Received: 2 May 2022 | Revised: 20 May 2023 | Accepted: 11 July 2023

DOI: 10.1112/jlms.12804

RESEARCH ARTICLE

Journal of the London Mathematical Society

Derived categories of hearts on Kuznetsov components

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Funding information
Royal Society, Grant/Award Number: URF\R1\201129; PRIN 2017 Moduli; Lie Theory; Simons Collaborative, Grant/Award Number: 636187; NSF, Grant/Award Numbers: DMS-2101789, DMS-2052665

Abstract
We prove a general criterion that guarantees that an admissible subcategory $\mathcal{K}$ of the derived category of an abelian category is equivalent to the bounded derived category of the heart of a bounded t-structure. As a consequence, we show that $\mathcal{K}$ has a strongly unique dg enhancement, applying the recent results of Canonaco, Neeman, and Stellari. We apply this criterion to the Kuznetsov component $\mathcal{K}u(X)$ when X is a cubic fourfold, a GM variety, or a quartic double solid. In particular, we obtain that these Kuznetsov components have strongly unique dg enhancement and that exact equivalences of the form $\mathcal{K}u(X) \xrightarrow{\sim} \mathcal{K}u(X')$ are of Fourier–Mukai type when X, X' belong to these classes of varieties, as predicted by a conjecture of Kuznetsov.

MSC 2020
14F08, 14J30, 14J35, 14J45, 18G80 (primary)

1 | INTRODUCTION

The bounded derived category of coherent sheaves on a smooth projective variety X has a triangulated structure and encodes much information about the geometry of X. In 1997, Bondal and Orlov proved that smooth projective varieties with ample (anti)canonical bundle and equivalent bounded derived categories are isomorphic [13]. Similar reconstruction statements, called Categorical Torelli theorems, have been obtained for admissible subcategories of the bounded derived category, arising as residual components of exceptional collections in semiorthogonal decompositions, of certain Fano threefolds and fourfolds [3, 6, 10, 11, 19, 33, 46, 55] (see [54] for a survey on this topic).

It is often convenient to associate higher categorical structures to a triangulated category $\mathcal{T}$. The easiest one yields the notion of dg enhancement, which is a dg category with the same set of objects as $\mathcal{T}$ and whose homotopy category is equivalent to $\mathcal{T}$. One first advantage of passing to the dg level is that we gain a functorial notion of cone of a morphism [30, Paragraph 2.9].

Not all triangulated categories have a dg enhancement (see, e.g., [56] for counterexamples). However, if $\mathcal{A}$ is an abelian category, then an enhancement of $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ is given by the Drinfeld quotient of the dg category of bounded complexes in $\mathcal{A}$ over its full dg subcategory of acyclic complexes.

Once an enhancement exists, it is also natural to ask whether it is unique. This has been proved for $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ by Lunts and Orlov in 2009 when $\mathcal{A}$ is a Grothendieck abelian category with a small set of compact generators [45], generalized in 2015 by Canonaco and Stellari for any Grothendieck abelian category $\mathcal{A}$ in [28], and finally proved in 2018 for any abelian category by Antieau in [2]. Recently, Canonaco, Neeman, and Stellari have given a new proof of this result in [22].

The first result of this paper is a criterion that guarantees that an admissible subcategory $\mathcal{K}$ of the derived category of an abelian category is itself equivalent to the derived category of an abelian category. Clearly, [2] implies that $\mathcal{K}$ has a unique enhancement. Using the construction in [22], we can further show that $\mathcal{K}$ as in Theorem 1.1 has a unique enhancement in a strong sense (see Definition 2.2), as stated below.

Theorem 1.1 (Theorems 3.8 and 3.12). *Let $\mathcal{T}$ be the derived category of an abelian category. Assume that $\mathcal{T}$ is essentially small. Let $\mathcal{K}$ be an admissible subcategory of $\mathcal{T}$ having a stability condition $\sigma = (\mathcal{A}, Z)$, whose heart $\mathcal{A}$ is the restriction of a heart on $\mathcal{T}$ and satisfying Assumption 3.4. Then there is an exact equivalence $\mathrm{D}^{\mathrm{b}}(\mathcal{A}) \xrightarrow{\sim} \mathcal{K}$. Moreover, we have that $\mathcal{K}$ has a strongly unique enhancement.*

In the second part of this paper, we apply Theorem 1.1 to several interesting geometric examples defined over $\mathbb{C}$. The first and most famous is represented by the Kuznetsov component of a cubic fourfold $X \subset \mathbf{P}^5$, defined as the full admissible subcategory,

$$\mathcal{K}u(X) := \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^{\perp} = \{E \in \mathcal{K}u(X) \,:\, \mathrm{Hom}^{\bullet}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{O}_X(i), E) = 0 \text{ for every } i = 0, 1, 2\}$$

of $\mathrm{D}^{\mathrm{b}}(X)$, where $\mathcal{O}_X(n) := \mathcal{O}_{\mathbf{P}^5}(n)|_X$ for every $n \in \mathbb{Z}$ (see Example 2.8). We also consider the Kuznetsov components of GM varieties and of quartic double solids, which are defined analogously (see Examples 2.9, 2.10, and 2.11). We have the following main result.

Theorem 1.2 (Theorem 4.9). *Let $\mathcal{K}u(X)$ be the Kuznetsov component of a cubic fourfold or of a GM variety or of a quartic double solid defined over $\mathbb{C}$. Then there is an equivalence $\mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(\mathcal{A})$, where $\mathcal{A}$ is the heart of a stability condition on $\mathcal{K}u(X)$. Moreover, we have that $\mathcal{K}u(X)$ has a strongly unique enhancement.*

Theorem 1.2 has interesting consequences on the characterization of exact equivalences between these Kuznetsov components as functors of Fourier–Mukai type (see Definition 3.13). In fact, the most important exact functors in the geometric context are of Fourier–Mukai type. In 1996, Orlov proved that every exact fully faithful functor with adjoints between the bounded derived categories of coherent sheaves on smooth projective varieties is of Fourier–Mukai type [49]. Since then this result has been further generalized, see [23, 25, 34, 45]. In particular, the key point in [45] was showing the existence of a dg lift of the functor (see Definition 2.3) to the enhancements, which implies that it is of Fourier–Mukai type by the work of Toën [57].

In our setting, we can prove a version of Orlov's result for the studied Kuznetsov components.

Theorem 1.3. *Let X_1, X_2 be two cubic fourfolds or GM varieties of even dimension defined over $\mathbb{C}$. Then every fully faithful exact functor $\mathcal{K}u(X_1) \to \mathcal{K}u(X_2)$ is of Fourier–Mukai type.*

Theorem 1.4. *Let X_1, X_2 be two GM varieties of odd dimension or quartic double solids defined over $\mathbb{C}$. Then every exact equivalence $\mathcal{K}u(X_1) \to \mathcal{K}u(X_2)$ is of Fourier–Mukai type.*

Motivations and related works. In [38, Definition 3.1], Kuznetsov defined the notion of splitting functor that is a generalization of that of fully faithful functor. Motivated by Orlov's result, he conjectured in [38, Conjecture 3.7] that a splitting functor between bounded derived categories of coherent sheaves on smooth projective varieties is of Fourier–Mukai type. Note that equivalences among Kuznetsov components, composed with the embedding functor of the Kuznetsov component in the derived category and its left adjoint, are splitting functors. Thus, Theorems 1.3 and 1.4 prove the above-mentioned conjecture in the considered geometric cases.

In the case of the quartic double solid, it was observed in [19, Theorem 7.2] that Theorem 1.4 implies the failure of original Fano threefolds Kuznetsov's Conjecture [39, Conjecture 3.7]. Note that Fano threefolds Conjecture has been disproved in [58] and [15], independently, in a stronger sense, namely, that the Kuznetsov component of a quartic double solid is never equivalent to that of a GM threefold.

Making a speculation, there could be a connection between Theorem 1.2 and a proof of the formality conjecture for polystable objects in the Kuznetsov components of cubic fourfolds and GM varieties of even dimension. Recall that the formality conjecture, formulated for the first time by Kaledin and Lehn, states that the differential graded algebra of derived endomorphisms of polystable objects in the bounded derived category of a K3 surface is formal. In [21], the authors proved this conjecture, using Orlov's result on strong uniqueness of the enhancement. In the case of cubic fourfolds and GM varieties of even dimension, the formality conjecture follows from the general results in [29]. Nevertheless, Theorem 1.2 could be useful to provide a direct and simpler proof of this conjecture in these cases. Moreover, the description in Theorem 1.2 as the bounded derived category of a heart of a stability condition makes the Kuznetsov component much more explicit and manageable.

An interesting question arisen in [22] is whether there exist admissible subcategories of the bounded derived category of coherent sheaves on a smooth projective scheme over a field with a nonunique enhancement. Theorem 1.1 could be helpful to find an answer to this question.

Finally, we believe that Theorem 1.1 could be applied to the Kuznetsov component of a cubic threefold, although we cannot yet show this, because of the lack of a control of the semistable objects (see Remark 4.10).

Strategy of the proofs. In [7], Beilinson constructed a functor, known as realization functor, from the derived category of an abelian category in a triangulated category $\mathcal{T}$ to $\mathcal{T}$. Note that this construction makes use of the structure of filtered derived category on $\mathcal{T}$, see Remark 3.2. To prove the first part of Theorem 1.1, we show that the realization functor is an equivalence under suitable assumptions, which are listed in Assumption 3.4. To summarize, we require the existence of a stability condition on the admissible subcategory $\mathcal{K}$ with 'sufficiently many' semistable objects, namely, Assumption 3.4 (b), and whose heart has homological dimension $\leqslant 2$. Assuming this, we show that the $\mathrm{Hom}^2_{\mathcal{K}}$ between objects in $\mathcal{A}$ are generated by elements in $\mathrm{Ext}^1_{\mathcal{A}}$ through Yoneda's composition, which implies that the realization functor is an equivalence by [32, IV, Exercise 2].

The second part of Theorem 1.1 follows from the first part of Theorem 1.1, the construction in [22] of the quasi-isomorphism between the enhancements, and a criterion in [26] for the extension of isomorphisms of functors.

We prove that Assumption 3.4 holds for the Kuznesov component of a cubic fourfold and of a GM variety of even dimension using [9, 10, 51]. In the case of GM varieties of odd dimension and quartic double solids, stability conditions are known to exist by [10]. We make use of [53] and [55] to control the homological dimension of the heart, and of [52] where we show the density of the set of semistable objects. This provides the proof of Theorem 1.2.

We remark in Proposition 3.14 that the strongly uniqueness of the enhancement implies that equivalences have a dg lift, and these are of Fourier–Mukai type. This implies Theorems 1.3 and 1.4. □

Plan of the paper. In Section 2, we recollect the introductory material on enhancements, Fourier–Mukai functors, and stability conditions we need in the next, and the definitions of the Kuznetsov components of cubic fourfolds, GM varieties, and quartic double solids. In Sections 3.1 and 3.2, we prove Theorem 1.1. In Section 3.3, we explain how to deduce from the second part of Theorem 1.1 the characterization of equivalences as Fourier–Mukai functors. Section 4 is devoted to the proof of Theorems 1.2, 1.3, and 1.4.

Convention. Throughout the paper, we assume that all triangulated categories are essentially small, that is, they are equivalent to categories in which the class of objects is a set. In particular, in Section 3, we assume that $\mathcal{T}$ is essentially small.

2 | PRELIMINARIES ON DG ENHANCEMENTS, STABILITY CONDITIONS, AND KUZNETSOV COMPONENTS

In this section, we recollect some definitions and known results on dg enhancements and stability conditions. Finally, we list the examples of geometric categories to investigate in this paper.

2.1 | Enhancements and Fourier–Mukai functors

Let $\mathbb{K}$ be a field. Recall that a differential graded (dg) category is a $\mathbb{K}$-linear category $\mathcal{E}$ such that for every pair of objects $A, B \in \mathcal{E}$ the space of morphisms $\mathrm{Hom}_{\mathcal{E}}(A, B)$ has the structure of $\mathbb{Z}$-graded $\mathbb{K}$-module with differential $d: \mathrm{Hom}_{\mathcal{E}}(A, B) \to \mathrm{Hom}_{\mathcal{E}}(A, B)$ of degree 1 and such that the composition maps $\mathrm{Hom}_{\mathcal{E}}(B, C) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathcal{E}}(A, B) \to \mathrm{Hom}_{\mathcal{E}}(A, C)$ are morphisms of complexes for every $A, B, C \in \mathcal{E}$.

The homotopy category of a dg category $\mathcal{E}$, denoted by $\mathrm{H}^0(\mathcal{E})$, is the category with the same set of objects as $\mathcal{E}$ and such that $\mathrm{Hom}_{\mathrm{H}^0(\mathcal{E})}(A, B) = \mathrm{H}^0(\mathrm{Hom}_{\mathcal{E}}(A, B))$ for every $A, B \in \mathcal{E}$.

Note that if $\mathcal{E}$ is a pretriangulated dg category (see [35, Section 4.5]), then $\mathrm{H}^0(\mathcal{E})$ is triangulated. In this case, we have the following definitions.

Definition 2.1. A (dg) enhancement of a triangulated category $\mathcal{T}$ is a pair $(\mathcal{E}, \epsilon)$, where $\mathcal{E}$ is a pretriangulated dg category and $\epsilon: \mathrm{H}^0(\mathcal{E}) \to \mathcal{T}$ is an exact equivalence.

Recall that a dg functor $F: \mathcal{E} \to \mathcal{E}'$ between two dg categories $\mathcal{E}, \mathcal{E}'$ is a functor such that for every pair of objects $A, B \in \mathcal{E}$, the map $F_{A,B}: \mathrm{Hom}_{\mathcal{E}}(A, B) \to \mathrm{Hom}_{\mathcal{E}'}(F(A), F(B))$ is a morphism of complexes of $\mathbb{K}$-modules. A dg functor F is a quasi-equivalence if $F_{A,B}$ is a quasi-isomorphism for every $A, B \in \mathcal{E}$ and $\mathrm{H}^0(F)$ is an equivalence.

We denote by Hqe the localization of the category of small dg categories with respect to quasi-equivalences. Morphisms in Hqe are called quasi-functors.

Definition 2.2. A triangulated category $\mathcal{T}$ has a unique enhancement if given two enhancements $(\mathcal{E}, \epsilon), (\mathcal{E}', \epsilon')$, there exists a quasi functor $F: \mathcal{E} \to \mathcal{E}'$ such that the induced exact functor $\mathrm{H}^0(F)$ is an equivalence. We say that $\mathcal{T}$ has a strongly unique enhancement if in addition F can be chosen with the property that there is an isomorphism of functors $\epsilon' \circ \mathrm{H}^0(F) \cong \epsilon$.

Definition 2.3. Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories with enhancements $(\mathcal{E}, \epsilon)$ and $(\mathcal{E}', \epsilon')$, respectively. Let $\Phi: \mathcal{T} \to \mathcal{T}'$ be an exact functor. A quasi-functor $F: \mathcal{E} \to \mathcal{E}'$ is a dg lift of Φ if there is an isomorphism of exact functors $\Phi \cong \epsilon' \circ \mathrm{H}^0(F) \circ \epsilon^{-1}$.

Let $\Phi: \mathrm{D^b}(X) \to \mathrm{D^b}(X')$ be an exact functor between the bounded derived categories of two smooth projective $\mathbb{K}$-schemes X and X'. Recall that Φ is of Fourier–Mukai type if there exists $K \in \mathrm{D^b}(X \times X')$ and an isomorphism of functors

$$\Phi(-) \cong p'_*(K \otimes p^*(-)), \tag{1}$$

where $p: X \times X' \to X$, $p': X \times X' \to X'$ are the projections. All functors here are derived. Let $(\mathcal{E}, \epsilon)$ and $(\mathcal{E}', \epsilon')$ be enhancements of $\mathrm{D^b}(X)$ and $\mathrm{D^b}(X')$, respectively. By [47, 57] (see also [27, Proposition 6.1]), we have that $\Phi: \mathrm{D^b}(X) \to \mathrm{D^b}(X')$ is a Fourier–Mukai functor if and only if Φ has a dg lift.

We suggest the interested readers to consult the excellent survey [27] for more details and examples on these topics.

2.2 | Stability conditions on triangulated categories

Let $\mathcal{T}$ be a triangulated category. In this section, we recall the notions and first properties of stability conditions on $\mathcal{T}$, introduced by Bridgeland in [17].

Definition 2.4. A t-structure on $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ satisfying the following conditions:

1. $\mathcal{T}^{\leqslant 0} \subseteq \mathcal{T}^{\leqslant 0}[-1]$ and $\mathcal{T}^{\geqslant 0} \supseteq \mathcal{T}^{\geqslant 0}[-1]$;
2. $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for $X \in \mathcal{T}^{\leqslant 0}$, $Y \in \mathcal{T}^{\geqslant 0}[-1]$;
3. for any object $T \in \mathcal{T}$, there exists an exact triangle $X \to T \to Y \xrightarrow{+}$, where $X \in \mathcal{T}^{\leqslant 0}$ and $Y \in \mathcal{T}^{\geqslant 0}[-1]$.

The heart of a t-structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ is the full subcategory $\mathcal{A} := \mathcal{T}^{\geqslant 0} \cap \mathcal{T}^{\leqslant 0}$. A t-structure is bounded if

$$\mathcal{T} = \bigcup_{a \leqslant b} \mathcal{T}^{[a,b]},$$

where $\mathcal{T}^{[a,b]} := \mathcal{T}^{\leqslant 0}[-b] \cap \mathcal{T}^{\geqslant 0}[-a]$.

By [5], the heart of a t-structure is an abelian category. Given a t-structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$, there exist functors

$$\tau_{\leqslant a} : \mathcal{T} \to \mathcal{T}^{\leqslant 0}[-a], \quad \tau_{\geqslant a} : \mathcal{T} \to \mathcal{T}^{\geqslant 0}[-a]$$

called truncation functors, which are right adjoint and left adjoint to the inclusion functors $\mathcal{T}^{\leqslant 0}[-a] \to \mathcal{T}$ and $\mathcal{T}^{\geqslant 0}[-a] \to \mathcal{T}$, respectively. For every object $T \in \mathcal{T}$, there exists an exact triangle of the form

$$\tau_{\leqslant 0}(T) \to T \to \tau_{\geqslant 1}(T) \xrightarrow{+}$$

(see [32, IV.4.5.Lemma]).

Fix a finite rank lattice Λ with a surjective morphism $\omega : K(\mathcal{T}) \twoheadrightarrow \Lambda$, where $K(\mathcal{T})$ denotes the Grothendieck group of $\mathcal{T}$.

Definition 2.5. A stability condition (with respect to Λ) on $\mathcal{T}$ is a pair $\sigma = (\mathcal{A}, Z)$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{T}$ and $Z : \Lambda \to \mathbb{C}$ is a group morphism called central charge, satisfying the following properties:

1. For any $0 \neq E \in \mathcal{A}$, we have $\Im Z\omega(E) \geqslant 0$, and in the case that $\Im Z\omega(E) = 0$, we have $\Re Z\omega(E) < 0$ (we will write $Z(-)$ instead of $Z\omega(-)$ for simplicity).
   The slope of a nonzero object $E \in \mathcal{A}$ is defined as
$$\mu_Z(E) = \begin{cases} -\frac{\Re Z(E)}{\Im Z(E)} & \text{if } \Im Z(E) > 0, \\ +\infty & \text{otherwise.} \end{cases}$$
   An object $E \in \mathcal{T}$ is σ-(semi)stable if E is nonzero, $E[k] \in \mathcal{A}$ for some $k \in \mathbb{Z}$, and for every nonzero proper subobject $F \subset E[k]$ in $\mathcal{A}$, we have $\mu_Z(F) < (\leqslant)\ \mu_Z(E[k]/F)$.
2. Every object of $E \in \mathcal{A}$ has a unique filtration
$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots E_{m-1} \hookrightarrow E_m = E,$$
   where $A_i := E_i/E_{i-1}$ is σ-semistable and $\mu_Z^+(E) := \mu_Z(A_1) > \ldots > \mu_Z(A_m) =: \mu_Z^-(E)$.
3. (Support Property) There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that the restriction of Q to the kernel of Z is negative definite and $Q(E) \geqslant 0$ for all σ-semistable objects E in $\mathcal{A}$.

The objects A_i in Definition 2.5 are called Harder–Narasimhan factors of E.

Given a stability condition $\sigma = (\mathcal{A}, Z)$ on $\mathcal{T}$, we can associate a slicing as follows. Recall that the phase of a nonzero object $E \in \mathcal{A}$ is

$$\phi(E) = \begin{cases} \frac{1}{\pi}\mathrm{Arg}(Z(E)) & \text{if } \Im Z(E) > 0, \\ 1 & \text{otherwise.} \end{cases}$$

If $F = E[k]$ for $E \in \mathcal{A}$, then $\phi(F) = \phi(E) + k$. We define the collection $\mathcal{P}_\sigma = \{\mathcal{P}_\sigma(\phi)\}$ of full additive subcategories $\mathcal{P}_\sigma(\phi) \subset \mathcal{T}$ for $\phi \in \mathbb{R}$ such that:

1. if $\phi \in (0,1]$, the subcategory $\mathcal{P}_\sigma(\phi)$ is the union of the zero object and all σ-semistable objects with phase ϕ;
2. for $\phi + n$ with $\phi \in (0,1]$ and $n \in \mathbb{Z}$, set $\mathcal{P}_\sigma(\phi + n) := \mathcal{P}_\sigma(\phi)[n]$.

We write $\mathcal{P}_\sigma(I)$, where $I \subset \mathbb{R}$ is an interval, to denote the extension-closed subcategory of $\mathcal{T}$ generated by the subcategories $\mathcal{P}_\sigma(\phi)$ with $\phi \in I$. Note that $\mathcal{P}_\sigma((0,1]) = \mathcal{A}$.

Note that $\mathcal{P}_\sigma(\phi)$ has finite length for every $\phi \in \mathbb{R}$.[†] In particular, every object $E \in \mathcal{P}_\sigma(\phi)$ has a (nonunique) finite filtration in σ-stable objects of the same phase ϕ, which are called Jordan–Hölder factors.

Now let X be a smooth projective variety defined over the field of complex numbers $\mathbb{C}$. Assume that $\mathcal{T}$ is a full admissible subcategory of the bounded derived category $\mathrm{D^b}(X)$, in other words, the inclusion functor $\mathcal{T} \to \mathrm{D^b}(X)$ is fully faithful and has left and right adjoint. The Grothendieck group $\mathrm{K}_0(\mathcal{T})$ comes equipped with a well-defined Euler pairing $\chi : \mathrm{K}_0(\mathcal{T}) \times \mathrm{K}_0(\mathcal{T}) \to \mathbb{Z}$ defined as follows:

$$([E],[F]) := \sum_{n\in\mathbb{Z}} (-1)^n \dim \operatorname{Hom}(E, F[n]).$$

The numerical Grothendieck group $\mathrm{K}_{\mathrm{num}}(\mathcal{T}) := \mathrm{K}_0(\mathcal{T})/\ker(\chi)$ is isomorphic to a subgroup of $\mathrm{K}_{\mathrm{num}}(X)$ that is a finitely generated free abelian group. Let $\sigma = (\mathcal{A}, Z)$ be a stability condition on $\mathcal{T}$ with respect to the numerical Grothendieck group $\mathrm{K}_{\mathrm{num}}(\mathcal{T})$ of $\mathcal{T}$. For $v \in \mathrm{K}_{\mathrm{num}}(\mathcal{T})$, consider the functor

$$\mathcal{M}_\sigma(\mathcal{T}, v) \colon (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{Gpd}$$

from the category of schemes over $\mathbb{C}$ to the category of groupoids, which associates to a scheme S the groupoid $\mathcal{M}_\sigma(\mathcal{T}, v)(S)$ of all perfect complexes $E \in \mathrm{D}(X \times S)$ such that, for every $s \in S$, the restriction E_s of E to the fiber $X \times \{s\}$ belongs to $\mathcal{T}$, is σ-semistable of phase ϕ and $\mathbf{v}(E_s) = v$. In the examples we will consider in this paper, the functor $\mathcal{M}_\sigma(\mathcal{T}, v)$ admits a good moduli space $M_\sigma(\mathcal{T}, v)$, in the sense of [1], which is a proper algebraic space over $\mathbb{C}$. We will denote by $M^s_\sigma(\mathcal{T}, v)$ the locus of classes of σ-stable objects in $M_\sigma(\mathcal{T}, v)$.

Denote by $\operatorname{Stab}(\mathcal{T})$ the set of stability conditions on $\mathcal{T}$ with respect to $\mathrm{K}_{\mathrm{num}}(\mathcal{T})$. By Bridgeland deformation theorem [17], the set $\operatorname{Stab}(\mathcal{T})$ (given that it is nonempty) has the structure of complex manifold of dimension equal to the rank of $\mathrm{K}_{\mathrm{num}}(\mathcal{T})$.

Denote by $\mathrm{GL}^+_2(\mathbb{R}) := \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det(g) > 0\}$. Let $\widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ be the universal cover of $\mathrm{GL}^+_2(\mathbb{R})$. We have the following right group action of $\widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ on $\operatorname{Stab}(\mathcal{T})$. Given $\widetilde{g} = (g, M) \in \widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ with $M \in \mathrm{GL}^+_2(\mathbb{R})$ and $g : \mathbb{R} \to \mathbb{R}$ increasing with $g(\phi + 1) = g(\phi) + 1$, the action on $\sigma = (\mathcal{P}_\sigma((0,1]), Z) \in \operatorname{Stab}(\mathcal{T})$ is given by

$$\sigma \cdot \widetilde{g} = (\mathcal{P}_\sigma((g(0), g(1)]), M^{-1} \circ Z).$$

In particular, σ and $\sigma \cdot \widetilde{g}$ have the same set of semistable objects but with different phases.

[†] See [12, Lemma A.4] for the equivalent definitions of support property. Definition A.2 in [12] implies that the slice is Artinian directly.

2.3 | Semiorthogonal decompositions and Kuznetsov components

Let $\mathcal{T}$ be a $\mathbb{K}$-linear triangulated category, where $\mathbb{K}$ is a field.

Definition 2.6. A semiorthogonal decomposition for $\mathcal{T}$, denoted by $\mathcal{T} = \langle \mathcal{T}_1, \dots, \mathcal{T}_m \rangle$, is a sequence of full triangulated subcategories $\mathcal{T}_1, \dots, \mathcal{T}_m$ of $\mathcal{T}$ such that:

1. $\mathrm{Hom}_{\mathcal{T}}(E, F) = 0$, for all $E \in \mathcal{T}_i$, $F \in \mathcal{T}_j$ and $i > j$;
2. for any $E \in \mathcal{T}$, there is a sequence of morphisms

$$0 = E_m \to E_{m-1} \to \cdots \to E_1 \to E_0 = E,$$

such that $\mathrm{Cone}(E_i \to E_{i-1}) \in \mathcal{T}_i$ for $1 \leqslant i \leqslant m$.

Definition 2.7. An object $E \in \mathcal{T}$ is exceptional if $\mathrm{Hom}_{\mathcal{T}}(E, E[k]) = 0$ for all integers $k \neq 0$, and $\mathrm{Hom}_{\mathcal{T}}(E, E) \cong \mathbb{K}$. An exceptional collection is a collection of objects $E_1, \dots, E_m$ in $\mathcal{T}$ such that E_i is an exceptional object for all i, and $\mathrm{Hom}_{\mathcal{T}}(E_i, E_j[k]) = 0$ for all k and $i > j$.

Assume that $\mathcal{T}$ is a proper $\mathbb{K}$-linear triangulated category, that is, for every $A, B \in \mathcal{T}$, the vector space $\bigoplus_i \mathrm{Hom}(A, B[i])$ is finite-dimensional. Given an exceptional collection $E_1, \dots, E_m$ in $\mathcal{T}$, we have the semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{K}, E_1, \dots, E_m \rangle,$$

where $\mathcal{K} := \langle E_1, \dots, E_m \rangle^{\perp} = \{F \in \mathcal{T} \,:\, \mathrm{Hom}_{\mathcal{T}}(E_i, F) = 0 \text{ for all } i = 1, \dots, m\}$.

We now recall some explicit examples of semiorthogonal decompositions associated to exceptional collections, which define the Kuznetsov components we will consider in the next. In all of them, we assume that X is a variety defined over $\mathbb{C}$.[†]

Example 2.8. Let $X \subset \mathbb{P}^5$ be a cubic fourfold, in other words, a smooth cubic hypersurface in $\mathbb{P}^5$. By [40], the bounded derived category of X has a semiorthogonal decomposition of the form

$$\mathrm{D^b}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle,$$

where $\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)$ are an exceptional collection of line bundles on X and

$$\mathcal{K}u(X) := \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^{\perp}$$

is known as the Kuznetsov component of X. The Serre functor of $\mathcal{K}u(X)$ satisfies $S_{\mathcal{K}u(X)} \cong [2]$. In addition, $\mathcal{K}u(X)$ has Hochschild cohomology isomorphic to that of the bounded derived category of a K3 surface [42, Proposition 4.1]. For these reasons, we say that $\mathcal{K}u(X)$ is a noncommutative K3 surface. Stability conditions on $\mathcal{K}u(X)$ have been constructed in [10] and the associated moduli spaces of stable objects have been studied in [9].

[†] The following examples can be stated more generally over an algebraically closed field of characteristic 0 or large enough positive characteristic. However, we need to work over $\mathbb{C}$ to have the results on moduli spaces which we will use in Section 4.

Example 2.9. The second example of noncommutative K3 surface is given by the Kuznetsov component of a GM variety of even dimension. Recall that a GM variety of dimension $2 \leqslant n \leqslant 6$ is a smooth intersection of the form

$$X = \mathrm{CG}(2,5) \cap Q \subset \mathbb{P}^{10},$$

where $\mathrm{CG}(2,5)$ denotes the cone over the Grassmannian $\mathrm{G}(2,5)$ embedded via the Plücker embedding in a 10-dimensional projective space $\mathbb{P}^{10}$, and Q is a quadric hypersurface in a projective space $\mathbb{P}^{n+4} \subset \mathbb{P}^{10}$ of dimension $n+4$. By [36], the bounded derived category of X has the semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{U}_X^{\vee}, \dots, \mathcal{O}_X(n-3), \mathcal{U}_X^{\vee}(n-3)\rangle.$$

Here, $\mathcal{U}_X^{\vee}$ denotes the pullback of the dual of the rank-two tautological bundle on the Grassmannian. If n is even, then the Serre functor of $\mathcal{K}u(X)$ is isomorphic to the homological shift [2]. Moreover, stability conditions and their related moduli spaces have been constructed and studied in [51].

Example 2.10. We can also consider the Kuznetsov component of a GM variety X of odd dimension, namely, a GM threefold or fivefold. In this case, the Serre functor of $\mathcal{K}u(X)$ is the composition of an involutive autoequivalence and the homological shift by 2, see [36]. We call $\mathcal{K}u(X)$ a 2-Enriques category, or simply Enriques category, see [44, Definition 4.2] for more details. Stability conditions on $\mathcal{K}u(X)$ have been constructed in [10, Section 6].

Example 2.11. Another example of Enriques category is given by the Kuznetsov component of a quartic double solid X, which is the double cover of $\mathbb{P}^3$ ramified in a smooth quartic surface. By [39, Corollary 3.5], there is a semiorthogonal decomposition of the form

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{K}u(X), \mathcal{O}_X, \mathcal{O}_X(1)\rangle,$$

where $\mathcal{K}u(X)$ is the Kuznetsov component. Its Serre functor is the composition of an involutive autoequivalence and the homological shift by 2 by [43, Corollary 4.6]. Again, stability conditions on $\mathcal{K}u(X)$ have been constructed in [10, Section 6].

3 | PROOF OF THE GENERAL RESULTS

In this section, we prove Theorem 1.1 that is split in Theorems 3.8 and 3.12.

3.1 | Admissible subcategories and hearts

Let $\mathcal{T}$ be the derived category of an abelian category, and let $\mathcal{K}$ be an admissible subcategory of $\mathcal{T}$. Suppose that there exists a heart $\mathcal{A}_{\mathcal{T}}$ of a bounded t-structure on $\mathcal{T}$, and the intersection of $\mathcal{A}_{\mathcal{T}}$ with $\mathcal{K}$ is a heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{K}$. We denote by $(\mathcal{K}^{\leqslant 0}, \mathcal{K}^{\geqslant 0})$ the bounded t-structure on $\mathcal{K}$ whose heart is $\mathcal{A}$.

The following lemma is a direct consequence of [7].

Lemma 3.1. *There exists a t-exact functor $F: \mathrm{D^b}(\mathcal{A}) \to \mathcal{K}$, whose restriction to the heart $\mathcal{A} \subset \mathrm{D^b}(\mathcal{A})$ is identity to the heart $\mathcal{A} \subset \mathcal{K}$.*

Proof. Since $\mathcal{T}$ is the derived category of an abelian category, there exists a filtered derived category over $\mathcal{T}$. Then by [7, Statement A 6] (see [5, Proposition 3.1.10] for the proof), there is an exact functor $F_{\mathcal{T}}: \mathrm{D^b}(\mathcal{A}_{\mathcal{T}}) \to \mathcal{T}$, which is t-exact with respect to the standard t-structure on $\mathrm{D^b}(\mathcal{A}_{\mathcal{T}})$ and that defining the heart $\mathcal{A}_{\mathcal{T}}$ in $\mathcal{T}$, whose restriction to $\mathcal{A}_{\mathcal{T}}$ is the identity.

Now the inclusion $\mathcal{A} \subset \mathcal{A}_{\mathcal{T}}$ induces a natural exact functor $G: \mathrm{D^b}(\mathcal{A}) \to \mathrm{D^b}(\mathcal{A}_{\mathcal{T}})$. Since by definition $\mathcal{A} = \mathcal{A}_{\mathcal{T}} \cap \mathcal{K}$, the composition $F := F_{\mathcal{T}} \circ G: \mathrm{D^b}(\mathcal{A}) \to \mathcal{T}$ is t-exact and is the identity on $\mathcal{A}$.

It remains to show that F factors through $\mathcal{K}$. Equivalently, we show that $\mathrm{R}_{\mathcal{K}} \circ F = 0$, where $\mathrm{R}_{\mathcal{K}}$ is the right mutation functor with respect to $\mathcal{K}$. Note that if $A \in \mathcal{A}$, then $\mathrm{R}_{\mathcal{K}} F(A) = \mathrm{R}_{\mathcal{K}}(A) = 0$, as F is the identity on $\mathcal{A}$ and $A \in \mathcal{K}$. Since $\mathcal{A}$ is the heart of a bounded t-structure on $\mathrm{D^b}(\mathcal{A})$, this implies that $\mathrm{R}_{\mathcal{K}} \circ F = 0$ as we wanted. □

Remark 3.2. The assumption that $\mathcal{T}$ is the derived category of an abelian category is used to ensure the existence of a filtered derived category over it, which allows constructing the functor using [7]. Alternatively, we can assume that $\mathcal{T}$ is the homotopy category of a stable ∞-category to obtain a similar result.

To see when the functor F is an equivalence, we will use the following lemma, which is well known to the experts, also see in [32, IV.4, Exercise 2, p. 286].

Lemma 3.3. *The functor $F: \mathrm{D^b}(\mathcal{A}) \to \mathcal{K}$ constructed in Lemma 3.1 is an equivalence if and only if for any two objects A, B in $\mathcal{A}$, and any morphism $f \in \mathrm{Hom}_{\mathcal{K}}(A, B[n])$ for $n \geqslant 2$, there exist objects $A_0 = A$, A_1, $A_2, \dots, A_n = B$ in $\mathcal{A}$, and morphisms $f_i \in \mathrm{Hom}_{\mathcal{K}}(A_{i-1}, A_i[1])$ for $i = 1, 2, \dots, n$, such that f is the composition of the f_i's.*

Proof. We outline the proof for the sake of completeness. Assume that F is an equivalence. Then F induces an isomorphism

$$F^n_{A,B}: \mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(A, B[n]) \cong \mathrm{Hom}_{\mathcal{K}}(A, B[n])$$

for every pair of objects A, B in $\mathrm{D^b}(\mathcal{A})$, $n \in \mathbb{Z}$. If $A, B \in \mathcal{A}$, then by definition,

$$\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(A, B[n]) = \mathrm{Ext}^n_{\mathcal{A}}(A, B)$$

and by Yoneda interpretation (see [32, III.5.Theorem (c)]), every extension $f' \in \mathrm{Ext}^n_{\mathcal{A}}(A, B)$ is generated by extensions in $\mathrm{Ext}^1_{\mathcal{A}}$, in other words, there exist objects $A_0 = A, A_1, A_2, \dots, A_n = B$ in $\mathcal{A}$, and morphisms $f'_i \in \mathrm{Ext}^1_{\mathcal{A}}(A_{i-1}, A_i)$ for $i = 1, 2, \dots, n$, such that

$$f' = f'_n[n-1] \circ \dots f'_2[1] \circ f'_1.$$

Thus, for every $f \in \mathrm{Hom}_{\mathcal{K}}(A, B[n])$, there exists $f' \in \mathrm{Ext}^n_{\mathcal{A}}(A, B)$ such that $f = F^n_{A,B}(f')$ and f' is the composition of extensions f'_i as above. Setting $f_i := F^1_{A,B}(f'_i)$, since F is a functor, we have

$$f = F^n_{A,B}(f') = F^1_{A,B}(f'_n)[n-1] \circ \dots F^1_{A,B}(f'_2)[1] \circ F^1_{A,B}(f'_1) = f_n[n-1] \circ \dots f_2[1] \circ f_1,$$

where $f_i \in \mathrm{Hom}_{\mathcal{K}}(A_{i-1}, A_i[i])$. This proves the first implication.

On the other hand, assume the second condition holds. We first show that F is fully faithful. Indeed, by definition of $\mathrm{D^b}(\mathcal{A})$, it is enough to show that $F^n_{A,B}$ is an isomorphism for every A, B in $\mathcal{A}$, $n \in \mathbb{Z}$. Note that

$$\mathrm{Hom}_{\mathcal{K}}(A, B[-n]) = 0 = \mathrm{Ext}^{-n}_{\mathcal{A}}(A, B) \text{ for } n > 0$$

since $\mathcal{A}$ is the heart of a bounded t-structure, and

$$\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(A, B) \cong \mathrm{Hom}_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{\mathcal{K}}(A, B)$$

since $\mathcal{A}$ is a full subcategory of both $\mathrm{D^b}(\mathcal{A})$ and $\mathcal{K}$. Now note that every $f \in \mathrm{Hom}_{\mathcal{K}}(A, B[1])$ corresponds to a triangle

$$A \xrightarrow{f} B[1] \to \mathrm{Cone}(f) \xrightarrow{+} .$$

Then $C := \mathrm{Cone}(f)[-1]$ is in $\mathcal{A}$, since A and B are. Thus, C corresponds to the extension

$$0 \to B \to C \to A \to 0$$

in $\mathcal{A}$ and defines $f' \in \mathrm{Ext}^1_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(A, B[1])$. Since F is the identity on $\mathcal{A}$, it follows that $F^1_{A,B}(f') = f$. If $n \geqslant 2$, by assumption, every $f \in \mathrm{Hom}_{\mathcal{K}}(A, B[n])$ can be written as a composition

$$f = f_n[n-1] \circ \dots f_2[1] \circ f_1$$

with $f_i \in \mathrm{Hom}_{\mathcal{K}}(A_{i-1}, A_i[1])$ and $A_0 = A$, A_1, $A_2, \dots, A_n = B$ in $\mathcal{A}$. Since F induces an identification $\mathrm{Hom}_{\mathcal{K}}(A_{i-1}, A_i[1]) \cong \mathrm{Ext}^1_{\mathcal{A}}(A_{i-1}, A_i)$, there exists $f'_i \in \mathrm{Ext}^1_{\mathcal{A}}(A_{i-1}, A_i)$ such that $f_i = F^1_{A,B}(f'_i)$. Setting $f' := f'_n[n-1] \circ \dots f'_2[1] \circ f'_1$, we have

$$F^n_{A,B}(f') = F^1_{A,B}(f'_n)[n-1] \circ \dots F^1_{A,B}(f'_2)[1] \circ F^1_{A,B}(f'_1) = f_n[n-1] \circ \dots f_2[1] \circ f_1 = f.$$

This shows that

$$F^n_{A,B} : \ \mathrm{Ext}^n_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(A, B[n]) \to \mathrm{Hom}_{\mathcal{K}}(A, B[n])$$

is surjective for every n.

Note that $F^n_{A,B}$ is also injective. To show this, we argue by induction on n. The case $n \leqslant 1$ has already been shown. Let $n \geqslant 2$ and assume that $F^m_{A,B}$ is injective for every $m < n$. Let $f' \in \mathrm{Ext}^n_{\mathcal{A}}(A, B)$ such that $F^n_{A,B}(f') = 0$. By Yoneda interpretation, f' is of the form

$$f' = f'_2[1] \circ f'_1 : \ A \to A_1[1] \to B[n],$$

where $f'_1 \in \mathrm{Ext}^1_{\mathcal{A}}(A, A_1)$, $A_1 \in \mathcal{A}$ and $f'_2 \in \mathrm{Ext}^{n-1}_{\mathcal{A}}(A_1, B)$. Set $K := \mathrm{Cone}(f'_1)[-1]$. Then we have the short exact sequence

$$0 \to A_1 \xrightarrow{a} K \to A \to 0$$

in $\mathcal{A}$ and the exact triangle

$$A \xrightarrow{f'_1} A_1[1] \xrightarrow{a[1]} K[1] \xrightarrow{+}$$

in $\mathrm{D^b}(\mathcal{A})$. Then, we have

$$0 = F^n_{A,B}(f') = F^n_{A,B}(f'_2[1] \circ f'_1) = F^{n-1}_{A,B}(f'_2)[1] \circ F^1_{A,B}(f'_1).$$

It follows that $F^{n-1}_{A,B}(f'_2)[1]$ lifts to a morphism $g[1] \in \mathrm{Hom}_{\mathcal{K}}(K[1], B[n])$ such that

$$F^{n-1}_{A,B}(f'_2)[1] = g[1] \circ F^0_{A,B}(a)[1].$$

Since $F^{n-1}_{A,B}$ is surjective, there exists $g' \in \mathrm{Ext}^{n-1}_{\mathcal{A}}(K, B)$ such that $F^{n-1}_{A,B}(g') = g$. It follows that

$$F^{n-1}_{A,B}(f'_2)[1] = F^{n-1}_{A,B}(g')[1] \circ F^0_{A,B}(a)[1] = F^{n-1}_{A,B}(g' \circ a)[1].$$

By induction hypothesis, the map $F^{n-1}_{A,B}$ is injective, so we deduce that $f'_2 = g' \circ a$. Then, we have

$$f' = f'_2[1] \circ f'_1 = g'[1] \circ a[1] \circ f'_1 = 0$$

since $a[1] \circ f'_1 = 0$. We conclude that $F^n_{A,B}$ is an isomorphism for every n, and thus, F is fully faithful.

We now show that F is essentially surjective. We argue as in [5, Section 3.1.15]. By definition of bounded t-structure, we have

$$\mathcal{K} = \bigcup_{a \leqslant b} \mathcal{K}^{[a,b]},$$

thus if $K \in \mathcal{K}$, then $K \in \mathcal{K}^{[a,b]}$ for some $a \leqslant b$. We argue by induction on $l = b - a \geqslant 0$. If $l = 0$, then $K = A[a]$ for some $A \in \mathcal{A}$. Since F is the identity on $\mathcal{A}$, the object K is in the essential image of F. If $l \geqslant 1$, assume that the statement holds for every nonnegative integer $< l$. Take $a \leqslant c < b$ and consider the truncation functors $\tau_{\leqslant c}$ and $\tau_{>c}$. Then we have the triangle

$$\tau_{\leqslant c}K \to K \to \tau_{>c}K \xrightarrow{f} \tau_{\leqslant c}K[1].$$

By the induction hypothesis, there exist $K_1, K_2 \in \mathrm{D^b}(\mathcal{A})$ such that $F(K_1) = \tau_{\leqslant c}K$ and $F(K_2) = \tau_{>c}K$. Since F is fully faithful, there exists $f' : K_2 \to K_1[1]$ such that $F(f') \cong f$. Then applying F to the triangle

$$\mathrm{Cone}(f')[-1] \to K_2 \xrightarrow{f'} K_1[1] \xrightarrow{+},$$

we get the commutative diagram

$$\begin{tikzcd} F(\mathrm{Cone}(f')[-1]) \arrow[r] \arrow[d, dashed] & F(K_2) \arrow[r, "F(f')"] \arrow[d, "\cong"] & F(K_1)[1] \arrow[r, "+"] \arrow[d, "\cong"] & {} \\ K \arrow[r] & \tau_{>c}K \arrow[r, "f"] & \tau_{\leq c}K[1] \arrow[r, "+"] & {}. \end{tikzcd}$$

By Axiom TR3 of triangulated categories, we have an induced morphism $F(\mathrm{Cone}(f')[-1]) \to K$, which is an isomorphism. We conclude that F is essentially surjective, and thus, F is an equivalence as we wanted. □

The key observation is that to ensure the condition in Lemma 3.3, it suffices to have a stability condition on $\mathcal{K}$ with certain special properties. More precisely, suppose that there exists a stability condition σ on $\mathcal{K}$ with heart $\mathcal{A}$. Denote by Z the central charge of σ and by μ the associated slope. Further assume the following holds for σ:

Assumption 3.4.

(a) The image of the central charge $Z : \mathrm{K}_{\mathrm{num}}(\mathcal{K}) \to \mathbb{C}$ is discrete.
(b) For every nonzero object E in $\mathcal{A}$ and every real number s_0, there exists a σ-stable object F in $\mathcal{A}$ satisfying $\mu(F) < s_0$ and $\mathrm{Hom}_{\mathcal{K}}(F, E) \neq 0$.
(c) For any σ-stable objects E and F in $\mathcal{A}$, we have $\mathrm{Hom}_{\mathcal{K}}(E, F[m]) = 0$ for $m \geqslant 3$. If $\mu(E) < \mu(F)$ in addition, then we have $\mathrm{Hom}_{\mathcal{K}}(E, F[2]) = 0$.

We will write $Z = -\deg + i\mathrm{rk}$. Denote by $\delta_0(\sigma) := \inf\{\mathrm{rk}(E) | E \in \mathcal{K}, \mathrm{rk}(E) > 0\}$. Note that when $\delta_0(\sigma) \neq 0$, the image of rk in $\mathbb{R}$ consists of integral multiples of $\delta_0(\sigma)$, hence discrete.

We say that a stability condition σ satisfies the Assumption 3.4 if

- $\delta_0(\sigma) \neq 0$ and σ satisfies (a-c) as above;
- or $\delta_0(\sigma) = 0$ and there exists an open neighborhood U in $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ of the lift of the identity such that $\sigma \cdot \tilde{g}$ satisfies (a–c) for every $\tilde{g} \in U$.

Our goal is to show that under the above assumptions, the condition in Lemma 3.3 is satisfied, and the functor $F : \mathrm{D^b}(\mathcal{A}) \to \mathcal{K}$ is an equivalence.

Lemma 3.5. *Let σ be a stability condition on $\mathcal{K}$ satisfying the Assumption 3.4. Then for every nonzero object A in $\mathcal{A}$ and every real number s, there exists an object C in $\mathcal{A}$ satisfying:*

1. *$\mu^+(C) < s$;*
2. *there exists a surjective morphism $C \to A$ in $\mathcal{A}$.*

We first prove a lemma that will be used twice in the proof of Lemma 3.5.

Lemma 3.6. *Let σ be a stability condition on $\mathcal{K}$ satisfying the Assumption 3.4. Let A_1 be a subobject of A in $\mathcal{A}$ and denote the quotient by $A_2 = A/A_1$. Then for every real number s, if the statement in Lemma 3.5 holds for A_1 with respect to s and for A_2 with respect to all real numbers, then it holds for A with respect to s.*

Proof. By assumptions, there is an object C_1 with $\mu^+(C_1) < s$ and a surjective morphism $f_1 : C_1 \to A_1$ in $\mathcal{A}$. Denote by $K_1 := \ker f_1$. Then in particular, $\mu^+(K_1) < s$ as well.

By assumptions, there exists C_2 in $\mathcal{A}$ with $\mu^+(C_2) < \mu^-(K_1)$ and a surjective morphism $f_2 : C_2 \to A_2$ in $\mathcal{A}$. Denote by $K_2 := \ker f_2$.

Note that for all Harder–Narasimhan factors C_2' of C_2 and K_1' of K_1, we have $\mu(C_2') < \mu(K_1')$. By Assumption (c), $\mathrm{Hom}_{\mathcal{K}}(C_2', K_1'[2]) = 0$. It follows that $\mathrm{Hom}_{\mathcal{K}}(C_2, K_1[2]) = 0$. Therefore, the composition $e : C_2 \to A_2 \to A_1[1]$ lifts to $\tilde{e} : C_2 \to C_1[1]$. In particular, we get the commutative diagram:

$$\begin{tikzcd} & K_2 \arrow[d] & K_1[1] \arrow[d] \\ C \arrow[r] & C_2 \arrow[r, "\tilde{e}"] \arrow[d] & C_1[1] \arrow[d] \\ A \arrow[r] & A_2 \arrow[r] & A_1[1]. \end{tikzcd} \tag{2}$$

By the octahedral axiom (see [5, Proposition 1.1.11] or [48, Section 2]), the commutative square can be completed to a 3×3 (possibly noncommutative) diagram of distinguished triangles. We have the distinguished triangle

$$K \to C \xrightarrow{f} A \to K[1],$$

where $C = \mathrm{Cone}(C_2[-1] \xrightarrow{\tilde{e}[-1]} C_1)$ and $K \cong \mathrm{Cone}(K_2[-1] \to K_1)$. In particular, both objects C and K are in $\mathcal{A}$. It follows that f is surjective. One also has

$$\mu^+(C) < \max\{\mu^+(C_1), \mu^+(C_2)\} < s.$$

So, the statement in Lemma 3.5 holds for A with respect to s. $\square$

Proof for Lemma 3.5. We first prove the case of $\delta_0 \neq 0$. Take an element $\tilde{g} = (g, M)$ of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that $g(0) = 0$ and the image of $g^{-1} \circ Z$ is contained in $\mathbb{Z} + i\mathbb{Z}$ by Assumption (a). Note that the heart of $\sigma \cdot \tilde{g}$ is the same as that of σ and $\sigma \cdot \tilde{g}$ still satisfies Assumption 3.4. Without loss of generality, we may assume that the image of the central charge of σ is contained $\mathbb{Z} + i\mathbb{Z}$.

Make induction on $(\mathrm{rk}, \deg)$ of A with lexicographic order. When $(\mathrm{rk}(A), \deg(A)) = (0, 1)$, then A is stable. By Assumption (b), there exists a σ-stable object C in $\mathcal{A}$ with $\mu(C) < s$ and $\mathrm{Hom}_{\mathcal{K}}(C, A) \neq 0$. Since A is a simple object, it has no nontrivial quotient object in $\mathcal{A}$. Therefore, every nonzero morphism from C to A is surjective.

Now assume that the statement holds for all objects with $\mathrm{rk} = 0$ and $\deg < m$. Then when A has $(\mathrm{rk}, \deg) = (0, m)$, by Assumption (b), for every real number s, there is a σ-stable object C_1 with $\mu(C_1) < s$ and $\mathrm{Hom}_{\mathcal{K}}(C_1, A) \neq 0$. Choose a nonzero morphism and denote its image in A as A_1.

If $A_1 = A$, then there is nothing to prove. Otherwise, we may write A as a short exact sequence

$$0 \to A_1 \to A \to A_2 \to 0$$

in $\mathcal{A}$, where both A_1 and A_2 have $\mathrm{rk} = 0$ and $\deg < m$.

By induction, the statement holds for both A_i. By Lemma 3.6, the statement holds for A.

Now we have finished the induction for the case of $\mathrm{rk} = 0$. We may assume that the statement holds for all objects with $\mathrm{rk} < r$. When $\mathrm{rk}(A) = r$, we have the short exact sequence

$$0 \to A_+ \to A \to A_- \to 0$$

in $\mathcal{A}$, where A_+ is the Harder–Narasimhan factor with $\mu = +\infty$. In particular, we have $\mathrm{rk}(A_+) = 0$ and $\mu^+(A_-) < +\infty$. By induction, the statement holds for A_+. So, if the statement holds for A_-, then by Lemma 3.6, it will hold for A.

Now we may assume $\mu^+(A) < +\infty$. By Assumption (b), for every real number s, there exists a σ-stable object C_1 with $\mu^+(C_1) < s$ and $\mathrm{Hom}_{\mathcal{K}}(C_1, A) \neq 0$. Choose a nonzero morphism, and denote its image in A as A_1 and the quotient as A_2. In particular, the statement holds for A_1 with respect to s.

Since $\mu^+(A) < +\infty$, we must have $\mathrm{rk}(A_1) > 0$. Hence, $\mathrm{rk}(A_2) < \mathrm{rk}(A)$. By induction, the statement holds for A_2. By Lemma 3.6, the statement holds for A with respect to s. As the s can be an arbitrary real number, the statement holds for A. We finish the induction.

We then prove the case of $\delta_0 = 0$. Let $t \in (0, 1)$ be a real number sufficiently small such that:

1. $-\cot(\pi t) < \min\{s, \mu^-(A)\}$;
2. the image of $e^{-\pi i t} Z$ is infinite on the real axis.

Denote by $\sigma_t := (\mathcal{P}_\sigma((t, t+1]), e^{-\pi i t} Z)$. In particular, the image of $e^{-\pi i t} Z$ can be $\mathbb{Z}$-linear spanned by one of its images on the real axis and another image not on the real axis with the smallest absolute value of the imaginary part. So, σ_t satisfies Assumption 3.4 with $\delta_0 \neq 0$.

Since $-\cot(\pi t) < \mu^-(A)$, the object A is in $\mathcal{P}_\sigma((t, 1]) \subset \mathcal{P}_\sigma((t, t+1])$. By the statement in the $\delta_0 \neq 0$ case, there exists an object C in $\mathcal{P}_\sigma((t, t+1])$ with $\mu^+_{\sigma_t}(C) < -\cot(\cot^{-1}(-s) - t\pi)$ and a surjective morphism $f : C \to A$ in $\mathcal{P}_\sigma((t, t+1])$.

Note that the object C is in $\mathcal{P}_\sigma((t, 1]) \subset \mathcal{A}$ and $\mu^+_\sigma(C) = -\cot(\cot^{-1}(-\mu^+_{\sigma_t}(C)) + t\pi) < s$. The kernel of f is in $\mathcal{P}_\sigma((t, \frac{1}{\pi}\cot^{-1}(-s)))$. Therefore, the morphism f is surjective in $\mathcal{A}$ as well. We finish the proof of the statement. □

Corollary 3.7. *Let σ be a stability condition on $\mathcal{K}$ satisfying the Assumption 3.4. Then, for every A, B in $\mathcal{A}$, we have that $\mathrm{Hom}_{\mathcal{K}}(A, B[2])$ is generated by compositions of extensions between objects in $\mathcal{A}$.*

Proof. Let $s = \mu^-(B) > -\infty$, we may pick C as that in Lemma 3.5. Let $f : C \to A$ be the surjective morphism and denote by K the kernel of f in $\mathcal{A}$. Applying $\mathrm{Hom}_{\mathcal{K}}(-, B)$ to the short exact sequence $0 \to K \to C \to A \to 0$, we get

$$\cdots \to \mathrm{Hom}_{\mathcal{K}}(K, B[1]) \to \mathrm{Hom}_{\mathcal{K}}(A, B[2]) \to \mathrm{Hom}_{\mathcal{K}}(C, B[2]) \to \dots .$$

By the choice of C and Assumption (c), $\mathrm{Hom}_{\mathcal{K}}(C, B[2]) = 0$. In particular, the last map $\mathrm{Hom}_{\mathcal{K}}(K, B[1]) \to \mathrm{Hom}_{\mathcal{K}}(A, B[2])$ is a surjection and the claim holds. □

As a consequence of the previous computations, we get our first main result.

Theorem 3.8. *Let $\mathcal{T}$ be the derived category of an abelian category. Let $\mathcal{K}$ be an admissible subcategory of $\mathcal{T}$ having a stability condition $\sigma = (\mathcal{A}, Z)$, whose heart $\mathcal{A}$ is induced from a heart $\mathcal{A}_{\mathcal{T}}$*

on $\mathcal{T}$ and satisfying the Assumption 3.4. Then the functor $F : \mathrm{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{K}$ defined in Lemma 3.1 is an equivalence.

Proof. By Lemma 3.3, the functor F defined in Lemma 3.1 is an equivalence if and only if for every A, B in $\mathcal{A}$, $n \geqslant 2$, we have that $\mathrm{Hom}_{\mathcal{K}}(A, B[n])$ is generated by degree 1 extensions of objects in $\mathcal{A}$. First note that $\mathrm{Hom}_{\mathcal{K}}(A, B[n]) = 0$ for every $n \geqslant 3$. Indeed, up to pass to the stable factors, it is enough to have this vanishing for every pair of σ-stable objects A, $B \in \mathcal{A}$, which holds by Assumption (c).

On the other hand, by Corollary 3.7, we have that $\mathrm{Hom}_{\mathcal{K}}(A, B[2])$ is generated by compositions of extensions between objects in $\mathcal{A}$. We conclude that the condition in Lemma 3.3 is satisfied, and thus, F is an equivalence. □

3.2 | Enhancements

Assume that $\mathcal{K}$ satisfies the conditions in Theorem 3.8. By [2], the bounded derived category of an abelian category has a unique enhancement (see also [22] for the same result without the boundedness condition). Together with Theorem 3.8, this directly implies that $\mathcal{K}$ has a unique enhancement.

Using [22, 26], we further prove in this section that $\mathcal{K}$ has a strongly unique enhancement, namely, the second part of Theorem 1.1.

Let us first recall the notion of almost ample sequence from [26].

Definition 3.9 [26, Definition 2.9]. Given an abelian category $\mathcal{A}$ and a set I, a subset $\{C_i\}_{i\in I}$ of objects $C_i \in \mathcal{A}$ is an *almost ample set* if, for every $A \in \mathcal{A}$, there exists $i \in I$ satisfying:

(i) there exist $k \in \mathbb{N}$ and a surjection $C_i^{\oplus k} \twoheadrightarrow A$;
(ii) $\mathrm{Hom}_{\mathcal{A}}(A, C_i) = 0$.

If $\mathcal{A}$ is the heart of $\mathcal{K}$ as in Theorem 3.8, by Lemma 3.5 for any $A \in \mathcal{A}$, there is $C_A \in \mathcal{A}$ satisfying conditions (i) and (ii). Indeed, it is enough to choose $s < \mu^-(A)$ and apply Lemma 3.5 to construct C_A together with a surjection to A; then $\mathrm{Hom}(A, C_A) = 0$, as $\mu^+(C_A) < s$.

Now note that the object C_A only depends on the isomorphism class $[A]$ of $A \in \mathcal{A}$, namely, for $A \cong A' \in \mathcal{A}$, we have that C_A satisfies (i) and (ii) of Definition 3.9 with respect to A'. So, we change the notation to $C_{[A]}$ and set

$$I := \{[A], A \in \mathcal{A}\}.$$

Since $\mathcal{A}$ is a full subcategory of $\mathcal{K}$ that is essentially small as $\mathcal{T}$ is, it follows that $\mathcal{A}$ is essentially small, so I is a set. Thus, the collection

$$\{C_{[A]}\}_{[A]\in I} \tag{3}$$

is an almost ample set.

The notion of almost ample set plays a key role in the extension of isomorphisms of functors. The next result is a special case of [26, Proposition 3.3].

Proposition 3.10 [26, Proposition 3.3]. *Let $\mathcal{A}$ be an abelian category with finite homological dimension. Assume that $\{C_i\}_{i\in I}$ is an almost ample set and let C be the corresponding full subcategory of $\mathrm{D^b}(\mathcal{A})$. Let F be an autoequivalence of $\mathrm{D^b}(\mathcal{A})$ such that there is an isomorphism of functors*

$$f:\ F|_C \xrightarrow{\simeq} \mathrm{id}_C.$$

Then there exists an isomorphism of functors $F \xrightarrow{\simeq} \mathrm{id}_{\mathrm{D^b}(\mathcal{A})}$ extending f.

Recall that the Drinfeld quotient $\mathrm{D^b_{dg}}(\mathcal{A}) := \mathrm{C^b_{dg}}(\mathcal{A})/\mathrm{Ac^b_{dg}}(\mathcal{A})$ is an enhancement of $\mathrm{D^b}(\mathcal{A})$ (see [30, Section 3] for the definition of the quotient), and thus, by Theorem 3.8 of $\mathcal{K}$. Here, $\mathrm{C^b_{dg}}(\mathcal{A})$ denotes the dg category of bounded complexes in $\mathcal{A}$, and $\mathrm{Ac^b_{dg}}(\mathcal{A}) \subset \mathrm{C^b_{dg}}(\mathcal{A})$ its full dg subcategory of acyclic complexes. In fact, the homotopy category of $\mathrm{C^b_{dg}}(\mathcal{A})$ is the homotopy category of complexes $\mathrm{H}^0(\mathrm{C^b_{dg}}(\mathcal{A})) = \mathrm{K^b}(\mathcal{A})$. This implies the natural identification $\mathrm{H}^0(\mathrm{D^b_{dg}}(\mathcal{A})) = \mathrm{D^b}(\mathcal{A})$ (see, e.g., [27, Section 1.2]).

By [22], if $(\mathcal{E}, \epsilon)$ is another enhancement of $\mathrm{D^b}(\mathcal{A})$, then there exists a quasi-functor

$$F:\ \mathrm{D^b_{dg}}(\mathcal{A}) \to \mathcal{E}, \tag{4}$$

whose construction is explained below, such that $\mathrm{H}^0(F)$ is an equivalence (thus F is an isomorphism in Hqe). In order to recall the definition of F, we need to introduce some technical notions from [22].

Let $\mathrm{V^b}(\mathcal{A}) \subset \mathrm{K^b}(\mathcal{A})$ be the full subcategory whose objects have zero differential. Let

$$Q:\ \mathrm{K^b}(\mathcal{A}) \to \mathrm{D^b}(\mathcal{A})$$

be the quotient functor and set

$$\mathrm{B^b}(\mathcal{A}) := Q(\mathrm{V^b}(\mathcal{A})),$$

which is a full subcategory of $\mathrm{D^b}(\mathcal{A})$, having the same objects as $\mathrm{V^b}(\mathcal{A})$ (but with different morphisms), see [22, Section 1.1]. We will use the notation A^* for objects in $\mathrm{V^b}(\mathcal{A})$ and thus of $\mathrm{B^b}(\mathcal{A})$. Note that the full dg subcategory $\mathrm{V^b_{dg}}(\mathcal{A})$ of $\mathrm{C^b_{dg}}(\mathcal{A})$ consisting of complexes with trivial differential is in a natural way an enhancement of $\mathrm{V^b}(\mathcal{A})$.

In [22, Section 4], the authors construct a dg enhancement $\mathrm{B^b_{dg}}(\mathcal{A})$ of $\mathrm{B^b}(\mathcal{A})$, whose definition depends on the pair $(\mathcal{E}, \epsilon)$ (but we omit this from its notation for simplicity), and such that $\mathrm{Perf}(\mathrm{B^b_{dg}}(\mathcal{A})) \cong \mathcal{E}$ in Hqe (see [22, Remark 4.3]).[†] Using this construction, the isomorphism $\mathrm{Perf}(\mathrm{V^b_{dg}}(\mathcal{A})) \cong \mathrm{C^b_{dg}}(\mathcal{A})$ in Hqe, and the morphism $\mathrm{Perf}(\mathrm{V^b_{dg}}(\mathcal{A})) \to \mathrm{Perf}(\mathrm{B^b_{dg}}(\mathcal{A}))$ in Hqe, they define in [22, (5.2)] the functor

$$g:\ \mathrm{C^b_{dg}}(\mathcal{A}) \to \mathcal{E} \tag{5}$$

[†] If C is a dg category, we denote by $\mathrm{Perf}(C)$ the smallest full dg subcategory of the dg category of h-projective dg C-modules, containing the image of the dg Yoneda embedding, closed under homotopy equivalences, shifts, cones, and direct summands in the homotopy category.

by composing the previous (iso)morphisms in Hqe. Moreover, as checked in [22, Section 5.3], the functor g factors through the quotient $\mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(\mathcal{A})$, so that g is the composition

$$g\colon\ \mathrm{C}^{\mathrm{b}}_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{dg}}(\mathcal{A}) \xrightarrow{F} \mathcal{E}. \tag{6}$$

They finally show that $\mathrm{H}^0(F)\colon\ \mathrm{D}^{\mathrm{b}}(\mathcal{A}) \to \mathrm{H}^0(\mathcal{E})$ is an equivalence.

Our goal is to show the following lemma that implies $\mathrm{D}^{\mathrm{b}}(\mathcal{A}) \cong \mathcal{K}$ has a strongly unique enhancement.

Lemma 3.11. *Let $\mathcal{T}$ be the derived category of an abelian category. Let $\mathcal{K}$ be an admissible subcategory of $\mathcal{T}$ having a stability condition $\sigma = (\mathcal{A}, Z)$, whose heart $\mathcal{A}$ is induced from a heart $\mathcal{A}_{\mathcal{T}}$ on $\mathcal{T}$ and satisfying the Assumption 3.4. Let $(\mathcal{E}, \epsilon)$ be an enhancement of $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$. Then there is an isomorphism of functors $\epsilon \circ \mathrm{H}^0(F) \cong \mathrm{id}_{\mathrm{D}^{\mathrm{b}}(\mathcal{A})}$, where F is defined in (4).*

Proof. Set $G' := \epsilon \circ \mathrm{H}^0(F)$. By Proposition 3.10, to prove the statement, it is enough to show that there is an isomorphism between the restriction functors $G'|_C \cong \mathrm{id}_C$, where C is the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ defined by the almost ample set (3). This isomorphism is a consequence of the construction of F in [22].

Indeed, set $G := \epsilon \circ \mathrm{H}^0(g)$, where g is defined in (5). Since $\mathrm{H}^0(g) = \mathrm{H}^0(F) \circ Q$ by (6), we have $G = G' \circ Q$. By [22, Lemma 5.1], there is an isomorphism of functors

$$\theta\colon\ G|_{\mathrm{V}^{\mathrm{b}}(\mathcal{A})} \xrightarrow{\sim} Q|_{\mathrm{V}^{\mathrm{b}}(\mathcal{A})}$$

(in their notation, G is $\mathbf{F}_1$ and Q is $\mathbf{F}_2$). As a consequence, for every $A^* \in \mathrm{B}^{\mathrm{b}}(\mathcal{A})$, we have that θ_{A^*} induces an isomorphism $G'(A^*) \cong A^*$ as objects in $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$.

We now claim that θ induces an isomorphism of functors between the restriction of G' to $\mathrm{B}^{\mathrm{b}}(\mathcal{A})$ and the identity. Indeed, consider $\alpha \in \mathrm{Hom}_{\mathrm{B}^{\mathrm{b}}(\mathcal{A})}(A_1^*, A_2^*) = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{A})}(A_1^*, A_2^*)$ for $A_i^* \in \mathrm{B}^{\mathrm{b}}(\mathcal{A})$. We can represent α as a roof in $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ of the form

$$\begin{tikzcd} & P \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ A_1^* & & A_2^* \end{tikzcd}$$

where $P \in \mathrm{V}^{\mathrm{b}}(\mathcal{A})$ and $f_i \in \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(\mathcal{A})}(P, A_i^*)$ for $i = 1, 2$. Now by [22, Corollary 5.3], there is an isomorphism $\theta'_{A_i^*}\colon G(A_i^*) \cong A_i^*$ such that the diagram

$$\begin{tikzcd} G(P) \arrow[r, "G(f_i)"] \arrow[d, "\theta_P"'] & G(A_i^*) \arrow[d, "\theta'_{A_i^*}"] \\ P \arrow[r, "Q(f_i)"] & A_i^* \end{tikzcd}$$

commutes in $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ for $i = 1, 2$. This implies that in $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$, we have the commutative diagram

$$\begin{tikzcd} G'(A_1^*) \arrow[r, "G'(\alpha)"] \arrow[d, "\theta'_{A_1^*}"] & G'(A_2^*) \arrow[d, "\theta'_{A_2^*}"] \\ A_1^* \arrow[r, "\alpha"] & A_2^* \end{tikzcd} .$$

Thus, θ induces a natural transformation $\theta' : G'|_{\mathrm{B^b}(\mathcal{A})} \cong \mathrm{id}_{\mathrm{B^b}(\mathcal{A})}$, which is an isomorphism of functors. Since $\mathcal{C}$ is a full subcategory of $\mathrm{B^b}(\mathcal{A})$, it follows that θ' induces an isomorphism of functors $f : G'|_{\mathcal{C}} \cong \mathrm{id}_{\mathcal{C}}$. This implies the statement. □

We are now ready to prove the second part of Theorem 1.1.

Theorem 3.12. *Let $\mathcal{T}$ be the derived category of an abelian category. Let $\mathcal{K}$ be an admissible subcategory of $\mathcal{T}$ having a stability condition $\sigma = (\mathcal{A}, Z)$, whose heart $\mathcal{A}$ is induced from a heart on $\mathcal{T}$ and satisfying Assumption 3.4. Then $\mathcal{K}$ has a strongly unique enhancement.*

Proof. Let $(\mathcal{E}, \epsilon)$ be an enhancement of $\mathcal{K}$. Consider the quasi-functor F defined in (4). By Lemma 3.11, there is an isomorphism of functors $\epsilon \circ \mathrm{H}^0(F) \cong \mathrm{id}_{\mathrm{D^b}(\mathcal{A})}$, giving the statement. □

3.3 | Fourier–Mukai functors

Let X_1 and X_2 be smooth projective schemes over a field $\mathbb{K}$. Let $\mathcal{K}_1 \subset \mathcal{T}_1 := \mathrm{D^b}(X_1)$ and $\mathcal{K}_2 \subset \mathcal{T}_2 := \mathrm{D^b}(X_2)$ be admissible subcategories that are components of a semiorthogonal decomposition. For $j = 1, 2$, denote by $i_j^* : \mathcal{T}_j \to \mathcal{K}_j$ the left adjoint functor of the inclusion $i_j : \mathcal{K}_j \hookrightarrow \mathcal{T}_j$.

Definition 3.13. A functor $\Phi : \mathcal{K}_1 \to \mathcal{K}_2$ is of Fourier–Mukai type if the composition

$$\mathcal{T}_1 \xrightarrow{i_1^*} \mathcal{K}_1 \xrightarrow{\Phi} \mathcal{K}_2 \xrightarrow{i_2} \mathcal{T}_2$$

is a Fourier–Mukai functor as in (1).

Note that if Φ is an equivalence, then the composition $i_2 \circ \Phi \circ i_1^*$ is a splitting functor in the sense of [38, Definition 3.1]. We now remark the following property that is probably well known to the experts.

Proposition 3.14. *Assume that $\mathcal{K}_1$ has a strongly unique enhancement. Then every equivalence $\Phi : \mathcal{K}_1 \xrightarrow{\sim} \mathcal{K}_2$ is of Fourier–Mukai type.*

Proof. Let $(\mathcal{E}_j, \epsilon_j)$ be the natural enhancement of $\mathcal{T}_j$ for $j = 1, 2$. Denote by $(\mathcal{F}_j, \delta_j)$ the enhancement of $\mathcal{K}_j$ induced from $(\mathcal{E}_j, \epsilon_j)$. By definition, $\mathcal{F}_j$ is the dg subcategory of $\mathcal{E}_j$ whose objects belong to $\mathcal{K}_j$ via the equivalence ϵ_j and is a full admissible subcategory of $\mathcal{E}_j$. The functor i_j^{dg} is the natural embedding of $\mathcal{F}_j$ in $\mathcal{E}_j$ and $\epsilon_j \circ \mathrm{H}^0(i_j^{\mathrm{dg}})$ factors through $\mathcal{K}_j$ defining δ_j. Note also that the composition $i_j \circ i_j^* : \mathcal{T}_j \to \mathcal{T}_j$ is a Fourier–Mukai functor by [41, Theorem 7.1]. In particular, it has a dg lift Ψ_j^{dg} by [57] and [27, Proposition 6.11]. By definition, Ψ_j^{dg} factors through $\mathcal{F}_j$ defining the projection $i_j^{*\mathrm{dg}} : \mathcal{E}_j \to \mathcal{F}_j$ such that $\Psi_j^{\mathrm{dg}} = i_1^{\mathrm{dg}} \circ i_j^{*\mathrm{dg}}$ and which is a dg lift of i_j^*.

Note that $(\mathcal{F}_2, \delta_2)$ is an enhancement of $\mathcal{K}_1$, because of the equivalence $\mathrm{H}^0(\mathcal{F}_2) \xrightarrow{\delta_2} \mathcal{K}_2 \xrightarrow{\Phi^{-1}} \mathcal{K}_1$. Since $\mathcal{K}_1$ has a strongly unique enhancement, there exists a quasi-functor $F : \mathcal{F}_1 \xrightarrow{\sim} \mathcal{F}_2$ such that

$\mathrm{H}^0(F)$ is an equivalence sitting in the following commutative diagram:

$$\begin{tikzcd} \mathrm{H}^0(\mathcal{F}_1) \arrow[r, "\mathrm{H}^0(F)", "\sim"'] \arrow[d, "\delta_1", "\sim"'] & \mathrm{H}^0(\mathcal{F}_2) \arrow[d, "\delta_2", "\sim"'] \\ \mathcal{K}_1 \arrow[r, "\Phi", "\sim"'] & \mathcal{K}_2 \end{tikzcd}.$$

On the other hand, by definition of $(\mathcal{F}_2, \delta_2)$, we have the commutative diagram

$$\begin{tikzcd} \mathrm{H}^0(\mathcal{F}_2) \arrow[r, "\mathrm{H}^0(i_2^{\mathrm{dg}})"] \arrow[d, "\delta_2", "\sim"'] & \mathrm{H}^0(\mathcal{E}_2) \arrow[d, "\epsilon_2", "\sim"'] \\ \mathcal{K}_2 \arrow[r, "i_2"] & \mathcal{T}_2 \end{tikzcd}.$$

Analogously, we have the commutative diagram

$$\begin{tikzcd} \mathrm{H}^0(\mathcal{E}_1) \arrow[r, "\mathrm{H}^0(i_1^{*\mathrm{dg}})"] \arrow[d, "\epsilon_1", "\sim"'] & \mathrm{H}^0(\mathcal{F}_1) \arrow[d, "\delta_1", "\sim"'] \\ \mathcal{T}_1 \arrow[r, "i_1^*"] & \mathcal{K}_1 \end{tikzcd}.$$

Putting everything together, we have

$$i_2 \circ \Phi \circ i_1^* = \epsilon_2 \circ \mathrm{H}^0(i_2^{\mathrm{dg}}) \circ \mathrm{H}^0(F) \circ \mathrm{H}^0(i_1^{*\mathrm{dg}}) \circ (\epsilon_1)^{-1}.$$

Thus $i_2^{\mathrm{dg}} \circ F \circ i_1^{*\mathrm{dg}}$ is a dg lift of $i_2 \circ \Phi \circ i_1^*$. By [57] and [27, Proposition 6.11], we conclude that the latter is of FM type. □

From the previous results, we deduce the following characterization.

Corollary 3.15. *Let X_1 and X_2 be smooth projective schemes over a field $\mathbb{K}$. Let $\mathcal{K}_1$ be an admissible subcategory of $\mathcal{T}_1 := \mathrm{D}^{\mathrm{b}}(X_1)$ having a stability condition $\sigma = (\mathcal{A}, Z)$ whose heart $\mathcal{A}$ is induced from a heart $\mathcal{A}_{\mathcal{T}_1}$ on $\mathcal{T}_1$ and satisfying the Assumption 3.4. Let $\mathcal{K}_2$ be an admissible subcategory of $\mathrm{D}^{\mathrm{b}}(X_2)$. Then every equivalence $\mathcal{K}_1 \xrightarrow{\sim} \mathcal{K}_2$ is of Fourier–Mukai type.*

Proof. This is a consequence of Theorem 3.12 and Proposition 3.14. □

4 | GEOMETRIC APPLICATIONS

In this section, we apply the general results proved in the previous section to interesting geometric situations, listed in Examples 2.8–2.11, providing the proof of Theorems 1.2–1.4. The key point is to show that the Kuznetsov components of the varieties in these examples satisfy Assumption 3.4.

To make a universal argument for most of the cases at once, we make the following assumption on the stability conditions that turns out to be easy to check. Denote by $\mathrm{K}_{\mathrm{num}}(\mathcal{A})$ the numerical Grothendieck group. Recall that the Euler pairing $\chi([E],[F]) := \sum_{i\in\mathbb{Z}}(-1)^i \dim \mathrm{Hom}(E, F[i])$ is well defined on $\mathrm{K}_{\mathrm{num}}(\mathcal{A})$. Let $\sigma = (\mathcal{A}, Z)$ be a stability condition on $\mathcal{K}$ with heart $\mathcal{A}$. We make the following assumption on σ.

Assumption 4.1.

(a0) There exists λ_1 and λ_2 in $\mathrm{K}_{\mathrm{num}}(\mathcal{A})$ such that the image of Z is a rank 2 lattice spanned by $Z(\lambda_1)$ and $Z(\lambda_2)$.
(b1) The Euler paring χ is symmetric on $\mathrm{K}_{\mathrm{num}}(\mathcal{A})$ and is negative definite on $\mathrm{span}_{\mathbb{Z}}\{\lambda_1, \lambda_2\}$.
(b2) There exists $c \in \mathbb{Z}$ such that for any primitive v in $\mathrm{K}_{\mathrm{num}}(\mathcal{A})$ with $\chi(v, v) < c$, there exists σ-stable object E in $\mathcal{K}$ with $[E] = v$.
(c) Same as that of Assumption 3.4 (c).

Recall that we write $Z = -\mathrm{deg} + i\mathrm{rk}$ and $\delta_0 := \inf\{\mathrm{rk}(E) | E \in \mathcal{K}, \mathrm{rk}(E) > 0\}$.

Lemma 4.2. *Let σ be a stability condition on $\mathcal{K}$ satisfying the Assumption 4.1. Then, for every nonzero object E in $\mathcal{A}$, real number s and $\delta > \delta_0$, there exists a σ-stable object F in $\mathcal{A}$ with $\mu(F) < s$ and $\mathrm{rk}(F) < \delta$ such that $\chi(F, E) \neq 0$.*

Proof. In the case of $\delta_0 > 0$, by Assumption (a0), the numbers $\mathrm{rk}(\lambda_1)$ and $\mathrm{rk}(\lambda_2)$ are $\mathbb{Q}$-linear dependent, so there exists nonzero v with $\mathrm{rk}(v) = 0$. Among all such v's with $\mathrm{deg}(v) < 0$, we may choose one with the largest deg.

So, there exist nonzero w and v in $\mathrm{span}_{\mathbb{Z}}\{\lambda_1, \lambda_2\}$ such that

$$\mathrm{rk}(w) = \delta_0, \mathrm{rk}(v) = 0, \text{ and } \mathrm{deg}(v) < 0.$$

In particular, their images $Z(v)$ and $Z(w)$ also span the image of Z.

Note that $\mathrm{K}_{\mathrm{num}}(\mathcal{A})$ can be spanned by w, v and elements in $\mathrm{Ker}(Z)$. Since χ is nondegenerate on $\mathrm{K}_{\mathrm{num}}(\mathcal{A})$ by definition, if $\chi(v, E) = 0$, then there exists $\kappa_E \in \mathrm{Ker}(Z)$ satisfying $\chi(w + \kappa_E, E) \neq 0$. If $\chi(v, E) \neq$ or $\chi(w, E) \neq 0$, then we may set $\kappa_E = 0$.

Let n be sufficiently negative such that

1. $\chi(nv + w + \kappa_E, nv + w + \kappa_E) < c$ and $nv + w + \kappa_E$ is primitive;
2. $\chi(nv + w + \kappa_E, E) \neq 0$;
3. $\mu(nv + w + \kappa_E) < s$.

The first condition can be satisfied because the left-hand side of the inequality is

$$n^2\chi(v, v) + 2n\chi(v, w + \kappa_E) + \chi(w + \kappa_E, w + \kappa_E),$$

and tends to $-\infty$ when n tends to $-\infty$ by Assumption (b1). As $Z(nv + w + \kappa_E) = Z(nv + w)$ is primitive in the lattice of the image of Z, the character $nv + w + \kappa_E$ is primitive.

The second condition can be satisfied since either $\chi(v, E) \neq 0$ or $\chi(w + \kappa_E, E) \neq 0$ by the choice of κ_E.

The third condition can be satisfied since $\lim_{n\to-\infty} \mu(nv + w + \kappa_E)/n = \mathrm{deg}(v)/\delta_0$.

By Assumption (b2) and the first condition, there exists a σ-stable object $F \in \mathcal{A}$ with numerical class $nv + w + \kappa_E$. By the choice of n, we have $\mu(F) < s$, $\mathrm{rk}(F) = \delta_0 < \delta$, and $\chi(F, E) \neq 0$.

In the case of $\delta_0 = 0$, there exists a sequence of primitive characters $w_n = a_n\lambda_1 + b_n\lambda_2$ such that

$$\mathrm{rk}(w_n) > 0,\ \deg(w_n) < 0 \text{ and } \lim_{n\to+\infty} \mathrm{rk}(w_n) = 0.$$

As $\delta_0 = 0$, it follows that

$$\lim_{n\to+\infty} \deg(w_n) = -\infty,\ \lim_{n\to+\infty} \frac{b_n}{a_n} = -\frac{\mathrm{rk}(\lambda_1)}{\mathrm{rk}(\lambda_2)} =: q \text{ and } \lim_{n\to+\infty} |a_n| = +\infty.$$

Note that if $q = \frac{n}{m}$ is rational, then $\delta_0 > |\frac{\mathrm{rk}(\lambda_2)}{m}|$. So $q \notin \mathbb{Q}$.

Note that $\mathrm{K}_{\mathrm{num}}(\mathcal{A})$ is spanned by λ_1, λ_2 and elements in $\mathrm{Ker}(Z)$. Since χ is nondegenerate on $\mathrm{K}_{\mathrm{num}}(\mathcal{A})$ by definition, if $\chi(\lambda_1, E) = \chi(\lambda_2, E) = 0$, then there exists $\kappa_E \in \mathrm{Ker}(Z)$ satisfying $\chi(\kappa_E, E) \neq 0$. If $\chi(\lambda_1, E)$ or $\chi(\lambda_2, E) \neq 0$, then we set $\kappa_E = 0$.

Let n be sufficiently large such that

1. $\chi(w_n + \kappa_E, w_n + \kappa_E) < c$;
2. $\chi(w_n + \kappa_E, E) \neq 0$;
3. $\mu(w_n + \kappa_E) < s$ and $\mathrm{rk}(w_n + \kappa_E) < \delta$.

Note that the left-hand side of the inequality in the first condition is $\chi(w_n, w_n) + 2\chi(w_n, \kappa_E) + \chi(\kappa_E, \kappa_E)$. Divided by $|a_n|^2$, it tends to $\chi(\lambda_1 + q\lambda_2, \lambda_1 + q\lambda_2)$ when n tends to ∞. This value is negative by Assumption (b1). Therefore, the first condition can be satisfied.

Since q is not a rational number, if $\chi(\lambda_1, E)$ or $\chi(\lambda_2, E) \neq 0$, then $\chi(w_n, E)$ is not constantly zero. If both $\chi(\lambda_1, E)$ and $\chi(\lambda_2, E) = 0$, then $\chi(\kappa_E, E) \neq 0$ by the choice of κ_E. Therefore, the second condition can be satisfied.

The third condition can be satisfied by the choice of w_n.

By Assumption (b2) and the first condition, there exists a σ-stable object $F \in \mathcal{A}$ with numerical class $w_n + \kappa_E$. By the choice of n , we have $\mu(F) < s$, $\mathrm{rk}(F) < \delta$, and $\chi(F, E) \neq 0$. $\square$

Proposition 4.3. *Let σ be a stability condition satisfying the Assumption 4.1, then for every $\tilde{g} \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, the stability condition $\sigma \cdot \tilde{g}$ satisfies the Assumption 3.4.*

Proof. By definition, Assumption 3.4 (a) and (c) hold automatically. Note that the conditions in Assumption 4.1 are preserved by the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action. We only need to check that σ satisfies Assumption 3.4 (b).

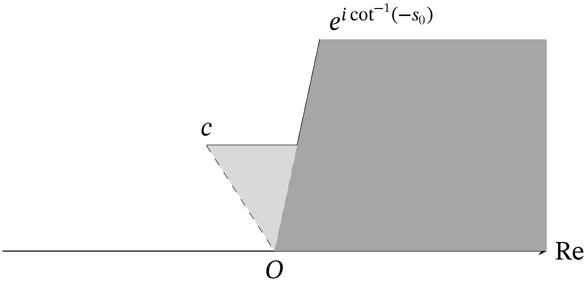
By taking the Harder–Narasimhan filtration of E with respect to σ, we only need to prove that Assumption 3.4 (b) holds for every σ-semistable object E in $\mathcal{A}$. By taking a Jordan–Hölder factor that is also a subobject of E, we only need to prove that Assumption 3.4 (b) holds for every σ-stable object E in $\mathcal{A}$. Namely, we are going to prove the statement that for every σ-stable object E in $\mathcal{A}$ and every real number s_0, there exists a σ-stable object F in $\mathcal{A}$ satisfying $\mu(F) < s_0$ and $\mathrm{Hom}_{\mathcal{K}}(F, E) \neq 0$.

Fix a real number s_0, we define the order $\prec_{s_0}$ for complex numbers in $\mathbb{R}_{>0} \cdot e^{i(0,\pi]}$ as follows:

- If $c_1 \in \mathbb{R}_{>0} \cdot e^{i(0,\cot^{-1}(-s_0))}$ and $c_2 \in \mathbb{R}_{>0} \cdot e^{i[\cot^{-1}(-s_0),\pi]}$, then $c_1 \prec_{s_0} c_2$.

- If both $c_j = a_j + ib_j \in \mathbb{R}_{>0} \cdot e^{i[\cot^{-1}(-s_0),\pi]}$ for $j = 1, 2$, then

$$c_1 \prec_{s_0} c_2 \iff b_1 \leqslant b_2 \text{ and } -\frac{a_1}{b_1} < -\frac{a_2}{b_2}.$$



Numbers in both light and dark gray areas are $\prec_{s_0} c$.

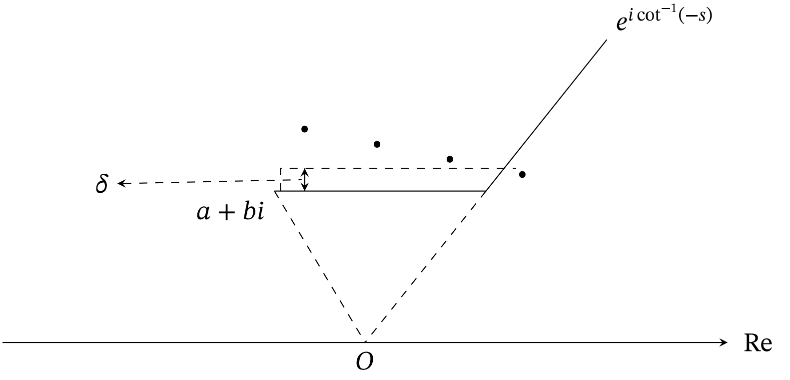
Let us go back to the proof of the statement. Note that if $\mu(E) < s_0$, namely, $Z(E) \in \mathbb{R}_{>0} \cdot e^{i(0,\cot^{-1}(-s_0))}$, then we may just let $F = E$ and there is nothing to prove.

By Assumption (a), for any complex number $c \in \mathbb{R}_{>0} \cdot e^{i[\cot^{-1}(-s_0),\pi]}$, the area of the light gray part is finite. There are only finitely many $Z(v)$'s of numerical characters $v \in \mathrm{K}_{\mathrm{num}}(\mathcal{A})$ satisfying $Z(v) \notin \mathbb{R}_{>0} \cdot e^{i(0,\cot^{-1}(-s_0))}$ and $Z(v) \prec_{s_0} c$. We may make induction on $Z(E)$ with respect to the order $\prec_{s_0}$.

Assume that for every σ-stable object $E' \in \mathcal{A}$ satisfying $Z(E') \prec_{s_0} Z(E)$, the statement holds. In other words, there exists a σ-stable object F' satisfying $\mu(F') < s_0$ and $\mathrm{Hom}_{\mathcal{K}}(F', E') \neq 0$.

In the case that $\delta_0 > 0$, we set $\delta = \frac{3}{2}\delta_0$ and $s < \min\{s_0, \frac{1}{\delta_0}((\mathrm{rk}(E) + \delta_0)s_0 - \deg(E))\}$. In the case that $\delta_0 = 0$, we set $s = s_0$ and sufficiently small $\delta > 0$ such that the image of Z does not intersect the area

$$\{a + ib \mid \mathrm{rk}(E) \leqslant b \leqslant \mathrm{rk}(E) + \delta, s \leqslant -\frac{a}{b} < \mu(E)\}.$$



The choice of δ when $\delta_0 = 0$.

Bullet points stand for the image of Z.

Apply Lemma 4.2 for E, s and δ, we get a σ-stable object F_0 in $\mathcal{A}$ such that $\mu(F_0) < s$, $\mathrm{rk}(F_0) < \delta$ and $\chi(F_0, E) \neq 0$. Note that this in particular forces $\mathrm{rk}(F_0) = \delta_0$ when $\delta_0 > 0$.

If $\chi(F_0, E) > 0$, then by Assumption (c), $\mathrm{Hom}_{\mathcal{K}}(F_0, E[m]) = 0$ for all $m \neq 0, 1$. Hence, $\mathrm{Hom}_{\mathcal{K}}(F_0, E) \neq 0$ and the statement holds.

Otherwise, by Assumption 4.1 (b1), we have $\chi(E, F_0) = \chi(F_0, E) < 0$ and by Assumption (c), $\mathrm{Hom}_{\mathcal{K}}(E, F_0[m]) = 0$ for all $m \neq 0, 1, 2$. Hence, we have $\mathrm{Hom}_{\mathcal{K}}(E, F_0[1]) \neq 0$. Let e be a nonzero morphism in $\mathrm{Hom}_{\mathcal{K}}(E[-1], F_0)$ and G be $\mathrm{Cone}(E[-1] \xrightarrow{e} F_0)$. So, there is a short exact sequence

$$0 \to F_0 \to G \to E \to 0, \tag{7}$$

and the object G is in $\mathcal{A}$.

If G is σ-stable, then we may let $F = G$. Indeed, we always have $\mathrm{Hom}_{\mathcal{K}}(G, E) \neq 0$. In the case of $\delta_0 > 0$, by the choice of s and δ, we have $\mathrm{rk}(F_0) = \delta_0$ and $\deg(F_0) = \mu(F_0)\mathrm{rk}(F_0) < s\delta_0$. It follows that $\mu(G) = \frac{\deg(E)+\deg(F_0)}{\mathrm{rk}(E)+\mathrm{rk}(F_0)} = \frac{\deg(E)+\deg(F_0)}{\mathrm{rk}(E)+\delta_0} < \frac{\deg(E)+s\delta_0}{\mathrm{rk}(E)+\delta_0} < s_0$. In the case of $\delta_0 = 0$, by the choice of s and δ, we must have $\mu(G) < s_0$. The statement holds.

If G is not σ-stable, then we consider one of the Jordan–Hölder factors G^+ of G with a maximal slope that is also a subobject of G. Note that $\mu(G^+) \geqslant \mu(G) > \mu(F_0)$, it follows that $\mathrm{Hom}(G^+, F_0) = 0$. Apply $\mathrm{Hom}(G^+, -)$ to (7), we get $0 \neq \mathrm{Hom}_{\mathcal{K}}(G^+, G) \hookrightarrow \mathrm{Hom}_{\mathcal{K}}(G^+, E)$. We may let f be a nonzero morphism in $\mathrm{Hom}_{\mathcal{K}}(G^+, E)$. The image of f is a nonzero subobject E_0 of E in $\mathcal{A}$. Let E_0^+ be one of the Jordan–Hölder factors of E_0 with a maximal slope that is a subobject of E_0. Then E_0^+ is also a subobject of E in $\mathcal{A}$ as well.

In the case of $\delta_0 > 0$, we have $\mathrm{rk}(G^+) \leqslant \mathrm{rk}(G) - \delta_0 = \mathrm{rk}(E)$. Since $e \neq 0$, the sequence (7) does not split. It follows that $\mu(G^+) < \mu(E)$. Hence, the object E_0^+ is a proper subject of E. In the case of $\delta_0 = 0$, it follows by the choice of s and δ that $\mathrm{rk}(G^+) < \mathrm{rk}(E)$. Hence, the object E_0^+ is a proper subject of E in this case as well.

If $\mu(E_0^+) < s_0$, then we may let $F = E_0^+$ and the statement holds. Otherwise, we have $Z(E_0^+) \prec_{s_0} Z(E)$. The statement holds by induction. □

Example 4.4. Let X be a K3 surface and $\sigma_{\omega,\beta}$ be a stability condition as that defined in [18]. When $\omega, \beta \in \mathbb{Q} \cdot H$ for an ample divisor H, the stability condition $\sigma_{\omega,\beta}$ satisfies Assumption 4.1. Hence, $\mathrm{D}^{\mathrm{b}}(\mathcal{A}_{\omega,\beta}) = \mathrm{D}^{\mathrm{b}}(X)$. Note that this statement has already been proved in [14, Corollary A.4] by a very different argument. We would like to thank Georg Oberdieck to point out this example for us.

Lemma 4.5. *Let X be a cubic fourfold defined over $\mathbb{C}$. Then the stability conditions $\sigma = (\mathcal{A}, Z)$ on $\mathcal{K}u(X)$ constructed in [10] satisfy Assumption 4.1.*

Proof. Recall that the numerical Grothendieck group $\mathrm{K}_{\mathrm{num}}(\mathcal{K}u(X))$ contains two classes λ_1 and λ_2 spanning an A_2-lattice

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

with respect to the Euler pairing χ by [4]. The image of the central charge is spanned by the image of λ_1 and λ_2. As the Serre duality on $\mathcal{K}u(X)$ is [2], the Euler pairing on $\mathrm{K}_{\mathrm{num}}(\mathcal{K}u(X))$ is symmetric. Hence, Assumptions (a0) and (b1) hold.

Assumption (b2) follows by [9, Theorem 1.6].

Finally, Assumption (c) follows by Serre duality: for any σ-stable objects E and F in $\mathcal{A}$, we have

$$\mathrm{Hom}_{\mathcal{K}}(E, F[m]) \cong \mathrm{Hom}_{\mathcal{K}}(F, E[2-m]) = 0$$

for $m \geqslant 3$, and if $\mu(E) < \mu(F)$, for $m \geqslant 2$. $\square$

Lemma 4.6. *Let X be a GM fourfold defined over $\mathbb{C}$. Then the stability conditions σ on $\mathcal{K}u(X)$ constructed in [51] satisfy Assumption 4.1.*

Proof. The numerical Grothendieck group $\mathrm{K}_{\mathrm{num}}(\mathcal{K}u(X))$ contains two classes λ_1 and λ_2 spanning an $A_1^{\oplus 2}$-lattice

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

with respect to the Euler pairing χ by [36, Lemma 2.27] and [50]. The image of the central charge is spanned by the image of λ_1 and λ_2, see [51, Section 4]. As the Serre duality on $\mathcal{K}u(X)$ is [2], the Euler pairing on $\mathrm{K}_{\mathrm{num}}(\mathcal{K}u(X))$ is symmetric. Hence, Assumptions (a0) and (b1) hold.

Assumption (b2) follows by [51, Theorem 1.5].

Finally, Assumption (c) follows from Serre duality as that in Lemma 4.5. $\square$

Lemma 4.7. *Let X be a GM threefold defined over $\mathbb{C}$. Then the stability conditions σ on $\mathcal{K}u(X)$ constructed in [10] satisfy Assumption 4.1.*

Proof. Recall that $\mathrm{K}_{\mathrm{num}}(\mathcal{K}u(X))$ has rank 2 by [39, Proposition 3.9] and a basis is given by

$$\lambda_1 := 1 - \frac{1}{5}H^2, \quad \lambda_2 := 2 - H + \frac{5}{6}P,$$

where H is the class of a hyperplane and P is the class of a point in X. The intersection matrix with respect to χ is

$$\langle \lambda_1, \lambda_2 \rangle = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Assumptions (a0) and (b1) hold as the central charge is not degenerate.

Assumption (b2) follows by [52, Theorem 1.3].

Assumption (c) follows from Serre-duality and the fact that the property of being stable with respect to σ is preserved by the Serre functor. Indeed, if E is σ-stable, then $S_{\mathcal{K}u(X)}(E)$ is σ-stable by [53, Theorem 1.1]. Since $S^2_{\mathcal{K}u(X)} = [4]$, it is easy to check (see, e.g., [55, Lemma 5.9]) that $\phi(E) < \phi(S_{\mathcal{K}u(X)}(E)) \leqslant \phi(E) + 2$. Then

$$\mathrm{Hom}_{\mathcal{K}}(E, F[m]) \cong \mathrm{Hom}_{\mathcal{K}}(F[m], S_{\mathcal{K}u(X)}(E)) = 0$$

for $m \geqslant 3$, and if $\mu(E) < \mu(F)$, for $m \geqslant 2$. $\square$

Lemma 4.8. *Let X be a quartic double solid defined over $\mathbb{C}$. Then the stability conditions σ on $\mathcal{K}u(X)$ constructed in [10] satisfy Assumption 4.1.*

Proof. Denote by H the class of a hyperplane and by P the class of a point. By [39, Proposition 3.9], we have that

$$\lambda_1 := 1 - \frac{1}{2}H^2, \quad \lambda_2 := H - \frac{1}{2}H^2 - \frac{2}{3}P$$

is a basis for $\mathrm{K}_{\mathrm{num}}(\mathcal{K}u(X))$ with intersection form

$$\langle \lambda_1, \lambda_2 \rangle = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$$

with respect to χ. Then, Assumptions (a0) and (b1) hold as the central charge is not degenerate.

Assumption (b2) follows by [52, Theorem 1.1].

Assumption (c) can be proved as in Lemma 4.7 using that σ is Serre invariant by [55, Proposition 5.7]. □

We are now ready to prove Theorem 1.2–1.4.

Theorem 4.9. *Let $\mathcal{K}u(X)$ be the Kuznetsov component of a cubic fourfold or of a GM variety or of a quartic double solid defined over $\mathbb{C}$. Then there is an equivalence $F: \mathcal{K}u(X) \cong \mathrm{D}^{\mathrm{b}}(\mathcal{A})$, where $\mathcal{A}$ is the heart of a stability condition on $\mathcal{K}u(X)$ and F is defined in Lemma 3.1. Moreover, we have that $\mathcal{K}u(X)$ has a strongly unique enhancement.*

Proof. Note that if X is a GM fivefold (resp. sixfold), then its Kuznetsov component $\mathcal{K}u(X)$ is equivalent to that of a GM threefold (resp. fourfold). This is a consequence of the duality conjecture proved in [37, Theorem 1.6], as explained in [51, Proof of Theorem 4.18]. Thus, we reduce to prove the statement in this case. Note that the heart of the stability conditions constructed in [10, 51] is induced on the Kuznetsov component from the heart of $\mathrm{D}^{\mathrm{b}}(X)$ obtained by double-tilting $\mathrm{Coh}(X)$ (see [10, Theorem 6.9, Proof of Theorem 1.2] and [51, Theorem 4.12]). Then, it is a consequence of the above lemmas, Proposition 4.3, Theorem 3.8, and Theorem 3.12. □

Proof of Theorem 1.3 and Theorem 1.4. The proof of Theorem 1.4 is a direct consequence of Corollary 3.15. Now assume that X_1, X_2 are cubic fourfolds or GM varieties of even dimension. Let $F: \mathcal{K}u(X_1) \to \mathcal{K}u(X_2)$ be a fully faithful exact functor. We claim that F is an equivalence. Indeed, first note that the 0th Hochschild cohomology of $\mathcal{K}u(X_i)$ is $\mathrm{HH}^0(\mathcal{K}u(X_i)) = \mathbb{C}$, as computed in [42, Proposition 4.1] and [36, Corollary 2.11]; thus, $\mathcal{K}u(X_i)$ is connected. Since $\mathcal{K}u(X_i)$ is Calabi–Yau, by [43, Proposition 5.5], it follows that $\mathcal{K}u(X_i)$ is indecomposable.

Note also that $\mathcal{K}u(X_i)$ is (right) saturated,[†] as it is an admissible subcategory of $\mathrm{D}^{\mathrm{b}}(X_i)$ that is saturated (see [20] and [8, Proposition 2.8]). Then F admits left and right adjoints. Indeed, since $\mathcal{K}u(X_1)$ is saturated, for every $A_2 \in \mathcal{K}u(X_2)$, the functor $\mathrm{Hom}(F(-), A_2)$ is representable by a unique $A_1 \in \mathcal{K}u(X_1)$. By Yoneda lemma, this defines a functor $G: \mathcal{K}u(X_2) \to \mathcal{K}u(X_1)$ such that $G(A_2) = A_1$, which is right adjoint to F (see [24, Proposition 3.5]). Since $Ku(X_i)$ has Serre functor, denoted as $S_{\mathcal{K}u(X_i)}$, the left adjoint of F is $H := S^{-1}_{\mathcal{K}u(X_1)} \circ G \circ S_{\mathcal{K}u(X_2)}$. By [16, Theorem 3.3], this implies that F is an equivalence. The result then follows from Corollary 3.15. □

[†] We say that a triangulated category $\mathcal{T}$ is right saturated if every contravariant cohomology functor of finite type is representable.

Remark 4.10. Note that Theorem 3.8 could potentially be applied to the Kuznetsov component of a cubic threefold. The main missing ingredient is the nonemptiness of moduli spaces of stable objects for the constructed stability conditions. This is part of the work in progress [31].

ACKNOWLEDGEMENTS

We wish to thank Paolo Stellari for many useful conversations and for suggesting the strategy in Section 3.2 and the application to the characterization of equivalences. We are also grateful to the referee for careful reading of the paper and for useful comments.

C.L. is supported by the Royal Society URF\R1\201129 "Stability condition and application in algebraic geometry." L.P. is supported by the national research project PRIN 2017 Moduli and Lie Theory. X.Z. is partially supported by the Simons Collaborative Grant 636187, NSF grant DMS-2101789, and NSF FRG grant DMS-2052665.

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The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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