Entropic and trace-distance-based measures of non-Markovianity

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(Received 26 July 2022; accepted 29 September 2022; published 17 October 2022)

We analyze and compare different measures for the degree of non-Markovianity in the dynamics of open quantum systems. These measures are based on the distinguishability of quantum states, which is quantified, on the one hand, by the trace distance or, more generally, by the trace norm of the Helstrom matrix and, on the other hand, by entropic quantifiers: the Jensen-Shannon divergence and the Holevo or quantum skew divergence. We explicitly construct a qubit dynamics for which the trace-norm-based non-Markovianity measure is nonzero, while all the entropic measures turn out to be zero. This leads to the surprising conclusion that the non-Markovianity measure which employs the trace norm of the Helstrom matrix is strictly stronger than all entropic non-Markovianity measures.

DOI: 10.1103/PhysRevA.106.042212

I. INTRODUCTION

The study of quantum non-Markovian dynamics involves the investigation of the very notion of stochastic processes in the quantum realm, as well as the characterization of memory effects in open quantum system dynamics [1–4]. Memory effects in the dynamics of a quantum system interacting with an external environment can be uniquely traced back to local retrieval of exchanged information in the approach to non-Markovianity based on the nonmonotonic behavior in time of the distinguishability of quantum states. This strategy was introduced in [5] and validated for different distinguishability quantifiers of quantum states. In particular, while the original approach was focused on the trace distance, it was later put into evidence that invariance under translations of this quantifier led to failure in assessing memory features in certain dynamics [6]. To avoid this difficulty, the trace norm of the Helstrom matrix was used as a generalized trace distance also sensitive to translations [7,8]. A crucial feature associated with the trace norm of the Helstrom matrix is the fact that its nonmonotonicity in time is equivalent to the lack of P divisibility of the considered dynamics, provided the evolution is invertible as a linear transformation. In such a way a direct relation could be established between a divisibility and a distinguishability criterion.

More recently, entropic distinguishability quantifiers have also been introduced and directly connected to the notion of non-Markovianity due to information backflow [9]. To this aim, suitable regularizations of the quantum relative entropy have been considered, which, at variance with the quantum relative entropy, remain finite for any pair of states and allow us to introduce trianglelike inequalities which connect revivals of the quantifier to information backflow, even in the absence of a true triangle inequality like for distances. Furthermore, these entropic quantifiers are also sensitive to translations. In this framework, a special role is played by the Jensen-Shannon divergence, whose square root is a true distance [10–12]. We remark that the measure of non-Markovianity defined by the Jensen-Shannon divergence is not the only possible quantifier of non-Markovianity linked to the concept of entropy. Indeed, measures based on the nonmonotonic behavior of entanglement or of quantum mutual information have been proposed [13,14] and linked to a backflow of information [15,16], although within a different perspective with respect to the one proposed in this paper. These two measures have been compared in the literature and have been shown to be inequivalent in detecting memory effects [17].

Given that entropic distinguishability quantifiers are contractions under positive trace-preserving maps which are not necessarily completely positive, which happens for the trace distance and the trace norm of the Helstrom matrix, a natural question is the role of P divisibility in this context. Importantly, we show by means of an example that the Jensen-Shannon divergence, as well as the other entropic quantifiers, might fail in detecting breaking of P divisibility.

Our results imply that the non-Markovianity measure employing the trace norm of the Helstrom matrix is strictly stronger than all of the entropic non-Markovianity measures, leading to a nonzero value even for dynamics for which the entropic measures are zero, while the opposite cannot happen.

This paper is organized as follows. In Sec. II we introduce and exemplify the general framework for the treatment of non-Markovianity based on distinguishability quantifiers, together with the associated measures. In Sec. III we outline the

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connection between non-Markovianity and the divisibility of the dynamics and explore this relationship via its dependence on the considered distinguishability quantifier. In particular, we construct an example of non-P-divisible evolution whose non-Markovianity measure is zero according to entropic quantifiers. We summarize and discuss the conclusions of our work in Sec. IV.

II. ENTROPIC AND TRACE-DISTANCE-BASED DISTINGUISHABILITY QUANTIFIERS

Let us begin by introducing the general framework of non-Markovianity for the dynamics of open quantum systems. The main aim is to compare the well-known measure of memory effects based on the trace distance with other measures using alternative different distinguishability quantifiers between quantum states, in particular those related to the quantum relative entropy.

A. Trace distance and Helstrom matrix

In the framework of quantum information and statistics there are many different quantifiers of distinguishability between two quantum states ρ and σ. A very important one is given by the trace distance (TD) [18]

$$D(\rho,\sigma) = \tfrac{1}{2}\|\rho-\sigma\|, \tag{1}$$

where the trace norm of any trace-class operator A is defined as $\|A\| = \operatorname{tr}\sqrt{A^\dagger A}$. The TD is bounded, $0 \leqslant D(\rho,\sigma) \leqslant 1$, with $D(\rho,\sigma)=0$ if and only if $\rho=\sigma$ and $D(\rho,\sigma)=1$ if and only if $\rho\perp\sigma$. Additionally, the TD obeys the triangle inequality

$$D(\rho,\sigma) \leqslant D(\rho,\tau)+D(\tau,\sigma) \tag{2}$$

and is contractive under the action of any completely positive trace-preserving (CPTP) map Λ, as well as of any positive trace-preserving map [19]

$$D(\Lambda\rho,\Lambda\sigma) \leqslant D(\rho,\sigma), \tag{3}$$

and it is invariant under unitary and antiunitary transformations [20]. It is also invariant under translations in the sense that

$$D(\rho+A,\sigma+A) = D(\rho,\sigma) \tag{4}$$

for any operator A. This follows directly from the fact that the TD depends on the difference between its two arguments.

It is possible to interpret the TD as the bias in favor of a correct identification between two quantum states upon performing a single measurement. Let us suppose that Alice prepares the state ρ or σ, each with probability $\frac{1}{2}$, and sends it to Bob; the TD is linked to Bob's maximal probability of correctly distinguishing between the two as [21]

$$P_{\text{dist}}(\rho,\sigma) = \tfrac{1}{2}[1+D(\rho,\sigma)]. \tag{5}$$

This feature, combined with the contractivity of the TD under CPTP maps (3), tells us that CPTP maps cannot increase the probability of distinguishing between quantum states.

The idea of using the trace norm $\|\cdot\|$ to quantify the bias in favor of a correct identification can be generalized also to the case in which the two states ρ and σ are not prepared with the same *a priori* probability. In fact, if one supposes that Alice prepares ρ with probability p and σ with probability $1-p$, then Bob's maximal probability of distinguishing between the two is given by [22]

$$P_{\text{dist}}(\rho,\sigma) = \tfrac{1}{2}(1+\|\Delta\|), \tag{6}$$

where

$$\Delta = p\rho-(1-p)\sigma \tag{7}$$

is known as the Helstrom matrix [23]. The trace norm of Δ represents the bias in favor of a correct identification, and Eq. (6) reduces to (5) in the unbiased case $p=\frac{1}{2}$. The Helstrom matrix can be seen as a generalization of the TD to generic ensembles $\{(p,\rho),(1-p,\sigma)\}$, and it inherits properties such as boundedness and contractivity from the TD.

B. Jensen-Shannon and skew divergences

The TD is not the only possible quantifier of distinguishability between quantum states. A particularly interesting distinguishability quantifier is the relative entropy

$$S(\rho,\sigma) = \begin{cases} \operatorname{tr}[\rho\log_2\rho-\rho\log_2\sigma] & \text{if } \operatorname{supp}\rho\subseteq\operatorname{supp}\sigma, \\ \infty & \text{otherwise.} \end{cases} \tag{8}$$

The relative entropy, just like the TD, is contractive under both CPTP maps and positive trace-preserving maps [24]. However, as is evident from the definition, it is not bounded.

The relative entropy can also be naturally associated with a distinguishability task. In particular, let us suppose we are able to prepare and measure the states an arbitrarily large number N of times. The relative entropy $S(\rho,\sigma)$ represents the maximal asymptotic rate at which the probability of erroneously concluding that the state is ρ when it is actually σ decays with the size N of the sample over which a measurement is performed, so that for large enough N the probability of correctly identifying the state is [25–27]

$$P_{N,\text{dist}}(\rho,\sigma) = 1-e^{-NS(\rho,\sigma)}. \tag{9}$$

Unboundedness here arises naturally: whenever $\operatorname{supp}\rho\nsubseteq\operatorname{supp}\sigma$, one might distinguish with certainty ρ from σ with only a finite number of measurements, and hence, the rate is infinite (see [28] for a more detailed discussion).

It is possible to define a smoothed version of the relative entropy, namely, the Jensen-Shannon divergence (JSD), according to [29]

$$\begin{aligned} J(\rho,\sigma) &= \frac{1}{2}S\Big(\rho,\frac{\rho+\sigma}{2}\Big)+\frac{1}{2}S\Big(\sigma,\frac{\rho+\sigma}{2}\Big) \\ &= H\Big(\frac{\rho+\sigma}{2}\Big)-\frac{1}{2}H(\rho)-\frac{1}{2}H(\sigma), \end{aligned} \tag{10}$$

where H denotes the von Neumann entropy $H(\rho) = -\operatorname{tr}\rho\log_2\rho$. This definition ensures that the JSD inherits the contractivity under CPTP maps from the relative entropy and, additionally, it is bounded according to $0\leqslant J(\rho,\sigma)\leqslant 1$, with $J(\rho,\sigma)=0$ if and only if $\rho=\sigma$, while $J(\rho,\sigma)=1$ if and only if $\rho\perp\sigma$. In particular, it can be bounded by monotonic functions of the TD as [30,31]

$$\tfrac{1}{2}D(\rho,\sigma)^2 \leqslant J(\rho,\sigma) \leqslant D(\rho,\sigma). \tag{11}$$

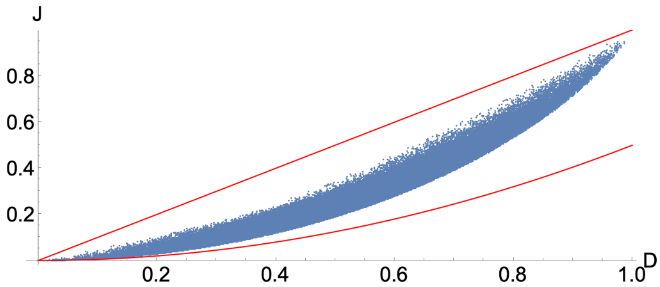

FIG. 1. Plot of the JSD and the TD for 10^5 randomly generated pairs of qubits. The red lines are the upper and lower bounds of Eq. (11), which are also valid for arbitrarily dimensional Hilbert spaces.

The lower bound directly follows from the Pinsker inequality [26]. Figure 1 shows these bounds together with the value of the TD and the JSD for randomly chosen pairs of states. The JSD is not invariant under translations in the sense of (4) since, unlike the TD, it does not depend solely on the difference $\rho - \sigma$. This fact is visualized in Fig. 2.

The JSD, unlike the TD, is not a distance since it does not obey the triangle inequality. However, it has been proven that its square root ($\text{JSD}^{\frac{1}{2}}$) does obey this inequality and is indeed a distance [10–12]. Even if it does not obey the triangle inequality, the JSD obeys a trianglelike inequality,

$$J(\rho,\sigma) - J(\rho,\tau) \leqslant \sqrt[4]{2J(\sigma,\tau)}, \tag{12}$$

which follows from the inequalities presented in [30].

It is possible to generalize the JSD to generic ensembles $\{(\mu,\rho),(1-\mu,\sigma)\}$; however, unlike for the TD, such generalization is not unique. As suggested in [32], we point to two distinct generalizations based on a skewed version of the relative entropy, also called telescopic relative entropy in the quantum setting [33,34]. We therefore introduce the Holevo skew divergence

$$K_\mu(\rho,\sigma) = \frac{\chi_\mu(\rho,\sigma)}{h(\mu)}, \tag{13}$$

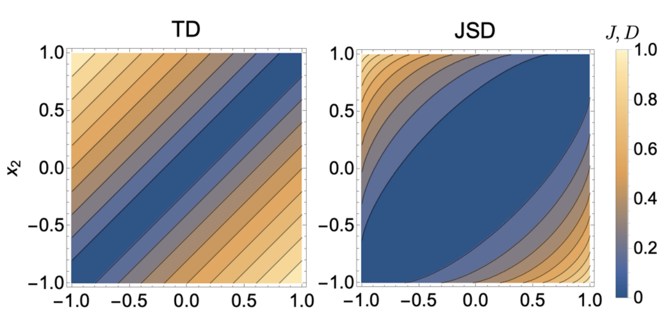

FIG. 2. Plots of the TD (left) and of the JSD (right) for qubit states represented by Bloch vectors of the form $\boldsymbol{r}_\rho = (x_1, 0, 0)^\top$, $\boldsymbol{r}_\sigma = (x_2, 0, 0)^\top$. The translational invariance of the TD is reflected by the fact that the plot on the left depends on only the difference $x_1 - x_2$. Note further the different sensitivities in the central and corner regions.

where

$$h(p) = -p\log_2 p - (1-p)\log_2(1-p) \tag{14}$$

is the binary entropy for the distribution $\{p, 1-p\}$ and

$$\chi_\mu(\rho,\sigma) = H[\mu\rho + (1-\mu)\sigma] - \mu H(\rho) - (1-\mu)H(\sigma) \tag{15}$$

is the Holevo χ quantity [35] for the considered ensemble, as well as the quantum skew divergence

$$S_\mu(\rho,\sigma) = \frac{\mu}{\log_2(1/\mu)} S(\rho, \mu\rho + (1-\mu)\sigma) + \frac{1-\mu}{\log_2[1/(1-\mu)]} S(\sigma, (1-\mu)\sigma + \mu\rho). \tag{16}$$

Both quantities are bounded, and they reduce to the JSD in the unbiased case $\mu = \frac{1}{2}$. Furthermore, they both obey trianglelike inequalities similar to the ones that hold for the unbiased case (12), namely [9,32],

$$S_\mu(\rho,\sigma) - S_\mu(\rho,\tau) \leqslant \eta_\mu^S \sqrt[4]{S_\mu(\sigma,\tau)}, \tag{17}$$

$$K_\mu(\rho,\sigma) - K_\mu(\rho,\tau) \leqslant \eta_\mu^K \sqrt[4]{K_\mu(\sigma,\tau)}, \tag{18}$$

with

$$\eta_\mu^S = \log_2\left(\frac{1}{\mu(1-\mu)}\right)\sqrt[4]{\frac{\mu(1-\mu)}{2\,h(\mu)\log_2^3(\mu)\log_2^3(1-\mu)}},$$
$$\eta_\mu^K = \sqrt[4]{\frac{8\mu(1-\mu)}{h(\mu)^3}}. \tag{19}$$

C. Non-Markovianity measures from distinguishability quantifiers

The unavoidable interaction between a quantum system and its surroundings leads to system-environment correlations and nonunitary time evolution of the state. Assuming that at the initial time $t = 0$ the global system-environment state is factorized, the dynamics is described by a one-parameter family of CPTP dynamical maps $\Phi = \{\Phi_t \mid 0 \leqslant t \leqslant T, \Phi_0 = \mathbb{1}\}$ such that $\rho(t) = \Phi_t\rho(0)$. We always assume that the dynamics is invertible, in the sense that Φ_t^{-1} exists at all times $t \geqslant 0$. Under this assumption, it is possible to define a two-parameter family of maps as

$$\Phi_{t,s} = \Phi_t\Phi_s^{-1}, \quad t \geqslant s \geqslant 0, \tag{20}$$

such that $\Phi_{t,0} = \Phi_t$, describing the evolution of the state from time s to time t. The dynamics is said to be (C)P divisible if $\Phi_{t,s}$ is (completely) positive for all times $t \geqslant s \geqslant 0$.

The interaction between the system and the environment can lead to memory effects during the dynamics of the state. If this happens, the dynamics is said to be non-Markovian. Following [1,5], we define a family of non-Markovianity measures based on some distinguishability quantifier d as

$$\mathcal{N}^d(\Phi) = \sup \int_{\sigma_d(t)>0} dt\, \sigma_d(t), \tag{21}$$

where

$$\sigma_d(t) = \frac{d}{dt} d(\rho_1(t), \rho_2(t)) \tag{22}$$

and the maximization is performed over all possible pairs of initial states $\rho_{1,2}(0)$ and any eventual parameter defining the distinguishability quantifier d, such as the skewing parameter μ defining S_μ or K_μ. A certain pair of initial states $\rho_{1,2}(0)$ is said to be optimal if the maximum of Eq. (21) is attained on this pair. Thus, a dynamical map Φ is Markovian according to the quantifier d if and only if $\mathcal{N}^d(\Phi) = 0$ or, equivalently, if $d(\rho_1(t), \rho_2(t))$ is a monotonic function of time for any initial pair of states $\rho_{1,2}(0)$. Alternative approaches are, indeed, possible, such as violations of divisibility of the dynamical map [1–3,36–38].

Following [32], in order to have a well-defined measure of non-Markovianity we require the quantifier d to obey three properties.

(1) The first is the boundedness and indistinguishability of identical states:

$$0 \leqslant d(\rho, \sigma) \leqslant 1, \tag{23}$$

with $d(\rho, \sigma) = 1$ if and only if $\rho \perp \sigma$ and $d(\rho, \sigma) = 0$ if and only if $\rho = \sigma$. Considering bounded distinguishability quantifiers allows us to perform the maximization in Eq. (21), thus warranting that the measure of non-Markovianity is well defined.

(2) The second one is contractivity under CPTP maps:

$$d(\Lambda\rho, \Lambda\sigma) \leqslant d(\rho, \sigma) \tag{24}$$

for any CPTP map Λ. This property is crucial because any revival in d must necessarily correspond to violations of divisibility of the dynamical map. In fact, if Φ is CP divisible, d must be monotonically decreasing since the map $\Phi_{t,s}$ describing the evolution from s to $t > s$ is always CPTP. Therefore, a revival in d is possible only if Φ violates the divisibility.

(3) The last property is the trianglelike inequalities:

$$d(\rho, \sigma) - d(\rho, \tau) \leqslant \phi(d(\sigma, \tau)), \tag{25}$$

$$d(\rho, \sigma) - d(\tau, \sigma) \leqslant \phi(d(\rho, \tau)), \tag{26}$$

where $\phi(x)$ is a strictly positive concave function for $x > 0$, with $\phi(0) = 0$. This property allows for a microscopic interpretation of the revivals of d as a twofold exchange of information, which is at first stored in external degrees of freedom and later retrieved in the open system.

TD, JSD, $\mathrm{JSD}^{\frac{1}{2}}$, and their generalizations all obey properties 1–3 and hence lead to a well-defined measure of non-Markovianity. For the TD and $\mathrm{JSD}^{\frac{1}{2}}$, which are actually distances, the function ϕ is given by the identity, while, for the JSD and the other entropic quantities, the function ϕ is proportional to the fourth root following Eq. (12), as well as (17) and (18).

Given two distinguishability quantifiers d_1 and d_2 satisfying properties 1–3, we say that $\mathcal{N}^{d_1}$ is stronger than $\mathcal{N}^{d_2}$ if, for any dynamical map Φ such that $\mathcal{N}^{d_2}(\Phi) > 0$, $\mathcal{N}^{d_1}(\Phi) > 0$. Furthermore, $\mathcal{N}^{d_1}$ is strictly stronger than $\mathcal{N}^{d_2}$ if it is stronger and Φ exists such that $\mathcal{N}^{d_2}(\Phi) = 0$ and $\mathcal{N}^{d_1}(\Phi) > 0$. Conversely, $\mathcal{N}^{d_1}$ is (strictly) weaker than $\mathcal{N}^{d_2}$ if $\mathcal{N}^{d_2}$ is (strictly) stronger than $\mathcal{N}^{d_1}$. Two measures are said to be equivalent if $\mathcal{N}^{d_1}$ is both stronger and weaker than $\mathcal{N}^{d_2}$.

An important distinguishability quantifier obeying the three above-mentioned properties is the TD. Optimal pairs for this measure must always be orthogonal and therefore on the border of the set of states [39]. Additionally, the triangle inequality (2) allows an upper bound on the revival of the TD from s to a later time $t > s > 0$ as [40–42]

$$\begin{aligned}\Delta D(t, s) &= D\big(\rho_S^1(t), \rho_S^2(t)\big) - D\big(\rho_S^1(s), \rho_S^2(s)\big) \\ &\leqslant D\big(\rho_{SE}^1(s), \rho_S^1(s) \otimes \rho_E^1(s)\big) \\ &\quad + D\big(\rho_{SE}^2(s), \rho_S^2(s) \otimes \rho_E^2(s)\big) \\ &\quad + D\big(\rho_E^1(s), \rho_E^2(s)\big),\end{aligned} \tag{27}$$

where $\rho_{SE}^i(s)$ for $i = 1, 2$ is the global system-environment state and $\rho_S^i(s) = \mathrm{tr}_E\, \rho_{SE}^i(s)$ and $\rho_E^i(s) = \mathrm{tr}_S\, \rho_{SE}^i(s)$ are, respectively, the reduced system and environmental states at time s. This allows for a microscopic interpretation of the measure of non-Markovianity [40,41]: a revival in the TD is possible only if at time s the two environments are different or if correlations have built up during the dynamics. Therefore, information is stored as correlations or as difference between the environmental states and can later flow back into the open system. A similar interpretation also holds for the measure of non-Markovianity arising from the Helstrom matrix [8], as well as for entropic distinguishability quantifiers [9,32]. In particular, the non-Markovianity measure obtained according to Eq. (21) when the quantifier d is the trace norm of the Helstrom matrix Eq. (7), which we denote as $\mathcal{N}^\Delta(\Phi)$, is positive if and only if Φ is not P divisible, which was shown in [7,8] building on results in [43,44]. The measure based on the TD, instead, is strictly weaker than $\mathcal{N}^\Delta$ since it can equal zero even for non-P-divisible dynamics due to its translational invariance [6]. Given that both the TD and the quantum relative entropy are contractive under positive trace-preserving maps and the equivalence between a non-Markovianity measure and a divisibility property was obtained by considering positivity, from now on we will concentrate our attention simply on positivity.

Let us now focus our attention on the entropic distinguishability quantifiers in Sec. II A. Except for the relative entropy, which is unbounded, all the other quantifiers obey properties 1–3 and hence can be used to define a measure of non-Markovianity. In particular, we want to investigate whether these measures of non-Markovianity are equivalent to $\mathcal{N}^\Delta$. Namely, we want to know whether the equivalence between the positivity of $\mathcal{N}^d(\Phi)$ and lack of P divisibility of Φ also holds when choosing d as one of the previously introduced entropic quantifiers. We will show in Sec. III C that this is not the case: we will use a counterexample to point out that $\mathcal{N}^\Delta$ is strictly stronger. Let us focus in particular on the JSD since $\mathrm{JSD}^{\frac{1}{2}}$ is just a monotonic function of it and hence $\mathcal{N}^J$ and $\mathcal{N}^{\sqrt{J}}$ are equivalent.

D. Behavior on unital models

Let us now focus our discussion on qubits since it suffices to consider the simplest nontrivial case to prove that $\mathcal{N}^\Delta$ is strictly stronger than $\mathcal{N}^J$. For qubits, a generic state ρ can be represented by means of a real three-dimensional Bloch vector

$\boldsymbol{r}_\rho$ with $|\boldsymbol{r}_\rho| \leqslant 1$ in the form

$$\rho = \tfrac{1}{2}(\mathbb{1} + \boldsymbol{r}_\rho \cdot \boldsymbol{\sigma}), \tag{28}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^\top$ is the vector of the Pauli matrices. Under the action of a generic trace- and Hermiticity-preserving linear map, the Bloch vector associated with the state transforms according to

$$\boldsymbol{r} \mapsto \boldsymbol{r}(t) = D(t)\boldsymbol{r} + \boldsymbol{\kappa}(t), \tag{29}$$

where $D(t) = \mathrm{diag}\{\lambda_1(t), \lambda_2(t), \lambda_3(t)\}$ is a real diagonal 3×3 matrix and $\boldsymbol{\kappa}(t)$ is a real three-dimensional vector [45]. The representation (29) is valid up to orthogonal transformations, which in the present context play no role, given that both TD and JSD are invariant under unitary transformations, and the measure of non-Markovianity in Eq. (21) is obtained by maximizing over the possible initial states. Let us focus in particular on unital maps, which are the maps that preserve the maximally mixed state at any time $t \geqslant 0$: $\Phi_t[\frac{\mathbb{1}}{2}] = \frac{\mathbb{1}}{2}$. Alternatively, employing the representation (29), they are the maps for which $\boldsymbol{\kappa}(t) = 0$ at all times $t \geqslant 0$. Assuming that the dynamics is invertible, i.e., Φ_t^{-1} exists at all times, a dynamics of this kind is not P divisible if and only if at least one of the functions $\lambda_i(t)$ does not decrease monotonically. Invertibility is a necessary assumption since there exist noninvertible dynamics with monotonic $\lambda_i(t)$ that violate divisibility [46].

An important feature of unital dynamics is that $\mathcal{N}^D(\Phi) > 0$ if and only if Φ violates P divisibility: any backflow of information, corresponding to a violation of P divisibility, is witnessed by the TD, without the need to generalize it to the Helstrom matrix. Interestingly, this feature also holds for the JSD. Let $\rho_{1,2}(0)$ be the optimal pair for the TD. Since they must be pure and orthogonal, we have $\rho_1(0) + \rho_2(0) = \mathbb{1}$, and the Bloch vectors representing the states obey $\boldsymbol{r}_1(0) = -\boldsymbol{r}_2(0)$. Thanks to unitality, the transformed average state $[\rho_1(t) + \rho_2(t)]/2$ remains the maximally mixed state, so that $\boldsymbol{r}_1(t) = -\boldsymbol{r}_2(t)$ holds at all times. Thus, the TD between the two states reads $D(\rho_1(t), \rho_2(t)) = r(t)$, and both evolved states have the same von Neumann entropy,

$$H(\rho_1(t)) = H(\rho_2(t)) = h\left(\frac{1 - r(t)}{2}\right), \tag{30}$$

where h is the binary entropy, introduced in Eq. (14). It is therefore possible to rewrite the JSD using Eq. (10) as

$$\begin{aligned} J(\rho_1(t), \rho_2(t)) &= 1 - H(\rho_1(t)) = 1 - H(\rho_2(t)) \\ &= 1 - h\left(\frac{1 - D(\rho_1(t), \rho_2(t))}{2}\right). \end{aligned} \tag{31}$$

This expression is a monotonic function of the TD, and thus, a revival in the JSD is witnessed if and only if it is witnessed by the TD. Therefore, like what occurs for the TD, $\mathcal{N}^J(\Phi) > 0$ if and only if Φ violates P divisibility. Unlike the TD, the characterization of optimal pairs for the JSD is still an open problem. For unital maps acting on qubits, numerical evidence suggests that they must be pure and orthogonal, just like for the TD. This feature allows for an interesting interpretation of $\mathcal{N}^J$ in terms of the von Neumann entropy. By employing the first line of Eq. (31), which holds for any pair of pure and orthogonal initial states, it is therefore possible to rewrite the measure of non-Markovianity as

$$\mathcal{N}^J(\Phi) = \max_{\rho(0)} \int_{\Gamma_{\rho(0)}} dt \frac{d}{dt}[-H(\rho(t))], \tag{32}$$

where $\Gamma_{\rho(0)} = \{t \in \mathbb{R} \mid \frac{d}{dt}H(\rho(t)) < 0\}$. The measure of non-Markovianity for the JSD in the case of unital dynamics is given by the total decrease of entropy for a single state, maximized over all possible initial states. This is clearly linked to violations of P divisibility of Φ since any unital positive map acting on qubits increases the entropy of the state [26], so that any revival in the entropy must necessarily correspond to a violation of P divisibility. Unfortunately, this feature is true only for qubits since in higher dimensions orthogonal states do not need to have the same eigenvalues. Additionally, no similar interpretation holds for $\mathrm{JSD}^{\frac{1}{2}}$ or for the two generalizations to ensembles K_μ and S_μ.

E. Robustness of optimal pairs

Let us now study the robustness of optimal pairs, i.e., how the measure of non-Markovianity changes when moving away from the optimal pair, for the different distinguishability quantifiers. We will illustrate this by considering a simple, but paradigmatic, model: the dephasing model. This model consists of a modification of the coherences without a corresponding change in the populations:

$$\rho(t) = \begin{pmatrix} \rho_{00} & \rho_{01}\gamma(t)e^{-i\omega_S t} \\ \rho_{10}\gamma^*(t)e^{i\omega_S t} & \rho_{11} \end{pmatrix}, \tag{33}$$

where γ is called the decoherence function. For this model, non-Markovianity corresponds to a nonmonotonic behavior of $|\gamma|$. It is worth stressing that considerations similar to the ones for this model are also valid for other models such as the phase-covariant model which will be introduced in Sec. III B.

Optimal pairs are all the pairs of pure and orthogonal states corresponding to antipodal vectors on the equator of the Bloch sphere since the x and y directions are the only ones in which the dynamics is not trivial. In order to evaluate the robustness of the optimal pairs, Fig. 3 shows the behavior of the measure of non-Markovianity when moving away from the equatorial plane but still considering pure and orthogonal states. It is possible to notice that the TD and $\mathrm{JSD}^{\frac{1}{2}}$, which are both distances, behave very similarly, with the maximum of the measure of non-Markovianity on the equatorial plane that quickly decreases when moving towards the poles of the Bloch sphere. For the JSD, on the other hand, the situation is qualitatively different, with a broader region around the equator with a measured value of non-Markovianity similar to the maximal one, which is about one order of magnitude smaller than the value obtained for the other quantifiers.

III. NON-MARKOVIANITY AND DIVISIBILITY

In the definition of a measure of non-Markovianity for a generic distinguishability quantifier d, the condition of contractivity (24) implies that every P divisible dynamics leads to a zero measure of non-Markovianity. On the other hand, using the Helstrom matrix, as soon as the dynamics violates P divisibility, one has a nonzero measure of non-Markovianity, $\mathcal{N}^\Delta(\Phi) > 0$. In other words, $\mathcal{N}^\Delta$ is stronger than $\mathcal{N}^d$ for

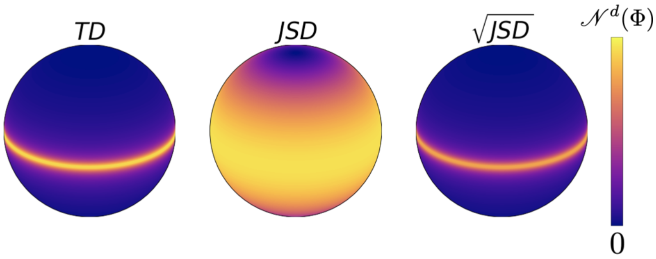

FIG. 3. Measure of non-Markovianity for pure and orthogonal states for TD (left), JSD (middle), and JSD$^{\frac{1}{2}}$ (right). The color shows the value of the measure of non-Markovianity, rescaled according to the maximum value reached by the measure for the considered distinguishability quantifier d, obtained considering as initial states the corresponding point on the surface of the Bloch sphere and its antipodal point. Brighter colors correspond to higher values of the revivals. The reference values are taken to be 1×10^{-2} for $\mathcal{N}^D(\Phi)$ as well as $\mathcal{N}^{\sqrt{J}}(\Phi)$ and 1×10^{-3} for $\mathcal{N}^J(\Phi)$, reflecting the different scales of the revivals. The two distances (TD and JSD$^{\frac{1}{2}}$) behave very similarly. For the JSD, on the other hand, the value of the measure of non-Markovianity decreases more slowly when moving away from the optimal pair, i.e., all the pairs of pure and orthogonal vectors on the equator of the Bloch sphere. The dynamics Φ is given by the dephasing model of Eq. (33) with the decoherence function $\gamma(t)$ corresponding to a bosonic bath described by a spectral density with an exponential cutoff of the form $J(\omega) = \lambda(\omega^s/\Omega^{s-1})\exp(-\omega/\Omega)$ as considered in [47,48], with $s = 3$, $\lambda = 3$, and $\Omega = 1$ in inverse units of time.

any other quantifier d. We now want to study whether other quantifiers d exist that lead to measures that are equivalent to the one arising from the Helstrom matrix, with a particular focus on the entropic quantifiers.

We already know that properties 1–3 are not sufficient to have a measure of non-Markovianity equivalent to $\mathcal{N}^\Delta$ since it is strictly stronger than $\mathcal{N}^D$. In Sec. III C we will show with a counterexample that the same also holds for the JSD and its generalizations.

A. Positivity and noncontractivity domain

Let us first tackle the question of the behavior of the JSD under nonpositive maps. We already know that the JSD is contractive under any positive map. We now want to investigate if the reverse is also true; namely, we want to clarify whether, for any nonpositive map Λ, a pair of states for which the JSD is strictly noncontractive exists. Nonpositivity of Λ implies that some state ρ exists which is mapped to a nonpositive operator $\Lambda\rho$. However, the JSD, unlike the TD, cannot be extended to nonpositive operators since it involves the logarithm of the eigenvalues. Therefore, the search for a noncontractive pair for Λ must be restricted to the set of states that are mapped to states after the action of the map, i.e., to the positivity domain

$$\mathcal{PD}_\Lambda = \{\rho \in \mathcal{S}(\mathscr{H}) \mid \Lambda\rho \in \mathcal{S}(\mathscr{H})\}, \tag{34}$$

where $\mathcal{S}(\mathscr{H})$ is the set of quantum states on a Hilbert space $\mathscr{H}$. In the following, we will consider only qubits $\mathscr{H} = \mathbb{C}^2$ since this will turn out to be sufficient to show that $\mathcal{N}^J$ is strictly weaker than $\mathcal{N}^\Delta$. We denote the set of all qubit states, i.e., the Bloch sphere, as $\mathcal{S}(\mathbb{C}^2) = \mathcal{S}$.

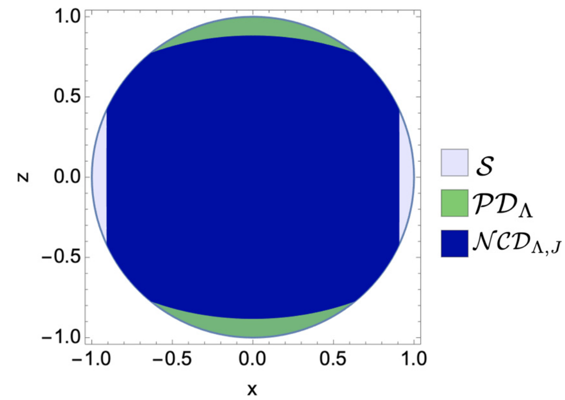

FIG. 4. Section at $y = 0$ of the Bloch sphere [light blue (light gray)] for an example of a nonpositive map Λ for which the noncontractivity domain $\mathcal{NCD}_{\Lambda,J}$ [blue (dark gray)] is strictly included in the positive domain $\mathcal{PD}_\Lambda$ [green (medium gray)]. This map acts on a Bloch vector $\boldsymbol{r} = (x, y, z)^\top$ as $\boldsymbol{r} \mapsto (\lambda_x x, \lambda_y y, \lambda_z z)^\top$, with $\lambda_x = \lambda_y = 1.1$ and $\lambda_z = 0.1$.

Considering unital nonpositive maps Λ, it is easy to show that a noncontractive pair always exists inside $\mathcal{PD}_\Lambda$. Such maps act on Bloch vectors according to Eq. (29) with $\boldsymbol{\kappa} = 0$, and nonpositivity implies that some $\lambda_i > 1$, which we take to be λ_1, without loss of generality. The noncontractive pair is the one represented by the Bloch vectors $\boldsymbol{r}_\rho = (\lambda_1^{-1}, 0, 0)^\top = -\boldsymbol{r}_\sigma$. In fact, by direct calculation it is easy to show that $J(\rho, \sigma) < J(\Lambda\rho, \Lambda\sigma) = 1$. In the general case, an analytic proof for the existence of a noncontractive pair is missing. However, by parameterizing the nonpositive map Λ as in Eq. (29) and performing a sample on all the parameters, we observed numerically that for any such map it is always possible to find a pair of states $\rho, \sigma \in \mathcal{PD}_\Lambda$ such that $J(\Lambda\rho, \Lambda\sigma) > J(\rho, \sigma)$.

Turning back to the dynamical point of view, however, the search for the noncontractive pair might not be extended to all $\mathcal{PD}_\Lambda$. In fact, not all the domain of positivity of $\Lambda = \Phi_{t,s}$ is available; only the image at time s of the Bloch sphere $\Phi_s(\mathcal{S})$ is. We stress that $\Phi_s(\mathcal{S})$ is, in general, only a subset of $\mathcal{PD}_\Lambda$. Thus, in order to have $\mathcal{N}^J(\Phi) > 0$ for all non-P-divisible processes, we would need to be able to find a noncontractive pair for the JSD inside $\Phi_s(\mathcal{S})$. Let us now define the set of states in which it is possible to find a noncontractive pair as the noncontractivity domain

$$\mathcal{NCD}_{\Lambda,J} = \{\rho \in \mathcal{PD}_\Lambda \mid \exists\sigma \in \mathcal{PD}_\Lambda,\; J(\Lambda\rho, \Lambda\sigma) > J(\rho, \sigma)\}. \tag{35}$$

Therefore, in order to have non-Markovianity for all non-P-divisible dynamical maps, we would need to have $\mathcal{NCD}_{\Lambda,J} = \mathcal{PD}_\Lambda$: for any state in $\Phi_s(\mathcal{S}) \cap \mathcal{PD}_\Lambda$ it is always possible to find another state such that noncontractivity holds. However, this is not the case, which is clear from the example in Fig. 4. There, in fact, $\mathcal{NCD}_{\Lambda,J}$ is a proper subset of $\mathcal{PD}_\Lambda$. Therefore, if we were able to construct a dynamics Φ such that for times $t > s > 0$ it acts as this nonpositive map ($\Phi_{t,s} = \Lambda$), with a dynamics prior to time s that is P divisible and with a Bloch sphere that is mapped at time s inside $\mathcal{PD}_\Lambda$ but outside $\mathcal{NCD}_{\Lambda,J}$, i.e., $\Phi_s(\mathcal{S}) \subset \mathcal{PD}_\Lambda \setminus \mathcal{NCD}_{\Lambda,J}$, we would construct a non-P-divisible dynamics but with $\mathcal{N}^J(\Phi) = 0$. This is, indeed, feasible, as we will show in Sec. III C by providing explicitly a model which is similar in spirit to the one just described.

B. Phase-covariant dynamics

In order to construct the counterexample in Sec. III C, let us first set the theoretical background of the considered dynamics, namely, phase-covariant dynamics. They contain a broad class of dynamics, and they involve maps Φ that satisfy covariance with respect to phase transformations, namely [49],

$$e^{-i\sigma_z\theta}\Phi_t[\rho]e^{i\sigma_z\theta} = \Phi_t[e^{-i\sigma_z\theta}\rho e^{i\sigma_z\theta}] \tag{36}$$

for all real θ and for all states $\rho \in \mathcal{S}(\mathbb{C}^2)$. Phase-covariant dynamics have the form [50,51]

$$\Phi_t\rho = \tfrac{1}{2}[\mathbb{1} + \eta_\perp(t)(v_x\sigma_x + v_y\sigma_y) + \eta_\parallel(t)v_z\sigma_z + \kappa_z(t)\sigma_z], \tag{37}$$

where $v_i = \mathrm{tr}[\rho\sigma_i]$ for $i = x, y, z$. The complete-positivity conditions reads

$$\eta_\parallel \pm \kappa_z \leqslant 1, \quad 1 + \eta_\parallel \geqslant \sqrt{4\eta_\perp^2 + \kappa_z^2}. \tag{38}$$

The dynamics can be reformulated in terms of a master equation of the form

$$\begin{aligned}\frac{d\rho}{dt} &= \gamma_+(t)\left(\sigma_+\rho\sigma_- - \frac{1}{2}\{\rho, \sigma_-\sigma_+\}\right)\\ &\quad + \gamma_-(t)\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\rho, \sigma_+\sigma_-\}\right) + \gamma_z(t)(\sigma_z\rho\sigma_z - \rho),\end{aligned} \tag{39}$$

where

$$\gamma_\pm(t) = \frac{\eta_\parallel(t)}{2}\frac{d}{dt}\left(\frac{1 \pm \kappa_z(t)}{\eta_\parallel(t)}\right), \quad \gamma_z(t) = \frac{1}{4}\frac{d}{dt}\ln\frac{\eta_\parallel(t)}{\eta_\perp^2(t)}. \tag{40}$$

The dynamics is CP divisible if and only if $\gamma_\pm(t) \geqslant 0$ and $\gamma_z(t) \geqslant 0$. P divisibility, instead, is satisfied whenever [49]

$$\gamma_\pm(t) \geqslant 0, \quad \sqrt{\gamma_+(t)\gamma_-(t)} + 2\gamma_z(t) > 0. \tag{41}$$

The composition of two phase-covariant dynamics is again phase covariant. If we suppose that the system undergoes a first phase-covariant dynamics Φ^1 from $t = 0$ to $t = t_1$ and later it evolves following Φ^2, then the total dynamics $\Phi = \Phi^2 \circ \Phi^1$, defined as

$$\Phi_t = \begin{cases}\Phi_t^1 & \text{if } t \leqslant t_1,\\ \Phi_{t-t_1}^2\Phi_{t_1}^1 & \text{if } t > t_1,\end{cases} \tag{42}$$

is again phase covariant, described by the functions

$$\eta_{\parallel,\perp}(t) = \begin{cases}\eta_{\parallel,\perp}^1(t) & \text{if } t \leqslant t_1,\\ \eta_{\parallel,\perp}^2(t-t_1)\eta_{\parallel,\perp}^1(t_1) & \text{if } t > t_1,\end{cases} \tag{43}$$

$$\kappa_z(t) = \begin{cases}\kappa_z^1(t) & \text{if } t \leqslant t_1,\\ \kappa_z^2(t-t_1) + \eta_\parallel^2(t-t_1)\kappa_z^1(t_1) & \text{if } t > t_1,\end{cases} \tag{44}$$

where the superscripts 1 and 2 label the functions defining, respectively, Φ^1 and Φ^2. The composition of two phase-covariant dynamics is not commutative since, in general, $\Phi^1 \circ \Phi^2 \neq \Phi^2 \circ \Phi^1$, which is evident from (43) and (44). Furthermore, the family of all phase-covariant dynamics does not form a group since, in general, the inverse of the dynamics Φ^{-1} is not positive.

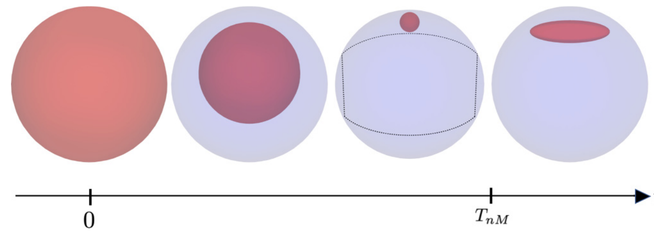

FIG. 5. Schematic representation of the dynamics of the counterexample, obtained by visualizing the Bloch sphere (light blue) and the evolved ellipsoid (dark red) at different time steps. After time $t = 0$ until time T_{nM} the image of the Bloch sphere is a uniformly contracted sphere translated in the z direction, without violating P divisibility. After T_{nM}, the image of the sphere is further contracted and translated in the z direction but also expanded in the x and y directions, thus violating P divisibility. In the third panel, the dashed line represents the border of the noncontractivity domain for the dynamical map from T_{nM} onwards, corresponding to the green area in Fig. 4. It thus clearly appears that the Bloch sphere at T_{nM} is mapped outside the noncontractivity domain. In this way, even if the second part of the dynamics is not P divisible, there is no pair of states available for the JSD to witness a revival.

C. Example showing that $\mathcal{N}^J$ is strictly weaker than $\mathcal{N}^\Delta$

Let us now employ the previously introduced phase-covariant model to build a counterexample of a dynamics which is not P divisible but leads to a zero measure of non-Markovianity for the JSD. It follows from this counterexample that $\mathcal{N}^J$ is a strictly weaker measure of non-Markovianity than $\mathcal{N}^\Delta$ since $\mathcal{N}^\Delta > 0$ for all non-P-divisible dynamics. We will actually consider a dynamics for which the memory effects are already detected by the TD, without the need to generalize to the Helstrom matrix.

The dynamical map Φ of such a counterexample is described by the functions

$$\begin{aligned}\eta_{\parallel,\perp}(\tau) &= e^{-\mu_1\tau}\sigma(1-\tau)\\ &\quad + e^{-\mu_1}e^{-\mu_2(\tau-1)}\sigma(\tau-1)\sigma(2-\tau)\\ &\quad + e^{-\mu_1-\mu_2}[(3-\tau) + A_{\parallel,\perp}(\tau-2)]\sigma(\tau-2),\end{aligned} \tag{45}$$

$$\begin{aligned}\kappa_z(\tau) &= A_\kappa\tau\sigma(2-\tau)\\ &\quad + 2A_\kappa[(3-\tau) + A_\parallel(\tau-2)]\sigma(\tau-2),\end{aligned} \tag{46}$$

where σ is the sigmoid function

$$\sigma(\tau) = \frac{1}{1+e^{-\alpha\tau}},$$

which is a smooth version of the Heaviside θ function and $\tau = t/T$ is a dimensionless time parameter, where T is a reference time determining the duration of the different stages depicted in Fig. 5. These functions, together with the corresponding rates $\gamma_\pm$ and γ_z obtained from Eq. (40), are shown in Fig. 6.

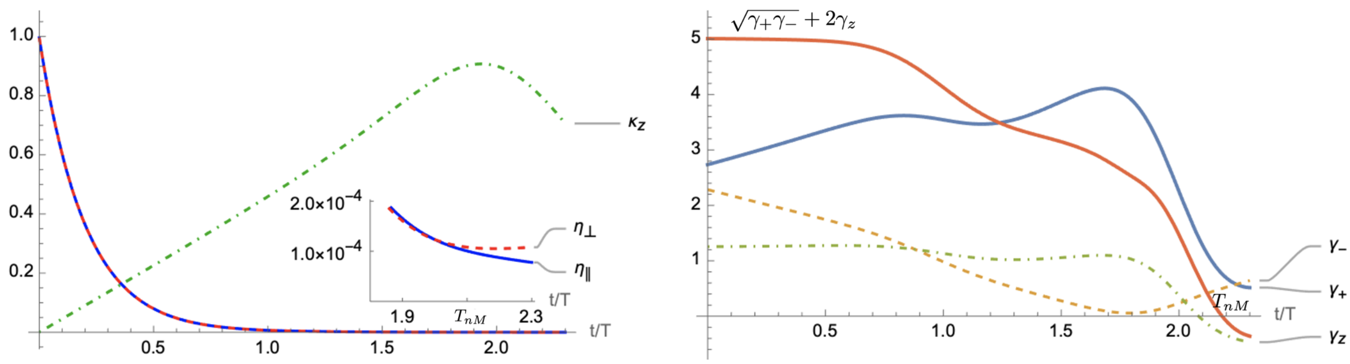

FIG. 6. Left: The functions of Eqs. (45) and (46). Non-Markovianity is present after $T_{nM} \approx 2.2T$ and corresponds to a nonmonotonic behavior of $\eta_\perp$. Right: Rates $\gamma_\pm$ and γ_z for the corresponding dynamics. The red thick line corresponds to $\sqrt{\gamma_+\gamma_-} + 2\gamma_z$, whose positivity is necessary for the dynamics to be P divisible, together with the conditions $\gamma_\pm > 0$. Non-Markovianity for $t > T_{nM}$ corresponds to a violation of divisibility. We considered the choice of the parameters $\mu_1 = 5$, $\mu_2 = 4$, $A_\parallel = 0.01$, $A_\perp = 1.01$, $A_\kappa = 0.45$, and $\alpha = 5$.

The idea of the counterexample follows from the considerations in Sec. III A: the dynamics in the time interval in which the memory effects arise will consist of a nonpositive map $\Phi_{t,s}$ similar to the one described in Fig. 4, for which the noncontractivity domain (35) is strictly smaller than the positivity domain (34). Prior to this time interval, the dynamics is P divisible such that it maps the whole Bloch sphere inside $\mathcal{PD}_{\Phi_{t,s}}$ but outside $\mathcal{NCD}_{\Phi_{t,s},J}$, so that there is no pair of states available for the JSD to witness the violations of P divisibility. A schematic representation of such dynamics is shown in Fig. 5.

The violation of P divisibility takes place for $t > T_{\mathrm{nM}}$, with $T_{\mathrm{nM}} \approx 2.2T$ as (41) is violated, and is due to the positivity of the time derivative of $\eta_\perp$, in turn leading to the negativity of the second condition appearing in (41). This behavior is shown in Fig. 6. Such a violation corresponds to a revival in the coherences, without a corresponding revival in the population, thus building on a genuine quantum effect. The fact that memory effects are due to a unital feature of the map, and not to the translation κ_z, implies that $\mathcal{N}^D(\Phi) > 0$, which, in turn, leads to $\mathcal{N}^\Delta(\Phi) > 0$.

On the other hand, we evaluated numerically that $\mathcal{N}^J(\Phi) = 0$, as can be seen in Fig. 7. The numerical analysis was performed considering all the possible initial pairs of states on the Bloch sphere, studying their time evolution and evaluating any eventual revival of the JSD, but none was found. Therefore, we can conclude that the measure of non-Markovianity arising from the JSD is strictly weaker than the one arising from the Helstrom matrix. In other words, non-P-divisible dynamics leading to a zero measure of non-Markovianity exists.

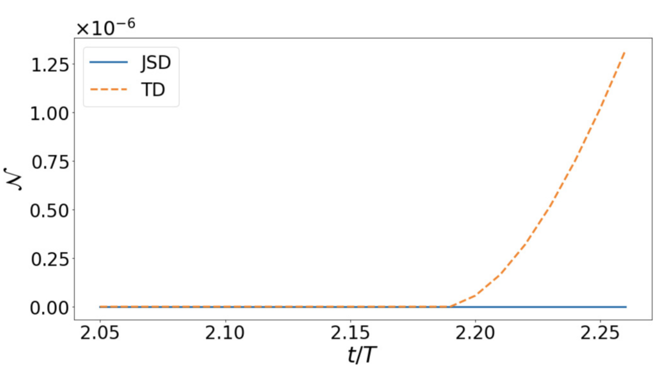

FIG. 7. Non-Markovianity measure for the JSD (solid line) and for the TD (dashed line) for the considered counterexample as a function of time. Clearly, $\mathcal{N}^D(\Phi) > 0$ since a revival in the TD is witnessed. The JSD, on the other hand, is always a monotonic function, and hence, $\mathcal{N}^J(\Phi) = 0$.

Nevertheless, this fact is also true for the TD. In order to be able to capture any violation of P divisibility as a revival of some quantifier, one has to generalize the TD to ensembles, introducing a bias parameter. One might wonder whether something similar also happens for the JSD: if we generalize it to ensembles as the Holevo skew divergence (13) or as the quantum skew divergence (16), would we be able to witness all the violations of P divisibility? Unfortunately, the answer to this question is no. In fact, considering the same counterexample, one has again $\mathcal{N}^{K_\mu}(\Phi) = \mathcal{N}^{S_\mu}(\Phi) = 0$. This fact actually is unsurprising: for the TD, the generalization to ensembles breaks the translational symmetry and makes us able to detect violations of P divisibility due to the translational components of the dynamics; for the JSD, on the other hand, there is no symmetry to break, and thus generalizing it to ensembles does not lead to any qualitative difference. In particular, one can consider a time evolution $\tilde{\Phi}$ such that $\mathcal{N}^J(\tilde{\Phi}) > 0$, while $\mathcal{N}^D(\tilde{\Phi}) = 0$, as shown in [9] considering a different phase-covariant model.

A natural question is what additional constraints d has to obey in order to have a measure of non-Markovianity equivalent to $\mathcal{N}^\Delta$. Building on our counterexample, a necessary condition that d must obey is naturally the existence of a strictly noncontractive pair of states for any nonpositive map Λ. A second condition, which is crucial for entropic distinguishability quantifiers, is the relation between the noncontractivity domain $\mathcal{NCD}_{\Lambda,d}$, depending on both the map and the quantifier, and the positivity domain $\mathcal{PD}_\Lambda$ of the map. As we have shown, if $\mathcal{NCD}_{\Lambda,d}$ is a proper subset of $\mathcal{PD}_\Lambda$, detection of the violation of divisibility after a time s can fail for maps whose image at time s, by necessity within the positivity domain, is strictly outside the noncontractivity domain. This is exactly the feature we exploited to provide the counterexample.

IV. CONCLUSIONS AND OUTLOOK

In this work, we have compared different measures for the degree of non-Markovianity in the dynamics of open systems based on distinguishability quantifiers between quantum states. In particular, we have provided evidence that the measure based on the trace norm of the Helstrom matrix is strictly stronger than all of the measures of non-Markovianity based on entropic distinguishability quantifiers, as well as stronger than the measure based on the trace distance, which is neither stronger nor weaker than the entropic ones. This is our central result. It means that the value of the measure based on the trace norm of the Helstrom matrix associated with a dynamical map Φ, namely, $\mathscr{N}^{\Delta}(\Phi)$, is greater than zero whenever this happens for the measures associated with entropic distinguishability quantifiers or with the trace distance, while the reverse is not true, as we have demonstrated here. Thus, we can conclude that the different distinguishability measures exhibit a quite different performance in the detection of nonpositive maps, which is surprising in view of similar physical interpretations outlined in Secs. II A and II B.

This result was obtained while considering the explicit expression of a qubit dynamics which is not P divisible, so that it is non-Markovian according to $\mathscr{N}^{\Delta}$, although strong numerical evidence shows that the associated entropic non-Markovianity measure $\mathscr{N}^{J}$, based on the Jensen-Shannon divergence as the distinguishability quantifier, is equal to zero. A purely analytic proof of this property, in the present or a different counterexample, would provide further insights into the relationship between the positivity of a map and its contractivity property with respect to entropic distinguishability quantifiers obtained from the quantum relative entropy, such as the Jensen-Shannon divergence.

Moreover, it might be very interesting to clarify whether the Helstrom-based quantifier is unique or whether other distinguishability quantifiers are equivalent to it as measures for quantum non-Markovianity.

ACKNOWLEDGMENTS

The authors thank A. Smirne for many fruitful discussions and careful reading of the manuscript.

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