeping the height lines parallel to a side of the sandbox—to the parameter a or b in the equation. The first stage of embodied design is to develop a degree of fluency in such motor skills; in the next, perceptions that facilitate the execution of these actions are developed into mathematical constructs. While our case student showed progress in making sense of the plane equation and the gradient, the visual projection of coordinate systems and symbols was not facilitated within the AR Sandbox as a learning environment, and here we find our next challenge.

Current study

As the AR Sandbox was initially developed for geography demonstrations, the software mostly allowed projecting “landscape information”. To show more mathematical information, we are developing new features. These features include projecting a coordinate system; when a stick is positioned in the sandbox, projecting a vector presentation associated with this stick as a line; when a plastic plane is positioned in the sandbox, projecting an equation or a vector presentation associated with it. While in our initial design students switched from AR Sandbox to paper and pen as soon as the symbolic presentations of the plane and the gradient were addressed, these improvements should enrich the AR Sandbox as a learning environment to facilitate further investigations of the connection between directed actions and symbolic notations, like plane rotations and translations, and parameters in the plane equation $z = a x + b y + c$.

The embodied learning environment we are developing allows for a unique situation consisting of real-world tactile objects augmented with projected real-time digital mathematical feedback of symbolic and non-symbolic nature. We aim to contribute to the questions of what opportunities such an environment has for teaching calculus and geometry, and how students’ learning is affected by working in such an environment. Our poster will show our initial designs and preliminary results of pilot studies.

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Programming tasks and activities promoting mathematical competencies and subject matter

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Keywords: Mathematical competencies, Computational and algorithmic thinking, Praxis and logos

Introduction

In many countries, mathematics has been given a particular responsibility for students' knowledge and skills in programming. In Norway, the curriculum in mathematics provides a set of competence aims for each grade level, and one of these, the last one in each grade, is clearly connected to programming. Additionally, the curriculum includes a set of *core elements*, similar to the mathematical competencies proposed by the Danish KOM-project (Niss & Højgaard, 2019). The core elements specify what should be most important in the subject (Smestad, 2018), including competencies such as exploration and problem solving, modelling, reasoning, representation, generalization and abstraction. Despite the name “core elements”, most teachers seem to focus their teaching on subject matter. Some may find subject matter more important, while others find it difficult to teach the core elements. Hence, when I recently challenged a group of in-service teachers in primary education to design, plan and test a teaching session where programming was to be used to promote mathematical reasoning competence, all the teachers prepared teaching sessions with other goals, focusing on subject matter, and not on reasoning.

Theoretical framework

This study utilizes Chevallard's (2006) framework of how knowledge is transposed between different instances of the educational system (Kilhamn et al. 2022). Chevallard's (2006) notion of praxeology describes how human doing, including the didactics of mathematics, can be categorized into “two main, interrelated components” described as (1) *praxis*, meaning “*what* people do, and *how* they do it” and (2) *logos* meaning “*what* they think, and *how* they do so”. The idea in the current study is that “depending on the relation between praxis and logos, a task can induce quite different kinds of learning” (Kilhamn et al. 2022). When I challenged the in-service teachers to create a teaching plan focusing on mathematical reasoning, reasoning was “my” logos, but I soon realized that my logos was different from the teachers'. Their logos was subject matter, and in particular multiplication as repeated addition. Rather than initiating a (potentially endless) discussion on which logos are better, we decided to investigate if we can “do it all”: *develop teaching sessions that uses programming to teach subject matter in a way that also promotes development of mathematical competencies*.

Demonstration and results

One group of teachers had created a teaching session centered around an algorithm that would “program” a student to draw a rectangle of crosses on the floor corresponding to the multiplication problem $3 \cdot 4$ (Algorithm 1). This was their *praxis*, and multiplication as repeated addition was their *logos*. What can we do to facilitate mathematical competence development as well?

Design of a dashboard in Calcularis to support teaching numbers and operations at primary school

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Keywords: Dashboard, number concept, computation, primary school.

Context

This research is part of an ongoing project aiming at evaluating the contribution of Calcularis³, an online adaptive computer-based program, to teaching and learning basic arithmetic skills (number concept and calculation) at primary school. Previous research (Rauscher et al., 2016; Gardes et al., 2022) shows positive effects of the use of Calcularis on students' development of spatial representation of numbers and mental calculation. Calcularis also embeds a learning analytics dashboard (LAD) providing information about students' behavior and learning, based on collected data such as students' connection time, learning time, tasks worked out, and provided answers. However, Calcularis being designed for an individual use mostly by students with mathematics learning disabilities, the LAD turned out not to be fully adapted for a classroom use. For this reason, a new LAD is being designed, drawing on the concepts of *usefulness*, *usability* and *acceptability*.

Usefulness, usability and acceptability of a digital tool

Tricot et al. (2003) highlight three different, though interrelated, dimensions that need to be considered when evaluating a digital tool: *usefulness*, i.e., a possibility to achieve the goal for which it has been designed, *usability*, i.e., ease of its manipulation, its user-friendliness, and *acceptability*, i.e., the decision to use it. Clearly, perceptions of usefulness and usability of a tool affect its acceptability, but the latter is also sensitive to factors such as culture, values, motivations, or practices in which it will be integrated. These dimensions guide the design of the new LAD in Calcularis.

Calcularis dashboard – design choices

Our aim is to design LAD that is useful, usable, and acceptable for the primary school teachers. For the sake of space limitation, we only focus on the usefulness (the poster will document design choices underpinned by all three dimensions).

We consider that an LAD is useful for teachers when it allows monitoring the class and each student without overloading teachers and supports their decision making. To this end, the LAD should detect students struggling at the behavior and the cognitive levels. At the level of the behavior, collected data allow detecting students with unusual behavior, either compared to their own previous behavior, or compared to students' average. It has been decided to call for teacher's attention when a student shows too many short sessions (less than 5 minutes, a full session lasting 20 minutes), low productivity (too few answers provided) or too many incorrect answers compared to either the individual value or the general reference value. In either of these cases, the teacher will be warned by a message such as "Need support in Activity, 5 out of 17 learners" (Figure 1, upper left). At the

³ <https://www.mathetmots.com/eu-en/calcularis>

cognitive level, a student is detected as struggling when she is not progressing anymore. This can be detected thanks to the personalized learning path computed by the application. When a student fails in solving a task, let us say T, related to a given skill, the application provides her with another tasks in which pre-requisite skills are trained. When these tasks are successfully solved, the task T is proposed again. When the student fails solving it three times and remains in a loop, teacher’s intervention is crucial. Therefore, a message such as “Need support in Progress, 3 out of 17 learners” is displayed calling attention to students falling in loops (Figure 1, upper right).

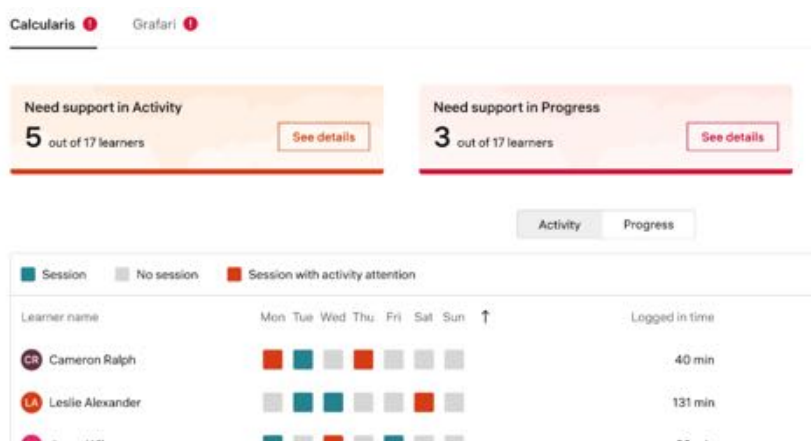


Figure 1. Screenshot of the Calcularis learning analytics dashboard

Pilot experimentation of Calcularis with the new LAD will be carried out from March to June 2024 in two primary schools in Switzerland. Preliminary results will be presented at the conference.

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LSA Gestinv Database as a tool for the professional development of mathematics teachers.

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Key words: Large scale assessment, teachers' professional development, Gestinv Database.

Our study fits within the strand of research linking large-scale assessments and mathematics education. In particular, our research is based on the use, in teacher professional development courses, of an online database, Gestinv and on the ways in which its informed use can contribute to increasing the mathematics specialized knowledge of teachers and future teachers. The main outcome of the research is the design of a model for teacher professional development.

The Gestinv database contains all the data from the Italian INVALSI large-scale assessments. In details, Gestinv includes items administered in INVALSI standardized assessments assembly together with results, comments, educational insights, metadata, statistics and analysis. Regarding the mathematical area, the database, to date, contains 1469 mathematics items from the INVALSI surveys conducted from 2008 to the present. Within the database, for each survey there is a PDF of the complete test and each item is accompanied by detailed results, statistical data, and classifications into different categories. Referring to each item there is the figure of the question, the content area, the process, the reference to the National Guidelines, some key words characterizing the content involved, the text of the question, the correct answer or the figure of the correct answer, the national response rates and other statistical data. The ways in which the database can be used are manifold. By entering in the mathematics section, it is possible to search by National Guidelines, by keyword (there are about 200 keywords that identify the main content for each item), by full-text (it is possible to do a full-text search of all questions, processes, keywords) and by “guided search” (it is possible to do a cross-search, with connectors and/or, of all parameters in reference to each item and other characteristics, such as national response rates). Up to now are currently enrolled 410631 pre-service and in-service teacher.

Research shows how the use of Gestinv can help increase specialized knowledge for teaching mathematics (i.e. Ferretti, Martignone & Viola, 2023); starting the main literature findings, a professional development model for teachers and prospective teachers, based on informed use of the database, has been designed and implemented. What also guided the design of the teacher professional development model is the analysis of the responses to a questionnaire designed to investigate training needs in this area. To date, 150 teachers and prospective teachers of mathematics from all school grades have responded to the questionnaire.

The teachers' professional development model with the use of Gestinv

The macro-phenomena highlighted and quantified by INVALSI test results - that are relevant from an educational perspective - which can also be observed by teachers, and related to their own

classroom experience. Our model, which is developed from the studies of Ferretti, Gambini and Santi (2019) uses the Gestinv database within a Jaworski's *Community of Inquiry* (Jaworski, 2006). The focus underpinning the design of our model concern the influence that the informed use of Gestinv can have on mathematics teachers' expert knowledge by critically reflecting on the complexity of standardized assessments with the support of research findings in mathematics education. The theoretical framework by which we frame the specialized knowledge of teachers and prospective teachers of mathematics is the Mathematics Teacher Specialized Knowledge model of Carrillo-Yañez and colleagues (2018). During the training activities, teachers can question and confront each other, discussing mathematical content and it is teaching vertically in different school grades, assessment methods and results, teaching strategies and methodologies, cognitive processes, and student errors and difficulties. In developing our model, we adopt an evolutionary approach that stimulates teachers and educators to a deeper awareness of their own actions, motivations and goals. The practices were designed in order to improve and complement teachers' awareness of both subdomains of the MTSK model.

This model is the core of the design for the research activities of the Project "Mathematics standardized assessment as tool for teachers' professional development", that won the PRIN 2022, funded by the European Union and the Italian Ministry of Education and University and it is currently the subject of two research-training tracks carried out within the activities of the Ferrara Mathematics PLS.

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Students' difficulties in standardized assessment: a comparative study on the impact of cultural, educational, and technological factors

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Keywords: International comparative study, technology integration, assessment practices, standardized tests.

Introduction and rationale

In today's rapidly evolving world, the emphasis on rote knowledge has shifted towards cultivating competencies such as independent and critical thinking, creativity, and adaptability. Moreover, the potential of standardized tests to assess and promote students' mathematical learning is extensively recognized. However, teaching is inherently intertwined with culture, and its complexity extends also to mathematics education, which varies significantly from one country to another. International comparisons of mathematics education may shed light on these differences and provide insights into how to enhance mathematics achievement globally (e.g., Sacristan et al., 2023). With these motivations, we have developed a project which embarks on a comparative analysis of school systems, traditions, and cultures in Italy and Portugal, with a particular focus on teaching mathematics and improving students' mathematical learning, especially through technology. In particular, to ignite discussions on enhancing students' mathematical learning experiences, and by analysing our data, we endeavour to identify guidelines that harness the potential of technology to enhance mathematics education through standardized assessment. At its core, this research delves into the perspectives and beliefs of teachers regarding assessment in mathematics education. Drawing upon selected items from standardized tests, our study aims to elucidate how standardized assessment data can serve as effective tools to enhance teaching and learning processes. In this work, we present early findings from this comparative study focusing on standardized assessment and the impact of cultural, educational, and technological factors on students' difficulties.

Elements of comparative studies and theoretical framework

Comparative analysis allows researchers to estimate theory generalizability and understand the complexity of reality by comparing interpretations of different cases (Jablonka et al., 2018). This approach offers a nuanced understanding of educational phenomena beyond traditional case studies. In our study, special emphasis was placed on document analysis, providing a comprehensive description of the process and detailing the types of documents that proved beneficial in augmenting the research data. Document analysis involves systematically reviewing documents to elicit meaning and develop empirical knowledge. It is often used in qualitative research, particularly in mixed-methods studies and qualitative case studies. Documents serve multiple purposes in research, including providing context, suggesting research questions, offering supplementary data, tracking change, and verifying findings. In our study, materials from Eurydice and national education ministry websites provided crucial background information. Descriptions of national curriculum guidelines in Italy and Portugal highlighted unique educational contexts. Overall, document analysis complemented our study, supporting theory-building and enhancing our understanding of the research

context. The study is also framed by research results concerning the role of errors in mathematics teaching and learning: confronting difficult problems and learning from mistakes is essential for knowledge development and recognizing the types of errors students make is crucial for targeted teaching strategies (Gagatsis & Kyriakides, 2000).

Research questions and methods

This study aims to investigate two critical questions regarding the influence of cultural, educational, and technological factors on standardized test performance. Firstly, we seek to determine whether varied cultural and educational backgrounds, coupled with enhanced technology literacy among both students and teachers, can lead to improved outcomes in standardized testing. Secondly, we aim to explore how teachers' beliefs and educational choices impact students' problem-solving abilities, as they can be elicited by standardized tests. By addressing these questions, we aim to contribute to a deeper understanding of how cultural, educational, and technological factors intersect to shape student achievement in mathematics education. Methodologically, the research employs a combination of literature review, document analysis, and data collection (students' protocols and a survey for teachers). To collect our data, we developed and implemented across multiple classrooms in Italy and Portugal, a research tool to investigate 10th-grade students' responses to standardised tests and discern their mathematics teachers' perceptions of test difficulty and student errors.

Results and discussion

Preliminary qualitative analysis reveals insights into the needs of students and teachers, and explores the influence of cultural and technological contexts on test performance. We observed that Portuguese students outperformed Italian students on standardized tests, possibly due to their exposure to technology and alternative teaching methods. Despite being allowed to use graphic calculators during the test, Portuguese students demonstrated varied problem-solving strategies, while Italian students struggled with non-standard questions. The qualitative data also aligned with teachers' opinions, indicating a need to rethink traditional teaching practices and incorporate technological tools to foster problem-solving skills. We conclude with implications for leveraging technology integration and assessment strategies to foster improved mathematics learning outcomes internationally.

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An analysis of a commonly used Norwegian professional platform for mathematics teachers and its reported use by those teachers

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Keywords: Digital resources, professional platforms, connectivity, lesson planning.

Introduction

In the European-funded project MaTeK (<https://projectmatek.eu/>), a survey across the five countries was conducted concerning mathematics teachers' use of resources for their teaching (Çakıroğlu, et al. 2023). The survey revealed that, from a sample of 161 mathematics teachers in Norway, professional platforms were the most used digital resources by those teachers (Figure 1). When asked how often they use different resources when preparing their lessons, this was the most frequently used digital resource. Follow-up interviews were conducted to see how teachers (n=10) integrated those resources in their teaching, in particular how they used them for their lesson planning. From the interviews it appeared that a particular platform, called Campus Inkrement (<https://campus.inkrement.no>), was most commonly used. This was analyzed by the authors using the 'connectivity framework' (Gueudet et al. 2018) and associated with what teachers said, i.e. in which ways the different features of the platform were used and beneficial (or not) for teachers. This was done in order to answer the research question:

How do professional (digital) platforms support mathematics teachers in and for their teaching?

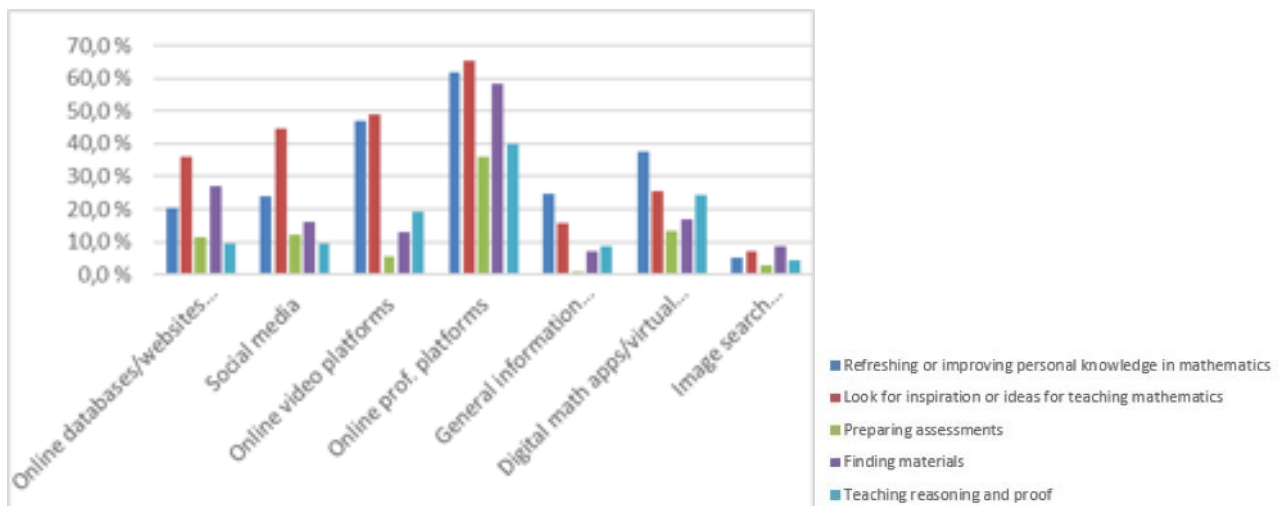


Figure 1: Percentage of respondents (n=161) reporting the use of digital resources in/for their teaching

Results

Results from the 'connectivity analysis' of the platform revealed the following features (Table 1):

Table 1: Connections at macro and micro levels

Connections (C) @ macro level (refers to the potential of linking to and between users and resources outside the platform; it includes the potential to create	C @ micro level
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virtual communities, connecting users with users (both teachers and students), as well as users and designers, and the platform's interaction with other resources, via web links, for example.) (Gueudet et al. 2018)	(refers to connections made inside the platform; it concerns the specific mathematical content, i.e., that the platform offers different kinds of combined materials (e.g., definitions, properties, exercises) and also software files, videos, etc.). (Gueudet et al. 2018)
C across grades: different topics appear again at different grades	C between different topic areas: rarely, but in videos
C to the National curriculum: explicit	C between different semiotic representations (e.g. text, figures, static, and dynamic): yes
C to assessment systems: previous national examinations	C between different software/s for carrying out a particular task: yes, e.g. Geogebra and Scratch
C to other resources: files to download	C in terms of different strategies for problem solving (e.g., procedural vs problem-solving tasks): yes, class activities and in particular the discussion feature where students discuss different answers to given tasks
C to other teacher resource systems: explicit	C in terms of different moments of appropriating a given concept (e.g., spiral progression, progressively deepening a concept instead of proposing a complete presentation of it in the same chapter): yes, e.g. functions and equations are both in year 8 and 10, with deepening of the concepts in year 10
C to other teachers for collective work: yes	C to assessments (concerning the particular mathematical content studied): yes, self-assessment, auto-assessment of tasks, examination training module, designing tests
C between teachers and students: yes	C to different students' needs: yes, tasks at different difficulty levels and the examination training module

The teachers interviewed, and in particular teacher 2, used Campus Inkrement like an e-textbook, in and out of the classroom: in the classroom for working with pupils and providing differentiated tasks adapted for individual levels, and for automatic analyzing/overseeing their results; outside the classroom for letting pupils prepare the following lesson via videos (flipped classroom) and allowing the teacher to prepare or choose tests. In particular, the videos were highly appreciated. At the same time teachers found it difficult to work on problem solving and open-ended tasks, as the automatic task grading function in Campus Inkrement does not assess these kinds of tasks. In the poster we will present quotes from teachers, how they used the platform: for example, teacher 2 used the entire platform for his teaching and lesson preparation, no other textbook was used. Another teacher, teacher 6, selected certain features of the platform and integrated them into his/her teaching.

We claim that Campus Inkrement offers teachers rich possibilities for planning their lessons (e.g. including videos), analyzing student work and keeping track of their progress, offering tasks and tests adapted to individual student needs, and providing students with opportunities for self-evaluation.

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3D printing: leveraging three approaches

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Keywords: 3D printing, mathematics, digital fabrication, spatial visualization, technology.

Introduction

The integration of 3D printers offers novel opportunities for enhancing teaching and learning, particularly in mathematics classrooms. Through our work with pre-service teachers, in-service teachers, and K-12 students, we recognize that this integration varies dramatically. In this article, we will share three key approaches to integrating 3D printing in classroom instruction, including examples of each approach for mathematics classrooms. We recognize that the integration of 3D printing technology varies dramatically. Subsequently, we encourage teachers' judicious integration in support of the curriculum standards by purposefully make choices about the use and availability of tools to enhance teaching and learning. This article is intended to serve as a resource for teachers seeking to extend their practice beyond their current integration of 3D printing to include additional approaches and increase opportunities for teaching and learning.

Approaches

In a classroom setting, the mathematics content is the focus of the activities, not 3D printing technology. Classrooms without 3D printers can still use computer aided design (CAD) programs to create models. With the lens of focus on the content through years of work with K-12 teachers, K-12 students, preservice teachers, and teacher educators, Wan and Ivy (2021) categorized activities utilizing 3D modeling and printing to enhance teaching and learning of content areas like mathematics into three approaches: machine function, teacher-created models, and student-created models.

Machine Function

“*Machine function*” approaches to activities use 3D printer parts, 3D printer construction, and 3D printing processes like servo motor movements and slicer settings to enhance content area instruction. Although it would be useful to have a 3D printer in front of the students, there are plenty of activities like comparing print volumes of 3D printers, accomplished through a web search, slicer settings through free print slicing software like CURA and Slic3r, and temperature settings for various 3D printers and filaments.

Teacher-Created Models

In the *teacher-created model* approach, teachers produce the 3D model for student use. (i) Teachers can print existing models from websites like Thingiverse, where no knowledge of CAD is required. (ii) Teachers can modify existing models with CAD programs or slicing software and print for student use. (iii) Teachers can design in a CAD program. Models that were created by teachers can then be printed for student use or students can interact with the models virtually in the CAD program.

Student-Created Models

In the *student-created model* approach, students produce 3D models, either in a CAD program or a physical print. Students may use sites like Thingiverse, like *teacher-created models* to print a model that demonstrates curriculum content. Students go through guided modeling to create models in CAD with general guidance like modeling a fidget spinner or creating models of simple machines. In design focus modeling, students create a model in CAD with criteria like modeling a fidget spinner that fits with a specific sized ball bearing or a battery-powered 3D printed scale model car.

Takeaways

In considering your commitment to enhancing teaching and learning through 3D printing, keep in mind the thoughtful and judicious integration of 3D printing. Ball and Stacey (2005) proposed that technology use should be used for “reasons that include increased opportunities for learning, increased opportunity for real-world problem solving, and orientation to the future” (p. 1). We challenge you to expand and extend your use of 3D printing in your classroom beyond the approaches you most frequently use. If you do not currently integrate 3D printing, teacher-created models are a great place to start. If you are already creating and adapting tools for your classroom, we encourage you to allow your students to create and refine their own models.

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Augmented reality vs. 3D printing technologies – A comparison of students' mathematical modelling

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Keywords: Augmented reality, 3D printing technology, mathematical modelling.

3D print and augmented reality in mathematical modelling

Mathematical modelling allows students to experience mathematics in different ways: There are unique artifacts in many places which might be a site of interest, such as landmarks, roadways, or areas (Zheng et al., 2016) where mathematics can be experienced in a real-life context. To stay up with current mobile technological breakthroughs, this connection of reality and mathematics lessons may be enhanced using Augmented Reality (AR) and 3D printing technologies (Cahyono & Lavicza, 2023). Students can use digital simulations in different stages of the modelling cycle to study the impact of changing variables in the system or environment, as well as in terms of visualization of a real situation (Greefrath & Siller, 2018). For the latter, AR technology enables computer-generated virtual image information to be layered onto a live direct or indirect real-world environment in real time (Azuma, 1997). Also, 3D printing technology as a rapid prototyping technology has made its way into education in various contexts (Ford & Minshall, 2019), besides others modelling. It opens further visualizing possibility through enabling the analyzation of 3D models, e.g., the dimensions, aspect ratios or elements involved in the model.

Research goal

Based on previous research in the scope of different visualizations of reality in modelling tasks, it could be seen that different modelling activities can be observed during the students' solution processes (Jablonski, 2023). This is especially evident in terms of the data collection and the interpretation of perspective and scale when working with these representations. Still, the results are limited to the visualizations of real objects themselves, photos and 3D printed models. So, it is possible to make assumptions about the comparison between different sizes and dimensions in the representations. Still, all visualizations are analogue – an involvement of digital visualizations and their role in the modelling process of students is still pending. It is the research goal of the planned study to focus on a comparison of the 3D printed model and a technological model realized with AR.

The pilot study

To gain first insights into the work processes of students working with a 3D print model and an augmented reality model, a pilot study with two grade six students (13 years) was conducted in January 2024. The students solved two geometric modelling tasks – one using a 3D print and the other through an augmented reality model (see Figure 1 left and middle). In the first tasks, the students were asked to determine to volume of St. Paul's church in reality. The second task asked students to determine the surface area of Candi Borobudur in reality. In both cases, the models were enriched by a cube with side lengths 1 cm to allow estimations of sizes and a scale was given. While solving the

tasks, the students were filmed by a student assistant. The videos are the basis for the data analysis in terms of the students' undertaken modelling steps and activities therein.



Figure 1: 3D print model of St. Paul's church (left), AR model of Candi Borobudur (middle) and students working with the 3D model (right)

First results and outlook

From the first analysis of data, the following main differences emerge. Concerning the students' movement, we can see that they actively manipulate the 3D print model and the cube. For example, they rotate the cube along the church tower to determine its height (cf. Figure 1 right). In addition, they use the 3D print model to draw the ground plan of the church and determine its area on a piece of sheet. In contrast, the AR model cannot be manipulated in the same way, which leads the students to change their own perspective to the object and walk around it. This resembles the research findings of Jablonski (2023) in the work on a real object outside the classroom. In addition, the relation between the object and the cube cannot be changed – therefore, data collection happens more by means of estimations, using the fingers on the screen as well as zooming. The first results are taken up and particularly analyzed in the context of modalities and embodiment during modelling. These results are – together with plans for a follow-up study with a larger sample – presented in the poster.

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PerspeCTivO – Design-based research in perspective taking and computational thinking by using the robot Ozobot

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Keywords: Perspective taking, Computational thinking, Programming robot, Ozobot

One possibility to improve the spatial reasoning seems to be programming movements of robots (Francis et al., 2021). This makes it possible to profitably combine the important issue of promoting spatial thinking with promoting computational thinking. Following the design science approach in our study we develop a learning environment for third and fourth grade primary school students (age 8 to 10) using the educational robot Ozobot and reconstruct in an interpretative approach the learning processes of the children regarding perspective taking and computational thinking. So, the study aims to develop a learning environment to enhance students' perspective taking and computational thinking skills by using the educational robot Ozobot and evaluate the Ozobot as a form of digital media in terms of perspective taking and computational thinking. It is designed in the tradition of design-based research.

Theoretical and empirical background

An important component of spatial reasoning is the ability to take perspective. To describe spatial constellations, a frame of reference must be chosen. There are three types of spatial reference systems: object-centred (intrinsic), viewer-centred (relative) or environment-centred (absolute) (Bender & Beller, 2013). For spatial perspective taking tasks, the intrinsic reference system is used, which means that the perspective of another object is taken to describe spatial constellations. Findings suggest that a common mistake of students by spatial perspective taking tasks is the using of the relative reference system (e.g. Rost, 1978).

Computational thinking involves the thought process of solving a problem. The solution is represented as computational steps and algorithms (Aho, 2012). The skill of computational thinking is fundamental for everyone (Wing, 2006) and the four key computational thinking techniques (decomposition, pattern recognition, abstraction and algorithms) have interactions to elementary mathematics (Pugalee, 2023). The use of educational robots offers potential for promoting computational thinking, because educational robots can be used as a tool to justify or disprove conjectures in problem solving activities (Yanik et al., 2017).

The empirical results therefore suggest that the promotion of perspective taking can be profitably combined with the promotion of computational thinking through the use robots. Initial considerations for learning environments already exist (Eilerts et al., 2022). However, it has not yet been systematically investigated how exactly a learning environment for inclusive primary school classes can be designed so that the children benefit on the basis of their different potentials. Also, little is known about the extent to which the findings in relation to both learning subjects are mutually dependent and complementary.

Insight in the learning environment

In essence, the learning environment is about programming paths in a city that consist of horizontal and vertical streets (Wittmann et al., 2022), to compare programmed paths and to find starting points and finishing points with given instructions in the town. Differentiated tasks encourage the children to program paths with the instructions turn right, turn left, straight ahead and stop so that the Ozobot arrives at the desired destination. By programming paths, students must choose the right instruction and must program the color code in the correct relation from the perspective of the Ozobot, so that the students (mentally) take the perspective of the Ozobot. As the Ozobot can locate in different positions on the map and can move in different directions (own view direction, sideways $90^\circ/270^\circ$, opposite 180°), the students continuously realign the intrinsic reference system during programming ways. When the Ozobot executes the program, the students receive feedback to their perspective taking. Simultaneous, determining whether or not a solution or algorithm was successful in relation to the goal(s), is one of the components of computational thinking. Integrated into mathematics lessons, thus students IT skills are enhanced.

Outlook

The evaluation of the design experiments in the teaching and learning laboratory at Paderborn University serves to elaborate students' individual conceptual understanding of perspective taking and computational thinking, evaluate the Ozobot as a form of digital media in terms of perspective taking and computational thinking and give indications for prospective planned and taught lessons to perspective taking und computational thinking in primary schools.

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From novices to navigators: the impact of micro:bit on pre-service teachers

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Keywords: Pre-service teachers, block-based programming, micro:bit, professional development.

A growing number of countries are integrating programming into their curricula, recognizing its importance across subjects. This study examines how pre-service teachers (PSTs) benefit from workshops introducing them to block-based programming with micro:bit. Further, we measure the impact of micro:bit on four dimensions: problem-solving skills, motivation, collaboration, and learning by discovery. Data were collected via online questionnaires, showing strong positive trends in these areas.

Introduction and focus

Computational thinking (CT) is a crucial skill for problem-solving and system design and is therefore an essential component of education (Wing, 2006). CT involves the decomposition of problems, pattern recognition, abstraction, algorithm design, and debugging (Hsu et al., 2018). Programming, particularly with tools like micro:bit, enhances CT and supports mathematical learning and teaching (Sacristán et al., 2022). The integration of programming into curricula on a global scale serves to highlight its importance (Kaufmann & Stenseth, 2021; Weintrop et al., 2016).

This study employed workshops at the University of Education Schwäbisch Gmünd, Germany, which were attended by 50 PSTs who were previously unfamiliar with micro:bit. The workshops were structured into three segments and were designed to guide participants from basic to advanced levels. Data was collected through pre- and post-surveys, which assessed the PSTs' perceptions of micro:bit's impact on problem-solving, motivation, discovery-based learning, and collaboration.

Findings

The findings revealed that PSTs rated micro:bit as significantly enhancing problem-solving skills. Activities such as creating LED displays, simulating dice rolls, and developing step counters required algorithmic thinking and practical programming skills. PSTs also recognized micro:bit's potential to foster these skills, which are essential for both teaching and learning in the modern educational landscape. Moreover, micro:bit was seen as a tool to increase student motivation. PSTs found the device engaging and noted its potential to inspire students to explore programming beyond the classroom. This motivation is critical for sustaining interest in programming and related fields. The workshops also emphasized teamwork, with PSTs acknowledging micro:bit's role in promoting collaborative learning environments. The activities required cooperative problem-solving, which fostered teamwork skills crucial for educational and professional success. Micro:bit facilitated hands-on, discovery-based learning. PSTs valued the opportunity to explore and experiment, finding that such approaches significantly enhanced their engagement and understanding of programming concepts. The PSTs rated micro:bit as engaging and reported an increase in their self-confidence to

teach block-based coding after the workshop. It is our contention that such experiences are necessary for PSTs to effectively pass on these skills and confidence to their future students.

Outlook

Furthermore, the integration of micro:bit in educational settings aligns with broader educational goals, such as preparing students for future careers in STEM fields. By engaging with micro:bit, PSTs not only learn programming skills but also develop critical thinking, creativity, and collaboration skills, which are transferable to various aspects of their professional lives. The workshops provided a platform for PSTs to experience the joy of learning through discovery, thereby reinforcing the value of hands-on, experiential learning in education. The positive feedback from PSTs indicates that micro:bit can be a valuable tool in modern education, facilitating the transition from theoretical knowledge to practical application. As our study indicates, when teachers are confident and motivated, they are more likely to inspire the same qualities in their students. Thus, equipping PSTs with the skills and confidence to teach programming is an essential step towards fostering a generation of learners who are proficient in CT and prepared for the challenges of the future.

In conclusion, our study suggests that micro:bit workshops may be an effective way to enhance PSTs' problem-solving abilities, motivation, collaboration, and engagement in discovery-based learning. These workshops could potentially prepare PSTs to integrate programming into their teaching, fostering their personal and professional development. By developing CT and related skills, PSTs may be better equipped to engage their future students in programming, ultimately supporting broader educational goals.

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Analysis of GPT-4's responses: an unconventional student

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Keywords: Logical thinking, artificial intelligence, mathematics education, semiotic mediation.

Introduction

In this poster, we explore the mathematical capabilities of the Large Language Model GPT-4 and its interface, the chatbot ChatGPT. Some recent works focus on answering questions as Bellettini et al. (2023) in computer science domain: they analyzed tasks with images that could be described with a text to submit to the model. Given the relevance of graphical representations in mathematics, we want to submit images themselves and discuss here, from a Mathematics education point of view: is ChatGPT able to generate a coherent, student-comparable answer when challenged with a picture based mathematical reasoning task? We compared its first answers to some tasks extracted from the Italian semi-final phase of the individual, out-of-school math competition for students "Campionati Internazionali di Giochi Matematici 2023", with the ones given by participants.

Theoretical Framework

GPT can be seen as a learner in the environment of a didactic conversation, gaining information from the question itself, from the context (the prompt) and from previous similar situations (internal weights are adapted in the neural network according to users' previous searches). We underline that the model lacks semantic (human-comparable) capacity and doesn't permanently absorb the contents provided by users; therefore, every semantic and learning operation here represents a search path in the training database and in the conversation. User's interaction with the interface produces what we can call signs (i.e. new paths in the model's "conversational memory"), like the Vygotskian notion in the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008).

The test

We chose and submitted five tasks (Tasks 5, 6, 8-10, available here https://giochimatematici.unibocconi.it/images/campionati/2023/Semifinali/Testi_Semifinali23.pdf) to ChatGPT with the prompt: "You are a participant in a math competition. You have to solve the task by looking at the picture.": we wanted the model to have information that participants get from the context, as it works better if provided with such knowledge (Marvin et al., 2024).

Table 1: GPT-4's answers and student's correct answers in tasks

	Task 5	Task 6	Task 8	Task 9	Task 10
13-15	71 (100%)	39 (55%)	8 (11%)	3 (4,2%)	39 (54%)
GPT-4	Not given	Wrong	Not given	Correct	Not given

In Table 1 we compared correct answers of 71 participants aged 13-15 with first answers of GPT, classified as correct, wrong or not given; the latter refers to GPT stopping its computation, probably

due to automatic shutdown. Pictures in the tasks include graphs (tasks 5 and 10), tokens numbered, shaped and colored in different ways (task 6), numbered tokens (task 9), sequences of digits that, when flipped, still represent a digit (task 8).

Preliminary results and analysis

In task 5, that all students got right, GPT inconsistently described connections between elements in the given graph, and then stopped computing. We did not get any attempt for tasks 10 and 8. In task 6 it extracted data from the picture coherently, but at some point, as if determined to give an answer, contradicts itself and arrives at a false conclusion. In task 9, the most difficult for students, GPT correctly identified numbers in the tokens, the problem request and how to solve it, and produced a correct calculation. According to what we observed we can state that GPT-4, when the task doesn't cause an overload, can provide an adequate answer to a picture based mathematical reasoning task: in its explanations, even the partial one in task 5, it finds the aim of the task and a sensible approach to solve it, even if incomplete and if the final computation is incorrect. On the other hand, we observed that GPT-4 makes mistakes counting and connecting elements in a picture (graph in task 5). Regarding GPT's overload, it would be appropriate to seek other strategies and perhaps to design a prompt with some hints for pictures' interpretation. These initial results raise a further research question: is GPT capable of engaging in a conversation and interacting with a user about a task as described? The chatbot offers the opportunity to give suggestions to GPT, producing new signs that can be followed by the model to change or correct its answers; besides, its tendency to accommodate the user's requests even in a nonsensical way (task 6) represents a limit for its use as a bearer of knowledge, showing similarities with students who feel obliged to provide an answer by an implicit didactic contract (Brousseau, 1986). We can say that these features reveal GPT-4's potential in terms of promotion of critical thinking, enabling the detection of mistakes and development of argumentative skills, but only if students and teachers use it with awareness.

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Use of digital learning resources in a flipped tertiary calculus class

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Keywords: College mathematics, flipped classroom, learning analytics, student use of resources, digital curriculum resources.

First year students of “service mathematics” (here: any tertiary mathematics class for students who are not majoring nor minoring in mathematics) have a plethora of resources to their avail in addition to lectures and recitations: textbooks, digital learning environments, software, videos, tutoring, etc. On the other hand, students often drop out of STEM programmes because they fail mathematics, and failure follows from mathematics being “scary”, from low perceived self-efficacy, from lower resilience to stressors by students with social disadvantage (Tariq, 2005; Schwarzer & Warner, 2013).

Inglis et al. (2011) show that — in their particular context of undergraduate multivariate calculus — students that relied mainly on online lectures (OLs) performed worse than the students who mainly attended live lectures (LLs) but better than students that did not make heavy use of any available learning resource.

Upkeeping digital resources that were created for emergency pandemic teaching is costly (in term of time or money): is it worth it? I.e., do students actually make use of them? Are they helping frail students, say those with low perceived self-efficacy or with math anxiety? Does the use of resources correlate positively with success (defined as: passing the exam within the year — students are allowed to sit an exam as many times as they wish, with our university imposing at least six dates a year)?

Digital curriculum resources

I have been teaching calculus in the “Natural Sciences” programme at the University of Milan since 2015; the programme is known for attracting students which are quite weak in mathematics. Because of pandemic teaching, 35 short (10’–18’) videos have been created to allow the implementation of the flipped classroom, with lecture time devoted in part to a formative assessment cycle based on short quizzes, and mostly to working on problems relevant to the students (Rizzo, 2023).

Besides the 35 mentioned videos used as OLs, students had access to class notes, an asynchronous version of the Wooclap quizzes used during LLs, past (unsolved and solved) tests; online exercises are being created (only exercises on differential calculus were available at the end of the term). The OL videos, besides being assigned lecture by lecture through the Learning Management System (LMS), are available on YouTube. This implies that on one side, students could reach to them at any point, even before they were assigned, on the other that the number of visualisations does not correspond to the number of visualisations by the class students.

Method

I collected the log of all activities on the LMS from the first day of class up to the day of the second high stake partial exam (which was after the end of classes), and the log of YouTube visualisations in the same time frame. Wooclap asynchronous quizzes were not considered as the number of respondents is too low ($N=5$ for most of them). Only the data from LMS carry the identity of students, so YouTube visualisations can be used only in aggregate form. Students were submitted non

anonymous Likert scale questionnaires about perceived self-efficacy, math anxiety, attitude towards the flipped classroom. Correlation between the use of curriculum resources, results in tests, and answers to a questionnaire on perceived self-efficacy, Math anxiety, attitude towards the flipped classroom was computed: the data set was too small to draw sound statistical conclusions, so data will have to be aggregated over the years.

Results

With 75 students enrolled, LL attendance drops from 65 to 28 midterm before stabilising; numbers are similar with respect to views to OLs. The sharp decrease in attendance had been observed also in previous years and, according to informal discussion with students, can be attributed both to students giving up and to fatigue. The latter can be confirmed by the distribution of views of OL during the term: while OLs were assigned for specific dates, only half the class did so for the first (out of five) part of the syllabus, but numbers were much lower for the other four parts; views rebounded in the week prior to partial tests and after the midterm (high stake) partial exam.

Correlation results hint that the availability of OLs affects students in rather different ways:

- Students who suffer from low perceived self-efficacy and high math anxiety are possibly profiting at the beginning of the class from OLs but could be led to believe that *watching* the videos is enough to get a good score for the later part of the class.
- Students who enjoy high perceived self-efficacy and low math anxiety are possibly showing a strong inverse correlation between the use of OLs and results of the first partial exam (the p -value is too high to afford more than an educated guess): possibly they underestimate the difficulty of the class, but a low result in the first partial puts them back to work, and they are able to make good use of digital resources.

Acknowledgment

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Theoretical background for studying geometric practices related to geometric construction in dynamic geometry environments

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Keywords: Geometric construction, Dynamic geometry, Geometrical practices.

Introduction

The emergence of Dynamic Geometry Environments (DGE) has led to a shift in perspective in geometry education research and geometry teaching and learning (Sinclair et al., 2017). While it has become a mainstream of research, there are still issues that have not been addressed (Sinclair et al., 2016), most notably the “little research that has focused on geometric constructions, despite their central role in geometry” (Sinclair et al., 2017, p. 480).

Due to its relevance and the reported gap in research, this study focuses on geometric construction, which we consider both a process and an object, as pointed out by Martin (1998), who refers to this duality as a geometric algorithm or a drawing that illustrates a theorem.

While dragging has been a key feature in investigations using DGE, other elements have also been recognized as important in broadening the scope and depth of investigations of geometric constructions, such as dynamic behavior, dependency relationships, construction sequence, geometric relationships, and geometric properties (Hölzl et al., 1994; Jones, 2001; Talmon & Yerushalmy, 2002).

In addition to the above, it is also necessary to move from the “typical passive emphasis on vocabulary (naming and classifying shapes according to their properties) towards a more active orientation in producing geometric meanings” (Sinclair & Bruce, 2015, p. 320) in geometry education research. Accordingly, we adopt a social theoretical approach focused on people’s practices, specifically we use our own theoretical proposal, called the Geometric Practices Model, which is based on the abstract-concrete dialectic that characterizes geometry, as described in its historical development (Ostermann & Wanner, 2012), in the method used in Euclid's Elements (Martin, 1998), in the properties of geometric objects (Laborde, 2005) and in geometric reasoning (Arzarello et al., 2002).

For geometric construction in DGE, this Model states that the practice of construction allows the transition from the abstract to the concrete, and the practice of deconstruction allows the transition from the concrete to the abstract. In general terms, practices are a set of organized doings—what people perform directly with their bodies—and are materially mediated (Schatzki, 2002).

In the case of this research, the geometric practices of construction and deconstruction are made up of a set of geometric doing associated with the elements that characterize geometric construction in DGE, which are defined as follows:

The Practice of constructing:

1. State or identify the geometric properties of the construction or diagram.
2. Establish the geometric relationships that link the objects of the construction and operationalize the geometric properties in the diagram.
3. Establish a construction sequence appropriate

The Practice of deconstructing:

1. Determine the dynamic behavior of the diagram and its constituent objects.
2. Conjecture about the dependency relationships between the objects in the diagram.
3. Conjecture a sequence of diagram

- | | |
|---|---|
| <p>to the geometric properties of the diagram.</p> <ol style="list-style-type: none"> 4. Establish dependency relationships between the objects that make up the diagram to ensure that the geometric properties are represented correctly. 5. Explore the dynamic behavior of the diagram and verify its geometric properties. | <p>construction.</p> <ol style="list-style-type: none"> 4. Infer the geometric relationships between the objects in the diagram. 5. Determine and dynamically verify the geometrical properties of the diagram. |
|---|---|

Based on these geometric practices, the present research aims to elaborate a design to study geometric doings and their organization when developing tasks on geometric constructions in DGE, which corresponds to the following stages of the research.

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From problems to progress: LLM-based individual support in digital mathematical escape games

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Keywords: LMMs, digital escape games, support-element, problem-solving.

Introduction and theoretical background

A digital escape game is an interactive online problem-solving experience where players use logic and teamwork to solve problems and complete tasks within a digital environment to "escape" from a themed scenario in a limited amount of time. Digital escape games hold many potentials to support the learning of mathematics. For example, the playful story-based approach can support motivation, engagement, immersion, and flow experience (Sidekerskienė & Damaševičius, 2023). Moreover, the integration of interactive tasks and visualizations as well as automatic feedback can support increased cognitive activation and a stronger focus on conceptual understanding (rather than mere rote learning) (Sidekerskienė & Damaševičius, 2023). However, one major challenge is to provide individual support to students in digital escape games, especially, when working on problem-solving tasks. To date, support elements in digital escape games (as in many digital learning environments in general) are usually "hard-coded" and, therefore, limited in flexibility and adaptivity (for example, pre-programmed hints that appear after a predefined time or after a certain number of failed attempts).

This research project aims to investigate how large language models (LLMs) can be used in digital escape games to offer highly individualized support for students when working on problem-solving tasks. LLMs are a kind of artificial intelligence (AI) that are trained to process and generate natural language. They are an innovative digital technology for mathematics education that offer different opportunities, for example, by using the natural language to present questions and answers associated with the mathematical challenge (Matzakos et al., 2023). A further development of LLMs is retrieval-augmented generation (RAG), an AI framework that improves the quality of LLM-generated responses by grounding on external sources of knowledge. Such RAG allows us to tailor the LLM to specific domains of functioning. However, so far, there has been little research on the design and integration of RAG-enhanced LLM-based support elements (LLM-SE) and students' use of them in digital mathematical escape games (and in mathematics education in general).

Research goal and methodology

We will develop a RAG-enhanced LLM-SE that takes the form of a game character (for example a magician) that students can interact with through text-chatting if they require assistance in their problem-solving process. We utilize a design-based research approach (Prediger et al., 2015) and address the following research questions.

1. What are relevant design principles for the LLM-SE to support the students' problem-solving processes? How can these design principles be realized in specific design elements in the digital escape game?

2. How do students use the LLM-SE in their problem-solving process while playing a digital mathematical escape game and which potentials and limitations can be identified?

We develop a digital mathematical escape game for grade 6 in the field of geometry to answer these research questions. This escape game focuses on various mathematical problems that deal with the context of symmetry (in the plane). For one of these mathematical problems, an RAG-enhanced LLM-SE (in the form of a game character) is designed according to the following initial design principles: A) “Principle of providing minimal help”: The LLM-SE should progress from motivational and more strategic support gradually to more content-specific support. B) “Multiple representations to foster conceptual focus” (Prediger et al., 2022): The LLM-SE should focus on deepening and connecting the mathematical content and the underlying concepts by offering helpful visualizations and encouraging students to make connections between them. C) “Enhancing language for developing conceptual understanding” (Prediger et al., 2022): The LLM-SE should engage students in rich discourse practices and connect different language registers, for example, academic and everyday language.

The study will utilize a clinical setting where two students will play the game together. The problem-solving process, the interaction between the students and the LLM-SE, and the communication with the LLM-SE are videographed and transcribed for the following analysis. Based on the results, we will refine and research the LLM-SE in further design cycles.

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Teaching and learning geometric constructions: a multimodal approach

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Keywords: Multimodality, artefact, instrument, geometry, mediation.

Introduction

The study presented in this text is part of an ongoing doctoral research that investigates the didactic potential underlying the use of multiple artefacts in teaching and learning geometric constructions. We hypothesize that the synergy emerging from such an articulation might influence students' comprehension and would help them to better grasp mathematical ideas. We are concerned with exploring the types of articulations, scrutinizing emerging mediations and unveiling particular inscriptions and gestures associated with geometric constructions. We illustrate our work by presenting one out of three case studies whose analysis will provide us with more focused answers to our research questions and will enrich the discussion during MEDA conference.

Theoretical frameworks

The overarching theoretical framework for our study is the instrumental approach wherein instruments occupy an intermediary mediating position between the subject and the object of his activity (Rabardel & Bourmaud, 2003). We focus on object-oriented mediations that relate to the mathematics' teacher and the students when integrating multiple artefacts within their activities. The double approach adapted to technologies (DAaT) will allow us to better grasp teachers' practices in technology-based environments (Abboud-Blanchard, 2014). As teaching and learning form a holistic unit, we examine the semiotic sets activated by the teacher and the students through the lens of the semiotic bundle (Arzarello, 2006).

Methodology

The data for the case study comes from grade 7 wild class observations in a French middle school. The math topic is "Constructing parallelograms". Seven periods were videotaped: one camera focused on the whiteboard and the teacher, Laure, and three other cameras were focused on three students. We also had an a priori and a posteriori interviews with Laure who provided us with her scope and sequence as well as her lesson plan.

Case study

We distinguish three analysis levels: macro, meso and micro. The macro level highlights the global synchronic and diachronic articulation of artefacts by Laure and her students in class and outside class over the seven periods taking into account different classroom environments that we break down into non-technological (NT), techno-neutral (TNE) and techno-operant (TOE) (see Figure 1). Our preliminary findings indicate that this articulation varies according to the *intended cognitive route* of the teacher (Abboud-Blanchard, 2014).

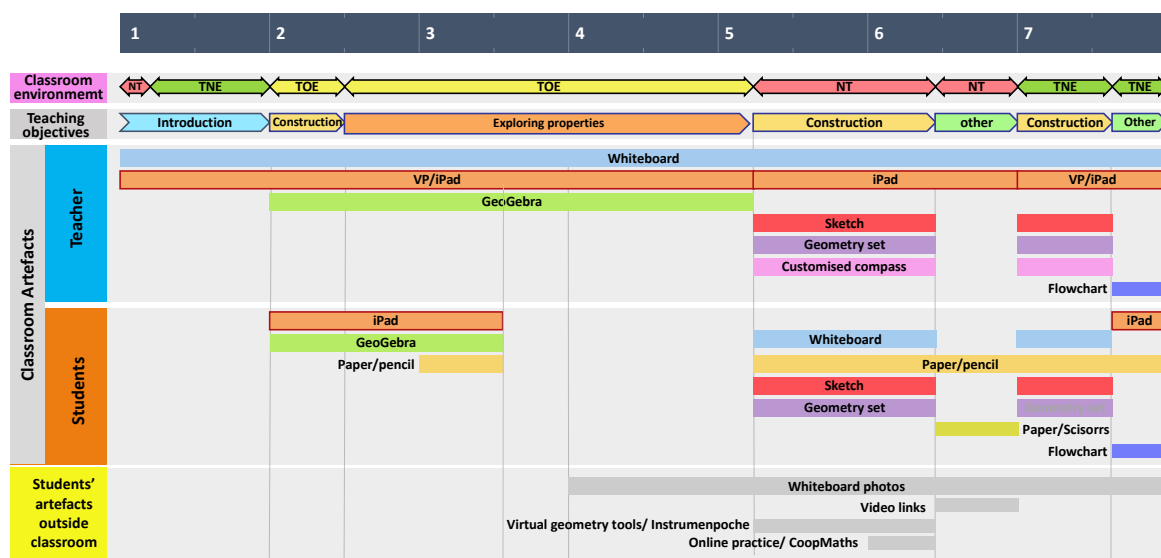


Figure 1: Macro synoptic diagram

The meso level analysis is performed over one chosen period. Every task is divided into subtasks in order to spot key moments pertinent to the fine-grained micro analysis. We examine closely different instrumental functions. We recognize a *synergic* aspect related to the multiple functions attributed to GeoGebra. We further spot a *resonance* aspect related to different subjects using different instruments simultaneously in order to achieve the same objective. Moreover, we discern some *inter-instrumental* mediations emanating from the continuous dynamic interactions between the combined use of different instruments within a subject’s activity. These mediations help to bridge the gap between traditional and virtual artefacts on one hand and physical experience and formal mathematics on the other hand.

The micro-analysis of key moments addresses the articulation of artefacts as part of the semiotic sets activated by both actors. We shed light on some difficulties encountered by students during their construction activities and on the solutions provided by the teacher. We perceive the double role of “sketching” for students as a simple inscription and as a symbolic artefact mediating their construction task. We identify a particular type of writing gestures emerging when the teacher elaborates a construction procedure and simulates the action produced by an evoked artefact to foster students’ instrumental manipulation.

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Exploring technology-enhanced learning and pedagogical practices using bar model virtual manipulatives in solving word problems

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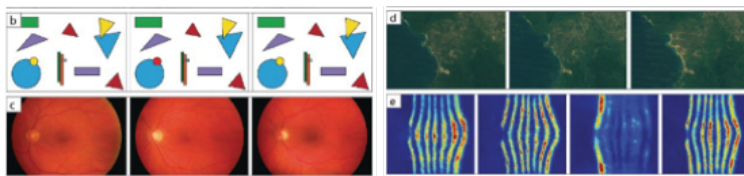
Key words: bar models, virtual manipulatives, side-by-side screen, screencast.

Introduction

Technology integration in classrooms has significantly introduced fresh possibilities for mathematics learning. We examine its impact on learning and pedagogical practices in integrated classrooms.

Theoretical Framework

Mayer’s (2002) Cognitive Theory of Multimedia Learning posits that learning is enhanced through multiple modalities, interaction, and framing questions for cognitive transfer.



Side-by-side comparison technique by Schmidt et al. (2013), Visual Analysis for Image Comparison (VAICo), offers effective data

Figure 1: Image datasets (Schmidt et al., 2013, p. 6)

analysis across domains using similarity metrics (Fig 1). Research by Rittle-Johnson et al. (2017) shows comparison boosts conceptual learning. Singapore mathematics model method, inspired by Greeno's schemas (Nesher et al., 1982; Kintsch & Greeno, 1985), uses rectangular bars to visualize mathematical relationships, aiding abstraction (Table 1).

Table 1: Comparison model (Kho et al., 2014, p. 227)

Larger quantity <input type="text"/> <input type="text"/> <input type="text"/> Smaller quantity <input type="text"/>	The larger quantity is 3 times the smaller, making the smaller one-third of the larger, e.g., if the larger is 3 units, the smaller is 1 unit, totaling 4 units, with a difference of 2 units.
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Research Questions: 1) How does using side-by-side screens support student-led learning? 2) How can teachers leverage technology to analyse students’ conceptual deficiencies?

Methods

A program piloted at a Czech elementary school involved nine Grade 8 participants using bar model virtual manipulative apps on tablets for word problems. Activities were recorded via screencast.

Results



RQ1: Side-by-side screens provide opportunity for student-led learning through discussion. Screenshots are selected using similarity metrics. Analysis used reflective prompts i.e. cognitive transfer and image comparison (Table 2a & 2b).

Figure 2: Tereza’s & Jakub’s screens

Note. From MathsBot.com (Hall, 2013)



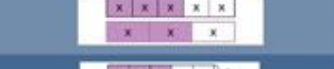
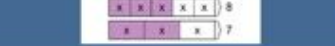
Table 2a: Analysis of students' contributions

Student contribution	Cognitive Transfer	Image Comparison
That (model) is incorrect	Critical analysis displayed	Debate over perspectives on errors
$5x$ is not equal to $3x$	Grasp of relationships	Debate on accuracy of Tereza's model.
One bar should be longer, and one bar should be shorter	Ability to evaluate visual	Discussion revealed 4 more boys. Some students adjusted constructions.

Table 2b: Analysis of teacher's contributions

Teacher contribution	Cognitive Transfer	Image Comparison
Equal number of girls in both classes	Restating problems led to clarity	Aligned model meant equal girl count
Show $\frac{3}{5}$ from G8 and $\frac{2}{3}$ from G7 are equal	Equivalent values sparked learning	Tereza's model as adjustment reference
Make rectangles equal on screen	Models convey abstract concepts	More discussions, revision of constructions

Table 3: Reflections on significant interactions

Transformations of Bar Model	Significant interactions
	Girls' count aligns with problem conditions visually
	Applied comparison model correctly
	Confusion due to 'x' notation
	Inconsistent unit sizes reveal illogical reasoning

RQ2: How can teachers leverage classroom connectivity to effectively analyse students' conceptual deficiencies in word problem solving utilising bar model virtual manipulatives? (Table 3).

Discussion and initial findings

Unobtrusive monitoring of student learning offers authenticity. Gradual screen capture showcased Jakob's conceptual potential, not only errors. Side-by-side screen prompted students' contributions.

Conclusion

Our research identified constraints in screencast scalability in larger classrooms. App's limited tools hinder students' authentic mathematical practices and full digital problem-solving expression.

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3D modelling and printing in mathematics education: Enhancing pre-service teachers' knowledge through task design

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Keywords: 3D Modelling and Printing, pre-service mathematics teachers, teacher knowledge.

Introduction

3D Modelling and Printing (3DMP) presents a novel approach to creating educational materials (Ng, 2017) and integrating them into pre-service mathematics teachers (PMTs) training. However, there is a need for research on how the integration of this technology affects the knowledge of PTMs (Ford & Minshall, 2019). As part of the initial training of mathematics teachers in Upper Austria, a course on 3DMP for PMTs (1 ECTS) was designed and implemented during two semesters in the academic year 2022/2023. The course introduced PMTs to the technological and didactical aspects of the 3DMP process and required the design of a task integrating this digital tool (DT) as a final assignment. A design-based research approach (Fowler et al., 2023) was employed to refine the course design and implementation. This practical application of 3DMP in teacher education accompanies the broader narrative on task design with DT (Günster & Weigand, 2020), emphasising the transformative potential of DT in enhancing mathematical education. By focusing on the development of innovative technological tasks, such as those involving 3DMP, teacher education programs can equip PMTs with the skills and knowledge necessary to incorporate DT into their future teaching practices effectively.

This study delves into the integration of 3DMP within mathematics teacher education, guided by the research question: *Which specific areas of the Knowledge for Teaching Mathematics with Technology are mobilised by a PMT when creating a lesson plan on functions incorporating 3DMP?* It not only aims to examine the variety and depth of knowledge that the PMT engages with and develops when designing a task with 3DMP technology but also to contribute to the refinement of our 3DMP course. Identifying the specific knowledge areas used in task creation helps to refine course content, pedagogical strategies, and the integration of other digital technologies.

Theoretical framework

The *Knowledge for Teaching Mathematics with Technology* (KTMT; Rocha, 2023) model was considered to analyse the lesson plan (LP). This model is selected for its focus on the intersection of four basic domains, delineating categories such as *mathematics and technology knowledge* (MTK) and *teaching and learning and technology knowledge* (TLTK). The MTK emphasises how DT influences mathematics, for example, by looking at the representations created by the DT and its fidelity to the mathematical objects. The TLTK focuses on how DT affects the teaching and learning process, for instance, considering the affordances that DT brings to the teaching process.

Task analysis

A LP section created by a 22-year-old female PMT in the 3DMP course was selected to illustrate the analysis. The LP dealt with power functions for a 6th-grade middle school class. It contemplated using

function stamps that the PMT modelled and 3D printed, analysing function properties, and having students create their own stamps through GeoGebra and TinkerCAD.

The MTK appears in the PMT work when she analyses the sequence of content and the representational fluency, as shown in this extract from her LP: “The pupils go through different forms of representation while working out the power functions. [...] According to Bruner’s E-I-S principle, they go through three levels: enactive, iconic and symbolic.” It is also possible to see her reflection on the issues that technology requires students to deal with, the potential of technology in teaching and learning mathematics and the lack of mathematical fidelity of the 3D printed object, showing the relevance of the TLTK and the MTK in the process of integrating 3DMP for this lesson, as it appears in the following text from her LP: “The discussion about the advantages and disadvantages of a model should show the students that a model is a simplified representation of reality. This can promote or support understanding”.

Discussion

As shown in the two extracts from the LP, the PMT’s reflections and knowledge about 3DMP in connection with teaching and learning (TLTK), power functions, and the characteristics of the printed object (MTK) allowed her to anticipate possible topics of discussion for the implementation of the lesson regarding the lack of mathematical fidelity (Rocha, 2023) and the student’s process on the task. This investigation contributes to the broader discourse on task design with DT and teachers’ professional development (Günster & Weigand, 2020). It aims to inform effective practices for integrating emerging technologies into mathematics education, thus enriching the knowledge base of mathematics teachers and the theoretical foundations of teacher programs (Ford & Minshall, 2019).

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AI digital technologies for Mathematics Education: the use of social robots for personalized learning and formative assessment

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Keywords: Artificial Intelligence, formative assessment, student feedback, educational robotics, Pepper.

Introduction

With the development of AI many aspects of daily life have been changed. Social robots are part of that development. They are tools with embodied AI capable of collecting, producing and analyzing data from the surrounding reality and interacting with it. For this reason, social robots can also be an excellent tool to be used in educational contexts (AIED, Artificial intelligence in Education), for formative assessment. Social robots, through their ability to interact with users and provide personalized responses can revolutionize the formative assessment process (Black & Wiliam, 1998). One challenging point is to use these social robots as short- and long-term observers of the learners' learning process. It is critical, therefore, to understand whether and how social robots are able to produce personalized actions for the individual learner. For example, they could offer suggestions for solving a specific task, autonomously understand when a particular skill has been achieved and, automatically, move on to the next task. Furthermore, they could use an appropriate emotional support strategy according to the user's state of mind, assist with a meta-cognitive learning strategy, decide when to take a break and encourage appropriate help-seeking behavior.

A research project

Our research project fits into the AIED paradigm. This research work seeks to make a contribution on the progress of AI (social robots) in education with the aim of improving school practices focused on assessment. Our aim is to use social robots for developing adaptive learning environments that are inclusive and effective. Some of the key points we focused on are: tasks for personalized learning; intelligent tutoring systems and tools-assistants for learners with special educational needs. In 2014, Brown and Howard used the humanoid robot DARwIn - OP (Darwin) as a SIRT (Socially Interactive Robot Tutor) in an educational setting. They analyzed learner engagement in interactive activities with Darwin compared to standard (non-interactive) methods. Their results show that the use of the social robot Darwin promoted student engagement through the use of adaptive learning techniques. Our research is still in her early stage and closely follows the work proposed by Brown and Howard (2014). The idea is to design intelligent tutoring systems that are able to monitor learners' progress. Our intent is to address both technological and pedagogical aspects in an integrative way, especially considering the pedagogical ones, that is, the relationship between the user (teacher/learner) and the

robot and how the user perceives this relationship. We are working with the social robot Pepper: it is a semi-humanoid robot designed for interacting with people. Through the operation of sensors, Pepper can interact with the user multimodally at various levels: through a tablet, using gestures, words, sounds. It can interact with users by orienting its head, it can make funny gestures with its arms, and, thanks to the presence of lights that turn on following well-defined specifications, it can provoke in the user emotions of joy, serenity, interest, motivation, confidence, enthusiasm, and amusement. Pepper's potential to provide appropriate feedback to students could be exploited. Moreover, Pepper looks very much like a person and can make 'social' movements (Brignone et al., 2021). Due to its graceful and reassuring appearance, it could provide not only situational but also emotional feedback (Lehmann & Svarny, 2021). Thus, the aim is to work with students also on affective aspects (Hannula, 2020) by exploiting Pepper's potential. From the context described, our research questions are as follows: to what extent is the robot Pepper able to identify students difficulties at an early stage and adapt the mathematical task according to their responses (RQ1); to offer personalized feedback from a formative assessment perspective (RQ2); to encourage students from an affective point of view (RQ3); to promote students' meta-cognitive reflections (RQ4). To answer these questions, we will carry out experimentations with students from different school grades. We will implement personalized mathematics sessions (learning activities) with Pepper, focusing on student interaction and learning. The sessions will be recorded using Pepper's sensor. The robot will provide dynamic tasks and continuous feedback. Some hypothetical Pepper's feedback could be: "Are you sure about your answer?"; "Could you explain your reasoning?". If Pepper notices a level of difficulty that is too high for the student, it might say: "This task was complicated for you, now I will show you an easier one". At the end of the sessions, Pepper could propose metacognitive questions, for example: "What was the key idea that enabled you to solve the task?"; "What were your difficulties?"; "How did you overcome them?". We will analyze students' behavior, emotional reactions, participation and task performance. We will engage the teachers of the classes involved. They will discuss the results with us and long-term work sessions will be set up to monitor progress and evaluate effectiveness and adapt personalized programmes.

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Beyond the screen: Types of student notes for video learning

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Keywords: Instructional mathematics videos, notetaking, mathematical notes

Introduction

In the digital age, students prefer to use educational tools that resemble their everyday lives, such as instructional videos, e.g. on YouTube (Wetzel & Ludwig, 2024). A common criticism regarding these videos is that they tend to force students into the role of passive recipients not actively constructing their own knowledge (Bersch et al., 2020). Some authors suggest that certain strategies can enable students to become more active in the process of learning with this medium such as controlling the flow of the video (rewinding, fast-forwarding, pausing), solving related tasks or taking notes (e.g. Bergmann & Sams, 2012). While some studies have investigated whether learners use these strategies, there is a lack of in-depth analyses, especially in the field of mathematics education. As a first step to address this issue, we aim to answer the following research question:

What kind of notes (if any) do students take while watching an instructional mathematics video?

Methods

To examine the effects of interactive tasks and a comprehension-oriented design of instructional mathematics videos, a pretest-posttest study was conducted with 20 classes of German ninth graders in 2023. A description of the study and the materials used as well as the results of a pilot study can be found in Wetzel and Ludwig (2024). Participating students watched a ten-minute video on the topic of systems of linear equations on an iPad with headphones. The video focuses on the meaning and determination of the solution set, the latter especially regarding the substitution method. Students were given a blank sheet of paper and instructed that they could use it to take notes while watching the video if they wished, but that there was no obligation to take notes. The sheets were collected after the students had finished watching the video. We analysed the notes of eight classes ($n = 177$) in depth. Using an inductive approach, we first identified note type categories. We then assigned each student sheet to one or more categories by dividing the note into meaningful subparts. Each subpart was then assigned exactly one note type.

Results

Of the 177 note sheets collected from the students, 76 were returned blank, so 101 (57,1%) of the students had taken notes. We deduced three main content-related categories and 14 subcategories⁴, see Table 1. The mean number of segments into which we divided a single sheet of notes was $M = 3.2$ ($SD = 2.1$, $Min = 1$, $Max = 10$). The frequencies in Table 1 indicate unique occurrences of a note type, so if more than one segment on a sheet was assigned the same type, it was counted only once. The mean number of unique types assigned per sheet is $M = 2.6$ ($SD = 1.6$, $Min = 1$, $Max = 7$). All three main types of notes occurred almost equally often. The most common types of notes were copying a calculation from the screen and summarising the relationship between the orientation of

⁴ We also deduced three off-topic subcategories such as doodling which are excluded from the analysis.

the lines of the graphs of the equations and the number of solutions, which was the central theme of the video. Reviewing the results on a class-level provides further insights: The number of students in each class who took notes varies enormously from 20.0% to 90.4% ($M = 55.3\%$, $SD = 24.2\%$). This suggests that there may be some kind of note-taking culture or other norms or rules within a class that influence the note-taking behaviour of the whole class.

Copy of content (52)	Summary of content (55)	Calculation (50)
Copy of calculation (30)	Orientation of lines in relation to number of solutions (30)	Solving an equation (21)
Copy of larger section of video (18)	Interpretation of set of solutions (26)	Solving a system of linear equations (19)
Incomplete copy of a section (17)	Summary of audio-only content (17)	Doing a solution check (16)
Copy of the steps of the method of substitution in words (7)	Own definition of a term (15)	Other kind of calculation (11)
	Own words for the steps of the method of substitution (13)	
	What is a solution check (5)	

Table 1: Deduced types of content-related notes and frequency of unique occurrences

Discussion and outlook

The presented results are preliminary findings that must be analysed in more detail in future work. Regarding the study context in which the data was collected, it will be interesting to investigate whether the amount and type of notes taken predict success in the posttest. Furthermore, given that some students received an interactive video, it will be interesting to see whether this is a predictor variable for the observed note-taking behaviour. Overall, it is a promising result that more than half of the students took notes. However, we cannot conclusively say whether this reflects students' typical video-watching behaviour or whether this stems from the formal setting of the study.

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A DecisionTheatreLab on infection disease spreading: Connecting school mathematics to societal challenges in a digital environment

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Keywords: Mathematical models, science and society, communicable diseases.

Introduction

Many students struggle to grasp the relevance of abstract mathematical concepts when they seem disconnected from real-world applications. In response, a *DecisionTheatreLab* aims to bridge this gap between theoretical mathematics education and real-world societal challenges. By immersing high school students in the exploration of pressing issues using computer-aided mathematical models and large-scale data visualizations, the program offers a three-part format comprising a Decision Theatre and supplementary workshops in mathematics and computer science. This innovative IT-supported approach not only promotes students' awareness of the impact of mathematically informed decisions but also fosters active participation. We present a practical implementation of a DecisionTheatreLab on infection spreading.

Decision Theatre: Behind the scenes of infection control

In our Decision Theatre on the spread of infections we created an interactive web application⁵ to engage students with a scientific model enabling them to discuss and eventually agree on political measures in a decision-making situation. The underlying model is the GGeoReferenced Demographic Agent-based model (GERDA) that takes into account a selection of different non-pharmaceutical interventions including various lockdown options, use of face masks, social distancing and acceptance or disobedience of measures (Goldenbogen et al., 2022).



Figure 1: decision theatre and school labs

⁵ <https://pandemie.streamlit.app>

In a first step the students gain insight into the basic mechanics and outputs of GERDA and perform a guided sensitivity analysis. Then students are allowed to simulate up to three scenarios in small groups and examine the results before deciding on infection control measures.

Supported by large-scale visualizations of simulation results of GERDA the students reconstruct and discuss the effects of different political measures on potential future evolutions. The students finally evaluate the benefits and limitations of the model against the background of a complex set of societal interests.

Mathematics and computer science school labs

The use of GERDA to support a political exchange requires mathematical competencies only for the stochastic analysis of the generated SIRD curves. To highlight the importance of mathematics for the development of such a model, the Decision Theatre is accompanied by two school labs. In a mathematics course, the students first develop a propaedeutic understanding of agent-based modelling in an interactive experiment (Figure 1, center-right). They are then instructed to develop a simple infection model based on elementary probability theory, reconstructing the essential mechanics of the professional model.

In addition to the mathematical basics, students can also gain an insight into the implementation of an agent-based model in a special computer science course. There they receive the basic scaffold of an agent-based infection model in StarLogo, which they can modify to solve various coding challenges (Figure 1, right).

In our prior work, we established a DecisionTheatreLab addressing sustainable mobility (Gürtler et al., 2023). Our ongoing efforts are dedicated to crafting distinct modules within the DecisionTheatreLab that are tailored for seamless integration into school lessons and online accessibility.

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Learning numbers, place values, and CT skills with a spreadsheet

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Keywords: computational thinking, mathematics education, spreadsheet, numbers

Introduction

According to Wing (2006), computational thinking (CT) is an essential skill that everyone should possess, just like reading, writing, and arithmetic. Additionally, there have been efforts to operationalize CT in order to make it more accessible in the classrooms (Bocconi et al., 2016). Furthermore, Bocconi et al. (2016) found that CT was taught in a variety of ways across countries; some include it as a separate subject or include it as a component of a subject.

Over the last five years, there has been an increase in research into integrating CT into mathematics education (Subramaniam et al., 2022). Subramaniam et al. (2022) discovered that coding and programming are the most popular ways to incorporate CT into mathematics lessons, and it is argued that this is the best strategy for improving CT skills while learning mathematics. We implement this strategy by programming in Microsoft Excel (MS Excel).

Ye et al. (2023) conducted a literature review that summarized general programming software like Scratch, Arduino, and Phyton, as well as specific mathematics software like GeoGebra, Geometer Sketchpad, MATLAB, and spreadsheets to enhance CT in mathematics lessons. These programs have been shown to help students learn CT and mathematics (Wang et al., 2022; Ye et al., 2023). In this study, we used MS Excel to assist students in learning CT and mathematics.

Theoretical Framework

Shute et al. (2017) proposed a CT framework that guided the development of our math CT lessons. Their framework has six facets: decomposition, abstraction, algorithm, debugging, iteration, and generalization. In addition to this framework, we utilized Kynigos' (2007) half-baked approach. It allows students to work on an incomplete design but requires them to finish it. Our spreadsheet file was already filled with formulas, but they did not perform optimally, requiring students to fix them.

Methodology

This study employed the educational design research (EDR) approach proposed by McKenney and Reeves (2018). This method is suitable for conducting developmental innovation research. We selected a math topic for junior high school students in Indonesia: numbers and place values. Prior to this study, in an interview with junior high school students, we asked them to read specific numbers, such as 723 and 794231. Our preliminary findings revealed that students struggled to read numbers and place values. As a result, we used this context to prompt students to write a program that reads numbers in Microsoft Excel. For example, if we type the number "1" into one cell, another cell will display the text "One". Students received a spreadsheet file and began working on the sheets. The file contained three sheets: Section 1, Section 2, and Section 3. Section 1 was to improve a program that

could read numbers one through nine but not four. Students had to identify and fix the problem. Later, students had to modify Section 1 to read number 10. Section 2 was similar to Section 1, but in order to improve the program, read 1 to 20, and the modulo formula " $\text{mod}(\text{number}, \text{divisor})$ " was introduced to them. Section 3 extended students' reading numbers from 1 to 30.

Report and Analysis

The report will highlight the students' performances in using the MS Excel program to read numbers. Some groups found it difficult to read the formulas. Section 1 was straightforward for students to correct the formula. Meanwhile, students struggled with sections 2 and 3.

Proposal

The poster will display the formulas that students used to complete the spreadsheet program that reads numbers from 1 to 30. Furthermore, CT facets by Shute et al. (2017), would be elaborated on what students have been working on in our math CT lesson using spreadsheets.

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