

# Time Dynamics in Quantum Field Theory Systems

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# Zusammenfassung

In dieser Dissertation entwickeln und untersuchen wir mathematische Werkzeuge, welche es zum Ziel haben, eine rigorose Beschreibung der nicht-perturbativen Dynamik in Quantenfeldtheorien (QFT) zu ermöglichen. Der Begriff QFT ist hier zu verstehen als ein Quantensystem, welches Teilchenerzeugung und -vernichtung involviert, sowie mit der speziellen Relativitätstheorie verträglich sein kann, aber nicht muss. Die Werkzeuge zielen auf Fälle ab, in denen ein formaler Hamiltonian existiert, aber nicht mathematisch definiert ist.

Das erste untersuchte Werkzeug ist ein vor Kurzem von Lienert und Tumulka eingeführtes axiomatisches System namens Hyperflächenentwicklung (hypersurface evolution). Dieses kann als eine Alternative zu den etablierten Wightman- und Haag-Kastler-Axiomensystemen betrachtet werden, da es Rahmenbedingungen für relativistische, nicht-perturbative QFT-Systeme vorgibt. Im Gegensatz zu letzteren beiden Axiomensystemen arbeitet das Hyperflächenentwicklungs-System jedoch nicht im Heisenbergbild, sondern im Schrödingerbild. Der physikalische Zustand wird hierbei durch eine Familie von Vektoren  $\Psi_\Sigma$  beschrieben; zu jeder Cauchy-Fläche  $\Sigma$  gehört ein Vektor  $\Psi_\Sigma$ . Diese Situation ähnelt derjenigen in einer von Tomonaga und Schwinger vorgeschlagenen QFT-Beschreibung über Cauchy-Flächen-abhängige Vektoren  $\Psi_\Sigma$  im Wechselwirkungsbild.

Das Hyperflächenentwicklungs-System befindet sich in einem vergleichsweise frühen Forschungsstadium. Wir entwickeln es in dieser Arbeit weiter und diskutieren Möglichkeiten zur Modifikation, sowie zu einem Vergleich mit bestehenden Axiomensystemen der nicht-perturbativen QFT.

Eine Besonderheit des Hyperflächenentwicklungs-Systems ist, dass die Bornsche Regel nicht einfach für alle  $\Psi_\Sigma$  postuliert werden kann, sondern vielmehr als Theorem bewiesen werden muss: Unter der Voraussetzung, dass die Bornsche Regel nur auf einer gewissen Teilmenge aller Cauchy-Flächen  $\Sigma$  gilt (z.B. nur auf ebenen Cauchy-Flächen), lassen sich bereits Detektionswahrscheinlichkeiten für sämtliche Cauchy-Flächen rekonstruieren. Diese rekonstruierten Wahrscheinlichkeiten müssen nicht zwangsläufig mit den von der Bornschen Regel vorhergesagten Wahrscheinlichkeiten übereinstimmen, allerdings erscheint eine derartige Übereinstimmung sehr natürlich. In dieser Dissertation beweisen wir nun, dass für bestimmte Rekonstruktionswege tatsächlich eine Übereinstimmung vorliegt.

## Zusammenfassung

Ein wesentlicher Zweck dieser Dissertation besteht in der Einführung zahlreicher neuer mathematischer Werkzeuge im Rahmen sogenannter „erweiterter Zustandsräume“ (extended state spaces, ESSs). Bei diesen Werkzeugen handelt es sich um Vektorräume, die eine rigorose Behandlung gewisser unendlicher Größen ermöglichen, welche in formalen Berechnungen zur Cutoff-freien Renormierung von QFT-Modellen auftreten. Wir präsentieren ein allgemeines Schema, welches die Konstruktion dieser Vektorräume erlaubt. Unter den generierten Räumen finden sich insbesondere zwei Erweiterungen  $\overline{\mathcal{F}}$ ,  $\overline{\mathcal{F}}_{\text{ex}}$  eines dichten Unterraums des Fockraums, die eine mathematische Beschreibung „virtueller Teilchenzustände“ ermöglichen. Das Schema ist inspiriert von einer kürzlich entwickelten Cutoff-freien nicht-perturbativen Renormierungstechnik, genannt „Innere-Rand Bedingungen“ (interior-boundary conditions, IBC), sowie der Cutoff-freien perturbativen Epstein-Glaser-Renormierung.

Anschließend stellen wir zwei konkrete Konstruktionen vor, welche diesem Schema folgen. Die erste ist für eine nicht-perturbative Renormierung in Polaronmodellen konzeptioniert, die zweite ist zur Behandlung von Bogoliubov-Transformationen vorgesehen.

Für die erste Konstruktion beweisen wir, dass eine Cutoff-freie nicht-perturbative Renormierung für  $M$  ruhende Fermionen, die linear an ein Bosonenfeld gekoppelt sind, tatsächlich möglich ist. Dieser Fall entspricht mehreren gekoppelten Van Hove Modellen.

Die zweite Konstruktion wird verwendet, um gewisse Bogoliubov-Transformationen in einem erweiterten Sinne zu implementieren, obwohl diese die Shale-Stinespring-Bedingung verletzen und somit nicht im klassischen Sinne (sprich: auf dem Fockraum) implementiert werden können.

Für Bogoliubov-Transformationen untersuchen wir zusätzlich von Neumanns unendliche Tensorprodukträume (infinite tensor product spaces, ITP spaces)  $\widehat{\mathcal{H}}$ , welche eine weitere Fockraumerweiterung darstellen. Hier beweisen wir, dass bestimmte Bogoliubov-Transformationen, welche die Shale-Stinespring-Bedingung verletzen, unter Benutzung von  $\widehat{\mathcal{H}}$  dennoch implementiert werden können. Anschließend geben wir Beispiele an, in denen eine erfolgreiche Diagonalisierung quadratischer Hamiltonians durch eine Bogoliubov-Transformation möglich ist, wobei die Shale-Stinespring-Bedingung verletzt, aber eine Implementierung mittels  $\overline{\mathcal{F}}$  oder  $\widehat{\mathcal{H}}$  erfolgen kann.

# Summary

In this doctoral thesis, we develop and investigate new mathematical tools that are intended to allow for a rigorous description of non-perturbative quantum field theory (QFT) dynamics. Here, the term QFT is to be understood as describing a quantum system with particle creation and annihilation that can, but does not need to, comply with special relativity. The tools aim at cases where a formal Hamiltonian exists but is ill-defined.

The first investigated tool is an axiomatic setting called hypersurface evolution, which has recently been introduced by Lienert and Tumulka. One may view it as an alternative to the well-established Wightman and the Haag–Kastler axiom systems, as it sets up a framework for relativistic non-perturbative QFT dynamics. In contrast to these two systems, the hypersurface evolution setting works in the Schrödinger picture, instead of the Heisenberg picture. The state of the system is described by a family of vectors  $\Psi_\Sigma$ , one for each Cauchy surface  $\Sigma$ . This situation is similar to a QFT description suggested by Tomonaga and Schwinger, which works via Cauchy surface-dependent vectors  $\Psi_\Sigma$  in the interaction picture. The hypersurface evolution setting is at a comparably early stage of development. We further refine it in this thesis and briefly discuss, how it might be modified and related to existing axiomatic frameworks in non-perturbative QFT. It is a peculiarity of this setting, that the Born rule for all  $\Psi_\Sigma$  cannot simply be postulated, but must be proven as a theorem: Provided that the Born rule holds on a certain subset of all Cauchy surfaces  $\Sigma$  (e.g., only on flat  $\Sigma$ ), one may reconstruct detection probabilities on the set of all Cauchy surfaces. These reconstructed probabilities may or may not coincide with those predicted by the Born rule. In this dissertation, we prove that for certain reconstructions, both expressions indeed agree.

The main set of new tools, which is introduced in this thesis, is given within the “extended state space” (ESS) framework. We provide a construction scheme for vector spaces that allow for a rigorous treatment of certain infinite quantities, which appear in formal calculations concerning the cutoff-free renormalization of QFT models. Among these spaces, there are two extensions  $\overline{\mathcal{F}}, \overline{\mathcal{F}}_{\text{ex}}$  of a dense subspace of Fock space, which allow for a rigorous description of “virtual particle states”. The scheme has been inspired by a recently developed cutoff-free

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non-perturbative renormalization technique called “interior-boundary conditions” (IBC), as well as the cutoff-free perturbative Epstein–Glaser renormalization.

We present two concrete constructions following this scheme: One of them is designed to allow for a non-perturbative renormalization in polaron models, and the second is adapted to a treatment of Bogoliubov transformations.

For the first construction, we prove that a cutoff-free non-perturbative renormalization is indeed possible for  $M$  resting fermions linearly coupled to a boson field (i.e., several coupled Van Hove models).

The second construction is used to implement certain Bogoliubov transformations in an extended sense, although they violate the Shale–Stinespring condition and thus cannot be implemented on Fock space.

For Bogoliubov transformations, we also investigate a Fock space extension framework given by von Neumann’s infinite tensor product (ITP) space  $\widehat{\mathcal{H}}$ . Here, we prove that certain Bogoliubov transformations violating the Shale–Stinespring condition can nevertheless be implemented using  $\widehat{\mathcal{H}}$ . We then provide examples, where a successful diagonalization of quadratic Hamiltonians is possible by a Bogoliubov transformation, implemented using  $\widehat{\mathcal{F}}$  or  $\widehat{\mathcal{H}}$ , that violates the Shale–Stinespring condition.

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# Organization, Manuscripts and Personal Contribution

Chapter 1 is intended to give a detailed, but by no means exhaustive overview about QFT dynamics. We start with a short introduction, which provides more information about the motivation and content of this dissertation. After that, we explain the mathematical notation and provide examples for previous successful renormalizations.

Chapter 2 is derived from the article [1], which stems from a joint project with Roderich Tumulka. Both authors of [1] agreed on estimating the contribution per author in both scientific ideas and paper writing to 50%. The project can be seen as a sequel to a work by Lienert and Tumulka written in 2017 and published in 2020 [4].

While Sections 2.1 and 2.7 provide additional material elucidating the content of [1], Sections 2.2 and 2.3 contain parts of [1] that have been re-formulated and supplemented to fit the dissertation. Sections 2.4–2.6 can identically be found in [1] up to minor adaptations of the notation.

Chapter 3 introduces two Fock space extension frameworks, which play a central role in the thenceforth presented results: Von Neumann’s infinite tensor product (ITP) space– (Section 3.1) and the novel extended state space (ESS) framework (Section 3.2). This chapter has the role of explaining both frameworks and suggesting explicit choices for ITP spaces or ESSs. It is supplementary to the unpublished manuscripts [2] and [3], and only Section 3.1.1 is based on parts of those manuscripts, namely on Section 2.3 of [3].

Chapter 4 is based on the unpublished manuscript [2]. The introductory Sections 4.1 and 4.2 are adapted versions of Sections 1 and 2 of [2]. All further sections and appendices of [2] have been directly included into this dissertation as Sections 4.3–4.12 up to minor modifications.

The idea of introducing the rather unusual ESS construction was conceived in 2020, after several investigations on cutoff-free non-perturbative renormalization, as well as a comparison to perturbative renormalization techniques. Both involve

## *Organization, Manuscripts and Personal Contribution*

formal manipulations with infinite quantities, which led the author to the construction of mathematical tools that allow to directly make sense of such formal manipulations. When elaborating the construction, the author received numerous useful hints from Roderich Tumulka. Both agreed on estimating the contributions of the author in scientific ideas and paper writing to 100%.

Chapter 5 stems from the unpublished manuscript [3]. Sections 5.1 and 5.2 are modified versions of Sections 1 and 2 of [3]. In particular, Section 5.2 was greatly shortened. Here, various definitions have been abbreviated, as they are already explained in Chapter 3. All other sections and appendices of [3] have been inserted into Chapter 5 as Sections 5.8–5.12, up to minor modifications. Michał Wrochna suggested the idea of investigating Bogoliubov transformations and gave some useful references to the author. Both agreed on estimating the contributions of the author in scientific ideas and paper writing to 100%.

As customary in mathematical physics, the authors of the above-mentioned articles are ordered alphabetically.

# 1 Introduction

## 1.1 Setting, Scope and Structure

Since the first half of the 20th century, quantum models with particle creation and annihilation have found a plethora of applications in physics, ranging from quantum optics to condensed-matter and high-energy physics. Formally, the **dynamics** in the respective models are governed by a Schrödinger equation with some expression  $H$  for a Hamiltonian. Creation and annihilation of particles are described by including creation and annihilation operators  $a^\dagger$  and  $a$  into  $H$ . Mathematically, the formal expression  $H$  generates well-defined dynamics via Stone's theorem, if and only if it can be defined as a self-adjoint operator  $H : \mathcal{H} \supseteq \text{dom}(H) \rightarrow \mathcal{H}$ , where  $\mathcal{H}$  is a Hilbert space and the domain  $\text{dom}(H)$  is a dense subspace of  $\mathcal{H}$ . For systems with a variable number of particles,  $\mathcal{H}$  is a Fock space (denoted  $\mathcal{F}$ ) or a subspace of  $\mathcal{F}$ .

For systems with particle creation and annihilation, it is often comparably easy to establish a self-adjoint Hamiltonian operator  $H$ , if the number of degrees of freedom is finite (e.g., in finite volume on a lattice). This paves the way for making analytical predictions, numerical simulations with rigorous error estimates, as well as the justification of formulas describing effective dynamics. However, if one goes over to more fundamental models describing infinite volumes, a continuum, or including relativistic dispersion relations, then the formal expression for  $H$  may become ill-defined. The main reason is that certain quantities, such as integrals, grow infinite as the integration domain becomes infinitely large or reaches a pole. A careful manipulation procedure called **non-perturbative renormalization** is then needed to make sense of  $H$  as a self-adjoint operator on  $\mathcal{H}$ . Finding a renormalization procedure can be a challenging task and there are several physically interesting models, where until now, no such procedure is known. In these models, a formal  $H$  exists, and there may even be procedures that allow for recovering verifiable physical predictions, e.g., by cutoffs or by perturbation theory. But it is not known how to establish a self-adjoint operator that corresponds to  $H$ .

In fact, due to the lack of a self-adjoint  $H$ , most physical predictions in relativistic QFT are made by perturbative methods. These methods also include a manipulation procedure for establishing formally infinite quantities in a well-defined way, called **perturbative renormalization**, see also Section 1.5. When talking about

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QFT or renormalization, physicists often refer to perturbative QFT methods or renormalization, such as in standard textbooks [5, 6, 7, 8]. It is important to carefully distinguish between perturbative and non-perturbative renormalization. For various physically interesting QFT models, a perturbative renormalization procedure is well-known, while finding a non-perturbative one can be an enormous mathematical challenge. We will mainly focus on non-perturbative renormalization in this dissertation.

Further, the term QFT is not only used for denoting the perturbative or non-perturbative manipulation and proof methods, but also for the models, where these methods are employed. There exist, in turn, several senses, in which a “QFT” can be understood as a model:

In a narrower sense, a QFT is a relativistic quantum model that satisfies a certain set of axioms, such as the Haag–Kastler or the Wightman axioms presented in Section 1.2.3. Within this sense, to our best knowledge, no QFT model in 3 space dimensions has been renormalized non-perturbatively so far. This includes the standard model of particle physics, as well as its constituents: quantum electrodynamics (QED), quantum chromodynamics (QCD) and Yang–Mills theory (which is itself part of QCD).

In a wider sense, a QFT can be any quantum model with particle creation and annihilation, including non-relativistic ones. Also in the non-relativistic case, there are several physically interesting situations, where a non-perturbative renormalization procedure is, to our best knowledge, unknown. An example is the Pauli–Fierz model (see Section 1.3.7), which serves as a starting point for deriving various effective models in quantum optics.

The title of this dissertation refers to the second, wider sense. It is our aim to develop new tools that allow for describing dynamics in quantum systems with particle creation and annihilation, which are not necessarily relativistic. However, these tools are also designed to overcome divergence issues that arise in relativistic environments.

One tool is an axiomatic setting called **hypersurface evolution**, that can be seen as a Schrödinger picture-based alternative to the Wightman or the Haag–Kastler setting. The hypersurface evolution setting was recently introduced by Lienert and Tumulka [4] and is further investigated in this dissertation. We hope that this alternative approach may lead to new ideas concerning the non-perturbative renormalization of QFT models.

Further tools are provided within the construction of Fock space extensions. We both investigate the employment of von Neumann’s **infinite tensor product (ITP) space**  $\widehat{\mathcal{H}}$  and present a new construction scheme for two so-called **exten-**

**ded state spaces (ESS)**  $\overline{\mathcal{F}}, \overline{\mathcal{F}}_{\text{ex}}$ . Throughout the latter construction, we define several vector spaces that accommodate formally infinite quantities in a rigorous way.

Let us quickly sketch, how a direct renormalization using the ITP or ESS setting works on Fock space  $\mathcal{H} = \mathcal{F}$ .

A common way to perform a non-perturbative renormalization starts from a formal and ill-defined Hamiltonian  $H$ , to which position or momentum cutoffs are applied. For simplicity, we index those cutoffs by a single  $\Lambda \in [0, \infty)$ , here. The cutoff  $\Lambda$  renders divergent quantities finite, such that the cut-off Hamiltonians  $H_\Lambda$  are well-defined and generate dynamics on  $\mathcal{F}$ . In the limit  $\Lambda \rightarrow \infty$ , where the cutoff is removed,  $H_\Lambda$  formally goes over into  $H$ . However, after adding an operator  $c_\Lambda$ , called “counterterm”, and applying a “dressing transformation”  $W_\Lambda : \mathcal{F} \rightarrow \mathcal{F}$ , the limit

$$\tilde{H}_{\text{cutoff}} = \lim_{\Lambda \rightarrow \infty} W_\Lambda^{-1}(H_\Lambda + c_\Lambda)W_\Lambda \quad (1.1)$$

may exist as a self-adjoint operator on  $\text{dom}(\tilde{H}) \subseteq \mathcal{F}$ .

Our direct renormalization via ITPs or ESSs, by contrast, does not involve any cutoffs. We directly define the expression

$$\tilde{H} = W^{-1}(H + c)W, \quad (1.2)$$

where  $W$  maps from a suitable dense domain  $\mathcal{D}_{\mathcal{F}} \subset \mathcal{F}$  into the Fock space extension  $\overline{\mathcal{F}}$  or  $\widehat{\mathcal{H}}$ , and  $(H + c)$  is rigorously interpreted as an operator, mapping  $W[\mathcal{D}_{\mathcal{F}}]$  into itself. See also Figure 1.1. In Chapter 4, we will also define a smaller domain  $\tilde{\mathcal{D}}_{\mathcal{F}} \subseteq \mathcal{D}_{\mathcal{F}}$ , such that  $\tilde{H} : \tilde{\mathcal{D}}_{\mathcal{F}} \rightarrow \mathcal{D}_{\mathcal{F}}$ .

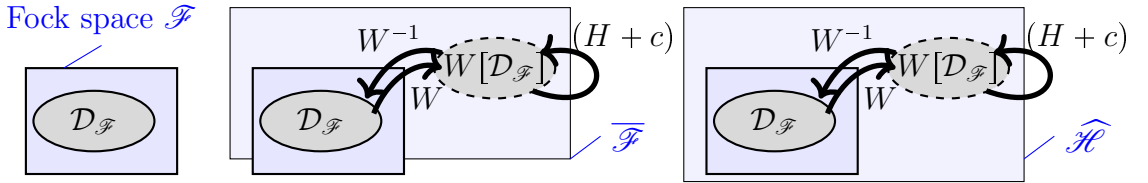


Figure 1.1: Direct renormalization of  $H$  using Fock space extensions. Color online.

The main challenge is to define the Fock space extension  $\overline{\mathcal{F}}$  or  $\widehat{\mathcal{H}}$  in such a way that  $(H + c)$  makes sense as a well-defined operator on at least a subspace of the extension. This involves handling infinite quantities, for which additional tools may be necessary, such as the second ESS  $\overline{\mathcal{F}}_{\text{ex}}$ . See Section 3.1 for a definition of the ITP space  $\widehat{\mathcal{H}}$  and Section 3.2 for a presentation of the tools constructed within the ESS framework, including  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$ .

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Note that there is no need that  $\widehat{\mathcal{H}}$ ,  $\overline{\mathcal{F}}$  or  $\overline{\mathcal{F}}_{\text{ex}}$  extend all of Fock space for the direct renormalization to work. Indeed,  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$  generally only extend the dense subspace  $\mathcal{D}_{\mathcal{F}} \subseteq \mathcal{F}$ , while  $\widehat{\mathcal{H}}$  extends all of  $\mathcal{F}$ .

The idea of a direct renormalization is not entirely new. Similar results have been recently achieved by a renormalization technique called “interior–boundary conditions” (IBC) or “abstract boundary conditions”. However, IBC renormalization works without Fock space extensions. In particular, there exists a famous theorem due to Haag [9], [10, Sect. II.1], which implies that a direct renormalization on Fock space is not possible for a relativistic QFT satisfying certain axioms. This is the main reason, why we investigate Fock space extensions in this dissertation. See also Section 1.4.3 for a discussion.

Put in an algebraic language (see also Section 1.2.3), the Fock space extensions help us find the correct counterterms  $c$  and the correct (GNS) representation on which the algebraic expression  $(H+c)$  can be interpreted as a self–adjoint operator on some Hilbert space  $\mathcal{H}$ .

### Structure of this Dissertation

In order to compare the employment of our tools with already existing non–perturbative renormalization techniques, we start by introducing some mathematical notation in Section 1.2. This includes the Fock space formalism in the Schrödinger picture, creation and annihilation operator products, but also the algebraic formulation of quantum mechanics in the Heisenberg picture. We then briefly discuss the Haag–Kastler and the Wightman axioms, which are both formulated in the algebraic framework.

In Section 1.3, we discuss some techniques and results concerning non–perturbative renormalization via cutoffs. The literature on this topic is vast and we are only able to present a fraction of all results and techniques that have ever been established on this kind of renormalization. In particular, many results have been achieved in the setting of Euclidean QFT, including those obtained by a renormalization technique called stochastic quantization, which has recently gained some considerable attention. A discussion of results in this domain is beyond the scope of this dissertation, and we can only provide references containing further information in the end of Section 1.2.3.

Section 1.4 is devoted to the cutoff–free IBC renormalization. Results using this method have strongly inspired our ESS construction. We therefore give an extensive discussion of results concerning the IBC methods and its limits, which the ESS construction is intended to overcome.

Another inspiration for the ESS construction came from a cutoff–free perturbative

renormalization method called Epstein–Glaser construction. For this reason, and since perturbative methods are commonly used for extracting predictions in QFT, we provide a brief discussion on perturbative renormalization in Section 1.5.

Chapter 2 is concerned with the presentation of results in an axiomatic setting called hypersurface evolution, that can be seen as a Schrödinger picture alternative to the Haag–Kastler and the Wightman setting. We give a short derivation of this setting from the Schrödinger picture formulation of relativistic quantum mechanics via multi–time wave functions (MTWFs) in Sections 2.1 and 2.2. The hypersurface evolution axioms can be found in Section 2.2.2.

Our **main result of this chapter** is Theorem 2.3.7, presented in Section 2.3. It concerns the derivation of Born’s rule on arbitrary Cauchy surfaces if Born’s rule is only assumed to hold on flat Cauchy surfaces. We prove it in Sections 2.4–2.6. The last Section 2.7 is devoted to a discussion of how the Wightman axioms might be derived from a hypersurface setting in future works.

In Chapter 3, we then present the general construction schemes for ITP spaces (Section 3.1) and ESSs (Section 3.2).

Von Neumann’s ITP space definition is briefly explained in Section 3.1.1, followed by a discussion, what concrete realizations of an ITP space for applications in quantum dynamics could look like.

Since the ESS construction is novel, we motivate it in Section 3.2.1, before explaining the construction scheme in Section 3.2.2. Again, a discussion on concrete realizations for applications in quantum dynamics follows.

Chapter 4 concerns the application of Fock space extensions to a class of polaron models that can be “undressed” by a certain simple Gross transformation  $W$ . This  $W$  is very similar to a Weyl transformation, where the rigorous implementation on an ITP space is known to work (see Sections 3.1.2 and 3.1.3). We therefore only apply the ESS construction to this kind of polaron models. After some introductory remarks in Sections 4.1 and 4.2, we carry out the ESS construction in Section 4.3. The following Sections 4.4 and 4.5 concern the extension of operators using  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$ .

Our **main result of this chapter** is Theorem 4.6.1 in Section 4.6, which asserts that  $\tilde{H}$ , as in (1.2), is indeed well–defined. Section 4.7 is then devoted to the establishment of self–adjoint extensions for  $\tilde{H}$ , and Sections 4.8–4.12 provide further material and proofs.

Finally, in Chapter 5, we apply both Fock space extension frameworks to Bogoliubov transformations  $\mathcal{V}$ , implemented by  $W = \mathbb{U}_{\mathcal{V}}$ . After a short introduction in

## 1 Introduction

Sections 5.1 and 5.2, we recap the well-known implementation process for Bogoliubov transformations on  $\mathcal{F}$  in Section 5.3. An implementer  $\mathbb{U}_\gamma : \mathcal{F} \rightarrow \mathcal{F}$  exists, if and only if the so-called Shale–Stinespring condition holds, see Sections 5.1 and 5.3.2. The recapped material is vital for a definition of concrete ITP spaces  $\widehat{\mathcal{H}}$  and ESSs  $\overline{\mathcal{F}}, \overline{\mathcal{F}}_{\text{ex}}$  in Section 5.4.

Our **main results of this chapter** are Theorems 5.5.5–5.5.8 in Section 5.5, which assert that a suitable implementer  $\mathbb{U}_\gamma : \mathcal{D}_{\mathcal{F}} \rightarrow \widehat{\mathcal{H}}$  or  $\mathbb{U}_\gamma : \mathcal{D}_{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$  exists, even in certain cases where the Shale–Stinespring condition fails to hold. In Section 5.6, we use these results to derive conditions for a successful diagonalization of quadratic Hamiltonians  $H$ , using the ITP and the ESS framework. These conditions are given in Propositions 5.6.2 and 5.6.3, and they assure that  $\tilde{H}$ , as in (1.2), is well-defined. Section 5.7 contains three applications for Bogoliubov transformations on Fock space extensions and Sections 5.8–5.12 provide additional material.

$\mathbb{U}_\gamma$ exists	ITP	ESS
bosonic	Thm. 5.5.5	Thm. 5.5.6
fermionic	Thm. 5.5.7	Thm. 5.5.8

$\tilde{H}$ exists	ITP and ESS
bosonic	Prop. 5.6.2
fermionic	Prop. 5.6.3

## 1.2 Mathematical Notions

### 1.2.1 Fock Space Notions

One way to describe the dynamics of a quantum system is by a family of vectors in a Hilbert space  $(\Psi_t)_{t \in \mathbb{R}} \subseteq \mathcal{H}$ , where  $\Psi_t$  represents the state of the system at time  $t$ . Throughout the entire dissertation, we will use  $\mathcal{H}$  as a placeholder for the Hilbert space representing a quantum system, which depends on the exact model. If  $\mathcal{H}$  bears a certain structure, we will call it a **Fock space** and denoted it by  $\mathcal{H} = \mathcal{F}$ . We explain this structure in the following.

First, we consider a system with an indefinite number  $N \in \mathbb{N}_0$  of particles belonging to one species, which are at positions  $\mathbf{x}_j = (x_j^1, \dots, x_j^d) \in X \subseteq \mathbb{R}^d$ . For  $X$ , we assume the existence of a measure  $\mu_X$ , for instance the Lebesgue or a spectral measure, such that  $L^p$ -spaces can be defined. The configuration vector for these  $N$  particles is given by

$$q = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in X^N. \quad (1.3)$$

All possible configurations for  $N \in \mathbb{N}_0$  make up the **ordered configuration space**

$$\mathcal{Q}(X) := \bigsqcup_{N \in \mathbb{N}_0} \mathcal{Q}^{(N)}(X) := \bigsqcup_{N \in \mathbb{N}_0} X^N, \quad (1.4)$$

where  $\mathcal{Q}^{(N)}(X)$  is called the  $(N)$ -sector of  $\mathcal{Q}$  and where the  $(0)$ -sector only consists of a single so-called vacuum configuration  $X^0 = \{\emptyset\}$ . The measure  $\mu_X$  on  $X$  implies a measure  $\mu_{\mathcal{Q}}$  on  $\mathcal{Q}(X)$ , with respect to which we can define the **Fock space (for one particle species)**

$$\mathcal{F}(X) := L^2(\mathcal{Q}(X), \mathbb{C}, \mu_{\mathcal{Q}}). \quad (1.5)$$

This is a Hilbert space with scalar product  $\langle \Phi, \Psi \rangle = \int_{\mathcal{Q}(X)} \overline{\Phi(q)} \Psi(q) dq$ , where  $\Phi(\cdot)$  and  $\Psi(\cdot)$  are representative functions  $\mathcal{Q} \rightarrow \mathbb{C}$  for the Fock space vectors  $\Phi, \Psi \in \mathcal{F}$ . In the following, we will drop the  $(X)$  whenever it is not explicitly needed.

For physical systems, the vector  $\Psi \in \mathcal{F}(X)$  now has to satisfy certain symmetry conditions. These can be expressed using the symmetrization operators  $S_+, S_- : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$  given by

$$(S_{\pm} \Psi)(\mathbf{x}_1, \dots, \mathbf{x}_N) := \frac{1}{N!} \sum_{\sigma \in S_N} (\pm 1)^{(1-\text{sgn}(\sigma))/2} \Psi(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)}), \quad (1.6)$$

where  $S_N$  denotes the permutation group and  $(1-\text{sgn}(\sigma))/2$  is 0 if the permutation is even, and 1 if the permutation is odd. The symmetry condition is now the requirement that vectors be elements of the **symmetrized Fock spaces**

$$\mathcal{F}_{\pm}(X) := S_{\pm}[\mathcal{F}(X)], \quad (1.7)$$

which are sub-Hilbert spaces of  $\mathcal{F}$ . In case of  $\Psi \in \mathcal{F}_+(X)$ , the particle of the species is called a **boson** and in case  $\Psi \in \mathcal{F}_-(X)$ , it is called a **fermion**.

An alternative description of symmetry is given by using the **unordered configuration space**  $\Gamma(X)$ , whose elements  $q_{\Gamma}$  are not a tuple of  $N$  coordinates, but rather a finite set  $q_{\Gamma} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ :

$$\Gamma(X) := \{q_{\Gamma} \subset X \mid |q_{\Gamma}| < \infty\}, \quad (1.8)$$

with sectors

$$\Gamma^{(N)}(X) := \{q_{\Gamma} \subset X \mid |q_{\Gamma}| = N\}, \quad \Gamma(X) = \bigsqcup_{N \in \mathbb{N}_0} \Gamma^{(N)}(X). \quad (1.9)$$

The measure  $\mu_X$  implies a measure  $\mu_{\Gamma}$  on  $\Gamma(X)$ . One may embed the corresponding space  $L^2(\Gamma(X), \mathbb{C}, \mu_{\Gamma})$  into  $\mathcal{F}_{\pm}$  as follows: For each  $q_{\Gamma} = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$  we fix an arbitrary ordering of the coordinates  $\mathbf{x}_1, \dots, \mathbf{x}_N$ , such that it becomes unique what is meant by the configuration  $(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)})$ . Then for each  $\Psi \in \mathcal{F}_{\pm}$  we may

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choose a representative function  $\Psi(\cdot)$  and identify with  $\Psi$  the vector  $\Psi_\Gamma \in \Gamma(X)$  whose representative function is<sup>1</sup>

$$\Psi_\Gamma(\{\mathbf{x}_1, \dots, \mathbf{x}_N\}) := \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (\pm 1)^{(1-\text{sgn}(\sigma))/2} \Psi(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)}). \quad (1.10)$$

This identification is a surjective, but generally not injective linear map: Consider the set of collision configurations  $\mathcal{Q}_{\text{col}}$  and of non-collision configurations  $\mathring{\mathcal{Q}}$

$$\begin{aligned} \mathcal{Q}_{\text{col}}(X) &:= \{q \in \mathcal{Q}(X) \mid \mathbf{x}_i = \mathbf{x}_j \text{ for some } i, j \in \{1, \dots, N\}\}, \\ \mathring{\mathcal{Q}}(X) &:= \mathcal{Q}(X) \setminus \mathcal{Q}_{\text{col}}(X). \end{aligned} \quad (1.11)$$

Each  $q \in \mathring{\mathcal{Q}}$  is associated with a unique  $q_\Gamma(q) \in \Gamma(X)$  and, by symmetry of  $\Psi \in \mathcal{F}_\pm$ , fixing  $\Psi(q)$  specifies the value of  $\Psi(\cdot)$  at exactly all  $q' \in \mathring{\mathcal{Q}}$  with  $q_\Gamma(q') = q_\Gamma(q)$  (see Figure 1.2), namely

$$\Psi(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)}) = \frac{(\pm 1)^{(1-\text{sgn}(\sigma))/2}}{\sqrt{N!}} \Psi_\Gamma(\{\mathbf{x}_1, \dots, \mathbf{x}_N\}). \quad (1.12)$$

So for a given  $\Psi_\Gamma$ , the required values for a  $\Psi$  to be identified with  $\Psi_\Gamma$  are given at all  $q \in \mathring{\mathcal{Q}}$ . It is easy to find a preimage of  $\Psi_\Gamma$  by fixing  $\Psi(q)$  according to (1.12) at  $q \in \mathring{\mathcal{Q}}$  and choosing arbitrary values at  $\mathcal{Q}_{\text{col}}$ . So the identification  $\Psi \mapsto \Psi_\Gamma$  is surjective. Its kernel is given by

$$\left\{ \Psi \in \mathcal{F}_\pm \mid \Psi(q) = 0 \quad \forall q \in \mathring{\mathcal{Q}} \right\}. \quad (1.13)$$

If  $\mu_X$  is absolutely continuous with respect to the Lebesgue measure, then  $\mathcal{Q}_{\text{col}}$  is a null set, since on each  $(N)$ -sector, it is a finite union of codimension- $d$  hyperplanes (which have Lebesgue measure 0). In that case, the kernel (1.13) is  $\{0\}$  and the identification is bijective, so  $L^2(\Gamma(X)) \cong \mathcal{F}_\pm$ . For more general measures  $\mu_X$ , the identification  $\Psi \mapsto \Psi_\Gamma$  is not necessarily bijective. Only for fermions, we can guarantee bijectivity, since symmetry enforces  $\Psi(q) = 0$  at  $q \in \mathcal{Q}_{\text{col}}$ , so the kernel (1.13) is  $\{0\}$ .

A system with  $n$  species, each being a boson or a fermion, is described by a

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<sup>1</sup>The identification is independent of the choice of the  $L^2$ -representative function, since a modification of  $\Psi(\cdot)$  on a null set results in a modification of  $\Psi_\Gamma(\cdot)$  on a null set, which leaves the represented vector  $\Psi_\Gamma \in L^2(\Gamma(X))$  invariant.

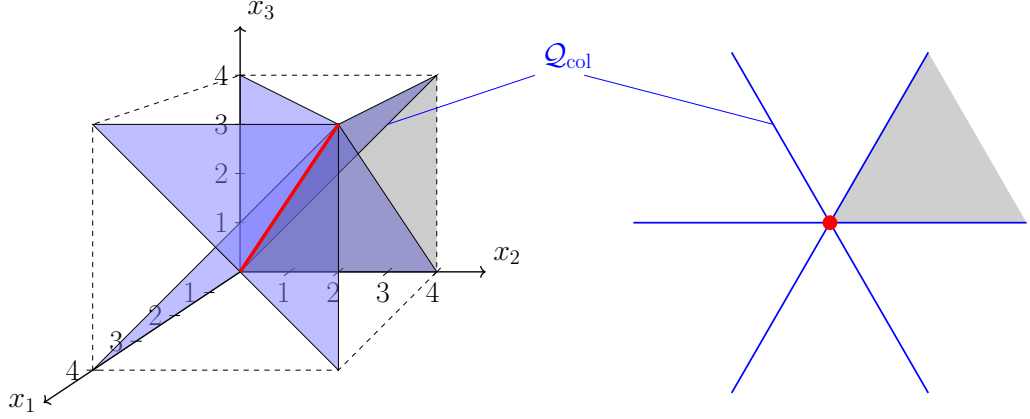


Figure 1.2: On the (3)-sector, the set of collision configurations  $\mathcal{Q}_{\text{col}}$  divides the configuration space  $\mathcal{Q}^{(3)}$  into  $3! = 6$  sectors. Specifying  $\Psi(q)$  on one sector determines  $\Psi(q)$  for  $\Psi \in \mathcal{F}_{\pm}$  on all other sectors. Color online.

tensor product Hilbert space

$$\mathcal{H} = \underbrace{\mathcal{F}_+ \otimes \dots \otimes \mathcal{F}_+ \otimes \mathcal{F}_- \otimes \dots \otimes \mathcal{F}_-}_{n \text{ factors}}. \quad (1.14)$$

In that case, the configuration space of the system is given by

$$\mathcal{Q}(X) := \bigsqcup_{N_1, \dots, N_n \in \mathbb{N}_0} \mathcal{Q}^{(N_1, \dots, N_n)}(X) = \bigsqcup_{N_1, \dots, N_n \in \mathbb{N}_0} X^{N_1} \times \dots \times X^{N_n}, \quad (1.15)$$

which is a generalization of  $\mathcal{Q}(X)$  in (1.4) to many particle species. A measure  $\mu_{\mathcal{Q}}$  can then again be naturally defined on  $\mathcal{Q}(X)$  and the corresponding **Fock space (for  $n$  particle species)**  $\mathcal{F}$  is again given by (1.5). Just as  $\mathcal{Q}$ , also  $\mathcal{F}$  can be decomposed into sectors as

$$\mathcal{F}(X) := \bigoplus_{N_1, \dots, N_n \in \mathbb{N}_0} \mathcal{F}^{(N_1, \dots, N_n)}(X). \quad (1.16)$$

Since physical operations preserve symmetry, it is also customary to drop the symmetrization by setting  $\mathcal{H} = \mathcal{F}$ , having in mind that it could be imposed at any time by an application of  $S_{\pm}$  to  $\Psi \in \mathcal{H}$ .

Particles with spin  $s$  (and correspondingly  $2s + 1$  spin degrees of freedom) can

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be treated as  $2s + 1$  spinless particle species, so the **Fock space with spin** is then

$$\mathcal{H} = \underbrace{\mathcal{F}_\pm \otimes \dots \otimes \mathcal{F}_\pm}_{2s+1 \text{ times}}. \quad (1.17)$$

A second way to treat particle species with spin is to define  $\Psi \in \mathcal{H}$  as a vector-valued function, where on each sector  $\mathcal{Q}^{(N)}$ , the vector has  $(2s + 1)^N$  entries corresponding to any tuple  $\mathbf{r} = (r_1, \dots, r_N)$  of  $N$  spin indices  $r_j \in \{1, \dots, 2s + 1\}$ . We set

$$\mathcal{F} = \bigoplus_{N \in \mathbb{N}_0} L^2(\mathcal{Q}^{(N)}, \mathbb{C}^{(2s+1)^N}) = \bigoplus_{N \in \mathbb{N}_0} L^2(\mathcal{Q}^{(1)}, \mathbb{C}^{(2s+1)})^{\otimes N}, \quad (1.18)$$

and obtain  $\mathcal{H}$  after imposing suitable symmetry conditions. A third equivalent way of treating spins is to introduce a **spin-configuration space**

$$\mathcal{Q}_s(X) := \bigsqcup_{N \in \mathbb{N}_0} \mathcal{Q}_s^{(N)}(X) := \bigsqcup_{N \in \mathbb{N}_0} \bigsqcup_{r_1, \dots, r_N=1}^{2s+1} X^N. \quad (1.19)$$

So  $\mathcal{Q}_s^{(N)}(X)$  consists of  $(2s + 1)^N$  identical copies of  $X^N$ . Configurations with spin can then be written as  $q_s = (q, \mathbf{r}) \in \mathcal{Q}_s(X)$ , with  $\mathbf{r}$  indicating in which copy of  $X^N$  the configuration  $q_s$  lies. The Fock space with spin is then given by  $\mathcal{F} = L^2(\mathcal{Q}_s, \mathbb{C})$  with sectors  $\mathcal{F}^{(N)} = L^2(\mathcal{Q}_s^{(N)}, \mathbb{C})$ . The corresponding unordered spin-configuration space is defined as

$$\Gamma_s(X) := \{ \{(\mathbf{x}_1, r_1), \dots, (\mathbf{x}_N, r_N)\} \mid \mathbf{x}_j \in X, r_j \in \{1, \dots, 2s + 1\}, N \in \mathbb{N}_0 \}. \quad (1.20)$$

Of course, it is also possible to consider several particle species with spin, in which case the generalization of  $\mathcal{Q}_s, \Gamma_s$  and  $\mathcal{F}$  via (1.15) is straightforward.

We will also consider systems that are restricted to a subset of sectors, for instance if particle numbers are fixed or have a maximal value. In that case,  $\mathcal{H}$  is a subspace of  $\mathcal{F}$ . Other cases will require  $\mathcal{H}$  being an abstract Hilbert space without an a priori identification of vectors with elements of  $\mathcal{F}$ .

The Hilbert space  $\mathcal{H}$  will be called a **Fock space**<sup>2</sup> and denoted  $\mathcal{F}$ , whenever it comes with at least one decomposition into sectors  $\mathcal{F} = \bigoplus_{N \in \mathbb{N}_0} \mathcal{F}^{(N)}$ .

Creation and annihilation operators  $a_\pm^\dagger(f), a_\pm(f)$  for one species of bosons (+) or fermions (−) and with form factor  $f \in L^2(X)$  can be defined on a dense subspace

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<sup>2</sup>We use the term “Fock space” in a comparably wide sense, here. Other authors only call  $\mathcal{H} = \mathcal{F}$  a Fock space, if  $\mathcal{F} = \bigoplus_{N \in \mathbb{N}_0} \mathfrak{h}^{\otimes N}$  with  $\mathfrak{h}$  being a “one-particle Hilbert space”. Or, even more exclusively, only one specific space  $\mathcal{F}$  of this form is called “the Fock space”.

of  $\mathcal{F}$  by stating its actions on configuration space functions

$$\begin{aligned} (a_{\pm}^{\dagger}(f)\Psi)(q) &= \sum_{j=1}^N \frac{(\pm 1)^j}{\sqrt{N}} f(\mathbf{x}_j) \Psi(q \setminus \mathbf{x}_j), \\ (a_{\pm}(f)\Psi)(q) &= \sqrt{N+1} \int_X \overline{f(\mathbf{x})} \Psi(q, \mathbf{x}) d\mathbf{x}. \end{aligned} \quad (1.21)$$

Here  $(q \setminus \mathbf{x}_j) \in \mathcal{Q}^{(N-1)}$  is the configuration  $q = (\mathbf{x}_1, \dots, \mathbf{x}_N)$  with entry  $\mathbf{x}_j$  removed. By a similar null-set argument as above, the definition in (1.21) is independent of the choice of a representative function  $\Psi(\cdot)$  for the respective  $\Psi \in \mathcal{F}$ .

The extension of  $a_{\pm}^{\dagger}(f), a_{\pm}(f)$  to many particle species with spins (which can in turn be treated as several particle species without spin) works as follows: If an operator  $A$  (for instance  $a_{\pm}^{\dagger}(f)$  or  $a_{\pm}(f)$ ) is defined on  $\text{dom}(A) \subset \mathcal{H}_1$ , then its extension to  $\mathcal{H}_1 \otimes \mathcal{H}_2$  is given by<sup>3</sup>  $A \otimes 1 : \text{dom}(A) \otimes_a \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$  and will, for simplicity, also be denoted by  $A$ .

We remark that the bosonic operators  $a_{+}^{\dagger}, a_{+}$  cannot be defined on all of  $\mathcal{F}$ , but only a dense subspace, while the fermionic operators  $a_{-}^{\dagger}, a_{-}$  are bounded and hence defined on all of  $\mathcal{F}$ . Moreover,  $a_{\pm}^{\dagger}, a_{\pm}$  preserve symmetry, i.e., they map the respective space  $\mathcal{F}_{\pm}$  into itself, where the restriction of  $a_{+}^{\dagger}, a_{+}$  to  $\mathcal{F}_{+}$  is still a densely defined operator.

Definition (1.21) directly implies the canonical commutation/anticommutation relations (CCR/CAR) as strong operator identities on a dense domain in  $\mathcal{F}$ :

$$[a_{\pm}(f), a_{\pm}^{\dagger}(g)]_{\pm} = \langle f, g \rangle, \quad [a_{\pm}(f), a_{\pm}(g)]_{\pm} = [a_{\pm}^{\dagger}(f), a_{\pm}^{\dagger}(g)]_{\pm} = 0, \quad (1.22)$$

with commutator  $[A, B]_{+} = AB - BA$  and anticommutator  $[A, B]_{-} = \{A, B\} = AB + BA$ . We remark that for fermions, the anticommutation relations (1.22) imply  $a^{\dagger}(f)a^{\dagger}(f)\Psi = 0$ , so there are no Fock space vectors containing more than one particle with the same one-particle wavefunction  $f \in L^2(X)$  (which is also called the Pauli exclusion principle).

The set of creation and annihilation operators generate a  $*$ -algebra

$$\mathcal{A} := \mathcal{A}_{\pm} \quad \text{generated by} \quad \{a_{\pm}^{\dagger}(f), a_{\pm}(f) \mid f \in L^2(X)\}, \quad (1.23)$$

with involution  $*$  defined by  $a_{\pm}(f)^* = a_{\pm}^{\dagger}(f)$  and  $(a_{\pm}^{\dagger}(f))^* = a_{\pm}(f)$ . That means,

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<sup>3</sup>Here, the algebraic tensor product  $\otimes_a$  denotes all finite linear combinations of tensor products. So  $\mathcal{H}_1 \otimes_a \mathcal{H}_2$  contains all  $\Psi = \sum_{j=1}^{j_{\max}} \psi_{j,1} \otimes \psi_{j,2}$  with  $\psi_{j,1} \in \mathcal{H}_1, \psi_{j,2} \in \mathcal{H}_2$ . By contrast, the above-used Hilbert space tensor product  $\mathcal{H}_1 \otimes \mathcal{H}_2$  may also contain infinite linear combinations and again renders a Hilbert space.

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$\mathcal{A}_\pm$  contains all sums of products of expressions  $a_\pm^\dagger(f), a_\pm(f)$ . In the fermionic case all operators in  $\mathcal{A}_-$  are bounded<sup>4</sup>, so  $\mathcal{A}_-$  is even a  $C^*$ -algebra.

It is also customary to write

$$a_\pm^\dagger(f) = \int a_{\pm, \mathbf{x}}^\dagger f(\mathbf{x}) d\mathbf{x}, \quad a_\pm(f) = \int a_{\pm, \mathbf{x}} f(\mathbf{x}) d\mathbf{x}, \quad (1.24)$$

where the expressions  $a_{\pm, \mathbf{x}}^\dagger, a_{\pm, \mathbf{x}}$  are operator-valued functionals. That means, they each correspond to a linear map  $L^2(X) \rightarrow \mathcal{A}_\pm$ , which maps  $f \mapsto a_{\pm, \mathbf{x}}^\dagger(f)$  or  $f \mapsto a_{\pm, \mathbf{x}}(f)$ .

For fermions, boundedness of  $a_-^\dagger(f), a_-(f)$  implies that these functionals are continuous from the usual Hilbert space topology on  $L^2(X)$  induced by  $\langle \cdot, \cdot \rangle$  to  $\mathcal{A}_-$  equipped with the operator norm topology.

However, for open  $X \subseteq \mathbb{R}^d$ , it is nevertheless customary to restrict  $f$  to the space  $\mathcal{E}(X) = C^\infty(X)$  of smooth functions, to the space  $\mathcal{D}(X) = C_c^\infty(X)$  of smooth functions with compact support, or to the Schwartz space (for  $X = \mathbb{R}^d$ )

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in \mathcal{E} \mid \sup_{\mathbf{x} \in \mathbb{R}^d} |\mathbf{x}^\beta (D^\alpha f)(\mathbf{x})| < \infty \right\}, \quad (1.25)$$

where  $\alpha, \beta$  run through all multi-indices of the form  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$  and characterize the monomial  $\mathbf{x}^\beta = (x^1)^{\beta_1} \dots (x^d)^{\beta_d}$  and the derivative  $D^\alpha = \partial_{x^1}^{\alpha_1} \dots \partial_{x^d}^{\alpha_d}$ . Note that  $\mathcal{D} \subseteq \mathcal{S} \subseteq \mathcal{E}$  and for  $X = \mathbb{R}^d$  one has  $\mathcal{D} \subseteq \mathcal{E}$ . The topologies on  $\mathcal{D}, \mathcal{S}$  and  $\mathcal{E}$  are induced by the seminorms

$$\|f\|_{m, K} = \sup_{\mathbf{x} \in K, |\alpha| \leq m} |D^\alpha f(\mathbf{x})|, \quad (1.26)$$

with  $K \subset X$  running through all compact subsets and  $m \in \mathbb{N}_0$ , and where  $|\alpha| = \sum_{j=1}^d \alpha_j$ . With these seminorms,  $\mathcal{D}, \mathcal{S}$  and  $\mathcal{E}$  are locally convex spaces, which allow for a convenient mathematical treatment [11, Part III], [12, 13].

Elements of either of the topological dual spaces  $\mathcal{D}' \supseteq \mathcal{S}' \supseteq \mathcal{E}'$  are called **distributions**, while those in  $\mathcal{S}'$  bear the name **tempered distribution**.

The map  $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{A}_-, f \mapsto a_-(f)$  is now an **operator-valued distribution**, which means that there is a dense domain  $\mathcal{D}_{\mathcal{F}} \subseteq \mathcal{F}(\mathbb{R}^d)$ , such that for all  $\Psi_1, \Psi_2 \in \mathcal{D}_{\mathcal{F}}$ , the map

$$\mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}, \quad f \mapsto \langle \Psi_1, a_-(f) \Psi_2 \rangle \quad (1.27)$$

---

<sup>4</sup>This can be seen from  $\|a_-^\dagger(f)\Psi\|^2 = \langle \Psi, a_-(f) a_-^\dagger(f) \Psi \rangle = \|f\|^2 - \langle \Psi, a_-^\dagger(f) a_-(f) \Psi \rangle \leq \|f\|^2$ . A similar estimate holds for  $a_-(f)$  and implies bounds for arbitrary operator products.

is a tempered distribution, i.e., an element of  $\mathcal{S}'$ . It is easy to see that also the maps  $f \mapsto a_-^\dagger(f)$  and  $f \mapsto (a_-^\dagger(f) + a_-(f))$  are operator-valued distributions.

For bosons, the operator-valued functionals are not continuous, since  $a_+^\dagger(f), a_+(f)$  are unbounded (for  $f \neq 0$ ). However, one may easily see that they are operator-valued distributions: Consider the vacuum vector  $\Omega \in \mathcal{F}$  defined by  $\Omega^{(0)} = 1$  and  $\Omega^{(N)} = 0$  for  $N \geq 1$  and consider the domain<sup>5</sup>

$$\mathcal{D}_{\mathcal{F}} = \text{span}\{\Psi \in \mathcal{F}(X) \mid \Psi = a^\dagger(f_1) \dots a^\dagger(f_N)\Omega, f_j \in \mathcal{D} = C_c^\infty(X)\} \quad (1.28)$$

(this is a special choice for the domain  $\mathcal{D}_{\mathcal{F}}$ , mentioned in the introductory Section 1.1, although we will go over to more general definitions of  $\mathcal{D}_{\mathcal{F}}$ , later). So each  $\Psi \in \mathcal{D}_{\mathcal{F}}$  can be written as a finite linear combination  $\Psi = \sum_{m=1}^M \Psi_m$  with

$$(\Psi_m)^{(N_m)}(\mathbf{x}_1, \dots, \mathbf{x}_{N_m}) = \sqrt{N_m!} S_\pm f_1(\mathbf{x}_1) \dots f_{N_m}(\mathbf{x}_{N_m}) \quad (1.29)$$

for some  $N_m \in \mathbb{N}_0$  and  $(\Psi_m)^{(N)} = 0$  on all other sectors. Now an explicit calculation shows that for  $\Psi_1, \Psi_2 \in \mathcal{D}_{\mathcal{F}}$ , the scalar product  $\langle \Psi_1, a_+(f)\Psi_2 \rangle$  is a finite linear combination of expressions of the form  $\langle f, f_j \rangle, f_j \in \mathcal{D}$ . Since each map  $f \mapsto \langle f, f_j \rangle$  is a tempered distribution on  $f \in \mathcal{S}$ , so is  $f \mapsto \langle \Psi_1, a_+(f)\Psi_2 \rangle$ . So  $a_+(f)$  is an operator-valued distribution and it is easy to establish a similar statement for  $f \mapsto a_+^\dagger(f)$  and  $f \mapsto (a_+^\dagger(f) + a_+(f))$ .

It is also possible to construct bounded operator-valued functionals by introducing the (unbounded) field operators  $\phi(f)$  and the (bounded) Weyl operators  $W(f)$ :

$$\phi(f) = a_+^\dagger(f) + a_+(f), \quad W(f) = e^{-i\phi(if)} = e^{a_+^\dagger(f) - a_+(f)}. \quad (1.30)$$

In that case, all  $W(f)$  are bounded, so the **Weyl algebra**

$$\mathcal{A}_W \quad \text{generated by} \quad \{W(f) \mid f \in L^2(X)\} \quad (1.31)$$

is even a  $C^*$ -algebra instead of just a  $*$ -algebra, and the corresponding operator-valued function  $L^2(X) \rightarrow \mathcal{A}_W, f \mapsto W(f)$  is bounded.

In the following, we will drop the indices “ $\pm$ ” if no explicit distinction is needed.

### 1.2.2 Operator Products and Hamiltonian Formalism

The algebra  $\mathcal{A}$  contains operator products of the form  $a^\sharp(f_1) \dots a^\sharp(f_N)$  with  $\sharp \in \{\cdot, \dagger\}$ . This is, however, not sufficient to describe certain physically desirable observables, such as the particle number, which is represented by the **number operator**

<sup>5</sup>By span, we mean the set of all **finite** linear combinations.

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$N : \mathcal{F} \supset \text{dom}(N) \rightarrow \mathcal{F}$  defined by

$$(N\Psi)(\mathbf{x}_1, \dots, \mathbf{x}_{N'}) = N'\Psi(\mathbf{x}_1, \dots, \mathbf{x}_{N'}). \quad (1.32)$$

It is formally expressed as

$$N = \int_X a_{\mathbf{x}}^\dagger a_{\mathbf{x}} d\mathbf{x} = \int_{X \times X} a_{\mathbf{x}_1}^\dagger a_{\mathbf{x}_2} \delta(\mathbf{x}_1 - \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2. \quad (1.33)$$

The tensor product of operator-valued distributions  $a_{\mathbf{x}_1}^\dagger a_{\mathbf{x}_2} = a_{\mathbf{x}_1}^\dagger \otimes a_{\mathbf{x}_2}$  can again be considered as a functional on elements of<sup>6</sup>  $\mathcal{D}(X) \otimes \mathcal{D}(X) = \mathcal{D}(X^2)$  or  $\mathcal{S}(\mathbb{R}^d) \otimes \mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^{2d})$  (in case  $X = \mathbb{R}^d$ ). However,  $a_{\mathbf{x}_1}^\dagger a_{\mathbf{x}_2}$  is not defined on the distribution  $\delta(\mathbf{x}_1 - \mathbf{x}_2) \notin \mathcal{S}(\mathbb{R}^{2d})$ . Similar issues arise for operator products of the form

$$A = \int_{X \times \dots \times X} a_{\mathbf{x}_1}^\# \dots a_{\mathbf{x}_N}^\# f(\mathbf{x}_1, \dots, \mathbf{x}_N) d\mathbf{x}_1 \dots d\mathbf{x}_N. \quad (1.34)$$

If  $f$  is a distribution with  $f \notin \mathcal{D}(X)$  (or with  $f \notin \mathcal{S}(\mathbb{R}^{Nd})$  in case  $X = \mathbb{R}^d$ ), then the expression  $A$  is a priori not a densely defined operator on  $\mathcal{F}$ . However, the case  $f \in \mathcal{S}'(\mathbb{R}^{Nd})$  still allows for taking the Fourier transform of  $f$ , where we use the notation and convention

$$\hat{f}(\mathbf{p}_1, \dots, \mathbf{p}_N) = (2\pi)^{-\frac{Nd}{2}} \int_{\mathbb{R}^{Nd}} f(\mathbf{x}_1, \dots, \mathbf{x}_N) e^{-i\sum_{j=1}^N \mathbf{x}_j \mathbf{p}_j} d\mathbf{x}_1 \dots d\mathbf{x}_N. \quad (1.35)$$

The inverse Fourier transform of a function will be denoted using a check, i.e.,  $\check{v}(\mathbf{x})$  is the Fourier inverse of  $v(\mathbf{p})$ . For operators  $a_{\pm}^\#(f)$ , we will call both  $\hat{f}$  and  $f$  a “form factors”. In the same way, also  $\Psi \in \mathcal{F}(\mathbb{R}^d)$  can be Fourier-transformed to  $\hat{\Psi} \in \mathcal{F}(\mathbb{R}^d)$ . Another notation we use is  $\hat{f} = \mathcal{F}(f)$  for the Fourier transform and  $\check{v} = \mathcal{F}^{-1}(v)$  for the Fourier inverse. Further, we will reserve the variables  $\mathbf{x}$  and  $\mathbf{y}$  for position coordinates, whereas  $\mathbf{p}$  and  $\mathbf{k}$  are used to denote the corresponding momentum coordinates. The expression (1.34) can then formally be written as

$$A = \int_{\mathbb{R}^{Nd}} a_{\mathbf{p}_1}^\# \dots a_{\mathbf{p}_N}^\# \hat{f}(\mathbf{p}_1, \dots, \mathbf{p}_N) d\mathbf{p}_1 \dots d\mathbf{p}_N, \quad (1.36)$$

where  $a_{\mathbf{p}}^\#$  is defined such that for  $f \in L^2(\mathbb{R}^d)$ ,

$$\int a_{\mathbf{p}}^\# \hat{f}(\mathbf{p}) d\mathbf{p} \hat{\Psi} = \mathcal{F} \left( \int a_{\mathbf{x}}^\# f(\mathbf{x}) d\mathbf{x} \Psi \right). \quad (1.37)$$

---

<sup>6</sup>While for Hilbert spaces,  $\otimes$  is used to denote the Hilbert space tensor product, the same symbol in conjunction with  $\mathcal{D}$ ,  $\mathcal{E}$  or  $\mathcal{S}$  denotes the topological tensor product. The spaces  $\mathcal{D}$ ,  $\mathcal{E}$  and  $\mathcal{S}$  allow for an easy identification of topological tensor products as  $\mathcal{D}(X) \otimes \mathcal{D}(Y) \cong \mathcal{D}(X \times Y)$ ,  $\mathcal{E}(X) \otimes \mathcal{E}(Y) \cong \mathcal{E}(X \times Y)$  and  $\mathcal{S}(\mathbb{R}^{d_1}) \otimes \mathcal{S}(\mathbb{R}^{d_2}) \cong \mathcal{S}(\mathbb{R}^{d_1+d_2})$ , see [11, Chap. 50].

We will also consider expressions like (1.36) with momentum coordinates  $\mathbf{p}, \mathbf{k} \in X \subseteq \mathbb{R}^d$  where the set  $X$  possibly does not allow for taking a Fourier inverse.

An easy special case where operator products with  $f$  being a distribution can nevertheless be defined on Fock space is the **second quantization** of a multiplication operator: Let  $\omega : X \rightarrow \mathbb{C}$  be a sufficiently regular function, such that the equally denoted operator  $\omega$ , which maps  $\psi(\mathbf{k}) \mapsto \omega(\mathbf{k})\psi(\mathbf{k})$ , is defined on a dense domain in  $L^2(X)$ . Then, the second quantization of  $\omega$  is given by the operator product

$$d\Gamma(\omega) = \int_X a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \omega(\mathbf{k}) d\mathbf{k}, \quad (1.38)$$

so within (1.36),  $\hat{f}(\mathbf{k}_1, \mathbf{k}_2) = \omega(\mathbf{k}_1)\delta(\mathbf{k}_1 - \mathbf{k}_2)$ . Physically, adding an operator  $d\Gamma(\omega)$  to the Hamiltonian assigns a dispersion relation  $\omega$  to each particle. In this notation, also the number operator can be written as a second quantization  $N = d\Gamma(\mathbb{1})$ , and can be densely defined on a suitable domain in  $\mathcal{F}$ . More generally, if an operator  $\omega$  is defined on  $\text{dom}(\omega)$ , then  $d\Gamma(\omega)$  can at least be defined on the (dense) domain

$$\bigoplus_{N \in \mathbb{N}_0} \text{dom}(\omega)^{\otimes_a N} \subseteq \mathcal{F}. \quad (1.39)$$

On configuration space functions,  $d\Gamma(\omega)$  acts as

$$(d\Gamma(\omega)\Psi)(\mathbf{x}_1, \dots, \mathbf{x}_N) = \sum_{j=1}^N \omega(\mathbf{x}_j)\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N). \quad (1.40)$$

If there are two particle species, e.g.,  $x$ - and  $y$ -particles described by  $\mathcal{H} = \mathcal{F}_x \otimes \mathcal{F}_y$ , with different dispersion relations  $\omega_x, \omega_y$ , then the operator  $d\Gamma(\omega_x)$  defined by (1.40) on a domain in  $\mathcal{F}_x$  is naturally extended to  $\mathcal{H}$  by

$$d\Gamma_x(\omega_x) := d\Gamma(\omega_x) \otimes \mathbb{1}. \quad (1.41)$$

Likewise,  $d\Gamma_y(\omega_y) = \mathbb{1} \otimes d\Gamma(\omega_y)$  can be defined on  $\mathcal{H}$  and the extension to more than two particle species works analogously.

However, more complicated expressions of the form (1.34) often appear in the formal description of dynamics within QFT or many-body models. For instance, if dynamics are to be described by a family of state vectors  $(\Psi_t)_{t \in \mathbb{R}} \subseteq \mathcal{H}$  satisfying the Schrödinger equation

$$i\partial_t \Psi_t = H\Psi_t, \quad (1.42)$$

then the formal Hamiltonian  $H$  might be a sum of operator products as in (1.34) that is not defined on  $\mathcal{H}$ . However, it is crucial to have a self-adjoint operator

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$H : \mathcal{H} \supseteq \text{dom}(H) \rightarrow \mathcal{H}$  in order to obtain a solution via Stone's theorem for any initial state vector  $\Psi_0 \in \mathcal{H}$  by

$$\Psi_t = U(t)\Psi_0. \quad (1.43)$$

Here,  $(U(t))_{t \in \mathbb{R}}, U(t) = e^{-itH}$  is a family of unitary operators defined by spectral calculus. This way of describing dynamics of a quantum system is also called the ‘‘Schrödinger picture of quantum mechanics’’.

Other approaches for describing time dynamics of quantum models start from the Heisenberg or the interaction picture, which we briefly explain in the following. In order to make physical predictions, one associates to each observable an appropriate self-adjoint operator  $A : \mathcal{H} \supseteq \text{dom}(A) \rightarrow \mathcal{H}$ . The expectation value for a measurement at time  $t$  is then given by

$$\langle A \rangle = \langle \Psi_t, A\Psi_t \rangle. \quad (1.44)$$

In the Heisenberg picture, one now shifts the time dynamics from  $\Psi$  to  $A$  by defining  $A_t = U(t)^*AU(t)$  and using that

$$\langle A \rangle = \langle \Psi_0, U(t)^*AU(t)\Psi_0 \rangle = \langle \Psi_0, A_t\Psi_0 \rangle. \quad (1.45)$$

That means, the time dependence is carried by the family of operators  $(A_t)_{t \in \mathbb{R}}$  which satisfy the Heisenberg equations of motion

$$\partial_t A_t = i[H, A_t], \quad (1.46)$$

whereas the state of the system is described by a single time-independent vector  $\Psi_0 \in \mathcal{H}$ .

It is also common to consider the interaction picture for describing quantum dynamics: The Hamiltonian  $H$  is split into a free and an interaction part  $H = H_0 + H_I$  and only the free evolution  $U_0(t) = e^{-itH_0}$  is shifted into the operators  $A$ . That means, one defines  $\Psi_{I,t} := U_0(t)^*\Psi_t$  and  $A_{I,t} := U_0(t)^*AU_0(t)$  and obtains

$$\langle A \rangle = \langle \Psi_t, A\Psi_t \rangle = \langle \Psi_{I,t}, A_{I,t}\Psi_{I,t} \rangle. \quad (1.47)$$

So both the vectors  $\Psi_{I,t}$  and the operators  $A_{I,t}$  carry the time dependence of the system.

By means of (1.45) and (1.47), all three pictures yield the same physical predictions.

The task of **non-perturbative renormalization** is now to derive a self-

adjoint Hamiltonian  $\tilde{H}$  from a formal expression  $H$  containing operator products. There exist several techniques for accomplishing this goal, both in the Schrödinger and the Heisenberg picture. We present some of them in Section 1.3. The tools developed in Chapter 3 also aim at simplifying non-perturbative renormalization. Another popular approach to QFT dynamics is to circumvent the problem of constructing  $\tilde{H}$  by assuming that Heisenberg dynamics already exist and satisfy a certain set of reasonable axioms. Conclusions may then be drawn from these axioms. This approach is called **axiomatic** or **algebraic QFT**, or for short **AQFT**. We outline some of the main notions of this approach in Section 1.2.3, which also turn out useful for non-perturbative renormalization in the Heisenberg picture.

Another common way to extract physical predictions from a problematic expression  $H$  is given by **perturbative QFT** or for short **pQFT**. Here, the formal time evolution operator in the interaction picture  $U_I(t) = e^{-itH_I}$  is expanded in a so-called Dyson series (1.155) which is then truncated. From these approximate dynamics, one may derive physical predictions after a series of formal manipulations, which may involve subtracting infinite quantities, and are called **perturbative renormalization**. These techniques are briefly discussed in Section 1.5.

### 1.2.3 Algebraic and Axiomatic Notions

Sometimes, expressions for operator products (1.34) cannot be given meaning as a Fock space operator. However, they still allow for employing formal manipulations, such as addition, multiplication, complex linear combination or taking adjoints. Mathematically, these manipulations can be made meaningful by interpreting the expressions as elements of a  $*$ -algebra  $\mathcal{A}$ , which does not a priori contain operators on a Hilbert space. One may even formulate quantum dynamics without making reference to a fixed Hilbert space  $\mathcal{H}$ . This formulation is called algebraic quantum mechanics, for a thorough introduction see [14, 15]. Algebraic QFT (AQFT) is a framework within algebraic quantum mechanics, that additionally assumes a set of axioms which a physically reasonable relativistic QFT should satisfy. For an introduction, see [15, 10, 16].

The algebraic approach to quantum dynamics starts from a  $*$ -**algebra of observables**  $\mathcal{A}_{\text{obs}}$ , which is related to physical predictions either at a fixed time  $t \in \mathbb{R}$  or at all times. It is assumed that  $\mathcal{A}_{\text{obs}}$  is a unital  $*$ -algebra, meaning there is a unit element  $\mathbb{1} \in \mathcal{A}_{\text{obs}}$  with  $\mathbb{1}A = A$  for all  $A \in \mathcal{A}_{\text{obs}}$ . In a narrower sense,  $\mathcal{A}_{\text{obs}}$  only allows for self-adjoint elements as observables. We adopt the more general convention of [15] and also allow for non-self-adjoint elements in  $\mathcal{A}_{\text{obs}}$ , which simplifies the description. It is often required that  $\mathcal{A}_{\text{obs}}$  is a  $C^*$ -algebra, so on its elements, there is a norm  $\|\cdot\|$  respecting the algebraic relations, which allows for

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more conclusions to be drawn [10, 17, 16]. The transition from a  $*$ -algebra with unbounded operators to a  $C^*$ -algebra with bounded operators might, for instance, be done by taking resolvents  $R_A := (A - z)^{-1}$ ,  $z \in \mathbb{C}$  or by exponentiation as in (1.30).

In an algebraic description, the dynamics are encoded within **states**, which are linear functionals  $\omega : \mathcal{A}_{\text{obs}} \rightarrow \mathbb{C}$  that are positive and normalized:

$$\omega(A^*A) \geq 0 \quad \forall A \in \mathcal{A}_{\text{obs}}, \quad \omega(\mathbb{1}) = 1. \quad (1.48)$$

The value

$$\langle A \rangle = \omega(A) \quad (1.49)$$

is interpreted as the expectation value for a measurement of the observable  $A$ . It corresponds to the value  $\langle \Psi_0, A\Psi_0 \rangle$  in (1.45), so there is no explicit need to refer to Hilbert space vectors  $\Psi \in \mathcal{H}$  in this description. If  $\mathcal{A}_{\text{obs}}$  refers to all times  $t \in \mathbb{R}$ , then the state  $\omega$  describes all measurement expectations at any time and hence encodes the dynamics of the system. If  $\mathcal{A}_{\text{obs}}$  is related to a fixed time  $t$ , then dynamics are given by a state  $\omega$  together with an automorphism  $\alpha_t : \mathcal{A}_{\text{obs}} \rightarrow \mathcal{A}_{\text{obs}}$  describing the Heisenberg evolution<sup>7</sup>. However, also in the case of  $\mathcal{A}_{\text{obs}}$  referring to all times, an automorphism  $\alpha_t : \mathcal{A}_{\text{obs}} \rightarrow \mathcal{A}_{\text{obs}}$  is useful in order to identify observables of the same physical meaning but appearing at different times.

It is important to distinguish states  $\omega$  yielding measurement predictions via (1.44) from state vectors  $\Psi \in \mathcal{H}$  yielding predictions via (1.49). States  $\omega$  are in the same spirit, but more general than state vectors  $\Psi$ : While  $\Psi$  provides objective (i.e., observer-independent) information about the system, the state  $\omega$  may also include information describing the state of knowledge of an observer, e.g., by choosing  $\omega$  according to a probability distribution  $\mathbb{P}$  on state vectors  $\Psi \in \mathcal{H}$ . That is,  $\omega(A) = \int_{\mathcal{H}} \langle \Psi, A\Psi \rangle d\mathbb{P}(\Psi)$ .

The bridge between the algebraic and the Fock space-based description of dynamics is spanned by representations  $\pi$ , i.e., by maps that associate to each  $A \in \mathcal{A}_{\text{obs}}$  an operator  $\mathcal{H} \supseteq D \rightarrow \mathcal{H}$  with  $D$  being a dense subspace of  $\mathcal{H}$ , such that  $\pi(\mathbb{1}) = \mathbb{1}|_D$  and algebraic relations are respected. Given a state  $\omega$  on a  $C^*$ -algebra, it is always possible to find at least one representation, for instance the **GNS representation**  $\pi_\omega$ . This representation associates to  $\omega$  a Hilbert space  $\mathcal{H}_\omega$  together

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<sup>7</sup>It would also be possible to describe dynamics by a time-dependent family of states  $(\omega_t)_{t \in \mathbb{R}}$ , i.e., in an “algebraic Schrödinger picture”. However, when applying the GNS construction presented below, this would result in a distinct Hilbert space  $\mathcal{H}_t$  at each time  $t \in \mathbb{R}$ , which makes the approach rather inconvenient. Nevertheless, time-dependent Hilbert spaces  $\mathcal{H}_t$  can indeed serve for a reasonable description of quantum dynamics in an abstract setting, as demonstrated in Section 2.2.

with a distinguished “vacuum” state vector  $\Omega \in D \subset \mathcal{H}_\omega$  such that

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle \quad \forall A \in \mathcal{A}_{\text{obs}}. \quad (1.50)$$

All further state vectors  $\Psi \in D$  correspond to states  $\omega_\Psi$  with  $\omega_\Psi(A) = \langle \Psi, \pi(A)\Psi \rangle$ . One may construct even more states by considering all density matrices  $\rho : \mathcal{H} \rightarrow \mathcal{H}$ ,  $\text{tr}(\rho) = 1$ ,  $\rho \geq 0$  and setting  $\omega_\rho(A) = \text{tr}(\rho\pi(A))$ . However, there may still exist even more states on  $\mathcal{A}_{\text{obs}}$  that cannot be described by a density matrix  $\rho$ . An example are state vectors  $\Psi'$  in a representation  $\pi'$  inequivalent to  $\pi$  (where inequivalence of states is discussed below). Those states  $\omega'$  which can indeed be described by a density matrix  $\rho$  are given the name **normal state** (with respect to  $\omega$  or  $\pi_\omega$ ) and the set of all normal states is called the **folium** of  $\omega$ .

The advantage of the algebraic formulation is that a given  $\mathcal{A}_{\text{obs}}$  admits several representations that may not be **unitarily inequivalent**. Unitary equivalence of two representations  $\pi_1$  and  $\pi_2$  on Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2$  with domains  $D_1, D_2$  means that there exists a unitary operator  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  with  $UD_1 = D_2$  and  $\pi_1(A) = U^*\pi_2(A)U$  for all  $A \in \mathcal{A}_{\text{obs}}$ . So unitarily inequivalent representations do not work on a common Hilbert space and the disattachment from a fixed Hilbert space permits a more general description. Further, the algebraic framework is convenient for describing finite-temperature thermal states  $\omega$  that cannot simply be modeled by a single state vector  $\Psi \in \mathcal{H}$ . Nevertheless, in order to interpret the theory in terms of particles appearing in detectors, the algebra  $\mathcal{A}_{\text{obs}}$  needs to contain a set of projection observables  $P(B)$ . That is,  $P(B) = P(B)^*$  and  $P(B)^2 = P(B)$ , for each (Borel-) measurable subset  $B$  of the configuration space  $\mathcal{Q}(X)$  or  $\Gamma(X)$  (or at least a union of several sectors from it) and  $P$  is required to be an “observable-valued measure”. Only then, one may interpret  $\omega(P(B))$  as the probability to find the system in a particle-configuration  $q \in B$ . By contrast, on a Hilbert space with Fock space structure  $\mathcal{H} \subseteq \mathcal{F}$ , such a particle interpretation is readily implied by the projection-valued measure (PVM):

$$B \mapsto P(B), \quad (P(B)\Psi)(q) = \begin{cases} \Psi(q) & \text{if } q \in B \\ 0 & \text{else} \end{cases}, \quad (1.51)$$

for  $B$  being any measurable subset of  $\mathcal{Q}$ . The probability to find the system in any configuration  $q \in B$  is then given by  $\|P(B)\Psi\|^2 = \langle \Psi, P(B)\Psi \rangle$ , which is also called **Born’s rule** (for position measurements).

In relativistic QFT, an algebraic formulation is especially convenient for two reasons: First, Haag’s theorem [9], [10, Sect. II.1] forbids an establishment of dynamics for a relativistic, interacting QFT in that particular representation on a

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Fock space, where creation and annihilation operators indeed act as in (1.21). Second, Haag and Kastler [17] observed that under certain natural requirements on a QFT, all faithful representations of  $\mathcal{A}_{\text{obs}}$  are physically equivalent. Here, faithful means that  $\pi$  is injective and physical equivalence of  $\pi_1$  and  $\pi_2$  means that if  $\Omega_1, \Omega_2$  are the sets of states representable by density matrices with respect to  $\pi_1, \pi_2$  (i.e., their folia), then each  $\omega_1 \in \Omega_1$  can be written as a weak limit of a sequence of states in  $\Omega_2$ . The “natural requirements” above are also called “Haag–Kastler axioms”. Before presenting them, we need to introduce some notation concerning special relativity.

As the set of allowed spacetime points  $x = (t, \mathbf{x}) = (t, x^1, \dots, x^d)$ , we consider Minkowski space  $\mathbb{M} \cong \mathbb{R}^{d+1}$ , which is equipped with the metric

$$\eta = dt^2 - (dx^1)^2 - \dots - (dx^d)^2. \quad (1.52)$$

Sometimes, we will also write  $t = x^0$ .  $\eta$  then induces the metric distance

$$d(x, y) = \begin{cases} \sqrt{|x^0 - y^0|^2 - \|\mathbf{x} - \mathbf{y}\|^2} & \text{if } |x^0 - y^0| \geq \|\mathbf{x} - \mathbf{y}\| \\ -\sqrt{-|x^0 - y^0|^2 + \|\mathbf{x} - \mathbf{y}\|^2} & \text{if } |x^0 - y^0| < \|\mathbf{x} - \mathbf{y}\|. \end{cases} \quad (1.53)$$

Two points  $x, y \in \mathbb{M}$  are called

$$\begin{array}{lll} \text{spacelike} & \text{if } d(x, y) < 0 & \Leftrightarrow |x^0 - y^0| < \|\mathbf{x} - \mathbf{y}\|, \\ \text{lightlike} & \text{if } d(x, y) = 0 & \Leftrightarrow |x^0 - y^0| = \|\mathbf{x} - \mathbf{y}\|, \\ \text{timelike} & \text{if } d(x, y) > 0 & \Leftrightarrow |x^0 - y^0| > \|\mathbf{x} - \mathbf{y}\|. \end{array} \quad (1.54)$$

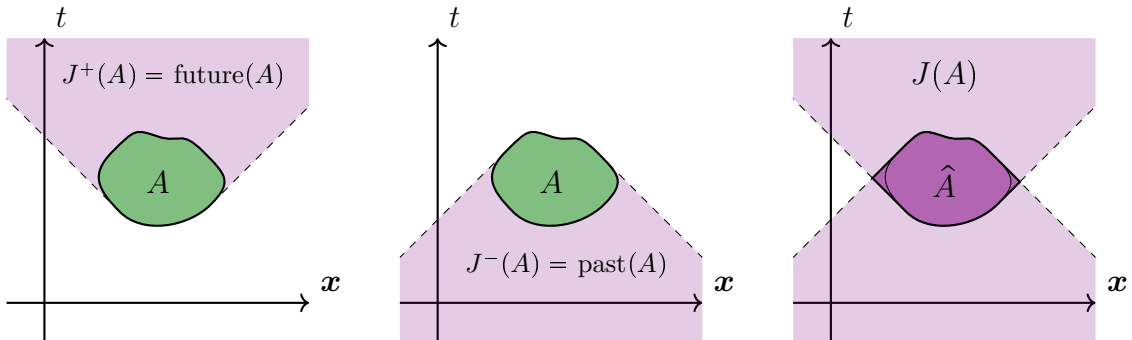


Figure 1.3: Depiction of the causal future  $J^+(A)$ , past  $J^-(A)$  and completion  $\hat{A}$ , as well as the domain of dependence  $J(A)$ . Color online.

Two sets  $A, B \subset \mathbb{M}$  are called spacelike (separated), if all pairs  $x, y$  with  $x \in A, y \in B$  are spacelike, in which case  $d = -\sup_{x \in A, y \in B} d(x, y)$  is called the

**spacelike distance** of  $A$  and  $B$ . For some  $A \subset \mathbb{M}$ , we define the

$$\begin{aligned}
 \text{causal future} & \quad J^+(A) := \{y \in \mathbb{M} \mid \exists x \in A : d(x, y) > 0 \wedge y^0 \geq x^0\}, \\
 \text{causal past} & \quad J^-(A) := \{y \in \mathbb{M} \mid \exists x \in A : d(x, y) > 0 \wedge y^0 \leq x^0\}, \\
 \text{causal completion} & \quad \hat{A} := J^+(A) \cap J^-(A), \\
 \text{domain of dependence} & \quad J(A) := J^+(A) \cup J^-(A),
 \end{aligned} \tag{1.55}$$

see Figure 1.3. We will also write  $J^+(A) = \text{future}(A)$  and  $J^-(A) = \text{past}(A)$ .

The set of all isometries on  $\mathbb{M}$  is called the **Poincaré group**  $\mathcal{P}$  and  $\mathcal{P}_+^\uparrow$  is its identity-connected component, which is a subgroup of  $\mathcal{P}$ .

The **Haag–Kastler axioms**, within their original definition [17], now read as follows:

- (1) To each bounded open region  $\mathcal{O}$  in Minkowski spacetime, we associate a unital  $*$ -algebra<sup>8</sup>  $\mathcal{A}(\mathcal{O})$ .
- (2) **Isotony**: For  $\mathcal{O}_1 \subseteq \mathcal{O}_2$ , we have  $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$ <sup>9</sup>.
- (3) **Local commutativity**: If  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are spacelike separated (as in Figure 1.4), then  $[A_1, A_2] = 0$  for all  $A_1 \in \mathcal{A}(\mathcal{O}_1)$  and  $A_2 \in \mathcal{A}(\mathcal{O}_2)$ .
- (4) There is a unital  $*$ -algebra  $\mathcal{A}_{\text{qloc}}$ , called the **algebra of quasilocal observables**, that contains all  $\mathcal{A}(\mathcal{O})$  as subalgebras<sup>10</sup>.
- (5) **Poincaré covariance**: For each element of the Poincaré group  $g \in \mathcal{P}$ , there exists an automorphism  $\alpha_g : \mathcal{A}_{\text{qloc}} \rightarrow \mathcal{A}_{\text{qloc}}$  with  $\alpha_g \mathcal{A}(\mathcal{O}) = \mathcal{A}(g\mathcal{O})$ . Sometimes, the group  $\mathcal{P}$  is also called “inhomogeneous Lorentz group” and the corresponding property is called “Lorentz covariance”. It is also customary to only consider elements of the identity-connected component  $g \in \mathcal{P}_+^\uparrow$  [16] or, in  $d = 3$ , to take elements from the inhomogeneous  $\text{SL}(2, \mathbb{C})$ -group, which provides a double covering of  $\mathcal{P}_+^\uparrow$ .
- (6) **Primitivity**: There exists a faithful, irreducible<sup>11</sup> representation of  $\mathcal{A}_{\text{qloc}}$ .

<sup>8</sup>In its original formulation [17],  $\mathcal{A}(\mathcal{O})$  is assumed to be a  $C^*$ -algebra and not necessarily unital. However, the formulation via  $*$ -algebras as in [15] is more general.

<sup>9</sup>In the original formulation without unitality, it is further required that  $\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)$  either have a common unit element or both no unit element at all.

<sup>10</sup>If  $\mathcal{A}(\mathcal{O})$  are  $C^*$ -algebras then  $\mathcal{A}_{\text{qloc}}$  is chosen as the completion of  $\mathcal{A}_{\text{loc}} := \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$  in the norm topology. Usually,  $\mathcal{A}_{\text{qloc}}$  is denoted  $\mathcal{A}$ . The notation  $\mathcal{A}_{\text{qloc}}$  here is just in order to avoid confusion with (1.23).

<sup>11</sup>Irreducibility of  $\pi$  means that a subspace of  $\mathcal{H}$  which is invariant under all  $\pi(A)$  must be either  $\{0\}$  or dense in  $\mathcal{H}$ .

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- (7) **Causality:** If  $\mathcal{O}_2$  is in the causal completion<sup>12</sup> of  $\mathcal{O}_1$ , then  $\mathcal{A}(\mathcal{O}_2) \subseteq \mathcal{A}(\mathcal{O}_1)$ .
- (8) **Locality:** As the spacelike distance of  $\mathcal{O}_1$  and  $\mathcal{O}_2$  (see Figure 1.4) is increased to  $\infty$  by translations, the partial states on  $\mathcal{A}(\mathcal{O}_1)$  and  $\mathcal{A}(\mathcal{O}_2)$  become decoupled. Here, a partial state  $\omega_j$  is a positive linear functional on  $\mathcal{A}(\mathcal{O}_j)$  and decoupling means that for any pair of partial states  $\omega_1, \omega_2$  there is a state  $\omega$  on  $\mathcal{A}_{\text{qloc}}$  with  $\omega(A_j) = \omega_j(A_j) \quad \forall A_j \in \mathcal{A}(\mathcal{O}_j)$ . However, there is no precise mathematical formulation in [17] how decoupling is to be interpreted in the limit, so this axiom is rather heuristic<sup>13</sup>.

There exist modified formulations of the Haag–Kastler axioms in the more recent literature. For instance, [16, 15] only postulate axioms (1), (2), (3) and (5), while considering (4) as emergent. Both additionally postulate the

- (9) **Time slice axiom:** If  $\mathcal{O}_1 \subset \mathcal{O}_2$  contains a Cauchy surface of  $\mathcal{O}_2$ , then  $\mathcal{A}(\mathcal{O}_1) \cong \mathcal{A}(\mathcal{O}_2)$ .

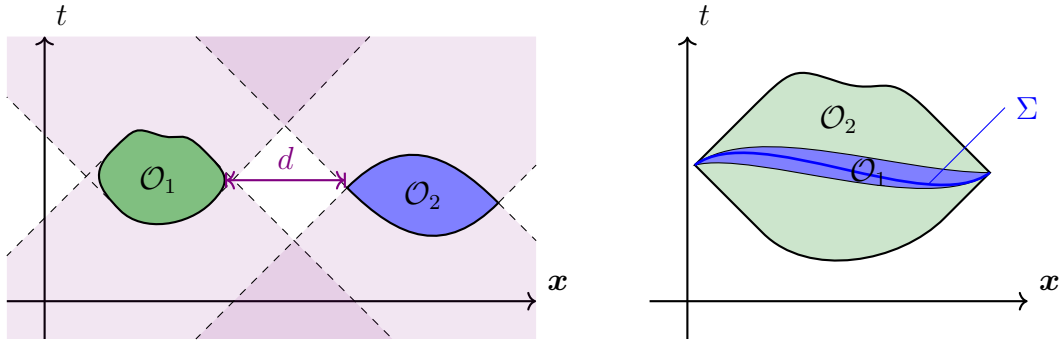


Figure 1.4: Left: Two open regions  $\mathcal{O}_1, \mathcal{O}_2$  with spacelike distance  $d > 0$ .

Right: The time slice  $\mathcal{O}_1 \subset \mathcal{O}_2$  contains a Cauchy surface  $\Sigma$  of  $\mathcal{O}_2$ . Hence, all observables in  $\mathcal{A}(\mathcal{O}_2)$  can be reconstructed from  $\mathcal{A}(\mathcal{O}_1)$ .  
Color online.

Here, a **Cauchy surface** is a set  $\Sigma \subset \mathbb{M}$ , which is intersected exactly once by every inextendible causal (i.e., timelike–or–lightlike) curve. This definition is the

<sup>12</sup>The original formulation is “causal shadow” without a precise definition of this expression. We interpret it as the “causal completion”, here, since isotony would then make axiom (7) equivalent to  $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\hat{\mathcal{O}})$ , which is the axiom formulated in [10, III 1.10].

<sup>13</sup>A precise mathematical formulation of a similar requirement called **split property** can be found in [10, Sect. V.5.2], which requires a full decoupling via  $\omega(A_1 A_2) = \omega_1(A_1) \omega_2(A_2) \quad \forall A_1 \in \mathcal{A}(\mathcal{O}_1), A_2 \in \mathcal{A}(\mathcal{O}_2)$ .

same as used by Wald [18]. See also Figure 1.4.

The time slice axiom ensures that, given initial data about a quantum system on an arbitrarily thin slice  $\mathcal{O}_1$  around some Cauchy surface of a region  $\mathcal{O}_2$ , predictions can be made for all of  $\mathcal{O}_2$ . So local quantum dynamics exist.

The algebras  $\mathcal{A}(\mathcal{O})$  may for instance be generated by formal expressions  $\phi(f)$ ,  $f \in \mathcal{S}(\mathbb{R}^{d+1})$  or  $W(f) = e^{-i\phi(if)}$ , called **fields**, such that  $\text{supp } f \subset \mathcal{O}$ . It is also possible to use fields with  $f \in \mathcal{S}(\mathbb{R}^d)$  at a fixed time, similar to the field operators (1.30). However,  $\phi(f)$ ,  $W(f)$  are no operators and only become **field operators** when represented by  $\pi$  on a Hilbert space  $\mathcal{H}$ . In that case, there is no need that  $\mathcal{H}$  bears a Fock space structure or that  $\pi(\phi(f))$ ,  $\pi(W(f))$  are of the form (1.30). We remark that the term “field” is also used in the literature to denote the operators  $\pi(\phi(f))$ ,  $\pi(W(f))$ , or the maps  $\phi(x)$ ,  $\phi(\mathbf{x})$  which send  $f \in \mathcal{S}(\mathbb{R}^{d+1})$  or  $f \in \mathcal{S}(\mathbb{R}^d)$  to  $\phi(f)$ .

There are also similar sets of axioms for different QFT settings, e.g., on curved spacetime [19, 20], [16, Chap. 4] or for conformal field theory (CFT) [16, Chap. 8]. The name for specific axioms may vary depending on the literature resource. For instance, in the above-mentioned references, sometimes (3) instead of (8) is called “locality” or (3) is given the name “(Einstein) causality”.

Further, in the original work by Haag and Kastler, the axiom set (1)–(8) is not considered fixed, but rather in an “experimental stage” [17, p. 849], where, for instance, the necessity of axiom (8) is subject to discussion [17, p. 852]. The set of “structural assumptions” stated later by Haag [10, p. 110] consists of axioms (1), (3) and (5) together with a version of the time slice axiom<sup>14</sup>. Haag himself writes about axiomatic frameworks of QFT [10, p. 58]:

[...] the word “axiom” suggests something fixed, unchangeable. This is certainly not intended here. Indeed, some of the assumptions are rather technical and should be replaced by more natural ones as deeper insight is gained.

However, this quote is not given in the context of the Haag–Kastler axioms, but a different set of axioms, which have been proposed by Streater and Wightman [21, 22]. These Wightman axioms are directly related to a fixed representation on a Hilbert space and are commonly accepted as a criterion for a proposed non-perturbative QFT model to be physically reasonable [23].

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<sup>14</sup>In this version,  $\mathcal{A}(\mathcal{O}_1) \cong \mathcal{A}(\mathcal{O}_2)$  is required whenever  $\mathcal{O}_2$  is the causal completion of  $\mathcal{O}_1$ .

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These axioms contain a condition that refers to **fiber decompositions**: Assume that the Poincaré group is represented by a continuous family of unitary operators  $U(g)$  on a Hilbert space  $\mathcal{H}$ , where the time translation subgroup  $(U(t))_{t \in \mathbb{R}}$  is generated by a Hamiltonian  $\tilde{H}$  and the space translation subgroup by momentum operators  $P_{\text{op}}^1, \dots, P_{\text{op}}^d$ . As a physically reasonable interaction is momentum-conserving, the operators  $\tilde{H}$  and  $P_{\text{op}}^j$  are expected to commute and one may use the spectral theorem in the form [24, Thm. 10.9] to write  $\mathcal{H}$  and  $\tilde{H}$  as a direct integral over the spectral values  $P = (P^1, \dots, P^d)$  with  $P^j \in \sigma(P_{\text{op}}^j)$ :

$$\mathcal{H} = \int_{\mathbb{R}^d}^{\oplus} \mathcal{H}_P dP, \quad \tilde{H} = \int_{\mathbb{R}^d}^{\oplus} \tilde{H}_P dP. \quad (1.56)$$

Each  $\Psi \in \mathcal{H}$  is here translated into a cross-section  $P \mapsto \Psi_P \in \mathcal{H}_P$  up to modifications on a null set in  $P \in \mathbb{R}^d$ . The **joint spectrum** of the tuple of operators  $P^\mu = (\tilde{H}, P_{\text{op}}^1, \dots, P_{\text{op}}^d)$  is now given by all points  $(E, P) \in \mathbb{M}$ , such that  $E \in \mathbb{R}$  is within the spectrum  $\sigma(\tilde{H}_P)$ .

The **Wightman axioms** now read as follows [21, Chap. 3-1]:

### (0) Assumptions on the Hilbert space:

- There exists a Hilbert space  $\mathcal{H}$  of state vectors, where a quantum state corresponds to a ray  $\{c\Psi \mid c \in \mathbb{C}\}$  with  $\Psi \in \mathcal{H}$ .
- For the group  $\mathcal{P}_+^\uparrow$ , there exists a continuous representation on  $\mathcal{H}$  by unitary operators  $U(g)$ ,  $g \in \mathcal{P}_+^\uparrow$ . The original formulation in 3 + 1 dimensions uses the inhomogeneous  $\text{SL}(2, \mathbb{C})$ -group, which is a double covering of  $\mathcal{P}_+^\uparrow$ .
- **Spectral positivity**: The joint spectrum of  $P^\mu$  lies in the forward light-cone  $V^+ = J^+(\{0\})$ , see Figure 1.5.
- **Unique vacuum state**: There exists a vector  $\Omega \in \mathcal{H}$  that is invariant under all  $U(g)$  and unique up to a phase.

### (1) Assumptions on the fields:

- For each Schwartz function  $f \in \mathcal{S}(\mathbb{R}^{d+1})$ , there exists a set of field operators  $\phi_r(f)$ ,  $r \in \{1, \dots, 2s + 1\}$  together with its adjoints  $\phi_r(f)^*$ .
- All operators  $\phi_r(f)$ ,  $\phi_r(f)^*$  are defined on a common dense domain  $D \subseteq \mathcal{H}$  with  $\Omega \in D$  and  $D$  is invariant under all  $\phi_r(f)$ ,  $\phi_r(f)^*$  and  $U(g)$ . Further, for  $\Psi_1, \Psi_2 \in D$ , the map  $f \mapsto \langle \Psi_1, \phi_r(f)\Psi_2 \rangle$  is a distribution.

### (2) Covariance of the fields: For all $g \in \mathcal{P}_+^\uparrow$ (or $g \in \text{SL}(2, \mathbb{C})$ ) there is a spinor transformation matrix $S_{jk}(g)$ with $U(g)\phi_r(f)U(g)^{-1} = \sum_r S_{rr'}(g)\phi_{r'}(f \circ g^{-1})$ .

- (3) **Causality:** If  $\text{supp } f, \text{supp } g$  are spacelike, then  $[\phi_r(f), \phi_{r'}(g)^*]_{\pm} = 0$ .
- (4) **Completeness:**  $\text{span}\{\phi_r(f)\Omega, \phi_r(f)^*\Omega \mid f \in \mathcal{S}, r \in \{1, \dots, 2s + 1\}\}$  is dense in  $\mathcal{H}$ , i.e.,  $\Omega$  is cyclic.

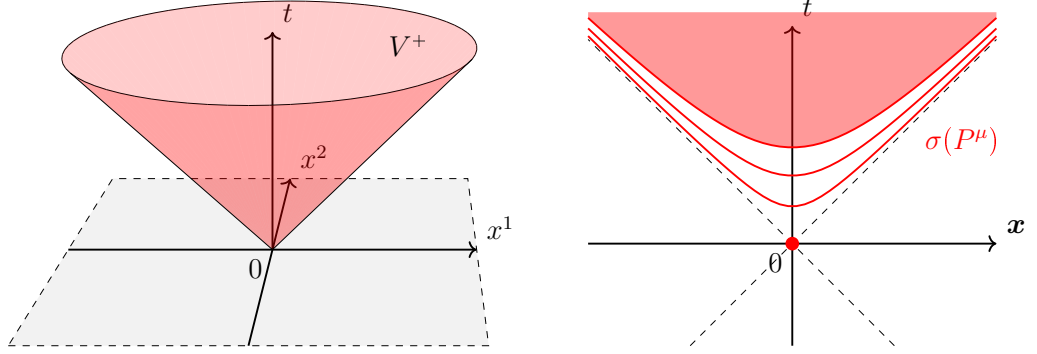


Figure 1.5: Left: The forward light cone  $V^+$  with 2 of 3 spatial dimensions drawn. Right: A typical joint spectrum  $\sigma(P^\mu)$ . It has to lie in  $V^+$ , be Lorentz-invariant and contain  $0 \in \mathbb{M}$ , as  $\Omega$  is a corresponding eigenvector. Color online.

In the following, we will often drop the spin indices  $r, r'$ , if a generalization to many spins is obvious.

It is customary to call a physical model a **field theory** if and only if it satisfies the above axioms. Additionally, Wightman and Streater state a criterion for a physically reasonable QFT, which however does not enter the definition of a field theory: If within a scattering theoretic setting, ingoing and outgoing state vectors can be defined that make up the corresponding Hilbert spaces  $\mathcal{H}^{\text{in}}$  and  $\mathcal{H}^{\text{out}}$ , then one additionally requires

- (5) **Asymptotic completeness:**  $\mathcal{H} = \mathcal{H}^{\text{in}} = \mathcal{H}^{\text{out}}$ .

Haag and Ruelle have shown that the construction of  $\mathcal{H}^{\text{in}}$  and  $\mathcal{H}^{\text{out}}$  together with one-particle creation- and annihilation operators  $a_{\text{in/out}}^\dagger(f)$  is indeed possible, if one assumes a Wightman field theory featuring a mass gap [25, 26]. In this context, having a **mass gap** means that the joint spectrum of  $P^\mu$ , after removing the origin  $0 \in \mathbb{M}$  (corresponding to the vacuum  $\Omega$  as an eigenvector of  $\tilde{H}$  and  $P_j^{\text{op}}$ ), lies within one of the sets  $\{(E, P) \in \mathbb{M} \mid E^2 \geq |P|^2 + M^2\}$  for some  $M > 0$ . This is equivalent to  $\tilde{H}_P$  having an isolated eigenvalue 0.

More generally, the aim of scattering theory is to construct a unitary operator  $S : \mathcal{H}^{\text{in}} \rightarrow \mathcal{H}^{\text{out}}$ , called **S-matrix**, which describes the time evolution from  $t \rightarrow -\infty$  to  $t \rightarrow \infty$ . The S-matrix allows to easily derive predictions for

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scattering experiments (such as scattering cross-sections), which can be used to empirically verify a QFT model. One definition involves the **Møller operators**  $\Omega^{\text{in/out}} : \mathcal{H}^{\text{in/out}} \rightarrow \mathcal{H}$  (or also called wave operators) and reads  $S = (\Omega^{\text{out}})^{-1}\Omega^{\text{in}}$ . For instance, if the Hamiltonian  $H$  takes the form  $H = H_0 + H_I$ , as in Section 1.3.1, then a possible definition for the Møller operators is [27, Chap. 12]:

$$\Omega^{\text{in}} = \lim_{t \rightarrow -\infty} e^{itH} e^{-itH_0}, \quad \Omega^{\text{out}} = \lim_{t \rightarrow \infty} e^{itH} e^{-itH_0}. \quad (1.57)$$

There are cases in which a definition different from (1.57) is used, e.g., if Dollard modifiers are employed, see (3.18) in Section 3.1.2.

Instead of asymptotic completeness, Haag [10] considers a version of the **time-slice axiom** as a necessary addition to the Wightman axioms, namely that all fields can be expressed by fields with  $f$  supported on an arbitrarily thin horizontal slice  $\mathcal{O}_{t,\varepsilon} = \{x \in \mathbb{M} \mid |x^0 - t| < \varepsilon\}$ .

It seems natural to identify  $\mathcal{A}(\mathcal{O})$  with the algebra generated by all field operators  $\phi(f)$  or all Weyl operators  $e^{-i\phi(if)}$  with  $\text{supp } f \in \mathcal{O}$ , in order to relate the Wightman- to the Haag-Kastler axioms. However, even with this identification, both axiomatic settings are not equivalent. For a comparison, see [10].

There are further axiomatic settings, which allow for recovering a Wightman field theory. One of these settings is given in the context of the **Wightman reconstruction theorem** [21, Thm. 3–7] (originally in  $d = 3$  space dimensions): For any given Wightman field theory, one may define the family of **Wightman distributions**  $(\mathfrak{W}_N)_{N \in \mathbb{N}_0}$ ,  $\mathfrak{W}_N \in \mathcal{S}'(\mathbb{R}^{N(d+1)})$  via

$$\mathfrak{W}_N : f_1 \otimes \dots \otimes f_N \mapsto \langle \Omega, \phi(f_1) \dots \phi(f_N) \Omega \rangle. \quad (1.58)$$

The Wightman reconstruction theorem now asserts that if only a family of Wightman distributions  $(\mathfrak{W}_N)_{N \in \mathbb{N}_0}$  satisfying a set of axioms is given, then a Wightman field theory (comprising  $(\phi(f))_{f \in \mathcal{S}}$ ,  $\mathcal{H}$ ,  $(U(g))_{g \in \mathcal{P}_+^\uparrow}$ ,  $D \subseteq \mathcal{H}$  and  $\Omega \in \mathcal{H}$ ) can be reconstructed such that the  $\mathfrak{W}_N$  are given by (1.58). The reconstructed field theory is then unique, up to unitary equivalence.

A slightly stronger setting is given by the Osterwalder-Schrader axioms. Given the Wightman axioms, the Wightman distributions  $\mathfrak{W}_N : \mathbb{R}^{4N} \rightarrow \mathbb{C}$  in 3+1 dimensions can be written as boundary values of complex functions  $\mathbb{C}^{4N} \supset V_T^{(4N)} \rightarrow \mathbb{C}$  with  $V_T^{(4N)}$  being the future tube [28]:

$$V_T^{(4N)} := \{(x_1, \dots, x_N) \in \mathbb{C}^{4N} \mid -\text{Im}(x_j - x_{j+1}) \in V^+ \subset \mathbb{M} \quad \forall j \in \{1, \dots, n-1\}\}. \quad (1.59)$$

The functions on  $V_T^{(4N)}$  which have  $\mathfrak{W}_N$  as its boundary value are called **Wightman functions**<sup>15</sup> and are also denoted  $\mathfrak{W}_N$ . Their domains  $V_T^{(4N)}$  contain all coordinates of the kind

$$z_j = (ix_j^0, x_j^1, x_j^2, x_j^3) \in i\mathbb{R} \oplus \mathbb{R}^3, \quad (1.60)$$

such that  $x_j^0 < x_{j+1}^0$ . Hence,  $\mathfrak{W}_N$  can be restricted to these coordinates, which results in the so-called **Schwinger functions**  $(\mathfrak{S}_N)_{N \in \mathbb{N}_0}$ . Applying the Minkowski metric to a pair of  $z, z' \in \mathbb{M}$  as in (1.60) then gives the same result as if one would apply the Euclidean metric to the respective vectors  $x, x'$ :

$$\eta(z, z') = x^0 x'^0 + x^1 x'^1 + x^2 x'^2 + x^3 x'^3 = x \cdot x'. \quad (1.61)$$

So the transition  $\mathfrak{W}_N \mapsto \mathfrak{S}_N$  corresponds to a change from Minkowski to Euclidean space.

The **Osterwalder–Schrader axioms** [29] (in  $3 + 1$  dimensions) are now a set of conditions on a family of Schwinger functions  $(\mathfrak{S}_N)_{N \in \mathbb{N}_0}$  which allow for recovering a corresponding family of Wightman functions  $(\mathfrak{W}_N)_{N \in \mathbb{N}_0}$  that fulfill the conditions within the Wightman reconstruction theorem. So establishing the Osterwalder–Schrader axioms results in the existence of non-perturbative dynamics in a Wightman field theory. Conversely, each family  $(\mathfrak{W}_N)_{N \in \mathbb{N}_0}$  satisfying the assumptions in the Wightman reconstruction theorem gives rise to a family  $(\mathfrak{S}_N)_{N \in \mathbb{N}_0}$  fulfilling the Osterwalder–Schrader axioms [29].

The area of research concerning the construction of a set of Schwinger functions is called **Euclidean field theory**. Sometimes, also non-relativistic models are investigated, and sometimes, Euclidean field theory concerns the construction of a probability measure on a space of functions (e.g., on all  $f \in \mathcal{S}(\mathbb{R}^{d+1})$ ), which can in turn be used to generate all Schwinger functions. There have been many successful constructions using this method, and naming them all would go beyond the scope of this dissertation. We refer the reader to [30] for an overview about earlier works on Euclidean field theory (including the first investigations by Symmanzik that appeared well before Osterwalder and Schrader’s reconstruction theorem) and to [31, Sect. 1.1] for an overview about more recent results.

More generally, the area of research concerning the construction of a field theory satisfying one of the three axiomatic frameworks above, e.g., from a given formal Hamiltonian  $H$ , is called **constructive quantum field theory** (CQFT). Besides Euclidean field theory, there are several other approaches to CQFT, see [32].

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<sup>15</sup>Sometimes, also a Wightman distribution  $\mathfrak{W}_N$  is called a Wightman function.

## 1.3 Cutoff Renormalization and Techniques

### 1.3.1 Hamiltonians with Interactions

The aim of cutoff renormalization is to construct a self-adjoint Hamiltonian  $\tilde{H}$  out of a formal expression  $H$ , which is of the form

$$H = H_0 + H_I. \quad (1.62)$$

The **kinetic** or **free Hamiltonian**  $H_0$  is described in Fourier space with  $X = \mathbb{R}^d$  as

$$H_0 = \sum_j d\Gamma_j(\omega_j) = \sum_j \int_{\mathbb{R}^d} a_{j,\mathbf{k}}^\dagger a_{j,\mathbf{k}} \omega_j(\mathbf{k}) d\mathbf{k}, \quad (1.63)$$

with  $j$  indexing the particle species,  $d\Gamma_j$  being the second quantization (compare (1.38)) with respect to particle species  $j$  and with  $\omega_j$  being the respective **dispersion relation**. Typically, the function  $\omega_j$  is continuous and gives rise to a self-adjoint operator  $\omega_j$  on some domain  $\text{dom}(\omega_j)$  dense in  $L^2(\mathbb{R}^d)$ . The operator  $d\Gamma_j(\omega_j)$  is then essentially self-adjoint on the dense domain  $\bigoplus_{N \in \mathbb{N}_0} \text{dom}(\omega_j)^{\otimes_a N} \subseteq \mathcal{F}_j$  [22, Sect. X.7] and naturally extends to the Hilbert space of the model  $\mathcal{H} \subseteq \bigotimes_j \mathcal{F}_j$ .

The **interaction Hamiltonian**<sup>16</sup>  $H_I$  is typically a sum of operator products as in (1.34) with  $f$  being a distribution. In order to describe local interactions, it is convenient to take

$$f(\mathbf{x}_1, \dots, \mathbf{x}_N) = \tilde{v}(\mathbf{x}_1) \delta(\mathbf{x}_1 - \mathbf{x}_2) \dots \delta(\mathbf{x}_{N-1} - \mathbf{x}_N). \quad (1.64)$$

Here,  $\tilde{v} \in \mathcal{S}'$  is defined by stating its Fourier transform  $v : \mathbb{R}^d \rightarrow \mathbb{C}$  called **form factor**. So the Fourier transform of  $f$  is

$$\hat{f}(\mathbf{k}_1, \dots, \mathbf{k}_N) = (2\pi)^{-\frac{(N-1)d}{2}} v(\mathbf{k}_1 + \dots + \mathbf{k}_N). \quad (1.65)$$

In some cases, an operator product can be defined on a dense domain in  $\mathcal{F}$ , even if  $f \notin L^2$ , as it is the case for certain  $d\Gamma_j(\omega_j)$ . However, in most cases,  $H_I$  contains ill-defined operator products. An easy example is  $a^\dagger(v)$  with  $v \notin L^2(X)$  such that the  $L^2$ -integral diverges at  $|\mathbf{k}| \rightarrow 0$  (IR-regime) or  $|\mathbf{k}| \rightarrow \infty$  (UV-regime). In that case,  $(a^\dagger(v)\Psi)^{(N)} = \sqrt{N}\tilde{v} \otimes \Psi^{(N-1)}$ , and since  $\tilde{v} \notin L^2$ , the operator  $a^\dagger(v)$  is defined only on  $\Psi = 0$ . Cutoff renormalization circumvents this problem by replacing  $v$  by

<sup>16</sup>In some relativistic QFT models, instead of the Hamiltonian  $H = H_0 + H_I$ , a Lagrangian  $L = L_0 + L_I$  or a Lagrangian density  $\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_I(x)$  with  $x = (t, \mathbf{x})$  is given. In that case,  $H_I = -L_I = -\int_{\mathbb{R}^d} \mathcal{L}(0, \mathbf{x}) d\mathbf{x}$ , if the Schrödinger picture is used, and  $H_0$  can be extracted from  $L_0$ , see any standard physics QFT book, e.g., [5, Sect. 2.2] or [6, 7, 8].

the cut-off form factors

$$\begin{aligned} v_\sigma(\mathbf{k}) &:= \chi_{\sigma < |\mathbf{k}|}(\mathbf{k})v(\mathbf{k}), & v_\Lambda(\mathbf{k}) &:= \chi_{|\mathbf{k}| < \Lambda}(\mathbf{k})v(\mathbf{k}), \\ v_{\sigma,\Lambda}(\mathbf{k}) &:= \chi_{\sigma < |\mathbf{k}| < \Lambda}(\mathbf{k})v(\mathbf{k}). \end{aligned} \tag{1.66}$$

The indicator function  $\chi$  is sometimes replaced by a smooth cutoff function in order to sustain regularity of  $v$ . The parameter  $\sigma \in [0, \infty)$  at small momenta (long wavelengths) is called infrared- or **IR-cutoff** and the parameter  $\Lambda \in [0, \infty)$ ,  $\Lambda > \sigma$  at large momenta (short wavelengths) is called ultraviolet- or **UV-cutoff**.

Corresponding operators or formal operator products are also given a subscript  $\sigma$  or  $\Lambda$ , such as  $H_\sigma, H_\Lambda$  or  $H_{\sigma,\Lambda}$ , which can be well-defined even if  $H$  is not.

The process of removing one or both cutoffs by taking the limits  $\sigma \rightarrow 0$  or  $\Lambda \rightarrow \infty$  in order to construct a well-defined limit Hamiltonian  $\tilde{H}$  is called **cutoff renormalization**. It is customary to manipulate  $H_{\sigma,\Lambda}$  before taking the limit or to construct  $\tilde{H}$  indirectly, see Section 1.3.6.

In the following Sections 1.3.2–1.3.6, we present, without claiming to be exhaustive, some techniques for establishing a well-defined  $\tilde{H}$  within cutoff renormalization. Examples from the literature, where they have been applied to concrete models, are discussed in Sections 1.3.7–1.3.8. Sometimes, also a set of fiber Hamiltonians  $\tilde{H}_P$  as in (1.56) is constructed using the mentioned techniques.

### 1.3.2 Interpreting $H$ as a Bilinear Form

This technique is useful in less singular cases. If  $H_{\sigma,\Lambda}$  is defined on the dense domain  $\text{dom}(H_{\sigma,\Lambda})$ , one may consider the bilinear form  $(\Psi_1, \Psi_2) \mapsto \langle \Psi_1, H_{\sigma,\Lambda} \Psi_2 \rangle$ , which is at least defined for  $\Psi_1, \Psi_2 \in \text{dom}(H_{\sigma,\Lambda})$ . In certain models, the limit

$$\lim_{\substack{\sigma \rightarrow 0 \\ \Lambda \rightarrow \infty}} \langle \Psi_1, H_{\sigma,\Lambda} \Psi_2 \rangle =: b(\Psi_1, \Psi_2) \in \mathbb{C} \tag{1.67}$$

may now exist for  $\Psi_1, \Psi_2$  within a dense form domain containing  $\text{dom}(H_{\sigma,\Lambda})$ , so (1.67) defines a bilinear form  $b$ . If this form is bounded from below, then the Friedrichs extension theorem [27, Sect. 2.3] allows for finding a unique self-adjoint operator  $\tilde{H}$  with  $b(\Psi_1, \Psi_2) = \langle \Psi_1, \tilde{H} \Psi_2 \rangle$  for  $\Psi_1, \Psi_2 \in \text{dom}(\tilde{H})$ . This  $\tilde{H}$  is the renormalized Hamiltonian.

### 1.3.3 Counterterms

In some cases,  $H_{\sigma,\Lambda}$  can be split into a particularly simple divergent part  $c_{\sigma,\Lambda}$  and a part that converges to an operator as  $\sigma \rightarrow 0, \Lambda \rightarrow \infty$ . The term  $c_{\sigma,\Lambda}$  is then subtracted from  $H_{\sigma,\Lambda}$  within an ad hoc modification, which corresponds to adding a so-called **counterterm**  $-c_{\sigma,\Lambda}$ . A self-adjoint Hamiltonian may then be constructed by taking the limit:

$$\tilde{H} = \lim_{\substack{\sigma \rightarrow 0 \\ \Lambda \rightarrow \infty}} (H_{\sigma,\Lambda} - c_{\sigma,\Lambda}). \quad (1.68)$$

A particularly simple kind of counterterm occurs if  $c_{\sigma,\Lambda} = E_{\sigma,\Lambda}$  is just a constant with  $E_{\sigma,\Lambda} \rightarrow \infty$  as  $\sigma \rightarrow 0$  or  $\Lambda \rightarrow \infty$ . In that case,  $-E_{\sigma,\Lambda}$  is called an **energy counterterm** or a **self-energy**. From a physical point of view, a success of the construction (1.68) means that the formal expression  $H$  was chosen “too large by an infinite constant”. Formal algebraic calculations with  $H$  may then nevertheless produce sensible results, as a constant commutes with all operators, so formally  $[H, A] = [\tilde{H}, A]$ . Thus,  $H$  and  $\tilde{H}$  generate the same formal Heisenberg dynamics. In the related and equally denoted case of  $c_{\sigma,\Lambda} = E_{\sigma,\Lambda}$  being a multiple of a number operator  $N_j = d\Gamma_j(1)$ , the term  $E_{\sigma,\Lambda}$  is also called an energy counterterm and has the same heuristic interpretation.

It may also appear that the divergent part contains a kinetic term

$$\delta m_{\sigma,\Lambda} = d\Gamma(\omega_{\sigma,\Lambda}), \quad (1.69)$$

with  $\omega_{\sigma,\Lambda}$  being an operator multiplying by a function  $\omega_{\sigma,\Lambda} : \mathbb{R}^d \rightarrow \mathbb{R}$  in momentum space. That means, it alters the dispersion relation, which is typically accompanied by a change of the particle mass. These terms are called **mass counterterms**. As energy counterterms proportional to  $N$  also take the form  $E_{\sigma,\Lambda} = d\Gamma(z_{\sigma,\Lambda})$ , with  $z_{\sigma,\Lambda} \in \mathbb{R}$ , they are sometimes also called “mass counterterms”. Sometimes, also quadratic operator products that contain  $a^\dagger a^\dagger$ - and  $aa$ -terms are called “mass counterterms”.

There exist further types of counterterms that may be subtracted from  $H_{\sigma,\Lambda}$ , such as **charge counterterms**, which are proportional to summands within the interaction part of the Hamiltonian  $H_{I,\sigma,\Lambda}$ .

Sometimes, counterterms are also called (energy, mass or charge) “renormalizations”, in which case they share one name with the process of removing certain divergences from  $H$ .

Which counterterms are needed might heuristically be inferred from a careful

investigation of the divergence of  $H_{\sigma,\Lambda}$ . The heuristic interpretation of these ad hoc–added terms is that the initial and simple guess  $H$  for a Hamiltonian describing the physical system a Hamiltonian was actually wrong and additional terms must be included in order to make sense of it. It is however desirable to justify why physical predictions derived from a formal  $H$  can still be achieved from  $\tilde{H}$  under the choice of a certain set of counterterms. For instance, it might happen that in perturbative QFT (see Section 1.5), correct physical predictions are obtained from  $H$  under addition of perturbative counterterms, whose effects can be analogous to the non–perturbative  $c_{\sigma,\Lambda}$  added above.

### 1.3.4 Dressing Transformations

Another ad hoc modification for constructing an operator  $\tilde{H}$  out of a formal  $H$  is the conjugation with a **dressing transformation**  $W, W_\sigma, W_\Lambda$  or  $W_{\sigma,\Lambda}$ . In the following, we present the general dressing procedure while using both cutoffs  $\sigma$  and  $\Lambda$ , although in many examples,  $H_{\sigma,\Lambda}, W_{\sigma,\Lambda}$  are independent of  $\sigma$  or  $\Lambda$ . Certain cases allow to directly make sense of the limit

$$\tilde{H} = \lim_{\substack{\sigma \rightarrow 0 \\ \Lambda \rightarrow \infty}} W_{\sigma,\Lambda}^{-1} H_{\sigma,\Lambda} W_{\sigma,\Lambda}. \quad (1.70)$$

In other cases, it turns out necessary to add counterterms before the conjugation:

$$\tilde{H} = \lim_{\substack{\sigma \rightarrow 0 \\ \Lambda \rightarrow \infty}} W_{\sigma,\Lambda}^{-1} (H_{\sigma,\Lambda} - c_{\sigma,\Lambda}) W_{\sigma,\Lambda}. \quad (1.71)$$

It is common to choose  $W_{\sigma,\Lambda}$  as a unitary operator of the form

$$W_{\sigma,\Lambda} = e^{B_{\sigma,\Lambda}}, \quad (1.72)$$

where  $iB_{\sigma,\Lambda}$  is self–adjoint. Examples are

- **Weyl transformations:** For a form factor  $s_{\sigma,\Lambda} \in L^2(X)$ , with  $X \subseteq \mathbb{R}^d$  denoting momentum space, we have

$$B_{\sigma,\Lambda} = \int_X \left( s_{\sigma,\Lambda}(\mathbf{k}) a^\dagger(\mathbf{k}) - \overline{s_{\sigma,\Lambda}(\mathbf{k})} a(\mathbf{k}) \right) d\mathbf{k} = a^\dagger(s_{\sigma,\Lambda}) - a(s_{\sigma,\Lambda}). \quad (1.73)$$

In general, conjugation of  $a^\dagger(f), a(f), f \in L^2(X)$  with a Weyl transformation  $W = e^{a^\dagger(s) - a(s)}, s \in L^2(X)$  leads to a formal addition of a constant

$$W^* a^\dagger(f) W = a^\dagger(f) + \langle s, f \rangle, \quad W^* a(f) W = a(f) + \langle f, s \rangle. \quad (1.74)$$

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The above replacement  $a(f) \mapsto a(f) + \langle f, s \rangle$  may even be well-defined if  $s \notin L^2(X)$ , provided that  $f$  is sufficiently regular (e.g., if  $f \in \mathcal{D}(X)$  and  $s \in \mathcal{D}'(X)$ ). Sometimes, also the replacement procedure is called a “Weyl transformation”. In any case it preserves the CCR/CAR (1.22).

- **Bogoliubov transformations:** These are named after Bogoliubov’s seminal paper [33] and exist in a bosonic and a fermionic version. For bosons, one has

$$B_{\sigma,\Lambda} = \frac{1}{2} \int_X \left( \xi_{\sigma,\Lambda}(\mathbf{k}) a_+^\dagger(\mathbf{k}) a_+^\dagger(-\mathbf{k}) + \overline{\xi_{\sigma,\Lambda}(\mathbf{k})} a_+(\mathbf{k}) a_+(-\mathbf{k}) \right) d\mathbf{k}, \quad (1.75)$$

with form factor  $\xi_{\sigma,\Lambda} : X \rightarrow \mathbb{C}$ . The fermionic case is more delicate since pairs of fermions (so-called Cooper pairs) typically appear. For further details, see also Section 5.3.

We denote the unitary operator corresponding to a Bogoliubov transformation by  $\mathbb{U} := e^{B_{\sigma,\Lambda}}$ . In both the bosonic and fermionic case, a conjugation with  $\mathbb{U}$  maps creation and annihilation operators  $a^\dagger(f), a(f)$  into new operators

$$b^\dagger(f) := \mathbb{U}^* a^\dagger(f) \mathbb{U} = a^\dagger(uf) + a(v\bar{f}), \quad b(f) := \mathbb{U}^* a(f) \mathbb{U} = a(uf) + a^\dagger(v\bar{f}), \quad (1.76)$$

such that  $b^\dagger(f), b(f)$  still satisfy the CCR/CAR (1.22), where  $\bar{f}$  is the complex conjugate function of  $f$ , and with certain operators  $u, v$  densely defined on  $L^2(X)$ .

The term “Bogoliubov transformation” often denotes the above algebraic replacement  $a^\sharp \mapsto b^\sharp$ , whereas the operator  $\mathbb{U}$  is called **implementer** of the transformation. Sometimes the term “Bogoliubov transformation” is also used in a wider sense, denoting any operator replacement that preserves the CCR/CAR. In that sense, Weyl transformations can be seen as a particular kind of Bogoliubov transformations [34]. For an introduction into Bogoliubov transformations (both bosonic and fermionic), we refer the reader to [34] and [35].

- **Gross transformations:** Originally introduced by E.P. Gross in 1962 [36], these transformations are derived from a cubic interaction between fermions (–) and bosons (+):

$$B_{\sigma,\Lambda} = \int_X \left( s_{\sigma,\Lambda}(\mathbf{p}, \mathbf{k}) a_+^\dagger(\mathbf{k}) a_-^\dagger(\mathbf{p}) a_-(\mathbf{p} + \mathbf{k}) + \text{h.c.} \right) d\mathbf{p} d\mathbf{k}, \quad (1.77)$$

with “h.c.” denoting the “Hermitean conjugate”. Gross transformations are

especially useful if the interaction  $H_I$  in the physical model is also a cubic term of the above form. This is the case for polaron models described in Section 1.3.7.

The Weyl transformation can also be seen as a special kind of Gross transformation: If one restricts the Hilbert space  $\mathcal{F}_- \otimes \mathcal{F}_+$  to the one-fermion space  $\mathcal{H}_1 = L^2(X) \otimes \mathcal{F}_+$  and splits it into fibers  $\mathcal{H}_{1,P}$  as in (1.56), then  $W_{\sigma,\Lambda}|_{\mathcal{H}_{1,P}}$  just amounts to a Weyl transformation.

But also more sophisticated unitary and non-unitary dressing transformations may be used. An example are non-unitary dressings transformations in constructive QFT (often denoted  $T$ ) as described in Section 1.3.8. A useful diagrammatic formalism for evaluating these dressing transformations has been proposed by Friedrichs [37] and applied in many QFT models. These Friedrichs diagrams look similar to Feynman diagrams as mentioned in Section 1.5. However, Friedrichs diagrams encode operator products, whereas Feynman diagrams encode integrals occurring in perturbation theory.

The formal limit  $W = \lim_{\substack{\sigma \rightarrow 0 \\ \Lambda \rightarrow \infty}} W_{\sigma,\Lambda}$  is often ill-defined. Sometimes  $W_{\sigma,\Lambda}$  contains a constant factor of the form  $e^{z_{\sigma,\Lambda}}$  with  $|z_{\sigma,\Lambda}| \rightarrow \infty$  as the cutoffs are removed, which is called an (infinite) **wave function renormalization**. Factors  $e^{iz_{\sigma,\Lambda}}$  with the same divergence behavior are interpreted as **infinite phases**.

A further heuristic interpretation of formal dressing transformations  $W$ , that are ill-defined on  $\mathcal{H}$ , is that  $\tilde{H}$  determines the dynamics for a state vector  $\Psi \in \mathcal{H}$  describing “true” or “physical” particles, which are mapped by  $W$  to states of **virtual particles**. The formal  $H$  then describes dynamics on these virtual particles. So these operations “dress” the vacuum vector (for Weyl- and Bogoliubov transformations) or a one-fermion vector (for Gross transformations) with a “cloud of particles”. Mathematically, by this “cloud”, we mean

- a **coherent state** for Weyl transformations
- a **squeezed** or **quasi-free state** for Bogoliubov transformations
- a **sum of coherent states** (one per fermion) for Gross transformations.

### 1.3.5 Converging States and GNS Construction

Finding a suitable dressing transformation such that the limit (1.71) makes sense can be a challenging task. It may be circumvented by an algebraic construction: Consider a sequence of cutoffs (for instance  $\sigma \rightarrow 0$ ) where for each  $\sigma$  a representation  $\pi_\sigma$  of the  $C^*$ -algebra  $\mathcal{A}$  corresponding to the system is known, such that

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$H_\sigma$  is self-adjoint on  $\mathcal{H}_\sigma$ . One now picks a state  $\omega_\sigma$  for each  $\sigma$ , for instance by taking a ground state  $\Psi_\sigma$  of  $H_\sigma$  (if it exists) and setting  $\omega_\sigma(A) = \langle \Psi_\sigma, A\Psi_\sigma \rangle$  for all  $A \in \mathcal{A}$ . By means of the Banach–Alaoglu theorem,  $\omega_\sigma$  has a weakly convergent subsequence with  $\omega_\sigma \rightharpoonup \omega$ . This  $\omega$  allows for a GNS construction, resulting in a representation  $\pi$  on a Hilbert space  $\mathcal{H}$  with “vacuum state”  $\Omega \in \mathcal{H}$ . It remains to construct the dynamics generated by some renormalized  $\tilde{H}$  on  $\mathcal{H}$ , for instance by defining  $\pi_\omega(e^{it\tilde{H}})$  as the limit of  $\pi_\sigma(e^{itH_\sigma})$  or  $\langle \pi(A)\Omega, \tilde{H}\pi(B)\Omega \rangle$  as the limit of  $\langle \pi_\sigma(A)\Psi_\sigma, H\pi_\sigma(B)\Psi_\sigma \rangle$  for suitable  $A, B \in \mathcal{A}$ .

### 1.3.6 Segal’s Theorem: Restriction to Local Algebras

There is another algebraic trick to circumvent the tedious adjustment of  $W_{\sigma,\Lambda}$  and  $c_{\sigma,\Lambda}$ , if the model has only an IR-divergence and is in  $d = 1$  space dimensions [38]. Instead of defining  $\tilde{H}$  directly, one may define the action of  $\tilde{H}$  on all local time-zero algebras  $\mathcal{A}(\{0\} \times C)$  with open and bounded space region  $C \subset \mathbb{R}$ . For each pair  $t \in \mathbb{R}, C \subset \mathbb{R}$ , the form factor  $\check{v}(\mathbf{x})$  is multiplied by a spatial cutoff function  $h : \mathbb{R}^d \rightarrow [0, 1]$  with  $h(\mathbf{x}) = 1$  near  $|\mathbf{x}| = 0$  and  $h(\mathbf{x}) \rightarrow 0$  as  $|\mathbf{x}| \rightarrow \infty$ . Assume that a UV-renormalized Hamiltonian  $\tilde{H}_h$  can be found with  $h(\mathbf{x}) = 1$  in a region larger than the region  $C$  enlarged by  $|t|$ . The latter enlarged region is called  $C_{|t|}$ , see Figure 1.6. Observables localized in  $C$  will be localized in  $C_{|t|}$  after the Heisenberg time evolution, so the cutoff should not affect the dynamics. Choosing a suitable cutoff for all  $C$  and  $t$ , one should then be able to recover the dynamics within the local time-zero algebra

$$\mathcal{A}_{\text{loc},0} = \bigcup_{\substack{C \subset \mathbb{R} \\ C: \text{ bounded, open}}} \mathcal{A}(\{0\} \times C). \quad (1.78)$$

And indeed, **Segal’s theorem** now asserts that if for each pair  $t, C$  as above a suitable spatial cutoff function  $h$  can be found, such that  $\tilde{H}_h$  is a unique self-adjoint operator, then there exists an automorphism group  $\alpha_t : \mathcal{A}_{\text{loc},0} \rightarrow \mathcal{A}_{\text{loc},0}$  describing the cutoff-free dynamics.

Of course, it remains to recover a renormalized Hamiltonian  $\tilde{H}$  from the automorphism group  $\alpha_t$ , which may require some considerable work. This can, for instance be done, by constructing a suitable state  $\omega$  on  $\mathcal{A}_{\text{loc},0}$  (e.g., as a weak limit  $\omega_h \rightharpoonup \omega$ ). If in the GNS representation  $\pi_\omega$ , the automorphisms  $\alpha_t$  can be implemented as  $\alpha_t(A) = U(t)^*AU(t)$  with  $(U(t))_{t \in \mathbb{R}}$  being a strongly continuous group of unitary operators, then Stone’s theorem renders the desired generator  $\tilde{H}$ .

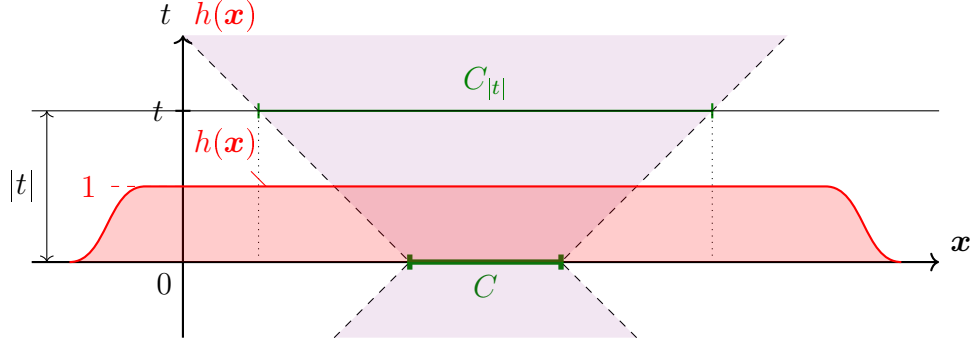


Figure 1.6: Within Segal's theorem, the spatial cutoff  $h(\mathbf{x})$  is chosen such that  $h \equiv 1$  on a region larger than  $C_{|t|}$ . Color online.

### 1.3.7 Non-Relativistic Models

There is a variety of relativistic and non-relativistic models with particle creation and annihilation where non-perturbative dynamics have been successfully defined or where a definition of dynamics would be highly desirable. We present some of them here and outline the employed techniques. The establishment of non-perturbative quantum dynamics in models with particle creation and annihilation is of course a vast field of research and our list is by no means meant to be exhaustive.

The first type of models we present are polaron-like models. By that we mean models with fermions which interact by an exchange of bosons. We denote position and momentum of fermions by  $\mathbf{x}$  and  $\mathbf{p}$ , as well as the boson position and momentum by  $\mathbf{y}$  and  $\mathbf{k}$  and index the fermionic and bosonic particle species (e.g., for particles with spin) by  $j \in \mathbb{N}$ . The free Hamiltonian then takes the form

$$H_0 = d\Gamma_{x,j}(\theta_j) + d\Gamma_{y,j}(\omega_j), \quad (1.79)$$

with  $\theta_j : \mathbb{R}^d \rightarrow \mathbb{R}$  being the dispersion relation of fermion species  $j$  with  $\omega_j : \mathbb{R}^d \rightarrow \mathbb{R}$  the dispersion relation of boson species  $j$ .

The interaction Hamiltonian describes the emission and absorption of bosons triggered by fermions:

$$H_I = \sum_{j,k} \int_{\mathbb{R}^d} \left( a_{j,-}^\dagger(\mathbf{p}) a_{j,-}(\mathbf{p} + \mathbf{k}) a_{k,+}^\dagger(\mathbf{k}) v_{j,k}(\mathbf{p}, \mathbf{k}) dp d\mathbf{k} + \text{h.c.} \right). \quad (1.80)$$

Interaction terms of this or a similar form appear in formal Hamiltonians of QFTs that are considered to be comparably fundamental, such as QED, QCD or the

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Standard Model of Particle Physics. Such terms also emerge from many-body models in condensed matter physics. In both situations, they may (at least heuristically) describe how attractive and repulsive forces appear between fermions that interact by exchanging bosons.

The following models fall under this category.

### Fröhlich Polaron

$\theta(\mathbf{p}) = |\mathbf{p}|^2$ ,  $\omega(\mathbf{k}) = 1$ ,  $v(\mathbf{k}) = |\mathbf{k}|^{-1}$ . This model has been proposed by Fröhlich, Pelzer and Zienau to describe the interaction of electrons by exchange of optical phonons in solid-state materials [39]. Self-adjointness of  $H$  can be established without counterterms or dressing transformations by interpreting  $H$  as a bilinear form [40]. However, a Gross transformation is useful to characterize the domain of  $H$ .

### Nelson Model

$\theta(\mathbf{p}) = \frac{|\mathbf{p}|^2}{2M}$ ,  $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m^2}$ ,  $v(\mathbf{k}) = g\omega(\mathbf{k})^{-1/2}$  with  $m \geq 0, g \in \mathbb{R}$ . This model was originally introduced by Nelson in order to describe the interaction of nucleons by meson exchange [41]. The renormalization for massive bosons  $m > 0$  requires an energy counterterm  $E_\Lambda$  that depends on the UV-cut-off  $\Lambda$  and is proportional to the fermion number  $N$ .

What makes this model particularly interesting is that in the massless case  $m = 0$ , there are several ways to perform a renormalization which result in different dynamics. The Hamiltonian renormalized without a dressing transformation

$$\tilde{H}^{(1)} := \lim_{\substack{\sigma \rightarrow 0 \\ \Lambda \rightarrow \infty}} (H_{\sigma,\Lambda} - E_\Lambda) \quad (1.81)$$

is self-adjoint [41], but does not have a ground state for  $m = 0$  and  $M = 1$  [42]. It is possible to generate an inequivalent Hamiltonian by fibering  $H_{\sigma,\Lambda} = \int_P H_{\sigma,\Lambda,P} dP$ , see (1.56), and conjugating each fiber Hamiltonian with a bosonic Weyl transformation  $W_{\sigma,P}$

$$\tilde{H}_P^{(2)} := \lim_{\substack{\sigma \rightarrow 0 \\ \Lambda \rightarrow \infty}} W_{\sigma,P}^* (H_{\sigma,\Lambda,P} - E_\Lambda) W_{\sigma,P}. \quad (1.82)$$

In that case  $\tilde{H}_P^{(2)}$  has a ground state [43]. Another Hamiltonian is constructed in [43] algebraically by considering the states  $\omega_{\sigma,P}$  associated with the ground state vector  $\Psi_{\sigma,P}$  of the UV-renormalized Hamiltonian  $H_{\sigma,P}$ . A limit state  $\omega_{\sigma,P} \rightarrow \omega_P$

exists in the weak sense and, using the GNS representation  $\pi_{\omega_P}$ , one may construct  $\tilde{H}_P^{(3)}$  via

$$e^{it\tilde{H}_P^{(3)}} := \lim_{\sigma \rightarrow 0} \pi_{\omega_{\sigma,P}}(e^{itH_{\sigma,P}}). \quad (1.83)$$

The sequence of Weyl transformations is given by  $W_{\sigma,P} = e^{a^\dagger(s_{\sigma,P}) - a(s_{\sigma,P})}$ , where the form factors  $s_{\sigma,P} \in L^2$  formally converge to some  $s_{\sigma,P} \notin L^2$ . So in the IR-limit, the corresponding dressing transformation  $W$  formally “leads out of Fock space”. An interpretation of the “dressed vacuum”  $W\Omega$  (with  $\Omega$  being the ground state of  $H_0$ ) as a vector in a Fock space extension  $\widehat{\mathcal{H}} \supset \mathcal{H}$  was given by Fröhlich [43], with  $\widehat{\mathcal{H}}$  being von Neumann’s ITP space [44] (see also Section 3.1). Now, when restricted to the one-fermion sector, all fiber Hamiltonians  $\tilde{H}_P^{(3)}$  have a ground state with energy  $E(P)$ . In case such a set of ground states exists, the function  $P \mapsto E(P)$  is also called a (proper) **one-fermion shell** or **mass shell** corresponding to  $\tilde{H}^{(3)}$ . By contrast, if the Hamiltonian  $\tilde{H}^{(1)}$  is restricted to the one-fermion sector and fibered, then none of the  $\tilde{H}_P^{(1)}$  has a ground state, which is also called an **infraparticle situation** or even **infrared catastrophe**, see also [45].

As the conjugation with a Weyl transformation formally adds a constant to bosonic operators (1.74), one may also replace  $a_-(f)$  by  $\tilde{a}_-(f) = a_-(f) + \langle f, s \rangle$ . Restricting to a dense subspace  $f \in \mathcal{S} \subseteq L^2$  allows for a well-defined transformation whenever  $s \in \mathcal{S}'$ , even if  $s \notin L^2$ . The replacement results in a change of the formal Hamiltonians

$$H_{\sigma,\Lambda} \rightarrow \tilde{H}_{\sigma,\Lambda}^{(4)}, \quad H_\Lambda \rightarrow \tilde{H}_\Lambda^{(4)}. \quad (1.84)$$

One may then show that  $\tilde{H}_\Lambda^{(4)}$  indeed generates the same Heisenberg dynamics on  $\tilde{a}_-^\sharp$  as  $H_\Lambda$  does on  $a_-^\sharp$ , and that  $\tilde{H}_\Lambda^{(4)}$  has a ground state [46, 47]. Note that the UV-cutoff  $\Lambda$  has not explicitly been removed, here. The so-constructed  $\tilde{H}_\Lambda^{(4)}$  may also be used for the construction of scattering states [48].

Another way to construct a renormalized Hamiltonian  $\tilde{H}^{(\text{euc})}$  is given by Euclidean field theory methods [42, 49, 50].

In case  $d \leq 2$ , a renormalized  $\tilde{H}^{(1)}$  can also be constructed as in (1.81), without a dressing transformation  $W$ , but with an energy renormalization  $E_\Lambda$  [51, 52]. A construction of  $\tilde{H}^{(\text{euc})}$  in  $d > 3$  is provided in [42].

It is also well-known that within the dressing process, a Yukawa potential (for  $m > 0$ ) or a Coulomb potential (for  $m = 0$ ) emerges, accompanied by radiative corrections that are higher-order in the coupling constant. See e.g., [41, (15)] in the case with cutoffs and  $m > 0$ . The emergence of a Coulomb potential for  $\tilde{H}^{(2)}$  without any cutoffs in case  $m = 0$  has been established in [53] for weak couplings (small  $g$ ) and two fermions out of which one is infinitely heavy and hence fixed. For the cut-off Hamiltonian  $\tilde{H}_\Lambda^{(2)} = W_\Lambda^* H_\Lambda W_\Lambda$  and  $m = 0$ , the emergence of a

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Coulomb potential and radiative corrections complying with the Larmor formula have been established in [54, 55].

For a comparison of the domains of  $\tilde{H}^{(1)}$  and  $\tilde{H}^{(2)}$ , see also [52]. Further references concerning the Nelson model in mathematical physics can be found in the introductions of [56] and [50].

### Pseudo-Relativistic Nelson Model

$\theta(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + M^2}$ ,  $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m^2}$ . For the (physically slightly undesirable, but mathematically more regular) form factor  $v(\mathbf{k}) \leq g(1 + |\mathbf{k}|)^{-1/2}\omega(\mathbf{k})^{-1/2}$ , the existence of a UV-renormalized Hamiltonian

$$H_\sigma = \lim_{\Lambda \rightarrow \infty} (H_{\sigma, \Lambda} - E_\Lambda) \quad (1.85)$$

and for  $m > 0$  the existence of a renormalized Hamiltonian

$$H = \lim_{\Lambda \rightarrow \infty} (H_\Lambda - E_\Lambda) \quad (1.86)$$

have been established in [57]. There, it was also proved that a ground state exists for each fiber Hamiltonian  $H_{P, \sigma}, H_P$ , which implies the existence of a mass shell. For  $m = 0$ , the IR-renormalized fiber Hamiltonian  $\tilde{H}_P^{(3)}$  as in (1.83) has been constructed in [43] and a mass shell has been proved to exist. Note that in both references, the form factor is not denoted  $v(\mathbf{k})$  but rather  $v(\mathbf{k})\omega(\mathbf{k})^{-1/2}$ .

However, for the more physical  $v(\mathbf{k}) = g\omega(\mathbf{k})^{-1/2}$  with small coupling  $g$  and for  $m > 1$ , it was proved that the mass shell of  $H_{\sigma, \Lambda}$  exists but becomes flat in the limit  $\Lambda \rightarrow \infty$  (without the need for constructing a limit Hamiltonian  $H_\sigma$ ) [58].

### Eckmann's Polaron

Eckmann [59] proposed a type of pseudo-relativistic Nelson model with  $v(\mathbf{p}, \mathbf{k}) = c|\theta(\mathbf{p} - \mathbf{k})\omega(\mathbf{k})\theta(\mathbf{p})|^{-1/2}$ ,  $c > 0$ . It can be renormalized as

$$\tilde{H}^{(1)} = \lim_{\Lambda \rightarrow \infty} (H_\Lambda - \delta m_\Lambda), \quad (1.87)$$

with a counterterm  $\delta m_\Lambda = \int m_\Lambda(\mathbf{k})a_+^\dagger(\mathbf{k})a_+(\mathbf{k}) d\mathbf{k}$  that acts as a simultaneous energy and mass renormalization. There are several choices of  $m_\Lambda(\mathbf{k})$  possible which can be used to adjust the renormalized mass shell  $E(P)$  [60]. A non-unitary dressing transformation was used in [60] to describe the domain  $\text{dom}(\tilde{H}^{(1)}) \subset \mathcal{H}$ ,

but is not needed for obtaining a renormalized Hamiltonian or a mass shell.

### Gross' Relativistic Polaron

$\theta(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + m_x^2}$ ,  $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m_y^2}$ ,  $v(\mathbf{k}) = g\omega(\mathbf{k})^{-1/2}$ . Here, a renormalized Hamiltonian has been constructed algebraically by L. Gross [61]. The renormalization process is similar to that of  $\tilde{H}_P^{(3)}$  in (1.83): A sequence of ground states  $\omega_\Lambda$  is constructed for  $(H_\Lambda - \delta m_\Lambda)$  such that  $\omega_\Lambda \rightarrow \omega$ . The weak convergence is understood with respect to the local  $C^*$ -algebra  $\mathcal{A}_{\text{loc}}$  of Weyl operators (1.30). So there exists a GNS representation  $\pi_\omega$  with respect to  $\omega$  on  $\mathcal{H}_0$  with vacuum  $\Omega$ . A mass renormalization  $\delta m_\Lambda$  is chosen such that it only modifies the bare mass  $m_x$  in a way that the infimum of the spectrum of  $(H_\Lambda - \delta m_\Lambda)$  restricted to the one-fermion Hilbert space is kept at a fixed value. The self-adjoint  $\tilde{H}^{(3)}$  is now constructed by defining the quadratic form  $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$  via

$$\langle \pi_\omega(A)\Omega, \pi_\omega(B)\Omega \rangle_{\mathcal{H}_0} = \lim_{\Lambda \rightarrow \infty} \langle \pi_\Lambda(A)\Psi_\Lambda, (H_\Lambda - \delta m_\Lambda)\pi_\Lambda(B)\Psi_\Lambda \rangle, \quad (1.88)$$

with  $A, B \in \mathcal{A}_{\text{loc}}$  and  $\pi_\Lambda$  being the standard representation of  $A, B$  by Weyl operators on  $\mathcal{F}$  and  $\Psi_\Lambda$  the ground state of  $H_\Lambda$ . This form is positive and hence corresponds to a self-adjoint operator  $\tilde{H}^{(3)}$  by Friedrichs' extension theorem.

In particular, the ground state vectors  $\Psi_\Lambda \in \mathcal{F}$  associated with  $\omega_\Lambda$  converge weakly to 0 in  $\mathcal{F}$ , so  $\mathcal{H}_0$  can heuristically be considered as being "outside the Fock space".

### Van Hove Model

In this model, fixed fermions are considered  $\theta(\mathbf{p}) = 0$ , which allows for fibering  $H$  in the fermion momenta  $(\mathbf{p}_1, \dots, \mathbf{p}_N)$  and thereby reducing considerations to bosonic fields. This reduced form is the one in which Van Hove originally introduced the model [62] with  $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m^2}$ ,  $v(\mathbf{k}) = g\omega^{-1/2}$ . In this paper, he also interpreted the ground state of the formal Hamiltonian (which is outside  $\mathcal{F}$ ) as an ITP vector.

Depending on the choice of  $\omega$  and  $v$ , the cutoff renormalization may require energy counterterms with  $E_{\sigma, \Lambda} \rightarrow \infty$  or dressing transformations  $W_{\sigma, \Lambda}$  that do not converge to a unitary operator on  $\mathcal{F}$  as  $\sigma \rightarrow 0$  or  $\Lambda \rightarrow \infty$  [63, 15]. In particular, both a UV-divergent self-energy  $E_\infty = -\int \frac{v(\mathbf{k})v(\mathbf{k})}{\omega(\mathbf{k})} d\mathbf{k}$  and a formal Weyl transformation  $W(s)$ ,  $s = -\frac{v}{\omega}$ , which heuristically leads out of Fock space, are required for the QED-typical scalings  $\omega(\mathbf{k}) = |\mathbf{k}|$ ,  $v = g|\mathbf{k}|^{-1/2}$ .

## 1 Introduction

There are also physically interesting models, with  $H_I$  of a form similar but not equal to (1.80):

### Pauli–Fierz Model

This more fundamental model is sometimes also called the **standard model of non-relativistic QED**. It describes the interaction between atoms and light, and many models in quantum optics can formally be derived from it by simplification [64]. Electrons are modeled by one fermion species with  $\theta(\mathbf{p}) = \frac{|\mathbf{p}|^2}{2m}$ . Photons are considered in the Coulomb gauge which entails two boson species (“transversal photons”) with  $\omega(\mathbf{k}) = |\mathbf{k}|$ ,  $v(\mathbf{p}, \mathbf{k}) \propto |\mathbf{k}|^{-1/2}$ . Physically, one would also expect a third species (scalar photons), which are not included in the Pauli–Fierz model. Instead, a Coulomb interaction between the fermions,  $V_C = g \sum_{k \neq k'} \frac{1}{|\mathbf{x}_k - \mathbf{x}_{k'}|}$ , is added. This is heuristically justified by the observation that interaction terms in  $H$  describing boson exchange produce pair potentials in the leading order of the coupling constant  $g$ , when conjugated with dressing transformations (see the Nelson model above).

The Pauli–Fierz model may or may not involve an external potential  $V_{\text{ext}}(\mathbf{x})$ . If no external potential is involved and only  $M = 1$  electron is considered, then the Pauli–Fierz model can be seen as a Polaron model in the sense of (1.62).

Self-adjointness of  $H_\Lambda$  with  $UV$ -cutoff,  $M \in \mathbb{N}$  electrons and arbitrary coupling strengths was established in [65]. Spectral dynamics of the Pauli–Fierz model have been thoroughly investigated, see for instance [66, 67, 68, 69] and the references therein.

As for the Nelson model, an infraparticle problem occurs: For  $M = 1$ , the fiber Hamiltonians  $H_{\Lambda, P}$  have no ground state for  $P \neq 0$ . Arai [70] showed that this problem can be resolved by constructing an alternative Hamiltonian  $\tilde{H}_{\Lambda, P}^{(5)}$  which has a ground state for all  $P$ : After applying an IR-cutoff  $\sigma > 0$ , the electron mass  $m_x$  within  $H_{\sigma, \Lambda, P}$  is replaced by  $m_x - \delta m_\sigma$ , which yields a cut-off Hamiltonian with mass renormalization, called  $H_{\sigma, \Lambda, P}^{(5)}$ . The Wightman functions of  $H_{\sigma, \Lambda, P}^{(5)}$  are explicitly evaluated and the limit Wightman functions for  $\sigma \rightarrow 0$  are constructed. It is then shown that these correspond to a new IR-renormalized Hamiltonian  $\tilde{H}_{\Lambda, P}^{(5)}$  that has a ground state. Further, Arai showed that the mass renormalization is necessary in order to obtain the correct Thomson scattering formula.

Similar to Fröhlich within the Nelson model above, Blanchard [71] also used the ITP space  $\widehat{\mathcal{H}}$  in order to describe the limit of a sequence of dressing transformations  $W_{\sigma, P} \rightarrow W_P$  where  $W_P$  is a formal expression ill-defined on  $\mathcal{H}_P$ , but can be given rigorous meaning as a unitary operator on the Fock space extension  $\widehat{\mathcal{H}}$ . However, this well-defined  $W_P$  was used to describe asymptotic Hilbert spaces appearing in scattering theory and not for the construction of an IR-renormalized

Hamiltonian.

To our best knowledge, no renormalized  $\tilde{H}$  without UV-cutoff has been constructed in the Pauli–Fierz model so far.

### Pseudo–Relativistic Pauli–Fierz Models

Here, the Pauli–Fierz model is modified such that the electron dispersion relation becomes pseudo–relativistic. That is,  $\theta(\mathbf{p}) \propto |\mathbf{p}|$  as  $|\mathbf{p}| \rightarrow \infty$ . Examples are the **semi–relativistic Pauli–Fierz model** with  $\theta(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + M^2}$  for spinless bosons and the **no–pair model** where spin–1/2–electrons are described by restricting a Dirac operator  $\boldsymbol{\alpha} \cdot \mathbf{p} + m\beta$  (Dirac–type operators are explained above (1.150)) to the positive–energy subspace [72]. Electrons may also be modeled as full **Dirac particles** involving positive– and negative–energy solutions of the Dirac operator [73].

Properties of the former two models have been investigated, for instance in [74, 75, 76, 77, 78] or the references therein. In all cited resources,  $M = 1$  electron is considered and an external potential  $V_{\text{ext}}(\mathbf{x})$  may or may not be involved. For  $V_{\text{ext}}(\mathbf{x}) = -\frac{g_{\text{ext}}}{|\mathbf{x}|}$ , smallness of the coupling constant  $g_{\text{ext}}$  must be assumed, since above a certain coupling threshold, several self–adjoint versions of the formal Hamiltonian  $H_\Lambda$  exist. This effect already occurs for a Dirac particle in a Coulomb potential, see also [79] and the last point in Section 1.4.4.

Just as the Nelson model, also the semi–relativistic Pauli–Fierz model allows for constructing an IR–regularized Hamiltonian on each fiber (see (1.56)) by a Weyl transformation  $W_{\sigma,P}$  as in (1.82)

$$\tilde{H}_{\Lambda,P}^{(2)} := \lim_{\sigma \rightarrow 0} W_{\sigma,P}^* H_{\sigma,\Lambda,P} W_{\sigma,P}, \quad (1.89)$$

which generates dynamics inequivalent to those of  $H_{\Lambda,P}$  [80]. Inequivalence manifests itself in the fact that  $H_{\Lambda,P}$  has only a (2–fold degenerate) ground state for  $P = 0$ , while  $\tilde{H}_{\Lambda,P}^{(2)}$  has a (2–fold degenerate) ground state for all  $P$  in a sufficiently small open ball around 0. So the dressing transformation removes the infraparticle problem of  $H_\Lambda$  at small total momentum. As for the Nelson model, the IR–limit  $W_P = \lim_{\sigma \rightarrow 0} W_{\sigma,P}$  does not define a unitary operator on  $\mathcal{H}_P$ , but Könenberg and Matte [80] could establish  $W_P$  as a map from  $\mathcal{H}_P$  into von Neumann’s ITP space  $\widehat{\mathcal{H}}$ .

However, to our best knowledge, for all above modifications of the Pauli–Fierz model, no removal of the UV–cutoff  $\Lambda$  has been achieved so far.

## Spin–Boson Model

This model describes a single spin coupled to a boson field, meaning that the fermion has only two allowed configurations  $\mathbf{x} \in \{0, 1\}$ . For bosons, a physically reasonable choice would be  $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m^2}$  with  $m \geq 0$  and  $v(\mathbf{k}) = |\omega(\mathbf{k})|^{-1/2}$  where a UV–cutoff is always used. But also other dispersion relations and form factors have been investigated [81]. The spin–boson mode can formally be derived from a one–electron Pauli–Fierz model in an external field by reducing the Hilbert space for the electron to a two–dimensional subspace. For instance the space spanned by the ground state and an excited state of an electron within an atomic potential can be used for this. The cut–off Hamiltonian  $H_\Lambda$  can be realized as a self–adjoint operator and even has a ground state for  $m \geq 0$ , so there is no need to avoid infrared problems by constructing a non–Fock representation of a suitable operator algebra [82, 83, 81].

It is also possible to consider **generalized spin–boson models** where the fermionic two–level system is replaced by a  $d$ –level system ( $\mathbf{x} \in \{1, \dots, d\}$ ), a harmonic oscillator ( $\mathbf{x} \in \mathbb{N}$ ) or even a finite collection of harmonic oscillators ( $\mathbf{x} \in \mathbb{N}^d$ ) [84, 85]. For  $m = 0$  and under certain assumptions, it has been proved that the Hamiltonian  $H_\Lambda$  has no ground state [86].

## Lee Model and Galilean Invariant Lee Model

The Lee model, introduced in [87], is another less polaron–like, but still non–relativistic QFT model, which treats two fermion species  $V$  and  $N$ , as well as one boson species  $\Theta$ . The interaction is of the kind  $V \leftrightarrow N + \Theta$ , meaning that a  $V$ –fermion transforms into an  $N$ –fermion under the emission of a  $\Theta$ –boson. For this model, formal expressions for generalized eigenfunctions have been derived already in the 1950s [88], which allowed for first comparisons between perturbative and non–perturbative renormalization in QFT [89].

A modified version of it, called the Galilean invariant Lee model, was non–perturbatively renormalized on a rigorous level by Schrader [90]. Here, the dispersion relations are  $\theta_V(\mathbf{p}) = U + \frac{|\mathbf{p}|^2}{2m_V}$ ,  $\theta_N(\mathbf{p}) = \frac{|\mathbf{p}|^2}{2m_N}$  and  $\omega_\Theta(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2m_\Theta}$  with  $U \in \mathbb{R}$  and masses  $m_V = m_N + m_\Theta$ . The cubic interaction Hamiltonian takes the form

$$H_I = g \int \left( a_V^\dagger(\mathbf{p}) a_N \left( \frac{m_N}{m_V} \mathbf{p} + \mathbf{q} \right) a_\Theta \left( \frac{m_\Theta}{m_V} \mathbf{p} - \mathbf{q} \right) + \text{h.c.} \right) d\mathbf{p} d\mathbf{q}. \quad (1.90)$$

In order to remove the UV–cutoff, Schrader added a self–energy counterterm  $E_\Lambda$  proportional to the number of  $V$ –particles (so  $E_\Lambda$  can also be called a “mass renormalization”) as well as a charge counterterm  $\delta V_\Lambda$ . For a reasonable choice of

$E_\Lambda$  and  $\delta V_\Lambda$ , non-perturbative scattering theory renders<sup>17</sup> a physical mass  $m_{V,\text{phys},\Lambda}$  and charge  $g_{\text{phys},\Lambda}$ . Correspondingly,  $E_\Lambda$  and  $\delta V_\Lambda$  can be adjusted such that the fixed values  $m_{V,\text{phys},\Lambda} = m_{V,\text{phys}}$  and  $g_{\text{phys},\Lambda} = g_{\text{phys}}$  are maintained as  $\Lambda \rightarrow \infty$ . It is then shown that the limit

$$\tilde{H} = \lim_{\Lambda \rightarrow \infty} (H_\Lambda - E_\Lambda + \delta V_\Lambda) \quad (1.91)$$

exists as a self-adjoint operator in the norm resolvent sense and that the resolvent  $\tilde{r}(z) = (z - \tilde{H})^{-1}$  can explicitly be expressed by a convergent Born series.

### 1.3.8 Relativistic Models

By relativistic, we mean that the models are considered as potential candidates for a Wightman field theory. Such models have been extensively studied in the context of CQFT.

In all models considered below, the interaction Hamiltonian  $H_I$  is constructed out of field operator-valued distributions, e.g., for bosons

$$\phi(\mathbf{x}) = \int \frac{1}{\sqrt{4\pi\omega(\mathbf{k})}} (a^\dagger(\mathbf{k}) + a(-\mathbf{k})) e^{-i\mathbf{x}\mathbf{k}} d\mathbf{k}, \quad (1.92)$$

with  $\omega(\mathbf{k})$  being the boson dispersion relation. The expressions for fermionic fields are related, but different, see below. The formal  $H_I$  is now a (possibly divergent) sum over **Wick products** of the kind

$$\int : \phi(\mathbf{x})^N : d\mathbf{x} = \int : \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_N) : \delta(\mathbf{x}_1 - \mathbf{x}_2) \dots \delta(\mathbf{x}_{N-1} - \mathbf{x}_N) d\mathbf{x}_1 \dots d\mathbf{x}_N, \quad (1.93)$$

where the two double dots  $::$  denote **Wick ordering**, which is the operation of moving all  $a^\dagger$  to the left and all  $a$  to the right inside the double dots. This makes (1.93) a sum over operator products of the form (1.34). Locality manifests itself in the fact that for all  $\phi(\mathbf{x}_j)$ , the same position  $\mathbf{x} = \mathbf{x}_j$  is used, so the interaction happens at a single point  $\mathbf{x}$ .

For each particle species, the dispersion relation in  $H_0$  is now chosen in such a way

<sup>17</sup>For recovering  $m_{V,\text{phys},\Lambda}$  and  $g_{\text{phys},\Lambda}$  from scattering theory, Schrader first constructs the Møller operators  $\Omega_\Lambda^{\text{in/out}}$ . The operator  $\tilde{H}_\Lambda = (H_\Lambda - E_\Lambda + \delta V_\Lambda)$  is then restricted to the  $P = 0$ -fiber of the one-fermion sector, with the restriction called  $\tilde{H}_\Lambda^S$ . For the resolvent  $\tilde{r}_\Lambda^S(z) = (z - \tilde{H}_\Lambda^S)^{-1}$  he then establishes  $\tilde{r}_\Lambda^S(z) \Omega_\Lambda^{\text{in/out}} = \Omega_\Lambda^{\text{in/out}} \tilde{r}^S(z)$  where  $r^S(z)$  is the resolvent of some Hamiltonian  $\tilde{H}^S$ . This  $\tilde{H}^S$  is of a form, which allows for directly recovering the “physical” mass  $m_{V,\text{phys},\Lambda}$  and charge  $g_{\text{phys},\Lambda}$ .

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that the **free propagator**  $G_2(t, t')(\cdot, \cdot) \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$  defined by

$$G_2(t, t') : f_t, g_{t'} \mapsto \langle \Omega, [\phi(f_t), e^{-i(t-t')H_0} \phi(g_{t'})] \Omega \rangle, \quad f_t, g_{t'} \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d), \quad (1.94)$$

with vacuum vector  $\Omega \in \mathcal{F}$ ,  $a(f_t)\Omega = 0$  is supported in the time- and lightlike separated coordinates

$$\text{supp } G_2(t, t') \subseteq \{(\mathbf{x}, \mathbf{x}') \in \mathbb{R}^{2d} \mid \|\mathbf{x} - \mathbf{x}'\| \leq |t - t'|\}, \quad \forall t, t' \in \mathbb{R}. \quad (1.95)$$

We remark that it is customary to conclude the family of distributions  $(G_2(t, t'))_{t, t' \in \mathbb{R}}$  into one single spacetime distribution  $G_2 \in \mathcal{S}'(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$  also called the “free propagator”. Condition (1.95) then becomes

$$\text{supp } G_2 \subseteq \{(\mathbf{x}, t, \mathbf{x}', t') \in \mathbb{R}^{2(d+1)} \mid \|\mathbf{x} - \mathbf{x}'\| \leq |t - t'|\}. \quad (1.96)$$

This condition assures that the **free field operators** smeared in space- and time-direction

$$\phi(f) = \int f(t, \mathbf{x}) e^{-itH_0} \phi(\mathbf{x}) e^{itH_0} dt d\mathbf{x}, \quad f \in \mathcal{S}(\mathbb{R}^{d+1}), \quad (1.97)$$

satisfy Wightman locality (see Section 1.2.3), i.e.,  $[\phi(f), \phi(g)]_{\pm} = 0$  if  $\text{supp } f$  and  $\text{supp } g$  are spacelike separated.

An example for a suitable dispersion relation is  $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m^2}$ . Although for this  $\omega(\mathbf{k})$ , the support of a wave packet  $f_t = e^{-i\omega(\mathbf{k})t} f_0$ ,  $f_0 \in C_c^\infty(\mathbb{R}^d)$  spreads faster than light, the propagator  $G_2$  still has causal support.

This causal support of  $G_2$  allows for constructing a causal perturbation theory by the Epstein–Glaser method. For further references, see Section 1.5.

### $\phi^4$ -Theory

This model describes a self-interacting boson field with dispersion relation  $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m^2}$ ,  $m > 0$ . The formal Hamiltonians with or without spatial cutoff  $h : \mathbb{R} \rightarrow [0, 1]$  read

$$\begin{aligned} H_h &= H_0 + H_{I,h}, & H &= H_0 + H_I \\ H_{I,h} &= g \int h(\mathbf{x}) : \phi(\mathbf{x})^4 : d\mathbf{x}, & H_I &= g \int : \phi(\mathbf{x})^4 : d\mathbf{x}, \end{aligned} \quad (1.98)$$

with  $\phi(\mathbf{x})$  given by (1.92). The Wick ordering of  $\phi(\mathbf{x})^4$  renders one operator product of the kind  $a^\dagger a^\dagger a^\dagger a^\dagger$ , 4 terms of the kind  $a^\dagger a^\dagger a^\dagger a$ , 6 terms  $a^\dagger a^\dagger a a$ , 4 terms  $a^\dagger a a a$  and one term  $a a a a$ . Note that sometimes, the product  $g \cdot h(\mathbf{x}) =: g(\mathbf{x})$  is

treated as a space-dependent coupling constant.

In 1+1 dimensions,  $\phi^4$ -theory has been renormalized by Glimm and Jaffe [91, 92, 93]. Existence of a self-adjoint realization of  $H_h$  without UV-cutoff can be inferred from [38, 94]. Essential self-adjointness of  $H_h$  on a suitable domain is established in [91], such that the uniqueness requirement of dynamics in Segal's theorem is satisfied and an automorphism group  $(\alpha_t)_{t \in \mathbb{R}}$  can be constructed (see Section 1.3.6). Several Haag-Kastler axioms are verified in [92] and a renormalized Hamiltonian  $\tilde{H}$  is recovered from  $(\alpha_t)_{t \in \mathbb{R}}$  in [93].

The renormalization of  $\phi^4$ -theory in 2 + 1 dimensions has been achieved by Glimm and Jaffe [95, 96]. Glimm first constructed a densely defined symmetric renormalized Hamiltonian of the kind

$$\tilde{H}_h = \lim_{\Lambda \rightarrow \infty} T_{\Lambda,h}^{-1} (H_{\Lambda,h} - c_{\Lambda,h}) T_{\Lambda,h}. \quad (1.99)$$

Here,  $T_{\Lambda,h}$  is a non-unitary dressing transformation [95, Sect. 3], and the counterterm  $c_{\Lambda,h}$  contains a self-energy  $E_{\Lambda,h}$ , as well as a mass counterterm  $\delta m_{\Lambda,h}$  consisting of quadratic operator products of the type  $a^\dagger a^\dagger$ ,  $aa$  and  $a^\dagger a$  [95, Sect. 4]. Later, Glimm and Jaffe [96] showed that  $\tilde{H}_h$  is bounded from below, which allows for finding a self-adjoint Friedrichs extension of  $\tilde{H}_h$  for any spatial cutoff  $h$ .

### $P(\phi)$ -Theories

This class of models, also called  $\lambda\phi^{2n}$ -theories, is given by a self-interacting boson field with  $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m^2}$ ,  $m > 0$  and a polynomial interaction

$$\begin{aligned} H_I &= g \int : P(\phi(\mathbf{x})) : d\mathbf{x}, & H_{I,h} &= g \int h(\mathbf{x}) : P(\phi(\mathbf{x})) : d\mathbf{x}, \\ P(\phi) &= \phi^{2n} + b_{2n-1} \phi^{2n-1} + \dots + b_0, \end{aligned} \quad (1.100)$$

with  $n \in \mathbb{N}$  and arbitrary coefficients  $b_j \in \mathbb{R}$ ,  $j \in \{0, \dots, 2n-1\}$ . Hence, it includes the  $\phi^4$ -theory.

A renormalized Hamiltonian in 1 + 1 dimensions with spatial cutoff  $\tilde{H}_h$  has been constructed by Rosen [97] as

$$\tilde{H}_h = \lim_{\substack{\sigma \rightarrow 0 \\ \Lambda \rightarrow \infty}} H_{\sigma,\Lambda,h}, \quad (1.101)$$

in the norm resolvent sense, where  $\sigma$  is an IR-cutoff imposed by forcing the system into a box of finite volume  $V$  and where  $\Lambda$  (called  $K$  in the original paper)

## 1 Introduction

is a UV–cutoff. As for  $\phi^4$ –theory in 1 + 1 dimensions, no counterterms or dressing transformations are involved. Further, Rosen discusses how to use Segal’s theorem to recover the dynamics automorphism group  $(\alpha_t)_{t \in \mathbb{R}}$  and a renormalized Hamiltonian  $\tilde{H}$  generating dynamics on a physical Hilbert space.

An alternative construction of a self–adjoint  $\tilde{H}_h$  has been given by Glimm and Jaffe in [98].

A set of self–adjoint field operators  $\phi(f)$ ,  $f \in \mathcal{S}(\mathbb{R}^2)$  has been established by Glimm and Jaffe in [99] via constructing all Wightman functions and using the Wightman reconstruction theorem.

### $e^{\alpha\phi}$ –Model

Also this model describes a single boson species with dispersion relation  $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m^2}$ ,  $m > 0$  and self–interaction

$$\begin{aligned} H_I &= g \int : V(\phi(\mathbf{x})) : d\mathbf{x}, & H_{I,h} &= g \int h(\mathbf{x}) : V(\phi(\mathbf{x})) : d\mathbf{x}, \\ V(\phi) &:= \int e^{\phi s} d\nu(s), \end{aligned} \tag{1.102}$$

with  $\nu$  being a measure on  $\mathbb{R}$ , supported in  $[-\sqrt{2\pi} + \varepsilon, \sqrt{2\pi} - \varepsilon]$  for  $\varepsilon > 0$  and  $\phi(\mathbf{x})$  as in (1.92).

Høegh–Krohn constructed a renormalized Hamiltonian with spatial cutoff in 1 + 1 dimensions [100] via

$$e^{-t\tilde{H}_h} = \lim_{\Lambda \rightarrow 0} e^{-tH_{\Lambda,h}}, \tag{1.103}$$

in the strong sense, which entails  $H_{\Lambda,h} \rightarrow \tilde{H}_h$  in the strong resolvent sense. No counterterms or dressings are involved.

The spatial cutoff is removed in [100] just as for the  $\phi^4$ –theory: By Segal’s theorem, an automorphism group  $(\alpha_t)_{t \in \mathbb{R}}$  is constructed. Then, the existence of a physical vacuum  $\omega = \lim_{h \rightarrow 1} \omega_h$  is established, whose GNS representation  $\pi_\omega$  allows for writing  $\pi_\omega(\alpha_t(A)) = U(t)^* \pi_\omega(A) U(t)$  where the unitary group  $(U(t))_{t \in \mathbb{R}}$  is generated by a self–adjoint renormalized Hamiltonian  $\tilde{H}$ .

### Yukawa Model

Usually considered in 1 + 1 dimensions, this model describes the interaction of two fermion species, whose creation/annihilation operators are denoted  $b^\dagger, b$  (particles) and  $b'^\dagger, b'$  (antiparticles), with one boson species with creation/annihilation

operators  $a^\dagger, a$ . The dispersion relations are  $\theta(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + M^2}$  (fermions) and  $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m^2}$  (bosons). The boson field operator-valued distribution  $\phi(\mathbf{x})$  is given by the usual expression (1.92), while the fermion field operator-valued distributions are defined as

$$\begin{aligned}\psi_1(\mathbf{x}) &= \int (v(-\mathbf{p})b'^\dagger(\mathbf{p}) + v(\mathbf{p})b(-\mathbf{p})) e^{-i\mathbf{p}\mathbf{x}} \frac{d\mathbf{p}}{\sqrt{4\pi\theta(\mathbf{p})}} \\ \psi_2(\mathbf{x}) &= \int (v(\mathbf{p})b'^\dagger(\mathbf{p}) - v(-\mathbf{p})b(-\mathbf{p})) e^{-i\mathbf{p}\mathbf{x}} \frac{d\mathbf{p}}{\sqrt{4\pi\theta(\mathbf{p})}},\end{aligned}\tag{1.104}$$

with form factor  $v(\mathbf{p}) = \sqrt{\theta(\mathbf{p}) + \mathbf{p}}$ . The formal Hamiltonian is then

$$\begin{aligned}H &= H_0 + H_I \\ H_I &= g \int : \psi_2(\mathbf{x})^* \psi_1(\mathbf{x}) + \psi_1(\mathbf{x})^* \psi_2(\mathbf{x}) : \phi(\mathbf{x}) d\mathbf{x}.\end{aligned}\tag{1.105}$$

As in (1.98), a spatial cutoff  $h : \mathbb{R} \rightarrow [0, 1]$  can be employed by putting a factor  $h(\mathbf{x})$  inside the integral in  $H_I$ , which yields a formal Hamiltonian with spatial cutoff  $H_h$ .

Glimm constructed a corresponding renormalized Hamiltonian  $\tilde{H}_h$  (called  $H_{\text{ren}}$  in his work) as a limit of quadratic forms [101]:

$$\langle \Psi, T_{\sigma,h}^* \tilde{H}_h T_{\sigma,h} \Psi \rangle = \lim_{\Lambda \rightarrow \infty} \langle \Psi, T_{\sigma,\Lambda,h}^* (H_{\Lambda,h} - c_{\Lambda,h}) T_{\sigma,\Lambda,h} \Psi \rangle,\tag{1.106}$$

with  $T_{\sigma,h}, T_{\sigma,\Lambda,h}$  being non-unitary dressing transformations and where  $\Psi$  is chosen from a form domain that is dense in  $\mathcal{H}$ . The counterterm  $c_{\Lambda,h}$  consists of a self-energy  $E_{\Lambda,h}$  and a mass renormalization term  $\delta m_{\Lambda,h}$  containing operator products of the type  $a^\dagger a^\dagger, aa$  and  $a^\dagger a$ . Self-adjointness of  $\tilde{H}_h$  was established by Glimm using a Friedrichs extension in [102]. Another proof of self-adjointness by norm resolvent convergence of  $(H_{\Lambda,h} - c_{\Lambda,h})$  together with independence from the cutoff function  $h$  was given in [103] and an automorphism group  $(\alpha_t)_{t \in \mathbb{R}}$  was constructed by Segal's theorem in [104]. The existence of a renormalized  $\tilde{H}$  generating dynamics corresponding to  $(\alpha_t)_{t \in \mathbb{R}}$  on a Hilbert space has been established by Schrader [105].

### Massless Thirring Model

Introduced by Thirring as early as 1958 [106], this 1 + 1-dimensional model is a prominent example of an integrable QFT model, meaning that the Heisenberg equations of motion can be solved explicitly. It concerns two fermion species: a particle with operators  $b^\dagger, b$  and its antiparticle with operators  $b'^\dagger, b'$ . The corre-

## 1 Introduction

sponding field operator-valued distributions are [34, Chap. IV]

$$\begin{aligned}\psi_1(\mathbf{x}) &= \int (\chi_{[0,\infty)}(\mathbf{p})b(\mathbf{p}) + \chi_{[0,\infty)}(-\mathbf{p})b'^{\dagger}(-\mathbf{p}))e^{i\mathbf{p}\mathbf{x}}\frac{d\mathbf{p}}{\sqrt{2\pi}} \\ \psi_2(\mathbf{x}) &= \int (\chi_{[0,\infty)}(-\mathbf{p})b(\mathbf{p}) + \chi_{[0,\infty)}(\mathbf{p})b'^{\dagger}(-\mathbf{p}))e^{i\mathbf{p}\mathbf{x}}\frac{d\mathbf{p}}{\sqrt{2\pi}}.\end{aligned}\tag{1.107}$$

The kinetic term is then

$$H_0 = -i \int (\psi_1(\mathbf{x})^* \partial_{\mathbf{x}} \psi_1(\mathbf{x}) - \psi_2(\mathbf{x})^* \partial_{\mathbf{x}} \psi_2(\mathbf{x})) d\mathbf{x},\tag{1.108}$$

which corresponds to a dispersion relation of  $\theta_1(\mathbf{p}) = \mathbf{p}$  for  $\psi_1$  and  $\theta_2(\mathbf{p}) = -\mathbf{p}$  for  $\psi_2$ . The interaction in its most general form is given by

$$H_I = g \int \psi_1(\mathbf{x}_1)^* \psi_1(\mathbf{x}_1) f(\mathbf{x}_1 - \mathbf{x}_2) \psi_2(\mathbf{x}_2)^* \psi_2(\mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2,\tag{1.109}$$

where for a local interaction, the form factor has to be chosen as  $f(\mathbf{x}_1 - \mathbf{x}_2) = \delta(\mathbf{x}_1 - \mathbf{x}_2)$ . The operator-valued distributions  $\psi_1$  and  $\psi_2$  can be interpreted as annihilating two new species of particles ( $\psi_1$ -fermions and  $\psi_2$ -fermions), for which the Heisenberg equations for  $H = H_0 + H_I$  can be solved explicitly [34, (9.7)]. The transition from  $\psi_1, \psi_2$  to  $b, b'$  via (1.107) is a Bogoliubov transformation like  $a \mapsto b$  in (1.76), but there does not exist a unitary implementer  $\mathbb{U}_{\mathcal{V}}$ <sup>18</sup>. However, performing a Bogoliubov transformation by operator replacement, i.e., plugging (1.107) into (1.108) and (1.109) renders a formal expression of the kind

$$H = \tilde{H} - E_{\infty},\tag{1.110}$$

where  $E_{\infty}$  is a formally infinite constant (self-energy) and  $\tilde{H}$  is indeed a self-adjoint Hamiltonian of the form  $\tilde{H} = \tilde{H}_0 + \tilde{H}_I$ , where the dispersion relation is  $\theta_b(\mathbf{p}) = \theta_{b'}(\mathbf{p}) = |\mathbf{p}|$  and  $\tilde{H}_I$  is a quartic interaction term [34, (9.21)].

One may construct even more formal Heisenberg fields by a bosonization approach, out of which some are nonlocal and others do not solve the Heisenberg equations on rigorous grounds, compare [107, Chap. 5].

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<sup>18</sup>Heuristically speaking, the Bogoliubov transformation is “too large” or leads “outside Fock space”. Mathematically, it violates the Shale–Stinespring condition. This is the case addressed in Chapter 5.

### Massive Thirring Model

It is also possible to assign a positive mass to the free dispersion relation of fermions in the 1 + 1 dimensional Thirring model. The corresponding QFT has been constructed by Benfatto, Falco and Mastropietro by a functional integral approach in Euclidean field theory [108]. More precisely, they constructed the Schwinger functions  $(\mathfrak{S}_{\sigma,\Lambda,n})_{n \in \mathbb{N}}$  for suitable cutoffs  $\sigma, \Lambda$ , proved that the limit Schwinger functions  $(\mathfrak{S}_n)_{n \in \mathbb{N}}$  exist as  $\sigma \rightarrow 0, \Lambda \rightarrow \infty$  and verified the Osterwalder–Schrader axioms.

### Federbush Model

While being closely related to the Thirring model, the Federbush model in 1 + 1 dimensions [109] is a further exactly integrable model with two pairs of fermion species  $\psi_{1,+}, \psi_{2,+}$  and  $\psi_{1,-}, \psi_{2,-}$  described each as in (1.107). It is convenient to conclude each pair into a vector with two entries

$$\psi_{\pm} = \begin{pmatrix} \psi_{1,\pm} \\ \psi_{2,\pm} \end{pmatrix}, \quad \overline{\psi_{\pm}} = \gamma^0 \psi_{\pm}^*, \quad (1.111)$$

with 1 + 1-dimensional Dirac matrix  $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . The formal free Hamiltonian can then conveniently be expressed as

$$H_0 = \sum_{s \in \{+, -\}} \int \overline{\psi_s(\mathbf{x})} (-i\gamma^5 \partial_{\mathbf{x}} + \gamma^0 m_s) \psi_s(\mathbf{x}) d\mathbf{x}, \quad (1.112)$$

with  $\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and fermion masses  $m_+, m_- > 0$ , so the particles have dispersion relation  $\theta_{\pm}(\mathbf{p}) = \gamma^5 \mathbf{p} + \gamma^0 m_{\pm}$ . The quartic interaction term reads

$$H_I = g \int \left( \overline{\psi_+(\mathbf{x})} \gamma^1 \psi_+(\mathbf{x}) \overline{\psi_-(\mathbf{x})} \gamma^0 \psi_-(\mathbf{x}) - \overline{\psi_+(\mathbf{x})} \gamma^0 \psi_+(\mathbf{x}) \overline{\psi_-(\mathbf{x})} \gamma^1 \psi_-(\mathbf{x}) \right) d\mathbf{x}. \quad (1.113)$$

Ruijsenaars applied a process called **bosonization** to this model. That is, the fermionic  $\Psi_{\pm}$  are used to construct formal (bosonic) Heisenberg operator-valued distributions  $\phi(\mathbf{x})$  [107, Chap. 4]. As for the Thirring model, some of them do not solve the Heisenberg equations on rigorous grounds or are nonlocal. However, for one specific choice of  $\phi(\mathbf{x})$  involving formal Bogoliubov implementers, Ruijsenaars could establish all Wightman axioms in the low-coupling regime [110]. So in a physical sense, the bosons described by  $\phi(\mathbf{x})$  are the “true” particles and the fermions described by  $\Psi_{\pm}$  are just a mathematical tool leading to a particularly nice and intuitive form of the Hamiltonian.

## Sine–Gordon Model

This model has also been considered in  $1 + 1$  dimensions and describes a self-interacting boson field with dispersion relation  $\omega(\mathbf{k}) = \sqrt{|\mathbf{k}|^2 + m^2}$ . Using the operator-valued distribution  $\phi(\mathbf{x})$  in (1.92), the interaction can be written as

$$H_I = g \int : \cos(\alpha\phi(\mathbf{x})) : d\mathbf{x}, \quad (1.114)$$

with  $\alpha \in \mathbb{R}$ . For  $m = 0$  and  $\alpha$  small enough, the Sine–Gordon model can be reduced to the massive Thirring model. This works by adding a divergent energy constant  $E_{\sigma,\Lambda}$  to the Thirring Hamiltonian and identifying bosonic operator expressions constructed out of  $\phi(\mathbf{x})$  from the Sine–Gordon Hamiltonian with bosonic operator expressions constructed out of the fermionic  $\Psi_1, \Psi_2$  from the Thirring Hamiltonian [111, (1.4)]. So the transition from the Sine–Gordon to the Thirring model can be seen as a kind of **fermionization**, while the converse transition is again a bosonization.

An equivalence of the resulting perturbation series has first been established by Coleman in the low-coupling regime [112]. A proof of non-perturbative equivalence has been given by Benfatto, Falco and Mastropietro in [111] via functional integral methods in Euclidean field theory.

## 1.4 Direct renormalization and Interior–Boundary Conditions (IBC)

In the recent years, a renormalization technique, that works entirely without cutoffs or limiting procedures, has gained attention. The technique starts from a formal Hamiltonian  $H = H_0 + A^\dagger + A$ , with  $A^\dagger$  containing ill-defined particle creation expressions. A modification of the formal  $H$  without using cutoffs then results in a renormalized Hamiltonian  $H_{IBC}$ , which is free from the ill-defined terms in  $A^\dagger$ , and which describes particle creation by a specific constraint equation included in the definition of its domain  $\text{dom}(H_{IBC})$ . In earlier considered models, this constraint equation relates interior points of the configuration space (for instance the non-collision configurations  $\mathring{\mathcal{Q}} := \mathcal{Q} \setminus \mathcal{Q}_{\text{col}}$  from (1.11)) to boundary points (for instance the collision configurations  $\partial\mathcal{Q} = \mathcal{Q}_{\text{col}}$ , at which a particle is created). Therefore, both the constraint equations and the renormalization technique have been dubbed **interior–boundary conditions (IBCs)**.

Later works, presented in Section 1.4.3, establish a self-adjoint  $H_{IBC}$  using abstract generalizations of IBCs, which are sometimes also denoted as “IBCs” and

sometimes called **abstract boundary conditions**. For a thorough introduction into the IBC technique, we refer the reader to [113, 114, 115, 116].

The works about IBCs have considerably inspired the Fock space extension approach described in Chapter 3. Both share the spirit of removing divergences without using cutoffs. The following Sections 1.4.1–1.4.3 provide a short outline of the construction principle for self–adjoint Hamiltonians via IBCs or abstract boundary conditions and point out difficulties, which the Fock space extension framework is designed to overcome.

### 1.4.1 Boundary Conditions and Self–Adjointness

The description of particle creation and annihilation by boundary condition relies on the well–known fact that for an open set  $\Omega \subset \mathbb{R}^d$  with boundary  $\partial\Omega$ , the Schrödinger equation on  $L^2(\Omega)$  has no unique solution. Some of these solutions do not conserve the  $L^2$ –norm, but instead describe wave packets leaving or entering the boundary.

A simple example illustrating this effect is given by [22, X.1 Example 1] (and in a similar form in [27, (2.29)]): Consider  $\Omega = (0, 1) \subset \mathbb{R}$  and the Schrödinger equation

$$i\partial_t\Psi_t(\mathbf{x}) = i\partial_{\mathbf{x}}\Psi_t(\mathbf{x}). \quad (1.115)$$

If (1.115) was posed on  $L^2(\mathbb{R})$ , then  $H = i\partial_{\mathbf{x}} = -\mathbf{p}$  would be a self–adjoint Hamiltonian whose domain is the first Sobolev space  $\text{dom}(H) = H^1(\mathbb{R})$ . The evolution would be given by  $U(t) = e^{ipt}$ , so for any initial data  $\Psi_0 \in L^2(\mathbb{R})$  we have  $\Psi_t = U(t)\Psi_0$  with<sup>19</sup>  $\Psi_t(\mathbf{x}) = \Psi_0(\mathbf{x} + t)$ . Therefore (1.115) describes the motion of a wavepacket to the left with velocity 1, while sustaining its shape. This situation is illustrated in Figure 1.7.

Now as (1.115) is restricted to  $\mathbf{x} \in (0, 1)$ , a wave packet  $\Psi \in L^2((0, 1))$  will also run to the left within the interval  $(0, 1)$ . Therefore, it eventually leaves the interval at  $\mathbf{x} = 0$ , which may lead to a decrease of  $\|\Psi_t\|$  in time. Heuristically, “probability will flow out of the left boundary of  $(0, 1)$ ”. On the other hand, an arbitrarily shaped wavepacket may enter the interval at  $\mathbf{x} = 1$  at any time, which might lead to an increase of  $\|\Psi_t\|$ . So “probability may flow into the right boundary of  $(0, 1)$ ”. The time evolution operator  $U(t)$  can now only be unitary, if it conserves the norm  $\|\Psi\|$ , so “the probabilities flowing in and out of the boundaries balance each other out”. In our example, probability balance formally amounts to the leaving probability density  $|\Psi(0)|^2$  and the entering probability density  $|\Psi(1)|^2$  being equally large,

<sup>19</sup>This can easily be checked by a Fourier decomposition:  $\Psi_0(\mathbf{x}) = (2\pi)^{-1/2} \int \hat{\Psi}_0(\mathbf{p})e^{i\mathbf{p}\mathbf{x}} d\mathbf{p}$ , so  $\Psi_t(\mathbf{x}) = e^{i\mathbf{p}t}\Psi_0(\mathbf{x}) = (2\pi)^{-1/2} \int \hat{\Psi}_0(\mathbf{p})e^{i\mathbf{p}(\mathbf{x}+t)} d\mathbf{p} = \Psi_0(\mathbf{x} + t)$ .

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which is realized by imposing the boundary condition

$$\Psi(0) = \alpha\Psi(1), \quad \alpha \in \mathbb{C}, \quad |\alpha| = 1, \quad (1.116)$$

see also Figure 1.7. It can indeed be proved that the PDE (1.115) together with boundary condition (1.116) then has a unique probability–conserving solution  $(\Psi_t)_{t \in \mathbb{R}}$ , whereas the PDE (1.115) alone allows for many solutions which do not necessarily conserve probability.

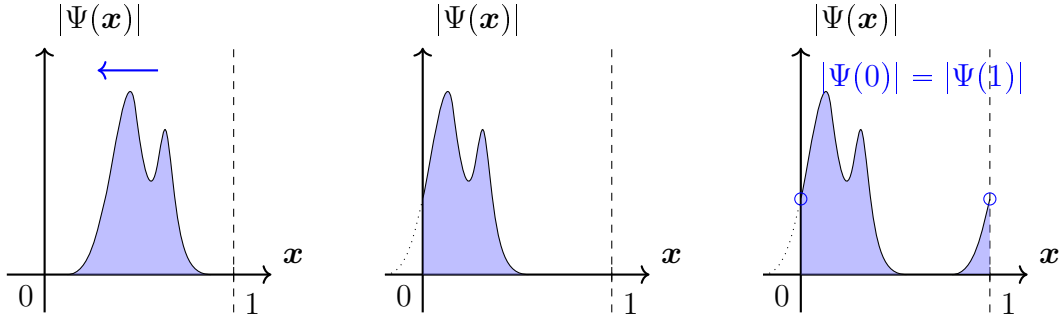


Figure 1.7: Left: A wave packet at  $t = 0$ .

Middle: The solution of (1.115) may leave the boundary.

Right: Boundary condition (1.116) ensures probability conservation.

Color online.

From a functional analytic point of view, the situation in the above example is the following: Each choice of a boundary condition (1.116) for  $|\alpha| = 1$  together with the PDE (1.115) corresponds to a distinct self–adjoint Hamiltonian  $H_\alpha$  with a distinct domain<sup>20</sup>

$$\text{dom}(H_\alpha) = \{\Psi \in H^1((0, 1)) \mid \Psi(0) = \alpha\Psi(1)\}. \quad (1.117)$$

If the boundary condition is a priori unknown, then the action of the Hamiltonian is only known on functions whose support has a sufficient distance to the boundary  $\Psi \in \mathcal{D}_0 = C_c^\infty((0, 1))$ , i.e., we a priori have an operator  $\mathring{H} = -\mathbf{p} : \mathcal{D}_0 \rightarrow L^2((0, 1))$ . This  $\mathring{H}$  is symmetric, but not self–adjoint. It allows for several self–adjoint extensions  $H_\alpha$  and is hence compatible with many different unitary dynamics  $(U_\alpha(t))_{t \in \mathbb{R}}$ ,  $U_\alpha(t) = e^{-itH_\alpha}$  on  $L^2((0, 1))$ . Also note that although  $\text{dom}(H_\alpha)$  in (1.117) is not

<sup>20</sup>The expressions  $\Psi(0), \Psi(1) \in \mathbb{C}$  are strictly speaking not function evaluations, but rather applications of the linear functionals  $\delta(\cdot), \delta(\cdot - 1)$  to  $\Psi$ . Both are generally ill–defined on  $\Psi \in L^2((0, 1))$ . However, they are well–defined on  $H^1((0, 1))$  by the Sobolev embedding theorem [117, Thm. 5.4, (8)], which implies that  $\Psi \in H^1((0, 1))$  is continuous. So  $\Psi(0) = \lim_{\varepsilon \rightarrow 0} \Psi(\varepsilon)$  and  $\Psi(1) = \lim_{\varepsilon \rightarrow 0} \Psi(1 - \varepsilon)$  exist as complex numbers.

### 1.4 Direct renormalization and Interior–Boundary Conditions (IBC)

the entire Hilbert space  $\mathcal{H}$ , the operator  $H_\alpha$  nevertheless defines dynamics on all of  $\mathcal{H}$ .

There are even non–symmetric extensions  $H_{\text{non}}$  possible, e.g., by only imposing the boundary condition  $\Psi(1) = 0$ , which allows “probability to leave but not to enter  $(0, 1)$ ”. A family of evolution operators  $(W(t))_{t \in [0, \infty)}$  can then be defined via

$$\Psi_t(\mathbf{x}) = \begin{cases} \Psi_0(t + \mathbf{x}) & \text{if } t + \mathbf{x} < 1 \\ 0 & \text{else} \end{cases} . \quad (1.118)$$

$(W(t))_{t \in [0, \infty)}$  is not a unitary group, but only a contraction semigroup, as  $\|\Psi\|$  may decrease in time and dynamics cannot be defined for  $t < 0$ . That is because, heuristically speaking, for backward running time, arbitrary wavepackets may enter at  $\mathbf{x} = 0$ . This “violation of probability conservation” under  $(W(t))_{t \in [0, \infty)}$  corresponds to the generator  $H_{\text{non}}$  of the semigroup not being symmetric on its domain

$$\text{dom}(H_{\text{non}}) = \{\Psi \in H^1((0, 1)) \mid \Psi(1) = 0\}, \quad (1.119)$$

although it might look symmetric at first glance. In general, only writing down a symmetrically–looking expression  $\mathring{H}$  (like  $\mathring{H} = -\mathbf{p}$ ) on a set with boundary  $\Omega \subset \mathbb{R}^d$  does not yet specify a self–adjoint Hamiltonian, but leaves an option for several extensions  $H$ , out of which some may even be non–symmetric. Only  $\mathring{H}$  together with a boundary condition determines a self–adjoint Hamiltonian  $H$ , where the boundary condition heuristically prescribes how probabilities “flowing in and out” at  $\partial\Omega$  are balanced.

#### 1.4.2 A Simple Example for an IBC

In order to model particle creation and annihilation by means of boundary conditions, one may choose  $\Omega$  as a subset of configuration space  $\mathcal{Q}$ , such that the configurations in  $\partial\Omega \subset \mathcal{Q}$  are associated with a particle just having been created. For instance, if a particle of one species may emit or absorb further particles of the same species at its current position, then one reasonable choice is given by taking the collision configurations  $\partial\Omega = \mathcal{Q}_{\text{col}}$  (compare (1.11)) and  $\Omega = \mathring{\mathcal{Q}} = \mathcal{Q} \setminus \mathcal{Q}_{\text{col}}$ . A creation process may then relate an interior configuration  $q = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathring{\mathcal{Q}}$  to the boundary configuration  $q' = (q, \mathbf{x}_j) \in \mathcal{Q}_{\text{col}}, j \in \{1, \dots, N\}$ , where a particle has been created at the position  $\mathbf{x}_j$ .

Another example is given by a single point source at  $\{0\}$ , which emits and absorbs

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particles. In that case,

$$\Omega = \mathcal{Q} \setminus \partial\Omega, \quad \partial\Omega = \{q = (\mathbf{x}_1, \dots, \mathbf{x}_N) \mid \mathbf{x}_j = 0 \text{ for some } j \in \{1, \dots, N\}\}. \quad (1.120)$$

So a particle creation at  $\mathbf{x} = 0$  relates interior configurations  $q \in \Omega$  to boundary configurations  $q' = (q, 0) \in \partial\Omega$ . In that context, it is also possible to restrict  $\mathcal{Q}$  to a subset of sectors, for instance, to  $\bigsqcup_{N=0}^{N_{\max}} \mathcal{Q}^{(N)}$ .

An IBC is now a constraint equation on  $\Psi \in L^2(\Omega)$ , which establishes probability balance between interior points  $q \in \Omega$  and corresponding boundary points  $q' \in \partial\Omega$  at a higher sector in configuration space.

As a simple example, consider the restriction of the configuration space to the vacuum- and the one-particle sector  $\Omega = \{\emptyset\} \sqcup \mathbb{R}^3 \setminus \{0\}$ , so  $\partial\Omega = \{0\}$  and  $\mathcal{H} = L^2(\Omega) = \mathbb{C} \oplus L^2(\mathbb{R}^3)$ . The boson dispersion relation is chosen as  $\omega(\mathbf{k}) = |\mathbf{k}|^2$  and the form factor is chosen proportional to a Dirac  $\delta$ -distribution in position space  $\check{v}(\mathbf{x}) = g\delta(\mathbf{x})$  with  $g$  being the coupling constant. This model has been considered for IBCs in that or a generalized form in [118, 119, 120, 121, 122, 114, 116] and can also be seen as a specific case of the spin-boson model presented in Section 1.3.7. The formal Hamiltonian reads:

$$\begin{aligned} H &= H_0 + g(a^\dagger(\delta) + a(\delta)) \\ H_0 &= 0 \oplus -\Delta \\ (a^\dagger(\delta)\Psi)^{(0)} &= 0, & (a^\dagger(\delta)\Psi)^{(1)}(\mathbf{x}) &= \delta(\mathbf{x})\Psi^{(0)} \\ (a(\delta)\Psi)^{(0)} &= \Psi^{(1)}(0), & (a(\delta)\Psi)^{(1)}(\mathbf{x}) &= 0. \end{aligned} \quad (1.121)$$

Obviously,  $a^\dagger(\delta)$  is ill-defined on  $\Psi \in \mathcal{H}$  with  $\Psi^{(0)} \neq 0$ , since  $\delta \notin L^2$ .

For a moment, let's drop the creation term and consider

$$\mathring{H} = H_0 + ga(\delta). \quad (1.122)$$

Although  $\mathring{H}$  does not look symmetric, it is indeed a symmetric operator if considered on  $\mathcal{D}_0 = \mathbb{C} \oplus C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ , since the annihilation term amounts to  $ga(\delta)\Psi = 0$  for  $\Psi \in \mathcal{D}_0$ . Alternatively, one may write  $\mathring{H} = H_0|_{\mathcal{D}_0}$ , which is known to allow for many self-adjoint extensions [123, Sect. I.1]. The aim is now to extend  $\mathring{H}$  to a self-adjoint operator, where symmetry is enforced by a constraint equation similar to (1.116). This is done by finding expressions for the probability rates that “leave” and “enter”  $\mathbb{R}^3 \setminus \{0\}$  at 0 and setting both equal. The probability current vector

### 1.4 Direct renormalization and Interior–Boundary Conditions (IBC)

field associated with  $-\Delta$  is given by

$$j_i(\mathbf{x}) = 2\text{Im} \left( \overline{\Psi^{(1)}(\mathbf{x})} \partial_{\mathbf{x}_i} \Psi^{(1)}(\mathbf{x}) \right), i \in \{1, 2, 3\}. \quad (1.123)$$

The rate at which probability leaves  $\mathbb{R}^3 \setminus \{0\}$  can now be expressed by considering the current flow through a scaled sphere  $rS^2 = \partial B_r(0)$ ,  $r \in [0, \infty)$  and shrinking  $r \rightarrow 0$ . For the integration over the sphere, it is convenient to use spherical coordinates, expressing  $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$  by  $(r, \omega) \in [0, \infty) \times S^2$ . The rate of probability leaving  $\mathbb{R}^3 \setminus \{0\}$  is then

$$\dot{\rho}_{\text{out}} = 2 \lim_{r \rightarrow 0} \int_{S^2} \text{Im} \left( \overline{\Psi^{(1)}(r, \omega)} \partial_r \Psi^{(1)}(r, \omega) \right) r^2 d\omega. \quad (1.124)$$

The ingoing current formally reads

$$\begin{aligned} \dot{\rho}_{\text{in}} &= -\partial_t |\Psi_t^{(0)}|^2 = -2\text{Re} \left( \overline{\Psi^{(0)}} \partial_t \Psi^{(0)} \right) = -2g \text{Im} \left( \overline{\Psi^{(0)}} (a(\delta)\Psi)^{(0)} \right) \\ &= -2g \text{Im} \left( \overline{\Psi^{(0)}} \Psi^{(1)}(0) \right) \\ &= -2 \frac{g}{4\pi} \lim_{r \rightarrow 0} \int_{S^2} \text{Im} \left( \overline{\Psi^{(0)}} \Psi^{(1)}(\omega, r) \right) d\omega. \end{aligned} \quad (1.125)$$

Probability balance  $\dot{\rho}_{\text{out}} = \dot{\rho}_{\text{in}}$  is now established by a constraint equation relating  $\Psi^{(0)}$  to  $\Psi^{(1)}$ , which is exactly the **interior–boundary condition**. Following [114, (27)] and [116, (17), (22)], one may choose as an IBC

$$\Psi^{(0)} = -\frac{4\pi}{g} \lim_{r \rightarrow 0} r \Psi^{(1)}(\omega, r). \quad (1.126)$$

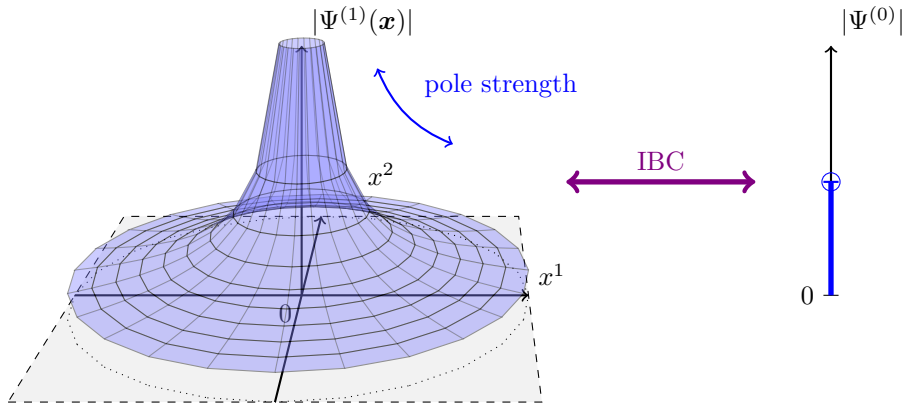


Figure 1.8: The IBC (1.126) relates the pole strength of  $\Psi^{(1)}$  to  $\Psi^{(0)}$ . Color online.

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That means,  $\Psi^{(1)}$  is forced to have a pole of the form  $-\frac{g\Psi^{(0)}}{4\pi|\mathbf{x}|}$  near  $\mathbf{x} = 0$ , see Figure 1.8. Since for  $\Psi^{(0)}$ , the value  $\Psi^{(1)}(0)$  is formally infinite, the annihilation operator  $a(\delta)$  does no longer make sense on all  $\Psi$  satisfying (1.126) and requires an **ad hoc modification** [116, (15)]: One replaces  $a(\delta)$  by  $A(\delta)$  given by

$$(A(\delta)\Psi)^{(0)} = \frac{1}{4\pi} \lim_{r \rightarrow 0} \partial_r \int_{S^2} r \Psi^{(1)}(\omega, r) d\omega, \quad (A(\delta)\Psi)^{(1)}(\mathbf{x}) = 0. \quad (1.127)$$

So the annihilation only acts on the regular part of  $\Psi^{(1)}$  and no longer on the pole  $\propto |\mathbf{x}|^{-1}$ . This replacement corresponds to the subtraction of a formal **infinite self-energy** from the Hamiltonian

$$E_\infty = -g^2 a(\delta) a^\dagger \left( \frac{1}{4\pi|\mathbf{x}|} \right). \quad (1.128)$$

The renormalized Hamiltonian then reads

$$\tilde{H} = H_{\text{IBC}} = H_0 + gA(\delta). \quad (1.129)$$

Although this expression does not look symmetric, it becomes a symmetric and self-adjoint operator, if considered on the IBC-domain

$$\text{dom}(H_{\text{IBC}}) = \left\{ \Psi \in \mathcal{H} \mid \left( 1 + ga^\dagger \left( \frac{1}{4\pi|\mathbf{x}|} \right) \right) \Psi \in \text{dom}(H_0) \right\}. \quad (1.130)$$

So the Hamiltonian (1.129) together with the IBC (1.126) provides a rigorous implementation of the formal expression

$$H = H_0 + g(a^\dagger(\delta) + a(\delta)) - E_\infty. \quad (1.131)$$

### 1.4.3 Abstract Boundary Conditions and Difficulties

The IBC method above can also be cast in a more abstract language as done in [116]. This allows for applications with more sophisticated  $H$ , as those presented in Section 1.4.5. Consider a formal Hamiltonian

$$H = H_0 + A^\dagger + A - E_\infty, \quad (1.132)$$

where  $H_0$  describes freely moving particles,  $A$  contains some annihilation terms, its formal adjoint  $A^\dagger$  contains creation terms (and is possibly ill-defined) and  $E_\infty$  is an infinite formal self-energy. We re-arrange the terms as follows (see also Section

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4.8.1):

$$\begin{aligned} H &= H_0 + A^\dagger + A + AH_0^{-1}A^\dagger - AH_0^{-1}A^\dagger - E_\infty \\ &= (1 + H_0^{-1}A^\dagger)^* H_0 (1 + H_0^{-1}A^\dagger) - (AH_0^{-1}A^\dagger + E_\infty) \\ &=: S^*S + T, \end{aligned} \tag{1.133}$$

with  $S = H_0^{1/2}(1 + H_0^{-1}A^\dagger)$  and  $T = -(AH_0^{-1}A^\dagger + E_\infty)$ . Under certain conditions on  $H_0$  and  $A^\dagger$ , the operator  $S^*S$  is self-adjoint on its domain

$$\text{dom}(H_{\text{IBC}}) = \{\Psi \in \mathcal{H} \mid (1 + H_0^{-1}A^\dagger)\Psi \in \text{dom}(H_0)\}. \tag{1.134}$$

Now if one can prove that  $T$  is a Kato–perturbation of  $S^*S$ , i.e.,

$$\|T\Psi\| \leq a\|S^*S\Psi\| + b\|\Psi\| \quad \forall \Psi \in \text{dom}(H_{\text{IBC}}), \tag{1.135}$$

with  $a \in [0, 1)$ ,  $b \in [0, \infty)$ , then the Kato–Rellich theorem [22, X.12] implies that

$$H_{\text{IBC}} = S^*S + T \tag{1.136}$$

is self-adjoint on  $\text{dom}(H_{\text{IBC}})$ .

Most of the examples in Section 1.4.5 apply to polaron models: For a system with  $N$  interacting fermions,  $A^\dagger$  and  $A$  make each fermion create or annihilate a boson. So  $AH_0^{-1}A^\dagger$  consists of  $N^2$  terms describing the interaction of fermion  $i$  with  $j$  by boson exchange, where  $i, j \in \{1, \dots, N\}$ . Terms with  $i \neq j$  (also called **off-diagonal terms**) amount to a pair potential interaction, where Kato–bounds with respect to a kinetic term  $H_0$  are well-known. Terms with  $i = j$  (also called **diagonal terms**) describe an infinite self-interaction, which is removed by the counterterm  $E_\infty$ .

One may now also imagine to include infinite **mass or charge counterterms** as in Section 1.3.3 into the Hamiltonian and reshuffle terms, in order to arrive at a self-adjoint  $H_{\text{IBC}}$ , although we are not aware of such an inclusion in the literature. It would be useful to have a formalism which puts term reshuffelings as in (1.133) on rigorous grounds, which was one of the central motivations for designing the Fock space extension framework presented in Section 3.2.

Further, the above formalism requires that  $\text{dom}(H_{\text{IBC}}) \subseteq \mathcal{H}$ , so within the formal manipulations, no dressing transformations leading out of Fock space are allowed, although these are expected in a cutoff-free setting for certain models. See, for instance [43, 61, 71, 80]. This is also a serious problem if one thinks of IBCs as a tool being involved in the construction of relativistic QFTs, since Haag’s theorem holds here [9], [10, Sect. II.1]. Thus, no renormalized Hamiltonian can be defined in the vacuum representation and a change of the representation is necessary,

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which corresponds to conjugating the formal  $H$  with a dressing transformation leading out of Fock space. One possible way out might be to perform a different ad hoc re-definition of creation or annihilation operators, e.g., a Weyl transformation  $a^\dagger(f) \mapsto a^\dagger(f) + \langle s, f \rangle$ , which can be well-defined for  $s \notin L^2$  even if the formal implementer  $W(s)$  is not defined on Fock space. However, the IBC method would then only be applicable after an appropriate operator replacement, which may be difficult to find and can involve handling infinite quantities. Without a suitable replacement, the IBC equation  $(1 + H_0^{-1}A^\dagger)\Psi \in \text{dom}(H_0)$  will formally produce vectors  $\Psi$  outside of Fock space.

A simple example illustrating this problem is the Van Hove model (see also Section 1.3.7), where we specifically consider a single point source at  $\mathbf{x} = 0$  emitting and absorbing bosons. The formal Hamiltonian, using the notation of Section 1.3.1, is

$$H = d\Gamma(\omega) + a^\dagger(v) + a(v), \quad (1.137)$$

where we choose  $\omega(\mathbf{k}) = |\mathbf{k}|$  and  $v(\mathbf{k}) = g|\mathbf{k}|^{-1/2}$ , which are the expected scalings of QED. It is well-known [63, 15], that this Hamiltonian can be given a rigorous meaning by performing an algebraic Weyl transformation, introducing the operators and operator-valued distributions

$$\begin{aligned} \tilde{a}^\dagger(f) &= a^\dagger(f) - \langle s, f \rangle, & \tilde{a}(f) &= a(f) - \langle f, s \rangle, \\ \tilde{a}^\dagger(\mathbf{k}) &= a^\dagger(\mathbf{k}) - \overline{s(\mathbf{k})}, & \tilde{a}(\mathbf{k}) &= a(\mathbf{k}) - s(\mathbf{k}) \\ \text{with } s(\mathbf{k}) &= -\frac{v(\mathbf{k})}{\omega(\mathbf{k})} = -g|\mathbf{k}|^{-3/2}, \end{aligned} \quad (1.138)$$

and subtracting the formal counterterm

$$E_\infty = \int \overline{v(\mathbf{k})}s(\mathbf{k}) d\mathbf{k} = - \int \frac{\overline{v(\mathbf{k})}v(\mathbf{k})}{\omega(\mathbf{k})} d\mathbf{k}. \quad (1.139)$$

In that case, the formal Hamiltonian amounts to

$$\begin{aligned} H - E_\infty &= \int \left( \omega(\mathbf{k})a^\dagger(\mathbf{k})a(\mathbf{k}) + v(\mathbf{k})a^\dagger(\mathbf{k}) + \overline{v(\mathbf{k})}a(\mathbf{k}) + \frac{\overline{v(\mathbf{k})}v(\mathbf{k})}{\omega(\mathbf{k})} \right) d\mathbf{k} \\ &= \int \omega(\mathbf{k}) \left( a^\dagger(\mathbf{k}) + \frac{\overline{v(\mathbf{k})}}{\omega(\mathbf{k})} \right) \left( a(\mathbf{k}) + \frac{v(\mathbf{k})}{\omega(\mathbf{k})} \right) d\mathbf{k} \\ &= \int \omega(\mathbf{k})\tilde{a}^\dagger(\mathbf{k})\tilde{a}(\mathbf{k}) d\mathbf{k}. \end{aligned} \quad (1.140)$$

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So a Weyl transformation  $\tilde{a}(\mathbf{k}) \mapsto \tilde{a}(\mathbf{k}) + s(\mathbf{k})$  results in the replacement  $\tilde{a} \mapsto a$  and hence in a renormalized Hamiltonian

$$\tilde{H} = d\Gamma(\omega) = H_0, \quad (1.141)$$

whose domain is well-known to be the first Sobolev space. However, the Weyl transformation does not have a unitary implementer  $W(s)$ , as defined by (1.72) and (1.73): The formal integral  $\int |s(\mathbf{k})|^2 d\mathbf{k}$  diverges both in the UV- and in the IR-regime, so  $s \notin L^2$ .

However, the formal operator  $(H - E_\infty)$  does not have a domain in Fock space. Formally, its domain would be given by  $W(s)[\text{dom}(H_0)] \subset W(s)[\mathcal{H}]$ , which can be given a rigorous meaning as a subspace of von Neumann’s ITP space  $\widehat{\mathcal{H}}$ , as in [71, 80]. See also Section 3.1.

Likewise, the formal Hamiltonian (1.137) results in a formal IBC domain (1.134) that contains vectors outside Fock space: The formal IBC reads

$$(1 + H_0^{-1}a^\dagger(v))\Psi \in \text{dom}(H_0), \quad (1.142)$$

which would, for instance, be satisfied by the “dressed vacuum” state<sup>21</sup>

$$\Omega_{\text{IBC}} = \sum_{N \in \mathbb{N}_0} \frac{(a^\dagger(s))^N}{N!} \Omega \quad \Rightarrow \quad \Omega_{\text{IBC}}^{(N)}(\mathbf{k}_1, \dots, \mathbf{k}_N) = \frac{1}{\sqrt{N!}} \prod_{\ell=1}^N s(\mathbf{k}_\ell). \quad (1.143)$$

Indeed, using  $s = -\frac{v}{\omega}$ , we have at each sector  $N \geq 1$ :

$$\begin{aligned} (H_0^{-1}a^\dagger(v)\Omega_{\text{IBC}})(\mathbf{k}_1, \dots, \mathbf{k}_N) &= \frac{1}{\sqrt{N!}} \frac{1}{\sum_{j=1}^N \omega(\mathbf{k}_j)} \sum_{j'=1}^N v(\mathbf{k}_{j'}) \prod_{\ell \neq j'} s(\mathbf{k}_\ell) \\ &= \frac{1}{\sqrt{N!}} \frac{1}{\sum_{j=1}^N \omega(\mathbf{k}_j)} \sum_{j'=1}^N (-\omega(\mathbf{k}_{j'})) \prod_{\ell=1}^N s(\mathbf{k}_\ell) \\ &= -\frac{1}{\sqrt{N!}} \prod_{\ell=1}^N s(\mathbf{k}_\ell) \\ &= -\Omega_{\text{IBC}}, \end{aligned} \quad (1.144)$$

so  $(1 + H_0^{-1}a^\dagger(v))\Omega_{\text{IBC}} = \Omega \in \text{dom}(H_0)$ . Obviously,  $\Omega_{\text{IBC}} \notin \mathcal{H}$  since the  $L^2$ -integral diverges on each sector. The expression (1.143) closely resembles the formal “Weyl-

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<sup>21</sup>This example was pointed out by Julian Schmidt to whom I am grateful for useful explanations concerning IBCs and its abstract generalizations.

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dressed vacuum”

$$\Omega_W = e^{-\frac{\|s\|^2}{2}} \sum_{N \in \mathbb{N}_0} \frac{(a^\dagger(s))^N}{N!} \Omega \quad \Rightarrow \quad \Omega_W^{(N)}(\mathbf{k}_1, \dots, \mathbf{k}_N) = \frac{e^{-\frac{\|s\|^2}{2}}}{\sqrt{N!}} \prod_{\ell=1}^N s(\mathbf{k}_\ell), \quad (1.145)$$

which is a generalized coherent state with  $\Omega_W \notin \mathcal{H}$ , but  $\Omega_W \in \widehat{\mathcal{H}}$ . However,  $\Omega_{\text{IBC}}$  lacks the **infinite wave function renormalization**  $e^{-\frac{\|s\|^2}{2}}$  which is formally “ $e^{-\infty} = 0$ ”. For this sake, the formal expression  $\Omega_W$  as in (1.145) would also amount to the configuration space function  $\Omega(q) \equiv 0$ .

As the examples in Sections 1.3.7 and 1.3.8 suggest, it would be useful for a direct renormalization procedure as the IBC method to allow for dealing with dressing transformations leading out of Fock space. In case of a relativistic QFT with  $s$  only diverging in the IR–region, it may also be possible to circumvent the non–Fock dressing by defining an  $H_{\text{IBC},h}$  with a spatial cutoff  $h$ , and to remove the cutoff via Segal’s theorem as explained in Section 1.3.6 and 1.3.7. We will however not pursue this approach, but provide tools that allow for treating dressing transformations leading out of Fock space. Those also work in a non–relativistic environment and are therefore more general. Before doing so, we quickly review some results about IBCs in the context of QFT.

### 1.4.4 Literature on Interior–Boundary Conditions

Interior–boundary conditions have already been described well before the onset of the investigations initiated by [114, 115, 116] and considering IBCs as a tool for non–perturbative renormalization:

- Already in the early 50s (and hence before the emergence of CQFT), Moshinsky considered IBCs for a description of nuclear reactions on a non–rigorous level [118, 119]. In [118], he provides IBC equations for 3 models:
  - A model with a compound particle dissociating into two particles [118, (17)], which is described in relative coordinates, so the Hilbert space is given by  $\mathcal{H} = \mathbb{C} \oplus L^2(\mathbb{R}^3)$ . This is a generalization of the example we presented in (1.121), allowing for a positive mass  $\mu$  and rest energies  $m_1, m_0$ . The formal free Hamiltonian then reads

$$H_0 = \begin{pmatrix} m_0 & 0 \\ 0 & -\frac{\Delta}{2\mu} + m_1 \end{pmatrix}. \quad (1.146)$$

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- An analogous model with  $N$  types of compound particles that can all dissociate into the same pair of particles, so  $\mathcal{H} = \mathbb{C} \oplus \dots \oplus \mathbb{C} \oplus L^2(\mathbb{R}^3)$ . This model can also be interpreted as describing a single compound particle with an internal structure that is modeled by a state vector in  $\mathbb{C}^N$ .
- A model with one compound particle that can dissociate into either of two types of particle pairs, so  $\mathcal{H} = L^2(\mathbb{R}^3) \oplus \mathbb{C} \oplus L^2(\mathbb{R}^3)$ .

A fourth model describing beta decay is also briefly and heuristically discussed.

In [119], Moshinsky provides the spectral resolution<sup>22</sup> (“generalized Hankel transformation”) in terms of generalized eigenfunctions for the first model above ( $\mathcal{H} = \mathbb{C} \oplus L^2(\mathbb{R}^3)$ )

- Pavlov [120] later considered a similar model on a mathematically rigorous level. Here, two particles recombine to a compound particle with internal structure, so  $\mathcal{H} = \mathcal{H}_0 \oplus L^2(\mathbb{R}^3)$ , where  $\mathcal{H}_0$  is either  $\mathbb{C}^N$  or  $\ell^2$  and describes the internal structure. The dispersion relation on  $L^2(\mathbb{R}^3)$  is again given by  $H_0 = -\Delta$ . It is then proved that all self-adjoint extensions of  $H_0$  on  $\mathcal{D}_0 = \{\Psi \in H^2(\mathbb{R}^3) \mid \Psi(0) = 0\}$  make up a one-parameter family  $(H_{\text{IBC},\alpha})_{\alpha \in \mathbb{C}}$ , where for each  $\alpha$ , the corresponding IBC and the spectral resolution of  $H_{\text{IBC},\alpha}$  in terms of generalized eigenfunctions is given.
- Thomas [121] independently considered a similar model with two particles recombining to one compound particle, with  $\mathcal{H} = \mathbb{C} \oplus L^2(\mathbb{R}^3)$  and the formal  $H_0$  from (1.146). He provided an IBC together with  $H_{\text{IBC}}$  and an eigenfunction expansion. Additionally the IBC was given for a model of three particles recombining into two, with  $\mathcal{H} = L^2(\Sigma_{12}) \oplus L^2(\Sigma_{13}) \oplus L^2(\Sigma_{23}) \oplus L^2(\mathbb{R}^6)$  where  $\Sigma_{12} \cong \Sigma_{13} \cong \Sigma_{23} \cong \mathbb{R}^3$  are the codimension-3 collision hyperplanes<sup>23</sup> in  $\mathbb{R}^6$ .

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<sup>22</sup>By a spectral resolution, we mean that  $H$  is explicitly diagonalized in the sense of the spectral theorem [24, Thm. 10.10]. That means, some spectral set  $X$  and unitary  $U : \mathcal{H} = L^2(\mathcal{Q}) \rightarrow L^2(X)$  is found, such that  $H = U^* \lambda U$ , where  $\lambda$  is the operator multiplying with the spectral value  $\lambda$ . For instance, a spectral resolution can be provided by stating the kernel  $U(q, x)$  of  $U$ , where  $\xi_x : q \mapsto U(q, x)$  is called a **generalized eigenfunction**.

<sup>23</sup> $\Sigma_{12}$  is obtained by taking the 6-dimensional hypersurface  $\{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \mid \mathbf{x}_1 = \mathbf{x}_2\} \subset \mathbb{R}^9$  and projecting it to  $\mathbb{R}^6$  by a suitable choice of center-of-mass coordinates.

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- Yafaev [122] also investigated the situation of two particles recombining to a compound particle with internal structure, both in 1 and 3 space dimensions. That means,  $\mathcal{H} = \mathbb{C}^N \oplus L^2(\mathbb{R})$  and  $\mathcal{H} = \mathbb{C}^N \oplus L^2(\mathbb{R}^3)$  were considered. All self-adjoint extensions of  $H_0 = -\Delta$  on  $\mathcal{D}_0 = \{\Psi \in H^2(\mathbb{R}^d) \mid \Psi(0) = 0\}$  were characterized, their resolvents determined and for  $d = 1$ , their spectra were analyzed.

Independently of the works above, a series of papers investigating IBCs for non-perturbative renormalization have been motivated by a work by Tumulka and Georgii, aiming at a description of quantum particle trajectories as stochastic processes. Here, an IBC was suggested in [124, (23)] as a condition ensuring probability balance. Works which make use of IBCs that actually relate interior points of configuration space with boundary points include the following:

- Teufel and Tumulka [114] established the self-adjointness of an IBC Hamiltonian  $H_{\text{IBC}}$  for one resting fermion interacting with a quantized boson field, where  $\mathcal{H} = \mathcal{F}(\mathbb{R}^3)$ ,  $H_0 = d\Gamma(\omega)$ ,  $\omega(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2m} + E_0$ . This can be seen as a generalization of Moshinsky's first model to an arbitrary particle number, while dropping the vacuum energy.

Teufel and Tumulka could prove that  $H_{\text{IBC}}$  coincides with the Hamiltonian obtained by cutoff-renormalization, up to a constant:

$$(H_{\text{IBC}} + c) = \tilde{H} = \lim_{\Lambda \rightarrow \infty} (H_\Lambda - E_\Lambda), \quad c \in \mathbb{C}. \quad (1.147)$$

Further, they suggested IBCs for a model with  $M \in \mathbb{N}$  resting fermions, a model with  $M$  moving fermions of dispersion relation  $\theta(\mathbf{p}) = \frac{|\mathbf{p}|^2}{2m_x}$ , and a model with a general configuration space whose boundary is of codimension 1.

In a related work [115], the same authors suggested IBCs, which couple the real line  $\mathbb{R}$  to the boundary of the upper half-plane in  $\mathbb{R}^2$ , so  $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R} \times [0, \infty))$ . An IBC for the model in [114] on  $\mathcal{F}(\mathbb{R}^3)$  restricted to the (0)- and the (1)-sector were proposed and an expression for  $H_{\text{IBC}}$  was given in all considered models.

- Lampart, Schmidt, Teufel and Tumulka [116] considered the above-mentioned case of  $M \in \mathbb{N}$  resting fermions and boson dispersion relation  $\omega(\mathbf{k}) = |\mathbf{k}|^2 + E_0$ . They could establish a 4-parameter family of IBCs corresponding to this model, indexed by  $v$ , which results in a family of distinct IBC Hamiltonians  $(H_{\text{IBC},v})$ . Further, all  $H_{\text{IBC}}$  satisfying a certain additional condition

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are physically equivalent to a free Hamiltonian, as they can be “undressed” by a Gross transformation

$$W_v^* H_{\text{IBC},v} W_v = d\Gamma(h) + c_v, \quad c_v \in \mathbb{C}, \quad (1.148)$$

where  $W_v$  and  $c_v$  depend on the choice of the IBC parameters  $v$ .

- Schmidt and Tumulka [125] showed that the above–mentioned extensions  $H_{\text{IBC}}$  are time–symmetric if and only if the coupling constant is real and that a Yukawa pair potential emerges from the ground state energy of  $M$  resting fermions.
- Keppeler and Sieber [126] established a self–adjoint  $H_{\text{IBC}}$  together with a spectral resolution in terms of generalized eigenfunctions for a variety of models in 1 space dimension:
  - The first model describes one point source emitting and absorbing bosons with  $\omega(\mathbf{k}) = |\mathbf{k}|^2$  on Fock space  $\mathcal{H} = \mathcal{F}(\mathbb{R})$ .
  - The second model describes two point sources with distance  $R$ , interacting with the boson field, which leads to an emergent one–dimensional Coulomb potential for small  $R$ .
  - The third model then generalizes to  $M \in \mathbb{N}$  point sources interacting with the boson field.
  - And the fourth model concerns one point source inside a box, so  $\mathcal{H} = \mathcal{F}((x, y))$ ,  $x < y$ , where the boson dispersion relation is still  $\omega(\mathbf{k}) = |\mathbf{k}|^2$ .

Further, they suggested IBCs and  $H_{\text{IBC}}$  for a “quantum graph” model, which is obtained by “gluing together” various boxes (i.e., lines as in the fourth model) at their endpoints. This way, one obtains a graph whose edges are the boxes, and where there is an IBC at each vertex balancing the probability flows between the adjacent edges.

- Lienert and Nickel [127] gave an example for the employment of IBCs in the so–called **multi–time framework** of QFT. This framework emerges when considering relativistic quantum models in the Schrödinger picture, as explained in Section 2.1. It describes quantum states of  $N$  particles by a wave function  $\Phi(t_1, \mathbf{x}_1, \dots, t_N, \mathbf{x}_N)$  with  $N$  time coordinates, which satisfies a system of  $N$  Schrödinger–like PDEs (one per time derivative  $\partial_{t_j}$ ). For an introduction into multi–time wave functions, see Section 2.1 or [128, 129].

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Lienert and Nickel now posed an initial value problem (IVP) consisting of a PDE system, an IBC for  $\Phi$  and initial values that satisfy an initial-value IBC. Two models with massless Dirac particles in 1 + 1 dimensions were treated, so the dispersion relation is given by the  $2 \times 2$ -matrix

$$\theta(\mathbf{p}) = \begin{pmatrix} \mathbf{p} & 0 \\ 0 & -\mathbf{p} \end{pmatrix}. \quad (1.149)$$

The first model is restricted to the one- and the two-particle sector in Fock space, so the corresponding single-time version would be formulated on  $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^2) \oplus L^2(\mathbb{R}^2, \mathbb{C}^4)$ .

The second model restricts to particle numbers between 1 and  $N$ , so  $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}^2) \oplus \dots \oplus L^2(\mathbb{R}^N, \mathbb{C}^{2^N})$ .

For both models, existence and uniqueness of the IVP including IBCs could be established.

- Tumulka [130] considered a general configuration space  $\mathcal{Q} = \bigcup_{N \in \mathbb{N}_0} \mathcal{Q}^{(N)}$ , where each  $\mathcal{Q}^{(N)}$  is a manifold with codimension-1 boundary. He provided a 3-parameter family of IBCs that preserve probability for a free Hamiltonian of the kind  $H_0 = -\Delta + V$ , where  $V$  is a multiplication operator by  $V(q)$ ,  $q \in \mathcal{Q}$ , i.e., an external potential.

The results were applied to a model with a spherical source or radius  $r > 0$ , emitting and absorbing bosons. So  $\mathcal{H} = \mathcal{F}(\mathbb{R}^3 \setminus B_r(0))$  and  $H_0 = d\Gamma(\omega)$  with  $\omega(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2m} + E_0$ .

- Schmidt, Teufel and Tumulka [131] provided an IBC and established self-adjointness of the corresponding  $H_{\text{IBC}}$  for a Dirac particle in  $d = 3$  dimensions<sup>24</sup>  $\theta(\mathbf{p}) = \boldsymbol{\alpha} \cdot \mathbf{p} + m\beta$ , which is confined to a bounded region  $\Omega \subset \mathbb{R}^3$  with codimension-1 boundary. So  $\mathcal{H} = \mathbb{C} \oplus L^2(\Omega, \mathbb{C}^4)$ .

A generalized case similar to [130] was also considered, with configuration space  $\mathcal{Q} = \bigcup_{N \in \mathbb{N}_0} \mathcal{Q}^{(N)}$  where  $\partial\mathcal{Q}^{(N)}$  is of codimension 1, and  $H_0$  is of ‘‘Dirac type’’:

$$H_0 = \sum_{a=1}^{d_N} A^a(q)(-i\partial_{q_a}) + B(q), \quad (1.150)$$

<sup>24</sup>Here  $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ , where  $\alpha_j, \beta \in \mathbb{C}^{4 \times 4}$  are Dirac matrices. Sometimes, one also writes  $\gamma_0 := \beta$  and  $\gamma_j = \gamma_0 \alpha_j$ . There are several representations for the Dirac matrices, for instance  $\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$  and  $\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$  with  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ,  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  being the Pauli matrices, see also [132].

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with  $d_N$  being the dimension of  $\mathcal{Q}^{(N)}$  at  $q = (q_1, \dots, q_N) \in \mathcal{Q}^{(N)}$  and  $A^a(q), B(q)$  being matrices of suitable size. A family of IBCs together with the corresponding  $H_{\text{IBC}}$  was suggested for this generalized case.

- The case of an IBC which does not conserve probability was considered by Teufel and Tumulka [133] under the name **absorbing boundary condition**. Absorption can be achieved by suppressing the probability flow out of the boundary of configuration space, which allows for  $\|\Psi_t\|$  to decrease in time. The corresponding dynamics are given by a contraction semigroup  $(W_t)_{t \in [0, \infty)}$ . For a region  $\Omega \subset \mathbb{R}^3$  with sufficiently regular boundary<sup>25</sup>  $\partial\Omega$ , the existence of a contraction semigroup  $(W_t)_{t \in [0, \infty)}$  has been established using the Hille–Yosida theorem (which is a generalization of Stone’s theorem to non–unitary dynamics, see [134, Sect. II.3]) in the following models:
  - One particle, so  $\mathcal{H} = L^2(\Omega)$ ,  $H_0 = -\Delta + V$ , where the multiplication operator  $V$  is Kato–bounded with respect to  $-\Delta$ .
  - $N$  particles, so  $\mathcal{H} = L^2(\Omega^N)$ ,  $H = \sum_{j=1}^N (-\Delta_j) + V$  where the multiplication operator  $V$  is Kato–bounded with respect to  $\sum_{j=1}^N (-\Delta_j)$ .
  - One Dirac particle, so  $\mathcal{H} = L^2(\Omega, \mathbb{C}^4)$ ,  $H = \boldsymbol{\alpha} \cdot \mathbf{p} + m\beta + V$ , where  $V$  is Kato–bounded with respect to the Dirac operator  $\boldsymbol{\alpha} \cdot \mathbf{p} + m\beta$ .

The boundary  $\partial\Omega$  is interpreted as a detector, absorbing particles with detection outcomes  $(t, \mathbf{x}) \in [0, \infty) \times \partial\Omega$ . A corresponding positive operator–valued measure (POVM) has then been constructed, which maps patches  $B \subset [0, \infty) \times \partial\Omega$  to positive operators  $E(B) : \mathcal{H} \rightarrow \mathcal{H}$ , such that  $\langle \Psi_0, E(B)\Psi_0 \rangle$  is the probability for a particle detection in  $B$ , given an initial state  $\Psi_0$  at  $t = 0$ .

- A no–go result on IBCs was established by Henheik and Tumulka [135] for  $\leq N_{\text{max}}$  Dirac particles at a point source. That means,  $\mathcal{H} = \mathbb{C} \oplus \dots \oplus L^2(\mathbb{R}^3, \mathbb{C}^4)^{\otimes N_{\text{max}}}$  and  $\theta(\mathbf{p}) = \boldsymbol{\alpha} \cdot \mathbf{p} + m\beta$ . The result states that for any extension of the Dirac operator  $d\Gamma(\theta)$  from

$$\mathcal{D} = \{0\} \oplus \dots \oplus \{0\} \oplus C_c^\infty \left( (\mathbb{R}^3 \setminus \{0\})^{N_{\text{max}}}, \mathbb{C}^{4^{N_{\text{max}}}} \right) \quad (1.151)$$

to some domain in  $\mathcal{H}$ , the  $(N_{\text{max}})$ –sector decouples from all lower sectors. That means,  $\langle \Phi, H_{\text{IBC}}\Psi \rangle = 0$  whenever  $\Phi \in \mathcal{F}^{(N_{\text{max}})}$ ,  $\Psi \perp \mathcal{F}^{(N_{\text{max}})}$ . So particle creation into the  $(N_{\text{max}})$ –sector via IBCs is ruled out. This is a consequence of the well–known fact, that the Dirac operator  $\theta(\mathbf{p})$  is already self–adjoint

<sup>25</sup>Here, “sufficiently regular” means locally Lipschitz and piecewise  $C^1$ .

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on  $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$  (which is not the case for  $-\Delta$  on  $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ ). So there is “no option for probability to enter or escape through the origin”.

However, it is also well-known that after introducing a Coulomb potential, the Hamiltonian

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta - \frac{g}{|\mathbf{x}|} \quad (1.152)$$

is no longer essentially self-adjoint, if the coupling is chosen larger than the critical value of  $g = \frac{\sqrt{3}}{2}$  [79]. Heuristically speaking, a sufficiently strong potential “opens the origin”, for “probability to escape or to enter”. This allows for probability flows between different sectors being balanced, e.g., by an IBC.

And indeed, for  $\frac{\sqrt{3}}{2} < |g| < 1$ , Henheik and Tumulka could establish a self-adjoint  $H_{\text{IBC}}$  describing dynamics with particle creation on  $\mathcal{H} = \mathbb{C} \oplus L^2(\mathbb{R}^3, \mathbb{C}^4)$ , which extends  $H|_{\mathcal{D}}$  on  $\mathcal{D} = \{0\} \oplus C_c^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}^4)$ .

### 1.4.5 Literature on Abstract Boundary Conditions

As explained in Section 1.4.3, it is also possible to directly define a renormalized Hamiltonian  $H_{\text{IBC}}$  by using IBC-like constraint equations that do not relate interior to boundary points. These “abstract boundary conditions” have been primarily applied to polaron-like models (compare Section 1.3.7) with fermion dispersion relation  $\theta(\mathbf{p})$ , boson dispersion relation  $\omega(\mathbf{k})$  and form factor  $v(\mathbf{k})$ . We will still use the name “IBC method” for the corresponding renormalization process and denote resulting self-adjoint Hamiltonians by  $H_{\text{IBC}}$ .

- Lampart and Schmidt [136] established self-adjointness of  $H_{\text{IBC}}$  for a general class of polaron models in 1, 2 or 3 space dimensions, with  $\theta(\mathbf{p}) = |\mathbf{p}|^2$ ,  $\omega(\mathbf{k}) \geq (1 + |\mathbf{k}|^2)^{\beta/2}$  and  $|v(\mathbf{k})| < |\mathbf{k}|^{-\alpha}$ , where  $\alpha, \beta \geq 0$  are scaling parameters. The constraints on  $\alpha$  and  $\beta$  are dimension-dependent and allow for treating (compare Section 1.3.7):
  - The Fröhlich polaron ( $\alpha = 1, \beta = 0$ ) in  $d \leq 3$ .
  - The massive (but not the massless) Nelson model ( $\alpha = \frac{1}{2}, \beta = 2$ ) in  $d \leq 3$ .
  - Moving point sources interacting by an exchange of non-relativistic bosons ( $\alpha = 0, \beta = 2$ ), but only in<sup>26</sup>  $d \leq 2$ . This corresponds to the

<sup>26</sup>Although only the case  $d = 2$  is explicitly discussed in the paper, the proof also applies to  $d = 1$ .

### 1.4 Direct renormalization and Interior–Boundary Conditions (IBC)

model established in [116], but with a nonzero dispersion relation for the fermions.

Further, Lampart and Schmidt could prove that  $H_{\text{IBC}}$  coincides with the Hamiltonian obtained by cutoff renormalization  $\tilde{H}^{(1)} = \lim_{\Lambda \rightarrow \infty} (H_{\Lambda} - E_{\Lambda})$  for a suitable choice of the self–energy  $E_{\Lambda}$ .

- Lampart [137] extended the above proof of self–adjointness for  $H_{\text{IBC}}$  within the moving point source model to  $d = 3$  (explicitly,  $\theta(\mathbf{p}) = \frac{|\mathbf{p}|^2}{2m}$ ,  $\omega(\mathbf{k}) = |\mathbf{k}|^2 + 1$ ,  $v(\mathbf{k}) = (2\pi)^{-3/2}$ ).
- Schmidt [138] established self–adjointness of  $H_{\text{IBC}}$  for a different choice of scaling parameters  $\alpha, \beta$  and  $\gamma$ , which also allows for pseudo–relativistic fermions  $\theta(\mathbf{p}) \geq |\mathbf{p}|^{\gamma}$ ,  $\omega(\mathbf{k}) \geq (1 + |\mathbf{k}|^2)^{\beta/2}$ . Further the form factor is allowed to depend on the fermion momentum:  $v(\mathbf{p}, \mathbf{k}) \leq |\mathbf{k}|^{-\alpha}$ . The constraints on  $\alpha, \beta$  and  $\gamma$  admit the treatment of
  - Eckmanns polaron model from [59] (compare Section 1.3.7).
  - L. Gross’ relativistic polaron from [61], but only in  $d = 2$  dimensions<sup>27</sup>.

Additionally, Schmidt could prove that under certain conditions,  $H_{\text{IBC}} = \tilde{H}^{(1)} = \lim_{\Lambda \rightarrow \infty} (H_{\Lambda} - E_{\Lambda})$  with a suitable self–energy counterterm  $E_{\Lambda}$ .

- Schmidt [139] also established self–adjointness for a class of polaron models including the  $d = 3$  massless Nelson model with one single fermion. More precisely,  $\theta(\mathbf{p}) = |\mathbf{p}|^2$ ,  $|\mathbf{k}|^{\beta} \leq \omega(\mathbf{k}) \leq |\mathbf{k}|^{\beta} + m$  and  $c(1 + |\mathbf{k}|^{\alpha})^{-1} \leq |v(\mathbf{k})| \leq |\mathbf{k}|^{-\alpha}$  for some constant  $c \in \mathbb{R}$  and where the scaling parameters  $\alpha$  and  $\beta$  are subject to certain constraints.
- Lampart [140] could establish self–adjointness of  $H_{\text{IBC}}$  for a model with a single fermion, closely related to the Fröhlich polaron and appearing in Bogoliubov theory. See also the referenecs given in [140]. The dispersion relations are  $\theta(\mathbf{p}) = \frac{|\mathbf{p}|}{2m}$ ,  $\omega(\mathbf{k}) = c|\mathbf{k}|\sqrt{1 + (|\mathbf{k}|\xi)^2/2}$  and the form factor is  $v(\mathbf{k}) = (2\pi)^{-2/3} \left( \frac{(|\mathbf{k}|\xi)^2}{2 + (|\mathbf{k}|\xi)^2} \right)$ . Here, the speed of sound  $c > 0$  and the healing length  $\xi > 0$  are arbitrary constants. The IBC renormalization involves

<sup>27</sup>Recall from Section 1.3.7 that the renormalized Hamiltonian for Gross’ relativistic polaron in  $d = 3$  is constructed within a non–Fock representation. Since IBC renormalization is constrained to the Fock representation, one may expect to obtain a formal IBC domain outside Fock space in this case.

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a two-step transformation of  $H$ , such that formally, two self-energy terms  $E_\infty^{(1)}$  and  $E_\infty^{(2)}$  are subtracted.

Lampart could also prove that  $H_{\text{IBC}} = \tilde{H}^{(1)}$  with

$$\tilde{H}^{(1)} = \lim_{\Lambda \rightarrow \infty} (H_\Lambda - E_\Lambda^{(1)} - E_\Lambda^{(2)}) \quad (1.153)$$

being the Hamiltonian obtained by cutoff renormalization.

- Posilicano [141] considered an abstract setting with a formal Hamiltonian  $H = H_0 + A^\dagger + A$ , where he extended  $H|_{\text{Ker}(A)}$  to a family of self-adjoint operators  $H_{\text{IBC},T}$  (called  $\hat{H}_T$ ), parametrized by a self-adjoint operator  $T$  defined on a subspace of  $\mathcal{F}$ . For these  $H_{\text{IBC},T}$ , resolvent formulas were provided. Further, for one of the self-adjoint extensions  $H_{\text{IBC},0}$ , agreement with the cutoff-renormalized Hamiltonian  $\tilde{H}^{(1)} = \lim_{\Lambda \rightarrow \infty} (H_\Lambda - E_\Lambda)$  were established.

## 1.5 Perturbative QFT as a Source of Heuristics

As mentioned in the introduction, a renormalized Hamiltonian  $\tilde{H}$  is often not available for relativistic QFTs, so one commonly resorts to perturbative methods for making physical predictions. We present some of them in this section, as they may also provide useful heuristics for non-perturbative renormalization. As an example, the use of distributions and the cutoff-free Epstein-Glaser method (described below) have inspired our ESS framework presented in Section 3.2, which is also designed to work with distributions outside  $L^2$  and without cutoffs. Further, it would be interesting to justify successful pQFT methods starting from non-perturbatively established models, which provides an additional motivation for the study of non-perturbative renormalization in QFT.

In a nutshell, perturbative renormalization can be motivated as follows (see also [142, Sect. 3.1], [6, Chap. 7], [143, Anhang]):

Suppose, we have a time-dependent family of self-adjoint Hamiltonian operators on  $\mathcal{H}$ :

$$H(t) = H_0 + H_I(t), \quad H_I(t) = \int H_I(t, \mathbf{x}) d\mathbf{x}, \quad (1.154)$$

where  $H_I(t, \mathbf{x}) d\mathbf{x}$  is to be understood as a  $t$ -dependent operator-valued measure on  $\mathbb{R}^d$ , called **Hamiltonian density**. The unitary operator evolving from time  $t_0$  to  $t$  in the interaction picture would then formally be given by a Dyson series [5,

Chap. 4], [6, Sect. 7.2]:

$$\begin{aligned}
 U_I(t, t_0) &= \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^t \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{n-1}} H_I(t_1) H_I(t_2) \dots H_I(t_n) dt_n \dots dt_2 dt_1 \\
 &\quad \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t \int_{t_0}^t \dots \int_{t_0}^t \mathcal{T}(H_I(t_1) H_I(t_2) \dots H_I(t_n)) dt_1 dt_2 \dots dt_n,
 \end{aligned} \tag{1.155}$$

where  $\mathcal{T}$  denotes the time ordering of operators, i.e., the operators at times  $t_j$  are sorted from earliest (right) to latest (left). In (1.155) we have replaced the integration over the time-ordered simplex  $\{(t_1, \dots, t_N) \mid t_0 \leq t_n \leq \dots \leq t_1 \leq t\}$  by an integration over the  $n!$ -times larger cube  $\{(t_1, \dots, t_N) \mid t_0 \leq t_j \leq t\}$ , see Figure 1.9. Formally, we may even write (1.155) by integrals over  $x_j = (t_j, \mathbf{x}_j)$ , so the integral measure does not make reference to a particular Lorentz frame. In particular, an integral over all  $x_j \in \mathbb{R}^{d+1}$  would be Poincaré-invariant:

$$\begin{aligned}
 U_I(t, t_0) &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_j \in [t_0, t]} \mathcal{T}(H_I(x_1) H_I(x_2) \dots H_I(x_n)) dx_1 \dots dx_n, \\
 S &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^{n(d+1)}} \mathcal{T}(H_I(x_1) H_I(x_2) \dots H_I(x_n)) dx_1 \dots dx_n,
 \end{aligned} \tag{1.156}$$

$$U_I(t, t_0) = \sum_{n=0}^{\infty} U_{I,n}(t, t_0), \quad S = \sum_{n=0}^{\infty} S_n. \tag{1.157}$$

Here,  $S$  is the (unitary) **S-matrix**  $S : \mathcal{H} \rightarrow \mathcal{H}$  similar<sup>28</sup> to the one introduced above (1.57). The operators  $S_n$  are called (perturbative) **S-matrix orders**.

We are now interested in the integral kernel of the operators  $U_I(t, t_0)$  and  $S$ , that is, the distribution which maps  $\overline{\Psi}_f \otimes \Psi_i \in \mathcal{S} \otimes \mathcal{S}$  to  $\langle \Psi_f, U_I(t, t_0) \Psi_i \rangle$  or  $\langle \Psi_f, S \Psi_i \rangle$ . Here, the initial and final state vectors  $\Psi_i, \Psi_f$  decay into sectors  $\Psi_i^{(N_i)} \in \mathcal{S}(\mathbb{R}^{N_i(d+1)})$ ,  $\Psi_f^{(N_f)} \in \mathcal{S}(\mathbb{R}^{N_f(d+1)})$ , so the required integral kernel consists of a countable family of distributions

$$(U_I(t, t_0)^{(N_i, N_f)})_{N_i, N_f \in \mathbb{N}_0}, (S^{(N_i, N_f)})_{N_i, N_f \in \mathbb{N}_0} \subset \mathcal{S}'(\mathbb{R}^{(N_i + N_f)(d+1)}). \tag{1.158}$$

Using the perturbation expansion (1.157), we can write  $U_I(t, t_0)^{(N_i, N_f)}, S^{(N_i, N_f)}$  as

<sup>28</sup>Above (1.57), we assumed  $H$  to be constant and defined the S-matrix, using limits of  $e^{-it(H_0 + H_I)} e^{itH_0}$ , which can be evaluated by the Baker–Campbell–Hausdorff (BCH) formula and is generally not identical to  $e^{-itH_I}$ .

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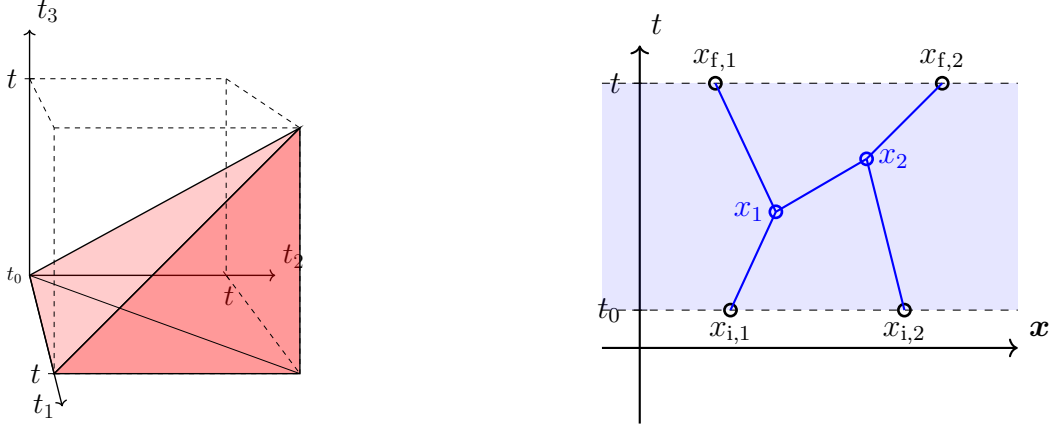


Figure 1.9: Left: The time-ordered simplex (red) takes up  $1/n!$  of the volume of the respective cube.

Right: A diagram representing one contribution to  $U_{I,n}(t, t_0)^{(N_i, N_f)}$  in (1.161) with  $n = 2$ ,  $N_i = 2$ ,  $N_f = 2$ . The integral runs over all  $x_1, x_2$  with  $t_0 \leq t_1, t_2 \leq t$  (shaded area). Color online.

a sum in  $n \in \mathbb{N}_0$  over distributions

$$U_{I,n}(t, t_0)^{(N_i, N_f)}, S_n^{(N_i, N_f)} \in \mathcal{S}'(\mathbb{R}^{(N_i + N_f)(d+1)}), \quad (1.159)$$

which are parts of the integral kernels of  $U_{I,n}(t, t_0)$  and  $S_n$ . The goal of pQFT is to make sense of these integral kernels, even if  $H(t)$  does not exist as a self-adjoint operator. Once  $U_{I,n}(t, t_0)^{(N_i, N_f)}$  or  $S_n^{(N_i, N_f)}$  are found, they serve for an approximate description of the dynamics generated by  $H_I(t)$ . This is particularly interesting, if no self-adjoint  $H(t)$  exists, but an approximation can nevertheless be defined. The failure of existence of a self-adjoint  $H_I(t)$  commonly occurs in relativistic models, which describe processes involving high particle energies and are often empirically verified in high-energy scattering experiments. Therefore, many pQFT methods focus on finding  $S_n$ , rather than  $U_{I,n}(t, t_0)$ . The distribution values

$$S_n^{(N_i, N_f)}(\mathbf{q}_i, \mathbf{q}_f) \quad \text{at} \quad (\mathbf{q}_i, \mathbf{q}_f) = (\mathbf{x}_{i,1}, \dots, \mathbf{x}_{i, N_i}, \mathbf{x}_{f,1}, \dots, \mathbf{x}_{f, N_f}), \quad (1.160)$$

provided that  $S_n^{(N_i, N_f)}$  can be written as a function at  $(\mathbf{q}_i, \mathbf{q}_f)$ , are then called **S-matrix elements**.

To heuristically derive expressions for  $U_{I,n}(t, t_0)$  and  $S_n$ , let us assume again, for a moment, that  $H_I(t)$  was well-defined, such that (1.156) would hold. The

operator  $H_I(t)$  is further assumed to be a local product of field operators<sup>29</sup> like  $\int_{\mathbb{R}^d} \phi(t, \mathbf{x})^p : d\mathbf{x}, p \in \mathbb{N}$ . Making use of the CCR/CAR and Wick's theorem [144], [6, Sect. 7.A.2], one arrives at a formal expression for the distribution  $U_{I,n}(t, t_0)$ , which is a sum over expressions proportional to (compare [143, p.228], [145, (1.2)] [146, (1.1), (2.7)]):

$$J_n(q_i, q_f) = \int_{q: t_j \in [t_0, t]} I_n(q_i, q, q_f) dq, \tag{1.161}$$

$$I_n(q_i, q, q_f) = \prod_{\ell=1}^L \Delta_\ell^F(x_{\ell,a} - x_{\ell,b}).$$

This expression needs some explanation: We have  $L \in \mathbb{N}_0$ , and  $\Delta_\ell^F \in \mathcal{S}'(\mathbb{R}^{2(d+1)})$  is a time-ordered propagator [145, (1.3)], [142, Sect. 2.3], also called Feynman propagator. The fixed configurations  $q_i, q_f$  can be derived from  $\mathbf{q}_i, \mathbf{q}_f$  by including time coordinates:

$$q_i = ((t_0, \mathbf{x}_{i,1}), \dots, (t_0, \mathbf{x}_{i,N_i})), \quad q_f = ((t_0, \mathbf{x}_{f,1}), \dots, (t_0, \mathbf{x}_{f,N_f})). \tag{1.162}$$

The integral runs over  $q = (x_1, \dots, x_n)$ , and  $x_{\ell,a}, x_{\ell,b}$  are either of the kind  $x_{i,j} = (t_0, \mathbf{x}_{i,j}), x_{f,j} = (t, \mathbf{x}_{f,j})$  or  $x_j \in q$ . Expression (1.161) can be represented by a diagram as in Figure 1.9, where all coordinates in  $q_i, q_f$  and  $q$  are represented by vertices at  $x_{i,j}, x_{f,j}$  or  $x_j$ , and each  $\ell \in \{1, \dots, L\}$  corresponds to a line connecting the vertices at  $x_{\ell,a}$  and  $x_{\ell,b}$ . The sum for obtaining  $U_{I,n}(t, t_0)$  then ranges over all diagrams with  $n$  “movable” vertices in  $q$ .

The expression for  $S_n$  is a similar sum over integrals corresponding to diagrams, as presented in Figure 1.10. As the integrals run over  $x_j$ , while only keeping the structure of the connections fixed, it is customary to drop the  $t$ - and the  $x$ -axis, and to only draw a graph representing the connection structure of the vertices. Such graphs for encoding integrals are also called **Feynman diagrams**.

Following some heuristic considerations [5, Sect. 4.6], only contributions corresponding to a special class of diagrams (so-called “amputated diagrams”, defined above [5, (4.103)] or in [6, Sect. 18.3.2]) are used for a calculation of  $S_n$ , and the external lines ending at the (formal) “infinite-time coordinates”  $x_{i,j}$  or  $x_{f,j}$  are translated into a distribution different from  $\Delta_\ell^F$ . The precise translation prescriptions from a diagram to an integral are called **Feynman rules** and can be found for various QFT models in the standard physics literature [6, Chap. 7], [7, Chap. 6], [5, 8].

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<sup>29</sup>By this, we mean operator products as in (1.34) with  $\int : \phi(x)^p : dx = \int : \phi(x_1) \dots \phi(x_p) : \delta(x_1 - x_2) \dots \delta(x_{p-1} - x_p) dx_1 \dots dx_p$ . Derivatives of  $x_j$  may also be included, as they can be evaluated locally.

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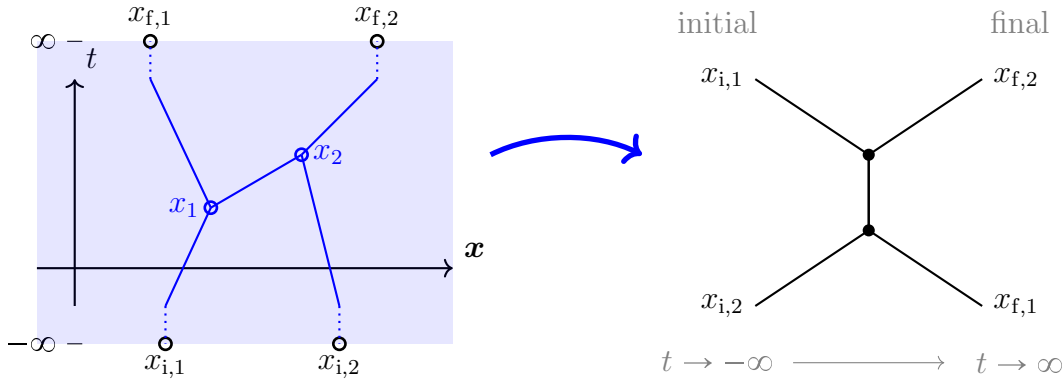


Figure 1.10: When computing the S–matrix orders  $S_n$ , integrals running over all  $x_j \in \mathbb{R}^{d+1}$  appear. Each integral is tracked by a Feynman diagram representing the connections of the vertices. Color online.

Now, the formal expressions for  $U_{I,n}(t, t_0)$  and  $S_n$  are ill–defined, as they include divergent expressions. To obtain finite results, called  $\tilde{U}_{I,n}(t, t_0)$  and  $\tilde{S}_n$ , one implies several ad hoc modifications to the formal expressions. Apart from the heuristically motivated ad hoc–restriction to amputated diagrams, these modifications include the following:

- A formal **Gell–Mann and Low formula** is used, which amounts to **omitting all disconnected diagrams**<sup>30</sup> in the sums that represent  $U_{I,n}(t, t_0)$  and  $S_n$ . This can heuristically be seen as a kind of “dressing transformation”.
- The expressions  $I_n$  as in (1.161) should be distributions in  $\mathcal{S}'$ , but are often ill–defined, as they contain ill–defined products<sup>31</sup> of distributions  $\Delta_\ell^F$ . Therefore, one has to **renormalize the distributions**  $I_n$ , which is done by a formal subtraction of infinite expressions and leads to a well–defined expression  $\tilde{I}_n \in \mathcal{S}'$ .
- Even after a successful establishment of  $I_n$ , it may happen that the integral over  $I_n$ , as in (1.161), still diverges. An example are **IR–problems**, for instance appearing in QED [5, Chap. 6]. Those are removed by a re–arrangement of sums, leading to so–called **inclusive cross–sections**, which

<sup>30</sup>A diagram is called “connected”, if one can reach any vertex from any other vertex by a path of consecutive lines. It is called “disconnected” if it is not connected. See Figure 1.11.

<sup>31</sup>A simple example of products of distributions being ill–defined is the following: Consider  $f(\mathbf{x}) = g(\mathbf{x}) = \delta(\mathbf{x})$  with  $\mathbf{x} \in \mathbb{R}^d$ . Then, there is no way to define  $(f \cdot g)(\mathbf{x})$  in a reasonable way, since the expression for its Fourier transform  $\mathcal{F}(f \cdot g)(\mathbf{p}) = (\hat{f} * \hat{g})(\mathbf{p}) = \int \hat{f}(\mathbf{p} - \mathbf{k})g(\mathbf{k}) d\mathbf{k} = \int (2\pi)^{-1} d\mathbf{k}$  contains a divergent integral. The divergences in pQFT are of the same spirit, but significantly more involved.

is formally equivalent to applying a second dressing transformation.

Depending on the QFT model, a combination of these three methods can be used to obtain well-defined expressions for the kernels of  $U_{I,n}(t, t_0)$  and  $S_n$ . Often, the second modification (renormalization of distributions) is considered the most important step in eliminating divergences and simply called “renormalization” in QFT books [5, 6, 7].



Figure 1.11: Left: Two disconnected Feynman diagrams for computing  $S_n, n = 4$ . Expressions corresponding to such diagrams are removed when assuming the Gell–Mann and Low formula to hold. Right: Two connected Feynman diagrams for  $S_n, n = 4$ .

In the following, we shortly explain, how these three ad hoc modifications are heuristically motivated and rigorously performed. Before we do so, let us shortly mention that for the first two modification methods, there exists a clean mathematical formulation in the framework of **perturbative algebraic quantum field theory (pAQFT)**. This approach encodes the above-mentioned distributions in functionals by identifying a distribution  $f \in \mathcal{S}'(\mathbb{R}^{n(d+1)})$  with the functional

$$F : \mathcal{S}(\mathbb{R}^{d+1}) \rightarrow \mathbb{C}, \quad F[\phi] = \int f(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n) dx_1 \dots dx_n. \quad (1.163)$$

The description of functionals is more general than the one by distributions and allows for taking functional derivatives. This way, one may also put various heuristic calculations from physics on rigorous grounds, which are deriving pQFT results by the so-called path integral approach, see [7, 8], [5, Chap. 9]. For an understandable introduction into pAQFT, we refer the reader to [147, 148].

We also remark, that there exist further perturbative techniques for extracting physical predictions, which we do not discuss here. An example are Wilson renormalization group (RG) techniques [6, Chap. 23], [5, Chap. 12].

## Gell–Mann and Low Formula

The test functions  $\Psi \in \mathcal{S}$  for obtaining the kernels of  $U_{I,n}(t, t_0), S_n$  can formally be written as

$$\Psi = \left( \sum_{N \in \mathbb{N}_0} \int \Psi^{(N)}(\mathbf{x}_1, \dots, \mathbf{x}_N) a_{\mathbf{x}_1}^\dagger \dots a_{\mathbf{x}_N}^\dagger d\mathbf{x}_1 \dots d\mathbf{x}_N \right) \Omega_0, \quad (1.164)$$

with  $\Omega_0$  being the “non–interacting vacuum”, i.e., the ground state of  $H_0$ . One would now like to perform a dressing operation, replacing  $\Omega_0$  by the “physical vacuum” or “dressed vacuum”  $\Omega$ , which is formally the ground state of  $H(t)$ . It is known that for a given  $H_I$  that is Kato–bounded against  $H_0$ , a transition from  $\Omega_0$  to  $\Omega$  can be achieved by evolving the system in time while slowly switching on the interaction [149]. That means, one sets  $H(t) = H_0 + g(\varepsilon t)H_I$  with a sufficiently regular **adiabatic switching function**  $g(t) : (-\infty, 0] \rightarrow [0, 1]$ , such that  $\lim_{t \rightarrow -\infty} g(t) = 0, g(0) = 1$ . Then, one takes  $\varepsilon \rightarrow 0$ .

In pQFT, this relation is heuristically assumed to hold, although mathematically,  $H_I$  is not even defined as an operator. A formal computation leads to the Gell–Mann and Low formula [150, (10)], [5, (4.29)], [6, (7.53)], as well as its equivalence to omitting all disconnected diagrams from the expansion [5, (4.27)–(4.57)], see Figure 1.11. This heuristically derived omission is rigorously realized by simply removing those diagrams from the sum over diagrams contributing to  $S_n$ .

However, also for non–perturbative renormalization, knowledge about this perturbative renormalization technique might become useful. For instance, one may imagine to use formal calculations in order to find a suitable physical vacuum state  $\omega : \mathcal{A} \rightarrow \mathbb{C}$  corresponding to  $\Omega$ , that serves for a GNS construction. Or one might find an expression for  $\Omega$ , that can be interpreted as an element of a Fock space extension. This could serve for defining a dressing transformation  $W$  beyond Fock space, mapping  $\Omega_0 \mapsto \Omega$ , and allowing for a renormalization as described in (1.2) and performed later in Chapters 4 and 5.

## Renormalization of Distributions

To make sense of the formal and ill–defined product of distributions  $I_n$  appearing in (1.161), Schwinger, Dyson, Feynman and Salam have proposed a subtraction mechanism [151, 152, 153, 154, 155] which consists of a systematic subtraction of infinite expressions.

One way to make mathematical sense of this subtraction formalism has been

established by Bogoliubov, Parasiuk, Hepp and Zimmermann [143, 145, 146] and is correspondingly called **BPHZ renormalization**. It relies on the fact that the distribution product  $I_n(q_i, q, q_f)$  is only ill-defined at configurations where  $x_{\ell,a} = x_{\ell,b}$ . That means, the formal expression  $I_n$  provides a well-defined linear functional  $\mathcal{S}_m \rightarrow \mathbb{C}$ , where  $\mathcal{S}_m$  is the space of Schwartz functions, which vanish together with all derivatives of order  $\leq m$  at configurations where  $x_{\ell,a} = x_{\ell,b}$  [143, Satz 2]<sup>32</sup>.

Establishing  $I_n \in \mathcal{S}'_m$  can be done by a process similar to non-perturbative cutoff renormalization: One replaces the Feynman propagator  $\Delta_\ell^F$  by a regularized expression  $\Delta_{\ell,M,\varepsilon}^F$ , such that the distribution multiplication gets well-defined at  $x_{\ell,a} = x_{\ell,b}$ , resulting in a well-defined functional  $I_{n,M,\varepsilon} \in \mathcal{S}' \subset \mathcal{S}'_m$ . The replacement of  $\Delta_\ell^F$  by  $\Delta_{\ell,M,\varepsilon}^F$  is also called **Pauli–Villars regularization** [156] and the original  $\Delta_\ell^F$  is recovered as the Limit  $M \rightarrow \infty, \varepsilon \rightarrow 0$ . On test functions  $f \in \mathcal{S}_m$ , one can then directly define

$$I_n(f) := \lim_{\substack{M \rightarrow \infty \\ \varepsilon \rightarrow 0}} I_{n,M,\varepsilon}(f), \quad (1.165)$$

i.e., we have a weak-\* convergence [143, Satz 2].

In order to extend the distribution  $I_n$  to all test functions  $f \in \mathcal{S}$ , one may use a subtraction map  $R$ . This  $R$  is defined on certain expressions corresponding to a diagram, such as  $I_n$  in (1.161) (for instance,  $I_{n,M,\varepsilon}$  is such an expression), and maps them to a distribution in  $\mathcal{S}'$  by subtracting certain “perturbative counterterms” based on a Taylor expansion in momentum space [143, §3], [145, Sect. 2], [146, Sect. 2]. By a sophisticated choice of  $R$ , the following limit exists in the weak-\* topology [143, Satz 5], [145]:

$$\tilde{I}_n := \lim_{\substack{M \rightarrow \infty \\ \varepsilon \rightarrow 0}} R(I_{n,M,\varepsilon}). \quad (1.166)$$

$\tilde{I}_n$  is then the renormalized distribution corresponding to a certain diagram, which may further be used to establish renormalized kernels  $\tilde{U}_{I,n}(t, t_0)$  or  $\tilde{S}_n$ . The statement of existence of  $\tilde{I}_n$  as in (1.166) is also called **BPHZ theorem**.

An alternative to BPHZ renormalization, which achieves the subtraction without cutoffs, has been established for  $\Delta_\ell^F$  corresponding to massive particles by Epstein and Glaser [157], and developed by Blanchard and Seneor [158] to treat also  $\Delta_\ell^F$  corresponding to massless particles. For a pedagogical introduction to this so-

<sup>32</sup>In the original formulation,  $I_n$  is even defined on the space of such test functions  $f$  with  $x^\alpha \partial^\beta f(x)$  being bounded only for multi-indices  $|\alpha| \leq r, |\beta| \leq q$  with sufficiently large  $r, q \in \mathbb{N}$ . This test function space, called  $\mathcal{D}_{r,q}$ , is larger than  $\mathcal{S}$ , as  $\mathcal{S} = \bigcup_{r,q \in \mathbb{N}} \mathcal{D}_{r,q}$ .

## 1 Introduction

called **Epstein–Glaser renormalization**, see [142, Chap. 3].

Here, one directly constructs the time–ordered operator–valued distributions

$$T_n(x_1, \dots, x_n) = \mathcal{T}(H_I(x_1) \dots H_I(x_n)), \quad (1.167)$$

appearing in (1.156), by an induction over  $n$ . Just as the BPHZ renormalization, the Epstein–Glaser construction also involves a removal of terms in a Taylor expansion in momentum space.

We remark that the renormalization of distributions is mathematically well–investigated [148, Sect. 7]. There even exists a “main theorem of renormalization” [147, Thm. 6], which classifies all renormalized distributions that can appear, when requiring a reasonable set of axioms.

The subtraction mechanism is sometimes also heuristically expressed by including infinite counterterms in the formal Lagrangian (which corresponds to the formal Hamiltonian  $H$ ) [5, Chap. 10], [8, Chap. 9], [152, (83)–(85)]. There exist **charge**, **mass**, and **self–energy renormalization** terms, as well as infinite **wave function renormalization factors**. These terms are similar to the counterterms in non–perturbative renormalization and may hence provide useful heuristics for finding suitable non–perturbative counterterms.

### IR Problem Remedy

Even when the formal  $I_n$  can be established as a well–defined distribution  $\tilde{I}_n$ , there is no guarantee that an integral over  $\tilde{I}_n$ , as in (1.161), converges. Formally, a contribution to a renormalized version of  $U_{I,n}(t, t_0)$  (or  $S_n$ , if the  $q_i, q_f$  are neglected) would be proportional to

$$\tilde{J}_n(q_i, q_f) = \int_{q: t_j \in [t_0, t]} \tilde{I}_n(q_i, q, q_f) dq = \int \chi_{\{q \mid t_j \in [t_0, t]\}} \tilde{I}_n(q_i, q, q_f) dq. \quad (1.168)$$

The indicator function  $\chi_{\{q \mid t_j \in [t_0, t]\}}$  is not in  $\mathcal{S}$ , but only in  $\mathcal{E} = C^\infty$ , so (1.168) does not necessarily render a well–defined distribution pairing. The integral may diverge at large  $|\mathbf{x}_j|$ , i.e., one might encounter an IR–divergence. This problem occurs, for instance, in QED. See [159] for a thorough mathematical discussion.

A common heuristic modification made to remedy this divergence problem is the use of **inclusive cross–sections** as introduced in [160], see also [5, Chap. 6]. Here, one introduces a suitable regularization indexed by  $\mu > 0$  (e.g., by adding a mass to particles as photons, that are physically expected to be massless), such

## 1.5 Perturbative QFT as a Source of Heuristics

that the integrals in  $\tilde{J}_{n,\mu}$  are finite and lead to a well-defined distribution  $\tilde{S}_{n,\mu}$ . Then, a new distribution  $\tilde{S}'_{n,\mu}(\mathbf{q}_i, \mathbf{q}_f)$  is defined as a linear combination of integrals over different orders  $n' > n$  of the original distribution,  $\tilde{S}'_{n',\mu}(\mathbf{q}_i, \mathbf{q}_f, \mathbf{x}_1, \dots, \mathbf{x}_{n'-n})$ . The integral is taken in momentum space over configurations with low momenta  $|\mathbf{p}_1|, \dots, |\mathbf{p}_{n'-n}|$ , which reminds strongly about a dressing transformation leading outside Fock space [159]. This dressing transformation was introduced by Chung, Kibble, Faddeev and Kulish and can be implemented on the ITP space. We further discuss it in 3.1.2, as it can also become interesting for a non-perturbative renormalization of QFT models.

In perturbative renormalization, it is known for small  $n$ , that the inclusive cross-section method produces well-defined results for  $\tilde{S}'_{n,\mu}$ . However, as of 2021, there is no mathematical proof that finite results can be achieved for all  $n \in \mathbb{N}$  [159, Sect. 2].



## 2 Hypersurface Evolution

This chapter contains the results concerning Born's rule on arbitrary Cauchy surfaces, which is obtained in an axiomatic framework called hypersurface evolution. Recent interest in this framework emerged from a series of investigations about multi-time wave functions (MTWFs), which describe pure quantum states within relativistic quantum dynamics in the Schrödinger picture. We shortly outline the concept of MTWFs in Section 2.1.

The extraction of physical predictions from a MTWF naturally leads to the framework of hypersurface evolution, presented and discussed in Section 2.2.

In the following Section 2.3, we explain why a proof of Born's rule on Cauchy surfaces is necessary in this setting and present our main Theorem 2.3.7, which directly implies our main result, Corollary 2.3.8. This result roughly states that assuming Born's rule on all flat Cauchy surfaces  $E$  implies Born's rule on any curved Cauchy surface  $\Sigma$ .

The proof is split in two steps: In Section 2.4, we prove that the flat Born rule on  $E$  implies the Born rule on triangular Cauchy surfaces  $\Upsilon$  (see Definition 2.3.1). Then we show in Section 2.5 that any Cauchy surface  $\Sigma$  can be approximated by a sequence of triangular Cauchy surfaces  $(\Upsilon_n)_{n \in \mathbb{N}}$ , which allows for recovering the Born rule on  $\Sigma$  from the Born rules on  $(\Upsilon_n)_{n \in \mathbb{N}}$  in Section 2.6.

It is an interesting open question, in which cases a hypersurface evolution allows for recovering a Wightman field theory. In Section 2.7, we provide some ideas of how this question might be answered in the future.

### 2.1 Multi-Time Wave Functions

Multi-time wave functions naturally arise when formulating relativistic quantum dynamics in the Schrödinger picture. One of the first descriptions of MTWFs was made as early as 1932 by Dirac [161], and by Dirac, Fock and Podolsky [162]. Recent mathematical results about MTWFs can be found in [163, 164, 165, 166, 167, 168]. More resources and a thorough discussion of MTWFs are given in [128, 169, 170]. For a pedagogical introduction to MTWFs, we refer the reader to [128].

## 2 Hypersurface Evolution

Consider a time-dependent family of Fock space vectors  $(\Psi_t)_{t \in \mathbb{R}} \subset \mathcal{F}$ . Each  $\Psi_t$  corresponds to an  $L^2$ -equivalence class of functions<sup>1</sup> on configurations  $q \in \mathcal{Q}$ ,  $q = (\mathbf{x}_1, \dots, \mathbf{x}_N)$ . At time  $t$ , with respect to one frame of reference, this  $q$  corresponds to the spacetime configuration

$$\tilde{q} = (x_1, \dots, x_N) = (t, \mathbf{x}_1, \dots, t, \mathbf{x}_N). \quad (2.1)$$

A change of the reference frame by a Poincaré transformation  $g \in \mathcal{P}_+^\uparrow$  now relates  $\tilde{q}$  to the spacetime configuration

$$\tilde{q}' = (gx_1, \dots, gx_N) = (t'_1, \mathbf{x}'_1, \dots, t'_N, \mathbf{x}'_N). \quad (2.2)$$

If a family of wave functions  $(\Psi'_t)_{t \in \mathbb{R}}$  was given in the new coordinate frame, then the function  $\Psi_t = U(g^{-1})\Psi'_t$  would intuitively be given by evaluating  $\Psi'$  at the Poincaré-transformed coordinates:

$$\Psi_t(\mathbf{x}_1, \dots, \mathbf{x}_N) = \Psi'(t'_1, \mathbf{x}'_1, \dots, t'_N, \mathbf{x}'_N). \quad (2.3)$$

The mathematical problems with this expression can easily be seen by the missing time index of  $\Psi'$ , replaced by several time coordinates  $t'_1, \dots, t'_N$ . In order to make sense of such formal transformation laws, it is necessary to introduce a wave function that depends on several time coordinates, i.e., a **multi-time wave function**

$$\Phi(x_1, \dots, x_N) = \Phi(t_1, \mathbf{x}_1, \dots, t_N, \mathbf{x}_N) = \Phi(\tilde{q}). \quad (2.4)$$

That means,  $\Phi$  is a complex-valued function on **configuration spacetime**  $\mathcal{Q}(\mathbb{M}) \cong \mathcal{Q}(\mathbb{R}^4)$  (compare (1.4)). Since after a Poincaré transformation (2.2), the spacetime coordinates are still spacelike, it suffices to define  $\Phi$  only on the set of **spacelike spacetime configurations** (see also Figure 2.1):

$$\mathcal{S} := \{\tilde{q} \in \mathcal{Q}(\mathbb{M}) \mid |t_j - t_k| < \|\mathbf{x}_j - \mathbf{x}_k\| \quad \forall j, k \in \{1, \dots, N\}, N \in \mathbb{N}_0\}, \quad (2.5)$$

in order to make sense of (2.3). Sometimes, in the definition (2.5), the “<” is replaced by a “≤”, depending on the literature resource. A given  $\Phi : \mathcal{S} \rightarrow \mathbb{C}$  allows for recovering the Fock space vector  $\Psi_t$  at any  $t \in \mathbb{R}$  via

$$\Psi_t(\mathbf{x}_1, \dots, \mathbf{x}_N) = \Phi(t, \mathbf{x}_1, \dots, t, \mathbf{x}_N). \quad (2.6)$$

So the MTWF  $\Phi$  provides a richer description of quantum dynamics than the family  $(\Psi_t)_{t \in \mathbb{R}}$ . However, this richness makes  $\Phi$  also more difficult to establish.

---

<sup>1</sup>As in Section 1.2.1, we denote both the Hilbert space vectors and the functions by  $\Psi_t$ . It will become clear from the context, which of both objects is meant.

Generally, for given initial data  $\Psi_0 \in \mathcal{F}$ , a MTWF is obtained as a solution to a set of Schrödinger-type PDEs: Since there are now  $N$  time coordinates on each  $(N)$ -sector, it is possible to take  $N$  distinct time derivatives  $\partial_{t_1}, \dots, \partial_{t_N}$ , each corresponding to a distinct Schrödinger equation

$$i\partial_{t_j}\Phi(\tilde{q}) = \left(\tilde{H}_j\Phi\right)(\tilde{q}). \quad (2.7)$$

Here,  $\tilde{H}_j$  is called the **partial Hamiltonian** belonging to particle  $j$ , and has to be defined on a space of sufficiently regular functions  $\Phi : \mathcal{S} \rightarrow \mathbb{C}$ , see for instance [168, Sect. 2.3.2] for the definition of such a regular MTWF space. In order to recover the single-time Schrödinger equation  $i\partial_t\Psi_t = H\Psi_t$ , it is necessary that

$$\left(\sum_{j=1}^N \tilde{H}_j\Phi\right)(t, \mathbf{x}_1, \dots, t, \mathbf{x}_N) = (H\Psi_t)(\mathbf{x}_1, \dots, \mathbf{x}_N), \quad (2.8)$$

for all  $\Phi$  entailing a sufficiently regular  $\Psi_t$ . In general, establishing sufficient regularity of  $\Phi$  is a considerable challenge when defining MTWFs, but indispensable for the recovery of  $\Psi_t$ , since an  $L^2$ -function can only be unambiguously evaluated at a point if sufficient regularity requirements hold. E.g., if  $\Phi \in C^0$ .

The equations (2.7) now define a PDE system, which is posed at all  $\tilde{q} \in \mathcal{S}$ . The number  $N$  of PDEs varies with  $\tilde{q}$ . Together with the initial condition

$$\Phi(0, \mathbf{x}_1, \dots, 0, \mathbf{x}_N) = \Psi_0(\mathbf{x}_1, \dots, \mathbf{x}_N), \quad (2.9)$$

this PDE system forms an **Initial Value Problem (IVP)**, whose solution  $\Phi : \mathcal{S} \rightarrow \mathbb{C}$  (if it exists) is the MTWF describing the complete dynamics of the quantum system.

## 2.2 Hypersurface Evolution

### 2.2.1 From MTWFs to Hypersurface Wave Functions

For a given MTWF  $\Phi : \mathcal{S} \rightarrow \mathbb{C}$  solving the above-explained IVP, the question now arises how to extract physical predictions. Generally,  $\Phi \notin L^2(\mathcal{S})$ , so  $\Phi$  does not directly serve for a probability interpretation via Born's rule. However, (2.6) allows for recovering  $\Psi_t \in \mathcal{F}$  for any  $t \in \mathbb{R}$ , where  $|\Psi_t(\mathbf{x}_1, \dots, \mathbf{x}_N)|^2$  can be interpreted as the probability density for a detection of the system in configuration  $q = (\mathbf{x}_1, \dots, \mathbf{x}_N)$  at time  $t$ .

## 2 Hypersurface Evolution

In fact, given any Cauchy surface  $\Sigma \subset \mathbb{M} \cong \mathbb{R}^4$ , and a sufficiently regular  $\Phi$ , it is possible to recover a Fock space vector [169, (8)]

$$\Psi_{\Sigma}(x_1, \dots, x_N) = \Phi(x_1, \dots, x_N) \quad x_j \in \Sigma. \quad (2.10)$$

Here,  $q_{\Sigma} := (x_1, \dots, x_N)$  can be considered an element of the configuration space  $\mathcal{Q}(\Sigma)$  and  $\Psi_{\Sigma}$  maps  $\mathcal{Q}(\Sigma) \rightarrow \mathbb{C}$ . Since  $\Sigma$  is a Cauchy surface, the points  $x_1, \dots, x_N$  are pairwise spacelike, meaning that  $\mathcal{Q}(\Sigma) \subset \mathcal{S}$ . So  $\Phi(q_{\Sigma}) \in \mathbb{C}$  is indeed well-defined.

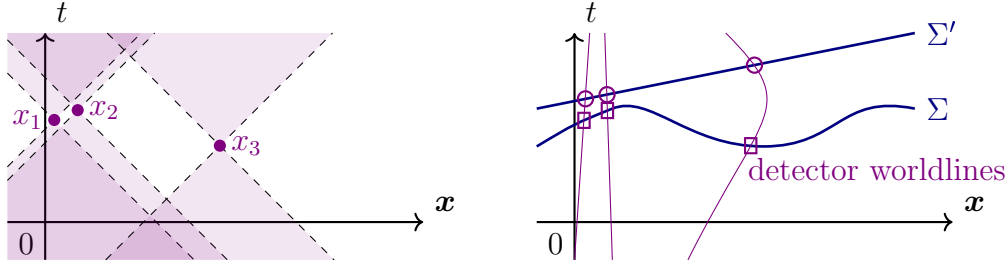


Figure 2.1: Left: Configurations  $(x_1, \dots, x_N)$  in a MTWF  $\Phi$  must be spacelike.

Right: Position measurement along hypersurfaces. Color online.

Physically,  $\Psi_{\Sigma}$  is related to **position measurements along the Cauchy surface**  $\Sigma$ . Such a measurement could be realized by taking a set of detectors moving on (timelike) world lines and initiating a detection on each of it, whenever its world line crosses  $\Sigma$ . This situation is depicted for a curved ( $\Sigma$ ) and a flat ( $\Sigma'$ ) Cauchy surface in Figure 2.1. The requirement that  $\Sigma$  be a Cauchy surface makes sure that this crossing occurs exactly once.

In order to give the  $\Psi_{\Sigma}$ , derived from a MTWF, a probability interpretation in the sense of a Born rule, it is necessary to define a measure  $\mu_{\Sigma}$  on  $\mathcal{Q}(\Sigma)$  for each  $\Sigma$ , such that

$$\mathbb{P}(B) := \int_B |\Psi_{\Sigma}(x_1, \dots, x_N)|^2 d\mu_{\Sigma}, \quad B \subset \mathcal{Q}(\Sigma) : \text{measurable} \quad (2.11)$$

can be seen as the probability of finding the system in a configuration  $q_{\Sigma} \in B$ . In particular,  $\int_{\Sigma} |\Psi_{\Sigma}|^2 d\mu_{\Sigma} = 1$  must hold for any  $\Sigma$  and any given  $\Phi$ . The definition of such a  $\mu_{\Sigma}$  is a non-trivial task. A possible definition for Dirac particles can easily be derived from [169, (10)] or [170, (1.60)], using a probability density  $\rho(x_1, \dots, x_N)$  with respect to the 3-metric on  $\Sigma$ , obtained as a pullback of the Minkowski metric  $\eta$  on  $\mathbb{M}$ . The measure  $\mu_{\Sigma}$  defines a scalar product on functions  $\Psi_{\Sigma}$ , allowing to interpret them as vectors in a Hilbert space  $\mathcal{H}_{\Sigma}$ . So a MTWF may

allow for recovering not just a family  $(\Psi_t)_{t \in \mathbb{R}}$  corresponding to dynamics, but an even larger family  $(\Psi_\Sigma)$  with  $\Sigma$  being any Cauchy surface.

The idea of expressing quantum dynamics by a hypersurface–dependent family of vectors  $(\Psi_\Sigma)$  was already put forward in the 1940s by Tomonaga [171] and Schwinger [151], both using the interaction picture. See also [172, Chap. 13]. In contrast to the usual description of dynamics by a family  $(\Psi_t)_{t \in \mathbb{R}}$  in the Schrödinger or the interaction picture, the hypersurface description by a family  $(\Psi_\Sigma)$  does not depend on a fixed frame of reference. Here, the evolution between two hypersurfaces, say from  $\Sigma$  to  $\Sigma'$ , is described by a unitary operator  $U_\Sigma^{\Sigma'}$  via  $\Psi_{\Sigma'} = U_\Sigma^{\Sigma'} \Psi_\Sigma$ . So the family  $(U_\Sigma^{\Sigma'})$  with  $\Sigma, \Sigma'$  running through all Cauchy surfaces replaces the family of evolution operators  $(U(t))_{t \in \mathbb{R}}$ . It is also possible to construct the family  $(U_\Sigma^{\Sigma'})$  without having a MTWF, but a given MTWF allows for a convenient reconstruction of this family, see for instance [169, Assertion 4], [170, (1.80)], [129, (13)].

### 2.2.2 Axiomatic Setting

Our hypersurface evolution framework now uses a further level of abstraction, which is the same as in [4]:  $\mathcal{H}_\Sigma$  is no longer required to be a Fock space, but can be a general Hilbert space. Due to the missing Fock space structure, position measurement probabilities  $\mathbb{P}(B)$  can no longer be defined by a measure  $\mu_\Sigma$  as in (2.11). Instead, it is necessary to define a PVM  $P_\Sigma(\cdot)$  sending each Borel–measurable set  $B$  to the projection operator  $P_\Sigma(B) : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_\Sigma$ . By permutation symmetry,  $P_\Sigma$  may also be defined on measurable subsets of the unordered configuration space  $\Gamma(\Sigma) \supset B$  (defined in (1.8)). Its elements will simply be denoted  $q = \{x_1, \dots, x_N\}$  in this chapter.

In order to define the Borel  $\sigma$ –algebra  $\mathcal{B}(X)$  on some set  $X$  (such as  $\Gamma(\Sigma)$ ), it is necessary to have a topology on  $X$ . On  $\Sigma$ , such a topology is induced by the Euclidean  $\mathbb{R}^4$ –norm on  $\mathbb{M}$ . Restricting the projection  $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  as in (2.32) to  $\Sigma$ , we obtain a homeomorphism  $\pi_\Sigma = \pi|_\Sigma : \Sigma \rightarrow \mathbb{R}^3$ , which can be used to identify  $\mathcal{B}(\Sigma)$  with  $\mathcal{B}(\mathbb{R}^3)$ : For  $R \subseteq \Sigma$ , we have that  $R \in \mathcal{B}(\Sigma) \Leftrightarrow \pi(R) \in \mathcal{B}(\mathbb{R}^3)$ . By Rademacher’s theorem,  $\Sigma$  possesses a tangent plane almost everywhere [4, Sect. 3]. If a tangent plane exists at  $x \in \Sigma$ , the pullback of  $\eta$  under the embedding  $\Sigma \hookrightarrow \mathbb{M}$  is either degenerate or a Riemann 3–metric. This metric can be used to define a volume measure  $\mu_\Sigma$  on  $(\Sigma, \mathcal{B}(\Sigma))$ , as well as a volume measure  $\mu_{\Gamma(\Sigma)}$  on  $(\Gamma(\Sigma), \mathcal{B}(\Gamma(\Sigma)))$ . Note that for disjoint sets  $A \cap B = \emptyset$ , we have

$$\Gamma(A \cup B) \cong \Gamma(A) \times \Gamma(B), \tag{2.12}$$

## 2 Hypersurface Evolution

with bijective identification map  $q \mapsto (q \cap A, q \cap B)$ . The following notation for sets on  $\Gamma(\Sigma)$  will be convenient: for any subset  $A \subseteq \Sigma$ , let

$$\begin{aligned}\emptyset(A) &:= \{q \in \Gamma(\Sigma) \mid q \cap A = \emptyset\} \\ \exists(A) &:= \{q \in \Gamma(\Sigma) \mid q \cap A \neq \emptyset\} \\ \forall(A) &:= \{q \in \Gamma(\Sigma) \mid q \subseteq A\}\end{aligned}\tag{2.13}$$

be the sets of configurations with no, at least one, or all particles in  $A$  (see Figure 2.2). Note that  $\exists(A)^c = \emptyset(A) = \forall(A^c)$ , where  $A^c$  means the complement of  $A$  with respect to  $\Sigma$ . We also briefly write  $\forall A$  for  $\forall(A)$ , and similarly  $\exists A$  and  $\emptyset A$ .

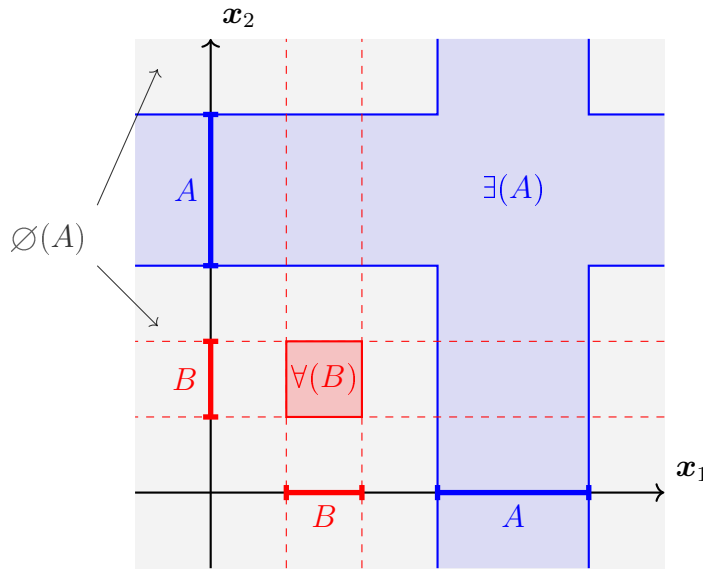


Figure 2.2: The sets  $\emptyset(\cdot)$ ,  $\exists(\cdot)$  and  $\forall(\cdot)$  on the (2)–sector of configuration space, visualized. Color online.

A probability for position measurements can then be extracted by the **Born rule**:

$$\mathbb{P}(B) = \langle \Psi_\Sigma, P_\Sigma(B) \Psi_\Sigma \rangle.\tag{2.14}$$

(2.14) is called a **curved Born rule**, if it holds on all (possibly curved) Cauchy surfaces  $\Sigma$ . So in this more abstract setting, the entire quantum dynamics are described by the three families

- $\mathcal{H}_\circ := (\mathcal{H}_\Sigma)$  of Hilbert spaces,
- $U_\circ^\circ := (U_\Sigma^{\Sigma'})$  of evolution operators,

- $P_\circ := (P_\Sigma)$  of PVMs,

with  $\Sigma, \Sigma'$  running through all Cauchy surfaces. Following [4], we call the triple  $\mathcal{E} = (\mathcal{H}_\circ, U_\circ, P_\circ)$  a **hypersurface evolution**, if it satisfies a set of desirable properties. Just as the Haag–Kastler or Wightman axioms, this “wishlist of properties” is not considered unchangeable, but may be subject to refinements as research on the subject progresses. Our list comprises the following properties:

- (1) **Absolute continuity of the PVMs:**  $P_\Sigma(B) = 0$  for all null sets<sup>2</sup>  $B \subset \Gamma(\Sigma)$ .
- (2) **Naturality:**  $U_{\Sigma'}^{\Sigma''} U_\Sigma^{\Sigma'} = U_\Sigma^{\Sigma''}$  and  $U_\Sigma^\Sigma = I_\Sigma$  with  $I_\Sigma$  being the identity on  $\mathcal{H}_\Sigma$ .
- (3) **Unique vacuum:**  $\text{Ran}(P_\Sigma(\{\emptyset\})) = \text{span}\{\Omega_\Sigma\}$  for some  $0 \neq \Omega_\Sigma \in \mathcal{H}_\Sigma$ .
- (4) **Factorization:** For each measurable  $A \subseteq \Sigma$ , there exists a unitary isomorphism  $T_{\Sigma,A} : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{\Sigma,A} \otimes \mathcal{H}_{\Sigma,\Sigma \setminus A}$  (“translation”), such that for any  $A, A' \subseteq \Sigma$ : measurable, we have<sup>3</sup>

$$P_\Sigma(\forall(A')) = T_{\Sigma,A}^{-1} [P_\Sigma(\forall(A \cap A')) \otimes P_\Sigma(\forall(A^c \cap A'))] T_{\Sigma,A}. \quad (2.15)$$

Here,  $\mathcal{H}_{\Sigma,A} := \text{Ran} P_\Sigma(\forall(A))$ .

- (5) **Permutation invariance:**  $T_{\Sigma,\Sigma \setminus A} = \Pi T_{\Sigma,A}$ .

Here,  $\Pi : \mathcal{H}_{\Sigma,A} \otimes \mathcal{H}_{\Sigma,\Sigma \setminus A} \rightarrow \mathcal{H}_{\Sigma,\Sigma \setminus A} \otimes \mathcal{H}_{\Sigma,A}$  with  $\Pi(\Psi_1 \otimes \Psi_2) = \Psi_2 \otimes \Psi_1$  is the permutation operator for two tensor factors.

In the original formulation of the hypersurface evolution setting [4], the factorization property is formulated without referring to a unitary isomorphism  $T$ . Correspondingly, the permutation invariance property does not appear. Our formulation above can hence be seen as a refinement of that one in [4].

The family of factorization maps  $(T_{\Sigma,A})$  could also be included into  $\mathcal{E}$ , as it is a further mathematical object necessary to define a hypersurface evolution. However, we rather interpret the factorization as an intrinsic property of the Hilbert spaces  $\mathcal{H}_\Sigma$  and do not include it explicitly in  $\mathcal{E}$ .

Further, we will often follow [4] and not make the isomorphism  $T$  explicit; that is, instead of saying “the given unitary isomorphism  $T_{\Sigma,A}$  maps  $\mathcal{H}_\Sigma$  to  $\mathcal{H}_{\Sigma,A} \otimes \mathcal{H}_{\Sigma,\Sigma \setminus A}$ ,” we simply say “ $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma,A} \otimes \mathcal{H}_{\Sigma,\Sigma \setminus A}$ .” Likewise, instead of (2.15), we simply write

$$P_\Sigma(\forall(B)) = P_A(\forall(A \cap B)) \otimes P_{A^c}(\forall(A^c \cap B)), \quad (2.16)$$

<sup>2</sup>The term “null set” is to be understood with respect to the volume measure  $\mu_{\Gamma(\Sigma)}$  defined above.

<sup>3</sup>Note that  $P_\Sigma$ , restricted to subsets of  $\forall(A)$ , maps  $\mathcal{H}_{\Sigma,A}$  to itself and in fact defines a PVM on  $\mathcal{H}_{\Sigma,A}$ .

## 2 Hypersurface Evolution

where  $P_A$  means the restriction of  $P_\Sigma$  to subsets of  $\forall(A)$  as in Footnote 3.

In order to express locality, we will add two separate properties on our wishlist, as in [4, Sect. 1.2.2]. To do so, we need to introduce some further notation:

**Definition 2.2.1.** Let  $\Sigma, \Sigma'$  be Cauchy surfaces and  $A \subseteq \Sigma$ . We then define the **grown set** of  $A$  in  $\Sigma'$  as (see Figure 2.3)

$$\text{Gr}(A, \Sigma') = [\text{future}(A) \cup \text{past}(A)] \cap \Sigma'. \quad (2.17)$$

Similarly, we define the **shrunk set** of  $A$  in  $\Sigma'$  as:

$$\text{Sr}(A, \Sigma') = \{x' \in \Sigma' \mid \text{Gr}(\{x'\}, \Sigma) \subseteq A\}. \quad (2.18)$$

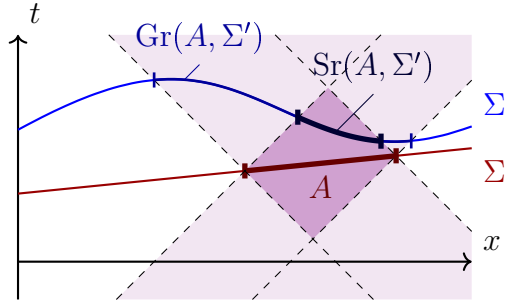


Figure 2.3: Grown and shrunk sets of  $A \subset \Sigma$ . Color online.

Our first locality assumption on  $\mathcal{E}$  is the following:

(6) **Propagation locality (PL)** asserts that

$$U_\Sigma^{\Sigma'} P_\Sigma(\forall A) U_{\Sigma'}^\Sigma \leq P_{\Sigma'}(\forall \text{Gr}(A, \Sigma')) \quad (2.19)$$

for all Cauchy surfaces  $\Sigma, \Sigma'$  and all  $A \subseteq \Sigma$ .

Heuristically, the projection  $P_\Sigma(\forall A)$  “does not spread faster than light”. Here,  $R \leq S$  means that  $S - R$  is a positive operator; if  $R$  and  $S$  are projections, then  $R \leq S$  is equivalent to  $\text{Ran}R \subseteq \text{Ran}S$ . In words, (PL) means that if  $\Psi_\Sigma$  is concentrated in  $A \subseteq \Sigma$ , i.e.,  $\Psi_\Sigma \in \mathcal{H}_{\Sigma, A}$ , then  $\Psi_{\Sigma'} = U_\Sigma^{\Sigma'} \Psi_\Sigma$  is concentrated in  $\text{Gr}(A, \Sigma')$ . This definition is equivalent to the “finite propagation speed” (FS) assumption given in [4].

Also, the definition of our second assumption, called “interaction locality”, was already given in [4], but will be formulated here in a more detailed way. We begin

with a summary of the condition: First, in a region  $A$  where  $\Sigma$  and  $\Sigma'$  overlap (see Figure 2.4),  $\mathcal{H}_{\Sigma,A}$  and  $\mathcal{H}_{\Sigma',A}$  can be identified with each other. The identification fits together with  $P$  and  $T$ . Second, the time evolution from  $\Sigma \setminus A$  to  $\Sigma' \setminus A$  (see Figure 2.4) is given by a unitary isomorphism  $V_{\Sigma \setminus A}^{\Sigma' \setminus A} : \mathcal{H}_{\Sigma \setminus A} \rightarrow \mathcal{H}_{\Sigma' \setminus A}$ , the ‘‘local evolution’’ replacing  $U_{\Sigma'}^{\Sigma'}$ . The fact that one can evolve from  $\Sigma \setminus A$  to  $\Sigma' \setminus A$  means in particular that this evolution does not depend on the state in  $A$ , that is, there is no interaction term in the evolution that would couple  $\Sigma \setminus A$  to  $A$ . Finally, we require that  $V_{\Sigma \setminus A}^{\Sigma' \setminus A}$  does not change when we deform  $A$  while keeping it spacelike from  $\Sigma \setminus A$ .

(7) **Interaction locality (IL)** asserts that  $\mathcal{E}$  is equipped in addition with, for all Cauchy surfaces  $\Sigma, \Sigma'$  and  $A \subseteq \Sigma \cap \Sigma'$ , a unitary isomorphism  $J_{A,\Sigma}^{\Sigma'} : \mathcal{H}_{\Sigma,A} \rightarrow \mathcal{H}_{\Sigma',A}$  (‘‘identification’’) such that

$$\begin{aligned} J_{A,\Sigma'}^{\Sigma''} J_{A,\Sigma}^{\Sigma'} &= J_{A,\Sigma}^{\Sigma''} \text{ whenever } A \subseteq \Sigma \cap \Sigma' \cap \Sigma'', \\ J_{A',\Sigma}^{\Sigma'} &= J_{A,\Sigma}^{\Sigma'} \Big|_{\mathcal{H}_{\Sigma,A'}} \text{ for } A' \subseteq A \subseteq \Sigma \cap \Sigma', \\ (J_{A,\Sigma}^{\Sigma'})^{-1} P_{\Sigma'}(\forall(A')) J_{A,\Sigma}^{\Sigma'} &= P_{\Sigma}(\forall(A')) \text{ for } A' \subseteq A, \\ T_{\Sigma',A} U_{\Sigma}^{\Sigma'} T_{\Sigma,A}^{-1} &= J_{A,\Sigma}^{\Sigma'} \otimes V_{\Sigma \setminus A, \Sigma}^{\Sigma'}, \end{aligned} \tag{2.20}$$

with some unitary isomorphism  $V_{\Sigma \setminus A, \Sigma}^{\Sigma'} : \mathcal{H}_{\Sigma, \Sigma \setminus A} \rightarrow \mathcal{H}_{\Sigma', \Sigma \setminus A}$  such that for all  $\tilde{\Sigma} \supseteq (\Sigma \setminus A)$ , setting  $\tilde{A} := \tilde{\Sigma} \setminus (\Sigma \setminus A)$  and  $\tilde{\Sigma}' := \tilde{A} \cup (\Sigma' \setminus A)$ ,

$$V_{\Sigma \setminus A, \tilde{\Sigma}}^{\tilde{\Sigma}'} = J_{\Sigma' \setminus A, \Sigma'}^{\tilde{\Sigma}'} V_{\Sigma \setminus A, \Sigma}^{\Sigma'} J_{\Sigma \setminus A, \tilde{\Sigma}}^{\Sigma}. \tag{2.21}$$

Henceforth, we will not mention the  $J$ -operators explicitly anymore and following [4], we simply write

$$\mathcal{H}_{\Sigma,A} = \mathcal{H}_{\Sigma',A} =: \mathcal{H}_A. \tag{2.22}$$

Further, we write  $V_{\Sigma \setminus A}^{\Sigma' \setminus A}$  in place of  $V_{\Sigma \setminus A, \Sigma}^{\Sigma'}$ , which is compatible with the Hilbert space identification.

Properties (1)–(7) set up the axiomatic framework for a relativistic description of quantum dynamics using a hypersurface evolution  $\mathcal{E}$ . For physical reasons, it is desirable to require a further Property (8) of Poincaré covariance described in Section 2.7.1. However, Poincaré covariance is not necessary for proving the results in this chapter (or those in [4]).

**Remarks.**

## 2 Hypersurface Evolution

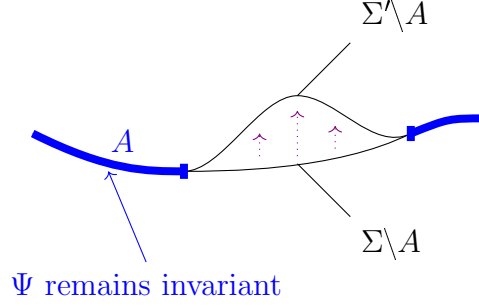


Figure 2.4: Depiction of interaction locality (IL). Color online.

1. *Uniqueness of the vacuum state.* Actually, the propositions and the theorem presented in this chapter do not make use of Property (3), the uniqueness of the vacuum state. The reason why we include this property into our wishlist is that it is part of the concept of a hypersurface evolution, as introduced in [4]. Further, this property may turn out useful in a future recovery of the Wightman axioms, as discussed in Section 2.7.
2.  *$P_\Sigma$  factorizes.* From (2.15) or (2.16) it follows that  $P_\Sigma$  factorizes not just for all-sets (i.e., sets of the form  $\forall(B)$ ) but for all product sets in configuration space: for all  $A \subseteq \Sigma$ ,  $B_A \subseteq \forall A$ , and  $B_{A^c} \subseteq \forall(\Sigma \setminus A)$ , we have

$$P_\Sigma(B_A \times B_{A^c}) = P_A(B_A) \otimes P_{A^c}(B_{A^c}), \quad (2.23)$$

with  $B_A \times B_{A^c}$  understood as a subset of  $\Gamma(\Sigma)$ . That is because, first,  $\forall(A') = \forall(A \cap A') \times \forall(A^c \cap A')$ , second, the all-sets  $\forall C$  form a  $\cap$ -stable generator of  $\mathcal{B}(\Gamma(\Sigma))$ , and third, it is a standard theorem in probability theory that measures (and hence also PVMs) agreeing on a  $\cap$ -stable generator of a  $\sigma$ -algebra agree on the whole  $\sigma$ -algebra; so, roughly speaking, relations true for all all-sets are true for all sets. Relation (2.23) is exactly the definition of the tensor product of two POVMs, so it can equivalently be expressed as

$$P_\Sigma = P_A \otimes P_{A^c}. \quad (2.24)$$

3. *Splitting into more than two regions.* The restriction  $T_{\Sigma, A', A}$  of  $T_{\Sigma, A}$  to  $\mathcal{H}_{\Sigma, A'}$  maps  $\mathcal{H}_{\Sigma, A'}$  unitarily to  $\mathcal{H}_{\Sigma, A \cap A'} \otimes \mathcal{H}_{\Sigma, A^c \cap A'}$ . Moreover, (2.15) for  $A \subseteq A'$  yields that  $P$  factorizes also in  $A'$ , i.e., for every  $A \subseteq A' \subseteq \Sigma$ ,  $B_A \subseteq \forall A$ , and  $B_{A' \setminus A} \subseteq \forall(A' \setminus A)$ ,

$$P_\Sigma(B_A \times B_{A' \setminus A}) = T_{\Sigma, A', A}^{-1} [P_\Sigma(B_A) \otimes P_\Sigma(B_{A' \setminus A})] T_{\Sigma, A', A}, \quad (2.25)$$

with  $B_A \times B_{A' \setminus A}$  understood as a subset of  $\forall(A')$ . Furthermore, it follows that  $T_{\Sigma, A', A' \setminus A} = \Pi T_{\Sigma, A', A}$ , and that an associative law holds for the  $T_{\Sigma, A', A}$ : For any partition  $A_1, A_2, A_3$  of  $A' \subseteq \Sigma$ ,

$$(I_{\Sigma, A_1} \otimes T_{\Sigma, A_2 \cup A_3, A_2}) T_{\Sigma, A', A_1} = (T_{\Sigma, A_1 \cup A_2, A_1} \otimes I_{\Sigma, A_3}) T_{\Sigma, A', A_1 \cup A_2}. \quad (2.26)$$

Hence, the Hilbert spaces and PVMs factorize also for finite partitions. The upshot is that it is OK to identify

$$\begin{aligned} \mathcal{H}_\Sigma &= \bigotimes_i \mathcal{H}_{\Sigma, A_i} \text{ and} \\ P_\Sigma &= \bigotimes_i P_{A_i}, \end{aligned} \quad (2.27)$$

for any finite partition  $\Sigma = \bigcup_i A_i$ .

4. *Examples for hypersurface evolutions  $\mathcal{E}$ .* Some examples for hypersurface evolutions can be found in [4]. As described there in Remark 15 and Section 4.1, the simplest example is provided by the non-interacting Dirac field without a Dirac sea, which also satisfies (IL) and (PL) as defined below. Further examples are provided by Tomonaga–Schwinger equations and MT-WFs, where a recovery procedure as in Section 2.2.1 is used. Explicit models include the emission–absorption model of [169] and the rigorous model with contact interaction in [165, 166].
5. *Other notions of locality.* There are several inequivalent (though not unrelated) concepts of locality; they often play important roles in selecting time evolution laws (e.g., [10, 142]).

In the Haag–Kastler and Wightman axioms (see Section 1.2.3), a locality condition appears that is different from both (IL) and (PL), viz., (anti-)commutation of field operators at spacelike separation. It seems clear that Wightman’s locality is closely related to (IL) and (PL). We discuss this relationship further in Section 2.7.3.

Another different locality condition is typically called Einstein locality or Bell locality or just locality. It implies (IL) and (PL) but is not implied by (IL) and (PL) together; it asserts that there are no influences between events in spacelike separated regions; that may sound similar to (IL), but it is not. In fact, Bell’s theorem [173, 174] shows that Bell locality is violated, whereas (IL) seems to be valid in our universe.

6. *Consistency condition.* It is known that multi-time equations require a con-

sistency condition (e.g., [128, Chap. 2]). We note here that neither (IL) nor (PL) follow from the consistency condition alone. Indeed, examples of (artificial) multi-time equations with an instantaneous interaction (violating (IL)) that leaves the multi-time equations consistent were given in Lemma 2.5 of [164], while the non-interacting multi-time equations with Schrödinger Hamiltonians  $-\Delta_j$  for each particle  $j$  provide an example of consistent multi-time equations violating (PL).

### 2.3 Deriving a Curved Born Rule

Note that none of the Properties (1)–(8) comprises a Born rule (2.14). In order to make physical predictions from  $\mathcal{E}$  by a probability interpretation, we may indeed postulate (2.14) on all Cauchy surfaces  $\Sigma$ , i.e., a curved Born rule. However, it is also possible to postulate (2.14) only on all flat Cauchy surfaces (also called Cauchy hyperplanes), i.e., a **flat Born rule**. In the rest of this chapter, we will reserve the letter  $E$  for Cauchy hyperplanes, with the special case of  $E_t = \{x \in \mathbb{M} \mid x^0 = t\}$  being a horizontal Cauchy surface. So the Born rule is assumed on all Poincaré transformed versions  $E$  of the time-zero surface  $E_0$ .

Our first objective is now to derive an expression for  $\mathbb{P}(B)$  on any Cauchy surface  $\Sigma$  from the flat Born rule. In that case, the flat Born rule together with the instruction how to derive  $\mathbb{P}(B)$  can be postulated as an alternative to the curved Born rule.

Our second objective is to prove that the “alternatively obtained” probabilities coincide with those predicted by the curved Born rule. We indeed establish a result of this kind in Corollary 2.3.8 as a direct consequence of Theorem 2.3.7.

#### 2.3.1 Previous Result

A theorem similar to ours has been proved by Lienert and Tumulka [4]; our result differs in what exactly is assumed, and how the detection process is modeled. The fact that the curved Born rule can be obtained through different models of the detection process and from different sets of assumptions suggests that it is a robust consequence of the flat Born rule.

In fact, our result was already conjectured by Lienert and Tumulka, who also suggested the essentials of the model of the detection process we use here, although their theorem concerned a different model. The biggest difference between their theorem and ours is that we assume the Born rule and collapse rule to hold on tilted hyperplanes, whereas Lienert and Tumulka assumed them only on horizontal hyperplanes in a fixed Lorentz frame.

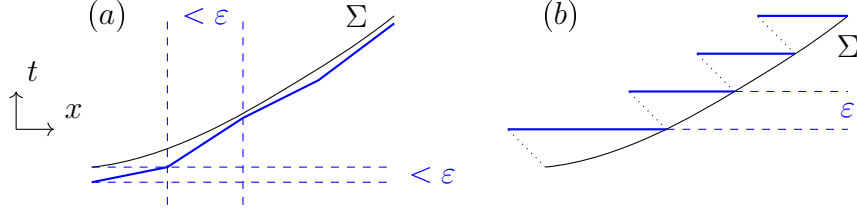


Figure 2.5: (a) Our detection process is based on approximating a curved surface  $\Sigma$  by a piecewise flat surface. (b) The detection process used by Lienert and Tumulka is based on approximating a curved surface  $\Sigma$  by disconnected pieces of horizontal surfaces. We have set the speed of light to  $c = 1$ . Color online.

Our model of the detection process is perhaps more natural than the one at the basis of Lienert and Tumulka’s theorem, as it approximates detectors on tilted surfaces through detectors on tilted hyperplanes, rather than on numerous small pieces of horizontal hyperplanes. On the other hand, the result of Lienert and Tumulka is stronger than ours in that it assumes the Born rule only on horizontal hyperplanes (“horizontal Born rule”) and not on all tilted spacelike hyperplanes (“flat Born rule”). Then again, our model allows for a somewhat simpler proof compared to that of Lienert and Tumulka, and the assumption of the Born and collapse rules on tilted hyperplanes seems natural if the workings of detectors are Lorentz invariant. Yet, our proof does not require the Lorentz invariance of the hypersurface evolution of the observed system (which follows from Property (8) in Section 2.7.1); in particular, the hypersurface evolution may involve external fields that break the Lorentz symmetry.

Other works in recent years dealing with a physical analysis of the quantum measurement process include [175, 176, 177, 178].

### 2.3.2 Detection Process

Our definition of the detection process is based on approximating any given Cauchy surface  $\Sigma$  by spacelike surfaces  $\Upsilon$  that are piecewise flat, and whose (countably many) flat pieces are 3d (non-regular) tetrahedra. See also Figure 2.6 for an illustration. This type of surface is defined as follows:

**Definition 2.3.1.** A **triangular surface** is a Cauchy surface  $\Upsilon \subset \mathbb{M}$  such that

$$\Upsilon = \bigcup_{k \in \mathcal{K}} \overline{\Delta_k}, \quad (2.28)$$

where  $\mathcal{K}$  is a countably infinite index set, each  $\Delta_k$  is a 3-open, non-degenerate,

## 2 Hypersurface Evolution

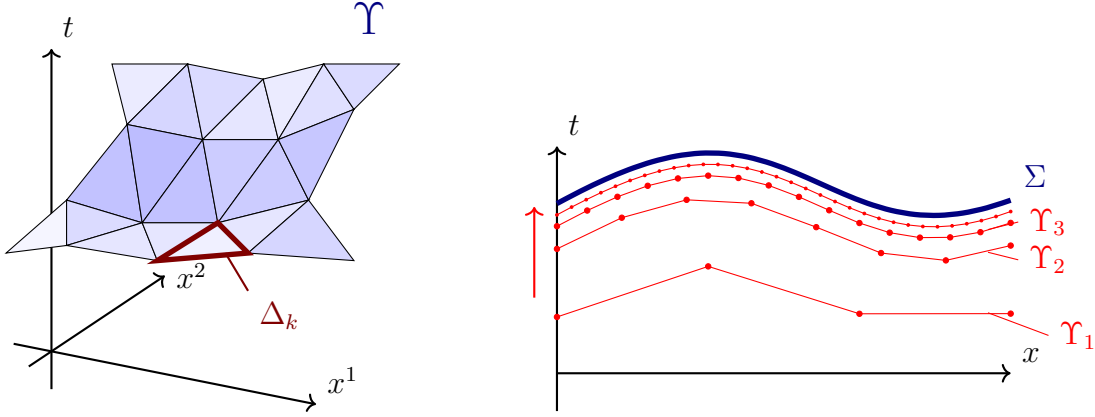


Figure 2.6: Left: Part of a triangular surface  $\Upsilon$  in  $1 + 2$  dim.

Right: A sequence of triangular surfaces  $\Upsilon_n$  covering increasingly and uniformly to  $\Sigma$  in  $1 + 1$  dim. Color online.

spacelike tetrahedron (i.e., the non-empty 3-interior of the convex hull of  $3 + 1$  points that are mutually spacelike), the  $\Delta_k$  are mutually disjoint ( $\Delta_{k_1} \cap \Delta_{k_2} = \emptyset$  for  $k_1 \neq k_2$ ), and every bounded region  $B \subset \Upsilon$  intersects only finitely many  $\Delta_k$ .

In Section 2.5, we will prove the following basic fact about triangular surfaces:

**Proposition 2.3.2.** *For every Cauchy surface  $\Sigma$  in Minkowski spacetime, there is a sequence  $(\Upsilon_n)_{n \in \mathbb{N}}$  of triangular Cauchy surfaces that converges increasingly and uniformly to  $\Sigma$ .*

Here, “increasingly” means that  $\Upsilon_{n+1} \subseteq \text{future}(\Upsilon_n)$  for all  $n$ ; see Figure 2.6. Uniform convergence in a given Lorentz frame means that for every  $\varepsilon > 0$ , all but finitely many  $\Upsilon_n$  lie in  $\{x + (s, 0, 0, 0) \mid x \in \Sigma, |s| < \varepsilon\}$ ; equivalently, since  $\Sigma$  is the graph of a function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $\Upsilon_n$  the graph of a function  $f_n : \mathbb{R}^3 \rightarrow \mathbb{R}$ , uniform convergence  $\Upsilon_n \rightarrow \Sigma$  means that  $f_n$  converges uniformly to  $f$ . It turns out that this notion is Lorentz invariant:

**Proposition 2.3.3.** *If a sequence  $(\Sigma_n)_{n \in \mathbb{N}}$  of Cauchy surfaces converges uniformly to a Cauchy surface  $\Sigma$  in one Lorentz frame, then also in every other.*

Again, the proof is given in Section 2.5.

We define the detection distribution on  $\Sigma$  as the limit of the detection distributions on the  $\Upsilon_n$ , and we show in Theorem 2.3.7 that this limit exists and agrees with  $|\Psi_\Sigma|^2$ . But to this end, we first need to talk about detection probabilities on

triangular surfaces  $\Upsilon$ .

So let  $\Delta_k$  be the open and disjoint tetrahedra such that

$$\Upsilon = \bigcup_{k \in \mathcal{K}} \overline{\Delta_k}. \quad (2.29)$$

We want to consider a detector in a bounded region  $B \subset \Upsilon$  that yields outcome 1 if there is at least one particle in  $B$  and outcome 0 if there is no particle in  $B$ . To this end, we imagine several smaller detectors, one in each region  $B_k := B \cap \Delta_k$ , and set the  $B$ -outcome equal to 1 whenever any of the small detectors clicked. Now each region  $B_k$ , being a subset of  $\Delta_k$ , lies in some hyperplane  $E_k$ , and on hyperplanes we assume the Born rule and collapse rule:

**Flat Born rule.** *If on the hyperplane  $E$  the state vector is  $\Psi_E \in \mathcal{H}_E$  with  $\|\Psi_E\| = 1$ , and a detection is attempted in the region  $B \subseteq E$ , then the probability of outcome 1 is  $\|P_E(\exists(B)) \Psi_E\|^2$  and that of outcome 0 is  $\|P_E(\exists(B)^c) \Psi_E\|^2$ .*

**Flat collapse rule.** *If the outcome is 1, then the collapsed wave function is*

$$\Psi'_E = \frac{P_E(\exists(B)) \Psi_E}{\|P_E(\exists(B)) \Psi_E\|}, \quad (2.30)$$

*otherwise*

$$\Psi'_E = \frac{P_E(\exists(B)^c) \Psi_E}{\|P_E(\exists(B)^c) \Psi_E\|}. \quad (2.31)$$

There are two natural possibilities for defining the detection probabilities on  $\Upsilon$  in terms of those on  $E_k$ : the sequential detection process and the parallel detection process. According to the **sequential detection process**, we choose an arbitrary ordering of the set  $\mathcal{K}$  indexing the tetrahedra or hyperplanes and carry out, in this order, a quantum measurement in each  $E_k$  representing the detection attempt in  $B_k$  including appropriate collapse and then use the unitary evolution  $U_{E_k}^{E_{k+1}}$  to evolve to the next hyperplane in the chosen order, here written as  $E_{k+1}$ . For the **parallel detection process**, consider the projection operators  $P_{E_k}(\exists(B_k))$  associated with attempted detection in  $B_k$ ; we show that they, after being transferred to  $\mathcal{H}_\Upsilon$  by means of  $U_{E_k}^\Upsilon$ , commute with each other if interaction locality holds, so they can be “measured simultaneously.” The simultaneous quantum measurement of these projections in  $\mathcal{H}_\Upsilon$  provides the parallel detection process for  $B \subset \Upsilon$  with outcome 1 whenever any of the quantum measurements yielded 1. It turns out that the sequential and the parallel process agree with each other and with the Born rule on  $\Upsilon$ :

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**Proposition 2.3.4.** *Fix a hypersurface evolution satisfying interaction locality (IL) (Property (7) in Section 2.2.2), a triangular Cauchy surface  $\Upsilon$ , a bounded subset  $B \subset \Upsilon$ , and a normalized quantum state  $\Psi$ , and assume the flat Born rule and the flat collapse rule. The sequential detection process in any order of the tetrahedra of  $\Upsilon$  yields the same detection probability, called  $\mathbb{P}_B^\Psi$ ; it agrees with the one given by the curved Born distribution on  $\Upsilon$ , which is  $\|P_\Upsilon(\exists(B))\Psi_\Upsilon\|^2$ . Moreover, the parallel detection process also yields the same detection probability.*

Proposition 2.3.4 will follow as a direct consequence of Proposition 2.4.3 in Section 2.4.

Actually, for either a triangular surface  $\Upsilon$  or a general Cauchy surface  $\Sigma$ , we want more than just to detect for a subset  $B$  whether there is a particle in  $B$ . We want to allow the use of several detectors, each covering a region  $P_1, \dots, P_r \subset \Sigma$ ; the outcome of the experiment is  $L = (L_1, \dots, L_r)$  with  $L_\ell = 1$  if a particle gets detected in  $P_\ell$  and  $L_\ell = 0$  otherwise. It seems physically reasonable that the region covered by a detector is bounded and has boundary of measure zero.

**Definition 2.3.5.** An **admissible partition**  $\mathcal{P} = (P_1, \dots, P_r)$  of  $\Sigma$  is defined by choosing finitely many subsets  $P_\ell$  of  $\Sigma$  that are mutually disjoint,  $P_\ell \cap P_m = \emptyset$  for  $\ell \neq m$ , and such that each  $P_\ell$  is bounded and has boundary in  $\Sigma$  of (invariant) 3-volume 0. Here, the term bounded refers to the Euclidean norm on  $\mathbb{R}^4$ . We set  $P_{r+1} = \Sigma \setminus (P_1 \cup \dots \cup P_r)$  to make  $(P_1, \dots, P_{r+1})$  a partition of  $\Sigma$ .

The idea is that there is no detector in  $P_{r+1}$ . Let  $M_{\mathcal{P}}(L)$  denote the set of configurations in  $\Gamma(\Sigma)$  such that, for each  $\ell = 1, \dots, r$ , there is no point in  $P_\ell$  if  $L_\ell = 0$  and at least one point in  $P_\ell$  if  $L_\ell = 1$ ; that is,  $M_{\mathcal{P}}(L)$  is the set of configurations compatible with outcome  $L$ .

Now the definition of detection probabilities on a triangular surface  $\Upsilon$  can straightforwardly be generalized from a bounded set  $B \subset \Upsilon$  to an admissible partition  $\mathcal{P} = (P_1, \dots, P_r)$  of  $\Upsilon$  in both the sequential and the parallel sense, and we find:

**Proposition 2.3.6.** *Fix a hypersurface evolution satisfying interaction locality, a triangular Cauchy surface  $\Upsilon$ , an admissible partition  $\mathcal{P} = (P_1, \dots, P_r)$  of  $\Upsilon$ , and a normalized quantum state  $\Psi$ , and assume the flat Born rule and the flat collapse rule. The joint distribution  $\mathbb{P}_{\mathcal{P}}^\Psi(L)$  of  $L = (L_1, \dots, L_r)$  according to the sequential detection process in any order of the tetrahedra of  $\Upsilon$  and according to the parallel detection process agree with each other and with the one given by the curved Born distribution on  $\Upsilon$ , which is  $\|P_\Upsilon(M_{\mathcal{P}}(L))\Psi_\Upsilon\|^2$ .*

Proposition 2.3.6 can be regarded as a statement of the Born rule on triangular surfaces. It follows from Proposition 2.4.3, which is proven in Section 2.4.

### 2.3.3 Main Result

The following aspect of our result requires some explanation: Once we have a triangular surface  $\Upsilon$  approximating a given Cauchy surface  $\Sigma$ , and once we are given an admissible partition  $\mathcal{P} = (P_1, \dots, P_r)$  on  $\Sigma$ , we want to approximate the sets  $P_\ell \subset \Sigma$  by sets  $B_\ell$  in  $\Upsilon$ . One may think of two natural possibilities of defining  $B_\ell$ : (i) project  $P_\ell$  downwards along the direction of the  $x^0$  axis of a chosen Lorentz frame; or (ii) take  $B_\ell = \text{Sr}(P_\ell, \Upsilon)$ , the smallest set on  $\Upsilon$  that in some sense corresponds to  $P_\ell$ . Our result holds in both variants; we formulate it in variant (i) (see Remark 14 in Section 2.6 about (ii)). That is, choose a Lorentz frame and let

$$\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3, \quad \pi(x^0, x^1, x^2, x^3) := (x^1, x^2, x^3), \quad (2.32)$$

be the projection to the space coordinates. It is known [179, p. 417] that the restriction  $\pi_\Sigma$  of the projection  $\pi$  to  $\Sigma$  is a homeomorphism  $\Sigma \rightarrow \mathbb{R}^3$ ; thus,  $\pi_\Sigma^\Upsilon := \pi_\Upsilon^{-1} \circ \pi_\Sigma$  is a homeomorphism  $\Sigma \rightarrow \Upsilon$ . We set

$$B_\ell := \pi_\Sigma^\Upsilon(P_\ell). \quad (2.33)$$

Of course, since we prove that the limiting probability distribution on  $\Gamma(\Sigma)$  is given by the curved Born distribution, the limiting probability distribution is independent of the choice of Lorentz frame used for defining  $\pi_\Sigma^\Upsilon$ .

We can now state our main result.

**Theorem 2.3.7.** *Let  $\Sigma$  be a Cauchy surface in Minkowski spacetime  $\mathbb{M}$  and  $(\Upsilon_n)_{n \in \mathbb{N}}$  a sequence of triangular Cauchy surfaces that converges increasingly and uniformly to  $\Sigma$ . Let  $\mathcal{E} = (\mathcal{H}_\circ, P_\circ, U_\circ^\circ)$  be a hypersurface evolution satisfying propagation locality and  $\Psi_0 \in \mathcal{H}_{\Sigma_0}$  with  $\|\Psi_0\| = 1$  for some  $\Sigma_0$  in the past of  $\Sigma$ . Then for any admissible partition  $\mathcal{P}$  of  $\Sigma$ ,  $\mathcal{B}_n := (\pi_{\Sigma}^{\Upsilon_n}(P_1), \dots, \pi_{\Sigma}^{\Upsilon_n}(P_r))$  is an admissible partition of  $\Upsilon_n$ , and*

$$\lim_{n \rightarrow \infty} \left\| P_{\Upsilon_n}(M_{\mathcal{B}_n}(L)) U_{\Sigma_0}^{\Upsilon_n} \Psi_0 \right\|^2 = \left\| P_\Sigma(M_{\mathcal{P}}(L)) U_{\Sigma_0}^\Sigma \Psi_0 \right\|^2, \quad (2.34)$$

for all  $L \in \{0, 1\}^r$ .

Together with Proposition 2.3.6, we obtain:

**Corollary 2.3.8.** *Assume the hypotheses of Theorem 2.3.7 together with the flat Born rule, the flat collapse rule, and interaction locality. Define the detection probabilities for  $\mathcal{P}$  on  $\Sigma$  as the limit of the detection probabilities for  $\mathcal{B}_n$  on  $\Upsilon_n$  and the*

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latter through either the sequential or the parallel detection process. Then the detection probabilities for  $\mathcal{P}$  on  $\Sigma$  are given by the curved Born rule,  $\|P_\Sigma(M_\mathcal{P}(L))\Psi_\Sigma\|^2$  for all  $L \in \{0, 1\}^r$ .

The proof of Theorem 2.3.7 (see Section 2.6) makes no special use of dimension  $3 + 1$  and applies equally in dimension  $d + 1$  for any  $d \in \mathbb{N}$ ; tetrahedra then need to be replaced by  $d$ -dimensional simplices.

### Remarks.

7. *Grown set*  $\text{Gr}(A, \Sigma')$  and *shrunk set*  $\text{Sr}(A, \Sigma')$ . Definition (2.17) is equivalent to saying that the grown set is the intersection  $\text{Gr}(A, \Sigma') = \Sigma' \cap J(A)$  with the domain of dependence  $J(A)$  defined in (1.55). Conversely, the shrunk set is defined such that  $A = \Sigma \cap J(\text{Sr}(A, \Sigma'))$ .
8. *Uniqueness of the measure on  $\Gamma(\Sigma)$* . It was shown in [4, Sect. 6 Prop. 3] that if two probability measures  $\mu, \mu'$  on  $\Gamma(\Sigma)$  agree on all detection outcomes,  $\mu(M_\mathcal{P}(L)) = \mu'(M_\mathcal{P}(L))$  for every  $L \in \{0, 1\}^r$  and every admissible partition  $\mathcal{P}$  of  $\Sigma$ , then  $\mu = \mu'$ . Thus, the whole  $|\Psi_\Sigma|^2$ -distribution is uniquely determined by the detection probabilities.

In fact, a probability measure  $\mu$  on  $\Gamma(\Sigma)$  is already uniquely determined by the values  $\mu(\emptyset(A))$ , where  $A$  runs through those subsets of  $\Sigma$  whose projection  $\pi(A)$  to  $\mathbb{R}^3$  is a union of finitely many open balls (see the proof of [4, Prop. 3]). This fact might suggest that, in order to prove the curved Born rule, it would have been sufficient to prove the statement of Theorem 2.3.7 only for a single detector (i.e., for partitions with  $r = 1$  consisting of  $P_1 = A$  and  $P_2 = \Sigma \setminus A$ ) in a region  $A$  of the type described. However, we prove the stronger statement for arbitrary  $r$  because it is not obvious that the detection probabilities for arbitrary  $r$  fit together to form a measure on  $\Gamma(\Sigma)$  (in other words, that detection probabilities for  $r > 1$  will agree with the Born distribution, given that detection probabilities for  $r = 1$  do).

9. *Curved collapse rule*. One can also consider a **curved collapse rule**: Suppose that  $r$  detectors are placed along  $\Sigma$ , that each detector (say the  $\ell$ -th) only measures whether there is a particle in the region  $P_\ell$ , where  $\mathcal{P} = (P_1, \dots, P_r)$  is an admissible partition, and that each detector acts immediately (i.e., is

### 2.3 Deriving a Curved Born Rule

*infinitely fast*). If the outcome was  $L = (L_1, \dots, L_r) \in \{0, 1\}^r$ , then the wave function immediately after detection is the collapsed wave function

$$\Psi'_\Sigma = \frac{P_\Sigma(M_\mathcal{P}(L)) \Psi_\Sigma}{\|P_\Sigma(M_\mathcal{P}(L)) \Psi_\Sigma\|}. \quad (2.35)$$

There is a sense in which the curved collapse rule also follows from our result and a sense in which it does not. To begin with the latter, our justification of the Born rule on triangular surfaces was based on the idea that on each tetrahedron  $\Delta_k$ , we apply a detector to  $B_{k\ell} = \Delta_k \cap B_\ell$  and deduce from the outcomes whether a particle has been detected anywhere in  $B_\ell$ . This detection process measures more than whether there is a particle in  $B_\ell$ , as it also measures which of the  $B_{k\ell}$  contain particles; as a consequence, this detection process would collapse  $\Psi$  more narrowly than (2.35).

However, if we assume that on triangular surfaces  $\Upsilon$  we can have detectors that *only* measure whether there is a particle in  $B_\ell$  for an admissible partition  $\mathcal{B} = (B_1, \dots, B_r)$ , so that the collapse rule (2.35) holds upon replacing  $\Sigma \rightarrow \Upsilon$  and  $\mathcal{P} \rightarrow \mathcal{B}$ , then sufficient approximation of an arbitrary Cauchy surface  $\Sigma$  by triangular surfaces leads to a collapsed wave function arbitrarily close to (2.35). Indeed, we have that (see Section 2.6 for the proof)

**Corollary 2.3.9.** *Under the hypotheses of Theorem 2.3.7,*

$$U_{\Upsilon_n}^\Sigma P_{\Upsilon_n}(M_{\mathcal{B}_n}(L)) U_{\Upsilon_n}^\Upsilon \xrightarrow{n \rightarrow \infty} P_\Sigma(M_\mathcal{P}(L)) \text{ strongly.} \quad (2.36)$$

10. *Other observables.* As the curved Born rule shows, the PVM  $P_\Sigma$  can be regarded as the totality of position observables on  $\Sigma$ . What about other observables? In a sense, all other observables are indirectly determined by the position observable. As Bell [180, p. 166] wrote:

[I]n physics, the only observations we must consider are position observation, if only the positions of instrument pointers. [...] If you make axioms, rather than definitions and theorems, about the ‘measurements’ of anything else, then you commit redundancy and risk inconsistency.

A detailed description of how self-adjoint observables arise from the Hamiltonian of an experiment, the quantum state of the measuring apparatus, and the position observable (of its pointer), can be found in [175, Sect. 2.7]. A conclusion we draw is that specifying a quantum theory’s hypersurface evolution is an informationally complete description.

## 2 Hypersurface Evolution

As another conclusion, the PVM  $P_{\Sigma}$  serves not only for representing detectors. When we want to argue that certain experiments are quantum measurements of certain observables, we may use it to link the quantum state with macro-configurations (say, of pointer positions), and in fact to obtain probabilities for pointer positions.

Coming back to the Bell quote, one may also note that for the same reason, making the curved Born rule an axiom in addition to the flat Born rule means to commit redundancy and to risk inconsistency. That is why we have made the curved Born rule a theorem.

Of course, we have still committed a little bit of the redundancy that Bell talked about by assuming the Born and collapse rules on all spacelike hypersurfaces while it suffices to assume them on horizontal hypersurfaces [4].

11. *Objections.* Some authors [181] have criticized the very idea of evolving states from one Cauchy surface to another on the grounds that such an evolution cannot be unitarily implemented for the free second-quantized scalar Klein-Gordon field. It seems to us that these difficulties do not invalidate the approach but stem from analogous difficulties with 1-particle Klein-Gordon wave functions, which are known to lack a covariantly-defined timelike probability current 4-vector field that could be used for defining a Lorentz-invariant inner product that makes the time evolution unitary (e.g., [172]). In contrast, a hypersurface evolution according to our definition can indeed be defined for the free second-quantized Dirac equation allowing negative energies [182, 183, 184, 4]. Other results ([132, Sect. 1.8], [185, 186]) may raise doubts about propagation locality; on the other hand, these results presuppose positive energy, which we do not require here; moreover, violations of propagation locality would seem to allow for superluminal signaling. Be that as it may, we simply assume here a propagation-local hypersurface evolution as given; further developments of this notion can be of interest for future works. See also Section 2.7.3 for a further discussion.
12. *Evolution Between Hyperplanes.* Following [4, Sect. 8], we conjecture that a hypersurface evolution  $\mathcal{E}$  satisfying interaction locality and propagation locality is uniquely determined up to unitary equivalence, see Remark 14 in [4, Sect. 3.2], by its restriction to hyperplanes. We conjecture further that a

hypersurface evolution that is in addition Poincaré covariant (see Property (8) in Section 2.7.1) is uniquely determined by its restriction to horizontal hyperplanes  $E_t$ . While we do not have a proof of these statements, a related statement follows from our results:

*Suppose two hypersurface evolutions  $\mathcal{E} = (\mathcal{H}_\circ, P_\circ, U_\circ^\circ)$  and  $\tilde{\mathcal{E}} = (\mathcal{H}_\circ, P_\circ, \tilde{U}_\circ^\circ)$  use the same Hilbert spaces and PVMs but potentially different evolution operators; suppose further that the evolution operators agree on hyperplanes,  $U_E^{E'} = \tilde{U}_E^{E'}$  for all spacelike hyperplanes  $E, E'$ ; finally, suppose that both  $\mathcal{E}$  and  $\tilde{\mathcal{E}}$  satisfy interaction locality and propagation locality. Then they yield the same Born distribution on every Cauchy surface  $\Sigma$ , i.e., for every  $\Psi_0 \in \mathcal{H}_{E_0}$  and every  $B \subseteq \Gamma(\Sigma)$ ,*

$$\|P_\Sigma(B) U_{E_0}^\Sigma \Psi_0\|^2 = \|P_\Sigma(B) \tilde{U}_{E_0}^\Sigma \Psi_0\|^2. \quad (2.37)$$

Indeed, by Remark 8, (2.37) holds for all  $B \subseteq \Sigma$  if it holds for all  $M_\mathcal{P}(L)$  for all admissible partitions  $\mathcal{P}$  of  $\Sigma$ . By Theorem 2.3.7, both sides can be expressed as the limits of detection probabilities on triangular surfaces. Those in turn can be expressed, using the sequential detection process, in terms of  $U_E^{E'}$  respectively  $\tilde{U}_E^{E'}$  only for hyperplanes  $E, E'$ , so they are equal.

## 2.4 Detection Process on Triangular Surfaces

We now give the detailed definitions of the sequential and parallel detection processes and prove Propositions 2.3.4 and 2.3.6.

To begin with, consider an admissible partition  $\mathcal{P} = (P_1, \dots, P_r)$  of a Cauchy surface  $\Sigma$  and a vector  $L = (L_1, \dots, L_r) \in \{0, 1\}^r$ . Actually, in this section we will not make use of the assumption in Definition 2.3.5 that the boundaries  $\partial P_\ell$  are null sets, an assumption we need for Theorem 2.3.7.

The set of configurations in  $\Gamma(\Sigma)$  compatible with the single outcome  $L_\ell$  at an attempted detection in  $P_\ell$  is

$$M_{\ell\Sigma}(L_\ell) := \begin{cases} \exists(P_\ell) & \text{if } L_\ell = 1 \\ \emptyset(P_\ell) & \text{if } L_\ell = 0 \end{cases}. \quad (2.38)$$

The set of configurations compatible with the measurement outcome vector  $L$  when detection is attempted in  $P_1, \dots, P_r$  is

$$M_\mathcal{P}(L) := \bigcap_{\ell=1}^r M_{\ell\Sigma}(L_\ell). \quad (2.39)$$

## 2 Hypersurface Evolution

Now consider a triangular surface  $\Upsilon = \bigcup_{k \in \mathcal{K}} \overline{\Delta_k}$  and an admissible partition  $\mathcal{B} = (B_1, \dots, B_r)$  of  $\Upsilon$ . For either the sequential or the parallel detection process on  $\Upsilon$ , we imagine a small detector checking for particles in each

$$B_{k\ell} := \Delta_k \cap B_\ell, \quad (2.40)$$

with outcome  $s_{k\ell} = 1$  if a particle was found and  $s_{k\ell} = 0$  otherwise.<sup>4</sup>

We say that the outcome matrix  $s$  is compatible with  $L$  (denoted  $s : L$ ) whenever

$$\forall \ell \in \{1, \dots, r\} : \begin{cases} \exists k \in \mathcal{K} : s_{k\ell} = 1 & \text{if } L_\ell = 1 \\ \forall k \in \mathcal{K} : s_{k\ell} = 0 & \text{if } L_\ell = 0 \end{cases}. \quad (2.41)$$

Let  $E_k$  be the hyperplane containing  $\Delta_k$ . The configurations in  $E_k$  compatible with outcomes  $s_{k\ell}$  or  $s_k := (s_{k1}, \dots, s_{kr})$  are then given by

$$M_{k\ell E_k}(s_{k\ell}) := \begin{cases} \exists(B_{k\ell}) \subset \Gamma(E_k) & \text{if } s_{k\ell} = 1 \\ \emptyset(B_{k\ell}) \subset \Gamma(E_k) & \text{if } s_{k\ell} = 0 \end{cases}, \quad M_{kE_k}(s_k) := \prod_{\ell=1}^r M_{k\ell E_k}(s_{k\ell}). \quad (2.42)$$

Likewise,

$$M_{k\ell \Upsilon}(s_{k\ell}) := \begin{cases} \exists(B_{k\ell}) \subset \Gamma(\Upsilon) & \text{if } s_{k\ell} = 1 \\ \emptyset(B_{k\ell}) \subset \Gamma(\Upsilon) & \text{if } s_{k\ell} = 0 \end{cases}, \quad M_{k\Upsilon}(s_k) := \prod_{\ell=1}^r M_{k\ell \Upsilon}(s_{k\ell}). \quad (2.43)$$

It follows that, based on the definition (2.39),

$$M_{\mathcal{B}}(L) = \bigcup_{s:L} \prod_{k \in \mathcal{K}} M_{k\Upsilon}(s_k) \quad \text{up to a set of measure 0,} \quad (2.44)$$

meaning that the symmetric difference between the two sets is a set of measure 0 in  $\Gamma(\Upsilon)$ . This is the case because, as described in Footnote 4, the configurations in the symmetric difference have at least one particle in the 2d set  $\partial\Delta_k$  for some  $k$ .

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<sup>4</sup>We could also have defined  $B_{k\ell}$  by  $\overline{\Delta_k} \cap B_\ell$  instead of (2.40), but that would have caused a bit of trouble because these sets would not have been disjoint. Our choice (2.40), on the other hand, has the consequence, which may at first seem like a drawback, that  $\cup_k B_{k\ell} \neq B_\ell$  because we have removed the points on the 2d triangles where two tetrahedra meet. However, the set removed, being a subset of a countable union of 2d triangles, has measure 0 on  $\Upsilon$ , and for any set  $A \subseteq \Sigma$  of measure 0,  $\exists(A)$  has measure 0 in  $\Gamma(\Sigma)$  and, by the absolute continuity property (1), also  $P_\Sigma(\exists(A)) = 0$ .

### 2.4.1 Sequential Detection Process

We now formulate the definition of the sequential detection process and prove agreement with the Born rule. Fix an ordering of  $\mathcal{K}$ , i.e., a bijection  $\mathcal{K} \rightarrow \mathbb{N}$ . For ease of notation, we will simply replace  $\mathcal{K}$  by  $\mathbb{N}$  using this particular ordering. The detection process is:

- Set  $E_0 = \{x^0 = 0\}$  and  $\Psi_0 = \Psi_{E_0}$ .
- For each  $k$  in the specified order, do:
  - Evolve  $\Psi_{k-1}$  to  $E_k$ .
  - Carry out detections of  $B_{k\ell}$  for all  $\ell = 1, \dots, r$ , i.e., quantum measurements of  $P_{E_k}(\exists(B_{k\ell}))$ , and collapse accordingly, resulting in the (normalized) state vector  $\Psi_k \in \mathcal{H}_{E_k}$ .
  - Repeat.

Note that by Definition 2.3.1, each  $B_\ell$  intersects only finitely many  $\Delta_k$ . Thus, from some  $K+1$  onwards, all  $B_{k\ell}$  are empty,  $s_{k\ell} = 0$ , and no quantum measurement needs to be carried out in  $\Delta_k$ . Hence, it suffices to consider finitely many repetitions in the above loop, namely those for  $k$  up to  $K$ .

From the flat Born rule and the flat collapse rule, we can now express the detection probabilities and the collapsed state vectors. Fix some  $k$  and  $\ell$ ; suppose that in the previous tetrahedra  $k' < k$  (i.e., none if  $k = 1$ ), the measurements have already been carried out with outcomes  $s_{k'\ell'}$ ; suppose further that in the previous detector regions  $B_{k'\ell'}$  with  $\ell' < \ell$  (i.e., none if  $\ell = 1$ ) in the same tetrahedron  $\Delta_{k'}$ , the measurements have already been carried out with outcomes  $s_{k'\ell'}$ ; suppose further that  $\Psi_{k,\ell-1}$  is the collapsed wave function after the previous measurements, which for  $\ell > 1$  is given by the previous step, for  $\ell = 1$  and  $k > 1$  is given by

$$\Psi_{k,0} = U_{E_{k-1}}^{E_k} \Psi_{k-1,r}, \quad (2.45)$$

(with  $\Psi_{k-1,r} = \Psi_{k-1}$  in the notation of the process description above), and for  $\ell = 1, k = 1$  is given by

$$\Psi_{1,0} = U_{E_0}^{E_1} \Psi_0. \quad (2.46)$$

Conditional on the previous detection outcomes, the probability distribution of the next detection outcome  $s_{k\ell}$  is, by the flat Born rule,

$$\mathbb{P}(s_{k\ell} = 1) = \|P_{E_k}(\exists(B_{k\ell})) \Psi_{k,\ell-1}\|^2, \quad (2.47)$$

and the state vector collapses, by the flat collapse rule, to

$$\Psi_{k\ell} = \frac{P_{E_k}(M_{k\ell E_k}(s_{k\ell})) \Psi_{k,\ell-1}}{\|P_{E_k}(M_{k\ell E_k}(s_{k\ell})) \Psi_{k,\ell-1}\|}. \quad (2.48)$$

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This completes the definition of the sequential detection process.

**Lemma 2.4.1.** *Assume the flat Born rule and the flat collapse rule. Conditional on the measurements in the tetrahedra  $k' < k$ , the joint distribution of all outcomes  $(s_{k\ell})_{\ell=1..r} = s_k$  in  $\Delta_k$  is*

$$\mathbb{P}(s_{k1}, \dots, s_{kr}) = \|P_{E_k}(M_{kE_k}(s_k)) \Psi_{k0}\|^2, \quad (2.49)$$

and the collapsed wave function after the  $kr$ -measurement, given  $s_k$  with nonzero probability, is

$$\Psi_{kr} = \frac{P_{E_k}(M_{kE_k}(s_k)) \Psi_{k0}}{\|P_{E_k}(M_{kE_k}(s_k)) \Psi_{k0}\|}. \quad (2.50)$$

*Proof.* It is well known general facts about PVMs  $P$  that

$$P(S_1)P(S_2) = P(S_2)P(S_1) = P(S_1 \cap S_2), \quad (2.51)$$

and that a quantum measurement of  $P(S_1)$  with outcome  $s_1$  on  $\Psi$ , followed by one of  $P(S_2)$  with outcome  $s_2$ , have joint Born distribution

$$\begin{aligned} \mathbb{P}(s_1 = 1, s_2 = 1) &= \mathbb{P}(s_2 = 1 | s_1 = 1) \mathbb{P}(s_1 = 1) \\ &= \left\| P(S_2) \frac{P(S_1)\Psi}{\|P(S_1)\Psi\|} \right\|^2 \|P(S_1)\Psi\|^2 = \left\| P(S_1 \cap S_2)\Psi \right\|^2, \end{aligned} \quad (2.52)$$

and collapsed state vector, given  $s_1 = 1, s_2 = 1$ ,

$$\Psi' = P(S_2) \frac{P(S_1)\Psi}{\|P(S_1)\Psi\|} / \left\| P(S_2) \frac{P(S_1)\Psi}{\|P(S_1)\Psi\|} \right\| = \frac{P(S_1 \cap S_2)\Psi}{\|P(S_1 \cap S_2)\Psi\|}. \quad (2.53)$$

Iteration with  $r$  sets rather than 2 and the definition of  $M_{kE_k}(s_k)$  yield Lemma 2.4.1.  $\square$

**Lemma 2.4.2.** *(IL) implies that*

$$U_{E_k}^\Upsilon P_{E_k}(M_{kE_k}(s_k)) U_{\Upsilon}^{E_k} = P_{\Upsilon}(M_{k\Upsilon}(s_k)). \quad (2.54)$$

*Proof.* Decompose  $\mathcal{H}_{E_k} = \mathcal{H}_{\Delta_k} \otimes \mathcal{H}_{E_k \setminus \Delta_k}$  and  $\mathcal{H}_{\Upsilon} = \mathcal{H}_{\Delta_k} \otimes \mathcal{H}_{\Upsilon \setminus \Delta_k}$ . By (IL), we have that

$$U_{\Upsilon}^{E_k} = I_{\Delta_k} \otimes V_{\Upsilon \setminus \Delta_k}^{E_k \setminus \Delta_k}. \quad (2.55)$$

## 2.4 Detection Process on Triangular Surfaces

We know that  $\Gamma(E_k) = \Gamma(\Delta_k) \times \Gamma(E_k \setminus \Delta_k)$ . The set  $M_{kE_k}(s_k) \subseteq \Gamma(E_k)$  factorizes in the same way:

$$M_{kE_k}(s_k) = N_{k\Delta_k}(s_k) \times \Gamma(E_k \setminus \Delta_k). \quad (2.56)$$

That is because whether a configuration  $q$  is compatible with the outcome  $s_k$ , i.e.,  $q \in M_{kE_k}(s_k)$ , does not depend on the points in  $q$  outside of  $\Delta_k$ . Here, the set  $N_{k\Delta_k}(s_k) \subseteq \Gamma(\Delta_k)$  is defined in the analogous way to  $M_{kE_k}(s_k)$ , i.e.,

$$N_{k\Delta_k}(s_k) := \bigcap_{\ell=1}^r N_{k\ell\Delta_k}(s_{k\ell}), \quad N_{k\ell\Delta_k}(s_{k\ell}) := \begin{cases} \exists_{\Delta_k}(B_{k\ell}) & \text{if } s_{k\ell} = 1 \\ \emptyset_{\Delta_k}(B_{k\ell}) & \text{if } s_{k\ell} = 0, \end{cases} \quad (2.57)$$

where  $\exists_A(B)$  means the set of all configurations in  $\Gamma(A)$  with at least one particle in  $B$ . Hence, the projection  $P_{E_k}(M_{kE_k}(s_k))$  decomposes into a tensor product

$$P_{E_k}(M_{kE_k}(s_k)) = P_{\Delta_k}(N_{k\Delta_k}(s_k)) \otimes I_{E_k \setminus \Delta_k}, \quad (2.58)$$

and by (2.55),

$$\begin{aligned} U_{E_k}^{\Upsilon} P_{E_k}(M_{kE_k}(s_k)) U_{\Upsilon}^{E_k} &= [I_{\Delta_k} \otimes V_{E_k \setminus \Delta_k}^{\Upsilon \setminus \Delta_k}] [P_{\Delta_k}(N_{k\Delta_k}(s_k)) \otimes I_{E_k \setminus \Delta_k}] [I_{\Delta_k} \otimes V_{\Upsilon \setminus \Delta_k}^{E_k \setminus \Delta_k}] \\ &= [I_{\Delta_k} \circ P_{\Delta_k}(N_{k\Delta_k}(s_k)) \circ I_{\Delta_k}] \otimes [V_{E_k \setminus \Delta_k}^{\Upsilon \setminus \Delta_k} \circ I_{E_k \setminus \Delta_k} \circ V_{\Upsilon \setminus \Delta_k}^{E_k \setminus \Delta_k}] \\ &= P_{\Delta_k}(N_{k\Delta_k}(s_k)) \otimes I_{\Upsilon \setminus \Delta_k} \\ &= P_{\Upsilon}(M_{k\Upsilon}(s_k)), \end{aligned} \quad (2.59)$$

for the same reasons as (2.58). □

**Proposition 2.4.3.** *Assume the flat Born rule, the flat collapse rule, and (II). The unconditional joint distribution of all outcomes, i.e., of the matrix  $s$  comprising all  $s_{k\ell}$ , agrees with the Born distribution on  $\Upsilon$ ,*

$$\mathbb{P}(s) = \left\| P_{\Upsilon} \left( \bigcap_{k \in \mathbb{N}} M_{k\Upsilon}(s_k) \right) \Psi_{\Upsilon} \right\|^2, \quad (2.60)$$

with  $\Psi_{\Upsilon} = U_{E_0}^{\Upsilon} \Psi_0$  (actually regardless of whether  $\partial B_{\ell}$  are null sets). In particular, the distribution of  $L = (L_1, \dots, L_r)$  is the Born distribution  $\|P_{\Upsilon}(M_{\emptyset\Upsilon}(L))\Psi_{\Upsilon}\|^2$ .

*Proof.* As noted before, all  $s_{k\ell}$  vanish from some  $K + 1$  onwards (and formulas below will take for granted they do), and we need consider only  $k \leq K$ . The fact,

## 2 Hypersurface Evolution

used before in (2.52), that for subsequent measurements the projections multiply, yields from Lemma 2.4.1 that

$$\mathbb{P}(s) = \left\| U_{E_K}^\Upsilon P_{E_K}(M_{KE_K}(s_K)) U_\Upsilon^{E_K} \cdots U_{E_1}^\Upsilon P_{E_1}(M_{1E_1}(s_1)) U_\Upsilon^{E_1} \Psi_\Upsilon \right\|^2. \quad (2.61)$$

Inserting (2.54) in (2.61) yields

$$\begin{aligned} \mathbb{P}(s) &= \left\| P_\Upsilon(M_{K\Upsilon}(s_K)) \cdots P_\Upsilon(M_{1\Upsilon}(s_1)) \Psi_\Upsilon \right\|^2 \\ &= \left\| P_\Upsilon \left( \bigcap_{k=1}^K M_{k\Upsilon}(s_k) \right) \Psi_\Upsilon \right\|^2 \\ &= \left\| P_\Upsilon \left( \bigcap_{k \in \mathbb{N}} M_{k\Upsilon}(s_k) \right) \Psi_\Upsilon \right\|^2, \end{aligned} \quad (2.62)$$

as claimed.  $\square$

Proposition 2.3.6, insofar as it concerns the sequential detection process, follows from Proposition 2.4.3 (actually regardless of whether  $\partial B_\ell$  are null sets), and Proposition 2.3.4 follows further as the special case in which  $r = 1$ ,  $B_1 = B$ , and  $B_{r+1} = B^c$ .

### 2.4.2 Parallel Detection Process

We now formulate the definition of the parallel detection process and prove the Born rule for it. Throughout the whole subsection,  $(IL)$  is assumed.

The proof of Lemma 2.4.2 also shows that, analogously to (2.54),

$$U_{E_k}^\Upsilon P_{E_k}(M_{k\ell E_k}(s_{k\ell})) U_\Upsilon^{E_k} = P_\Upsilon(M_{k\ell\Upsilon}(s_{k\ell})). \quad (2.63)$$

As outlined in Section 2.3.2, the idea is to think of the detection attempt in  $B_{k\ell}$  as a quantum measurement of the observable

$$U_{E_k}^\Upsilon P_{E_k}(\exists(B_{k\ell})) U_\Upsilon^{E_k} = P_\Upsilon(\exists(B_{k\ell})), \quad (2.64)$$

which is (2.63) for  $s_{k\ell} = 1$ . Since  $B_{k\ell}$  is non-empty only for finitely many  $k$  (for  $k = 1, \dots, K$ ), we are considering only finitely many observables. They commute because projections belonging to the same PVM always commute. Their simultaneous measurement is the definition of the parallel detection process.

We now prove the Born rule for the parallel detection process. When considering the simultaneous measurement of the operators (2.64), we need their joint

diagonalization; the joint eigenspace with eigenvalues  $(s_{k\ell})_{k\ell}$  is the range of

$$P_{\Upsilon} \left( \bigcap_{k=1}^K \bigcap_{\ell=1}^r M_{k\ell\Upsilon}(s_{k\ell}) \right) = P_{\Upsilon} \left( \bigcap_{k=1}^K M_{k\Upsilon}(s_k) \right), \quad (2.65)$$

so the probability of the outcomes  $(s_{k\ell})_{k\ell}$  is

$$\left\| P_{\Upsilon} \left( \bigcap_{k=1}^K M_{k\Upsilon}(s_k) \right) \Psi_{\Upsilon} \right\|^2, \quad (2.66)$$

and the probability of outcome  $L$  is

$$\begin{aligned} \sum_{s:L} \left\| P_{\Upsilon} \left( \bigcap_{k=1}^K M_{k\Upsilon}(s_k) \right) \Psi_{\Upsilon} \right\|^2 &= \left\| \sum_{s:L} P_{\Upsilon} \left( \bigcap_{k=1}^K M_{k\Upsilon}(s_k) \right) \Psi_{\Upsilon} \right\|^2 \\ &= \left\| P_{\Upsilon} \left( \bigcup_{s:L} \bigcap_{k=1}^K M_{k\Upsilon}(s_k) \right) \Psi_{\Upsilon} \right\|^2 \\ &= \left\| P_{\Upsilon} \left( M_{\mathcal{B}}(L) \right) \Psi_{\Upsilon} \right\|^2, \end{aligned} \quad (2.67)$$

because the sets  $\bigcap_{k=1}^K M_{k\Upsilon}(s_k)$  are mutually disjoint and thus their projections are mutually orthogonal, and because of (2.44) and the absolute continuity property (1). That is, the probability of outcome  $L$  agrees with the Born rule. This proves the statement about the parallel detection process in Proposition 2.3.6 and thus also in Proposition 2.3.4.

Another way of looking at the parallel detection process is based on tensor products: Since  $\Upsilon = \bigcup_{k=1}^K \Delta_k \cup R$  with remainder set  $R = \Upsilon \setminus \bigcup_{k=1}^K \Delta_k$ , we have from Remark 3 in Section 2.2.2 that

$$\mathcal{H}_{\Upsilon} = \bigotimes_{k=1}^K \mathcal{H}_{\Delta_k} \otimes \mathcal{H}_R. \quad (2.68)$$

By (IL), each  $\mathcal{H}_{\Delta_k}$  can be regarded as a factor in  $\mathcal{H}_{E_k} = \mathcal{H}_{\Delta_k} \otimes \mathcal{H}_{E_k \setminus \Delta_k}$ . With the flat Born rule in mind, or with the idea that  $P_{E_k}$  is the configuration observable on  $E_k$ , the attempted detection in  $B_{k\ell}$  can be regarded as a quantum measurement in  $\mathcal{H}_{E_k}$  of the observable  $P_{E_k}(\exists(B_{k\ell}))$ , which is of the form

$$P_{E_k}(\exists_{E_k}(B_{k\ell})) = P_{\Delta_k}(\exists_{\Delta_k}(B_{k\ell})) \otimes I_{E_k \setminus \Delta_k}. \quad (2.69)$$

Thus, the attempted detection in  $B_{k\ell}$  can *also* be regarded as a quantum measurement in  $\mathcal{H}_{\Delta_k}$  of the observable  $P_{\Delta_k}(\exists_{\Delta_k}(B_{k\ell}))$ . These observables commute

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for different  $\ell$  and equal  $k$  because they belong to the same PVM  $P_{\Delta_k}$ , and they commute for different  $k$  in  $\mathcal{H}_\Upsilon$  because of the tensor product structure (2.68). It follows that

$$P_\Upsilon(M_{\mathcal{O}}(L)) = \sum_{s:L} \bigotimes_{k=1}^K P_{\Delta_k}(N_{k\Delta_k}(s_k)) \otimes I_R, \quad (2.70)$$

with  $N_{k\Delta_k}$  as in (2.57), which agrees again with the Born rule on  $\Upsilon$ , as claimed in Proposition 2.3.6.

## 2.5 Approximation by Triangular Surfaces

In this section, we prove Propositions 2.3.2 and 2.3.3.

*Proof of Proposition 2.3.2.* Fix an  $n \in \mathbb{N}$  and set  $\varepsilon = 3^{-n}$ . We construct a  $3\varepsilon$ -approximation  $\Upsilon_n$  to  $\Sigma$ . First, consider the function  $f_t : \mathbb{M} \rightarrow \mathbb{M}$ ,  $(x^0, \mathbf{x}) \mapsto (x^0 - t, \mathbf{x})$ , which “lowers a point by an amount  $t$  in time.” We use  $f$  to define the sets (see Figure 2.7):

$$\Sigma_{2\varepsilon} := f_{2\varepsilon}[\Sigma], \quad \Sigma_{\varepsilon..3\varepsilon} := \bigcup_{\varepsilon < \varepsilon' < 3\varepsilon} f_{\varepsilon'}[\Sigma]. \quad (2.71)$$

So  $\Sigma_{2\varepsilon}$  is a version of  $\Sigma$ , lowered by  $2\varepsilon$  and  $\Sigma_{\varepsilon..3\varepsilon}$  is a slice below  $\Sigma$  of thickness  $2\varepsilon$ , centered at  $\Sigma_{2\varepsilon}$ .

We now choose a decomposition of  $\mathbb{R}^3$  into (non-regular) tetrahedra  $\mathbb{R}^3 = \bigcup_{k \in \mathbb{N}} \tilde{\Delta}_k^n$  with open  $\tilde{\Delta}_k^n$  such that each pair of vertices  $\mathbf{x}_{k,i}^n, \mathbf{x}_{k,j}^n, i, j \in \{1, 2, 3, 4\}$

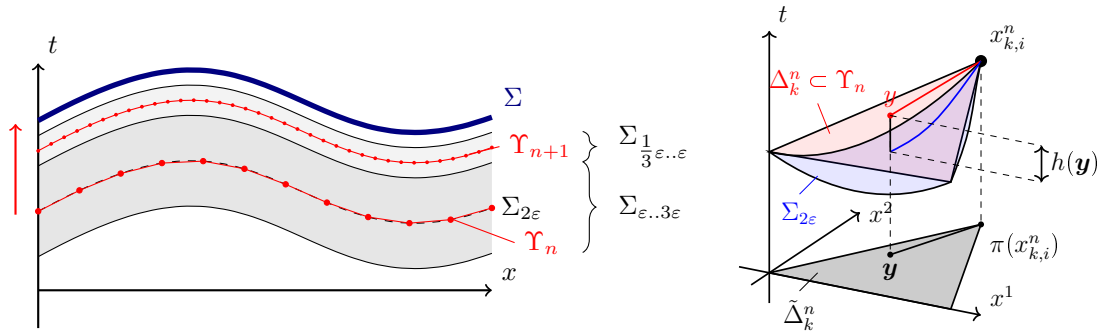


Figure 2.7: Left: Construction of the approximating sequence  $\Upsilon_n \nearrow \Sigma$ .

Right:  $|h(\mathbf{y})| < \varepsilon$  illustrated in 2+1 dim. Color online.

has a distance  $\|\mathbf{x}_{k,i}^n - \mathbf{x}_{k,j}^n\| \leq \varepsilon$  and such that every bounded region intersects only finitely many tetrahedra. For example, we may subdivide  $\mathbb{R}^3$  into axiparallel cubes with vertices on  $\frac{\varepsilon}{\sqrt{3}}\mathbb{Z}^3$  and subdivide each cube into  $3!$  tetrahedra with vertices on  $\frac{\varepsilon}{\sqrt{3}}\mathbb{Z}^3$ .

The four spacetime points  $x_{k,i}^n := \pi|_{\Sigma_{2\varepsilon}}^{-1} \mathbf{x}_{k,i}^n \in \mathbb{M}$  (obtained by lifting  $\mathbf{x}_{k,i}^n$  up to the  $2\varepsilon$ -surface, with  $i = 1, 2, 3, 4$ ) span a spacelike open tetrahedron  $\Delta_k^n$  in  $\mathbb{M}$ . Now set  $\Upsilon_n := \bigcup_{k \in \mathbb{N}} \overline{\Delta_k^n}$ .

Claim:  $\Upsilon_n$  is a uniform  $\varepsilon$ -approximation of  $\Sigma_{2\varepsilon}$ , i.e.,  $\Upsilon_n \subset \Sigma_{\varepsilon..3\varepsilon}$  (see Figure 2.7).

Proof: Regard the surfaces  $\Upsilon_n$  and  $\Sigma_{2\varepsilon}$  as the graphs of functions  $\mathbb{R}^3 \rightarrow \mathbb{R}$ , henceforth denoted simply by  $\Upsilon_n(\cdot)$  and  $\Sigma_{2\varepsilon}(\cdot)$ ; that is,  $(\Upsilon_n(\mathbf{x}), \mathbf{x}) \in \Upsilon_n$  for all  $\mathbf{x} \in \mathbb{R}^3$  and  $x = (\Upsilon_n(\pi(x)), \pi(x))$  for all  $x \in \Upsilon_n$ . Both functions are Lipschitz-continuous with Lipschitz constant 1. Further, there is always a vertex of  $\overline{\Delta_k^n}$  (possibly several ones) that maximizes  $\Upsilon_n(\cdot)$  on  $\overline{\Delta_k^n}$  (a “highest” vertex), and one (or several) that minimizes  $\Upsilon_n(\cdot)$  (a “lowest” vertex). Now consider the “height difference function”  $h(\mathbf{x}) = \Upsilon_n(\mathbf{x}) - \Sigma_{2\varepsilon}(\mathbf{x})$ . (It is Lipschitz continuous with Lipschitz constant 2.) For any vertex  $x_{k,i}^n$ , we have that  $h(\pi(x_{k,i}^n)) = 0$ . And for any other point  $y \in \Delta_k^n$ , we have that  $|\pi(x_{k,i}^n) - \pi(y)|_{\mathbb{R}^3} < \varepsilon$ , so by Lipschitz continuity,

$$\Sigma_{2\varepsilon}(\pi(x_{k,i}^n)) - \Sigma_{2\varepsilon}(\pi(y)) < \varepsilon. \quad (2.72)$$

If  $x_{k,i}^n$  is a highest vertex, then

$$\begin{aligned} & \Upsilon_n(\pi(x_{k,i}^n)) - \Upsilon_n(\pi(y)) > 0 \\ \Rightarrow & h(\pi(x_{k,i}^n)) - h(\pi(y)) > -\varepsilon \quad \Leftrightarrow \quad h(\pi(y)) < \varepsilon, \end{aligned} \quad (2.73)$$

(see Figure 2.7). The same reasoning with a lowest vertex yields  $h(\pi(y)) > -\varepsilon$ , so in total  $|h(\pi(y))| < \varepsilon$ , which proves the claim.  $\square$

Claim:  $\Upsilon_n$  is a Cauchy surface.

Proof: We need to show that  $\Upsilon_n$  is intersected exactly once by every causal inextendible curve  $\gamma : (-\infty, \infty) \rightarrow \mathbb{M}$ . We regard  $\Upsilon_n$  again as the graph of an equally denoted function  $\Upsilon_n : \mathbb{R}^3 \rightarrow \mathbb{R}$ . Now, consider the height difference function  $h(t) = \gamma^0(t) - \Upsilon_n(\pi(\gamma(t)))$ , which tells us “by how much  $\gamma$  is above  $\Upsilon_n$ .” Since  $\Upsilon_n$  consists of spacelike tetrahedra,  $\Upsilon_n$  is Lipschitz-continuous with Lipschitz constant  $\leq 1$ . As  $\gamma$  is timelike-or-lightlike and w.l.o.g. directed towards the future, we have that  $h$  is strictly increasing, so there can be at most one  $t$  with  $h(t) = 0$ . That is, there is at most one intersection of  $\gamma$  with  $\Upsilon_n$ .

On the other hand, an intermediate value argument yields that there must be at least one intersection point: Otherwise, either  $h(t) > 0$  for all  $t$  or  $h(t) < 0$  for all  $t$ ;

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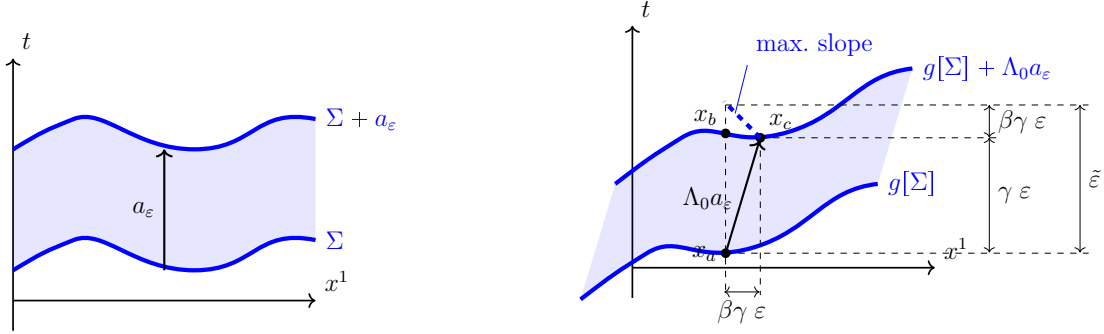


Figure 2.8: Left:  $\Sigma$  is being translated in the Proof of Proposition 2.5.1.  
Right: The same structure after a boost. Color online.

w.l.o.g., assume the former case. Since  $\Upsilon_n$  is an  $\varepsilon$ -approximation to  $\Sigma_{2\varepsilon}$ , we know that  $\gamma^0(t) > \Upsilon_n(\pi(\gamma(t))) > \Sigma_{2\varepsilon}(\pi(\gamma(t))) - \varepsilon = \Sigma_{3\varepsilon}(\pi(\gamma(t)))$ , which implies that  $\gamma$  does not intersect  $\Sigma_{3\varepsilon}$ , but that is impossible because  $\Sigma_{3\varepsilon}$  is a Cauchy surface.  $\square$

We can now complete the proof of Proposition 2.3.2. Since  $\Upsilon_n$  approximates  $\Sigma_{2\varepsilon}$  up to  $\varepsilon$ , it approximates  $\Sigma$  up to  $3\varepsilon$ . Furthermore,  $\Upsilon_n \subset \Sigma_{\varepsilon..3\varepsilon}$  and  $\Upsilon_{n+1} \subset \Sigma_{\frac{1}{3}\varepsilon..\varepsilon}$ , and since  $\Sigma_{\frac{1}{3}\varepsilon..\varepsilon}$  lies in the future of  $\Sigma_{\varepsilon..3\varepsilon}$  while being disjoint from it,  $\Upsilon_{n+1}$  lies in the future of  $\Upsilon_n$  (see Figure 2.7). This completes the proof of Proposition 2.3.2.  $\square$

Proposition 2.3.3 follows from the following statement:

**Proposition 2.5.1.** *Let  $\varepsilon > 0$ ,  $\Sigma$  be a Cauchy surface,  $a_\varepsilon := (\varepsilon, 0, 0, 0)$  the vertical 4-vector of length  $\varepsilon$ , and  $g : \mathbb{M} \rightarrow \mathbb{M}$ ,  $g \in \mathcal{P}_+^\uparrow$  a proper Poincaré transformation. Then*

$$g[\Sigma + a_\varepsilon] \subset \{x + (s, 0, 0, 0) \mid x \in g\Sigma, 0 < s < \tilde{\varepsilon}\}, \quad (2.74)$$

where

$$\tilde{\varepsilon} = (\beta\gamma + \gamma)\varepsilon, \quad (2.75)$$

with  $\beta \in [0, 1)$  the boost velocity of  $g$  and  $\gamma := (1 - \beta^2)^{-1/2}$  (the “Lorentz factor”).

*Proof of Proposition 2.5.1.* A Poincaré transformation  $g$  consists of a translation and a Lorentz transformation  $\Lambda$ , which in turn consists of a rotation and a subsequent boost  $\Lambda_0$ . The rotation leaves  $a_\varepsilon$  invariant. Thus,  $g[\Sigma + a_\varepsilon] = g\Sigma + \Lambda_0 a_\varepsilon$ . Without loss of generality,  $\Lambda_0$  is a boost in the  $x^1$  direction (see Figure 2.8),

$$\Lambda_0 = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \text{ so } \Lambda_0 a_\varepsilon = \begin{pmatrix} \gamma\varepsilon \\ \beta\gamma\varepsilon \\ 0 \\ 0 \end{pmatrix}. \quad (2.76)$$

Consider any point  $x_a = (x_a^0, \mathbf{x}_a) \in g\Sigma$ . Denote by  $x_b = (x_b^0, \mathbf{x}_b)$  the point on  $g[\Sigma + a_\varepsilon]$  right above  $x_a$ ,  $\mathbf{x}_b = \mathbf{x}_a$ . We want to show that  $x_b^0 \leq x_a^0 + \tilde{\varepsilon}$ . Set  $x_c := x_a + \Lambda_0 a_\varepsilon$ . Since  $g[\Sigma + a_\varepsilon]$  is a Cauchy surface, any two points on it (such as  $x_b$  and  $x_c$ ) must be spacelike separated, so

$$|x_b^0 - x_c^0| \leq |\mathbf{x}_b - \mathbf{x}_c| = |\mathbf{x}_a - \mathbf{x}_c| = \beta\gamma\varepsilon. \quad (2.77)$$

Now the triangle inequality implies the desired bound

$$|x_b^0 - x_a^0| \leq |x_b^0 - x_c^0| + |x_c^0 - x_a^0| \leq \beta\gamma\varepsilon + \gamma\varepsilon = \tilde{\varepsilon}. \quad (2.78)$$

□

## 2.6 Proof of Theorem 2.3.7

Here is a quick outline of the proof. We want to show that

$$\mathbb{P}_{\mathcal{B}_n}(L) := \|P_{\Upsilon_n}(M_{\mathcal{B}_n}(L))\Psi_{\Upsilon_n}\|^2, \quad (2.79)$$

converges, as  $n \rightarrow \infty$ , to

$$\mathbb{P}_{\mathcal{D}}(L) := \|P_{\Sigma}(M_{\mathcal{D}}(L))\Psi_{\Sigma}\|^2. \quad (2.80)$$

The proof is done by a squeeze–theorem argument: We will define two distributions  $\hat{\mathbb{P}}_n$  and  $\check{\mathbb{P}}_n$  on  $\{0, 1\}^r$  such that

$$\hat{\mathbb{P}}_n(L) \leq \mathbb{P}_{\mathcal{B}_n}(L) \leq \check{\mathbb{P}}_n(L), \quad \hat{\mathbb{P}}_n(L) \leq \mathbb{P}_{\mathcal{D}}(L) \leq \check{\mathbb{P}}_n(L), \quad (2.81)$$

and prove that  $\hat{\mathbb{P}}_n(L), \check{\mathbb{P}}_n(L)$  both converge to  $\mathbb{P}_{\mathcal{D}}(L)$  as  $n \rightarrow \infty$ .

We go through some preparations for the proof. To begin with, it is easy to see that  $\mathcal{B}_n = (B_{n1}, \dots, B_{nr})$  with

$$B_{n\ell} = \pi_{\Sigma}^{\Upsilon_n}(P_\ell), \quad (2.82)$$

is an admissible partition of  $\Upsilon_n$ : First,  $B_{n\ell} \cap B_{nm} = \emptyset$  for  $\ell \neq m$  because  $\pi_{\Sigma}^{\Upsilon_n}$  is a bijection. Second,  $B_{n\ell}$  is bounded because  $\pi_{\Sigma}^{\Upsilon_n}$  maps bounded sets to bounded sets. Third, the boundary  $\partial B_{n\ell}$  of  $B_{n\ell}$  in  $\Upsilon_n$  is  $\pi_{\Sigma}^{\Upsilon_n}(\partial P_\ell)$  because  $\pi_{\Sigma}^{\Upsilon_n}$  is a homeomorphism. Finally, in order to obtain that  $\mu_{\Upsilon_n}(\partial B_{n\ell}) = 0$  we note that  $\mu_{\Sigma}(\partial P_\ell) = 0$ , that  $\Sigma$  (and  $\Upsilon_n$ ) possesses a spacelike tangent plane almost everywhere (relative to Lebesgue measure  $\lambda$  on  $\mathbb{R}^3$ ), and that, at points with a spacelike tangent plane,  $\mu_{\Sigma}$  possesses a nonzero density relative to  $\lambda \circ \pi_{\Sigma}$ , so  $\mu_{\Sigma}$  and  $\lambda \circ \pi_{\Sigma}$  have the same

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null sets.

For the definition of  $\widehat{\mathbb{P}}_n, \check{\mathbb{P}}_n$  we introduce more notation:

We define

$$\widehat{C}_{nl} := \text{Sr}(B_{nl}, \Sigma), \quad \check{C}_{nl} := \text{Gr}(B_{nl}, \Sigma). \quad (2.83)$$

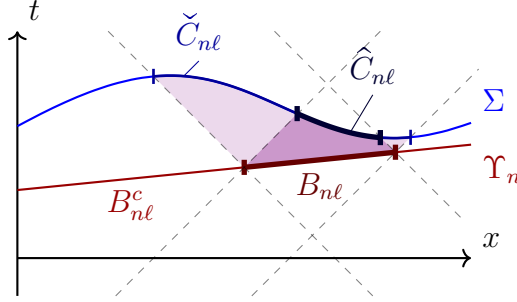


Figure 2.9: Definition of  $\widehat{C}_{nl}$  and  $\check{C}_{nl}$ . Color online.

The corresponding sets of compatibility in configuration space  $\Gamma(\Sigma)$  are

$$\widehat{M}_{nl}(L_\ell) := \begin{cases} \exists(\widehat{C}_{nl}) & \text{if } L_\ell = 1 \\ \emptyset(\check{C}_{nl}) & \text{if } L_\ell = 0, \end{cases} \quad \check{M}_{nl}(L_\ell) := \begin{cases} \exists(\check{C}_{nl}) & \text{if } L_\ell = 1 \\ \emptyset(\widehat{C}_{nl}) & \text{if } L_\ell = 0, \end{cases} \quad (2.84)$$

$$\widehat{M}_{n\Sigma}(L) := \bigcap_{\ell=1}^r \widehat{M}_{nl}(L_\ell), \quad \check{M}_{n\Sigma}(L) := \bigcap_{\ell=1}^r \check{M}_{nl}(L_\ell). \quad (2.85)$$

The probability distributions that serve for the squeeze–theorem bounds are defined by

$$\widehat{\mathbb{P}}_n(L) := \langle \Psi_\Sigma | P_\Sigma(\widehat{M}_{n\Sigma}(L)) | \Psi_\Sigma \rangle \quad \check{\mathbb{P}}_n(L) := \langle \Psi_\Sigma | P_\Sigma(\check{M}_{n\Sigma}(L)) | \Psi_\Sigma \rangle. \quad (2.86)$$

**Lemma 2.6.1** (Squeeze–theorem bound for  $\mathbb{P}_\varnothing$ ). *For all  $L \in \{0, 1\}^r$ ,*

$$\begin{aligned} \widehat{M}_{n\Sigma}(L) &\subseteq M_\varnothing(L) \subseteq \check{M}_{n\Sigma}(L), \\ \text{hence } P_\Sigma(\widehat{M}_{n\Sigma}(L)) &\leq P_\Sigma(M_\varnothing(L)) \leq P_\Sigma(\check{M}_{n\Sigma}(L)), \\ \text{and } \widehat{\mathbb{P}}_n(L) &\leq \mathbb{P}_\varnothing(L) \leq \check{\mathbb{P}}_n(L). \end{aligned} \quad (2.87)$$

*Proof.* The statement is actually true for any triangular surface  $\Upsilon$ , regardless of whether it belongs to a sequence converging to  $\Sigma$ . Since we need it for  $\Upsilon_n$ , we use here the notation that refers to  $\Upsilon_n$ .

The inclusion

$$\widehat{C}_{n\ell} \subseteq P_\ell \subseteq \check{C}_{n\ell}, \quad (2.88)$$

is obvious, since  $\pi[\widehat{C}_{n\ell}]$  is a shrunk version of  $\pi[P_\ell]$  (i.e., smaller) and  $\pi[\check{C}_{n\ell}]$  is a grown version of it (i.e., larger).

We “lift” those sets to configuration space, keeping in mind that

$$\text{if } A \subseteq B, \text{ then } \exists(A) \subseteq \exists(B) \text{ and } \emptyset(A) \supseteq \emptyset(B). \quad (2.89)$$

By definition (2.84) we then have:

$$\widehat{M}_{n\ell}(L_\ell) \subseteq M_{\ell\Sigma}(L_\ell) \subseteq \widetilde{M}_{n\ell}(L_\ell). \quad (2.90)$$

Inclusions persist under intersections, i.e.,

$$\text{if } A_\ell \subseteq B_\ell \text{ for all } \ell, \text{ then } \bigcap_\ell A_\ell \subseteq \bigcap_\ell B_\ell. \quad (2.91)$$

This yields the first line of (2.87). The transition from sets  $M$  to projections  $P(M)$  as in the second line of (2.87) is straightforward, and sandwiching between  $\Psi_\Sigma$ 's yields the third line of (2.87).  $\square$

**Lemma 2.6.2** (Squeeze–theorem bound for  $\mathbb{P}_{\mathcal{B}_n}$ ). *Assume (PL). Then, for all  $L \in \{0, 1\}^r$ ,*

$$\begin{aligned} P_\Sigma(\widehat{M}_{n\Sigma}(L)) &\leq U_{\Upsilon_n}^\Sigma P_{\Upsilon_n}(M_{\mathcal{B}_n}(L)) U_{\Sigma}^{\Upsilon_n} \leq P_\Sigma(\widetilde{M}_{n\Sigma}(L)), \\ \text{hence } \widehat{\mathbb{P}}_n(L) &\leq \mathbb{P}_{\mathcal{B}_n}(L) \leq \check{\mathbb{P}}_n(L). \end{aligned} \quad (2.92)$$

*Proof.* Also this statement is actually true for any triangular surface  $\Upsilon$ , regardless of whether it belongs to a sequence converging to  $\Sigma$ .

By (PL) (2.19),

$$U_{\Sigma}^{\Sigma'} P_\Sigma(\forall A) U_{\Sigma'}^\Sigma \leq P_{\Sigma'}(\forall \text{Gr}(A, \Sigma')). \quad (2.93)$$

Since  $(\exists A)^c = \emptyset A = \forall(A^c)$ , we have that

$$\begin{aligned} U_{\Sigma}^{\Sigma'} P_\Sigma(\exists A) U_{\Sigma'}^\Sigma &= U_{\Sigma}^{\Sigma'} (I - P_\Sigma((\exists A)^c)) U_{\Sigma'}^\Sigma \\ &= U_{\Sigma}^{\Sigma'} (I - P_\Sigma(\forall(A^c))) U_{\Sigma'}^\Sigma \\ &\geq I - P_{\Sigma'}(\forall \text{Gr}(A^c, \Sigma')) \\ &= I - P_{\Sigma'}(\forall(\text{Sr}(A, \Sigma')^c)) \\ &= I - P_{\Sigma'}((\exists \text{Sr}(A, \Sigma'))^c) \\ &= P_{\Sigma'}(\exists \text{Sr}(A, \Sigma')), \end{aligned} \quad (2.94)$$

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and

$$\begin{aligned}
U_{\Sigma}^{\Sigma'} P_{\Sigma}(\emptyset A) U_{\Sigma'}^{\Sigma} &= U_{\Sigma'}^{\Sigma'} P_{\Sigma}(\forall(A^c)) U_{\Sigma'}^{\Sigma} \\
&\leq P_{\Sigma'}(\forall \text{Gr}(A^c, \Sigma')) \\
&= P_{\Sigma'}(\forall(\text{Sr}(A, \Sigma')^c)) \\
&= P_{\Sigma'}(\emptyset \text{Sr}(A, \Sigma')).
\end{aligned} \tag{2.95}$$

Thus, inserting  $A \rightarrow B_{n\ell}$ ,  $\Sigma \rightarrow \Upsilon_n$ , and  $\Sigma' \rightarrow \Sigma$ ,

$$\begin{aligned}
U_{\Upsilon_n}^{\Sigma} P_{\Upsilon_n}(\exists B_{n\ell}) U_{\Sigma}^{\Upsilon_n} &\geq P_{\Sigma}(\exists \hat{C}_{n\ell}) \\
U_{\Upsilon_n}^{\Sigma} P_{\Upsilon_n}(\emptyset B_{n\ell}) U_{\Sigma}^{\Upsilon_n} &\leq P_{\Sigma}(\emptyset \hat{C}_{n\ell}).
\end{aligned} \tag{2.96}$$

On the other hand, inserting  $A \rightarrow \check{C}_{n\ell}$ ,  $\Sigma' \rightarrow \Upsilon_n$ , and  $\Sigma \rightarrow \Sigma$ ,

$$\begin{aligned}
U_{\Sigma}^{\Upsilon_n} P_{\Sigma}(\exists \check{C}_{n\ell}) U_{\Upsilon_n}^{\Sigma} &\geq P_{\Upsilon_n}(\exists \text{Sr}(\check{C}_{n\ell}, \Upsilon_n)) \\
U_{\Sigma}^{\Upsilon_n} P_{\Sigma}(\emptyset \check{C}_{n\ell}) U_{\Upsilon_n}^{\Sigma} &\leq P_{\Upsilon_n}(\emptyset \text{Sr}(\check{C}_{n\ell}, \Upsilon_n)).
\end{aligned} \tag{2.97}$$

Since for  $A \subseteq \Sigma$  always

$$A \subseteq \text{Sr}(\text{Gr}(A, \Sigma'), \Sigma), \tag{2.98}$$

and since  $A \subseteq B$  implies  $\exists(A) \subseteq \exists(B)$  and  $\emptyset(A) \supseteq \emptyset(B)$ , we have that

$$\begin{aligned}
P_{\Upsilon_n}(\exists \text{Sr}(\check{C}_{n\ell}, \Upsilon_n)) &\geq P_{\Upsilon_n}(\exists B_{n\ell}) \\
P_{\Upsilon_n}(\emptyset \text{Sr}(\check{C}_{n\ell}, \Upsilon_n)) &\leq P_{\Upsilon_n}(\emptyset B_{n\ell}).
\end{aligned} \tag{2.99}$$

Putting together (2.96), (2.97), (2.99),

$$\begin{aligned}
P_{\Sigma}(\emptyset \check{C}_{n\ell}) &\leq U_{\Upsilon_n}^{\Sigma} P_{\Upsilon_n}(\emptyset B_{n\ell}) U_{\Sigma}^{\Upsilon_n} \leq P_{\Sigma}(\emptyset \hat{C}_{n\ell}) \\
P_{\Sigma}(\exists \hat{C}_{n\ell}) &\leq U_{\Upsilon_n}^{\Sigma} P_{\Upsilon_n}(\exists B_{n\ell}) U_{\Sigma}^{\Upsilon_n} \leq P_{\Sigma}(\exists \check{C}_{n\ell}),
\end{aligned} \tag{2.100}$$

that is, in another notation,

$$P_{\Sigma}(\widehat{M}_{n\ell}(L_{\ell})) \leq U_{\Upsilon_n}^{\Sigma} P_{\Upsilon_n}(M_{\ell\Upsilon_n}(L_{\ell})) U_{\Sigma}^{\Upsilon_n} \leq P_{\Sigma}(\widetilde{M}_{n\ell}(L_{\ell})). \tag{2.101}$$

Now we want to conclude an analogous statement about  $L$  instead of  $L_{\ell}$ . Note that  $U_{\Upsilon_n}^{\Sigma} P_{\Upsilon_n}(\cdot) U_{\Sigma}^{\Upsilon_n}$  and  $P_{\Sigma}(\cdot)$  are two different PVMs that will in general not even commute with each other. The argument that we need has the following general

form: For two different PVMs  $P_1, P_2$ , the ranges satisfy the relations

$$\begin{aligned}
 & P_1(A_1) \leq P_2(A_2) \wedge P_1(B_1) \leq P_2(B_2) \\
 \Leftrightarrow & \text{Ran}(P_1(A_1)) \subseteq \text{Ran}(P_2(A_2)) \wedge \text{Ran}(P_1(B_1)) \subseteq \text{Ran}(P_2(B_2)) \\
 \Rightarrow & \underbrace{\text{Ran}(P_1(A_1)) \cap \text{Ran}(P_1(B_1))}_{=\text{Ran}(P_1(A_1)P_1(B_1))} \subseteq \underbrace{\text{Ran}(P_2(A_2)) \cap \text{Ran}(P_2(B_2))}_{=\text{Ran}(P_2(A_2)P_2(B_2))} \quad (2.102) \\
 \Leftrightarrow & P_1(A_1)P_1(B_1) \leq P_2(A_2)P_2(B_2) \\
 \Leftrightarrow & P_1(A_1 \cap B_1) \leq P_2(A_2 \cap B_2).
 \end{aligned}$$

Applying this argument to (2.101) and the finite intersection  $\bigcap_\ell$  yields the first line of (2.92).  $\square$

**Lemma 2.6.3.** Fix  $\ell \in \{1, \dots, r\}$ ;  $\check{C}_{n\ell}$  is a decreasing sequence of sets,  $\check{C}_{n\ell} \supseteq \check{C}_{n+1,\ell}$ , with

$$\bigcap_{n \in \mathbb{N}} \check{C}_{n\ell} \subseteq \overline{P_\ell}. \quad (2.103)$$

$\hat{C}_{n\ell}$  is an increasing sequence of sets,  $\hat{C}_{n\ell} \subseteq \hat{C}_{n+1,\ell}$ , with

$$\bigcup_{n \in \mathbb{N}} \hat{C}_{n\ell} \supseteq \text{interior}_\Sigma(P_\ell). \quad (2.104)$$

In particular,

$$\bigcap_{n \in \mathbb{N}} \check{C}_{n\ell} \setminus \hat{C}_{n\ell} \subseteq \partial P_\ell. \quad (2.105)$$

Moreover, equality holds in (2.103), (2.104), and (2.105) whenever  $\Upsilon_n \cap \Sigma = \emptyset$ .

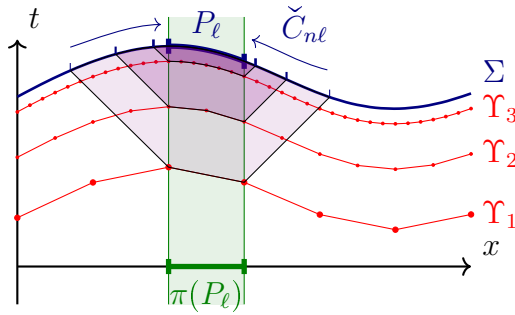


Figure 2.10: Convergence of the sets  $\check{C}_{n\ell}$  as  $n \rightarrow \infty$  for fixed  $\ell$  as in Lemma 2.6.3. Color online.

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*Proof.* The decreasing/increasing behavior of the sequence is a direct consequence of  $\Upsilon_{n+1} \subseteq \text{future}(\Upsilon_n)$  and the definition of grown and shrunk set. For demonstrating (2.103), since  $\pi_\Sigma$  is a homeomorphism  $\Sigma \rightarrow \mathbb{R}^3$ , it suffices to show that  $\bigcap_n \pi(\check{C}_{n\ell}) \subseteq \overline{\pi(P_\ell)}$  in  $\mathbb{R}^3$ . If  $\mathbf{y} \notin \overline{\pi(P_\ell)}$ , then it has positive distance to  $\pi(P_\ell)$  and  $\pi[[\Sigma - (\varepsilon, 0, 0, 0)] \cap \text{past}(\pi_\Sigma^{-1}(\mathbf{y}))]$  is disjoint from  $\pi(P_\ell)$  for sufficiently small  $\varepsilon > 0$ , so  $\mathbf{y} \notin \pi(\check{C}_{n\ell})$  for sufficiently large  $n$ . Similar arguments yield (2.104). Concerning the statement about equality, in that case for every  $x \in B_{n\ell}$ ,  $\text{future}(x) \cap \text{past}(\Sigma)$  has nonempty interior in  $\mathbb{M}$ , so  $\pi(\check{C}_{n\ell})$  contains an open neighborhood of  $\pi(P_\ell)$  and thus  $\overline{\pi(P_\ell)}$ . A similar statement holds for the interior.  $\square$

**Lemma 2.6.4.** *For every  $L \in \{0, 1\}^r$ ,  $\bigcap_{n \in \mathbb{N}} \widetilde{M}_{n\Sigma}(L) \setminus \widehat{M}_{n\Sigma}(L)$  is a null set w.r.t.  $\mu_{\Gamma(\Sigma)}$ .*

*Proof.* We make use here of the requirement  $\mu_\Sigma(\partial P_\ell) = 0$  in Definition 2.3.5. Consider first  $\widetilde{M}_{n\ell}(L_\ell)$  and  $\widehat{M}_{n\ell}(L_\ell)$ . In case  $L_\ell = 1$ , we have that

$$\begin{aligned} \widetilde{M}_{n\ell}(1) &= \exists \check{C}_{n\ell}, & \widehat{M}_{n\ell}(1) &= \exists \widehat{C}_{n\ell} \\ \Rightarrow \widetilde{M}_{n\ell}(1) \setminus \widehat{M}_{n\ell}(1) &= (\exists \check{C}_{n\ell}) \cap (\emptyset \widehat{C}_{n\ell}). \end{aligned} \quad (2.106)$$

In case  $L_\ell = 0$ , we have that

$$\widetilde{M}_{n\ell}(0) \setminus \widehat{M}_{n\ell}(0) = (\emptyset \widehat{C}_{n\ell}) \cap (\exists \check{C}_{n\ell}). \quad (2.107)$$

So either way,

$$\widetilde{M}_{n\ell}(L_\ell) \setminus \widehat{M}_{n\ell}(L_\ell) = (\emptyset \widehat{C}_{n\ell}) \cap (\exists \check{C}_{n\ell}) \subseteq \exists (\check{C}_{n\ell} \setminus \widehat{C}_{n\ell}). \quad (2.108)$$

Now we want to consider  $L$  instead of  $L_\ell$ . It is a general fact about sets that if  $A_\ell \subseteq B_\ell$  for all  $\ell$ , then

$$\left( \bigcap_{\ell} B_\ell \right) \setminus \left( \bigcap_{\ell} A_\ell \right) \subseteq \bigcup_{\ell} (B_\ell \setminus A_\ell). \quad (2.109)$$

Thus, for  $A_\ell = \widehat{M}_{n\ell}(L_\ell)$  and  $B_\ell = \widetilde{M}_{n\ell}(L_\ell)$ ,

$$\widetilde{M}_{n\Sigma}(L) \setminus \widehat{M}_{n\Sigma}(L) \subseteq \bigcup_{\ell=1}^r \widetilde{M}_{n\ell}(L_\ell) \setminus \widehat{M}_{n\ell}(L_\ell) \subseteq \bigcup_{\ell=1}^r \exists (\check{C}_{n\ell} \setminus \widehat{C}_{n\ell}) = \exists \left( \bigcup_{\ell=1}^r (\check{C}_{n\ell} \setminus \widehat{C}_{n\ell}) \right). \quad (2.110)$$

Now we want to take the intersection over all  $n \in \mathbb{N}$ . In this regard, we first note the following extension of (2.89): *if  $(A_n)_{n \in \mathbb{N}}$  is a decreasing sequence of sets, then*

$$\bigcap_n \exists A_n = \exists \left( \bigcap_n A_n \right). \quad (2.111)$$

After all, if  $q$  is a finite set that intersects every  $A_n$ , then it must contain a point from  $\bigcap_n A_n$ ; conversely, a finite set  $q$  intersecting  $\bigcap_n A_n$  trivially intersects every  $A_n$ .

Applying this to  $A_n = \bigcup_\ell (\check{C}_{n\ell} \setminus \hat{C}_{n\ell})$ , which is decreasing because  $\check{C}_{n\ell} \setminus \hat{C}_{n\ell}$  is, we obtain that

$$\bigcap_{n \in \mathbb{N}} \check{M}_{n\Sigma}(L) \setminus \widehat{M}_{n\Sigma}(L) \subseteq \exists \left( \bigcap_{n \in \mathbb{N}} \bigcup_{\ell=1}^r \check{C}_{n\ell} \setminus \hat{C}_{n\ell} \right). \quad (2.112)$$

It is another general fact about sets (not unrelated to (2.111)) that *if for every  $\ell \in \{1, \dots, r\}$ ,  $(A_{n\ell})_{n \in \mathbb{N}}$  is a decreasing sequence of sets, then*

$$\bigcap_{n \in \mathbb{N}} \bigcup_{\ell=1}^r A_{n\ell} = \bigcup_{\ell=1}^r \bigcap_{n \in \mathbb{N}} A_{n\ell}. \quad (2.113)$$

Thus, for  $A_{n\ell} = \check{C}_{n\ell} \setminus \hat{C}_{n\ell}$ ,

$$\bigcap_{n \in \mathbb{N}} \check{M}_{n\Sigma}(L) \setminus \widehat{M}_{n\Sigma}(L) \subseteq \exists \left( \bigcup_{\ell=1}^r \bigcap_{n \in \mathbb{N}} \check{C}_{n\ell} \setminus \hat{C}_{n\ell} \right) \subseteq \exists \left( \bigcup_{\ell=1}^r \partial P_\ell \right), \quad (2.114)$$

by Lemma 2.6.3 and (2.89). For any set  $A$  with  $\mu_\Sigma(A) = 0$  it follows that  $\exists A$  is, in every sector of configuration space  $\Gamma(\Sigma)$ , a finite union of null sets, so  $\mu_{\Gamma(\Sigma)}(\exists A) = 0$ . For  $A = \bigcup_\ell \partial P_\ell$  we obtain the statement of Lemma 2.6.4.  $\square$

*Proof of Theorem 2.3.7.* By Lemma 2.6.1 and Lemma 2.6.2, it suffices to show that for every  $L \in \{0, 1\}^r$ ,

$$\check{\mathbb{P}}_n(L) - \widehat{\mathbb{P}}_n(L) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.115)$$

From Lemma 2.6.4 and Property (1), according to which  $P_\Sigma$  must be absolutely continuous with respect to  $\mu_{\Gamma(\Sigma)}$ , we have that

$$P_\Sigma \left( \bigcap_{n \in \mathbb{N}} \check{M}_{n\Sigma}(L) \setminus \widehat{M}_{n\Sigma}(L) \right) = 0. \quad (2.116)$$

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The continuity property of measures  $\mu$  says that, for every decreasing sequence  $A_n$  of sets with  $\bigcap_n A_n =: A_\infty$ ,  $\mu(A_n) \rightarrow \mu(A_\infty)$  as  $n \rightarrow \infty$ . For every  $\Psi_\Sigma \in \mathcal{H}_\Sigma$ ,  $\mu(\cdot) := \langle \Psi_\Sigma | P_\Sigma(\cdot) | \Psi_\Sigma \rangle$  is a measure. We know from Lemma 2.6.1 that  $\widehat{M}_{n\Sigma}(L) \subseteq \widetilde{M}_{n\Sigma}(L)$ .

We show that for every  $L \in \{0, 1\}^r$ , the sequence  $A_n := \widetilde{M}_{n\Sigma}(L) \setminus \widehat{M}_{n\Sigma}(L)$  is decreasing: It suffices to show that  $\widetilde{M}_{n\Sigma}(L)$  is decreasing and  $\widehat{M}_{n\Sigma}(L)$  is increasing. We know from Lemma 2.6.3 that  $\check{C}_{n\ell}$  is decreasing and  $\hat{C}_{n\ell}$  is increasing, so by (2.89), both  $\exists \check{C}_{n\ell}$  and  $\emptyset \hat{C}_{n\ell}$  are decreasing, so  $\widetilde{M}_{n\ell}(L_\ell)$  (which is either  $\exists \check{C}_{n\ell}$  or  $\emptyset \hat{C}_{n\ell}$ , depending on  $L_\ell$ ) is decreasing, and so is

$$\widetilde{M}_{n\Sigma}(L) = \bigcap_{\ell=1}^r \widetilde{M}_{n\ell}(L_\ell). \quad (2.117)$$

Likewise,  $\widehat{M}_{n\ell}(L_\ell)$  (which is either  $\exists \hat{C}_{n\ell}$  or  $\emptyset \check{C}_{n\ell}$ , depending on  $L_\ell$ ) is increasing, and so is  $\widehat{M}_{n\Sigma}(L)$ . Therefore,  $A_n$  is decreasing, as claimed.

We can conclude that

$$\check{P}_n(L) - \hat{P}_n(L) = \langle \Psi_\Sigma | P_\Sigma(\widetilde{M}_{n\Sigma}(L) \setminus \widehat{M}_{n\Sigma}(L)) | \Psi_\Sigma \rangle \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.118)$$

This establishes the desired squeeze theorem argument and finishes the proof of Theorem 2.3.7.  $\square$

*Proof of Corollary 2.3.9.* It is well known that for a sequence  $P_n$  of projections, weak convergence to the projection  $P$  (i.e.,  $\langle \Psi | P_n | \Psi \rangle \rightarrow \langle \Psi | P | \Psi \rangle$  for every  $\Psi$ ) implies strong convergence (i.e.,  $P_n \Psi \rightarrow P \Psi$  for every  $\Psi$ ).<sup>5</sup> Set  $P_n = U_{\Upsilon_n}^\Sigma P_{\Upsilon_n}(M_{\mathcal{D}_n}(L)) U_{\Sigma}^{\Upsilon_n}$  and  $P = P_\Sigma(M_{\mathcal{D}}(L))$ . Then Theorem 2.3.7 provides the weak convergence, and the strong convergence was what we claimed.  $\square$

### Remarks.

13. *Type of convergence of  $(\Upsilon_n)_{n \in \mathbb{N}}$ .* The proof of Theorem 2.3.7 still goes through unchanged if the convergence of the sequence  $(\Upsilon_n)_{n \in \mathbb{N}}$  is not uniform but uniform on every bounded set.
14. *Alternative definition of  $B_{n\ell}$ .* In order to avoid the choice of a particular Lorentz frame in the definition of  $B_{n\ell}$  and thus of the detection probabilities,

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<sup>5</sup>For the sake of completeness, here is a proof: First,  $P_n^2 = P_n$  and  $P^2 = P$  imply that  $\|P_n \Psi\|^2 = \langle \Psi | P_n^2 | \Psi \rangle = \langle \Psi | P_n | \Psi \rangle \rightarrow \langle \Psi | P | \Psi \rangle = \|P \Psi\|^2$ . Second, since  $\langle \Psi | S | \Phi \rangle$  can be expressed through  $\langle \Psi \pm \Phi | S | \Psi \pm \Phi \rangle$  and  $\langle \Psi \pm i\Phi | S | \Psi \pm i\Phi \rangle$  (polarization identity [187, p. 63]), weak convergence implies  $\langle \Psi | P_n | \Phi \rangle \rightarrow \langle \Psi | P | \Phi \rangle$  for every  $\Psi$  and  $\Phi$ . Now  $\|P_n \Psi - P \Psi\|^2 = \langle \Psi | (P_n - P)^2 | \Psi \rangle = \langle \Psi | P_n^2 - P_n P - P P_n + P^2 | \Psi \rangle = \|P_n \Psi\|^2 - \langle \Psi | P_n | P \Psi \rangle - \langle P \Psi | P_n | \Psi \rangle + \|P \Psi\|^2 \rightarrow \|P \Psi\|^2 - \langle \Psi | P | P \Psi \rangle - \langle P \Psi | P | \Psi \rangle + \|P \Psi\|^2 = 0$ .

## 2.7 Ideas Towards the Reconstruction of a Wightman QFT

we could replace  $B_{n\ell}$  by

$$\check{B}_{n\ell} := \text{Sr}(P_\ell, \Upsilon_n). \quad (2.119)$$

(The use of Gr instead of Sr would lead to overlap among the  $B_{n\ell}$ , so they would no longer form a partition.) With this change, Theorem 2.3.7 remains valid. In the proof, we then need to modify the definition of  $\hat{C}_{n\ell}$  to

$$\hat{C}_{n\ell} := \text{Sr}(\check{B}_{n\ell}, \Sigma), \quad (2.120)$$

while the definition of  $\check{C}_{n\ell}$  is kept as it is. We would still use a preferred Lorentz frame for the definition of  $\check{C}_{n\ell}$ , but that is a matter of the method of proof, not of the statement of the theorem. The proof goes through as before, except that (2.104) needs to be checked anew: it is still true because for every  $x$  in the 3–interior of  $P_\ell$ ,  $\text{Gr}(\text{Gr}(x, \Upsilon_n), \Sigma) \subset P_\ell$  for sufficiently large  $n$ .

## 2.7 Ideas Towards the Reconstruction of a Wightman QFT

Recall that the three fundamental mathematical constituents for the Wightman setting were

- $\mathcal{H}$ : A single Hilbert space,
- $U(\circ) := (U(g))_{g \in \mathcal{P}_+^\uparrow}$ : A family of Poincaré transformation implementers,
- $\phi(\circ) := (\phi(f))_{f \in \mathcal{S}(\mathbb{R}^{d+1})}$ : A family of field operators.

In our case,  $d = 3$ . Further, Wightman axioms (0)–(4) in Section 1.2.3 require the existence of a distinct domain  $D \subset \mathcal{H}$  and a vacuum vector  $\Omega \in D$ .

### 2.7.1 Recovering the Hilbert Space

For a reconstruction of a Wightman QFT from  $\mathcal{E}$ , the first step is to define a Hilbert space  $\mathcal{H}$ , for instance by setting  $\mathcal{H} := \mathcal{H}_{E_0}$ . The vacuum vector is then evidently recovered as  $\Omega = \Omega_{E_0}$ .

Recovering the family  $U(\circ)$  from  $\mathcal{E}$  and Properties (1)–(7) does not work without further assumptions. It may be tempting to choose a definition of the kind  $U(g) = U_\Sigma^{g\Sigma}$  modulo identification of Hilbert spaces. Here,  $U_\Sigma^{g\Sigma}$  maps  $\mathcal{H}_\Sigma \rightarrow \mathcal{H}_{g\Sigma}$ , whereas  $U(g)$  has to map  $\mathcal{H} = \mathcal{H}_{E_0}$  onto itself. Given a hypersurface evolution  $\mathcal{E}$ , the only

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available identification of  $\mathcal{H}$  with  $\mathcal{H}_\Sigma$  is  $U_{E_0}^\Sigma$ , so the only possible operator that is equivalent to  $U_\Sigma^{g\Sigma}$  and acting on  $\mathcal{H}$  is

$$U_{g\Sigma}^{E_0} U_\Sigma^{g\Sigma} U_{E_0}^\Sigma = U_{E_0}^{E_0} = I_{E_0}. \quad (2.121)$$

So a recovery of the family  $U(\circ)$  necessarily requires additional assumptions on  $\mathcal{E}$ . A natural additional assumption allowing for such a recovery is given by Poincaré covariance: In order to gain additional structure on  $\mathcal{E}$ , we postulate the existence of a family of unitary operators  $(S_{g,\Sigma})$  with  $S_{g,\Sigma} : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{g\Sigma}$ , such that

$$U_{g\Sigma}^{g\Sigma'} = S_{g,\Sigma'} U_\Sigma^{g\Sigma'} S_{g,\Sigma}^{-1}, \quad (2.122)$$

where  $g$  runs through  $\mathcal{P}_+^\uparrow$  and  $\Sigma$  runs through all Cauchy surfaces. This allows for the recovery of a unique family  $U(\circ)$  via

$$U(g) := U_{gE_0}^{E_0} S_{g,E_0}. \quad (2.123)$$

The naturality assumptions

$$S_{0,\Sigma} = I_\Sigma, \quad S_{h,g\Sigma} S_{g,\Sigma} = S_{hg,\Sigma} \quad (2.124)$$

ensure that  $(U(g))_{g \in \mathcal{P}_+^\uparrow}$  is a group implementing  $\mathcal{P}_+^\uparrow$ . In analogy to the field covariance (Wightman axiom (2)), we further postulate that the PVMs transform covariantly and comply with the factorization  $T$ :

$$\begin{aligned} P_{g\Sigma}(\forall(gA)) &= S_{g,\Sigma} P_\Sigma(\forall(A)) S_{g,\Sigma}^{-1} \\ T_{g\Sigma,gA} S_{g,\Sigma} T_{\Sigma,A}^{-1} &= S_{g,\Sigma}|_{\mathcal{H}_{\Sigma,A}} \otimes S_{g,\Sigma}|_{\mathcal{H}_{\Sigma,\Sigma \setminus A}} \end{aligned} \quad (2.125)$$

for all measurable  $A \subset \Sigma$ . In total, we require the following additional property for hypersurface evolutions:

- (8) **Poincaré covariance:** There exists a family  $(S_{g,\Sigma})$ ,  $S_{g,\Sigma} : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{g\Sigma}$  of unitary identification operators, satisfying (2.122), (2.124) and (2.125).

This property entails that the structure of  $\mathcal{H}_\circ$ ,  $P_\circ$  and  $U_\circ^\circ$  is the same in all frames of reference. Generally, frame-independence of this kind is a central principle in many physical theories. But it would also be possible to obtain physical predictions in a relativistic setting without Property (8).

## 2.7.2 Recovering the Field Operators

Perhaps, the most difficult part in the reconstruction of a Wightman field theory from  $\mathcal{E}$  is the definition of suitable field operators  $\phi_r(f)$  with spin index  $r \in \{1, \dots, 2s + 1\}$ . There are two major difficulties, which have to be overcome for this:

- The PVMs  $P_\Sigma$  have to be used for a construction of field operators  $\phi_{\Sigma,r}(f_\Sigma)$  acting on a dense domain in  $\mathcal{H}_\Sigma$  with sufficiently regular  $f_\Sigma : \Sigma \rightarrow \mathbb{C}$
- Since  $f_\Sigma$  only establishes a smearing in space direction, we also have to smear in time direction in order to obtain  $\phi_r(f)$ ,  $f \in \mathcal{S}(\mathbb{M})$ .

The first step cannot be done without further assumptions, since the PVM does not reveal any information about the spin degrees of freedom hidden in the Hilbert space structure. In order to define creation- and annihilation operators, it would be necessary to have a Fock space structure on  $\mathcal{H}_\Sigma$ , which is compatible with  $P_\Sigma$ . That means,  $\Psi_\Sigma$  should be expressed as a vector in Fock space  $\mathcal{F} = \mathcal{F}(\Gamma_s(\Sigma))$ , with spin-configuration space  $\Gamma_s(\Sigma)$ , see (1.20), where a configuration contains pairs  $(x_j, r_j)$  with  $x_j \in \Sigma$  and spin index  $r_j \in \{1, \dots, 2s + 1\}$ . In that case,  $\mathcal{F}$  allows for a natural PVM  $P_{\mathcal{F},\Sigma}$ : For any measurable  $B \subseteq \Gamma(\Sigma)$ , the operator  $P_{\mathcal{F},\Sigma}(B)$  projects to all spin-configurations in

$$B_s := \{ \{(x_1, r_1), \dots, (x_N, r_N)\} \mid \{x_1, \dots, x_N\} \in B \} \subseteq \Gamma_s(\Sigma), \quad (2.126)$$

i.e.,  $B_s$  contains all  $\Gamma_s(\Sigma)$ -spin-configurations compatible with  $B$ .

The additional requirement now is a

- **Fock space structure assumption:** For certain<sup>6</sup> Cauchy surfaces  $\Sigma$ , there exists an isomorphism  $\mathcal{I} : \mathcal{H}_\Sigma \rightarrow \mathcal{F}(\Gamma_s(\Sigma))$ , such that  $\mathcal{I} P_\Sigma(B) \mathcal{I}^{-1} = P_{\mathcal{F},\Sigma}(B)$  for all measurable  $B \subset \Gamma(\Sigma)$ .

The definition of  $a_{\mathcal{F},\Sigma,r}^\dagger(f_\Sigma)$ ,  $a_{\mathcal{F},\Sigma,r}(f_\Sigma)$  then follows in analogy to (1.24), but for unordered configurations:

$$\begin{aligned} (a_{\mathcal{F},\Sigma,r}(f_\Sigma)\Psi)(q) &= \sum_{j=1}^N \frac{(\pm 1)^j}{\sqrt{N}} \delta_{rr_j} f_\Sigma(x_j) \Psi(q \setminus (x_j, r_j)), \\ (a_{\mathcal{F},\Sigma,r}^\dagger(f_\Sigma)\Psi)(q) &= \sqrt{N+1} \int_\Sigma \overline{f_\Sigma(x)} \Psi(q \cup (x, r)) dx, \end{aligned} \quad (2.127)$$

<sup>6</sup>This assumption is required to hold on all Cauchy surfaces  $\Sigma$ , on which the operators  $a_r^\sharp(f_\Sigma)$  have to be defined in order to recover the field operators  $\phi_r(f)$  below. For instance, this could be the foliation of  $\mathbb{M}$  by horizontal Cauchy planes  $(E_t)_{t \in \mathbb{R}}$  or any other foliation.

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where  $q = \{(x_1, r_1), \dots, (x_N, r_N)\}$ . Here,  $f_\Sigma \in L^2(\Sigma)$ , and a suitable dense domain is given by the space of wave functions with finite particle number

$$\mathcal{D}_\Sigma := \{\Psi \in L^2(\Gamma_s(\Sigma)) \mid \exists N_{\max} \in \mathbb{N}_0 : \Psi^{(N)} \equiv 0 \ \forall N > N_{\max}\}. \quad (2.128)$$

Definition (2.127) then entails the Heisenberg operators

$$a_r^\#(f_\Sigma) = U_\Sigma^{E_0} \mathcal{I}^{-1} a_{\mathcal{F}, \Sigma, r}^\#(f_\Sigma) \mathcal{I} U_{E_0}^\Sigma, \quad (2.129)$$

defined on

$$D_\Sigma := U_\Sigma^{E_0} \mathcal{I}^{-1} [\mathcal{D}_\Sigma] \subset \mathcal{H}_{E_0}, \quad (2.130)$$

which immediately allow for defining  $\phi_r(f_\Sigma) = a_r^\dagger(f_\Sigma) + a_r(f_\Sigma)$ .

The second step, namely smearing  $\phi_r(f_\Sigma)$  in time direction, also poses some difficulties. An intuitive way of smearing would be to consider a foliation of  $\mathbb{M}$ , given by the family of horizontal Cauchy surfaces  $(E_t)_{t \in \mathbb{R}}$ , and to formally define:

$$\phi_r(f) := \int_{\mathbb{R}} \phi_r(f_{E_t}) dt, \quad (2.131)$$

where a given  $f \in \mathcal{S}(\mathbb{M})$  allows for defining the single-time smearing functions  $f_{E_t}(\mathbf{x}) := f(t, \mathbf{x})$ . Boundedness and rapid decay of  $f$  imply  $f_{E_t} \in L^2(\Sigma)$ . We may then simply set  $\phi_r(f)^* := \phi_r(f)$ . For fermions,  $\phi_r(f_{E_t})$  is an operator bounded by a constant times  $\|f_{E_t}\|_2$ . So (2.131) also defines a bounded operator. However, for bosons,  $\phi_r(f_{E_t})$  is unbounded, and it is hence far from obvious that the operator  $\phi_r(f)$  is well-defined. In any case, the domains  $D_\Sigma$  considered above do not coincide for different  $\Sigma$ , in general. So it is necessary to explicitly construct a domain  $D \subset \mathcal{H}$ , which is invariant under all  $\phi_r(f)$ . The requirement by the Wightman axioms, that  $f \mapsto \langle \Psi_1, \phi_r(f) \Psi_2 \rangle$  be a distribution, must then be checked separately.

Further, (2.131) makes use of a specific choice for a foliation of  $\mathbb{M}$ , which is of course frame dependent. It would be desirable to give a definition of  $\phi_r(f)$  in any foliation, and to prove that all definitions indeed lead to the same operator. We expect this to be true, as a consequence of the Poincaré covariance assumption (8).

After a successful construction of  $\phi_r(f)$ , it would still be necessary to verify Wightman's covariance axiom, the causality axiom and the completeness axiom. We assert that, after a suitable definition of  $\phi_r(f)$ , Wightman covariance follows naturally from the Poincaré covariance assumption (8). Wightman causality is asserted to follow from (IL) and (PL) and completeness from the choice of  $\phi_r(f)$ . However, since we do not have a general definition for  $\phi_r(f)$  following from a hy-

persurface evolution  $\mathcal{E}$ , we are not in the position to give any general theorems or more precise assertions, yet.

### 2.7.3 On Spectral Positivity

The last Wightman axiom missing is spectral positivity. As indicated in Remark 11, this axiom may stand in conflict with (PL), which would make some further steps necessary until a Wightman QFT can be recovered from a hypersurface evolution. In the following, we first present some existing incompatibility results outlining this conflict and then discuss possible ways out.

An early result concerning the conflict between propagation locality and spectral positivity is given by the celebrated **Reeh–Schlieder theorem** [188]: Consider an open bounded region  $\mathcal{O} \subset \mathbb{M}$ , as well as the algebra  $\mathcal{A}(\mathcal{O})$  comprising all field operator products  $A$  smeared by a function supported<sup>7</sup> in  $\mathcal{O}$ . Then, under reasonable assumptions of relativistic QFT, the algebra  $\mathcal{A}(\mathcal{O})$  is cyclic with respect to  $\Omega$ . These assumptions include the existence of a continuous representation of the group  $\mathcal{P}_+^\uparrow$  by unitary operators  $U(g)$  on a Hilbert space  $\mathcal{H}$ , a unique vacuum  $\Omega \in \mathcal{H}$  and spectral positivity<sup>8</sup>. So the set of vectors obtained by applying field operator products  $A$  to  $\Omega$  must be dense in  $\mathcal{H} = \mathcal{H}_{E_0}$ . However, if field operator products were indeed defined as suggested above, and (PL) would hold, we expect  $A\Omega$  to be localized in the region  $R_0 := J(\mathcal{O}) \cap E_0$ , i.e.,  $A\Omega \in \text{Ran}(P_{E_0}(\mathbb{V}(R_0)))$ . This would even lead to a contradiction: The range of a projection is a closed subspace of  $\mathcal{H}$  and would have to be dense, meaning that  $\text{Ran}(P_{E_0}(\mathbb{V}(R_0))) = \mathcal{H}$ . Therefore, all projections to  $B \subseteq \mathbb{V}(R_0)^c$  would have to be 0. Now any set  $\mathbb{V}(A^c)$  for an open ball  $A = B_R(\mathbf{x})$  can be found in some  $\mathbb{V}(R_0)^c$  for a suitable  $R_0 = J(\mathcal{O}) \cap E_0 \subset A$  small enough. These sets  $\mathbb{V}(A^c)$  generate the Borel  $\sigma$ -algebra on  $\Gamma(\Sigma)$ <sup>9</sup>, so we indeed get  $P_{E_0}(B) = 0$  for any Borel  $B \subseteq \Gamma(\Sigma)$ , which is a contradiction to  $\text{Ran}(P_{E_0}(\mathbb{V}(R_0))) = \mathcal{H}$ .

In fact, if the assumptions for the Reeh–Schlieder theorem are valid, then for any position representation at time  $t$ , there must be a vector  $\Psi = A\Omega, A \in \mathcal{A}(\mathcal{O})$ , which is “leaking out” of  $J(\mathcal{O})$ . That means, if  $I_t : \mathcal{H} \rightarrow L^2(\mathbb{R}^3)$  is an isometric isomorphism, then  $\Psi_t := I_t\Psi$  has a support exceeding  $J(\mathcal{O}) \cap E_t$ , as depicted in Figure 2.11.

<sup>7</sup>By this, we mean  $\mathcal{A}(\mathcal{O})$  comprises all  $A = \int f(x_1, \dots, x_N) \phi_{r_1}(x_1) \dots \phi_{r_N}(x_N) dx_1 \dots dx_N$ , such that  $\text{supp } f \subset \mathcal{O}^N$ .

<sup>8</sup>In this and the upcoming no-go results, spectral positivity is characterized by the Hamiltonian  $H$  being bounded from below. For a Poincaré covariant theory, this is equivalent to Wightman’s condition of the joint spectrum of  $P^\mu$  lying in the closed forward lightcone.

<sup>9</sup>This fact is proved below [4, (100)].

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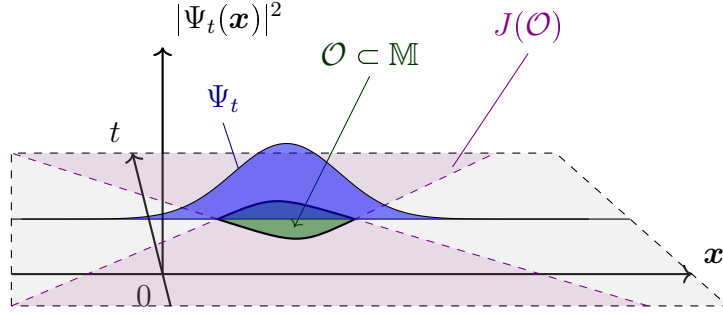


Figure 2.11: Heuristic depiction of the Reeh–Schlieder theorem: There are  $A \in \mathcal{A}(\mathcal{O})$ , such that  $\Psi = A\Omega$  “leaks out of  $J(\mathcal{O})$ ”. Color online.

Similar results outlining the conflict between propagation locality and spectral positivity have been provided by Hegerfeldt in a series of papers [189, 190, 185, 191]. These results (some of them are called **Hegerfeldt’s theorem**) concern a concrete spin–boson–like model, where two two–level atoms at positions  $0, \mathbf{x} \in \mathbb{R}^d$  are coupled to a common boson field (see Figure 2.12). Intuitively, if initially the boson field is in the vacuum and only one atom is excited, then one may expect the other atom to become excited only after a time delay of  $|t| \geq |\mathbf{x}|$ . However, Hegerfeldt showed that, assuming spectral positivity, there is a nonzero probability for the excitation to “jump” to the other atom at almost all times  $t \in \mathbb{R}$ , including  $|t| < |\mathbf{x}|$ . As indicated by Buchholz and Yngvason [192], this does not contradict causality in an algebraic setting, as the projection to the excited level of an atom is not required to be an element of any local algebra  $\mathcal{A}(\mathcal{O})$ . By contrast, in a hypersurface evolution setting, a simultaneous validity of (PL) and (IL) would certainly exclude such immediate jumps, if a projection to the excited level is assumed to act locally<sup>10</sup>.

A closely related no–go result on unifying locally propagating wavepackets with spectral positivity is **Malament’s theorem** [193]. Further no–go results of this type have been established and carefully compared to the above–mentioned ones by Halvorson and Clifton [186]. They suggest that a Poincaré–covariant hypersurface evolution may indeed be forced to violate spectral positivity.

<sup>10</sup>By a local action, we mean the following: Consider a projection  $P_{\mathbf{y},e}$  to the excited level of an atom at  $\mathbf{y} \in \{0, \mathbf{x}\}$ . Then, there has to exist a neighborhood  $R_{\mathbf{y}}$  of  $\mathbf{y}$ , small against  $|\mathbf{x}|$ , such that under the factorization  $\mathcal{H} \cong \mathcal{H}_{E_0, R_{\mathbf{y}}} \otimes \mathcal{H}_{E_0, E_0 \setminus R_{\mathbf{y}}}$ , the projection factorizes as  $P_{\mathbf{y},e} \cong P_{\mathbf{y},e, R_{\mathbf{y}}} \otimes \mathbb{1}_{E_0 \setminus R_{\mathbf{y}}}$ .

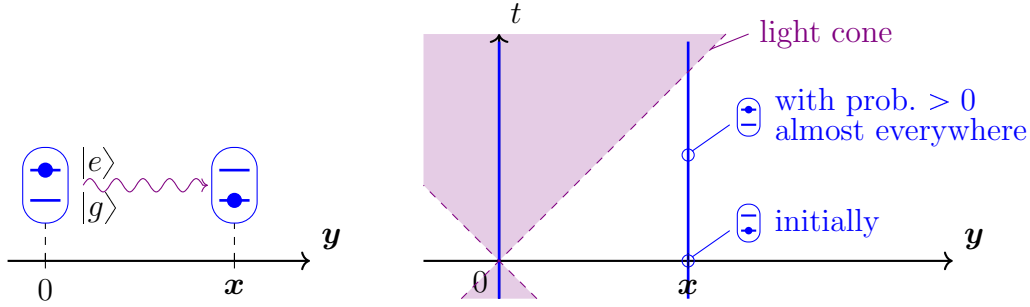


Figure 2.12: Depiction of Hegerfeldt's theorem: If spectral positivity holds, then the atom at  $\mathbf{x}$  gets excited at arbitrarily small times. Color online.

The no-go results above suggest, that for reconciling a hypersurface evolution with the Wightman setting, it will certainly be necessary to modify the properties required for a hypersurface evolution. As an obvious example, (PL) may be dropped to allow for spectral positivity. This may come along with a modified way of defining field operator products  $A$ , i.e., one that is schematically different from the one sketched above. The price, one would have to pay for this is that results above (or in [4]), which assume dropped properties, would no longer apply.

There is also a novel approach for reconciling (PL) and spectral positivity, if particle creation from the vacuum is not forbidden [194]. Mathematically, the violated property is [4]:

- **No particle creation from vacuum (NCFV):** For any pair of Cauchy surfaces  $\Sigma, \Sigma'$ , we have  $U_{\Sigma'}^{\Sigma} P_{\Sigma}(\{\emptyset\}) U_{\Sigma}^{\Sigma'} = P_{\Sigma'}(\{\emptyset\})$ .

This property is closely related to the Wightman vacuum  $\Omega$  being an eigenvector of the Hamiltonian for eigenvalue 0, which follows from the Wightman axiom that assumes  $\Omega$  to be invariant under all  $U(g)$ . Indeed, if uniqueness of the hypersurface evolution vacua  $\Omega_{\Sigma}$  is assumed, this entails  $U_{\Sigma'}^{\Sigma} \Omega_{\Sigma} = \Omega_{\Sigma'}$  up to a phase, so the vacuum vector effectively stays invariant under time evolution. Although (NCFV) seems like a natural assumption, it is not required within any of the Properties (1)–(8) for a hypersurface evolution and the results above and in [4] would still hold if (NCFV) was violated. This way, as [194] suggests, it might be possible to construct a hypersurface evolution simultaneously satisfying (PL) and spectral positivity.

Another option would be to investigate mathematical models that do not satisfy spectral positivity, but may comply with the required Properties (1)–(7) for a hypersurface evolution. An example is the free Dirac field given by the Hamiltonian  $H = d\Gamma(\boldsymbol{\alpha} \cdot \mathbf{p} + m\beta)$ , which allows for constructing a hypersurface evolution [4, Sect. 4]. The interacting 1 + 1-dimensional Dirac fields in [165, 166] also violate

## 2 Hypersurface Evolution

spectral positivity, since there exists a mode with dispersion relation unbounded from below. And even the Thirring model (1.108), (1.109) and the Federbush model (1.112), (1.113) feature such dispersion relations unbounded from below. Spectral positivity is then only obtained after “re-defining the particles” by a bosonization. Therefore, we indeed deem the study of spectrally non-positive relativistic systems to be a useful intermediate step towards the final description of physical systems.

Let us also clarify a possible misconception at this point: Unboundedness from below of a Hamiltonian is often associated with “instability of the system”. A common heuristic picture is that such a system will “continually drop down to arbitrarily negative energies”. This in turn would release an arbitrary amount of energy, which is a process that has never been observed in nature and fiercely contradicts the physical principle of inexhaustible energy sources being forbidden. As Malament writes about the implications of the occurrence of a negative-energy particle [193, pp.4–5]:

[. . .] the particle could serve as an infinite energy source [. . .] We could first tap the particle to run all the lights in Canada for a week. To be sure, in the process of doing so, we would lower its energy state. Then we could run all the lights for a second week, and lower the energy state of the particle still further. And so on. If the particle had no finite ground state, this process could continue forever.

However, this energy extraction requires the system to be coupled to an external system, just like an atomic or molecular system may transition from an excited to a lower level when being coupled to a radiation field or a heat bath. As long as a quantum system is left on its own, it just follows the evolution  $U(t) = e^{-itH}$ , which commutes with all spectral projections, so there is no transition to lower levels. This includes systems with Hamiltonians unbounded from below, as for the Dirac particles described above.

Nevertheless, the absence of spectral positivity within a QFT model can become a considerable drawback from the mathematical side, as it is essential for the analytic continuation from Wightman distributions to Schwinger functions and vice versa. Thus, it impedes the use of powerful Euclidean QFT methods. Haag–Ruelle scattering theory also relies on spectral positivity and will hence become inapplicable. Further, a thermal equilibrium state (formally given by  $\rho = \frac{1}{Z}e^{-\beta H}$  with  $\beta$  being the inverse temperature) will be difficult to define if  $H$  is unbounded from below. So there are plenty of incentives to restore spectral positivity after a successful definition of some non-spectrally positive hypersurface dynamics  $\mathcal{E}$ .

## 3 Fock Space Extensions

In this Section, we present the general construction schemes for the two Fock space extension frameworks: the infinite tensor product– (ITP) and the extended state space (ESS) construction.

The ITP construction yields a non–separable Hilbert space  $\widehat{\mathcal{H}}$ , called ITP space, which allows for finding uncountably many orthogonal copies of the original Fock space  $\mathcal{F}$  in it. ITP spaces have been introduced by von Neumann as early as 1939 [44] and thenceforth been applied to Weyl transformations in several cases. We outline the ITP construction and its application to Weyl transformations in Section 3.1. Since this application is already well–investigated, we do not discuss the ITP construction for the Weyl–like Gross transformations in Chapter 4, but only for Bogoliubov transformations in Chapter 5.

By contrast, the novel ESS construction, presented in Section 3.2, is used in both Chapters 4 and 5. As already mentioned in Section 1.1, it provides two vector spaces  $\overline{\mathcal{F}} \subset \overline{\mathcal{F}}_{\text{ex}}$ , which extend a dense subspace of  $\mathcal{F}$ . The generalized scheme presented in Section 3.2 includes the examples in Chapters 4 and 5 as special cases. As a byproduct of the ESS construction, we obtain vector spaces  $\text{Ren}_1$ ,  $\text{Ren}$  and a field  $\text{eRen}$ , that permit a rigorous description of formally infinite quantities. These spaces might even become useful in a purely algebraic treatment of QFT dynamics that does not refer to a fixed representation or vectors within a Fock space extension. Further, we discuss concrete realizations of the ESS construction, that may turn out advantageous in applications to quantum dynamics.

This chapter can also be seen as supplementary to the upcoming Chapters 4 and 5.

### 3.1 Infinite Tensor Products (ITPs)

In the following, we recap the construction of von Neumann’s ITP space  $\widehat{\mathcal{H}}$ . Further details on how  $\widehat{\mathcal{H}}$  can be decomposed in orthogonal subspaces and how to find a convenient basis, are provided in Sections 5.2.3 and 5.8. For a thorough discussion, we refer the reader to von Neumann’s original work [44].

### 3.1.1 Definition

The main goal of the ITP construction is to make sense of the tensor product

$$\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k, \quad (3.1)$$

where  $\mathcal{H}_k$  are separable Hilbert spaces indexed by a (not necessarily finite or countable) set  $I$ . Each  $\mathcal{H}_k$  has a scalar product  $\langle \cdot, \cdot \rangle_k$  and induced norm  $\| \cdot \|_k$ . Choosing one vector per Hilbert space,  $\Psi_k \in \mathcal{H}_k$ , it is easy to define a formal ITP as the family  $(\Psi) = (\Psi_k)_{k \in I}$ ,  $\Psi_k \in \mathcal{H}_k$ . If  $I$  is countable (so one may equivalently set  $I = \mathbb{N}$ ), then  $(\Psi) = (\Psi_1, \Psi_2, \dots)$  defines a sequence.

On the space spanned by these families, one may write down formal expressions for a bilinear form that would serve as a candidate for a scalar product inducing a norm. Formally, for two families  $(\Phi), (\Psi)$ ,

$$\langle (\Phi), (\Psi) \rangle = \prod_{k \in I} \langle \Phi_k, \Psi_k \rangle_k, \quad \|(\Psi)\| = \prod_{k \in I} \|\Psi_k\|_k. \quad (3.2)$$

In order to answer the question, whether the expressions in (3.2) indeed define complex numbers, one introduces the notion of convergence within a (possibly uncountable) sum or product:

- For  $z_k \in \mathbb{C}$ ,  $k \in I$ , we call  $\sum_{k \in I} z_k$  or  $\prod_{k \in I} z_k$  convergent to  $a \in \mathbb{C}$ , if for all  $\delta > 0$ , there exists some finite set  $I_\delta \subset I$ , such that for all finite sets  $J \subseteq I$  with  $I_\delta \subseteq J$ , we have

$$\left| a - \sum_{k \in J} z_k \right| \leq \delta \quad \text{or} \quad \left| a - \prod_{k \in J} z_k \right| \leq \delta, \quad \text{respectively.} \quad (3.3)$$

A simple consequence of this definition is that  $\sum_{k \in I} z_k$  can only converge if  $z_k \neq 0$  occurs for only countably many  $k \in I$ . So the question of convergence reduces to that of sequence convergence. Further, it is shown in [44] that  $\prod_{k \in I} z_k < \infty$  if and only if we have  $z_k = 0$  for at least one  $k \in I$  or if  $\sum_{k \in I} |z_k - 1| < \infty$ . The heuristic reason is that  $\prod_{k \in I} z_k = \exp(\sum_{k \in I} \ln z_k)$  and  $\ln z_k$  can be linearly approximated near 1 as  $\ln z_k = 1 - z_k + \mathcal{O}((1 - z_k)^2)$ .

If  $\prod_{k \in I} |z_k|$  converges to a nonzero number, then  $\prod_{k \in I} z_k$  converges if and only if no infinite phase variation occurs. That is, if  $\arg(z_k) \in (-\pi, \pi]$  is the phase of

the complex number  $z_k$ , then it is required that

$$\sum_{k \in I} |\arg(z_k)| < \infty. \quad (3.4)$$

In order to establish a notion of convergence, even when (3.4) is violated, one defines that

- $\prod_{k \in I} z_k$  is quasi-convergent, if and only if  $\prod_{k \in I} |z_k|$  converges.

The expression (3.2) may now contain divergent products, which formally correspond to “infinite scalar products” or “infinite norms”. In order to exclude them, one restricts the allowed families to the set Cseq of so-called “ $C$ -sequences”. More precisely, one calls a family  $(\Psi) = (\Psi_k)_{k \in I}$  a

- **$C$ -sequence**  $((\Psi) \in \text{Cseq})$  iff  $\prod_{k \in I} \|\Psi_k\|_k < \infty$ ,
- **$C_0$ -sequence** iff  $\sum_{k \in I} \|\Psi_k\|_k - 1 < \infty \iff \sum_{k \in I} \|\Psi_k\|_k^2 - 1 < \infty$ .

Each  $C_0$ -sequence is also a  $C$ -sequence. For all  $C$ -sequences, we have a well-defined value  $\|(\Psi)\| \in \mathbb{C}$  by (3.2) and each  $C$ -sequence, that is not a  $C_0$ -sequence, must automatically satisfy  $\|(\Psi)\| = 0$ .

Now, the bilinear form on  $\text{span}(\text{Cseq})$  in (3.2) renders finite values, but is generally not positive definite and hence no scalar product: For infinite cardinality of  $I$ , there exist  $(\Psi) \neq 0$  with<sup>1</sup>  $\langle (\Psi), (\Psi) \rangle = 0$ . So one has to mod out an equivalence relation  $\sim_C$ , which identifies all  $(\Psi)$  with 0 that satisfy  $\langle (\Psi), (\Psi) \rangle = 0$ .

Von Neumann implicitly constructs this equivalence relation by identifying  $C$ -sequences with functionals. For this identification, let  $\prod_{k \in I}^{\otimes} \mathcal{H}_k$  be the space of all conjugate-linear functionals  $\text{Cseq} \rightarrow \mathbb{C}$ . Following [44], we can embed  $\iota : \text{Cseq} \rightarrow \prod_{k \in I}^{\otimes} \mathcal{H}_k$  by identifying  $(\Phi) \in \text{Cseq}$  with the functional

$$\Phi = \iota((\Phi)) : (\Psi) \mapsto \prod_{k \in I} \langle \Phi_k, \Psi_k \rangle_k. \quad (3.5)$$

This identification essentially sets up an equivalence relation  $\sim_C$  on Cseq, where  $(\Phi) \sim_C (\Phi')$ , whenever  $\iota((\Phi)) = \iota((\Phi'))$ . In Proposition 5.8.1, we show that equivalence is given if and only if  $(\Phi)$  and  $(\Phi')$  just differ by a family of complex factors  $(c_k)_{k \in I}$  with  $\prod_{k \in I} c_k = 1$ . The functionals in  $\iota[\text{Cseq}]$  are then the equivalence classes

<sup>1</sup>As an example, consider any family  $(\Psi)$  scaled such that  $\|\Psi_k\|_k = 1/2$ . In fact, any  $(\Psi)$  with  $\|\Psi_k\|_k = c$ ,  $c \in (0, 1)$  being constant, serves as an example. However, for finite cardinality of  $I$ ,  $\langle (\Psi), (\Psi) \rangle = 0$  implies that  $\Psi_k = 0$  for one tensor factor, so  $(\Psi) = 0$ .

### 3 Fock Space Extensions

and the span of these functionals is denoted by [44]:

$$\prod_{k \in I}^{\iota \otimes} \mathcal{H}_k := \text{span}(\iota[\text{Cseq}]). \quad (3.6)$$

In the following, we will also drop the embedding map  $\iota$  and simply identify  $(\Phi)$  with  $\Phi$ . An inner product  $\langle \cdot, \cdot \rangle$  can uniquely be defined on  $\prod_{k \in I}^{\iota \otimes} \mathcal{H}_k$  via

$$\langle \Phi, \Psi \rangle = \prod_{k \in I} \langle \Phi_k, \Psi_k \rangle_k, \quad (3.7)$$

which makes  $\prod_{k \in I}^{\iota \otimes} \mathcal{H}_k$  a pre-Hilbert space and induces a norm  $\|\Phi\|$ . Both the inner product and the norm agree with the second expression in (3.2) under identification  $\|\Phi\| = \|(\Phi)\|$ . The completion of (3.6) with respect to the norm is exactly von Neumann's **ITP space**

$$\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k := \overline{\prod_{k \in I}^{\iota \otimes} \mathcal{H}_k}^{\|\cdot\|}. \quad (3.8)$$

Later in Section 5.8, we will use that  $\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k$  is the space of all  $\Phi \in \prod_{k \in I}^{\otimes} \mathcal{H}_k$ , such that there exists a Cauchy sequence  $(\Phi^{(r)})_{r \in \mathbb{N}} \subset \prod_{k \in I}^{\iota \otimes} \mathcal{H}_k$  which converges to  $\Phi$  in the weak-\* topology on  $\prod_{k \in I}^{\otimes} \mathcal{H}_k$ . That means,

$$\lim_{r, s \rightarrow \infty} \|\Phi^{(r)} - \Phi^{(s)}\| = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} \Phi^{(r)}((\Psi)) = \Phi((\Psi)) \quad (3.9)$$

for all  $(\Psi) \in \text{Cseq}$ .

Checking that  $\langle \cdot, \cdot \rangle$  is indeed an inner product on  $\prod_{k \in I}^{\iota \otimes} \mathcal{H}_k$ , and extends to  $\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k$ , such that one obtains a Hilbert space, is a technical task accomplished in [44]. Note that replacing “ $C$ -sequence” by “ $C_0$ -sequence” in the construction results in the same space  $\widehat{\mathcal{H}}$  after completion, since all  $C$ -sequences that are not  $C_0$ -sequences get identified by  $\iota$  with the same functional 0.

### 3.1.2 Infinite Tensor Products via Basis Choice

Consider now the formal expression (1.145) of a coherent state vector  $\Omega_W$  outside Fock space. Formally,

$$\Omega_W = W(s)\Omega = e^{-\frac{\|s\|^2}{2}} \sum_{N \in \mathbb{N}_0} \frac{(a^\dagger(s))^N}{N!} \Omega = e^{a^\dagger(s) - a(s)} \Omega. \quad (3.10)$$

One way to make sense of such non-Fock coherent states has been proposed by Chung [195], Kibble [196, 197, 198, 199], as well as Faddeev and Kulish [200]: We choose an orthonormal basis  $(e_k)_{k \in \mathbb{N}}$  of the one-particle Hilbert space  $\mathfrak{h} = L^2(\mathbb{R}^d)$ , such that the functions  $e_k$  (also called modes) are sufficiently regular. A suitable regularity could, for instance, be  $e_k \in \mathcal{D}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$  or  $e_k \in \mathcal{S}(\mathbb{R}^d)$ . The latter two choices would allow any displacement function  $s \in \mathcal{D}'(\mathbb{R}^d)$  or  $s \in \mathcal{S}'(\mathbb{R}^d)$  to be written in basis coefficients  $s_k := \langle e_k, s \rangle$ .

In particular, this includes the function  $s(\mathbf{k}) = -g|\mathbf{k}|^{-3/2}$  in  $d = 3$  from the Van Hove model (1.137). As an ITP space we choose

$$\widehat{\mathcal{H}} := \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{F}(\{e_k\}), \quad (3.11)$$

where  $\mathcal{H}(\{e_k\})$  is the Fock space over a single mode  $e_k$ , with creation and annihilation operators  $a_k^\dagger, a_k$  and vacuum  $\Omega_k$  satisfying  $a_k \Omega_k = 0$ . This allows us to expand the formal expression (3.10):

$$\Omega_W = \exp\left(\sum_{k \in \mathbb{N}} (s_k a_k^\dagger - \bar{s}_k a_k)\right) \Omega = \prod_{k \in \mathbb{N}} \exp\left(s_k a_k^\dagger - \bar{s}_k a_k\right) \Omega, \quad (3.12)$$

and make sense of it as in ITP in  $\widehat{\mathcal{H}}$ :

$$\Omega_W := \prod_{k \in \mathbb{N}} \Omega_{W,k}, \quad \Omega_{W,k} = \exp\left(s_k a_k^\dagger - \bar{s}_k a_k\right) \Omega_k \in \mathcal{F}(\{e_k\}). \quad (3.13)$$

One may even give a rigorous meaning to the displacement operator  $W(s)$  as

$$W(s) : \widehat{\mathcal{H}} \rightarrow \widehat{\mathcal{H}}, \quad W(s) = \prod_{k \in \mathbb{N}}^{\otimes} W_k(s_k), \quad W_k(s_k) = \exp\left(s_k a_k^\dagger - \bar{s}_k a_k\right). \quad (3.14)$$

Note that by the same arguments as in the proof of Lemma 5.4.5,  $W(s)$  does not depend on the choice of  $C$ -sequences used for representing  $\Psi \in \widehat{\mathcal{H}}$ . Here, all  $W_k(s_k)$  are unitary on  $\mathcal{F}(\{e_k\})$ , so  $W(s)$  maps ITPs to ITPs while preserving

### 3 Fock Space Extensions

their scalar product. And since the span of all ITPs is dense in  $\widehat{\mathcal{H}}$ , the operator  $W(s)$  can be continued to all of  $\widehat{\mathcal{H}}$  and is unitary on it. See also [71], where the construction is kept more general.

The operators  $W(s)$  with  $s \in \mathcal{S}'(\mathbb{R}^d)$  or  $s \in \mathcal{D}'(\mathbb{R}^d)$  (depending on the definition) now generate an **extended Weyl algebra**, which is given by

$$\mathcal{W}_* = \text{span}\{W(s_1) \dots W(s_N) \mid s_j \in *\}, \quad (3.15)$$

where  $*$  is a placeholder for either  $\mathcal{S}'(\mathbb{R}^d)$  or  $\mathcal{D}'(\mathbb{R}^d)$ . Also, other linear spaces could be used in place of  $*$ , as long as  $s_k = \langle e_k, s \rangle$  can be sensibly defined for all basis vectors  $e_k$  and all  $s \in *$ . The Weyl relations formally read

$$W(s)^{-1} = W(-s), \quad W(s_1)W(s_2) = e^{-\frac{i}{2}\sigma(s_1, s_2)}W(s_1 + s_2), \quad (3.16)$$

which serves at the same time as a definition of the unitary operator

$$e^{-\frac{i}{2}\sigma(s_1, s_2)} := W(s_1)W(s_2)W(s_1 + s_2)^{-1} : \widehat{\mathcal{H}} \rightarrow \widehat{\mathcal{H}}. \quad (3.17)$$

The formal expression for the symplectic form  $\sigma(s_1, s_2) = \langle s_1, s_2 \rangle - \langle s_2, s_1 \rangle$  is ill-defined and may, in certain cases, correspond to a divergent integral. Therefore the (well-defined) operator  $e^{-\frac{i}{2}\sigma(s_1, s_2)}$  can heuristically be interpreted as a **rotation by an infinite phase**.

We remark that [195, 196, 197, 198, 199, 200] all concern both perturbative and non-perturbative scattering theory for QED in  $d = 3$  space dimensions. So far, there is no self-adjoint Hamiltonian known for a cutoff-free relativistic QED model in  $d = 3$  and neither of the above references provides a self-adjoint QED Hamiltonian. Although the definitions of coherent ITP states are rigorous in these articles, and the authors provide explicit expressions for dressing operators  $W$  [195, (32)], [200, Sect. 4], the discussion of how the dressing acts is discussed either perturbatively or kept on a heuristic level. Nevertheless, these discussions may provide useful hints for a construction of non-perturbative renormalized Hamiltonians.

In case one intends to use dressing transformations employed in scattering theory to derive dressing transformations for non-perturbative renormalization, one needs to be aware of **Dollard modifiers** [201, 202, 159]. These are additional terms that enter the definition of the S-matrix in a way similar to a dressing transformation, but which are not needed for the definition of a renormalized Hamiltonian  $\tilde{H}$ . A prominent example where Dollard modifiers appear is given by the Coulomb Hamiltonian  $H = H_0 + H_I$  with  $H_0 = \frac{-\Delta}{2m}$  and  $H_I = g|\mathbf{x}|^{-1}$  in  $d = 3$  dimensions. The Møller operators  $\Omega^{\text{in/out}} = \lim_{t \rightarrow \mp\infty} e^{itH} e^{-itH_0}$  as in (1.57) do not exist. But one

can rigorously define the modified Møller operators [201, (29)–(33)]

$$\Omega_c^{\text{in/out}} = \lim_{t \rightarrow \mp\infty} e^{itH} e^{-iH_{0,c}(t)}, \quad H_{0,c}(t) = tH_0 + gm \frac{\text{sgn}(t)}{|\mathbf{k}|} \log \left( \frac{2|t|}{m} |\mathbf{k}|^2 \right), \quad (3.18)$$

with  $|\mathbf{k}|^2 = -\Delta$  being a multiplication operator in momentum space. The modification term in  $H_{0,c}(t)$  diverges logarithmically as  $|t| \rightarrow \infty$ , and can hence be seen as an infinite phase. However, this phase is not needed for the construction of a self-adjoint operator  $H$ , but only for the definition of the Møller operators  $\Omega_c^{\text{in/out}}$  and the S-matrix  $S_c = (\Omega_c^{\text{out}})^{-1} \Omega_c^{\text{in}}$ . Correspondingly, Faddeev and Kulish split their formal S-matrix modification into a dressing transformation (called  $e^{R(t)}$ ), leading out of Fock space, and a Dollard phase (called  $e^{i\Phi(t)}$ ) [200, (10), (11)]. In what follows, only an expression  $e^{R_f}$  of a form similar to  $e^{R(t)}$  is used as a dressing transformation that leads out of Fock space and is implemented on the ITP space  $\widehat{\mathcal{H}}$ . This transformation  $e^{R_f}$  is of a form similar to the bosonic Weyl operators  $W(s)$  above, but it acts on a system containing both bosons and fermions (similar to the polaron models in Section 1.3.7). Further, the form factor  $s(\mathbf{p}, \mathbf{k})$  also depends on the fermion momentum  $\mathbf{p}$ , as for the Pauli–Fierz model in Section 1.3.7. So  $e^{R_f}$  is a Gross and not a Weyl transformation according to the classification from Section 1.3.4.

### 3.1.3 Infinite Tensor Products via Patches

Another way to make sense of (3.10) has been proposed by Fröhlich [43], as well as Könenberg and Matte [80]:  $\mathbb{R}^d$  is decomposed into a sequence of patches  $\bigcup_{k \in \mathbb{N}} C_k = \mathbb{R}^d$ , such that  $C_k \cap C_{k'}$  is a null set for all  $k, k' \in \mathbb{N}$ . This allows for decomposing the one-particle space as  $L^2(\mathbb{R}^d) = \bigoplus_{k \in \mathbb{N}} L^2(C_k)$ . The corresponding ITP space is

$$\widehat{\mathcal{H}} := \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{F}(C_k). \quad (3.19)$$

For a given displacement function  $s \notin L^2(\mathbb{R}^d)$ , the patches  $C_k$  are now chosen such that  $s_{C_k} := s|_{C_k} \in L^2(C_k)$  for all  $k \in \mathbb{N}$ . For instance,  $s(\mathbf{k}) = -g|\mathbf{k}|^{-3/2}$  is radially symmetric and has an  $L^2$ -integral that diverges at  $|\mathbf{k}| \rightarrow 0$  and  $|\mathbf{k}| \rightarrow \infty$ , which suggests the use of

$$C_k := \{\mathbf{k} \in \mathbb{R}^d \mid R_k \leq |\mathbf{k}| \leq R_{k+1} \vee r_k \geq |\mathbf{k}| \geq r_{k+1}\}, \quad (3.20)$$

where a suitable choice of radii is given by  $R_k = 2^{k-1}$  and  $r_k = 2^{-k+1}$ . So each  $C_k, k \geq 2$  is a union of two concentric rings and all  $C_k$  are compact. Similar

### 3 Fock Space Extensions

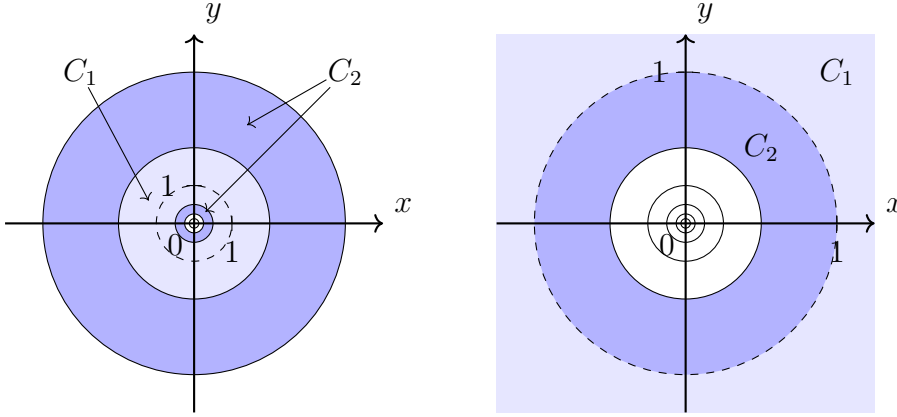


Figure 3.1: Left: Depiction of the compact patches  $C_k$  in (3.20).  
Right: The choice of  $C'_k$  as in [43]. Color online.

choices for  $C_k$  are used in [43, 80]. However, both references use displacement functions  $s$ , that are only IR-divergent. So one may choose concentric rings  $C'_k := \{\mathbf{k} \mid r_{k-1} \geq |\mathbf{k}| \geq r_k\}$ ,  $k \geq 2$  and set an infinite outer radius for the outer ring  $C'_1 := \{\mathbf{k} \mid |\mathbf{k}| \geq 1\}$  [43]. Or one may even exclude the outer ring from the definition of  $\widehat{\mathcal{H}}$ , if  $s(\mathbf{k}) = 0$  for large  $|\mathbf{k}|$  [80]. Blanchard [71] suggests a general choice of compact  $C_k$  covering  $\mathbb{R}^d$  (for  $d = 3$ ), while also considering the approach via a regular basis  $(e_k)_{k \in \mathbb{N}}$  in (3.11) as an alternative ITP construction. Since  $s_{C_k} \in L^2(C_k)$ , the dressing transformation  $W_k(s_{C_k}) : \mathcal{F}(C_k) \rightarrow \mathcal{F}(C_k)$  is a well-defined unitary operator, so

$$W(s) : \widehat{\mathcal{H}} \rightarrow \widehat{\mathcal{H}}, \quad W_k(s) = \prod_{k \in \mathbb{N}}^{\otimes} W_k(s_{C_k}) \quad (3.21)$$

is unitary, as well. The set of all families  $s = (s_{C_k})_{k \in \mathbb{N}}$ ,  $s_{C_k} \in L^2(C_k)$  forms a vector space (analogous to  $\mathcal{S}'(\mathbb{R}^d)$  or  $\mathcal{D}'(\mathbb{R}^d)$  in the previous section) and allows for defining an extended Weyl algebra in analogy to (3.15).

The ITP space construction (3.19) via patches  $(C_k)_{k \in \mathbb{N}}$  may seem more natural than the construction (3.11) via  $(e_k)_{k \in \mathbb{N}}$ , since it allows for respecting radial symmetry. However, the single-particle spaces  $L^2(C_k)$  are infinite-dimensional, which makes the approach more difficult to handle.

For Bogoliubov transformations, there is yet another complication with the patch-based ITP space construction (3.19), which forces us to pursue the basis approach (3.11) in Chapter 5. To demonstrate this issue, we turn a Weyl transformation into a Bogoliubov transformation by replacing the linear term  $\sum_{k \in \mathbb{N}} (s_k a_k^\dagger - \bar{s}_k a_k)$

in the exponential (3.12) by the quadratic term

$$Q = \sum_{k \in \mathbb{N}} (s_k a_k^\dagger a_k^\dagger - \overline{s_k} a_k a_k) = \sum_{k, k' \in \mathbb{N}} \delta_{kk'} (s_k a_k^\dagger a_{k'}^\dagger - \overline{s_k} a_k a_{k'}). \quad (3.22)$$

The coefficient matrix  $S_Q = (\delta_{kk'} s_k)_{k, k' \in \mathbb{N}}$  is diagonal, but will generally no longer be diagonal after a coordinate transformation. That means, for  $U = (U_{kk'})_{k, k' \in \mathbb{N}}$  : unitary, the matrix  $U^* S_Q U$  may contain off-diagonal terms. These correspond to cross-terms  $a_k^\dagger a_{k'}^\dagger$  and  $a_k a_{k'}$  with  $k \neq k'$ , which induce correlations between the modes and hence spoil the mode-wise split as in (3.14). Therefore, the choice of a basis  $(e_k)_{k \in \mathbb{N}}$  or patches  $(C_k)_{k \in \mathbb{N}}$  has to be done in a sophisticated way for Bogoliubov transformations, namely such that no correlations between different modes or patches occur. The cases discussed in Chapter 5 are chosen exactly such that they allow for a “sophisticated split” into a basis  $(e_k)_{k \in \mathbb{N}}$ , as we explain in Sections 5.3.2 and 5.4. But there might not exist a suitable split into patches  $(C_k)_{k \in \mathbb{N}}$  without (3.22) containing cross-terms between different patches<sup>2</sup>. Therefore, we choose an ITP space construction in Chapter 5, which relies on a basis  $(e_k)_{k \in \mathbb{N}}$  as in (3.11).

## 3.2 Extended State Space (ESS)

Both Chapters 4 and 5 concern two different pairs of extensions  $\overline{\mathcal{F}} \subset \overline{\mathcal{F}}_{\text{ex}}$  of a dense subspace of Fock space  $\mathcal{F}$ , which are constructed following the same scheme. In principle, it is possible to generate an even larger number of “extended state spaces”  $\overline{\mathcal{F}} \subset \overline{\mathcal{F}}_{\text{ex}}$ , using this construction scheme. There are two choices that can be made, which lead to different spaces  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$ :

- The choice of the one-particle space  $\mathcal{R}(X)$
- The choice of the  $N$ -particle space  $\mathcal{R}^{(N)}(X)$  complying with  $\mathcal{R}(X)$

The two choices are constrained by some Conditions (A), (B), (C) and (D) and have to be adapted to the respective model.

<sup>2</sup>An example for the impossibility of a split is the following: Consider an orthonormal basis  $(e_j)_{j \in \mathbb{N}}$  with  $\text{supp } e_j = \mathbb{R}^d$  for all  $j$ . Further, define the matrix  $S$  by  $S e_j = j e_j$ , so its spectrum is  $\sigma(S) = \mathbb{N}$ , with all eigenvalues being of multiplicity 1. Now assume, there exists a decomposition  $L^2(\mathbb{R}^d) = \bigoplus_{k \in \mathbb{N}} L^2(C_k)$ , such that  $S$  has no off-diagonal terms with respect to the split, i.e.,  $S$  maps each  $L^2(C_k)$  into itself. Then, the (orthogonal) projection  $P = P(C_k)$  to  $L^2(C_k)$  commutes with  $S$ , so  $S(P e_j) = P S e_j = j(P e_j)$ . Hence, either  $P e_j = 0$ , so  $e_j \perp L^2(C_k)$ , or  $P e_j$  is an eigenvector of eigenvalue  $j$ . Since some  $L^2(C_k)$  must contain a nonzero vector,  $e_j \perp L^2(C_k)$  cannot hold for all  $j, k \in \mathbb{N}$ . So for some  $j$  and  $k$ , the vector  $P e_j = P(C_k) e_j$  is indeed an eigenvector of  $j$ , and by the multiplicity assumption and  $P^2 = P$ , we have  $P(C_k) e_j = e_j \Rightarrow e_j \in L^2(C_k)$ , which contradicts the assumption  $\text{supp } e_j = \mathbb{R}^d$ .

### 3 Fock Space Extensions

In Section 3.2.2 we explain the general construction scheme for  $\overline{\mathcal{F}} \subset \overline{\mathcal{F}}_{\text{ex}}$ . Reasonable choices for  $\mathcal{R}(X)$  and  $\mathcal{R}^{(N)}(X)$  are discussed in Sections 3.2.3 (Van Hove example), 3.2.4 (polaron models as in Chapter 4) and 3.2.5 (Bogoliubov transformations as in Chapter 5).

#### 3.2.1 Motivation

The ESS construction is inspired by formal coherent state vectors outside Fock space, such as  $\Omega_W$  in (1.145), but also the “IBC–vacuum”  $\Omega_{\text{IBC}}$  in (1.143). Recall that

$$\Omega_W^{(N)}(\mathbf{k}_1, \dots, \mathbf{k}_N) = e^{-\frac{\|s\|^2}{2}} \Omega_{\text{IBC}}^{(N)}(\mathbf{k}_1, \dots, \mathbf{k}_N) = e^{-\frac{\|s\|^2}{2}} \frac{1}{\sqrt{N!}} \prod_{\ell=1}^N s(\mathbf{k}_\ell), \quad (3.23)$$

where the Van Hove example discussed in (1.137) requires  $s(\mathbf{k}) = -g|\mathbf{k}|^{-3/2}$  with  $s \notin L^2(\mathbb{R}^3)$ . The ITP space  $\widehat{\mathcal{H}}$  constructed in Section 3.1.2 allows to define  $\Omega_W \in \widehat{\mathcal{H}}$ , but not  $\Omega_{\text{IBC}}$ , since the formal renormalization factor  $e^{-\frac{\|s\|^2}{2}}$  cannot be treated separately. One goal of the ESS approach is to remedy this shortcoming.

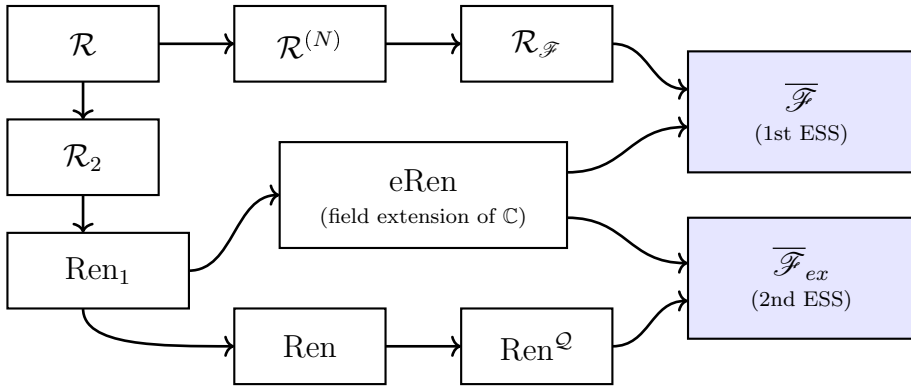


Figure 3.2: Construction scheme for the ESS approach. Color online.

The construction process is sketched in Figure 3.2. To make sense of  $\Omega_W$ , we start from a space  $\mathcal{R}(X)$  of generalized one–particle functions, that shall contain  $s$ . Via an  $N$ –particle space  $\mathcal{R}^{(N)}(X)$ , we arrive at  $\mathcal{R}_{\mathcal{F}}(X)$ , which contains configuration space functions  $\mathcal{Q}(X) \rightarrow \mathbb{C}$ , such as  $\Omega_{\text{IBC}}$ . This  $\mathcal{R}_{\mathcal{F}}(X)$  can be seen as a generalization of Fock space  $\mathcal{F}(X) = L^2(\mathcal{Q}(X))$ , although  $\mathcal{R}_{\mathcal{F}}(X)$  only extends a dense subspace of  $\mathcal{F}(X)$ .

The divergent integral  $\|s\|^2$  over the function product  $|s|^2$  is interpreted as an

element of a vector space  $\text{Ren}_1$ , and the “wave function renormalization”

$$e^\mathfrak{r} := \exp\left(-\frac{\|s\|^2}{2}\right) \quad (3.24)$$

is accommodated in a field  $e\text{Ren}$  extending  $\mathbb{C}$ . We then interpret the coherent state  $\Omega_W = e^\mathfrak{r}\Omega_{\text{IBC}}$  as an element of the first extended state space  $\overline{\mathcal{F}}$ , which is an  $e\text{Ren}$ -vector space containing elements like  $e^\mathfrak{r}\Psi_j$  with  $e^\mathfrak{r} \in e\text{Ren}$  and  $\Psi_j \in \mathcal{R}_{\mathcal{F}}(X)$ . For a suitable choice of  $\mathcal{R}(X)$ , the infinite counterterm  $E_\infty$  in (1.139) can be interpreted as an element  $\mathfrak{r}' = E_\infty \in \text{Ren}_1$ . In order to define  $E_\infty$  as a multiplication operator on  $\overline{\mathcal{F}}$ , we need to multiply vectors from  $\overline{\mathcal{F}}$  with an arbitrary number of factors  $\mathfrak{r}'_j \in \text{Ren}_1$ . These arbitrary products  $\mathfrak{R} := \mathfrak{r}'_1 \cdot \dots \cdot \mathfrak{r}'_n$ , are captured within an algebra  $\text{Ren}$ . We then allow for  $\mathfrak{R} = \mathfrak{R}(q)$  to depend on the configuration  $q \in \mathcal{Q}$  by introducing a space  $\text{Ren}^{\mathcal{Q}}$  and capture  $\mathfrak{R} \cdot e^\mathfrak{r}\Psi_j$  within a second extended state space  $\overline{\mathcal{F}}_{\text{ex}}$ .

### 3.2.2 General Construction Scheme

As mentioned above, our first aim is to construct a space  $\mathcal{R}_{\mathcal{F}}(X)$  of configuration space functions, that may go beyond Fock space  $\mathcal{F} = L^2(\mathcal{Q})$ . In order to reconstruct any Fock space vector from elements of  $\mathcal{R}_{\mathcal{F}}(X)$ , we require  $\mathcal{R}_{\mathcal{F}}(X)$  to contain at least a dense subspace of  $\mathcal{F}$ . This is ensured by Conditions (A), (B), (C) and (D), below. We proceed as follows:

- The construction begins with a **generalized one-particle space**<sup>3</sup>  $\mathcal{R}(X)$ . We require that
  - (A)  $\mathcal{R}(X)$  shall contain complex-valued, measurable functions over a measure space  $(X, \mu)$ , defined almost everywhere.
  - (B)  $\mathcal{R}(X)$  shall extend a dense subspace of  $L^2(X, \mu)$ .

The space  $\mathcal{R}(X)$  corresponds to the space  $\dot{\mathcal{S}}_1^\infty$  in Chapter 4, and is called  $\mathcal{E}(\mathbb{N})$  in Chapter 5. As usual, we will drop the parentheses  $(X)$ , if they are not explicitly needed.

- Starting from  $\mathcal{R}$ , one may construct the **generalized  $N$ -particle space**  $\mathcal{R}^{(N)}(X)$ , of which we require that

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<sup>3</sup>The names “generalized one-particle space” and “space of generalized one-particle wave functions” as in Chapter 5 are considered to be synonymous.

### 3 Fock Space Extensions

(C)  $\mathcal{R}^{(N)}(X)$  consists of complex-valued, measurable functions, defined almost everywhere on  $X^N$ .

(D)  $\mathcal{R}^{(N)}(X)$  contains the algebraic tensor product  $\mathcal{R}^{\otimes a}(X)$ .

In other words,  $\mathcal{R}^{(N)}(X)$  has to contain all finite linear combinations of tensor products  $\phi_1 \otimes \dots \otimes \phi_N$  with  $\phi_j \in \mathcal{R}$ . The measure on  $X^N$ , called  $\mu_N$ , is the product measure induced by  $\mu$  on  $X$ . This way, we assure that the following statement holds.

**Proposition 3.2.1.**  $\mathcal{R}^{(N)}(X)$  extends a dense subspace of the  $N$ -particle Hilbert space  $L^2(X, \mu)^{\otimes N}$ .

*Proof.* Every  $\Psi^{(N)} \in L^2(X, \mu)^{\otimes N} \cong L^2(X^N)$  can be written as

$$\Psi^{(N)} = \sum_{k \in \mathbb{N}} c_k \psi_{k,1} \otimes \dots \otimes \psi_{k,N}, \quad \|\psi_{k,j}\|_{L^2(X)} = 1, \quad (3.25)$$

with  $(c_k)_{k \in \mathbb{N}} \subset \mathbb{C}$  being an absolutely convergent series. Truncating the series at  $K \in \mathbb{N}$ , we obtain an approximating vector  $\Psi_K^{(N)}$ , with the sum in (3.25) running over  $k \leq K$ . For a given  $\varepsilon > 0$ , we can achieve  $\|\Psi^{(N)} - \Psi_K^{(N)}\| < \varepsilon/2$  by choosing  $K$  sufficiently large. Since  $\mathcal{R}(X)$  extends a dense subspace of  $L^2(X)$ , we may approximate all  $\psi_{k,j}$  for  $k \leq K$  by vectors  $\phi_{k,j} \in \mathcal{R}$  to arbitrary precision, obtaining

$$\Phi_K^{(N)} = \sum_{k=1}^K c_k \phi_{k,1} \otimes \dots \otimes \phi_{k,N}, \quad \Phi_K^{(N)} \in \mathcal{R}^{(N)}(X). \quad (3.26)$$

Choosing a sufficient approximation precision of  $\phi_{k,j} \in \mathcal{R}$ , we finally arrive at

$$\|\Psi_K^{(N)} - \Phi_K^{(N)}\| < \varepsilon/2 \quad \Rightarrow \quad \|\Psi^{(N)} - \Phi_K^{(N)}\| < \varepsilon. \quad (3.27)$$

□

Depending on the purpose, it may also be useful for  $\mathcal{R}^{(N)}$  to contain certain infinite linear combinations of tensor products. The space  $\mathcal{R}^{(N)}$  can be found under the name  $\mathcal{E}^{(N)}(\mathbb{N})$  in Chapter 5.

Within Chapter 4, two particle species are considered, which would, in this scheme, require the construction of spaces  $\mathcal{R}^{(M,N)}$  with  $M$  particles of the first kind and  $N$  particles of the second kind. More generally, a construction of  $\mathcal{R}^{(N_1, \dots, N_n)}$  for  $n \in \mathbb{N}$  particle species can be performed in analogy to  $\mathcal{R}^{(N)}$ .

### 3.2 Extended State Space (ESS)

The Conditions (C) and (D), as well as Proposition 3.2.1 extend analogously to this case:

- (C)  $\mathcal{R}^{(N_1, \dots, N_n)}$  consists of measurable functions, defined almost everywhere on  $X^{N_1 + \dots + N_n}$ .
- (D)  $\mathcal{R}^{(N_1, \dots, N_n)}$  contains the algebraic tensor product  $\mathcal{R}^{\otimes_a N_1} \otimes_a \dots \otimes_a \mathcal{R}^{\otimes_a N_n}$ .

**Proposition 3.2.2.**  $\mathcal{R}^{(N_1, \dots, N_n)}(X)$  extends a dense subspace of the sector Hilbert space  $L^2(X, \mu)^{\otimes(N_1 + \dots + N_n)}$ .

*Proof.* Apply Proposition 3.2.1 with  $N = \sum_{j=1}^n N_j$ . □

The construction in Chapter 4 slightly deviates from the above scheme: The space  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$ , defined there, decays into sectors, which we call  $\mathcal{R}_{xy}^{(M, N)}$ , here. As we explain in Section 3.2.4,  $\mathcal{R}_{xy}^{(M, N)}$  can be constructed starting from two different spaces  $\mathcal{R}_x$  and  $\mathcal{R}_y$ . However,  $\mathcal{R}_x \subset \mathcal{R}_y$ , and the space  $\mathcal{R}_{xy}^{(M, N)}$  can be seen as a subspace of some  $\mathcal{R}^{(M, N)}$  constructed from a single one-particle space  $\mathcal{R} = \mathcal{R}_y$  in the above-mentioned manner. So the general construction scheme described here also captures the space  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$  constructed in Chapter 4.

- Taking the orthogonal sum over all sectors, we obtain the space of **generalized Fock space functions**

$$\mathcal{R}_{\mathcal{F}}(X) := \bigoplus_{N \in \mathbb{N}_0} \mathcal{R}^{(N)}(X). \quad (3.28)$$

This  $\mathcal{R}_{\mathcal{F}}$  contains functions defined everywhere but on a null set on  $\mathcal{Q}$  and extends a dense subspace of Fock space  $\mathcal{F}(X)$ . The generalization to  $n$  particle species is given by  $\mathcal{R}_{\mathcal{F}} := \bigoplus_{N_1, \dots, N_n \in \mathbb{N}_0} \mathcal{R}^{(N_1, \dots, N_n)}$ , which also extends a dense subspace of the respective Fock space  $\mathcal{F}$ . In Chapter 4, the space  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$  corresponds to  $\mathcal{R}_{\mathcal{F}}$  for two particle species and in Chapter 5, the space  $\mathcal{E}_{\mathcal{F}}(\mathbb{N})$  corresponds to  $\mathcal{R}_{\mathcal{F}}$  for one particle species.

The extended state spaces  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$  shall now contain elements of the form

$$e^{\mathfrak{r}} \Psi_j \in \overline{\mathcal{F}} \quad \text{or} \quad e^{\mathfrak{r}} \mathfrak{r}'_1 \cdot \dots \cdot \mathfrak{r}'_p \Psi_j \in \overline{\mathcal{F}}_{\text{ex}}, \quad (3.29)$$

with  $p \in \mathbb{N}$ ,  $\Psi_j \in \mathcal{R}_{\mathcal{F}}$  and  $\mathfrak{r}, \mathfrak{r}'_k$  being well-defined versions of formal expressions like

$$\mathfrak{r} = \langle \phi_1, \phi_2 \rangle = \int_X \overline{\phi_1(x)} \phi_2(x) d\mu(x), \quad \phi_1, \phi_2 \in \mathcal{R}. \quad (3.30)$$

### 3 Fock Space Extensions

We now make sense of possibly divergent “renormalization integrals” as in (3.30):

- The product function  $r(x) := \overline{\phi_1(x)}\phi_2(x)$ , defined almost everywhere on  $X$ , is interpreted as an element of the **function product space**

$$\mathcal{R}_2 := \text{span}\{\phi_1\phi_2 \mid \phi_1, \phi_2 \in \mathcal{R}\}. \quad (3.31)$$

In Chapters 4 and 5,  $\mathcal{R}$  is chosen to be stable under function multiplication, so  $\mathcal{R}_2 = \mathcal{R}$  and no separate definition of a space  $\mathcal{R}_2$  is necessary. In any case, all functions in  $\mathcal{R}_2$  are measurable by means of (A).

- A formal integral  $\mathfrak{r} = \int_X r(x) d\mu(x)$  can now be expressed by the function  $r \in \mathcal{R}_2$ . However, we would like to think of  $\mathfrak{r}$  not as a function, but as the numerical value of the integral, so we may interpret  $\mathfrak{r}$  as a  $\mathbb{C}$ -number whenever  $r \in L^1(X)$ . In particular,  $r_1, r_2 \in \mathcal{R}_2$  should correspond to the same  $\mathfrak{r}$ , if their difference is an  $L^1$ -function with integral 0. This identification is established by introducing the equivalence relation

$$r_1 \sim_{\text{Ren}_1} r_2 \quad :\Leftrightarrow \quad (r_1 - r_2) \in L^1(X) \quad \wedge \quad \int_X (r_1(x) - r_2(x)) d\mu(x) = 0. \quad (3.32)$$

The **space of renormalization factors** is then defined as

$$\text{Ren}_1 := \mathcal{R}_2 / \sim_{\text{Ren}_1}. \quad (3.33)$$

We call it  $\text{Ren}_1$  in Chapter 4 and  $\text{Ren}_1(\mathbb{N})$  in Chapter 5. Heuristically,  $\mathfrak{r} \in \text{Ren}_1$  can be seen as a space containing “controlled infinitely large numbers”.

Whenever  $(r_1 - r_2) \in L^1$  for some functions  $r_1, r_2 \in \mathcal{R}_2$  representing  $\mathfrak{r}_1, \mathfrak{r}_2 \in \text{Ren}_1$ , we can identify  $(\mathfrak{r}_1 - \mathfrak{r}_2)$  with a unique complex number. This identification allows decomposing  $\text{Ren}_1$  into coarser equivalence classes

$$\text{Clas}_1 := \text{Ren}_1 / \sim_1, \quad \mathfrak{r}_1 \sim_1 \mathfrak{r}_2 \quad :\Leftrightarrow \quad (r_1 - r_2) \in L^1. \quad (3.34)$$

As discussed around (4.58), we can then decompose  $\text{Ren}_1 = V \oplus W$  with  $V \cong [0]_{\sim_1}$  and  $W$  containing one vector per class  $[\mathfrak{r}]_{\sim_1} \in \text{Clas}_1$ . Within each class, differences of  $\mathfrak{r}$ -factors can be translated into complex numbers, where the translation is one-to-one:

**Proposition 3.2.3.**  $[0]_{\sim_1} \cong \mathbb{C}$ .

*Proof.*  $[0]_{\sim_1}$  contains all  $\mathfrak{r} \in \text{Ren}_1$  that can be represented by  $r \in L^1$ . These  $\mathfrak{r}$  are identified with the unique integral  $\mathfrak{r} = \int_X r(x) d\mu(x)$ , so we can write

$\mathfrak{r} \in \mathbb{C}$ .

Conversely, each  $c \in \mathbb{C}$  corresponds to some  $c = \mathfrak{r} \in [0]_{\sim_1}$ : Since  $\mathcal{R}$  contains a dense subspace of  $L^2(X)$ , there must be some  $\phi \in \mathcal{R}$  with a finite  $L^2$ -norm  $0 \neq \|\phi\|_2 \in \mathbb{C}$ . Correspondingly, for  $\overline{\phi}\phi \in \mathcal{R}_2$  we have

$$\mathfrak{r}' := \int_X \overline{\phi(x)}\phi(x) d\mu(x) = \|\phi\|_2^2 \neq 0, \quad \mathfrak{r}' \in \mathbb{C} \quad \Rightarrow \quad \mathfrak{r}' \in [0]_{\sim_1}. \quad (3.35)$$

Now, since  $\mathfrak{r}' \neq 0$ , any  $c \in \mathbb{C}$  can be written as  $c = \tilde{c}\mathfrak{r}'$  with  $\tilde{c} \in \mathbb{C}$  and must hence be contained in the  $\mathbb{C}$ -vector space  $\text{Ren}_1$ . And since  $\tilde{c}\overline{\phi}\phi \in L^1$ , we have that  $c \in [0]_{\sim_1}$ . □

- The exponential  $e^{\mathfrak{r}}$  with  $\mathfrak{r} \in \text{Ren}_1$  is interpreted as an element of a further space  $\text{eRen}$ . Below in (3.57), we define an extended Weyl algebra  $\overline{\mathcal{W}}$ , similar to  $\mathcal{W}_*$  from (3.15), that allows for multiplication by  $\text{eRen}$ -elements. Therefore, it is useful to define  $\text{eRen}$  as a field extension of  $\mathbb{C}$ , so  $\overline{\mathcal{W}}$  can conveniently be made an algebra over the field  $\text{eRen}$  instead of  $\mathbb{C}$ .

So  $\text{eRen}$  shall contain all exponentials  $e^{\mathfrak{r}}$  and allow for multiplication, linear combination and inversion. The first two operations can be performed within the group algebra  $\mathbb{C}[\text{Ren}_1]$  containing all linear combinations  $c_1e^{\mathfrak{r}_1} + \dots + c_n e^{\mathfrak{r}_n}$  with  $c_j \in \mathbb{C}, \mathfrak{r}_j \in \text{Ren}_1$ . The multiplication law of the algebra is given by  $e^{\mathfrak{r}_1}e^{\mathfrak{r}_2} = e^{\mathfrak{r}_1+\mathfrak{r}_2}$ . Further, we would like to identify  $e^c e^{\mathfrak{r}}$  with  $e^{c+\mathfrak{r}}$ , which is done by modding out the ideal

$$\mathcal{I} \quad \text{generated by} \quad \{e^c e^{\mathfrak{r}} - e^{c+\mathfrak{r}} \mid c \in \mathbb{C}, \mathfrak{r} \in \text{Ren}_1\}. \quad (3.36)$$

The resulting algebra  $\mathbb{C}[\text{Ren}_1]/\mathcal{I}$  does not yet allow for taking inverses. To make this operation possible, we take the quotient field of  $\mathbb{C}[\text{Ren}_1]/\mathcal{I}$ , which is the final **field of wave function renormalizations**

$$\text{eRen} := \left\{ \mathfrak{c} := \frac{a_1}{a_2} \mid a_1, a_2 \in \mathbb{C}[\text{Ren}_1]/\mathcal{I} \right\}. \quad (3.37)$$

Definition (3.37) does only make sense, if  $\mathbb{C}[\text{Ren}_1]/\mathcal{I}$  has no proper zero divisors, i.e., it is an entire ring:

**Proposition 3.2.4.**  $\mathbb{C}[\text{Ren}_1]/\mathcal{I}$  has no proper zero divisors.

*Proof.* For a specific choice of  $\mathcal{R}$ , this fact is established in Propositions 4.3.7 and 4.3.8. The proof only makes use of  $\text{Ren}_1$  decaying into classes (3.34),

### 3 Fock Space Extensions

which are all isomorphic to  $\mathbb{C}$ . This is true for any choice of  $\mathcal{R}$  complying with (A) and (B), so Propositions 4.3.7 and 4.3.8 carry over to all  $\text{Ren}_1(X)$  considered in this section. □

Now, any element  $\mathbf{c} \in \text{eRen}$  can be written (non-uniquely) as

$$\mathbf{c} = \frac{a_1}{a_2} = \frac{c_1 e^{\mathbf{r}_1} + \dots + c_n e^{\mathbf{r}_n}}{c'_1 e^{\mathbf{r}'_1} + \dots + c'_{n'} e^{\mathbf{r}'_{n'}}}, \quad (3.38)$$

and if all  $\mathbf{r}_j, \mathbf{r}'_{j'} \in \mathbb{C}$ , then we can also identify  $\mathbf{c}$  with a  $\mathbb{C}$ -number. In general,  $\mathbf{c} \in \text{eRen}$  may heuristically be thought of as a “controlled, possibly infinitely large number” or, in some cases, as an “infinite phase”.

Having defined  $e^{\mathbf{r}} \in \text{eRen}$  and  $\Psi_j \in \mathcal{R}_{\mathcal{F}}$ , we are now in the position to define our first ESS, which contains elements of the form  $\Psi = e^{\mathbf{r}} \Psi_j \in \overline{\mathcal{F}}$  as in (3.29):

- Consider the free  $\text{eRen}$ -vector space

$$\overline{\mathcal{F}}_0 := \{ \mathbf{c}_1 \Psi_1 + \dots + \mathbf{c}_n \Psi_n \mid \mathbf{c}_j \in \text{eRen}, \Psi_j \in \mathcal{R}_{\mathcal{F}} \}, \quad (3.39)$$

with all sums being commutative. This space already contains expressions of the kind  $e^{\mathbf{r}} \Psi_j$ , but it does not allow for shifting complex factors from  $\mathbf{c}_j$  to  $\Psi_t$ . We allow for this shift by modding out the equivalence relation

$$\sim_{\mathbb{F}} \quad \text{generated by} \quad (c\mathbf{c})\Psi_j \sim_{\mathbb{F}} \mathbf{c}(c\Psi_j) \quad \forall c \in \mathbb{C}, \mathbf{c} \in \text{eRen}, \Psi_j \in \mathcal{R}_{\mathcal{F}}. \quad (3.40)$$

The **first extended state space** is then defined as

$$\overline{\mathcal{F}} := \overline{\mathcal{F}}_0 / \sim_{\mathbb{F}}. \quad (3.41)$$

In order to make sense of expressions  $\Psi = e^{\mathbf{r}} \mathbf{r}'_1 \cdot \dots \cdot \mathbf{r}'_p \Psi_j$  as in (3.29), we proceed as follows:

- First, we define a space  $\text{Ren}_P$  containing products  $\mathbf{r}'_1 \cdot \dots \cdot \mathbf{r}'_p$  of power  $p \leq P$  with  $\mathbf{r}'_k \in \text{Ren}_1$ . To do so, we consider the free  $\mathbb{C}$ -vector space

$$\text{Pol}_P := \text{span}\{ \mathbf{r}'_1 \cdot \dots \cdot \mathbf{r}'_p \mid \mathbf{r}'_k \in \text{Ren}_1, p \leq P \}, \quad (3.42)$$

with the products being commutative. We allow for shifting complex numbers between the  $\mathbf{r}'$ -factors and converting  $\mathbf{r}' \in \mathbb{C}$  into the respective complex

### 3.2 Extended State Space (ESS)

number, by defining the equivalence relation  $\sim_{\text{Ren}_P}$  generated by

$$\begin{aligned} (c_1 c_2) \mathbf{r}'_1 \cdots \mathbf{r}'_p &\sim_{\text{Ren}_P} c_1 (c_2 \mathbf{r}'_1) \cdots \mathbf{r}'_p \\ \mathbf{r}'_1 \mathbf{r}'_2 \cdots \mathbf{r}'_p &\sim_{\text{Ren}_P} c_1 \mathbf{r}'_2 \cdots \mathbf{r}'_p \quad \text{if } \mathbf{r}'_1 = c_1 \in \mathbb{C}, \end{aligned} \quad (3.43)$$

for  $c_j \in \mathbb{C}$ ,  $\mathbf{r}'_k \in \text{Ren}_1$ ,  $p \leq P$ . The **space of renormalization polynomials of degree  $P$**  is then defined by

$$\text{Ren}_P := \text{Pol}_P / \sim_{\text{Ren}_P}. \quad (3.44)$$

- The spaces  $\text{Ren}_P$  are contained in each other as  $\text{Ren}_P \subseteq \text{Ren}_{P'}$  for  $P \leq P'$ . Taking the union over all  $P$ , we arrive at the **space of renormalization polynomials**

$$\text{Ren} := \bigcup_{P \in \mathbb{N}} \text{Ren}_P. \quad (3.45)$$

In fact,  $\text{Ren}$  is even an algebra, as it allows for arbitrary multiplication of elements (which is not true for  $\text{Ren}_P$ ).

- The second ESS  $\overline{\mathcal{F}}_{\text{ex}}$  is now defined such that it allows for expressions like  $\mathfrak{R} \cdot e^{\mathfrak{r}} \Psi_j$ , where  $\mathfrak{R} \in \text{Ren}$  could, for instance, represent a product of self-energy terms  $\mathfrak{R} = \mathbf{r}'_1 \cdots \mathbf{r}'_p$ . Recall that  $\Psi_j \in \mathcal{R}_{\mathcal{F}}$  is a function on configuration space  $\mathcal{Q}(X)$ . Since we expect the self-energy to depend on the configuration  $q \in \mathcal{Q}(X)$ , we would like to allow for renormalization polynomials  $\mathfrak{R}(q)$  that depend on the configuration. This is realized by introducing the function space

$$\text{Ren}^{\mathcal{Q}} := \{ \Psi_j : \mathcal{Q}(X) \setminus \mathcal{N} \rightarrow \text{Ren} \mid \mu_{\mathcal{Q}}(\mathcal{N}) = 0 \}, \quad (3.46)$$

where the factor  $\mathfrak{R}$  has been included into the  $\Psi_j$  and  $\mu_{\mathcal{Q}}$  is the measure on  $\mathcal{Q}(X)$  induced by  $\mu$  on  $X$ . In Chapter 4, a similar space  $\text{Ren}^{\dot{\mathcal{Q}}}$  is defined, which consists of functions  $\dot{\mathcal{Q}} \rightarrow \text{Ren}$  with a fixed  $\dot{\mathcal{Q}} \subset \mathcal{Q}$ , such that  $\mathcal{Q} \setminus \dot{\mathcal{Q}} =: \mathcal{N}$  is a null set. So  $\text{Ren}^{\dot{\mathcal{Q}}}$  can be seen as a subspace of  $\text{Ren}^{\mathcal{Q}}$ .

Using  $\text{Ren}^{\mathcal{Q}}$ , the expression in (3.29), to be defined as an element of  $\overline{\mathcal{F}}_{\text{ex}}$ , now becomes  $e^{\mathfrak{r}} \Psi_j$  with  $\Psi_j \in \text{Ren}^{\mathcal{Q}}$ . We can interpret it as an element of the free eRen-vector space

$$\overline{\mathcal{F}}_{\text{ex},0} := \left\{ \sum_{j \in \mathbb{N}} \mathbf{c}_j \Psi_j \mid \mathbf{c}_j \in \text{eRen}, \Psi_j \in \text{Ren}^{\mathcal{Q}} \right\}. \quad (3.47)$$

Note that the countable union of all null sets  $\mathcal{N}$ , where some  $\Psi_j \in \text{Ren}^{\mathcal{Q}}$  is not

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defined, is again a null set. Again, all sums are considered to be commutative. In the slightly modified construction in Chapter 4, we choose the sum over  $j$  to be finite. As for  $\overline{\mathcal{F}}$ , we allow for shifting complex factors by modding out the equivalence relation

$$\sim_{\text{Fex}} \quad \text{generated by} \quad (c\mathbf{c})\Psi_j \sim_{\text{Fex}} \mathbf{c}(c\Psi_j) \quad \forall c \in \mathbb{C}, \mathbf{c} \in \text{eRen}, \Psi_j \in \text{Ren}^{\mathcal{Q}}. \quad (3.48)$$

We may then define the **second extended state space**

$$\overline{\mathcal{F}}_{\text{ex}} := \overline{\mathcal{F}}_{\text{ex},0} / \sim_{\text{Fex}}. \quad (3.49)$$

This space is also called  $\overline{\mathcal{F}}_{\text{ex}}$  in Chapters 4 and 5 and its definition concludes the ESS construction.

#### 3.2.3 Choices Within the Van Hove Model

To make sense of the coherent states outside Fock space (3.23) appearing in the Van Hove example, it is necessary to choose  $\mathcal{R}(X)$  such that it contains  $s(\mathbf{k}) = -g|\mathbf{k}|^{-3/2}$ . Here,  $X = \mathbb{R}^d$ . More generally, if a dispersion relation  $\omega(\mathbf{k})$  and a form factor  $v(\mathbf{k})$  are fixed in the Van Hove model (1.137), then  $\mathcal{R}(\mathbb{R}^d)$  has to contain  $s(\mathbf{k}) = -\frac{v(\mathbf{k})}{\omega(\mathbf{k})}$ . The case  $\omega(\mathbf{k}) \propto |\mathbf{k}|$  and  $v(\mathbf{k}) \propto |\mathbf{k}|^{-1/2}$  in  $d = 3$  is particularly interesting from a physical point of view, since these are the expected scalings for a relativistic light–matter interaction. After choosing  $\mathcal{R}$ , the space  $\mathcal{R}^{(N)}$  must be adapted correspondingly.

In the following, we discuss some candidates for  $\mathcal{R}$  and  $\mathcal{R}^{(N)}$ , such that  $\mathcal{R}$  contains  $s(\mathbf{k}) = -g|\mathbf{k}|^{-3/2}$  or other functions  $s(\mathbf{k})$  beyond  $L^2$ :

- $\mathcal{R} = \mathcal{E}(\mathbb{R}^d) = C^\infty(\mathbb{R}^d)$ : Smooth functions allow for arbitrary scaling of  $s(\mathbf{k})$  as  $|\mathbf{k}| \rightarrow \infty$ , including  $s \propto |\mathbf{k}|^{-3/2}$ . This makes the space  $\mathcal{E}$  ideal for absorbing UV–divergences. Further,  $\mathcal{E}$  is preserved under pointwise multiplication, so  $\mathcal{R} = \mathcal{E}$  entails  $\mathcal{R}_2 = \mathcal{R}$ . However,  $\mathcal{E}(\mathbb{R}^d)$  does not allow for poles in the IR–regime  $|\mathbf{k}| \rightarrow 0$ , which rules out  $s \propto |\mathbf{k}|^{-3/2}$  and strongly restricts the permitted choices for  $\omega(\mathbf{k}), v(\mathbf{k})$ . Nevertheless, it is easy to verify requirements (A) and (B), as  $\mathcal{E}(\mathbb{R}^d)$  contains the dense subspace  $\mathcal{D}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$ . A natural choice for the  $N$ –particle space is  $\mathcal{R}^{(N)} = \mathcal{E}(\mathbb{R}^{Nd})$ . By [203, Thm. 51.6], we can then identify the topological tensor product  $\mathcal{E}(\mathbb{R}^d)^{\otimes N}$  with  $\mathcal{E}(\mathbb{R}^{Nd})$ . Conditions (C) and (D) are satisfied for this choice, since  $\mathcal{E}(\mathbb{R}^{Nd})$  consists of measurable functions and the topological tensor products contain more elements than algebraic tensor products.

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- $\mathcal{R} = \mathcal{E}'(\mathbb{R}^d) \cap L_{\text{loc}}^1(\mathbb{R}^d)$ : For compactly supported distributions, also  $\mathcal{E}'(\mathbb{R}^{Nd}) \cong \mathcal{E}'(\mathbb{R}^d)^{\otimes N}$  [203, Thm. 51.6], which allows for the convenient choice  $\mathcal{R}^{(N)} = \mathcal{E}'(\mathbb{R}^{Nd}) \cap L_{\text{loc}}^1(\mathbb{R}^{Nd})$ . We need to take the intersection with  $L_{\text{loc}}^1$ , in order to obtain functions defined almost everywhere on  $\mathbb{R}^d$ , so Conditions (A) and (C) are satisfied. In fact, all distributions representable by a function  $\mathbb{R}^d \rightarrow \mathbb{C}$  must be in  $L_{\text{loc}}^1$ , since they could otherwise not be applied to all test functions in  $\mathcal{D} = C_c^\infty$ . Condition (B) is easily verified, since both  $\mathcal{E}'$  and  $L_{\text{loc}}^1$  contain  $\mathcal{D}$ . Condition (D) follows from the fact that  $\mathcal{E}'(\mathbb{R}^{Nd}) \cong \mathcal{E}'(\mathbb{R}^d)^{\otimes N}$ , and that  $L_{\text{loc}}^1(\mathbb{R}^{Nd})$  contains the  $N$ -fold algebraic tensor product of  $L_{\text{loc}}^1(\mathbb{R}^d)$ . However, compactness of  $\text{supp } s$  is a serious restriction, excluding  $s(\mathbf{k}) = -g|\mathbf{k}|^{-3/2}$ . Further, note that<sup>4</sup>  $\mathcal{R}_2 \neq \mathcal{R}$ .

Note that strictly speaking,  $\mathcal{R} = \mathcal{E}'(\mathbb{R}^d) \cap L_{\text{loc}}^1(\mathbb{R}^d)$  does not consist of functions, but rather equivalence classes of functions up to modifications on a null set. The same will be true for some of the spaces below. So the ESS construction has to be adapted in this case, such that  $\mathcal{R}, \mathcal{R}^{(N)}, \mathcal{R}_2$  and  $\text{Ren}^{\mathcal{Q}}$  are defined as equivalence classes of functions up to modifications on a null set. It is easy to see that the construction can be carried through analogously in that case: Sums, scalar products, tensor products and direct sums respect the equivalence up to modifications on a null set. The same holds for finite multiplication of functions, as in the definition of  $\mathcal{R}_2$ . Finally, the integral over a measurable function is not affected by a modification on a null set.

- $\mathcal{R} = \mathcal{S}'(\mathbb{R}^d) \cap L_{\text{loc}}^1(\mathbb{R}^d)$ : Tempered distributions  $\mathcal{S}' \supseteq \mathcal{E}'$  allow for a polynomial growth of  $s(\mathbf{k})$  as  $|\mathbf{k}| \rightarrow \infty$  and are therefore a good choice for absorbing UV-divergences. Again,  $\mathcal{S}'(\mathbb{R}^{Nd}) \cong \mathcal{S}'(\mathbb{R}^d)^{\otimes N}$  [203, Thm. 51.6] allows for the convenient choice  $\mathcal{R}^{(N)} = \mathcal{S}'(\mathbb{R}^{Nd}) \cap L_{\text{loc}}^1(\mathbb{R}^{Nd})$ , where Conditions (A), (B), (C) and (D) are fulfilled by the same arguments as in the case with  $\mathcal{E}'$  above. Another advantage is that  $\mathcal{S}'$  allows for taking the inverse Fourier transform  $\check{s}(\mathbf{y})$ , and hence for an easy change between momentum and position space. However, poles in  $\mathcal{R}$  are still restricted to order  $< d$  by the  $L_{\text{loc}}^1$ -condition. This puts restrictions on the IR-behavior of  $s$ , but still allows for  $s \propto |\mathbf{k}|^{-3/2}$  in  $d = 3$ . Nevertheless, as in the previous case,  $\mathcal{R}_2 \neq \mathcal{R}$  and the function  $|s(\mathbf{k})|^2 \propto |\mathbf{k}|^{-3}$ , over which the integral in (3.23) is taken, is no element of  $\mathcal{R}$ .
- $\mathcal{R} = \mathcal{D}'(\mathbb{R}^d) \cap L_{\text{loc}}^1(\mathbb{R}^d)$ : Also the distribution space  $\mathcal{D}' \supseteq \mathcal{S}'$  allows for the identification  $\mathcal{D}'(\mathbb{R}^{Nd}) \cong \mathcal{D}'(\mathbb{R}^d)^{\otimes N}$  [203, Thm. 51.6], and hence for the con-

<sup>4</sup>As a counterexample, consider some  $s \in \mathcal{R}$  with  $s(\mathbf{k}) \propto |\mathbf{k}|^{-d+\varepsilon}$  as  $|\mathbf{k}| \rightarrow 0$ . So  $|s(\mathbf{k})|^2 \propto |\mathbf{k}|^{-2d+2\varepsilon}$ , which is not in  $L_{\text{loc}}^1$  for  $\varepsilon$  small enough.

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venient choice  $\mathcal{R}^{(N)} = \mathcal{D}'(\mathbb{R}^{Nd}) \cap L_{\text{loc}}^1(\mathbb{R}^{Nd})$ . In comparison to  $\mathcal{S}' \cap L_{\text{loc}}^1$ , this space does not allow for taking Fourier transforms, but instead accommodates functions that diverge faster than polynomially in the UV–regime. Pole strengths are still limited to  $< d$ , allowing for the accommodation of  $s(\mathbf{k}) = -g|\mathbf{k}|^{-3/2}$  in  $d = 3$ , but ruling out  $|s(\mathbf{k})|^2 \propto |\mathbf{k}|^{-3}$ . Again,  $\mathcal{R}_2 \neq \mathcal{R}$ , and Conditions (A), (B), (C) and (D) are fulfilled by the same arguments as for  $\mathcal{E}'$ .

- **Hilbert space riggings:** It is possible to construct a triple of Hilbert spaces  $\mathcal{H}_+ \subseteq \mathcal{H}_0 \subseteq \mathcal{H}_-$  with  $\mathcal{H}_0 = L^2(\mathbb{R}^d)$  and a pairing  $\mathcal{H}_+ \times \mathcal{H}_- \rightarrow \mathbb{C}$  [204]. In that case, we may choose  $\mathcal{R} = \mathcal{H}_-$  and conveniently define  $\mathcal{R}^{(N)} = \mathcal{H}_-^{\otimes N}$  as the Hilbert space tensor product. However,  $\mathcal{H}_-$  may contain elements, that are no functions  $\mathbb{R}^d \rightarrow \mathbb{C}$ , so it might be necessary to intersect  $\mathcal{H}_-$  with a function space, such as  $L_{\text{loc}}^1$ , to define an  $\mathcal{R}$  satisfying Condition (A). In the following, we explain choices  $\mathcal{R} = \mathcal{H}_-$ , that do not necessarily satisfy (A). So an intersection might be necessary before the ESS construction.

A typical example for such a rigging are **Sobolev chains**  $H^m \subset L^2 \subset H^{-m}$  [204, Example 1.2]. With increasing derivative index  $m > 0$ , the allowed order of poles in  $H^m(\mathbb{R}^d)$  decreases to  $< \max(d/2 - m, 0)$ , which increases the allowed pole order in  $\mathcal{R} = H^{-m}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$  to  $< \min(d/2 + m, d)$ . Hence, all  $m > 0$  allow for accommodating the IR–divergence of  $s(\mathbf{k}) = -g|\mathbf{k}|^{-3/2}$ , but not that of  $|s(\mathbf{k})|^2 = g^2|\mathbf{k}|^{-3}$  in  $d = 3$ .

However, in contrast to  $\mathcal{S}'$ , the allowed UV–scalings for  $s \in H^{-m}(\mathbb{R}^d)$  are the same as for  $L^2(\mathbb{R}^d)$ . That means,  $s$  has to decay faster than  $|\mathbf{k}|^{-3/2}$ , irrespectively of  $m$ , which rules out  $s(\mathbf{k}) = -g|\mathbf{k}|^{-3/2}$  in the Van Hove example.

Another rigged Hilbert space [204, Example 1.3] is constructed by introducing the formal scalar product

$$\langle \phi, \psi \rangle_{\mathcal{H}_-} := \langle \phi, K\psi \rangle, \quad (3.50)$$

where  $K$  is an integral operator with bounded kernel  $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ .  $\mathcal{H}_-$  is then given by the set of all measurable functions

$$\phi : \mathbb{R}^d \rightarrow \mathbb{C} \quad \text{with} \quad \|\phi\|_{\mathcal{H}_-}^2 = \int \overline{\phi(\mathbf{k})} K(\mathbf{k}, \mathbf{k}') \phi(\mathbf{k}') d\mathbf{k} d\mathbf{k}' < \infty. \quad (3.51)$$

To obtain an actual scalar product,  $K$  must be positive definite, i.e.,  $\langle \phi, \phi \rangle_{\mathcal{H}_-} > 0$  for all  $0 \neq \phi \in L^2(\mathbb{R}^d)$ . There are many possible choices for  $K$ , such as the

**Gaussian kernel**<sup>5</sup> in [205]

$$K(\mathbf{k}, \mathbf{k}') = \exp\left(-\frac{|\mathbf{k} - \mathbf{k}'|^2}{2\sigma^2}\right), \quad \sigma > 0, \quad (3.52)$$

see also Figure 3.3. The corresponding operator  $K$  is then a convolution with a Gaussian function, which provides a good IR-regularization: If  $s \in \mathcal{S}'$ , then  $K$  acts as a multiplication by a Gaussian on the Fourier inverse  $\check{s} = \mathcal{F}^{-1}(s)$ :

$$\mathcal{F}^{-1}(Ks)(\mathbf{y}) = \sigma^d \exp\left(-\frac{|\mathbf{y}|^2\sigma^2}{2}\right) \check{s}(\mathbf{y}). \quad (3.53)$$

So in  $\mathcal{H}_-$  there are tempered distributions, whose Fourier inverses are functions with an arbitrary polynomial increase as  $|\mathbf{y}| \rightarrow \infty$ . However, (3.53) also shows that there is no UV-regularization, since the Gaussian scales like a constant as  $|\mathbf{y}| \rightarrow 0$ . In particular,  $s(\mathbf{k}) = -g|\mathbf{k}|^{-3/2}$  in  $d = 3$  cannot be accommodated into  $\mathcal{H}_-$ , since

$$\langle s, Ks \rangle = \langle \mathcal{F}^{-1}(s), \mathcal{F}^{-1}(Ks) \rangle = g^2 \int_{\mathbb{R}^3} \exp\left(-\frac{|\mathbf{y}|^2\sigma^2}{2}\right) |\mathbf{y}|^{-3} d\mathbf{x} \quad (3.54)$$

diverges logarithmically as  $|\mathbf{y}| \rightarrow 0$ .

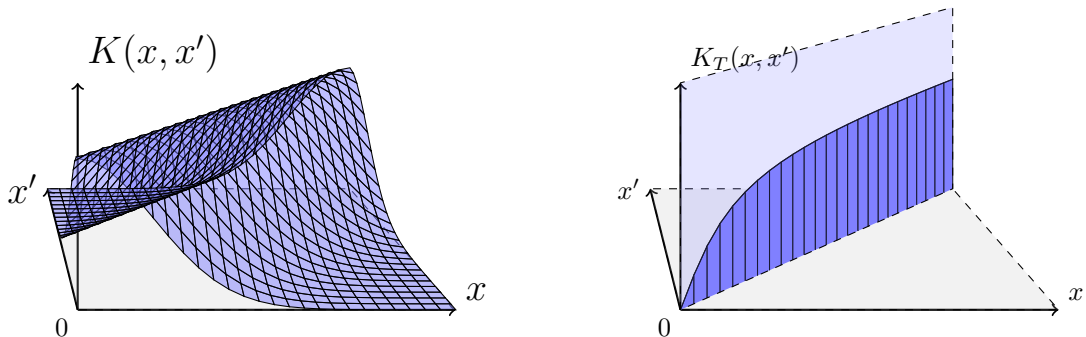


Figure 3.3: Left: The Gaussian kernel from (3.52).

Right: The multiplication operator  $T$  has a distribution  $K_T$  as its kernel, that is supported on the diagonal. Color online.

A further way to provide Hilbert spaces for a rigging [204, Example 1.1] is to introduce an operator  $T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  multiplying by a bounded, measurable function  $T : \mathbb{R}^d \rightarrow \mathbb{R}$  with  $T(\mathbf{k}) > 0$  almost everywhere, and to

<sup>5</sup>For normalization reasons, a factor of  $8\sigma$  instead of  $2\sigma$  is used in [205, (2)].

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define  $\mathcal{R} = \mathcal{H}_-$  via the scalar product

$$\langle \phi, \psi \rangle_{\mathcal{H}_-} := \langle \phi, T\psi \rangle. \quad (3.55)$$

The functions  $T^{1/2}(\mathbf{k})$  and  $T^{-1/2}(\mathbf{k})$  are then defined almost everywhere, and we can write

$$\begin{aligned} \mathcal{H}_- &= \{ \phi : \text{measurable} \mid T^{1/2}\phi \in L^2(\mathbb{R}^d) \} \\ \mathcal{H}_+ &= \{ \phi : \text{measurable} \mid T^{-1/2}\phi \in L^2(\mathbb{R}^d) \}, \end{aligned} \quad (3.56)$$

up to modifications on a null set<sup>6</sup>. The choice (3.55) can also be seen as a kernel-based scalar product, as in (3.50) with kernel  $K_T(\mathbf{k}, \mathbf{k}') = T(\mathbf{k})\delta(\mathbf{k} - \mathbf{k}')$  being a distribution, and hence unbounded. See also Figure 3.3.

The choice of the function  $T(\mathbf{k})$  now determines how well IR- and UV-divergences of  $s$  can be absorbed. For instance, one may choose a smooth function  $T(\mathbf{k})$ , scaling as  $|\mathbf{k}|^\alpha$  for  $|\mathbf{k}| \rightarrow 0$  and as  $|\mathbf{k}|^{-\beta}$  for  $|\mathbf{k}| \rightarrow \infty$  with  $\alpha, \beta > 0$ . This allows for  $s \in \mathcal{R}(\mathbb{R}^d)$ , which may have a pole at 0 of order  $< (d + \alpha)/2$  and are only required to decay faster than  $|\mathbf{k}|^{(\beta-d)/2}$  at  $|\mathbf{k}| \rightarrow \infty$ . Hence,  $\mathcal{R} = \mathcal{H}_-$  accommodates  $s(\mathbf{k}) = -g|\mathbf{k}|^{-3/2}$  in  $d = 3$  for all  $\alpha, \beta > 0$ .

One may even require  $T(\mathbf{k})$  to scale exponentially<sup>7</sup> at  $|\mathbf{k}| \rightarrow 0$  and  $|\mathbf{k}| \rightarrow \infty$ , which admits to have all  $s(\mathbf{k})$  within  $\mathcal{R}$ , which increase polynomially at  $|\mathbf{k}| \rightarrow \infty$  and have a pole of any order at  $\mathbf{k} = 0$ .

As a further generalization, one may consider a function  $T(\mathbf{k})$  that vanishes along the unit of finitely many submanifolds of co-dimension  $\geq 1$  in  $\mathbb{R}^d$  and scales exponentially, as it approaches a submanifold, or as  $|\mathbf{k}| \rightarrow \infty$ . This would even allow for  $s$ , which blow up polynomially as they approach a submanifold.

- It is even possible to define a **rigging by a limit of Hilbert spaces**. The final rigging is then  $\Phi \subset L^2(\mathbb{R}^d) \subset \Phi'$  [204, Sect. 1.2], where  $\Phi = \bigcap_{\tau \in \mathcal{T}} \mathcal{H}_\tau$ , with  $\mathcal{T}$  being any index set and the family  $(\mathcal{H}_\tau)_{\tau \in \mathcal{T}}$  being directed. That means, for any  $\mathcal{H}_\tau, \mathcal{H}_{\tau'}$ , one can still find some  $\tau''$  with  $\mathcal{H}_{\tau''} \subset \mathcal{H}_\tau, \mathcal{H}_{\tau''} \subset \mathcal{H}_{\tau'}$ . If  $\Phi$  is dense in each  $\mathcal{H}_\tau$ , then  $\Phi' = \bigcup_{\tau \in \mathcal{T}} \mathcal{H}_{-\tau}$  [204, Lemma 1.2]. The space  $\Phi'$  is equipped with the inductive limit topology, so it does not have to

<sup>6</sup>That means, just as in the standard construction of  $L^p$ -spaces,  $\mathcal{H}_-$  and  $\mathcal{H}_+$  as in (3.56) actually consist of equivalence classes of measurable functions, where two functions are considered equivalent if they agree everywhere except for a null set. So there is a slight abuse of notation in (3.56).

<sup>7</sup>By an exponential scaling at  $|\mathbf{k}| \rightarrow 0$ , we mean that  $|s(\mathbf{k})| < c_1 e^{-\frac{c_2}{|\mathbf{k}|}}$  for some constants  $c_1, c_2 > 0$  at small  $|\mathbf{k}|$ . For  $|\mathbf{k}| \rightarrow \infty$ , an exponential scaling means that  $|s(\mathbf{k})| < c_1 e^{-c_2 |\mathbf{k}|}$  for some  $c_1, c_2 > 0$  at high  $|\mathbf{k}|$ .

be a Hilbert space. This allows for even more general spaces  $\mathcal{R} = \Phi'$  than the above Hilbert space riggings. For instance, the rigging  $\mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$  can be obtained as a limit of Hilbert spaces [204, Example 1.8]. A definition of  $\mathcal{R}^{(N)}$  for  $\mathcal{R} = \Phi'$  can then be achieved as an algebraic or topological tensor product.

It is easy to verify that all above choices for  $\mathcal{R}$  and  $\mathcal{R}^{(N)}$  satisfy Conditions (B) and (C):  $\mathbb{R}^d$  is equipped with the Lebesgue measure, all  $\mathcal{R}$  contain the dense space  $C_c^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$  and  $\mathcal{R}^{(N)}$  always contains the algebraic tensor product. As explained above, Condition (A) is generally not satisfied, and requires the intersection with a function space. Whenever  $s \in \mathcal{R}$  for the Van Hove example, we have  $\Omega_W, \Omega_{\text{IBC}} \in \overline{\mathcal{F}}$  (compare (3.23)), since  $s^{\otimes N} \in \mathcal{R}^{(N)}$  and  $e^{-\frac{\|s\|^2}{2}} \in \text{eRen}$ .

As for the ITP space, it is also possible to construct an **extended Weyl algebra**

$$\overline{\mathcal{W}} := \text{span}_{\text{eRen}}\{W(s_1) \dots W(s_N) \mid s_j \in \mathcal{R}\}, \quad (3.57)$$

with  $W(s), s \in \mathbb{R}$  being formal algebraic expressions. This is a generalization of the Weyl algebra defined in Section 4.5.3 to a general ESS construction based on  $\mathcal{R}$ . Within the Weyl relations (3.16)

$$W(s)^{-1} = W(-s), \quad W(s_1)W(s_2) = e^{-\frac{i}{2}\sigma(s_1, s_2)}W(s_1 + s_2),$$

we can now interpret  $\sigma(s_1, s_2) = \langle s_1, s_2 \rangle - \langle s_2, s_1 \rangle \in \text{Ren}_1$  and  $e^{-\frac{i}{2}\sigma(s_1, s_2)} \in \text{eRen}$ . So in contrast to the ITP extended Weyl algebra  $\mathcal{W}_*$  (with  $*$  being a placeholder for a function space), we do not have to define the factor  $e^{-\frac{i}{2}\sigma(s_1, s_2)}$  via  $W(s)$ -operators, but can directly consider them as elements of the field eRen. Further,  $\overline{\mathcal{W}}$  is an eRen- instead of just a  $\mathbb{C}$ -algebra, which allows for more freedom of computation, and for  $* = \mathcal{R}$ , we obviously have  $\mathcal{W}_* \subseteq \overline{\mathcal{W}}$ .

However, the  $W(s)$  are not a priori defined as operators on a subspace of  $\overline{\mathcal{F}}$  or  $\overline{\mathcal{F}}_{\text{ex}}$ , and a suitable definition may require some work, as in Section 4.5. Further,  $W(s)$  defined on (a subspace of)  $\overline{\mathcal{F}}$  or  $\overline{\mathcal{F}}_{\text{ex}}$  are never unitary, since  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$  do not have a scalar product.

### 3.2.4 Choices within Polaron Models

In the class of polaron models with ‘‘Yukawa-type interaction’’ in Chapter 4, the construction process of  $\dot{\mathcal{S}}_{\mathcal{F}}^\infty$  treats fermions  $x$  and bosons  $y$  asymmetrically, and therefore slightly deviates from the one of  $\mathcal{R}_{\mathcal{F}}$ , described in Section 3.2.2. This

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asymmetric treatment can be seen as a refinement of the construction process in Section 3.2.2, with  $\mathcal{R}$  being based on Hilbert space riggings. In the following, we demonstrate how  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$ , which is defined in Section 4.3, can be embedded into a space  $\mathcal{R}_{\mathcal{F}}$  constructed exactly as in Section 3.2.2.

We start from two different one-particle spaces  $\mathcal{R}_x$  and  $\mathcal{R}_y$ . For bosons, we use  $\mathcal{R}_y = \dot{\mathcal{S}}_1^{\infty}$ , defined in (4.26), which contains smooth functions scaling “at most polynomially” at  $|\mathbf{k}| \rightarrow 0$  and  $|\mathbf{k}| \rightarrow \infty$ . This space can also be obtained as a limit of Hilbert space riggings

$$\mathcal{H}_{\alpha,\beta,+} \subset L^2(\mathbb{R}^d) \subset \mathcal{H}_{\alpha,\beta,-}, \quad \mathcal{R}_y = \bigcup_{\alpha,\beta>0} \mathcal{H}_{\alpha,\beta,-}, \quad (3.58)$$

with  $\mathcal{H}_{\alpha,\beta,-}$  defined as in (3.55) where  $T_{\alpha,\beta}$  multiplies by a suitable function  $T_{\alpha,\beta}(\mathbf{k})$  scaling like  $|\mathbf{k}|^{\alpha}$  as  $|\mathbf{k}| \rightarrow 0$  and  $|\mathbf{k}|^{-\beta}$  as  $|\mathbf{k}| \rightarrow \infty$ . It is natural to choose  $\mathcal{R}_y^{(N)}$  as the space of all functions on  $\mathbb{R}^{Nd}$  that are polynomially bounded as  $|\mathbf{k}_j| \rightarrow \infty$  and blow up at most polynomially, when approaching the set

$$\exists(\mathbf{k} = 0)_y := \{q = (\mathbf{k}_1, \dots, \mathbf{k}_N) \mid \mathbf{k}_j = 0 \text{ for some } j \leq N\}. \quad (3.59)$$

That means,  $(\Psi(q) \cdot \text{dist}(q, \exists(\mathbf{k} = 0)_y)^{\alpha}) \rightarrow 0$  for some  $\alpha > 0$ . For fermions, we choose as  $\mathcal{R}_x$  the space of smooth functions scaling at most polynomially at  $|\mathbf{p}| \rightarrow \infty$ . It is then natural to set  $\mathcal{R}_x^{(M)}$  to be the space of all functions polynomially bounded as  $|\mathbf{p}_j| \rightarrow \infty$ . For these choices,  $\mathcal{R}_x \subset \mathcal{R}_y$  and  $\mathcal{R}_x^{(M)} \subset \mathcal{R}_y^{(N)}$ . Note that  $\mathcal{R}_x$  can also be obtained as a limit of Hilbert space riggings

$$\mathcal{H}_{\beta,+} \subset L^2(\mathbb{R}^d) \subset \mathcal{H}_{\beta,-}, \quad \mathcal{R}_x = \bigcup_{\beta>0} \mathcal{H}_{\beta,-}, \quad (3.60)$$

with  $\mathcal{H}_{\beta,-}$  defined via (3.55) by a suitable family of multiplication functions  $T_{\beta}(\mathbf{p})$  scaling as  $|\mathbf{p}|^{-\beta}$  for  $|\mathbf{p}| \rightarrow \infty$ .

We now define  $\mathcal{R}_{xy}^{(M,N)}$  as a spaces of smooth functions on  $\mathbb{R}^{Md+Nd} \setminus \exists(\mathbf{k} = 0)$ , where

$$\exists(\mathbf{k} = 0) := \{q = (\mathbf{p}_1, \dots, \mathbf{p}_M, \mathbf{k}_1, \dots, \mathbf{k}_N) \mid \mathbf{k}_j = 0 \text{ for some } j \leq N\} \quad (3.61)$$

is the set of configurations where the momentum of one  $y$ -particle collides with 0. We assume these functions to be polynomially bounded as  $|\mathbf{k}_j| \rightarrow \infty$  or  $|\mathbf{p}_j| \rightarrow \infty$  and to blow up at most polynomially, as  $q$  approaches the set  $\exists(\mathbf{k} = 0)$ . With this

definition, each sector contains the algebraic tensor products

$$\mathcal{R}_{xy}^{(M,N)} \supset (\mathcal{R}_x^{(M)} \otimes_a \mathcal{R}_y^{(N)}) \supset (\mathcal{R}_x^{\otimes_a M} \otimes_a \mathcal{R}_y^{\otimes_a N}). \quad (3.62)$$

The corresponding space of generalized Fock space functions  $\mathcal{R}_{\mathcal{F}} = \bigoplus_{M,N \in \mathbb{N}_0} \mathcal{R}_{xy}^{(M,N)}$  is exactly the space  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$  from (4.40) in Chapter 4.

It is now easy to see how to embed  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$  into some space  $\mathcal{R}_{\mathcal{F}}$  constructed as in Section 3.2.2, i.e., starting from only a single one-particle space  $\mathcal{R} = \mathcal{R}_y$ . Evidently,  $\mathcal{R}$  satisfies Condition (A), and Condition (B) holds since  $C_c^{\infty}(\mathbb{R}^d \setminus \{0\})$  is dense in  $L^2(\mathbb{R}^d)$ . As a sector  $\mathcal{R}^{(M,N)}$ , we choose the space of all functions that are smooth on  $\mathbb{R}^{d(M+N)} \setminus \exists(\mathbf{p}, \mathbf{k} = 0)$  with

$$\exists(\mathbf{p}, \mathbf{k} = 0) := \{q \mid \mathbf{p}_j = 0 \text{ for some } j \leq M \text{ or } \mathbf{k}_j = 0 \text{ for some } j \leq N\}, \quad (3.63)$$

that are polynomially bounded as  $|\mathbf{k}_j|, |\mathbf{p}_j| \rightarrow \infty$  and blow up at most polynomially as  $q$  approaches  $\exists(\mathbf{p}, \mathbf{k} = 0)$ . Clearly, Condition (C) is satisfied and we have

$$\mathcal{R}^{(M,N)} \supset \mathcal{R}_{xy}^{(M,N)} \quad \text{and} \quad \mathcal{R}^{(M,N)} \supset \mathcal{R}^{\otimes_a(M+N)}. \quad (3.64)$$

So  $\mathcal{R}^{(M,N)}$  also satisfies Condition (D) and the corresponding space  $\mathcal{R}_{\mathcal{F}}$  contains  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$ .

In general, we do not consider the construction scheme presented in Section 3.2.2 as fixed. Modifications can be made, whenever needed. For instance, one may imagine further constructions based on two different one-particle spaces  $\mathcal{R}_x, \mathcal{R}_y$ . This may be particularly useful, if the Hilbert space of the model is  $\mathcal{H} = \mathcal{H}_x \otimes \mathcal{H}_y$  with  $\mathcal{H}_x \neq \mathcal{H}_y$ , e.g., within the spin-boson model.

For further considerations, it may also be useful to allow for poles of  $s \in \mathcal{R}_y$  not only at  $\mathbf{k} = 0$ , but anywhere. A possible realization of such poles could be by choosing  $\mathcal{R}_y$  as  $\mathcal{S}'$ ,  $\mathcal{D}'$ , or within a suitable rigging, as in Section 3.2.3.

### 3.2.5 Choices for Bogoliubov Transformations

In Chapter 5,  $X$  is not given by  $\mathbb{R}^d$ , but rather by  $\mathbb{N}$ , with the measure  $\mu$  assigning to  $A \subset \mathbb{N}$  its cardinality  $\mu(A) = |A|$ . This  $X$  corresponds to the discrete spectral set of some self-adjoint operator  $|C| = \sqrt{C^*C}$ . Therefore, no IR- but only UV-divergences can appear.

Since  $X = \mathbb{N}$ , it is natural to choose  $\mathcal{R}$  as the space of all sequences  $\mathbb{N} \rightarrow \mathbb{C}$ , as

### 3 Fock Space Extensions

we do in Chapter 5. Our notation  $\mathcal{R} = \mathcal{E}(\mathbb{N})$  in that chapter is motivated by the fact that functions  $\phi : j \mapsto \phi_j$  in this space may have arbitrary scaling behavior at  $j \rightarrow \infty$ , just as the space of smooth functions  $\mathcal{E}(\mathbb{R}^d)$ . Every other space of functions  $\mathbb{N} \rightarrow \mathbb{C}$  is included in  $\mathcal{E}(\mathbb{N})$ , so  $\mathcal{R}$  corresponds to the most general possible choice. It is easy to verify Conditions (A) and (B), since all functions  $\mathbb{N} \rightarrow \mathbb{C}$  are measurable and the sequence space  $\ell^2 = L^2(\mathbb{N})$  is contained in  $\mathcal{R}$ .

It is also natural to choose  $\mathcal{R}^{(N)} = \mathcal{E}^{(N)}(\mathbb{N})$ , by which we mean the space of all functions  $\mathbb{N}^N \rightarrow \mathbb{C}$  that obviously satisfies Condition (C). Again, this is the most general choice possible and Condition (D) is satisfied since  $\mathcal{R}^{(N)}$  contains the algebraic tensor product  $\mathcal{R}^{\otimes N}$ .

The case of Bogoliubov transformations with  $|C| = \sqrt{C^*C}$  having non-discrete spectrum is not treated in Chapter 5. It is an interesting question for future investigations, how an ESS description can be realized in that case. For bosons, the main challenge is to accommodate functions of the kind

$$f(x, x') = f_1(x)\delta(x - x') \quad (3.65)$$

in  $\mathcal{R}^{(2)}(X)$ , which corresponds to a bosonic pair generated by the transformation (1.75). As explained later in Section 5.4, we can write  $X \subset \mathbb{R} \times Y$  with  $Y \subset \mathbb{Z}$  and  $x = (\lambda, y)$ , where  $\lambda \in \mathbb{R}$  is a spectral value and  $y \in Y$  accounts for its multiplicity. The function of interest is

$$f_1(x) = f_1(\lambda, y) = \frac{\lambda}{1 + \sqrt{1 + 4\lambda^2}}, \quad (3.66)$$

which scales as  $\lambda$  as  $\lambda \rightarrow 0$  and converges to  $1/2$  as  $\lambda \rightarrow \infty$ . In the notation of Section 5.3.2, we may also write  $f_1(\lambda) = \frac{\nu}{2\mu} = \frac{\lambda}{2\mu^2}$ , which is the factor in (5.87) and (5.97). The fermionic case is more technical but leads to challenges of a similar kind, i.e., defining an  $\mathcal{R}^{(2)}$  that contains some  $f(x, x')$  involving a  $\delta$ -distribution. We suggest two possible approaches here, which may be pursued in future works concerning the implementation of Bogoliubov transformations using the ESS framework.

However, both approaches presented below generally violate condition (A), as they deal with distributions that cannot be represented by a function  $X \rightarrow \mathbb{C}$ . So it might become necessary to modify the ESS construction scheme by defining  $\mathcal{R}_2$  not as a space of functions  $X \rightarrow \mathbb{C}$ , but of pairs of distributions, similar to the space  $\text{Ren}_{01}$  introduced in Section 4.3.1. This space  $\text{Ren}_{01}$  is not explicitly necessary for the ESS construction in Chapter 4, but may turn out useful in future constructions.

### Distribution Spaces

A space like  $\mathcal{R} = \mathcal{S}'(X)$  or  $\mathcal{R} = \mathcal{D}'(X)$  could be used to accommodate the distribution  $f(x, x')$  in a space like  $\mathcal{R}^{(2)} = \mathcal{S}'(X^2)$  or  $\mathcal{R}^{(2)} = \mathcal{D}'(X^2)$ . A choice like  $\mathcal{E}'(X^2)$  for  $\mathcal{R}^{(2)}$  is not suitable, since  $f$  has no compact support and a choice like  $\mathcal{E}(X^2)$  for  $\mathcal{R}^{(2)}$  is also bound to fail, since  $f$  is not smooth.

We used the term “like” here, since there is a mathematical problem with defining  $\mathcal{S}'$ ,  $\mathcal{D}'$ ,  $\mathcal{E}'$  or  $\mathcal{E}$  on (powers of)  $X$ : It is not clear how to interpret the term “smooth”. For open  $X$ , this issue does not exist.  $\mathcal{D}'(X)$ ,  $\mathcal{D}(X)$ ,  $\mathcal{E}'(X)$  and  $\mathcal{E}(X)$  are then well-defined, and by [203, Thm. 51.6, 51.7] we have  $\mathcal{D}'(X)^{\otimes N} \cong \mathcal{D}'(X^N)$  (where the topological tensor product is used) and the same for  $\mathcal{D}$ ,  $\mathcal{E}'$ ,  $\mathcal{E}$ , which could serve for a definition of  $\mathcal{R}^{(N)}(X)$ . A similar statement holds true for  $\mathcal{S}'(\mathbb{R}^d)$  and  $\mathcal{S}(\mathbb{R}^d)$ . But  $X$ , as a spectral set, is always closed and can be quite “rugged”. So the question arises how to reasonably define  $\mathcal{D}(X)$ ,  $\mathcal{S}(X)$  or  $\mathcal{E}(X)$  and the respective dual spaces in that case.

Conditions for compact support of functions in  $\mathcal{D}(X)$  or rapid decay as  $\lambda \rightarrow \infty$  or  $y \rightarrow \infty$  for  $\mathcal{S}(X)$  can easily be generalized from the case  $X = \mathbb{R}^d$ . The main challenge when defining  $\mathcal{D}$ ,  $\mathcal{S}$  or  $\mathcal{E}$  is to give a notion of smoothness. In particular, if we can define  $\mathcal{E}(X)$ , then we obtain a definition of  $\mathcal{D}(X)$  and  $\mathcal{S}(X)$  straight away by adding a compact support or rapid decay condition.

One option is to define differentiability at each  $x_0 \in X$  by Taylor approximations, as illustrated Figure 3.4: We say that  $\phi : X \rightarrow \mathbb{C}$  is  $m$ -times differentiable at  $x_0 \in X$ , if there exists a function  $p(x)$ ,  $p : X \rightarrow \mathbb{C}$ , which is a polynomial of degree  $\leq m$  in  $(x - x_0)$ , such that

$$|\phi(x) - p(x)| = o(\|x - x_0\|^m) \quad \text{as } x \rightarrow x_0. \quad (3.67)$$

A possible definition would now be to consider  $\phi \in \mathcal{E}(X)$ , whenever it is  $m$  times differentiable at all  $x_0 \in X$  and for all  $m \in \mathbb{N}$ . The definition naturally extends to  $\mathcal{E}^{(N)} := \mathcal{E}(X^N)$ . This notion of smoothness can be used to define  $\mathcal{D}(X)$  and  $\mathcal{S}(X)$ , as well as  $\mathcal{D}(X^N)$  and  $\mathcal{S}(X^N)$ . In order to topologize  $\mathcal{D}$ ,  $\mathcal{S}$  by seminorms (1.26), it is necessary to define partial derivatives, which can be extracted from the total  $m$ -th derivative defined via  $p(x)$ . Definition (1.26) can then be used to obtain a family of seminorms  $(\|\cdot\|_{m,K})$  with  $m \in \mathbb{N}$  and  $K \subseteq X$  running through all compact subsets. These seminorms induce a topology that allows for constructing dual spaces  $\mathcal{D}'(X)$ ,  $\mathcal{S}'(X)$  and  $\mathcal{D}'(X^N)$ ,  $\mathcal{S}'(X^N)$ , which serve as  $\mathcal{R}(X)$  and  $\mathcal{R}^{(N)}(X)$ , respectively.

However, it is not obvious whether  $\mathcal{D}'(X^N) \cong \mathcal{D}'(X)^{\otimes N}$  or a similar statement for  $\mathcal{S}'$  holds true. A proof may become technical, involving the replication of proof

### 3 Fock Space Extensions

steps leading to  $\mathcal{D}'(X^N) \cong \mathcal{D}'(X)^{\otimes N}$  in [203, Thm. 51.6] for open  $X$ , and similar steps for  $\mathcal{S}'$ . In addition, verifying  $f_1(x, x') \in \mathcal{D}'$  or  $\mathcal{S}'$  may also become technical, as continuity of a functional with respect to an abstractly defined topology has to be checked. We postpone the investigation of whether these statements hold to future research.

Another option is to define smoothness by an equivalence relation

$$\mathcal{E}(X) := \mathcal{E}(\mathbb{R}^d) / \sim_{\mathcal{E}}, \quad \phi_1 \sim_{\mathcal{E}} \phi_2 :\Leftrightarrow (\phi_1 - \phi_2)(x) = 0 \quad \forall x \in X, \quad (3.68)$$

see Figure 3.4. It is then natural to set

$$\mathcal{E}(X)^{(N)} := \mathcal{E}(\mathbb{R}^{dN}) / \sim_{\mathcal{E},N}, \quad \Psi_1 \sim_{\mathcal{E},N} \Psi_2 :\Leftrightarrow (\Psi_1 - \Psi_2)(q) = 0 \quad \forall q \in X^N, \quad (3.69)$$

with  $q = (x_1, \dots, x_N)$ .

Adding a compact support or rapid decay condition, one may define  $\mathcal{D}(X)$ ,  $\mathcal{S}(X)$ , as well as  $\mathcal{D}(X^N)$  and  $\mathcal{S}(X^N)$ . However, there is no unique notion of a partial derivative at  $x \in X$  for  $\phi \in \mathcal{E}(X)$  as in (3.68), since there may be several functions  $\phi_j \in \mathcal{E}(\mathbb{R}^d)$  in one equivalence class, that have different partial derivatives at  $x$ . So one would first have to find a sensible way to topologize  $\mathcal{D}(X)$  and  $\mathcal{S}(X)$ , in order to define the dual spaces  $\mathcal{D}'(X)$ ,  $\mathcal{S}'(X)$ , that are used as  $\mathcal{R}$ .

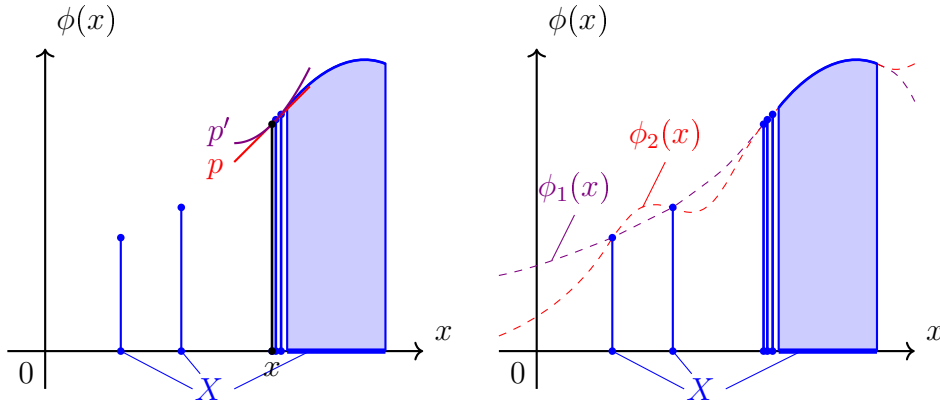


Figure 3.4: Left: Definition of  $\mathcal{E}(X)$  by Taylor approximation.

Right: Definition of  $\mathcal{E}(X)$  as an equivalence class of smooth functions.

Color online.

### Generalized Eigenfunctions

An interpretation of  $f \in \mathcal{R}^{(2)}$  by means of generalized eigenfunctions is also possible. Consider a self-adjoint operator (such as  $|C|$  in Chapter 5) with a spectral set  $X$ , spectral measure  $\mu$  on a  $\sigma$ -algebra  $\mathcal{A}(X)$  and a spectral PVM  $P : \mathcal{A}(X) \rightarrow \mathcal{B}(\mathcal{H})$ , where  $\mathcal{B}(\mathcal{H})$  is the  $C^*$ -algebra of bounded operators on  $\mathcal{H}$ . If  $\Phi \subset \mathcal{H} \subset \Phi'$  is a nuclear rigging, i.e.,  $\Phi$  is a nuclear space, then we may write [204, Thm. 2.5]:

$$P(A) = \int_A P_x d\mu(X). \quad (3.70)$$

Here, the operator  $P_x : \Phi \rightarrow \Phi'$  is defined for almost all  $x \in X$  and  $P(A)$  is also interpreted as an operator  $\Phi \rightarrow \Phi'$ . Heuristically,  $P_x = |e_x\rangle\langle e_x|$  is a “projection” to the generalized eigenvectors  $|e_x\rangle$ . Mathematically, by means of the Schwartz kernel theorem, we may associate to  $P_x$  an integral kernel in the topological tensor product:  $K_{P,x} \in \Phi' \otimes \Phi'$ .

So choosing  $\mathcal{R} = \Phi'$  and  $\mathcal{R}^{(N)} = (\Phi')^{\otimes N}$ , we can already accommodate  $K_{P,x} \in \mathcal{R}^{(2)}$ . And it is obvious, that Conditions (B), (C) and (D) are satisfied. (Note that elements of  $\Phi'$  are not necessarily functions, so Condition (A) is generally not satisfied.)

The accommodation of  $f$  in  $\mathcal{R}^{(2)}$  now works as follows: If we interpret  $f$  as the integral kernel of an operator  $F$  acting in spectral representation as

$$(F\phi)(x) = \int f(x, x')\phi(x') dx' = \int f_1(x)\delta(x - x')\phi(x') dx', \quad (3.71)$$

then  $F$  is just a spectral multiplication by the bounded function  $f_1(x)$ . So  $F$  maps  $\mathcal{H}$  into itself, and hence  $\Phi$  into  $\Phi'$ . So by the Schwartz kernel theorem, it has an integral kernel in  $\mathcal{R}^{(2)} = \Phi' \otimes \Phi'$ , which we identify with the function  $f$ .

There is large freedom in the choice of nuclear riggings. For instance, riggings with  $\mathcal{E}(X)$ ,  $\mathcal{S}(\mathbb{R}^d)$  or  $\mathcal{D}(X)$  as space  $\Phi$ , where  $X \subseteq \mathbb{R}^d$  is open, are all nuclear [203, Thm. 51.5]. Further information of when an eigenfunction expansion within a rigging is possible can be found in [206, 207].

Nevertheless, it remains to check all steps within the ESS construction and its application to bosonic Bogoliubov transformations, in order to arrive at an implementability result similar to Theorem 5.5.6 for generic  $|C|$ . For fermions, further peculiarities appear, such as the restriction to a finite number of particle-hole transformations in Theorem 5.5.8. A further and major complication is that within a Cooper pair both fermions have orthogonal state vectors. So the function

### *3 Fock Space Extensions*

$f(x, x')$  describing a Cooper pair is no longer supported on the diagonal  $x = x'$ .

We leave the implementability of a Bogoliubov transformation with general  $|C|$  on a Fock space extension as an interesting open question for future research.

# 4 Extended State Space for Describing Renormalized Fock Spaces in QFT

## 4.1 Overview and Main Results

In this chapter, we present a specific version of the ESS construction that is tailor-made for polaron-like models. More precisely, we consider a class of models with resting fermions that have the following formal Hamiltonian:

$$H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty. \quad (4.1)$$

That means the following: We consider a species of  $M \in \mathbb{N}_0$  fermions (associated with an index  $x$ ) and a species of  $N \in \mathbb{N}_0$  bosons (associated with an index  $y$ ). The operators  $A^\dagger(v)$ ,  $A(v)$  make each fermion create/annihilate a boson with some form factor  $v$  (in momentum space),  $H_{0,y}$  describes the free evolution of the bosons and  $E_\infty$  is an **infinite self-energy counterterm**.

In Section 4.8, we will also encounter formal Hamiltonians of the kind

$$H_0 + A^\dagger(v) + A(v) - E_\infty, \quad (4.2)$$

with  $H_0 = H_{0,x} + H_{0,y}$  including both a fermionic and a bosonic dispersion relation. As in Chapter 1, the fermionic dispersion relation is called  $\theta$  and the bosonic one  $\omega$ . Up to  $E_\infty$  and the missing fermionic dispersion relation  $H_{0,x}$ , (4.1) coincides with the formal “polaron-like Hamiltonian” introduced in the beginning of Section 1.3.7.

The additional term  $E_\infty$  is motivated by the finite counterterms within cutoff renormalization in Sections 1.3.7 and 1.3.8. We define  $E_\infty$  as a linear operator in Proposition 4.4.2 using the ESS framework.

Recall from Section 3.2 that the ESS construction provides us with the following main tools:

- A vector space of divergent integrals  $\mathfrak{t} \in \text{Ren}_1$ .

#### 4 Extended State Space for Describing Renormalized Fock Spaces in QFT

- A **field of** (possibly infinite) **wave function renormalizations**  $\text{eRen}$ , containing exponentials  $e^f$  and fractions of linear combinations thereof.
- An **extended state space**  $\overline{\mathcal{F}}$ , which is an  $\text{eRen}$ -vector space over complex-valued functions on configuration space  $\mathcal{Q}$ , which are not necessarily  $L^2$ -integrable.  $\overline{\mathcal{F}}$  extends a dense subspace of the Fock space,  $\mathcal{F}$ .
- A **second extended state space**  $\overline{\mathcal{F}}_{\text{ex}}$ , which allows for multiplying wave functions in  $\overline{\mathcal{F}}$  by elements  $\mathfrak{r} \in \text{Ren}_1$  (and not only by elements of  $\text{eRen}$ ).

In this framework, the Hamiltonian  $H$  in (4.1) will be defined as a linear operator on a subspace of  $\overline{\mathcal{F}}_{\text{ex}}$ . Physical meaning is assigned to  $H$  by a **dressing transformation**  $W(s)$ , which is a linear operator  $W(s) : \overline{\mathcal{F}}_{\text{ex}} \supset \mathcal{D}_W \rightarrow \overline{\mathcal{F}}_{\text{ex}}$  in form of a **Gross transformation** (compare Section 1.3.4). Its form factor  $s(\mathbf{k})$  does not depend on the fermion momentum  $\mathbf{p}$ . So one can fiber-decompose it into several Weyl transformations, one for each fermion configuration  $\mathbf{X} \in \mathbb{R}^{Md}$ . The “renormalized” or “undressed” Hamiltonian  $\tilde{H}$  is defined without cutoffs by

$$W(s)\tilde{H} = HW(s),$$

i.e., we require that the following diagram commutes:

$$\begin{array}{ccc} \overline{\mathcal{F}}_{\text{ex}} \supset W(s)[\mathcal{D}_{\mathcal{F}}] \supset W(s)[\tilde{\mathcal{D}}_{\mathcal{F}}] & \xrightarrow{H} & W(s)[\mathcal{D}_{\mathcal{F}}] \subset \overline{\mathcal{F}}_{\text{ex}} \\ & \uparrow W(s) & \uparrow W(s) \\ \mathcal{F} \supset \mathcal{D}_{\mathcal{F}} \supset \tilde{\mathcal{D}}_{\mathcal{F}} & \xrightarrow{\tilde{H}} & \mathcal{D}_{\mathcal{F}} \subset \mathcal{F} \end{array}$$

The large domain  $\mathcal{D}_{\mathcal{F}}$ , with the fermionic wave function being a Schwartz function, is defined in (4.132). It contains the small domain  $\tilde{\mathcal{D}}_{\mathcal{F}}$  defined in (4.147) via (4.146), with fermionic wave function in  $C_c^\infty$ , and its support avoiding the collision configurations. It is necessary to introduce two domains, since  $\tilde{H}$  does neither map  $\mathcal{D}_{\mathcal{F}}$  nor  $\tilde{\mathcal{D}}_{\mathcal{F}}$  to itself. Nevertheless,  $\tilde{H} : \tilde{\mathcal{D}}_{\mathcal{F}} \rightarrow \mathcal{D}_{\mathcal{F}}$  can be defined. By means of Lemmas 4.5.10 and 4.7.2, both  $\mathcal{D}_{\mathcal{F}}$  and  $\tilde{\mathcal{D}}_{\mathcal{F}}$  are dense in  $\mathcal{F}$ .

Our **main result** is Theorem 4.6.1, which establishes that  $\tilde{H}$  is indeed defined as a linear operator  $\tilde{H} : \mathcal{F} \supset \mathcal{D}_{WS} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$  (with definition of the domain  $\mathcal{D}_{WS}$  in (4.105)). By means of Lemma 4.7.2, under certain conditions, this  $\tilde{H}$  can indeed be interpreted as a Fock space operator  $\tilde{H} : \tilde{\mathcal{D}}_{\mathcal{F}} \rightarrow \mathcal{D}_{\mathcal{F}}$ , which allows for self-adjoint extensions by Corollary 4.7.3. More precisely, we have

$$\tilde{H} = H_{0,y} + V, \tag{4.3}$$

with  $V$  being a pair potential interaction between fermions.

The result (4.3) after an “undressing” is not too surprising: Our formal Hamiltonian (4.1) actually just corresponds to a direct integral of uncoupled Van Hove Hamiltonians, one for each fermion position configuration  $\mathbf{X} \in \mathbb{R}^{Md}$ . We have discussed these Hamiltonians in Section 1.4.3. There, it became clear that formally, a Weyl transformation together with subtraction of an infinite self-energy renormalization constant results in an operator  $H_{0,y}$ . This result is well-known and can easily be achieved by an algebraic computation without using Fock space extensions, see [15].

The main novelty in this chapter is that we are able to define certain products consisting of dressing operators  $W(s)$ ,  $s \notin L^2$ , creation operators  $A^\dagger(v)$ ,  $v \notin L^2$  and even divergent integrals  $\mathfrak{r} \in \text{Ren}_1$ , as linear operators on suitable domains. The rigorous treatment of divergent integrals may also become useful in a purely algebraic approach, which does not refer to any Fock space extensions, but nevertheless produces divergent integrals when evaluating commutation relations.

Let us add some remarks about the dressing operator  $W(s)$ : Gross transformations as described in Section 1.3.4 are usually of the form  $W(s) = e^{A^\dagger(s) - A(s)}$ , where  $A^\dagger(s) = \sum_{j=1}^M A_j^\dagger(s)$  describes a boson creation induced by all fermions  $j \in \{1, \dots, M\}$ . We will choose a slightly different dressing operator, which formally reads

$$W(s) = W_M(s) \dots W_1(s), \quad W_j(s) = e^{A_j^\dagger(s) - A_j(s)}. \quad (4.4)$$

The reason is that, when applying the Baker–Campbell–Hausdorff formula (as in (4.115)), formal calculations with  $e^{A^\dagger(s) - A(s)}$  would produce exponential operators of the kind  $e^{V_{jj'}}$ , with  $V_{jj'}$  being a potential interaction between fermions  $j$  and  $j'$ . Those expressions  $e^{V_{jj'}}$  cannot be defined as operators on the ESS  $\overline{\mathcal{F}}_{\text{ex}}$ , so we need to avoid their occurrence.

In general, we will define the extended dressing operators  $W_j(s)$  such that their action for  $s \in \mathfrak{h}$  slightly differs from that of the usual Fock space dressing operators  $W_{\mathcal{F},j}(s) : \mathcal{F} \rightarrow \mathcal{F}$ . That is, we drop any terms of the kind  $e^{V_{jj'}}$  in our definition. This ad-hoc modification is justified by the fact that, in a sufficiently regular case, the  $V_{jj'}$ -operators commute with  $W(s)$  (Lemma 4.5.8). So even if we could define them as operators, they would just act as an independent factor that can be pulled to the left. So dropping all  $e^{V_{jj'}}$ -terms can essentially be seen as a simplification of the bookkeeping.

Nevertheless, it would be interesting for future works to define  $e^{V_{jj'}}$  as an operator, mapping to some ESS  $\overline{\mathcal{F}}'_{\text{ex}}$ , which is constructed differently than  $\overline{\mathcal{F}}_{\text{ex}}$ . The current ESS  $\overline{\mathcal{F}}_{\text{ex}}$  also does not allow for defining general products of annihilation

#### 4 Extended State Space for Describing Renormalized Fock Spaces in QFT

operators  $A_{j_1}(v_1) \dots A_{j_n}(v_n)$  with  $n \geq 2$ . It is a further interesting question, how such products can be defined in the future by an alternative construction of some  $\overline{\mathcal{F}}'_{\text{ex}}$ .

The ESS construction is currently at an early stage of development. One may view the results of this chapter rather as a “proof of concept”, showing that the ESS construction generates reasonable outcomes in well–investigated environments. We do not yet attempt to produce renormalized Hamiltonians in models where non–perturbative renormalization has not succeeded before, although this is a clearly desirable objective for the future.

Although the spaces  $\overline{\mathcal{F}}$ ,  $\overline{\mathcal{F}}_{\text{ex}}$  do not have a topological structure, we may define a **renormalized scalar product** on  $W(s)[\mathcal{D}_{\mathcal{F}}]$  via:

$$\langle W(s)\Psi, W(s)\Phi \rangle_{\text{ren}} := \langle \Psi, \Phi \rangle \quad \forall \Psi, \Phi \in \mathcal{D}_{\mathcal{F}}. \quad (4.5)$$

The completion of  $W(s)[\mathcal{D}_{\mathcal{F}}]$  with respect to  $\langle \cdot, \cdot \rangle_{\text{ren}}$  defines a Hilbert space  $\mathcal{F}_{\text{ren}}$ , our **renormalized Fock space**. The map  $W(s)$  then uniquely extends to an isometric isomorphism between  $\mathcal{F}$  and  $\mathcal{F}_{\text{ren}}$ .

We remark that our result (4.3) is actually just the lowest–order approximation in a perturbation expansion within the weak–coupling regime, and completely decouples the fermions from the bosonic radiation field. The reason is that we both set  $\theta = 0$  and restrict to form factors  $v(\mathbf{k})$ , that only depend on the momentum  $\mathbf{k}$  of the emitted boson, and not on the momentum  $\mathbf{p}$  of the fermion emitting it. It is physically expected and confirmed for the Nelson model with UV–cutoff [54, 55], that  $W_{\Lambda}^* H_{\Lambda} W_{\Lambda}$  contains interactions between fermions and the radiation field, see also Section 1.3.7. And indeed, when changing to  $v(\mathbf{p}, \mathbf{k})$ , one may formally use a so–called “Lie–Schwinger series” (5.211), [208]:

$$\tilde{H} = e^A B a^{-A} = \sum_{n=0}^{\infty} \frac{\text{ad}^n(A)B}{n!},$$

where  $A := -A^{\dagger}(s) + A(s)$ ,  $B := H$  and with the  $n$ –fold commutator

$$\text{ad}^n(A)B := [A, [A, \dots [A, B] \dots]].$$

However, establishing well–definedness of  $\text{ad}^n(A)B$  for  $\theta \neq 0$  and  $v$  depending on  $\mathbf{p}$  is a rather involved task, so we postpone it to future investigations.

Another interesting objective for future works would be to introduce a mass renormalization term  $\delta m$ , as used for various polaron models in Section 1.3.7. In

constructive QFT (CQFT, see Section 1.3.8), where the Hamiltonian is of a form different from (4.1), mass renormalization terms can also be found in some works [95, 103, 104, 105, 209], while other CQFT renormalization procedures work without a mass renormalization [91, 92, 93, 98, 99, 100, 106, 110].

The rest of this chapter is structured as follows: After specifying some mathematical notation in Section 4.2, we conduct the ESS construction in Section 4.3, which slightly differs from the general scheme in Section 3.2, as announced. After the successful definition of the ESSs  $\overline{\mathcal{F}}$ ,  $\overline{\mathcal{F}}_{\text{ex}}$ , we establish  $H_0$ ,  $A^\dagger$ ,  $A$  and  $E_\infty$  as linear operators on  $\overline{\mathcal{F}}$  or  $\overline{\mathcal{F}}_{\text{ex}}$ , which is done in Section 4.4. In the following Section 4.5, we construct the dressing transformation  $W(s)$ . Further, we define an **extended Weyl algebra**  $\overline{\mathcal{W}}$ , which is a concrete realization of the ESS Weyl algebra  $\overline{\mathcal{W}}$  (3.57) in Section 3.2 with operators instead of algebraic expressions. In analogy to the  $\overline{\mathcal{W}}$  from Section 3.2, the concrete  $\overline{\mathcal{W}}$  in this section includes linear combinations of the operators  $W_j(s) = e^{A_j^\dagger(s) - A_j(s)}$ . A multiplication with infinite wave function renormalization factors  $\mathfrak{c} \in \text{eRen}$  is also allowed within  $\overline{\mathcal{W}}$ , and as in Section 3.2, the usual Weyl relations hold:

$$\begin{aligned} W_j(s)^{-1} &= W_j(-s) \\ W_j(s_1)W_j(s_2) &= e^{\text{Im}\langle s_1, s_2 \rangle} W_j(s_1 + s_2). \end{aligned} \tag{4.6}$$

However, note that  $\overline{\mathcal{W}}$  is generated by  $W_j(s)$  for a fixed  $j$ , instead of  $W(s)$  and the dressing operator  $W(s)$  is not contained in  $\overline{\mathcal{W}}$ .

In Section 4.6 we compute  $\tilde{H}$  such that  $W(s)\tilde{H} = HW(s)$ . This section contains the main result, Theorem 4.6.1.

The proof that under certain conditions  $\tilde{H}$  is a Fock space operator and allows for self-adjoint extensions follows in Section 4.7. Thus,  $\tilde{H}$  generates well-defined quantum dynamics, although they are not necessarily unique.

Finally, Section 4.8 concerns dressing transformations different from  $W(s)$ , which are inspired by examples from the literature and can successfully be defined on the ESSs  $\overline{\mathcal{F}}$ ,  $\overline{\mathcal{F}}_{\text{ex}}$ . This includes dressings  $(1 + H_0^{-1}A^\dagger)^{-1}$  within the IBC works presented in Section 1.4, as well as the dressing  $T = e^{-H_0^{-1}A^\dagger}$ , which is a simplified version of dressings used in the CQFT works like [209, 95].

Sections 4.9–4.12 can be seen as appendices which contain proofs.

## 4.2 The Mathematical Model

### 4.2.1 Formal Hamiltonian

In this chapter, we use the notation from Section 1.2, adapted to models with two particle species:

- There is one species of spinless fermionic particles ( $x$ ).
- These fermions interact by exchange of spinless bosons ( $y$ ).

By  $M$  and  $N$ , we denote the number of  $x$ - and  $y$ -particles, respectively. In the particle-position representation, a single particle is described by a Hilbert space vector

$$\varphi \in \mathfrak{h} = L^2(\mathbb{R}^d, \mathbb{C}), \quad (4.7)$$

with some coordinates denoted by a boldface symbol  $\mathbf{x} \in \mathbb{R}^d$ . So the system has  $d$  space dimensions and  $d + 1$  spacetime dimensions. The configuration of the entire system is given by an element  $(\mathbf{X}, \mathbf{Y})$  of the configuration space  $\mathcal{Q}$ ,

$$(\mathbf{X}, \mathbf{Y}) = (\mathbf{x}_1, \dots, \mathbf{x}_M, \mathbf{y}_1, \dots, \mathbf{y}_N) \in \mathcal{Q}_x \times \mathcal{Q}_y =: \mathcal{Q}, \quad (4.8)$$

where the  $x$ - and  $y$ -configuration spaces  $\mathcal{Q}_x, \mathcal{Q}_y$  and its sectors  $\mathcal{Q}_x^{(M)}, \mathcal{Q}_y^{(N)}$  are defined as

$$\bigsqcup_{M=0}^{\infty} \mathbb{R}^{Md} =: \bigsqcup_{M=0}^{\infty} \mathcal{Q}_x^{(M)} =: \mathcal{Q}_x \quad \bigsqcup_{N=0}^{\infty} \mathbb{R}^{Nd} =: \bigsqcup_{N=0}^{\infty} \mathcal{Q}_y^{(N)} =: \mathcal{Q}_y. \quad (4.9)$$

The state of the system at time  $t$  is described by a Fock space vector

$$\Psi \in L^2(\mathcal{Q}, \mathbb{C}). \quad (4.10)$$

For physical state vectors, we assume the function  $\Psi \in L^2(\mathcal{Q}, \mathbb{C})$  to be anti-symmetric under coordinate exchange for fermions  $x$  and symmetric under coordinate exchange for bosons  $y$ . The corresponding symmetrization and antisymmetrization operators  $S_-, S_+$  are defined on  $L^2(\mathcal{Q}_x, \mathbb{C})$  and  $L^2(\mathcal{Q}_y, \mathbb{C})$  as

$$\begin{aligned} (S_- \Psi)(\mathbf{x}_1, \dots, \mathbf{x}_M) &= \frac{1}{M!} \sum_{\sigma} \text{sgn}(\sigma) \Psi(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(M)}) \\ (S_+ \Psi)(\mathbf{y}_1, \dots, \mathbf{y}_N) &= \frac{1}{N!} \sum_{\sigma} \Psi(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)}), \end{aligned} \quad (4.11)$$

which is equivalent to (1.6). The **fermionic/bosonic/total Fock space** is then

$$\mathcal{F}_x = S_-[L^2(\mathcal{Q}_x, \mathbb{C})], \quad \mathcal{F}_y = S_+[L^2(\mathcal{Q}_y, \mathbb{C})], \quad \mathcal{F} = \mathcal{F}_x \otimes \mathcal{F}_y. \quad (4.12)$$

It is convenient to describe the action of the formal Hamiltonian  $H$  in the particle–momentum representation: For any  $\Psi \in \mathcal{F}$  with particle–position representation  $\Psi(\mathbf{X}, \mathbf{Y})$ , we define the Fourier transform, denoted  $\Psi(\mathbf{P}, \mathbf{K})$  in this chapter, with momentum configuration

$$(\mathbf{P}, \mathbf{K}) = (\mathbf{p}_1, \dots, \mathbf{p}_M, \mathbf{k}_1, \dots, \mathbf{k}_N) \in \mathbb{R}^{Md+Nd}, \quad (4.13)$$

via

$$\Psi(\mathbf{P}, \mathbf{K}) := (2\pi)^{-\frac{Md+Nd}{2}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^M} \Psi(\mathbf{X}, \mathbf{Y}) e^{-i\mathbf{P}\cdot\mathbf{X} - i\mathbf{K}\cdot\mathbf{Y}} d^{Md}\mathbf{X} d^{Nd}\mathbf{Y}. \quad (4.14)$$

In this chapter, the set of variables plugged into  $\Psi(\cdot, \cdot)$ , i.e.,  $(\mathbf{X}, \mathbf{Y})$  or  $(\mathbf{P}, \mathbf{K})$  will specify, which representation is meant.

As a formal Hamiltonian, we consider the following expression with zero fermion dispersion relation:

$$H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty. \quad (4.15)$$

In Section 4.8, we will also consider similar Hamiltonians of this kind that feature a nonzero fermion dispersion relation, i.e., where  $H_{0,y}$  is replaced by  $H_0 = H_{0,x} + H_{0,y}$ . The formal definitions of the relevant expressions read as follows:

- The kinetic term  $H_0$  is characterized by two dispersion relations, i.e., by real–valued functions  $\theta(\mathbf{p}), \omega(\mathbf{k}) \in C^\infty(\mathbb{R}^d \setminus \{0\})$  for fermions and bosons, respectively. We can decompose

$$\begin{aligned} H_0 &= H_{0,x} + H_{0,y}, \\ (H_{0,x}\Psi)(\mathbf{P}, \mathbf{K}) &= \sum_{j=1}^M \theta(\mathbf{p}_j) \Psi(\mathbf{P}, \mathbf{K}), \\ (H_{0,y}\Psi)(\mathbf{P}, \mathbf{K}) &= \sum_{\ell=1}^N \omega(\mathbf{k}_\ell) \Psi(\mathbf{P}, \mathbf{K}). \end{aligned} \quad (4.16)$$

As in Section 1.2.2, we use the symbols  $d\Gamma_x(\cdot)$  and  $d\Gamma_y(\cdot)$  for second quantization

$$H_{0,x} = d\Gamma_x(\theta), \quad H_{0,y} = d\Gamma_y(\omega), \quad (4.17)$$

#### 4 Extended State Space for Describing Renormalized Fock Spaces in QFT

with multiplication operators

$$\begin{aligned} \theta : \mathfrak{h} \ni \text{dom}(\theta) &\rightarrow \mathfrak{h} & \omega : \mathfrak{h} \ni \text{dom}(\omega) &\rightarrow \mathfrak{h} \\ \phi(\mathbf{p}) &\mapsto \theta(\mathbf{p})\phi(\mathbf{p}) & \phi(\mathbf{k}) &\mapsto \omega(\mathbf{k})\phi(\mathbf{k}). \end{aligned} \quad (4.18)$$

- The creation part  $A^\dagger(v)$  makes each fermion create a boson. It is specified by a form factor  $v$ . In this article, we will restrict to  $v \in C^\infty(\mathbb{R}^d \setminus \{0\})$  and put some further assumptions on the scalings of  $v$  for  $|\mathbf{k}| \rightarrow 0$  (IR–regime) and  $|\mathbf{k}| \rightarrow \infty$  (UV–regime). These assumptions are described in Section 4.2.2.

We may write  $A^\dagger(v)$  as a sum over operators  $A_j^\dagger(v), j \in \{1, \dots, M\}$ , which only make fermion  $j$  create a boson. The formal definition in particle–momentum representation reads

$$\begin{aligned} (A^\dagger(v)\Psi)(\mathbf{P}, \mathbf{K}) &= \left( \sum_{j=1}^M A_j^\dagger \Psi \right) (\mathbf{P}, \mathbf{K}) \\ &= \sum_{j=1}^M \frac{1}{\sqrt{N}} \sum_{\ell=1}^N v(\mathbf{k}_\ell) \Psi(\mathbf{P} + e_j \mathbf{k}_\ell, \mathbf{K} \setminus \mathbf{k}_\ell), \end{aligned} \quad (4.19)$$

where

- $\mathbf{K} \setminus \mathbf{k}_\ell = (\mathbf{k}_1, \dots, \mathbf{k}_{\ell-1}, \mathbf{k}_{\ell+1}, \dots, \mathbf{k}_N)$  denotes  $\mathbf{K}$  without  $\mathbf{k}_\ell$
- $\mathbf{P} + e_j \mathbf{k}_\ell = (\mathbf{p}_1, \dots, \mathbf{p}_{j-1}, \mathbf{p}_j + \mathbf{k}_\ell, \mathbf{p}_{j+1}, \dots, \mathbf{p}_M)$  is the shifted fermion momentum.

Here,  $e_j$  is meant to denote the  $j$ –th unit vector and, by slight abuse of notation,  $e_j \mathbf{k}_\ell = (0, \dots, \mathbf{k}_\ell, \dots, 0) \in \mathbb{R}^{Md}$  is used to denote the assignment of an additional momentum  $\mathbf{k}_\ell$  to fermion  $j$ , before boson  $\ell$  is emitted.

The corresponding definition in particle–position representation uses the Fourier inverse  $\check{v}$  of the form factor  $v$  and reads

$$(A^\dagger(v)\Psi)(\mathbf{X}, \mathbf{Y}) = \left( \sum_{j=1}^M A_j^\dagger(v)\Psi \right) (\mathbf{X}, \mathbf{Y}) = \sum_{j=1}^M \frac{1}{\sqrt{N}} \sum_{\ell=1}^N \check{v}(\mathbf{y}_\ell - \mathbf{x}_j) \Psi(\mathbf{X}, \mathbf{Y} \setminus \mathbf{y}_\ell). \quad (4.20)$$

Without the cutoff, the form factor is  $v \notin L^2$  in many physically relevant models. Its Fourier inverse  $\check{v}$  is therefore  $\notin L^2$ , as well, if it is even defined.

Note that there is a more general class of physically interesting models, which

we do not treat here: The form factor may depend on the fermion momentum  $\mathbf{p}$ , i.e., it may read  $v(\mathbf{p}, \mathbf{k})$  with  $v : \mathbb{R}^{2d} \rightarrow \mathbb{C}$ , such as in [59, 66, 210].

- The annihilation part  $A(v)$  in particle–momentum representation reads

$$\begin{aligned} (A(v)\Psi)(\mathbf{P}, \mathbf{K}) &= \left( \sum_{j=1}^M A_j(v)\Psi \right) (\mathbf{P}, \mathbf{K}) \\ &= \sum_{j=1}^M \sqrt{N+1} \int v(\tilde{\mathbf{k}})^* \Psi(\mathbf{P} - e_j \tilde{\mathbf{k}}, \mathbf{K}, \tilde{\mathbf{k}}) d\tilde{\mathbf{k}}. \end{aligned} \quad (4.21)$$

In particle–position representation,

$$\begin{aligned} (A(v)\Psi)(\mathbf{X}, \mathbf{Y}) &= \left( \sum_{j=1}^M A_j(v)\Psi \right) (\mathbf{X}, \mathbf{Y}) \\ &= \sum_{j=1}^M \sqrt{N+1} \int \check{v}(\tilde{\mathbf{y}} - \mathbf{x}_j)^* \Psi(\mathbf{X}, \mathbf{Y}, \tilde{\mathbf{y}}) d\tilde{\mathbf{y}}. \end{aligned} \quad (4.22)$$

Note that for  $v \in \mathfrak{h}$ , both  $A^\dagger(v)$  and  $A(v)$  can be defined as operators on a dense domain in  $\mathcal{F}$ , where  $A^\dagger(v)$  is the adjoint of  $A(v)$ .

- The self–energy  $E_\infty$  is a formal multiplication operator of the form  $E_\infty = d\Gamma_x(E_1)$  with  $E_1 : \mathbb{R}^d \rightarrow \mathbb{R}$ . In particle–momentum representation,

$$(E_\infty\Psi)(\mathbf{P}, \mathbf{K}) = \sum_{j=1}^M E_1(\mathbf{p}_j)\Psi(\mathbf{P}, \mathbf{K}). \quad (4.23)$$

We will consider the expression

$$E_1(\mathbf{p}) = \int -\frac{v(\mathbf{k})^* v(\mathbf{k})}{\omega(\mathbf{k})} d\mathbf{k}. \quad (4.24)$$

With cutoffs  $\sigma, \Lambda$  applied to  $v$ , this integral would be finite in many physically interesting situations. However, without cutoffs it is typically divergent and hence the operator  $E_\infty$  becomes a formal expression. We will define it as a map  $\overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$  in Proposition 4.4.2.

### 4.2.2 Scaling Degrees

The convergence of integrals appearing in formal calculations depends on how  $\theta(\mathbf{p}), \omega(\mathbf{k})$  and  $v(\mathbf{k})$  scale at  $|\mathbf{p}|, |\mathbf{k}|$  going to  $\infty$  or  $0$ . We assume a polynomial scaling, of which we keep track using scaling degrees  $m$  (UV) and  $\beta$  (IR). The scaling degree  $m$  in the UV regime is commonly used for symbols  $s \in \mathcal{S}^m$ . As we choose  $\theta, \omega, v \in C^\infty(\mathbb{R}^d \setminus \{0\})$ , an additional pole might appear at  $|\mathbf{k}| \rightarrow 0$ , which we assume to be of order  $\beta$ . Both scalings are assumed to be exact, i.e., polynomial bounds exist from above and below:

**Definition 4.2.1** (scaling). Consider  $s \in C^\infty(\mathbb{R}^d \setminus \{0\})$ . We say that  $s$  has a **polynomial scaling** if there are some scaling degrees  $m_s \leq \beta_s \in \mathbb{R}$  and constants  $C_1, C_2 \in \mathbb{R}$  such that

$$|s(\mathbf{k})| \leq C_1 |\mathbf{k}|^{\beta_s} + C_2 |\mathbf{k}|^{m_s} \quad \forall \mathbf{k} \in \mathbb{R}^d \setminus \{0\}. \quad (4.25)$$

The space of all **symbols with polynomial scaling** is denoted

$$\dot{\mathcal{S}}_1^\infty := \{s \in C^\infty(\mathbb{R}^d \setminus \{0\}) \mid (4.25) \text{ holds}\}. \quad (4.26)$$

**Definition 4.2.2** (exact scaling). Consider a symbol  $s \in C^\infty(\mathbb{R}^d \setminus \{0\})$ . We say that the symbol has an **exact polynomial scaling** (see Figure 4.1) if there are some  $0 < \underline{\varepsilon} < \bar{\varepsilon} \in \mathbb{R}$  such that

- There are a UV scaling degree  $m_s \in \mathbb{R}$  and  $c_1, C_1 > 0$  with

$$c_1 |\mathbf{k}|^{m_s} \leq |s(\mathbf{k})| \leq C_1 |\mathbf{k}|^{m_s} \quad \forall |\mathbf{k}| > \bar{\varepsilon}. \quad (4.27)$$

- There are an IR scaling degree  $\beta_s \in \mathbb{R}$  and  $c_2, C_2 > 0$  with

$$c_2 |\mathbf{k}|^{\beta_s} \leq |s(\mathbf{k})| \leq C_2 |\mathbf{k}|^{\beta_s} \quad \forall 0 < |\mathbf{k}| < \underline{\varepsilon}. \quad (4.28)$$

- There is a constant  $c_3 > 0$  with

$$c_3 \leq |s(\mathbf{k})| \quad \forall \underline{\varepsilon} \leq |\mathbf{k}| \leq \bar{\varepsilon}. \quad (4.29)$$

The space of all **symbols with exact polynomial scaling** is denoted

$$\dot{\mathcal{S}}_{1,>}^\infty := \{s \in C^\infty(\mathbb{R}^d \setminus \{0\}) \mid (4.27), (4.28) \text{ and } (4.29) \text{ hold}\} \subseteq \dot{\mathcal{S}}_1^\infty. \quad (4.30)$$

The “ $\succ$ ” refers to the fact that  $|s(\mathbf{k})| > 0$ , so  $\frac{1}{s}$  is defined everywhere except from  $\mathbf{k} = 0$  and also scales polynomially.

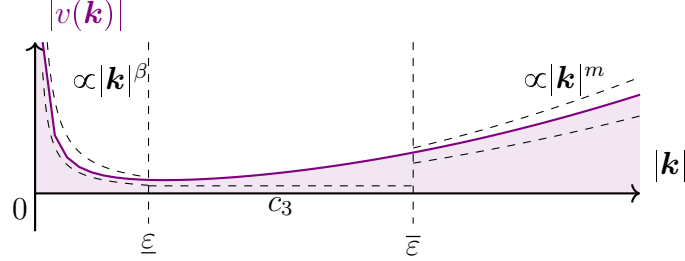


Figure 4.1: Scaling degrees in the special case of a radially symmetric, positive function  $v$  with  $|v(\mathbf{k})| > 0$ . Color online.

Obviously,  $C_c^\infty(\mathbb{R}^d) \subset \dot{\mathcal{S}}_1^\infty$ , so  $\dot{\mathcal{S}}_1^\infty \cap \mathfrak{h}$  is dense in  $\mathfrak{h}$ .

For obtaining a well-defined renormalized Hamiltonian  $\tilde{H}$  in Lemma 4.7.2, we will assume

$$\theta, \omega, v \in \dot{\mathcal{S}}_{1,\succ}^\infty \quad (4.31)$$

(see Figure 4.2) and denote their scaling degrees with  $m_\theta, m_\omega, m_v$  and  $\beta_\theta, \beta_\omega, \beta_v$ , respectively. Further, we will assume in Lemma 4.7.2:

$$m_\theta, m_\omega, \beta_\theta, \beta_\omega \geq 0. \quad (4.32)$$

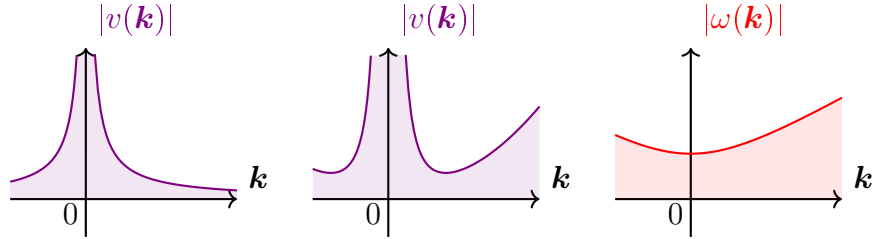


Figure 4.2: Examples for absolute values of functions  $v, \omega \in \dot{\mathcal{S}}_{1,\succ}^\infty$ . The functions  $v, \omega$  are complex-valued. Note that  $\beta_\omega, m_\omega \geq 0$ . Color online.

This implies, that for  $\theta$  and  $\omega$ , there is no pole at the origin. QFT models often use dispersion relations based on symbols with  $m_\theta, m_\omega, \beta_\theta, \beta_\omega \in \{0, 1, 2\}$ , which all satisfy this condition.

In order to obtain symmetric operators, we will also need to impose a symmetry condition in Lemma 4.7.2:

$$\theta(\mathbf{p}) = \theta(-\mathbf{p}), \quad \omega(\mathbf{k}) = \omega(-\mathbf{k}), \quad v(\mathbf{k}) = v(-\mathbf{k}). \quad (4.33)$$

## 4.3 Construction of the Extended State Space

### 4.3.1 Wave Function Renormalization Factors

As mentioned in Section 3.2, the motivation behind the introduction of the ESS setting is to rigorously define formal dressing transformations. Whereas the ESS construction in Section 3.2 was explained using a Weyl transformation in (3.23) as an example, we will explicitly conduct a similar ESS construction for defining the formal Gross transformation

$$W(s) = W_M(s) \dots W_1(s), \quad \text{with } W_j(s) = e^{A_j^\dagger(s) - A_j(s)}, \quad s = -\frac{v}{\omega}, \quad (4.34)$$

in the case where  $s \in \dot{\mathcal{S}}_1^\infty \subseteq C^\infty(\mathbb{R}^d \setminus \{0\})$ , but  $s \notin \mathfrak{h} = L^2(\mathbb{R}^d, \mathbb{C})$ . Here, the number  $M$  corresponds to the fermion sector, so the second term in (4.34) is a rather symbolic expression requiring us to adapt  $M$  in dependence of the fermion sector (similar to (4.16)). Formally, if we apply  $W(s)$  to a state vector  $\Psi_1$  where the boson field is in the vacuum  $\Omega_y$  and  $\Psi_x \in \mathcal{F}_x^{(1)}$  is a one-fermion state,

$$\Psi_1 = \Psi_x \otimes \Omega_y, \quad (4.35)$$

then we obtain the following expression in momentum space:

$$(W(s)\Psi_1)(\mathbf{p}, \mathbf{K}) = e^{-\frac{\|s\|^2}{2}} \frac{1}{\sqrt{N!}} \left( \prod_{\ell=1}^N s(\mathbf{k}_\ell) \right) \Psi_x \left( \mathbf{p} + \sum_{\ell=1}^N \mathbf{k}_\ell \right) =: e^{-\frac{\|s\|^2}{2}} \Psi_0(\mathbf{p}, \mathbf{K}). \quad (4.36)$$

For  $s \in \mathfrak{h}$ , the expression  $W(s)\Psi_1$  is a Fock space vector with norm  $\|W(s)\Psi_1\| = \|\Psi_1\|$ . However, if we set  $s \notin \mathfrak{h}$ , two problems arise:

- $\|s\|^2 = \langle s, s \rangle$  is formally a divergent integral. So the wave function renormalization factor  $e^{-\frac{\|s\|^2}{2}}$  is not well-defined.
- $\left( \prod_{\ell=1}^N s(\mathbf{k}_\ell) \right)$  (on every sector  $N \geq 1$ ) is a non-square integrable function, so even without the infinite renormalization factor  $e^{-\frac{\|s\|^2}{2}}$ , (4.36) does not describe an element of  $\mathcal{F}$ .

The second problem is tackled by defining a space  $\dot{\mathcal{S}}_{\mathcal{F}}^\infty$  containing non-square integrable functions, including the above product. As  $s \in \dot{\mathcal{S}}_1^\infty$ , the product above

### 4.3 Construction of the Extended State Space

is a smooth function apart from the zero boson momentum configuration set

$$\Xi(\mathbf{k} = 0) := \{q = (\mathbf{P}, \mathbf{K}) \in \mathcal{Q} \mid \exists \ell \in \{1, \dots, N\} : \mathbf{k}_\ell = 0\}. \quad (4.37)$$

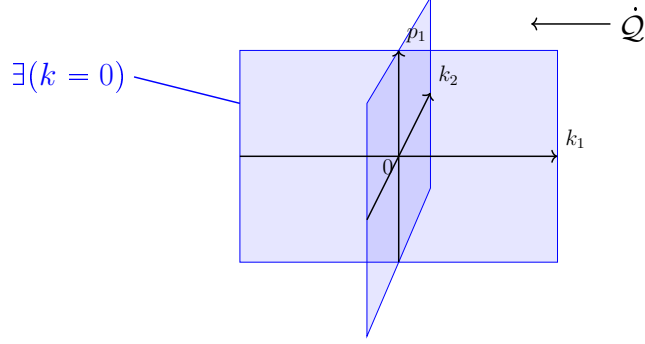


Figure 4.3: The set  $\Xi(\mathbf{k} = 0)$  on the  $(M, N) = (1, 2)$ -sector in configuration space  $\mathcal{Q}$  for  $d = 1$ . Color online.

The restriction of  $\Xi(\mathbf{k} = 0)$  to any sector is a union of hyperplanes (see Figure 4.3), which have Lebesgue measure 0 and hence, also  $\Xi(\mathbf{k} = 0) \subseteq \mathcal{Q}$  is null. Excluding this set from configuration space, we obtain

$$\dot{\mathcal{Q}} := \mathcal{Q} \setminus \Xi(\mathbf{k} = 0), \quad (4.38)$$

and on  $q \in \dot{\mathcal{Q}}$ , the function  $\Psi_0$  in (4.36) is smooth. For  $|\mathbf{k}| \rightarrow 0$  or  $|\mathbf{p}|, |\mathbf{k}| \rightarrow \infty$  we require the following scaling conditions to hold:

$$\lim_{|\mathbf{k}_\ell| \rightarrow 0} \frac{\Psi_0(\mathbf{P}, \mathbf{K})}{|\mathbf{k}_\ell|^\beta} = \lim_{|\mathbf{k}_\ell| \rightarrow \infty} \frac{\Psi_0(\mathbf{P}, \mathbf{K})}{|\mathbf{k}_\ell|^m} = \lim_{|\mathbf{p}_j| \rightarrow \infty} \frac{\Psi_0(\mathbf{P}, \mathbf{K})}{|\mathbf{p}_j|^m} = 0, \quad (4.39)$$

for all fixed  $q = (\mathbf{P}, \mathbf{K}) \in \dot{\mathcal{Q}}$  and some  $\beta, m \in \mathbb{R}$ . We hence interpret  $\Psi_0$  as an element of the following space:

$$\dot{\mathcal{S}}_{\mathcal{F}}^\infty := \left\{ \Psi \in C^\infty(\dot{\mathcal{Q}}) \mid (4.39) \text{ holds} \right\}. \quad (4.40)$$

Note that  $\dot{\mathcal{S}}_{\mathcal{F}}^\infty \cap \mathcal{F}$  contains  $C_c^\infty(\mathcal{Q})$ , which is a dense set in  $\mathcal{F}$ . So  $\dot{\mathcal{S}}_{\mathcal{F}}^\infty \cap \mathcal{F}$  is dense in  $\mathcal{F}$ . The Fourier transform of some  $\Psi_0 \in \dot{\mathcal{S}}_{\mathcal{F}}^\infty$  can be taken, if it is an element of the space

$$\dot{\mathcal{S}}_{\mathcal{F}, \text{loc}}^\infty := \dot{\mathcal{S}}_{\mathcal{F}}^\infty \cap L_{\text{loc}}^1(\mathcal{Q}). \quad (4.41)$$

In order to address the first problem, it is necessary to assign mathematical meaning to the expression  $\langle s, s \rangle$  (called **renormalization factor**). It may be convenient to take a Fourier transform of  $s \in \dot{\mathcal{S}}_1^\infty$ , for instance, in order to define

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a particle–position representation. This can be done whenever  $s \in L^1_{\text{loc}}(\mathbb{R}^d)$ , since then  $s \in \mathcal{S}'(\mathbb{R}^d)$ , i.e., it is a tempered distribution. We may hence distinguish two interesting function spaces for  $s$ :

- The generic case is given by  $s \in \dot{\mathcal{S}}_1^\infty$
- The special case is given by  $s \in \dot{\mathcal{S}}_1^\infty \cap L^1_{\text{loc}}(\mathbb{R}^d) =: \dot{\mathcal{S}}_{1,\text{loc}}^\infty$ , which allows for taking the Fourier transform.

In the following, we will present a construction based on the generic case, since the special one can be treated by the same means. Clearly,  $\mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty \subseteq \dot{\mathcal{S}}_{1,\text{loc}}^\infty$ , so even  $\dot{\mathcal{S}}_{1,\text{loc}}^\infty$  contains a dense subspace of  $\mathfrak{h}$ . If  $s \in \mathfrak{h}$ , then

$$\langle s, s \rangle = \int |s(\mathbf{k})|^2 d\mathbf{k} \in \mathbb{C}. \quad (4.42)$$

Otherwise, the renormalization factor  $\langle s, s \rangle$  is a symbolic expression, corresponding to a divergent integral. In contrast to Section 3.2.2, we do not directly interpret it as an element of  $\text{Ren}_1$ , but intermediately construct a space of function pairings:

**Definition 4.3.1.** Consider the free  $\mathbb{C}$ –vector space  $F(\dot{\mathcal{S}}_1^\infty \times \dot{\mathcal{S}}_1^\infty)$ . By a free vector space, we mean the set of all finite  $\mathbb{C}$ –linear combinations of pairings that are antilinear in the first component, denoted  $\mathfrak{r} = \sum_{m=1}^M c_m(s_m, t_m)$ , with  $s_m, t_m \in \dot{\mathcal{S}}_1^\infty$ ,  $c \in \mathbb{C}$  and the sum  $\sum_{m=1}^M$  being commutative. The **space of renormalization integrals** is defined as the quotient space

$$\text{Ren}_{01} := F(\dot{\mathcal{S}}_1^\infty \times \dot{\mathcal{S}}_1^\infty) / \sim_{\text{Ren}_{01}}, \quad (4.43)$$

where the equivalence relation  $\sim_{\text{Ren}_{01}}$  of formal equality is given by

$$\begin{aligned} \sum_{m=1}^M c_m(s_m, t_m) &\sim_{\text{Ren}_{01}} \sum_{m=1}^{\tilde{M}} \tilde{c}_m(\tilde{s}_m, \tilde{t}_m) \\ :\Leftrightarrow \sum_{m=1}^M c_m \overline{s_m(\mathbf{k})} t_m(\mathbf{k}) &= \sum_{m=1}^{\tilde{M}} \tilde{c}_m \overline{\tilde{s}_m(\mathbf{k})} \tilde{t}_m(\mathbf{k}) \quad \forall \mathbf{k} \in \mathbb{R}^d \setminus \{0\}. \end{aligned} \quad (4.44)$$

The deviation of constructing  $\text{Ren}_1$  via  $\text{Ren}_{01}$  generates a higher effort in our case, but it sets the foundation for a treatment of  $r, s$ , that are not necessarily functions (but, for instance, distributions), and may not be multiplied.

There is a natural one–to–one identification of renormalization integrals with functions  $\text{Ren}_{01} \cong \dot{\mathcal{S}}_1^\infty$ : It is easy to see that the following map is an embedding (by

### 4.3 Construction of the Extended State Space

definition of  $\sim_{\text{Ren}_{01}}$ ):

$$\begin{aligned} \iota_1 : \text{Ren}_{01} &\rightarrow \dot{\mathcal{S}}_1^\infty \\ \sum_{m=1}^M c_m(s_m, t_m) = \mathfrak{r} &\mapsto r(\mathbf{k}) = \sum_{m=1}^M \overline{c_m s_m(\mathbf{k})} t_m(\mathbf{k}). \end{aligned} \quad (4.45)$$

Conversely, for a given  $r \in \dot{\mathcal{S}}_1^\infty$ , the element  $(r, \chi_{\mathbb{R}^d \setminus \{0\}}) \in \text{Ren}_1$  (with  $\chi$  being the indicator function) is identified with  $r$ , so  $\iota_1$  is indeed an isomorphism.

In case  $\mathfrak{r} \in \text{Ren}_{01}$  with  $r = \iota_1(\mathfrak{r}) \in L^1(\mathbb{R}^d)$ , we can identify  $\mathfrak{r}$  with the  $\mathbb{C}$ -number

$$\mathfrak{r} = \int_{\mathbb{R}^d} \sum_{m=1}^M \overline{c_m s_m(\mathbf{k})} t_m(\mathbf{k}) d\mathbf{k} = \int_{\mathbb{R}^d} r(\mathbf{k}) d\mathbf{k} \in \mathbb{C}. \quad (4.46)$$

Now, several  $\mathfrak{r} \in \text{Ren}_{01}$  get identified with the same  $\mathbb{C}$ -number, e.g., all  $\mathfrak{r}$  corresponding to a function  $r$  with  $\int r(\mathbf{k}) d\mathbf{k} = 0$  are identified with 0. We remove this ambiguity by modding out another equivalence relation:

**Definition 4.3.2.** The renormalization factor space  $\text{Ren}_1$  is defined as

$$\text{Ren}_1 := \text{Ren}_{01} / \sim_{\text{Ren}_1}, \quad (4.47)$$

where for  $\mathfrak{r}_1, \mathfrak{r}_2 \in \text{Ren}_{01}$  we define

$$\mathfrak{r}_1 \sim_{\text{Ren}_1} \mathfrak{r}_2 \Leftrightarrow \iota_1(\mathfrak{r}_1) - \iota_1(\mathfrak{r}_2) \in L^1 \quad \text{and} \quad \int (\iota_1(\mathfrak{r}_1) - \iota_1(\mathfrak{r}_2))(\mathbf{k}) d\mathbf{k} = 0. \quad (4.48)$$

Elements of  $\text{Ren}_1$  will be denoted equally to a representative  $\mathfrak{r}$ . Note that we can identify  $\text{Ren}_1$  with the quotient space

$$\text{Ren}_1 \cong (\mathbb{C} \oplus \text{Ren}_{01}) / D \quad \text{with} \quad D := \{(-\int \iota_1(\mathfrak{r})(\mathbf{k}) d\mathbf{k}, \mathfrak{r}) \mid \iota_1(\mathfrak{r}) \in L^1(\mathbb{R}^d)\}, \quad (4.49)$$

where the isomorphism is given by identification of  $(c, \mathfrak{r})$  with that class  $[\mathfrak{r}'] \in \text{Ren}_1$  where  $\int (\iota_1(\mathfrak{r}')(\mathbf{k}) - \iota_1(\mathfrak{r})(\mathbf{k})) d\mathbf{k} = c$ .

We will encounter the special case where  $r(\mathbf{k})$  only takes values in  $[0, \infty)$ :

**Definition 4.3.3.** The **positive renormalization factor cone**  $\text{Ren}_{1+}$  is the set of all  $[\mathfrak{r}] \in \text{Ren}_1$ , where at least one representative  $\mathfrak{r} \in \text{Ren}_{01}$  is identified with a positive-valued function  $\iota_1(\mathfrak{r}) : \mathbb{R}^d \setminus \{0\} \rightarrow [0, \infty)$ .

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Scalar multiplication by  $c \in [0, \infty)$  is well-defined, making this indeed a cone.

It is also convenient to make sense of products and polynomials of factors  $\mathfrak{r} \in \text{Ren}_1$ . We hence define the following vector spaces, as in Section 3.2.2:

**Definition 4.3.4.** Consider the free  $\mathbb{C}$ -vector space of all finite linear combinations of products of up to  $P \in \mathbb{N}_0$  renormalization factors (i.e., formal polynomials of degree  $P$ )

$$\text{Pol}_P := \left\{ \mathfrak{R} = \sum_{m=1}^M c_m \mathfrak{r}_{m,1} \cdots \mathfrak{r}_{m,p_m} \mid 0 \leq p_m \leq P, c_m \in \mathbb{C}, \mathfrak{r}_{m,p} \in \text{Ren}_1 \right\}, \quad (4.50)$$

with the sum  $\sum_{m=1}^M$  and products being commutative. Then, the space of **renormalization factor polynomials** of order  $\leq P$  is the quotient space

$$\text{Ren}_P = \text{Pol}_P / \sim_{\text{Ren}_P}, \quad (4.51)$$

with the equivalence relation  $\sim_{\text{Ren}_P}$  of formal equality generated by

$$\begin{aligned} \mathfrak{r}_1 \mathfrak{r}_2 \cdots \mathfrak{r}_{p_1} + \mathfrak{R} &\sim_{\text{Ren}_P} c \mathfrak{r}_2 \cdots \mathfrak{r}_{p_1} + \mathfrak{R} && \text{if } \mathfrak{r}_1 = c \in \mathbb{C} \\ (c_1 c_2) \mathfrak{r}_1 \cdots \mathfrak{r}_{p_1} + \mathfrak{R} &\sim_{\text{Ren}_P} c_1 (c_2 \mathfrak{r}_1) \cdots \mathfrak{r}_{p_1} + \mathfrak{R}, \end{aligned} \quad (4.52)$$

with  $P_1 \leq P$ ,  $\mathfrak{R} \in \text{Pol}_P$ ,  $c_1, c_2 \in \mathbb{C}$  and  $\mathfrak{r}_m \in \text{Ren}_1$ .

As in Section 3.2.2, the bound  $P$  on the polynomial order is removed by taking the union over all orders.

**Definition 4.3.5.** The space of **renormalization factor polynomials** is given by

$$\text{Ren} = \bigcup_{P=1}^{\infty} \text{Ren}_P. \quad (4.53)$$

The spaces defined in this Section follow the hierarchy

$$\text{Ren}_{1+} \subseteq \text{Ren}_1 \subseteq \text{Ren}_2 \subseteq \cdots \subseteq \text{Ren}_P \subseteq \cdots \subseteq \text{Ren}. \quad (4.54)$$

### 4.3.2 Exponentials of Renormalization Factors and the Field $e\text{Ren}$

The state vector  $W(s)\Psi_1$  in (4.36) contains an exponential of a renormalization factor, i.e., an expression  $\mathfrak{c} = e^{\mathfrak{r}}$  with  $\mathfrak{r} \in \text{Ren}_1$ . More generally, we would like to give

### 4.3 Construction of the Extended State Space

a meaning to sums of exponentials with different (perhaps infinite) renormalization factors  $\mathfrak{r}_1, \mathfrak{r}_2 \in \text{Ren}_1$ . Formally, we would like to identify

$$e^{\mathfrak{r}_1} + e^{\mathfrak{r}_2} = e^{\mathfrak{r}_1}(1 + e^{\mathfrak{r}_2 - \mathfrak{r}_1}). \quad (4.55)$$

The bracket is a  $\mathbb{C}$ -number, whenever  $\mathfrak{r}_1 - \mathfrak{r}_2 \in \mathbb{C}$ , which defines an equivalence relation

$$\mathfrak{r}_1 \sim_1 \mathfrak{r}_2 \quad :\Leftrightarrow \quad \mathfrak{r}_1 - \mathfrak{r}_2 \in \mathbb{C}. \quad (4.56)$$

**Definition 4.3.6.** The space of **renormalization factor classes** is then given by the quotient space

$$\text{Clas}_1 = \text{Ren}_1 / \sim_1. \quad (4.57)$$

Whenever two elements  $\mathfrak{r}_1, \mathfrak{r}_2$  are of two different classes, we have  $\mathfrak{c} = e^{\mathfrak{r}_1 - \mathfrak{r}_2} \notin \mathbb{C}$  and may think of  $\mathfrak{c}$  as an “infinite constant”.

We may also interpret  $\text{Clas}_1$  as a subspace of  $\text{Ren}_1$ : The elements in the class of zero,  $b \in \text{Ren}_1 : b \sim_1 0$ , form a subspace  $V \subset \text{Ren}_1$ ,  $V \cong \mathbb{C}$ . Now we can find a basis  $B$  (containing one element) of  $V$  and, by the axiom of choice [211], extend it to a basis  $B \cup B'$  of  $\text{Ren}_1$ . By defining  $W := \text{span}(B')$ , we obtain a decomposition

$$\text{Ren}_1 = V \oplus W \quad \Rightarrow \quad W \cong \text{Ren}_1 / V. \quad (4.58)$$

Now, the following map is a bijection from  $W$  to  $\text{Clas}_1$ :

$$W \ni w \mapsto [w]_{\sim_{\text{Ren}_1}} \in \text{Clas}_1. \quad (4.59)$$

Note that this bijection is not unique, since it depends on the choice of  $B'$ , i.e., of representatives within each class.

We now consider elements of the group algebra  $\mathbb{C}[\text{Ren}_1] \ni c_1 e^{\mathfrak{r}_1} + \dots + c_M e^{\mathfrak{r}_M}$  with respect to the group with addition given by the vector space  $\text{Ren}_1$ . The formal exponentials make an addition in the group  $e^{\mathfrak{r}_1 + \mathfrak{r}_2} = e^{\mathfrak{r}_1} e^{\mathfrak{r}_2}$  appear as a multiplication. Here, we would like to consider two summands as equal, if “parts of the complex number can be pulled into the exponent”, i.e.,  $ce^{c' + \mathfrak{r}} = (ce^{c'})e^{\mathfrak{r}}$ . This is done by defining an ideal  $\mathcal{I} \subset \mathbb{C}[\text{Ren}_1]$  generated by all elements of the form

$$e^{c + \mathfrak{r}} - e^c e^{\mathfrak{r}} \quad \text{with} \quad c \in \mathbb{C}, \mathfrak{r} \in \text{Ren}_1. \quad (4.60)$$

Note that this ideal gives rise to an equivalence relation

$$u \sim_{\mathcal{I}} v \quad :\Leftrightarrow \quad (u - v) \in \mathcal{I}. \quad (4.61)$$

**Proposition 4.3.7.**

$$\mathbb{C}[\text{Ren}_1]/\mathcal{I} \cong \mathbb{C}[W]. \quad (4.62)$$

*Proof.* We define a map  $\pi : \text{Ren}_1 \rightarrow W$ , which assigns to  $\mathfrak{r} \in \text{Ren}_1$  the vector  $w$  within the decomposition  $\text{Ren}_1 = V \oplus W$  above. So  $\mathfrak{r} - \pi(\mathfrak{r}) \in \mathbb{C}$ . Now within,  $c_1 e^{\mathfrak{r}_1} + \dots + c_M e^{\mathfrak{r}_M} \in \mathbb{C}[\text{Ren}_1]$ , we can re-write each summand as

$$c_m e^{\mathfrak{r}_m} \sim_{\mathcal{I}} (c_m e^{\mathfrak{r}_m - \pi(\mathfrak{r}_m)}) e^{\pi(\mathfrak{r}_m)}. \quad (4.63)$$

This gives rise to a re-writing map

$$\begin{aligned} \Pi : \mathbb{C}[\text{Ren}_1] &\rightarrow \mathbb{C}[\text{Ren}_1] \\ c_1 e^{\mathfrak{r}_1} + \dots + c_M e^{\mathfrak{r}_M} &\mapsto (c_1 e^{\mathfrak{r}_1 - \pi(\mathfrak{r}_1)}) e^{\pi(\mathfrak{r}_1)} + \dots + (c_M e^{\mathfrak{r}_M - \pi(\mathfrak{r}_M)}) e^{\pi(\mathfrak{r}_M)}. \end{aligned} \quad (4.64)$$

Note that by the computation rules for group algebras, whenever  $e^{\pi(\mathfrak{r}_m)} = e^{\pi(\mathfrak{r}_m')}$  for two summands, they can be combined into one.

The map  $\Pi$  is an algebra homomorphism: it is linear and respects the multiplication:

$$\Pi(c_1 e^{\mathfrak{r}_1} + \dots + c_M e^{\mathfrak{r}_M}) \Pi(c'_1 e^{\mathfrak{r}'_1} + \dots + c'_M e^{\mathfrak{r}'_M}) = \Pi(c_1 e^{\mathfrak{r}_1} + \dots + c_M e^{\mathfrak{r}_M} + c'_1 e^{\mathfrak{r}'_1} + \dots + c'_M e^{\mathfrak{r}'_M}). \quad (4.65)$$

The latter is a consequence of  $\pi(\mathfrak{r}_1 + \mathfrak{r}_2) = \pi(\mathfrak{r}_1) + \pi(\mathfrak{r}_2)$  which does only hold true because we have chosen  $W$  as a vector space. Further,  $\Pi$  does not change the equivalence class with respect to  $\sim_{\mathcal{I}}$ . So  $\Pi(I) = 0$  implies  $I \in \mathcal{I}$ . Conversely, if  $I \in \mathcal{I}$  then  $I = I_1 + \dots + I_M$  with each  $I_i = A_i B_i C_i$  with  $A_i, C_i \in \mathbb{C}[\text{Ren}_1]$  and  $B_i = e^{c_i + \mathfrak{r}_i} - e^{c_i} e^{\mathfrak{r}_i}$ , so  $\Pi(B_i) = 0$  and  $\Pi(I_i) = 0$  and hence  $\Pi(I) = 0$ . So the kernel of  $\Pi$  is exactly  $\text{Ker}(\Pi) = \mathcal{I}$ .

Moreover, the image is  $\Pi[\mathbb{C}[\text{Ren}_1]] = \mathbb{C}[W]$ , since only elements of  $\mathbb{C}[W]$  appear in it and any element of  $\mathbb{C}[W] \subset \mathbb{C}[\text{Ren}_1]$  is mapped to itself.

By the isomorphism theorems, we now have that

$$\mathbb{C}[\text{Ren}_1]/\text{Ker}(\Pi) \cong \Pi[\mathbb{C}[\text{Ren}_1]] \quad :\Leftrightarrow \quad \mathbb{C}[\text{Ren}_1]/\mathcal{I} \cong \mathbb{C}[W], \quad (4.66)$$

as claimed.  $\square$

**Proposition 4.3.8.**  $\mathbb{C}[\text{Ren}_1]/\mathcal{I}$  has no proper zero divisors.

( $a, b \in \mathbb{C}[\text{Ren}_1]/\mathcal{I}$ ,  $a, b \neq 0$  are called proper zero divisors if and only if  $ab = 0$ .)

*Proof.* By Proposition 4.3.7, it suffices to show that  $\mathbb{C}[W]$  has no proper zero divisors.

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Now, the additive group  $W$  is an Abelian group that is torsion-free, i.e., for  $g \in W$  and  $n \in \mathbb{N}$ :

$$\underbrace{g + g + \dots + g}_{n \text{ times}} = 0 \quad \Rightarrow \quad g = 0. \quad (4.67)$$

Now, by Lemma 26.6 [212], the group  $G = W$  is ordered, so by Lemma 26.4 it is “t.u.p.,” so by Lemma 26.2 it is “u.p.” and  $K[G] = \mathbb{C}[W]$  has no proper zero divisors, i.e., it is an entire ring. □

Following [213, II. §3], it is a theorem that for every ring without proper zero divisors, the quotients form a field. So by Proposition 4.3.8 the following field extension of  $\mathbb{C}$  is well-defined:

**Definition 4.3.9** (and Corollary). The **field of (exponential) wave function renormalizations** is given by all fractions of linear combinations

$$\text{eRen} := \left\{ \mathbf{c} = \frac{a_1}{a_2} \mid a_1, a_2 \in \mathbb{C}[\text{Ren}_1]/\mathcal{I} \right\}. \quad (4.68)$$

By using representatives of  $\mathbb{C}[\text{Ren}_1]/\mathcal{I}$ , we can write any  $\mathbf{c} \in \text{eRen}$  as

$$\mathbf{c} = \frac{\sum_{m=1}^M c_m e^{\mathbf{r}_m}}{\sum_{m'=1}^{M'} c_{m'} e^{\mathbf{r}_{m'}}} \quad \text{with} \quad c_m, c_{m'} \in \mathbb{C}, \mathbf{r}_m, \mathbf{r}_{m'} \in \text{Ren}_1. \quad (4.69)$$

In particular, we can view a wave function renormalization  $\mathbf{c} \in \text{eRen}$  as an “extended complex number”.

#### 4.3.3 First Extended State Space

With the above definitions, we are able to give meaning to expressions like

$$\Psi = e^{-\mathbf{r}} \Psi_0 \quad \text{or even} \quad \Psi = \mathbf{c} \Psi_0, \quad (4.70)$$

with  $\mathbf{r} \in \text{Ren}_1$ ,  $\mathbf{c} \in \text{eRen}$  and  $\Psi_0 \in \dot{\mathcal{S}}_{\mathcal{F}}^\infty$ , as they appear in (4.36). We would like to take linear combinations of them and even handle expressions like  $\mathbf{c} \mathfrak{R} \Psi_0$  with  $\mathfrak{R} \in \text{Ren}$ . This is done by defining eRen-vector spaces including such expression, either without  $\mathfrak{R}$  (this will be the first ESS, called  $\overline{\mathcal{F}}$ ) or with  $\mathfrak{R}$  (this will be the second ESS, called  $\overline{\mathcal{F}}_{\text{ex}}$ ). The definitions are similar to those in Section 3.2.2.

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**Definition 4.3.10.** Consider the free eRen–vector space of all finite (commutative) sums of the form

$$\overline{\mathcal{F}}_0 := \left\{ \Psi = \sum_{m=1}^M \mathbf{c}_m \Psi_m \mid \Psi_m \in \dot{\mathcal{S}}_{\mathcal{F}}^{\infty}, \mathbf{c}_m \in \text{eRen} \right\}. \quad (4.71)$$

Then, the **first extended state space (ESS)** is the eRen–quotient space

$$\overline{\mathcal{F}} = \overline{\mathcal{F}}_0 / \sim_{\mathbb{F}}, \quad (4.72)$$

with equivalence relation  $\sim_{\mathbb{F}}$  generated by

$$(c\mathbf{c})\Psi_a + \Psi \sim_{\mathbb{F}} \mathbf{c}(c\Psi_a) + \Psi \quad \text{if } c \in \mathbb{C}, \quad (4.73)$$

for any  $\mathbf{c} \in \text{eRen}$ ,  $\Psi \in \overline{\mathcal{F}}_0$  and  $\Psi_a \in \dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$ .

The dense subspace  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty} \cap \mathcal{F}$  of the Fock space can be naturally embedded into  $\overline{\mathcal{F}}$ : Every  $\Psi_{\mathcal{F}} \in \dot{\mathcal{S}}_{\mathcal{F}}^{\infty} \cap \mathcal{F}$ , can be identified with

$$\Psi_{\mathcal{F}} = e^0 \Psi_{\mathcal{F}} \in \overline{\mathcal{F}}. \quad (4.74)$$

However, elements of  $\overline{\mathcal{F}}$  do not necessarily satisfy any symmetry conditions.

The coherent state in (4.36) can now be seen as an ESS element:

$$(W(s)\Psi_1)(\mathbf{p}, \mathbf{K}) = \underbrace{e^{-\frac{\|\mathbf{s}\|^2}{2}}}_{=e^{\mathbf{r}}} \underbrace{\frac{1}{\sqrt{N!}} \left( \prod_{\ell=1}^N s(\mathbf{k}_{\ell}) \right) \Psi_x \left( \mathbf{p} + \sum_{\ell} \mathbf{k}_{\ell} \right)}_{\in \dot{\mathcal{S}}_{\mathcal{F}}^{\infty}}.$$

For  $s \in \dot{\mathcal{S}}_{1,\text{loc}}^{\infty}$  and  $\Psi_x \in C_c^{\infty}$ , the second factor is even in  $\dot{\mathcal{S}}_{\mathcal{F},\text{loc}}^{\infty}$ .

#### 4.3.4 Second Extended State Space

Later in this work, we will encounter expressions like

$$\Psi(q) = e^{-\mathbf{r}\mathfrak{R}(q)} \Psi_0(q), \quad (4.75)$$

with  $q \in \dot{\mathcal{Q}}$ ,  $\mathfrak{R}(q) \in \text{Ren}$ ,  $\Psi_0(q) \in \mathbb{C}$ , or linear combinations of them. We interpret  $\Psi_{\mathfrak{R}} := \mathfrak{R}\Psi_0$  as a function in the function space

$$\Psi_{\mathfrak{R}} \in \text{Ren}^{\dot{\mathcal{Q}}} := \{ \Psi_{\mathfrak{R}} : \dot{\mathcal{Q}} \rightarrow \text{Ren} \}. \quad (4.76)$$

### 4.3 Construction of the Extended State Space

The second ESS then covers expressions (4.75) and linear combinations of them:

**Definition 4.3.11.** Consider the free eRen–vector space of all finite (commutative) sums of the form

$$\overline{\mathcal{F}}_{\text{ex},0} := \left\{ \Psi = \sum_{m=1}^M \mathbf{c}_m \Psi_m \mid \mathbf{c}_m \in \text{eRen}, \Psi_m \in \text{Ren}^{\dot{\mathcal{Q}}} \right\}. \quad (4.77)$$

Then, the **second extended state space (ESS)** is the eRen–quotient space

$$\overline{\mathcal{F}}_{\text{ex}} = \overline{\mathcal{F}}_{\text{ex},0} / \sim_{\text{Fex}}, \quad (4.78)$$

with equivalence relation  $\sim_{\text{Fex}}$  generated by

$$(c\mathbf{c})\Psi_a + \Psi \sim_{\text{Fex}} \mathbf{c}(c\Psi_a) + \Psi \quad \text{if } c \in \mathbb{C}, \quad (4.79)$$

where  $\mathbf{c} \in \text{eRen}$ ,  $\Psi \in \overline{\mathcal{F}}_{\text{ex},0}$  and  $\Psi_a \in \text{Ren}^{\dot{\mathcal{Q}}}$ .

The first ESS can be embedded into the second ESS  $\overline{\mathcal{F}} \hookrightarrow \overline{\mathcal{F}}_{\text{ex}}$  by interpreting all  $\Psi_m \in \dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$  as elements  $\Psi_m \in \text{Ren}^{\dot{\mathcal{Q}}}$ .

## 4.4 Operators on the Extended State Space

We first prove that creation and annihilation terms  $A^\dagger(v), A(v)$  as in (4.19) and (4.21) can be defined as operators using extended state spaces. Here, we even permit form factors  $v(\mathbf{p}, \mathbf{k})$ , that are allowed to depend on the fermion momentum  $\mathbf{p}$ . The momentum space definition reads

$$\begin{aligned} (A_j^\dagger(v)\Psi)(\mathbf{P}, \mathbf{K}) &= \frac{1}{\sqrt{N}} \sum_{\ell=1}^N v(\mathbf{p}_j, \mathbf{k}_\ell) \Psi(\mathbf{P} + e_j \mathbf{k}_\ell, \mathbf{K} \setminus \mathbf{k}_\ell), \\ (A_j(v)\Psi)(\mathbf{P}, \mathbf{K}) &= \sqrt{N+1} \int v(\mathbf{p}_j - \tilde{\mathbf{k}}, \tilde{\mathbf{k}})^* \Psi(\mathbf{P} - e_j \tilde{\mathbf{k}}, \mathbf{K}, \tilde{\mathbf{k}}) d\tilde{\mathbf{k}}, \\ A^\dagger(v) &= \sum_{j=1}^M A_j^\dagger(v), \quad A(v) = \sum_{j=1}^M A_j(v), \end{aligned} \quad (4.80)$$

which can be seen as a generalization of (4.19) and (4.21).

**Lemma 4.4.1** ( $A^\dagger, A$  are well-defined for  $v(\mathbf{p}, \mathbf{k})$ ). *Let  $v : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \rightarrow \mathbb{C}$  be smooth and satisfying the scaling condition (4.39). Then, (4.80) entails well-defined operators*

$$A_j^\dagger(v) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}, \quad A_j(v) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}, \quad (4.81)$$

which may be restricted<sup>1</sup> to

$$A_j^\dagger(v) : \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty \rightarrow \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty, \quad A_j(v) : \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty \rightarrow \text{Ren}^{\dot{\mathcal{Q}}}. \quad (4.82)$$

We may even extend  $A_j^\dagger(v) : \overline{\mathcal{F}}_{\text{ex}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$ .

*Proof.* Suppose,  $\Psi \in \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty$  and consider the expression  $A_j^\dagger(v)\Psi$  in (4.80). This is a finite sum over products consisting of two factors. By definition of  $\dot{\mathcal{S}}_1^\infty$ , the first factor  $v(\mathbf{p}_j, \mathbf{k}_\ell)$  in each product is smooth everywhere except where  $\mathbf{k}_\ell = 0$ . By definition of  $\dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty$ , the second factor  $\Psi(\mathbf{P} + e_j \mathbf{k}_\ell, \mathbf{K} \setminus \mathbf{k}_\ell)$  is smooth at all configurations, at which  $\mathbf{K} \setminus \mathbf{k}_\ell$  contains no coordinate  $\mathbf{k}_{\ell'} = 0$ . So the product is smooth on  $\dot{\mathcal{Q}}$ . In addition, the factors  $v$  and  $\Psi$  scale polynomially as in (4.39), so  $A^\dagger(v)\Psi \in \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty$ .

The expression for  $A(v)\Psi$  in (4.21) is a (possibly divergent) integral for each fixed  $(\mathbf{P}, \mathbf{K}) \in \dot{\mathcal{Q}}$ . Since both the functions  $\tilde{\mathbf{k}} \mapsto v(\mathbf{p}_j - \tilde{\mathbf{k}}, \tilde{\mathbf{k}})$  and  $\tilde{\mathbf{k}} \mapsto \Psi(\mathbf{P} - e_j \tilde{\mathbf{k}}, \mathbf{K}, \tilde{\mathbf{k}})$  are in  $\dot{\mathcal{S}}_1^\infty$ , the integral defines an element  $\mathfrak{R} \in \text{Ren}_1 \subset \text{Ren}$  for each fixed  $(\mathbf{P}, \mathbf{K})$ .

<sup>1</sup>We use the same notation for all extended or restricted versions of operators, here.

Thus,  $A(v)\Psi \in \text{Ren}^{\dot{\mathcal{Q}}}$ .

Both operators can be extended to  $\overline{\mathcal{F}}$  by taking eRen-linear combinations. For  $\Psi \in \text{Ren}^{\dot{\mathcal{Q}}}$ , the expression  $A_j^\dagger(v)\Psi$  in (4.80) is again a sum of products, that are all in  $\text{Ren}^{\dot{\mathcal{Q}}}$ . By eRen-linearity, we can then extend  $A_j^\dagger(v)$  to  $\overline{\mathcal{F}}_{\text{ex}}$ .  $\square$

We may also define the constituent operators of  $H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty$ , as well as  $H_{0,x}$  on  $\overline{\mathcal{F}}$ :

**Proposition 4.4.2** (Constituents of  $H$  are well-defined). *Consider the momentum space definitions of  $H_0$  (4.16),  $A^\dagger(v)$  (4.19),  $A(v)$  (4.21) and  $E_\infty$  (4.23), where we only assume  $\theta, \omega, v \in \dot{\mathcal{S}}_1^\infty$  and an arbitrary self-energy function  $E_1 : \mathbb{R}^d \rightarrow \text{Ren}$ . Then, the above four momentum space definitions entail well-defined operators*

$$\begin{aligned} H_0 : \overline{\mathcal{F}} &\rightarrow \overline{\mathcal{F}}, & A^\dagger(v) : \overline{\mathcal{F}} &\rightarrow \overline{\mathcal{F}}, \\ A(v) : \overline{\mathcal{F}} &\rightarrow \overline{\mathcal{F}}_{\text{ex}}, & E_\infty : \overline{\mathcal{F}} &\rightarrow \overline{\mathcal{F}}_{\text{ex}}. \end{aligned} \quad (4.83)$$

It is also possible to restrict

$$\begin{aligned} H_0 : \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty &\rightarrow \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty, & A^\dagger(v) : \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty &\rightarrow \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty, \\ A(v) : \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty &\rightarrow \text{Ren}^{\dot{\mathcal{Q}}}, & E_\infty : \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty &\rightarrow \text{Ren}^{\dot{\mathcal{Q}}} \end{aligned} \quad (4.84)$$

or to extend

$$H_0 : \overline{\mathcal{F}}_{\text{ex}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}, \quad A^\dagger(v) : \overline{\mathcal{F}}_{\text{ex}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}. \quad (4.85)$$

Further, the statements for  $H_0$  equally hold true for  $H_{0,x}$  and  $H_{0,y}$ .

*Proof.* Well-definedness and the mapping properties of  $A^\dagger(v)$  and  $A(v)$  are a consequence of Lemma 4.4.1: The function  $v(\mathbf{k})$  in (4.19) and (4.21) can be seen as a special case of  $v(\mathbf{p}, \mathbf{k})$  in (4.80). Taking the finite linear sum over  $j \in \{1, \dots, M\}$  sustains the mapping properties.

The operator  $H_0$  as in (4.16) just multiplies with a function in  $\dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty$  in momentum space, so for  $\Psi \in \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty$ , we also have  $H_0\Psi \in \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty$ . The same holds true for  $H_{0,x}$  and  $H_{0,y}$ . By an analogous argument,  $H_0$  can be extended to  $H_0 : \text{Ren}^{\dot{\mathcal{Q}}} \rightarrow \text{Ren}^{\dot{\mathcal{Q}}}$ .

Finally, we consider  $E_\infty\Psi$  for  $\Psi \in \dot{\mathcal{S}}_{\overline{\mathcal{F}}}^\infty$ . By (4.23), at each fixed  $(\mathbf{P}, \mathbf{K}) \in \dot{\mathcal{Q}}$  the expression  $E_\infty\Psi$  is defined as a finite sum over terms  $E_1(\mathbf{p})\Psi(\mathbf{P}, \mathbf{K}) \in \text{Ren}$ . So indeed,  $E_\infty\Psi \in \text{Ren}^{\dot{\mathcal{Q}}}$ .

#### 4 Extended State Space for Describing Renormalized Fock Spaces in QFT

Extensions to  $\overline{\mathcal{F}}$  or  $\overline{\mathcal{F}}_{\text{ex}}$  can again be done by eRen-linear combination.  $\square$

Thus, also the linear operator  $H : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$  is well-defined.

Finally, we prove that the momentum space definition (4.80) indeed entails certain canonical commutation relations on the extended state space:

**Lemma 4.4.3** (Extended CCR). *For  $\phi, \varphi \in \dot{\mathcal{S}}_1^\infty$ , the definitions (4.19) and (4.21) imply the commutation relations*

$$[A_j^\dagger(\varphi), A_{j'}^\dagger(\phi)] = 0, \quad [A_j(\varphi), A_{j'}^\dagger(\phi)] = \begin{cases} \langle \varphi, \phi \rangle & \text{for } j = j' \\ V_{jj'}(\varphi^* \phi) & \text{for } j \neq j' \end{cases}, \quad (4.86)$$

as a strong operator identity. That is, we have operators

$$[A_j^\dagger(\varphi), A_{j'}^\dagger(\phi)] : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}, \quad [A_j(\varphi), A_{j'}^\dagger(\phi)] : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}. \quad (4.87)$$

Here, the interaction potential  $V_{jj'} : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$  for momentum transfer from fermion  $j'$  to  $j$  is given by

$$V_{jj'}(\varphi^* \phi)(\mathbf{P}, \mathbf{K}) := \int \varphi^*(\tilde{\mathbf{k}}) \phi(\tilde{\mathbf{k}}) \Psi(\mathbf{P} + (e_{j'} - e_j) \tilde{\mathbf{k}}, \mathbf{K}) d\tilde{\mathbf{k}}. \quad (4.88)$$

*Proof.* By Lemma 4.4.1, the products  $A_j(\varphi)A_{j'}^\dagger(\phi)$  and  $A_{j'}^\dagger(\phi)A_j(\varphi)$  are well-defined as operators  $\overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$ . A momentum space evaluation renders

$$\begin{aligned} & ([A_j(\varphi), A_{j'}^\dagger(\phi)]\Psi)(\mathbf{P}, \mathbf{K}) \\ &= \int \varphi^*(\tilde{\mathbf{k}}) \phi(\tilde{\mathbf{k}}) \Psi(\mathbf{P} + (e_{j'} - e_j) \tilde{\mathbf{k}}, \mathbf{K}) d\tilde{\mathbf{k}} \\ &= \begin{cases} \langle \varphi, \phi \rangle \Psi(\mathbf{P}, \mathbf{K}) & \text{for } j = j' \\ \int \varphi^*(\tilde{\mathbf{k}}) \phi(\tilde{\mathbf{k}}) \Psi(\mathbf{P} + (e_{j'} - e_j) \tilde{\mathbf{k}}, \mathbf{K}) d\tilde{\mathbf{k}} & \text{for } j \neq j' \end{cases}. \end{aligned} \quad (4.89)$$

Similarly, Lemma 4.4.1 establishes that  $A_j^\dagger(\varphi)A_{j'}^\dagger(\phi)$  and  $A_{j'}^\dagger(\phi)A_j^\dagger(\varphi)$  are well-defined operators  $\overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$  and by a short momentum space calculation, one can verify that they are equal.  $\square$

The operator  $V_{jj'}$  defined above can be seen as an interaction potential operator. Under an inverse Fourier transform  $\mathcal{F}^{-1}$ , it amounts to a multiplication operator

in position space via

$$(V_{jj'}(\varphi^* \phi)\Psi)(\mathbf{X}, \mathbf{Y}) = \mathcal{F}^{-1}(\varphi^* s)(\mathbf{x}_j - \mathbf{x}_{j'})\Psi(\mathbf{X}, \mathbf{Y}), \quad (4.90)$$

provided that  $\mathcal{F}^{-1}(\varphi^* \phi)$  exists (e.g., for exact scaling degrees  $\beta_\varphi + \beta_s > -d \Rightarrow \varphi^* s \in \mathcal{S}'(\mathbb{R}^d)$ ).

The following property about  $V_{jj'}$  will become useful in later proofs:

**Lemma 4.4.4.** *If either of the functions  $\phi, \varphi \in \dot{\mathcal{S}}_1^\infty$  is an element of  $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ , then we even have*

$$V_{jj'}(\varphi^* \phi) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}. \quad (4.91)$$

*Proof.* First, let us consider  $\Psi \in \dot{\mathcal{S}}_{\mathcal{F}}^\infty$ . Without loss of generality, assume that  $\varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ , so  $\varphi$  is compactly supported, and the function

$$\tilde{\mathbf{k}} \mapsto \phi(\tilde{\mathbf{k}})\Psi(\mathbf{P} + (e_{j'} - e_j)\tilde{\mathbf{k}}, \mathbf{K}),$$

is smooth everywhere on that support. So the function

$$\tilde{\mathbf{k}} \mapsto \varphi^*(\tilde{\mathbf{k}})\phi(\tilde{\mathbf{k}})\Psi(\mathbf{P} + (e_{j'} - e_j)\tilde{\mathbf{k}}, \mathbf{K}) \quad \text{is in } C_c^\infty,$$

and the integral over it converges to a  $\mathbb{C}$ -number.

We now show that this number depends smoothly on  $(\mathbf{P}, \mathbf{K}) \in \dot{\mathcal{Q}}$ : Consider any multi-index  $\alpha$  corresponding to a derivative  $\partial^\alpha$  composed of arbitrarily many partial derivatives  $\partial_{\mathbf{p}_j}, \partial_{\mathbf{k}_\ell}$  with  $j, \ell \in \mathbb{N}$ . Then also  $\partial^\alpha \Psi \in \dot{\mathcal{S}}_{\mathcal{F}}^\infty$  and by the same arguments as above,

$$\tilde{\mathbf{k}} \mapsto \varphi^*(\tilde{\mathbf{k}})\phi(\tilde{\mathbf{k}})\partial^\alpha \Psi(\mathbf{P} + (e_{j'} - e_j)\tilde{\mathbf{k}}, \mathbf{K}) \quad \text{is in } C_c^\infty. \quad (4.92)$$

So the integral converges absolutely, derivative and integral commute, and we obtain  $\partial^\alpha (V_{jj'}(\varphi^* \phi)\Psi)(\mathbf{P}, \mathbf{K}) \in \mathbb{C}$  for any multi-index  $\alpha$ . Hence,  $(V_{jj'}(\varphi^* \phi)\Psi)(\mathbf{P}, \mathbf{K})$  is smooth at  $(\mathbf{P}, \mathbf{K}) \in \dot{\mathcal{Q}}$ .

Polynomial scaling of  $V_{jj'}(\varphi^* \phi)\Psi$  can be seen as follows: Under the coordinate rotation  $\mathbf{p}_+ := \mathbf{p}_j + \mathbf{p}_{j'}$  and  $\mathbf{p}_- := \mathbf{p}_j - \mathbf{p}_{j'}$ , the expression  $V_{jj'}\Psi$  becomes a convolution in  $\mathbf{p}_-$  of a polynomially scaling function with the function  $\varphi^* \phi \in C_c^\infty$ . Polynomial scaling bounds are neither affected by the coordinate rotation nor by the convolution with a  $C_c^\infty$ -function. So indeed,  $V_{jj'}(\varphi^* \phi)\Psi \in \dot{\mathcal{S}}_{\mathcal{F}}^\infty$ , and we have that  $V_{jj'}(\varphi^* \phi) : \dot{\mathcal{S}}_{\mathcal{F}}^\infty \rightarrow \dot{\mathcal{S}}_{\mathcal{F}}^\infty$ .

This mapping property extends to  $V_{jj'}(\varphi^* \phi) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$  by eRen-linearity. □

**Remarks.**

15. In (4.86), we have not included the commutation relation for annihilation operators  $[A_j(\varphi), A_{j'}(\phi)] = 0$ . The reason is that products of two or more annihilation operators are not necessarily defined, since we only have  $A_j(\varphi) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$ . An arbitrary product  $A_j(\varphi)A_{j'}(\phi)\Psi$  with  $\Psi \in \dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$  contains a double integral

$$\begin{aligned} & (A_j(\varphi)A_{j'}(\phi)\Psi)(\mathbf{P}, \mathbf{K}) \\ &= \sqrt{(N+1)(N+2)} \int \int \varphi(\tilde{\mathbf{k}})^* \phi(\tilde{\mathbf{k}}')^* \Psi(\mathbf{P} - e_j \tilde{\mathbf{k}} - e_{j'} \tilde{\mathbf{k}}', \mathbf{K}, \tilde{\mathbf{k}}, \tilde{\mathbf{k}}') d\tilde{\mathbf{k}} d\tilde{\mathbf{k}}', \end{aligned} \tag{4.93}$$

where the first integral produces a configuration space function  $\dot{\mathcal{Q}} \rightarrow \text{Ren}$ . And a second integral over such a function can generally not be interpreted as an element in  $\text{Ren}$ .

A definition of such operator products would require a modification of  $\overline{\mathcal{F}}_{\text{ex}}$  such that it also accommodates general divergent integrals over multiple coordinates of a  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$ -function as in (4.93). We postpone the investigation of such choices for  $\overline{\mathcal{F}}_{\text{ex}}$  to future investigations.

## 4.5 Dressing on the Extended State Space

Our next step is to define a dressing operator  $W(s)$  with  $s \in \dot{\mathcal{S}}_1^{\infty}$ . To do so, a naive approach would be to start from the expression  $W(s) = e^{A_M^\dagger(s) - A_M(s)} \dots e^{A_1^\dagger(s) - A_1(s)}$  (with  $M$  depending on the fermion sector) and expand the exponentials into series

$$e^{A_j^\dagger(s) - A_j(s)} = \sum_{n \in \mathbb{N}_0} \frac{(A_j^\dagger(s) - A_j(s))^n}{n!},$$

which can be multiplied out. There are two difficulties with this approach:

- Some terms in the resulting sum contain two or more annihilation operators  $A(s)$  (see Remark 15).
- There is an infinite number of such terms.

So with the current definition of  $\overline{\mathcal{F}}_{\text{ex}}$  we cannot simply define  $W(s)$  as an operator  $\overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$ . Instead, we pursue a different approach and define  $W(s) : \mathcal{D}_W \rightarrow \overline{\mathcal{F}}_{\text{ex}}$ . Here, we choose  $\mathcal{D}_W \subset \overline{\mathcal{F}}_{\text{ex}}$  in (4.131), such that  $\mathcal{D}_{\mathcal{F}}$  (which is a symmetrized version of  $\mathcal{D}_W \cap L^2$ ) is dense in  $\mathcal{F}$ .

If we consider  $W_{\mathcal{F}}(\varphi) : \mathcal{F} \rightarrow \mathcal{F}$ ,  $\varphi \in \mathfrak{h}$  as a unitary operator on Fock space, together with some suitable  $\Psi \in \mathcal{F}$ , then there is a well-defined expression (similar to (4.36)) for  $W_{\mathcal{F}}(\varphi)\Psi$  as an  $L^2$ -function on momentum-configuration space. For  $\varphi$  replaced by  $s \in \dot{\mathcal{S}}_1^\infty$ , we may then define  $W(s)\Psi \in \overline{\mathcal{F}}$  based on the momentum space expression of  $W_{\mathcal{F}}(\varphi)\Psi$ .

The domain  $\mathcal{D}_W$  in (4.131) is generated by vectors  $\Psi_C$  of the form

$$\Psi_C = W_1(\varphi)A_j^\dagger(v)\Psi_m \quad \text{or} \quad \Psi_C = XW_1(\varphi)\Psi_m,$$

where:

- $\Psi_m = \Psi_{mx} \otimes \Omega_y \in \mathcal{F}$  with  $\Psi_{mx} \in \mathcal{S}(\mathcal{Q}_x)$  (i.e., we have a Schwartz function) and  $\Omega_y$  describing the boson field in the vacuum.
- $W_1(\varphi) = e^{A_1^\dagger(\varphi) - A_1(\varphi)}$ ,  $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$  describes a dressing induced by only the first fermion.
- $A_j^\dagger(v)$ ,  $v \in \dot{\mathcal{S}}_1^\infty$  describes creation by only the fermion with number  $j \in \{1, \dots, M\}$ .
- $X$  is a linear combination of  $\text{Ren}_1$ -constants and operators  $V_{jj'}(v^*s)$  that formally commute with  $W(s)$ .

When setting  $X = 1$  (which formally commutes with  $W(s)$ ), we see that  $\mathcal{D}_W$  contains vectors of the kind  $W_1(\varphi)\Psi_m$ . We show in Lemma 4.5.6, that these are equal to  $W_{\mathcal{F},1}(\varphi)\Psi_m$  and, after symmetrization, span a dense subspace of  $\mathcal{F}$  (Lemma 4.5.2). This will later allow for a dense definition of  $\tilde{H}$ .

The definition of  $W(s)\Psi_C$  now exactly works as explained above: We establish a momentum space expression in case  $s, v \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$  using Lemma 4.5.3. Then, we generalize to  $s, v \in \dot{\mathcal{S}}_1^\infty$  by a suitable definition. As discussed in the introduction of his chapter, we remove certain exponential factors of the form  $e^{V_{jj'}}$  in an ad-hoc modification. Thus,  $W_j(\varphi_j)$  differs from  $W_{\mathcal{F},j}(\varphi_j)$  for  $\varphi_j \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ . However, we show in Lemma 4.5.8 that any  $V_{j'j''}$  commutes with  $W_{\mathcal{F},j}(\varphi_j)$ , so the omitted factor  $e^{V_{jj'}}$  can heuristically be “pulled into any position”. Heuristically, this factor disappears when performing an undressing, which justifies the omission within the computation of  $\tilde{H}$ .

### 4.5.1 Bosonic Dressing $W_y(\varphi)$

The upcoming proofs are based on some well-known facts about coherent states, where only bosons are present, i.e.,  $\Psi_y \in \mathcal{F}_y$ . The momentum space representation of the bosonic creation and annihilation operators  $a^\dagger(v), a(v)$  with form factor  $v \in \mathfrak{h}$  given by

$$\begin{aligned} (a^\dagger(v)\Psi_y)(\mathbf{K}) &:= \frac{1}{\sqrt{N}} \sum_{\ell=1}^N v(\mathbf{k}_\ell) \Psi_y(\mathbf{K} \setminus \mathbf{k}_\ell) \\ (a(v)\Psi_y)(\mathbf{K}) &:= \sqrt{N+1} \int v(\tilde{\mathbf{k}})^* \Psi_y(\mathbf{K}, \tilde{\mathbf{k}}) d\tilde{\mathbf{k}}. \end{aligned} \quad (4.94)$$

This definition implies that the commutation relations  $[a(v_1), a^\dagger(v_2)] = \langle v_1, v_2 \rangle$  hold as a strong operator identity on a dense domain in  $\mathcal{F}_y$ . These operators  $a^\dagger(v), a(v)$  substantially differ from  $A^\dagger(v), A(v)$  defined in (4.19), (4.21), which create or annihilate one boson at the position of each fermion, whereas in  $\mathcal{F}_y$ , there are no fermions.

Using  $a^\dagger(v), a(v)$ , we may define a set of displacement operators

$$W_y(\varphi) = e^{a^\dagger(\varphi) - a(\varphi)}, \quad (4.95)$$

and coherent states  $\Psi_y(\varphi) := W_y(\varphi)\Omega_y$ . Indeed,  $W_y(\varphi)$  is well-defined, since for  $\varphi \in \mathfrak{h}$ , we have the bounds

$$\|a^\dagger(\varphi)\Psi_y\| \leq \|(N+1)^{1/2}\Psi_y\| \|\varphi\|, \quad \|a(\varphi)\Psi_y\| \leq \|N^{1/2}\Psi_y\| \|\varphi\|, \quad (4.96)$$

so the exponential series (4.95) converges in norm:

$$\Psi_y(\varphi) = W_y(\varphi)\Omega_y := \sum_{k=0}^{\infty} \frac{1}{k!} (a^\dagger(\varphi) - a(\varphi))^k \Omega_y \quad (4.97)$$

$$\text{where } \left\| \frac{1}{k!} (a^\dagger(\varphi) - a(\varphi))^k \Omega_y \right\| \leq \frac{1}{k!} \|2^k (k!)^{1/2} \Omega_y\| \|\varphi\|^k = (k!)^{-1/2} \|2\varphi\|^k.$$

Here, we used in the second line that  $(a^\dagger(\varphi) - a(\varphi))^k \Omega_y$  occupies only sectors in Fock space with  $\leq k$  particles, so we can set  $N \leq (k-1)$  in (4.96). Subsequent application of (4.96) leads to the factor  $(k!)^{1/2}$ .

In momentum space representation and in terms of tensor products,

$$\Psi_y(\varphi)(\mathbf{K}) = e^{-\frac{\|\varphi\|^2}{2}} \frac{1}{\sqrt{N!}} \left( \prod_{\ell=1}^N \varphi(\mathbf{k}_\ell) \right) \Leftrightarrow \Psi_y(\varphi) = \sum_{N=0}^{\infty} \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \underbrace{\varphi \otimes_S \dots \otimes_S \varphi}_{N \text{ times}}. \quad (4.98)$$

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A calculation similar to (4.97) verifies that  $W_y(\varphi)$  can be defined on all  $\Psi_y$  with finite particle number, i.e.,  $\Psi_y \in \mathcal{F}_{\text{fin},y}$  with

$$\mathcal{F}_{\text{fin},y} := \{\Psi_y \in \mathcal{F}_y \mid \exists N_{\text{max}} \in \mathbb{N} : \Psi_y^{(N)} = 0 \forall N > N_{\text{max}}\}. \quad (4.99)$$

And by continuity, we can thus define  $W_y(\varphi)$  on all of  $\mathcal{F}_y$ .

Moreover, the  $W_y(\varphi)$  are unitary, so  $\|\Psi_y(\varphi)\| = 1$ , and they satisfy the Weyl relations. Further, it is a well-known fact that the span of the set of coherent states  $\{\Psi_y(\varphi) \mid \varphi \in \mathfrak{h}\}$  is dense in  $\mathcal{F}_y$  [116, Prop. 12]. In addition,

$$\langle \Psi_y(\varphi_1), \Psi_y(\varphi_2) \rangle_{\mathcal{F}_y} = e^{-\frac{\|\varphi_1\|^2 + \|\varphi_2\|^2}{2}} e^{\langle \varphi_1, \varphi_2 \rangle}, \quad (4.100)$$

so

$$\|\Psi_y(\varphi_1) - \Psi_y(\varphi_2)\|^2 = \|\Psi_y(\varphi_1)\|^2 + \|\Psi_y(\varphi_2)\|^2 - 2e^{-\frac{\|\varphi_1\|^2 + \|\varphi_2\|^2}{2}} \text{Re}(e^{\langle \varphi_1, \varphi_2 \rangle}). \quad (4.101)$$

As  $\mathfrak{h}$  is separable, we can find a countable dense set  $(\varphi_n)_{n \in \mathbb{N}}$  in  $\mathfrak{h}$ , such that  $(\Psi_y(\varphi_n))_{n \in \mathbb{N}}$  is also dense in the coherent states. So

$$\text{span}\{\Psi_y(\varphi_n) \mid n \in \mathbb{N}\} \quad \text{is dense in } \mathcal{F}_y. \quad (4.102)$$

Since  $\mathfrak{h} \cap \mathcal{S}_1^\infty \supset C_c^\infty$  is dense in  $\mathfrak{h}$  the above statements hold true, if we replace  $\varphi_n \in \mathfrak{h}$  by  $\varphi_n \in \mathfrak{h} \cap \mathcal{S}_1^\infty$ .

### 4.5.2 Dressing Induced by Fermions $W_1(\varphi)$

#### Density

Now, let us turn to the case with two particle species, i.e.,  $\mathcal{F} = \mathcal{F}_x \otimes \mathcal{F}_y$  and  $A^\dagger, A$  instead of  $a^\dagger, a$ . In order to make an analogous statement to (4.102) work, we restrict from  $A^\dagger(\varphi) = \sum_{j=1}^M A_j^\dagger(\varphi)$  to  $A_1^\dagger(\varphi)$ , i.e., creation by only the first fermion. Just as  $W_y(\varphi)$ , the Fock space operator  $W_{\mathcal{F},1}(\varphi) = e^{A_1^\dagger(\varphi) - A_1(\varphi)}$  can be defined in analogy to  $W_y(\varphi)$ . The operators  $A_1^\dagger(\varphi)$  and  $W_{\mathcal{F},1}(\varphi)$  break the fermionic symmetry, so they map  $\mathcal{F} \rightarrow L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$  (instead of  $\mathcal{F} \rightarrow \mathcal{F}$ ). We will therefore proceed by considering vectors  $\Psi \in L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ , i.e., with only bosonic exchange symmetry.

As a ‘‘cyclic set’’ of vectors  $\Psi_m$ , used for generating further domains, we choose

$$\mathcal{C}_{WS} := \mathcal{S}(\mathcal{Q}_x) \otimes \{\Omega_y\} \subset L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y. \quad (4.103)$$

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Since the boson field is in the vacuum,  $\mathcal{C}_{WS}$  is obviously not dense in  $L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ . However, it generates a dense subspace by applying operators  $W_{\mathcal{F},1}(\varphi)$  to it. The momentum representation after such an application is given by

$$(W_{\mathcal{F},1}(\varphi)\Psi_m)(\mathbf{P}, \mathbf{K}) = \frac{e^{-\frac{|\varphi|^2}{2}}}{\sqrt{N!}} \left( \prod_{\ell=1}^N \varphi(\mathbf{k}_\ell) \right) \Psi_{mx} \left( \mathbf{p}_1 + \sum_{\ell=1}^N \mathbf{k}_\ell, \mathbf{p}_2, \dots, \mathbf{p}_M \right). \quad (4.104)$$

**Definition 4.5.1.** The span of coherent states created by the first fermion is given by

$$\mathcal{D}_{WS} := \text{span}\{W_{\mathcal{F},1}(\varphi)\Psi_m \mid \varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty, \Psi_m \in \mathcal{C}_{WS}\}. \quad (4.105)$$

With (4.105), it is true that

**Lemma 4.5.2.**  $(S_- \otimes \mathbb{1})[\mathcal{D}_{WS}]$  is dense in  $\mathcal{F} = \mathcal{F}_x \otimes \mathcal{F}_y$

The proof is based on denseness of coherent states in  $\mathcal{F}_y$  and can be found in Section 4.9.

#### Dressed One–Boson States

Just as  $W_{\mathcal{F},1}(\varphi)$ , we may define  $W_{\mathcal{F},j}(\varphi)$ ,  $j \in \mathbb{N}$  and  $W_{\mathcal{F}}(\varphi) = W_{\mathcal{F},M}(\varphi) \dots W_{\mathcal{F},1}(\varphi)$  with  $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$  on each  $M$ -fermion sector. These operators are all unitary on  $L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$  and well-defined on  $\mathcal{F}$ .

We will now establish some useful commutation relations on the dense subspace

$$\mathcal{F}_{\text{fin}} := \mathcal{F}_x \otimes \mathcal{F}_{\text{fin},y} \subset \mathcal{F}, \quad (4.106)$$

with  $\mathcal{F}_{\text{fin},y}$  defined in (4.99).

**Lemma 4.5.3** (Commutation relations for  $W_{\mathcal{F}}$ ). For  $\varphi, \phi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ , we have the

following strong operator identities<sup>2</sup> on  $\mathcal{F}_{\text{fin}}$ :

$$\begin{aligned}
 W_{\mathcal{F},j}(\varphi)A_{j'}^\dagger(\phi) &= \begin{cases} (A_{j'}^\dagger(\phi) - \langle \varphi, \phi \rangle)W_{\mathcal{F},j}(\varphi) & \text{if } j = j' \\ (A_{j'}^\dagger(\phi) - V_{jj'}(\varphi^*\phi))W_{\mathcal{F},j}(\varphi) & \text{if } j \neq j' \end{cases} \\
 W_{\mathcal{F}}(\varphi)A_{j'}^\dagger(\phi) &= (A_{j'}^\dagger(\phi) - \langle \varphi, \phi \rangle - V_{\bullet j'}(\varphi^*\phi))W_{\mathcal{F}}(\varphi), \\
 W_{\mathcal{F},j}(\varphi)A^\dagger(\phi) &= (A^\dagger(\phi) - \langle \varphi, \phi \rangle - V_{j\bullet}(\varphi^*\phi))W_{\mathcal{F},j}(\varphi), \\
 W_{\mathcal{F}}(\varphi)A^\dagger(\phi) &= (A^\dagger(\phi) - M\langle \varphi, \phi \rangle - V(\varphi^*\phi))W_{\mathcal{F}}(\varphi),
 \end{aligned} \tag{4.107}$$

as well as

$$\begin{aligned}
 W_{\mathcal{F},j'}(\varphi)A_j(\phi) &= \begin{cases} (A_j(\phi) - \langle \phi, \varphi \rangle)W_{\mathcal{F},j'}(\varphi) & \text{if } j = j' \\ (A_j(\phi) - V_{jj'}(\phi^*\varphi))W_{\mathcal{F},j'}(\varphi) & \text{if } j \neq j' \end{cases} \\
 W_{\mathcal{F}}(\varphi)A_j(\phi) &= (A_j(\phi) - \langle \phi, \varphi \rangle - V_{j\bullet}(\phi^*\varphi))W_{\mathcal{F}}(\varphi), \\
 W_{\mathcal{F},j'}(\varphi)A(\phi) &= (A(\phi) - \langle \phi, \varphi \rangle - V_{\bullet j'}(\phi^*\varphi))W_{\mathcal{F},j'}(\varphi), \\
 W_{\mathcal{F}}(\varphi)A(\phi) &= (A(\phi) - M\langle \phi, \varphi \rangle - V(\phi^*\varphi))W_{\mathcal{F}}(\varphi),
 \end{aligned} \tag{4.108}$$

with  $V_{jj'}$  defined in (4.88) and<sup>3</sup>

$$\begin{aligned}
 V_{\bullet j'}(\varphi^*\phi) &:= \sum_{j:j \neq j'} V_{jj'}(\varphi^*\phi), & V_{j\bullet}(\varphi^*\phi) &:= \sum_{j':j \neq j'} V_{jj'}(\varphi^*\phi), \\
 V(\varphi^*\phi) &:= \sum_{j \neq j'} V_{jj'}(\varphi^*\phi).
 \end{aligned} \tag{4.109}$$

The proof of Lemma 4.5.3 is straightforward by applying the CCR. We present it in Section 4.10.

### 4.5.3 Extended Dressing $W(s)$

We would now like the relations in (4.107) to also hold true if we replace  $\varphi, \phi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$  by  $s, v \in \dot{\mathcal{S}}_1^\infty$ . In that case, the Fock space operators  $W_{\mathcal{F},j}$  turn into extended operators  $W_j$ . More precisely, it would be desirable to have

$$W_j(s)A_{j'}^\dagger(v)\Psi_m = \begin{cases} (A_{j'}^\dagger(v) - \langle s, v \rangle)W_j(s)\Psi_m & \text{if } j = j' \\ (A_{j'}^\dagger(v) - V_{jj'}(s^*v))W_j(s)\Psi_m & \text{if } j \neq j' \end{cases} \tag{4.110}$$

<sup>2</sup>By a strong operator identity  $A = B$  for  $A, B : \mathcal{F} \rightarrow L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ , we mean that  $A\Psi = B\Psi \forall \Psi \in \mathcal{F}$ , even if possibly  $A\Psi, B\Psi \notin \mathcal{F}$ .

<sup>3</sup>Here,  $\sum_{j:j \neq j'}$  is to be understood as a sum over only  $j$ , while  $\sum_{j \neq j'}$  is a sum over both  $j$  and  $j'$ . The second kind of sum will appear more often, so we give it a shorter notation.

#### 4 Extended State Space for Describing Renormalized Fock Spaces in QFT

for  $\Psi_m \in \mathcal{C}_{WS}$ . By Lemma 4.4.3, and since  $\langle s, v \rangle \in \text{Ren}_1$ , we may obviously interpret

$$V_{jj'}(s^*v) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}, \quad \langle s, v \rangle : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}. \quad (4.111)$$

So if we can define  $W_j(s)\Psi_m \in \overline{\mathcal{F}}$ , then the right-hand side of (4.110) serves as a definition for  $W_j(s)A_{j'}^\dagger(v)\Psi_m \in \overline{\mathcal{F}}_{\text{ex}}$ . And if we can further define products like  $W(s)W_1(\varphi)\Psi_m = W_M(s) \dots W_1(s)W_1(\varphi)\Psi_m \in \overline{\mathcal{F}}$ , then a generalization of (4.107) may even be used to define  $W(s)W_1(\varphi)A_{j'}^\dagger(v)\Psi_m \in \overline{\mathcal{F}}_{\text{ex}}$ .

However, before doing so, it is first necessary to specify what  $W_1(s)W_1(\varphi)$  is, which we will do by introducing some “extended Weyl relations”.

### Extended Weyl Relations

In order to treat products of factors  $W_j(s)$ ,  $s \in \dot{\mathcal{S}}_1^\infty$ , we introduce an extended Weyl algebra  $\overline{\mathcal{W}}$  that is generated by all  $W_j(s)$  and taken over the field  $\text{eRen}$  (as in Definition 4.3.9). Recall that  $\mathfrak{c} \in \text{eRen}$  is a fraction of linear combinations of exponentials  $e^\mathfrak{r}$ , with  $\mathfrak{r} \in \text{Ren}_1$  being a possibly divergent integral (see Definition 4.3.2). Multiplication on  $\overline{\mathcal{W}}$  is defined by the Weyl relations

$$\begin{aligned} W_j(s)^{-1} &= W_j(-s) \\ W_j(s_1)W_j(s_2) &= e^{-\frac{i}{2}\sigma(s_1, s_2)}W_j(s_1 + s_2), \end{aligned} \quad (4.112)$$

with symplectic form

$$\begin{aligned} \sigma &= \dot{\mathcal{S}}_1^\infty \times \dot{\mathcal{S}}_1^\infty \rightarrow \text{Ren}_1 \\ (s_1, s_2) &\mapsto \langle s_1, s_2 \rangle - \langle s_2, s_1 \rangle. \end{aligned} \quad (4.113)$$

Note that  $e^{-\frac{i}{2}\sigma(s_1, s_2)} = e^{\text{Im}\langle s_1, s_2 \rangle} \in \text{eRen}$  is not necessarily a complex number.

This  $\overline{\mathcal{W}}$  can be seen as an “almost-extension” of  $\mathcal{W}$  (strictly speaking, it only extends some Weyl algebra  $\mathcal{W}_0$  generated by  $\{W_j(s) \mid s \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty\}$ ).

The definition of the extended Weyl algebra now allows us to write

$$\begin{aligned} W(s)W_1(\varphi) &= W_M(s) \dots W_2(s)W_1(s)W_1(\varphi) \\ &= e^{\text{Im}\langle s, \varphi \rangle}W_M(s) \dots W_2(s)W_1(s + \varphi). \end{aligned} \quad (4.114)$$

So  $W(s)W_1(\varphi)$  can be brought into the form  $W_M(s_M) \dots W_1(s_1)$  times an  $\text{eRen}$ -factor, which is the same on each  $M$ -fermion sector.

**Extended Dressing on Coherent States**

In order to define vectors of the kind  $W_M(s_M) \dots W_1(s_1)\Psi_m \in \overline{\mathcal{F}}$ , we make use of the momentum space definition of  $W_{\mathcal{F},M}(\varphi_M) \dots W_{\mathcal{F},1}(\varphi_1)\Psi_m$  for  $\varphi_\ell \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ . For two dressing operators with  $j \neq j'$ , the Baker–Campbell–Hausdorff formula implies

$$\begin{aligned}
 W_{\mathcal{F},j}(\varphi_j)W_{\mathcal{F},j'}(\varphi_{j'})\Psi_m &= e^{A_j^\dagger(\varphi_j)-A_j(\varphi_j)}e^{A_{j'}^\dagger(\varphi_{j'})-A_{j'}(\varphi_{j'})}\Psi_m \\
 &= e^{-\frac{\|\varphi_j\|^2}{2}-\frac{\|\varphi_{j'}\|^2}{2}}e^{A_j^\dagger(\varphi_j)}e^{-A_j(\varphi_j)}e^{A_{j'}^\dagger(\varphi_{j'})}\underbrace{e^{-A_{j'}(\varphi_{j'})}\Psi_m}_{=\Psi_m} \\
 &= e^{-\frac{\|\varphi_j\|^2}{2}-\frac{\|\varphi_{j'}\|^2}{2}-[A_j(\varphi_j),A_{j'}^\dagger(\varphi_{j'})]}e^{A_j^\dagger(\varphi_j)}e^{A_{j'}^\dagger(\varphi_{j'})}e^{-A_j(\varphi_j)}\Psi_m \\
 &= e^{-\frac{\|\varphi_j\|^2}{2}-\frac{\|\varphi_{j'}\|^2}{2}-V_{jj'}(\varphi_j^*\varphi_{j'})}e^{A_j^\dagger(\varphi_j)+A_{j'}^\dagger(\varphi_{j'})}\Psi_m.
 \end{aligned} \tag{4.115}$$

Here, we used that  $V_{jj'}$  commutes with all  $A_{j''}^\dagger$  and  $A_{j''}$ , which follows by the same arguments as in the proof of Lemma 4.5.8 below. Thus, all double commutators between  $A$ - and  $A^\dagger$ -operators vanish. The generalization to arbitrarily many factors,  $W_{\mathcal{F},M}(\varphi_M) \dots W_{\mathcal{F},1}(\varphi_1)\Psi_m$  is straightforward, and we obtain an exponential of constants and  $V_{jj'}$ -terms, followed by  $e^{A_M^\dagger(\varphi_M)+\dots+A_1^\dagger(\varphi_1)}\Psi_m$ .

We now define  $W_M(\varphi_M) \dots W_1(\varphi_1)$  by dropping the  $V_{jj'}$ -terms in (4.115), which yields the following momentum space expression for  $\Psi_m = \Psi_{mx} \otimes \Omega_y \in \mathcal{C}_{WS}$

$$(W_M(\varphi_M) \dots W_1(\varphi_1)\Psi_m)(\mathbf{P}, \mathbf{K}) := \frac{1}{\sqrt{N!}}e^{-\sum_{j=1}^M \frac{\|\varphi_j\|^2}{2}} \sum_{\sigma} \left( \prod_{\ell=1}^N \varphi_{\sigma(\ell)}(\mathbf{k}_\ell) \right) \Psi_{mx}(\mathbf{P}'), \tag{4.116}$$

where the sum over  $\sigma$  over all  $M^N$  maps

$$\sigma : \{1, \dots, N\} \rightarrow \{1, \dots, M\}, \tag{4.117}$$

assigning each boson  $\ell$  to a fermion  $j = \sigma(\ell)$ . The shifted momentum, illustrated in Figure 4.4, is then

$$\mathbf{P}' := \mathbf{P} + \sum_{\ell} e_{\sigma(\ell)} \mathbf{k}_\ell. \tag{4.118}$$

**Lemma 4.5.4** (Products of  $W$  are well-defined). *Consider a sequence  $(s_j)_{j \in \mathbb{N}} \subset \dot{\mathcal{S}}_1^\infty$  and  $\Psi_m \in \mathcal{C}_{WS}$ . Then, the momentum space definition (4.116) renders a well-defined vector*

$$W_M(s_M) \dots W_1(s_1)\Psi_m \in \overline{\mathcal{F}}, \tag{4.119}$$

where (4.119) is to be interpreted as a sector-wise definition in  $M \in \mathbb{N}$ .

#### 4 Extended State Space for Describing Renormalized Fock Spaces in QFT

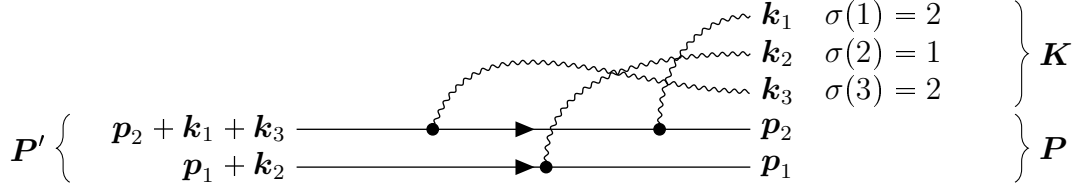


Figure 4.4: An example for momentum shift within the dressing.

*Proof.* Copying the momentum space definition (4.116), we obtain

$$(W_M(s_M) \dots W_1(s_1) \Psi_m)(\mathbf{P}, \mathbf{K}) := \frac{1}{\sqrt{N!}} e^{-\sum_{j=1}^M \frac{\|s_j\|^2}{2}} \sum_{\sigma} \left( \prod_{\ell=1}^N s_{\sigma(\ell)}(\mathbf{k}_{\ell}) \right) \Psi_{mx}(\mathbf{P}'). \quad (4.120)$$

Obviously,  $\left( \prod_{\ell=1}^N s_{\sigma(\ell)}(\mathbf{k}_{\ell}) \right) \Psi_{mx}(\mathbf{P}')$  defines a function in  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$ , which is still true after taking the finite sum over  $\sigma$ .

Further, we have  $\|s_j\|^2 = \langle s_j, s_j \rangle \in \text{Ren}_1$ , so  $e^{-\sum_{j=1}^M \frac{\|s_j\|^2}{2}} \in \text{eRen}$ .

Therefore, the expression (4.116) defines an element of  $\overline{\mathcal{F}}$ . □

This already allows us to define  $W_M(s_M) \dots W_1(s_1)$  on vectors  $\Psi_m \in \mathcal{C}_{WS}$  with the boson field in the vacuum. In order to define  $W_M(s_M) \dots W_1(s_1)$  also on a dense domain in  $\mathcal{F}$ , we extend the definition to vectors  $W_{\mathcal{F},1}(\varphi) \Psi_m \in \mathcal{D}_{WS}$ , whose symmetrized span, by Lemma 4.5.2, is dense in  $\mathcal{F}$ . This extension is done by assuming that  $W_{\mathcal{F},1}(\varphi)$  can be merged into  $W_1(s_1)$ , just as  $W_1(\varphi)$  in (4.114).

We will also allow for a treatment of state vectors by using the operator  $(S_- \otimes \mathbb{1})$ , which can obviously be extended to  $(S_- \otimes \mathbb{1}) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$  or  $(S_- \otimes \mathbb{1}) : \overline{\mathcal{F}}_{\text{ex}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$ , using the momentum space definition (4.11).

**Definition 4.5.5.** Let  $(s_j)_{j \in \mathbb{N}} \subset \dot{\mathcal{S}}_1^{\infty}$ . Then, by Lemma 4.5.4, copying the momentum space definition (4.116) results in a well-defined **product of dressing operators**

$$\begin{aligned} W_M(s_M) \dots W_1(s_1) &: \mathcal{D}_{WS} \rightarrow \overline{\mathcal{F}}, \\ W_M(s_M) \dots W_1(s_1) W_{\mathcal{F},1}(\varphi) \Psi_m &:= e^{\text{Im}\langle s_1, \varphi \rangle} W_M(s_M) \dots W_1(s_1 + \varphi) \Psi_m, \end{aligned} \quad (4.121)$$

where  $M$  is the respective fermion number on each sector. Further, we define the extension to **symmetrized vectors**

$$W_M(s_M) \dots W_1(s_1) : (S_- \otimes \mathbb{1})[\mathcal{D}_{WS}] \cup \mathcal{D}_{WS} \rightarrow \overline{\mathcal{F}}, \quad (4.122)$$

by imposing that  $W_M(s_M) \dots W_1(s_1)$  shall commute with the symmetrization operator  $(S_- \otimes \mathbb{1})$ .

With this definition, it is true that

**Lemma 4.5.6.** *For all  $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$  and  $\Psi_m \in \mathcal{C}_{WS}$ , it holds that*

$$W_{\mathcal{F},1}(\varphi)\Psi_m = W_1(\varphi)\Psi_m, \quad (4.123)$$

*in terms of momentum space functions.*

*Proof.* Consider (4.115) with  $j = 1$  and  $\varphi_{j'} = 0$ . Then, the  $V_{jj'}$ -term vanishes, so no  $V_{jj'}$ -terms are dropped when copying momentum space expressions in the transition  $W_{\mathcal{F},1} \rightarrow W_1$  and indeed  $W_{\mathcal{F},1}(\varphi)\Psi_m = W_1(\varphi)\Psi_m$ . □

### Remarks.

16. It may seem natural to extend Definition 4.5.5 to a general  $\Psi \in \mathcal{F}$ . By Lemma 4.5.2, we can write  $\Psi$  as a symmetrized version of  $\Psi' = \sum_{nn'} W_1(\varphi_n)\Psi_{n'}$  with  $\varphi_n \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$  and  $\Psi_{n'} \in \mathcal{C}_{WS}$ . In that case,  $W(s)\Psi = \sum_{nn'} W(s)W_1(\varphi_n)\Psi_{n'}$  contains a possibly infinite sum over functions  $\dot{\mathcal{Q}} \rightarrow \mathbb{C}$ , which may not converge.

However, our aim is to give a dense definition of  $\tilde{H} : \mathcal{F} \supset \text{dom}(\tilde{H}) \rightarrow \mathcal{F}$ , so it suffices to consider the action of  $W(s)$ , and  $HW(s)$  on a dense subset of  $\mathcal{F}$ , such as  $(S_- \otimes \mathbb{1})[\mathcal{D}_{WS}]$ .

17. Concerning the renormalization classes: Two ESS vectors  $W(s)W_1(\varphi)\Psi_m$  and  $W(s)W_1(\tilde{\varphi})\Psi_{m'}$  with  $\Psi_m$  and  $\Psi_{m'}$  concentrated on the same  $M$ -fermion sector can be added if the wave function renormalizations  $\mathbf{c} = e^{\mathbf{r}}$ ,  $\mathbf{r} = -\frac{\|s\|^2}{2}$

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belong to the same renormalization factor class, i.e.,

$$\begin{aligned}
\mathfrak{r} - \tilde{\mathfrak{r}} \in \mathbb{C} &\Leftrightarrow \left| \|s + \varphi\|^2 - \|s + \tilde{\varphi}\|^2 + 2\text{Im}\langle s, \varphi - \tilde{\varphi} \rangle \right| < \infty \\
&\Leftrightarrow \left| 2\text{Re}\langle s, \varphi - \tilde{\varphi} \rangle + 2\text{Im}\langle s, \varphi - \tilde{\varphi} \rangle + \underbrace{\|\varphi\|^2}_{< \infty} - \underbrace{\|\tilde{\varphi}\|^2}_{< \infty} \right| < \infty \\
&\Leftrightarrow |\langle s, \varphi - \tilde{\varphi} \rangle| < \infty.
\end{aligned} \tag{4.124}$$

That means, convergence of the integral  $\int s(\mathbf{k})^*(\varphi(\mathbf{k}) - \tilde{\varphi}(\mathbf{k})) d\mathbf{k}$  ensures that the renormalization classes coincide. Note that both Re and Im above may be infinite, but cancel each other out.

#### Extended Dressing on One-Boson States

Now, as announced, when replacing  $\varphi, \phi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$  by  $v, s \in \dot{\mathcal{S}}_1^\infty$  in (4.107), we obtain a well-defined right-hand side. This allows for the following extension of dressing operator products

**Definition 4.5.7.** Let  $v \in \dot{\mathcal{S}}_1^\infty, (s_j)_{j \in \mathbb{N}} \subset \dot{\mathcal{S}}_1^\infty$  and  $\Psi_m \in \mathcal{C}_{WS}$ . We extend the **product of dressing operators** to one-boson states via

$$W_M(s_M) \dots W_1(s_1) A_{j'}^\dagger(v) \Psi_m \in \overline{\mathcal{F}}_{\text{ex}}, \tag{4.125}$$

where  $M$  is the respective fermion number on each sector, via

$$\begin{aligned}
W_M(s_M) \dots W_1(s_1) A_{j'}^\dagger(v) \Psi_m &:= \left( A_{j'}^\dagger(v) - \sum_{j=1}^M X_j \right) W_M(s_M) \dots W_1(s_1) \Psi_m, \\
\text{with } X_j &= \begin{cases} \langle s, v \rangle & \text{if } j = j' \\ V_{jj'}(s^*v) & \text{if } j \neq j' \end{cases}.
\end{aligned} \tag{4.126}$$

This operator can further be extended to **symmetrized vectors** by imposing that  $W_M(s_M) \dots W_1(s_1)$  shall commute with the symmetrization operator  $(S_- \otimes \mathbb{1})$ .

It is easy to see that the right-hand side of (4.126) makes sense: By Lemma 4.5.4, we have  $W_M(s_M) \dots W_1(s_1) \Psi_m \in \overline{\mathcal{F}}$ . Lemma 4.4.1 implies that  $A_{j'}^\dagger(v) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$  and by Lemma 4.4.3 and  $\langle s, v \rangle \in \text{Ren}_1$ , we have that  $X_j : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$ .

Heuristically, the factors  $X_j$  now commute with  $W_{j'}(s)$ , since we have the following commutation relations.

**Lemma 4.5.8.** For  $\varphi, \varphi', \phi \in \mathfrak{h}$  it is true that

$$[W_{\mathcal{F},j}(\varphi'), V_{j'j''}(\varphi^* \phi)] = 0, \quad \text{and} \quad [W_{\mathcal{F},j}(\varphi'), \langle \varphi, \phi \rangle] = 0, \quad (4.127)$$

as a strong operator identity on  $\mathcal{F}$ .

*Proof.* Since  $\varphi, \phi \in L^2$ , we have  $\varphi^* \phi \in L^1$ , so after a Fourier transform, the operator  $V_{j'j''}(\varphi^* \phi)$  amounts to a multiplication by an  $L^\infty$ -function, and is hence bounded. Further,  $W_{\mathcal{F},j}(\varphi')$  is unitary on  $L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$  (and hence bounded). So the commutator is defined on all of  $L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$  and hence  $\mathcal{F}$ .

Now, in position space, both  $A_j^\dagger(\varphi')$  and  $A_j(\varphi')$  can be decomposed into a fiber integral by fiber-decomposing  $L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y = \int_{\mathcal{Q}_x} \mathcal{F}_y d\mathbf{X}$  (see (4.20) and (4.22)).

So we can also decompose  $W_{\mathcal{F},j}(\varphi') = e^{A_j^\dagger(\varphi') - A_j(\varphi')}$  into a fiber integral. And by (4.90), the operator  $V_{j'j''}(\varphi^* \phi)$  just amounts to a multiplication by a complex constant on each fiber Hilbert space. So the fiber operators commute on all fibers and hence the original operators commute on  $L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$  and  $\mathcal{F}$ .

The expression  $\langle \varphi, \phi \rangle$  is just a constant, so it trivially commutes with  $W_{\mathcal{F},j}(\varphi')$ .  $\square$

Mathematically, if we replace  $\varphi, \varphi', \phi \in \mathfrak{h}$  by  $s, s', v \in \dot{\mathcal{S}}_1^\infty$ , then the commutation relations (4.127) are not a priori valid, since  $W_j(s)$  is not necessarily defined on vectors of the kind  $V_{j'j''}(\varphi^* \phi)\Psi_m$  or  $\langle \varphi, \phi \rangle \Psi_m$ . We enforce their validity by taking (4.127) as a definition for an extension of  $W_j(s)$ :

**Definition 4.5.9.** Let  $(s_j)_{j \in \mathbb{N}} \subset \dot{\mathcal{S}}_1^\infty$ ,  $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$  and  $\Psi_m \in \mathcal{C}_{WS}$ , and let  $X$  be an element of the set of operators

$$\mathcal{X} := \text{span}_{\text{eRen}} \{ \langle s, v \rangle, V_{j'j''}(s^* v) \mid s, v \in \dot{\mathcal{S}}_1^\infty \}, \quad (4.128)$$

so  $X$  formally commutes with all  $W_j(s_j)$ . Then we extend the **product of dressing operators** via

$$\begin{aligned} W_M(s_M) \dots W_1(s_1) X \Psi_m &:= X W_M(s_M) \dots W_1(s_1) \Psi_m, \\ W_M(s_M) \dots W_1(s_1) X W_{\mathcal{F},1}(\varphi) \Psi_m &:= X W_M(s_M) \dots W_1(s_1) W_{\mathcal{F},1}(\varphi) \Psi_m \\ &= X W_M(s_M) \dots W_1(s_1) W_1(\varphi) \Psi_m, \end{aligned} \quad (4.129)$$

with  $M$  being the respective fermion number on each sector and where the last equality in (4.129) holds by Lemma 4.5.8. Again, we may extend the definition to **symmetrized vectors** by imposing that  $W_M(s_M) \dots W_1(s_1)$  shall commute with the symmetrization operator  $(S_- \otimes \mathbb{1})$ .

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Again, it is easy to see that this definition makes sense: By Lemma 4.5.4, we have  $W_M(s_M) \dots W_1(s_1)\Psi_m \in \overline{\mathcal{F}}$ . And since  $X \in \mathcal{X}$  maps  $\overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$ , indeed

$$XW_M(s_M) \dots W_1(s_1)\Psi_m \in \overline{\mathcal{F}}_{\text{ex}}, \quad (4.130)$$

so the right-hand sides of (4.129) are well-defined.

#### Remarks.

18. It seems natural to define (4.129) for all operators  $X$  which commute with  $A_j^\dagger(s')$  in a sufficiently regular case and  $A_j(s')$ . However, since we have only defined  $A_j(s') : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$  and  $V_{j'j''}(s^*v) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$ , it is not clear how to interpret the commutator  $[A_j(s'), V_{j'j''}(s^*v)]$ . So  $V_{j'j''}(s^*v)$  would then not be a valid  $X$ -operator, although it commutes with  $A_j^\dagger(s')$  and  $A_j(s')$  for  $s, s', v \in \mathfrak{h}$ .

If one succeeded to modify the definition of  $\overline{\mathcal{F}}, \overline{\mathcal{F}}_{\text{ex}}$  such that commutators as  $[A_j(s'), V_{j'j''}(s^*v)]$  are well-defined operators, then it seems reasonable to change the set of allowed  $X$  in Definition 4.5.9 to all  $X$  with  $[A_j^\dagger(s'), X] = [A_j(s'), X] = 0$ .

#### Final definition of $W(s)$

With Definitions 4.5.7 and 4.5.9, we may now provide the final domains for the product  $W_M(s_M) \dots W_1(s_1)$ : The **extended dressing domain**  $\mathcal{D}_W$  is defined as

$$\mathcal{D}_W := \text{span}_{\text{eRen}} \{W_1(\varphi)A_j^\dagger(v)\Psi_m, XW_1(\varphi)\Psi_m \mid \varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty, v \in \dot{\mathcal{S}}_1^\infty, X \in \mathcal{X}, \Psi_m \in \mathcal{C}_{WS}\}, \quad (4.131)$$

with  $\dot{\mathcal{S}}_1^\infty$  defined in (4.26),  $\mathcal{X}$  defined in (4.128) and  $\mathcal{C}_{WS}$  defined in (4.103). Well-definedness of  $W(s)$  on  $\mathcal{D}_W$  can be seen by combining (4.114) with Definitions 4.5.7 and 4.5.9. By imposing that  $W_M(s_M) \dots W_1(s_1)$  shall commute with  $(S_- \otimes \mathbb{1})$ , we extend  $W_M(s_M) \dots W_1(s_1)$  to  $(S_- \otimes \mathbb{1})[\mathcal{D}_W] \cup \mathcal{D}_W$ .

The maximal domain of  $W(s)$  in Fock space is now given by the **large domain**

$$\mathcal{D}_{\mathcal{F}} := (S_- \otimes \mathbb{1})[\mathcal{D}_W \cap (L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y)]. \quad (4.132)$$

The symmetrization operator  $(S_- \otimes \mathbb{1})$  ensures that indeed  $\mathcal{D}_{\mathcal{F}} \subset \overline{\mathcal{F}}$ . With this definition, it holds true that

**Lemma 4.5.10.** *We have the inclusion*

$$\mathcal{D}_{WS} \subset \mathcal{D}_W \cap (L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y), \quad (4.133)$$

and in particular,  $\mathcal{D}_{\mathcal{F}}$  is dense in  $\mathcal{F}$ .

*Proof.* Setting  $X = 1$  and using Lemma 4.5.6, we see that  $W_{\mathcal{F},1}(\varphi)\Psi_m = W_1(\varphi)\Psi_m \in \mathcal{D}_{WS}$  with  $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$  and  $\Psi_m \in \mathcal{C}_{WS}$  is also an element of  $\mathcal{D}_W$ . Further,  $W_{\mathcal{F},1}(\varphi)\Psi_m \in L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ , which yields the inclusion relation (4.133).

Hence, the symmetrized version  $(S_- \otimes 1)[\mathcal{D}_{WS}]$  is included in  $\mathcal{D}_{\mathcal{F}}$ . And since the former is dense in  $\mathcal{F}$  (Lemma 4.5.2), also the latter is.  $\square$

### Remarks.

19. In essence, we just transferred the commutation relations (4.107) for creation operators  $A_j^\dagger$  from Lemma 4.5.3 in a certain sense to extended dressing operators  $W_j(s)$ . This was done by imposing definitions such that these commutation relations still hold true. What about the commutation relations (4.108) for annihilation operators  $A_j$ ?

In fact, these relations cannot be imposed by definition, but one may show that they are an immediate consequence of Definition 4.5.5. This is proved in the following lemma:

**Lemma 4.5.11.** *Let  $s, v \in \dot{\mathcal{S}}_1^\infty$  and  $\Psi_m \in \mathcal{C}_{WS}$ . Then, we have the commutation relations*

$$\begin{aligned} W_{j'}(s)A_j(v)\Psi_m &= \begin{cases} (A_j(v) - \langle v, s \rangle)W_{j'}(s)\Psi_m & \text{if } j = j' \\ (A_j(v) - V_{jj'}(v^*s))W_{j'}(s)\Psi_m & \text{if } j \neq j' \end{cases} \\ W(s)A_j(v)\Psi_m &= (A_j(v) - \langle v, s \rangle - V_{j\bullet}(v^*s))W(s)\Psi_m \\ W_{j'}(s)A(v)\Psi_m &= (A(v) - \langle v, s \rangle - V_{\bullet j'}(v^*s))W_{j'}(s)\Psi_m \\ W(s)A(v)\Psi_m &= (A(v) - M\langle v, s \rangle - V(v^*s))W(s)\Psi_m. \end{aligned} \quad (4.134)$$

*Proof.* First, note that  $A_j(v)\Psi_m = 0$ . The first line in (4.134) then follows by

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momentum space definitions (4.21) and (4.120):

$$\begin{aligned}
& (A_j(v)W_{j'}(s)\Psi_m)(\mathbf{P}, \mathbf{K}) \\
&= \frac{e^{-\frac{\|s\|^2}{2}}}{\sqrt{N!}} \int v(\tilde{\mathbf{k}})^* s(\tilde{\mathbf{k}}) \left( \prod_{\ell=1}^N s(\mathbf{k}_\ell) \right) \Psi_{mx}(\mathbf{P}' + (e_{j'} - e_j)\tilde{\mathbf{k}}) d\tilde{\mathbf{k}} \\
&= \begin{cases} \langle v, s \rangle \Psi_m(\mathbf{P}, \mathbf{K}) & \text{if } j = j' \\ (V_{jj'}(v^*s)\Psi_m)(\mathbf{P}, \mathbf{K}) & \text{if } j \neq j' \end{cases}
\end{aligned} \tag{4.135}$$

with  $\mathbf{P}' = \mathbf{P} + e_{j'} \sum_{\ell=1}^N \mathbf{k}_\ell$ .

The second line in (4.134) is established similarly. We use again (4.21) and (4.120), yielding:

$$\begin{aligned}
& (A_j(v)W(s)\Psi_m)(\mathbf{P}, \mathbf{K}) \\
&= \frac{e^{-\frac{\|s\|^2}{2}}}{\sqrt{N!}} \sum_{\tilde{\sigma}} \int v(\tilde{\mathbf{k}})^* s(\tilde{\mathbf{k}}) \left( \prod_{\ell=1}^N s(\mathbf{k}_\ell) \right) \Psi_{mx}(\mathbf{P}' + (e_{\tilde{\sigma}(N+1)} - e_j)\tilde{\mathbf{k}}) d\tilde{\mathbf{k}},
\end{aligned} \tag{4.136}$$

where the sum runs over all  $\tilde{\sigma} : \{1, \dots, N+1\} \rightarrow \{1, \dots, M\}$  and we have set  $\mathbf{P}' = \mathbf{P} + \sum_{\ell=1}^N e_{\tilde{\sigma}(\ell)} \mathbf{k}_\ell$ , as well as  $\mathbf{k}_{N+1} = \tilde{\mathbf{k}}$ . We can split this sum into a sum over  $(\sigma, j)$  with  $\sigma : \{1, \dots, N\} \rightarrow \{1, \dots, M\}$ ,  $\sigma(\ell) = \tilde{\sigma}(\ell)$  and  $j' \in \{1, \dots, M\}$ ,  $j' = \tilde{\sigma}(N+1)$ :

$$\begin{aligned}
& (A_j(v)W(s)\Psi_m)(\mathbf{P}, \mathbf{K}) \\
&= \frac{e^{-\frac{\|s\|^2}{2}}}{\sqrt{N!}} \sum_{j'} \sum_{\sigma} \int v(\tilde{\mathbf{k}})^* s(\tilde{\mathbf{k}}) \left( \prod_{\ell=1}^N s(\mathbf{k}_\ell) \right) \Psi_{mx}(\mathbf{P}' + (e_{j'} - e_j)\tilde{\mathbf{k}}) d\tilde{\mathbf{k}}.
\end{aligned} \tag{4.137}$$

Now, the term with  $j' = j$  renders the contribution  $\langle s, v \rangle \Psi_m$  and all other  $M-1$  terms add up to  $\sum_{j': j' \neq j} V_{jj'}(v^*s)\Psi_m = V_{j\bullet}(v^*s)\Psi_m$ , which is exactly the desired contribution.

Lines three and four of (4.134) just follow by summing over  $j \in \{1, \dots, M\}$  in the first two lines. □

## 4.6 Pulling Back the Hamiltonian

This section is concerned with taking a formal Hamiltonian

$$H = H_{0,y} + A^\dagger(v) + A(v) - E_\infty,$$

and pulling it back under the dressing transformation  $W(s)$ , i.e., we compute

$$\tilde{H} : \mathcal{F}_{\text{ex}} \supset \mathcal{D}_{WS} \rightarrow \overline{\mathcal{F}}_{\text{ex}} \quad \text{with} \quad W(s)\tilde{H} = HW(s).$$

The computation is split into two steps. In Section 4.6.1, we compute the pullback of  $(A(v) - E_\infty)$ . Pulling back only  $A(v)$  will result in divergences which are canceled by  $E_\infty$ .

The pullback of  $(H_{0,y} + A^\dagger(v))$  is then computed in Section 4.6.2. Combining  $H_{0,y}$  and  $A^\dagger(v)$  yields a particularly easy result.

Our main theorem is the following:

**Theorem 4.6.1.** *Let  $s = -\frac{v}{\omega}$  with  $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$ . Then the pullback of the self-energy renormalized Hamiltonian*

$$\tilde{H} := H_{0,y} + V(v^*s) \quad \text{satisfies} \quad W(s)\tilde{H} = (H_{0,y} + A^\dagger(v) + A(v) - E_\infty)W(s), \quad (4.138)$$

which holds as a strong operator identity on  $\mathcal{D}_{WS}$  (defined in (4.105)), as well as on  $(S_- \otimes \mathbb{1})[\mathcal{D}_{WS}]$ .

Note that, the potential interaction  $V$  defined in (4.109) via (4.88) acts as

$$(V\Psi)(\mathbf{P}, \mathbf{K}) = \sum_{j \neq j'} \int v(\tilde{\mathbf{k}})^* s(\tilde{\mathbf{k}}) \Psi(\mathbf{P} + (e_j - e_{j'})\tilde{\mathbf{k}}, \mathbf{K}) d\tilde{\mathbf{k}}.$$

### Remarks.

20. We have not so far shown that  $W(s)$  is invertible on  $\mathcal{D}_{\mathcal{F}}$ . While we conjecture that this is true, we do not immediately see how to prove it, here. If this conjecture was true, we could write

$$W(s)\tilde{H} = HW(s) \quad \Leftrightarrow \quad \tilde{H} = W(s)^{-1}HW(s), \quad (4.139)$$

and for a given  $H$  and  $W(s)$ , the renormalized Hamiltonian  $\tilde{H}$  would be unique. Since we do not have an inverse  $W(s)^{-1}$  here, there might be several  $\tilde{H}$ , which satisfy  $W(s)\tilde{H} = HW(s)$ . So the direct renormalization procedure is yet not guaranteed to be unique, although it produces a well-defined renormalized Hamiltonian  $\tilde{H}$  that is physically reasonable.

### 4.6.1 Pulling Back $A - E_\infty$

We recall that by Proposition 4.4.2, one can define  $E_\infty : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$  with

$$(E_\infty \Psi)(\mathbf{P}, \mathbf{K}) = M \langle v, s \rangle \Psi(\mathbf{P}, \mathbf{K}) = \sum_{j=1}^M \int -\frac{v(\mathbf{k})^* v(\mathbf{k})}{\omega(\mathbf{k})} d\mathbf{k} \Psi(\mathbf{P}, \mathbf{K}), \quad (4.140)$$

even if  $\langle v, s \rangle \notin \mathbb{C}$ , but  $\langle v, s \rangle \in \text{Ren}_1$ .

**Lemma 4.6.2.** *Let  $\Psi_m \in \mathcal{C}_{WS}$  and  $s = -\frac{v}{\omega}$  with  $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$ . Then for  $\varphi \in \dot{\mathcal{S}}_1^\infty \cap \mathfrak{h}$ ,*

$$(A(v) - E_\infty)W(s)W_{\mathcal{F},1}(\varphi)\Psi_m = W(s)(\text{res}_1(\varphi) + V(v^*s))W_{\mathcal{F},1}(\varphi)\Psi_m \in \overline{\mathcal{F}}_{\text{ex}}, \quad (4.141)$$

where the *residual operator*

$$\text{res}_1(\varphi) = \underbrace{\langle v, \varphi \rangle}_{\in \text{Ren}_1} + V_{\bullet 1}(v^* \varphi), \quad (4.142)$$

is by Lemma 4.4.3 a well-defined mapping  $\overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$ .

The proof of Lemma 4.6.2 is given in Section 4.11.

### 4.6.2 Pulling Back $H_{0,y} + A^\dagger$

**Lemma 4.6.3.** *Let  $\Psi_m \in \mathcal{C}_{WS}$  and  $s = -\frac{v}{\omega}$  with  $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$ . Then for  $\varphi \in \dot{\mathcal{S}}_1^\infty \cap \mathfrak{h}$ ,*

$$(H_{0,y} + A^\dagger(v))W(s)W_{\mathcal{F},1}(\varphi)\Psi_m = W(s)(H_{0,y} - \text{res}_1(\varphi))W_{\mathcal{F},1}(\varphi)\Psi_m \in \overline{\mathcal{F}}_{\text{ex}}, \quad (4.143)$$

with the same residual operator  $\text{res}_1 = \langle v, \varphi \rangle + V_{\bullet 1}(v^* \varphi)$  as in Lemma 4.6.2.

As for Lemma 4.6.2, the proof of Lemma 4.6.3 is rather technical. It can be found in Section 4.12. With both lemmas at hand, Theorem 4.6.1 can directly be proved.

*Proof of Theorem 4.6.1.* This is a simple consequence of Lemmas 4.6.2 and 4.6.3. We put together (4.141) and (4.143) which yields

$$\begin{aligned} W(s)\tilde{H} &= (H_{0,y} + A^\dagger(v) + A(v) - E_\infty)W(s) = W(s)(H_{0,y} + V(v^*s) + \text{res}_1(\varphi) - \text{res}_1(\varphi)) \\ &= W(s)(H_{0,y} + V(v^*s)), \end{aligned} \quad (4.144)$$

as a strong operator identity on all  $\Psi = W_{\mathcal{F},1}(\varphi)\Psi_m, \varphi \in \dot{\mathcal{S}}_1^\infty \cap \mathfrak{h}$ . And these  $\Psi$  span  $\mathcal{D}_{WS}$ .

Since we imposed in Definitions 4.5.5, 4.5.7 and 4.5.9 that symmetrization  $(S_- \otimes \mathbb{1})$  shall commute with  $W(s)$ , the strong operator identity is also valid on  $(S_- \otimes \mathbb{1})[\mathcal{D}_{WS}]$ . □

## 4.7 Self-Adjointness

In this section, we prove that in certain cases,  $\tilde{H}$  can indeed be defined as a self-adjoint operator  $\tilde{H} : \mathcal{F} \supset \text{dom}(\tilde{H}) \rightarrow \mathcal{D}_{\mathcal{F}}$ . So far we have by Theorem 4.6.1 that  $\tilde{H} : (S_- \otimes \mathbb{1})[\mathcal{D}_{WS}] \rightarrow \overline{\mathcal{F}}_{\text{ex}}$  is well-defined.

In order for the image of  $\tilde{H}$  to be in  $\mathcal{D}_{\mathcal{F}}$ , we need to restrict the domain of  $\tilde{H}$  even further to some subspace  $\tilde{\mathcal{D}}_{\mathcal{F}} \subset \mathcal{D}_{\mathcal{F}}$ , defined in (4.147), and prove well-definedness of  $\tilde{H} : \tilde{\mathcal{D}}_{\mathcal{F}} \rightarrow \mathcal{D}_{\mathcal{F}}$  (Lemma 4.7.2). The existence of a self-adjoint extension on some  $\text{dom}(\tilde{H}) \supset \tilde{\mathcal{D}}_{\mathcal{F}}$  is then a simple consequence (Corollary 4.7.3).

### 4.7.1 Existence of Self-adjoint Extensions

First, we verify that  $\tilde{H} = H_{0,y} + V$  is well-defined and symmetric on a dense domain in  $\mathcal{F}$ .

**Definition 4.7.1.** Let  $\mathcal{Q}_{\text{col},x}$  be the set of **collision configurations**, i.e., all fermion position space configurations

$$\mathcal{Q}_{\text{col},x} := \{\mathbf{X} \in \mathcal{Q}_x \mid \exists j \neq j' : \mathbf{x}_j = \mathbf{x}_{j'}\}. \quad (4.145)$$

Denote by

$$\tilde{\mathcal{C}}_{WS} := \{\Psi_m = \Psi_{mx} \otimes \Omega_y \mid \mathcal{F}^{-1}(\Psi_{mx}) \in C_c^\infty(\mathcal{Q}_x \setminus \mathcal{Q}_{\text{col},x})\} \quad (4.146)$$

the “cyclic set” of functions whose support avoids the collision configurations (where  $\mathcal{F}^{-1}$  is the inverse Fourier transform). We define the **small domain**, on which  $\tilde{H}$  is initially defined as a Fock space operator as

$$\tilde{\mathcal{D}}_{\mathcal{F}} := (S_- \otimes \mathbb{1})[\text{span}\{W_{\mathcal{F},1}(\varphi)\Psi_m \mid \varphi \in C_c^\infty(\mathbb{R}^d), \Psi_m \in \tilde{\mathcal{C}}_{WS}\}], \quad (4.147)$$

see Figure 4.5. It is easy to see that  $\tilde{\mathcal{D}}_{\mathcal{F}} \subseteq \mathcal{D}_{\mathcal{F}}$  and  $\tilde{\mathcal{D}}_{\mathcal{F}} \subseteq (S_- \otimes \mathbb{1})[\mathcal{D}_{WS}]$ .

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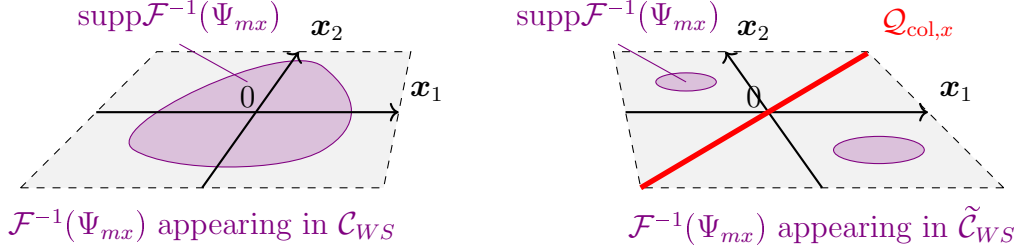


Figure 4.5: Within  $\mathcal{C}_{WS}$ , the fermionic wave functions must be in  $\mathcal{S}$ , allowing for any support (including compact ones). Within  $\tilde{\mathcal{C}}_{WS}$ , only  $C_c^\infty$ -functions are allowed with support avoiding the collision configurations  $\mathcal{Q}_{col,x}$ . Color online.

In the following Lemma, we will use that if  $\omega, v \in \dot{\mathcal{S}}_1^\infty$  satisfy (4.31) and (4.32) (so they scale polynomially), then the potential function

$$\hat{V} := v^* s = -\frac{v^* v}{\omega}, \quad (4.148)$$

also scales polynomially with

$$m_V = 2m_v - m_\omega \quad \beta_V = 2\beta_v - \beta_\omega. \quad (4.149)$$

Further, if  $\beta_V > -d$ , then the inverse Fourier transform  $V = \mathcal{F}^{-1}(\hat{V}) \in S'(\mathbb{R}^d)$  also exists, so we can make statements about the singular support of  $V$ .

**Lemma 4.7.2** ( $\tilde{H}$  is densely defined and symmetric). *The set  $\tilde{\mathcal{D}}_{\mathcal{F}}$  is dense in  $\mathcal{F}$ . Assume, that  $s, v, \omega \in \dot{\mathcal{S}}_1^\infty$  with  $\omega, v$  satisfying (4.31) and (4.32), as well as  $\beta_V > -d$ . If now*

$$\text{singsupp}(V) \subseteq \{0\}, \quad (4.150)$$

*then  $\tilde{H}$  maps  $\tilde{\mathcal{D}}_{\mathcal{F}} \rightarrow \mathcal{D}_{\mathcal{F}}$  and is thus densely defined. If in addition the symmetry condition (4.33) holds, then  $\tilde{H}$  is symmetric.*

*Proof.* Density of  $\tilde{\mathcal{D}}_{\mathcal{F}}$  in  $\mathcal{F}$  is established as density of  $(S_- \otimes \mathbb{1})\mathcal{D}_{WS}$  in Lemma 4.5.2 (proof in Section 4.9). We recall that by the last line of (4.174),

$$D_1 W_{\mathcal{F},1}(\varphi)\Psi_m = \Psi_{mx} \otimes W_y(\varphi)\Omega_y.$$

In the proof of Lemma 4.5.2, we argue that  $(S_- \otimes \mathbb{1})\mathcal{D}_{WS}$  is dense in  $\mathcal{F}$ , since  $\Psi_{mx} \otimes W_y(\varphi)\Omega_y$  approximates any  $\Psi \in L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$  arbitrarily well. The transition from  $(S_- \otimes \mathbb{1})\mathcal{D}_{WS}$  to  $\tilde{\mathcal{D}}_{\mathcal{F}}$  is achieved by a restriction to  $\Psi_{mx} \in \mathcal{F}[C_c^\infty(\mathcal{Q}_x \setminus \mathcal{Q}_{col,x})]$ . The set  $\mathcal{Q}_{col,x}$  is a union of hyperplanes on each sector in the fermionic configuration space, so  $C_c^\infty(\mathcal{Q}_x \setminus \mathcal{Q}_{col,x})$  is dense in  $L^2(\mathcal{Q}_x)$ . The Fourier transform  $\mathcal{F}$  is an

isometry, so the allowed set for  $\Psi_{mx}$  is dense in  $L^2(\mathcal{Q}_x)$ . Thus, we can still approximate any  $\Psi \in L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$  arbitrarily well by  $\Psi_{mx} \otimes W_y(\varphi)\Omega_y$  and by the same arguments as in the proof of Lemma 4.5.2,  $\tilde{\mathcal{D}}_{\mathcal{F}}$  is dense in  $\mathcal{F}$ .

Now we verify that  $H_{0,y}$  maps  $\tilde{\mathcal{D}}_{\mathcal{F}} \rightarrow \mathcal{D}_{\mathcal{F}}$ . By linearity, it suffices to show well-definedness on all vectors of the form  $W_{\mathcal{F},1}(\varphi)\Psi_m$ ,  $\varphi \in C_c^\infty(\mathbb{R}^d)$ . Denote by  $P_y^{(N)}$  the projection of  $\mathcal{F}_y$  to the  $N$ -boson sector  $\mathcal{F}_y^{(N)}$ , so  $\sum_{N \in \mathbb{N}_0} P_y^{(N)} = 1$ . The  $L^2$ -norm squares of  $W_{\mathcal{F},1}(\varphi)\Psi_m$  are Poisson-distributed over  $N$ , i.e.,

$$\|P_y^{(N)}W_{\mathcal{F},1}(\varphi)\Psi_m\|^2 = e^{-\|\varphi\|^2} \frac{\|\varphi\|^{2N}}{N!}, \quad (4.151)$$

so for any  $0 < q < 1$  they decay faster than  $q^N$  in  $N$ -direction. Now, define

$$\lambda := \max_{\mathbf{k} \in \text{supp}(\varphi)} |\omega(\mathbf{k})|. \quad (4.152)$$

Then,

$$\|P_y^{(N)}H_{0,y}W_{\mathcal{F},1}(\varphi)\Psi_m\|^2 \leq N^2\lambda^2\|P_y^{(N)}W_{\mathcal{F},1}(\varphi)\Psi_m\|^2, \quad (4.153)$$

which still decays faster than  $q^N$  in  $N$ -direction. Thus,  $\|H_{0,y}W_{\mathcal{F},1}(\varphi)\Psi_m\| < \infty$  and we have that  $H_{0,y}W_{\mathcal{F},1}(\varphi)\Psi_m \in L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ .

It remains to be shown that  $V = V(v^*s)$  is well-defined, which amounts to proving that

$$(V\Psi)(\mathbf{P}, \mathbf{K}) = \sum_{j \neq j'} \int \hat{V}(\tilde{\mathbf{k}})\Psi(\mathbf{P} + (e_j - e_{j'})\tilde{\mathbf{k}}, \mathbf{K}) d\tilde{\mathbf{k}},$$

defines an  $L^2$ -function on  $\mathcal{Q}$ . Since  $\beta_V > -d$ , we have that  $\hat{V} \in L_{\text{loc}}^1 \Rightarrow \hat{V} \in \mathcal{S}'$ , so we can take the Fourier transform as in (4.90):

$$(V\Psi)(\mathbf{X}, \mathbf{Y}) = \sum_{j \neq j'} V(\mathbf{x}_j - \mathbf{x}_{j'})\Psi(\mathbf{X}, \mathbf{Y}),$$

with  $V(\mathbf{x}) = \mathcal{F}^{-1}(\hat{V})(\mathbf{x})$ .

Now, for  $\Psi = W_{\mathcal{F},1}(\varphi)\Psi_m$  we obtain the position space representation by Fourier-transforming (4.104):

$$\Psi(\mathbf{X}, \mathbf{Y}) = \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \left( \prod_{\ell=1}^N \tilde{\varphi}(\mathbf{y}_\ell - \mathbf{x}_1) \right) \Psi_{mx}(\mathbf{X}), \quad (4.154)$$

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where  $\check{\varphi} = \mathcal{F}^{-1}(\varphi)$  is a Schwartz function, as  $\varphi \in C_c^\infty$  is Schwartz. So as  $\Psi_{mx}(\mathbf{X})$  is a smooth function with compact support apart from collision configurations in  $\mathcal{Q}_x$ , also  $\Psi$  is smooth, and it is zero at fermion collision configurations in  $\mathcal{Q}$ . Since the singular support of  $V(\mathbf{x})$  is at most  $\{0\}$ , the multiplication function  $\sum_{j \neq j'} V(\mathbf{x}_j - \mathbf{x}_{j'})$  is smooth on  $\text{supp}(\Psi_{mx})$  (which excludes collision configurations). And as  $\text{supp}(\Psi_{mx})$  is compact, there is some  $C_\Psi \in \mathbb{R}$  with

$$\max_{\mathbf{X} \in \text{supp}(\Psi_{mx})} \left| \sum_{j \neq j'} V(\mathbf{x}_j - \mathbf{x}_{j'}) \right| \leq C_\Psi. \quad (4.155)$$

Further, by compactness of support, a maximum occupied fermion sector  $\overline{M}$  exists. So

$$\|V\Psi\|^2 \leq \overline{M}^4 C_\Psi^2 \|\Psi\|^2 < \infty$$

for  $\Psi \in \tilde{\mathcal{D}}_{\mathcal{F}}$ . Thus,  $V\Psi \in L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ .

Symmetrization for fermions by  $(S_- \otimes \mathbb{1})$  is preserved by  $H_{0,y}$  and  $V$ . Hence, indeed  $\tilde{H}\Psi \in \mathcal{F}$ . And by Theorem 4.6.1, we have that  $\tilde{H}\Psi \in (S_- \otimes \mathbb{1})[\mathcal{D}_W]$  (otherwise, we could not apply  $W(s)$  to it).

Symmetry of  $\tilde{H}$  is an obvious consequence of the symmetry condition (4.33). And since  $\tilde{H}$  preserves symmetry, it maps  $\tilde{\mathcal{D}}_{\mathcal{F}} \rightarrow \mathcal{D}_{\mathcal{F}} = (S_- \otimes \mathbb{1})[\mathcal{D}_W \cap (L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y)]$  (compare (4.132)).

□

Note that by Lemma 4.7.2, we may extend  $\tilde{H}$  to any domain in  $\mathcal{F}$ , as long as its action on all vectors is well-defined.

**Corollary 4.7.3** (Existence of a self-adjoint extension).  *$\tilde{H} : \tilde{\mathcal{D}}_{\mathcal{F}} \rightarrow \mathcal{D}_{\mathcal{F}}$  as in Lemma 4.7.2 allows for a self-adjoint extension.*

*Proof.* This is a direct consequence of [22, Thm. X.3] (von Neumann's theorem): For a symmetric operator  $\tilde{H}$  (called  $A$  within [22]), this theorem asserts that there is a self-adjoint extension, provided that a conjugation operator  $C : \tilde{\mathcal{D}}_{\mathcal{F}} \rightarrow \tilde{\mathcal{D}}_{\mathcal{F}}$  can be found, such that

$$C\tilde{H} = \tilde{H}C. \quad (4.156)$$

As a conjugation, we choose  $(C\Psi)(\mathbf{K}) = \Psi(-\mathbf{K})^*$ , which amounts to complex conjugation in particle-position representation. By symmetry (4.33) and since  $\omega$  is real-valued,  $\hat{V}(\mathbf{k}) = \hat{V}(-\mathbf{k})^*$ , so  $VC = CV$ . And analogously,  $CH_{0,y} = H_{0,y}C$ . Thus, (4.156) holds, and we have at least one self-adjoint extension. □

## 4.8 Further Dressing Types

There exist also other types of (non-unitary) dressing operators replacing  $W(s)$ . An example is given by the IBC construction mentioned, presented in Section 1.4, where a dressing operator  $W = W_{\text{IBC}}^{-1} = (1 + H_0^{-1}A^\dagger)^{-1}$  defined on a subspace of  $\mathcal{F}$  is used. In this section, we extend this operator to  $\overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$ .

Another example is the dressing operator  $T = e^{-H_0^{-1}A^\dagger}$ , which is a strongly simplified version of certain operators used in CQFT.

### 4.8.1 IBC on the Extended State Space

In certain cases, the IBC renormalization renders a self-adjoint operator  $H$  with dense domain in  $\mathcal{F}$ , by using a formal undressing operator of the kind  $W_{\text{IBC}} = (1 + H_0^{-1}A^\dagger)$ . Within the construction, several divergent integrals appear, which get combined to convergent ones. Using the ESS construction, one can directly make sense of the divergent expressions and, in certain cases, perform the IBC renormalization in a particularly convenient way. Suppose,  $\theta(\mathbf{p}) > 0$  and  $\omega(\mathbf{k}) > 0$ . By Proposition 4.4.2,  $AH_0^{-1}A^\dagger : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$  is a well-defined operator. Using a self-energy operator  $E : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$ , we can define  $H_{\text{IBC}} : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}$  with  $H_{\text{IBC}} = H_0 + A^\dagger + A - E$  via (1.133):

$$\begin{aligned}
H_{\text{IBC}} &= H_0 + A^\dagger + A && - E \\
&= H_0 + A^\dagger + A + AH_0^{-1}A^\dagger && - AH_0^{-1}A^\dagger - E \\
&= \underbrace{(1 + AH_0^{-1})}_{=:S^*} H_0^{1/2} \underbrace{H_0^{1/2}(1 + H_0^{-1}A^\dagger)}_{=:S} && \underbrace{- AH_0^{-1}A^\dagger - E}_T \\
&= S^*S + T.
\end{aligned} \tag{4.157}$$

Clearly,  $S^*S$  is formally a symmetric and positive operator. If it can be densely defined on  $\mathcal{F}$  as a closed operator, then by [22, X.25] we have self-adjointness of  $S^*S$ . Using this argument, it is shown in [136, 137, 138, 139] that for certain dispersion relations and form factors,  $S^*S$  is self-adjoint on the domain

$$\text{dom}(S^*S) = \{\Psi \in \mathcal{F} \mid (1 + H_0^{-1}A^\dagger)\Psi \in \text{dom}(H_0)\}. \tag{4.158}$$

The condition  $(1 + H_0^{-1}A^\dagger)\Psi \in \text{dom}(H_0)$  is called interior- or abstract boundary condition.

Now, suppose there is a suitable  $E$  such that  $T : \mathcal{F} \supset \mathcal{D}(T) \rightarrow \mathcal{F}$  is a densely

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defined Kato–perturbation of  $S^*S$ , that is,

$$\|T\Psi\| \leq a\|S^*S\Psi\| + b\|\Psi\| \quad \forall \Psi \in \text{dom}(S^*S), \quad (4.159)$$

with  $a < 1$ . Then by the Kato–Rellich Theorem [22, X.12] we immediately obtain a self-adjoint  $H_{\text{IBC}}$  on the same domain  $\text{dom}(S^*S)$ .

Using the ESS construction, we may now rigorously define  $W_{\text{IBC}}$  and  $W_{\text{IBC}}^{-1}$ , even if they formally map out of Fock space:

**Proposition 4.8.1.** *The operator  $W_{\text{IBC}} : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$  is well-defined and bijective.*

*Proof.* We show that  $W_{\text{IBC}} : \dot{\mathcal{S}}_{\mathcal{F}}^{\infty} \rightarrow \dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$  is bijective. The extension to a bijective operator on  $\overline{\mathcal{F}}$  is then done by linearity with respect to the field eRen.

The operator  $H_0^{-1}A^\dagger$  maps  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$  to itself:  $A^\dagger$  maps  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty} \rightarrow \dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$  by Proposition 4.4.2 and  $H_0^{-1}$  is just a multiplication by a function that is smooth on  $\dot{\mathcal{Q}}$  and polynomially scaling. So  $W_{\text{IBC}}$  is well-defined on all of  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$ .

Bijectivity is shown by directly constructing  $W_{\text{IBC}}^{-1}$ . Formally, the inverse is given by a Neumann series:

$$W_{\text{IBC}}^{-1} = (1 + H_0^{-1}A^\dagger)^{-1} := \sum_{k=0}^{\infty} (-H_0^{-1}A^\dagger)^k. \quad (4.160)$$

Claim: The Neumann series (4.160) defines an operator  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty} \rightarrow \dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$ .

Proof of the Claim: Each  $-H_0^{-1}A^\dagger$  increases the boson number by 1. So  $(-H_0^{-1}A^\dagger)^k\Psi$  is only supported on configuration space sectors with  $N \geq k$ . Hence, on each  $(\mathbf{P}, \mathbf{K}) \in \dot{\mathcal{Q}}$  with  $\mathbf{K} \in \mathbb{R}^{Nd}$ , we have that

$$\begin{aligned} & \left( (-H_0^{-1}A^\dagger)^k\Psi \right) (\mathbf{P}, \mathbf{K}) = 0 \quad \text{for } k > N \\ \Rightarrow & \left( \sum_{k=0}^{\infty} (-H_0^{-1}A^\dagger)^k\Psi \right) (\mathbf{P}, \mathbf{K}) = \left( \sum_{k=0}^N (-H_0^{-1}A^\dagger)^k\Psi \right) (\mathbf{P}, \mathbf{K}). \end{aligned} \quad (4.161)$$

Since  $H_0^{-1}A^\dagger$  maps  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$  to itself, also all sums  $\sum_{k=0}^n (-H_0^{-1}A^\dagger)^k$  with  $n \in \mathbb{N}$  map  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty} \rightarrow \dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$ . So the Neumann series is defined on each  $N$ -boson sector and hence on all of  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$ .  $\diamond$

Claim: The Neumann series (4.160) is the inverse of  $(1 + H_0^{-1}A^\dagger)$ .

Proof of the Claim: We use the first line of (4.161) and perform a sector-wise verification:

$$\begin{aligned}
 & (1 + H_0^{-1}A^\dagger) \left( \sum_{k=0}^{\infty} (-H_0^{-1}A^\dagger)^k \Psi \right) (\mathbf{P}, \mathbf{K}) \\
 &= \left( \sum_{k=0}^{\infty} (-H_0^{-1}A^\dagger)^k \Psi - \sum_{k=1}^{\infty} (-H_0^{-1}A^\dagger)^k \Psi \right) (\mathbf{P}, \mathbf{K}) \\
 &\stackrel{(4.161)}{=} \left( \sum_{k=0}^N (-H_0^{-1}A^\dagger)^k \Psi - \sum_{k=1}^N (-H_0^{-1}A^\dagger)^k \Psi \right) (\mathbf{P}, \mathbf{K}) \\
 &= \Psi(\mathbf{P}, \mathbf{K}).
 \end{aligned} \tag{4.162}$$

So indeed  $(1 + H_0^{-1}A^\dagger) \left( \sum_{k=0}^{\infty} (-H_0^{-1}A^\dagger)^k \right) = 1$ .  $\diamond$

As there is a well-defined inverse of  $W_{\text{IBC}} = (1 + H_0^{-1}A^\dagger)$  on all of  $\dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$ , the operator  $W_{\text{IBC}} : \dot{\mathcal{S}}_{\mathcal{F}}^{\infty} \rightarrow \dot{\mathcal{S}}_{\mathcal{F}}^{\infty}$  must be bijective.  $\square$

With Proposition 4.8.1, we have a well-defined linear space  $W_{\text{IBC}}^{-1}[\dot{\mathcal{S}}_{\mathcal{F}}^{\infty} \cap \mathcal{F}]$  that can be equipped with a scalar product

$$\langle W_{\text{IBC}}^{-1}\Psi, W_{\text{IBC}}^{-1}\Phi \rangle_{\text{renI}} := \langle \Psi, \Phi \rangle \quad \text{for } \Psi, \Phi \in \dot{\mathcal{S}}_{\mathcal{F}}^{\infty} \cap \mathcal{F}. \tag{4.163}$$

The completion of  $W_{\text{IBC}}^{-1}[\dot{\mathcal{S}}_{\mathcal{F}}^{\infty} \cap \mathcal{F}]$  with respect  $\langle \cdot, \cdot \rangle_{\text{renI}}$  is a Hilbert space  $\mathcal{F}_{\text{renI}}$ , which we call the **IBC-renormalized Fock space**.  $H_{\text{IBC}}$  is then defined on  $\mathcal{F}_{\text{renI}}$ . The pullback to  $\mathcal{F}$  reads

$$\tilde{H}_{\text{IBC}} = W_{\text{IBC}} H_{\text{IBC}} W_{\text{IBC}}^{-1} \tag{4.164}$$

Whenever the expression (4.164) extends to a self-adjoint operator,  $H_{\text{IBC}}$  extends to a self-adjoint operator on  $\mathcal{F}_{\text{renI}}$ .

## 4.8.2 The $e^{-H_0^{-1}A^\dagger}$ -Transformation Inspired by CQFT

An operator  $T \propto e^{-H_0^{-1}A^\dagger}$  appears in a similar form in the CQFT literature [209, 95]. Within the latter reference, a renormalized scalar product is constructed by a procedure of the kind

$$\langle T\Psi, T\Phi \rangle_{\text{ren}} := \lim_{\Lambda \rightarrow \infty} \langle T_\Lambda \Psi, T_\Lambda \Phi \rangle e^{-\Lambda\Lambda} \quad \forall T\Psi, T\Phi \in T[\mathcal{D}], \quad \mathcal{D} \subset \mathcal{F}, \tag{4.165}$$

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where the operators  $T, T_\Lambda$  in [95] are, however, much more involved. Here,

$$\mathbf{\Lambda}_\Lambda = 4! \|\omega v_\Lambda\|_2^2, \quad (4.166)$$

and  $\Lambda$  is a UV-cutoff. The renormalized Hamiltonian is then constructed by a limiting procedure [95, Thm. 4.1.1]:

$$\langle T\Psi, H_{\text{ren}} T\Phi \rangle_{\text{ren}} = \lim_{\Lambda \rightarrow \infty} \langle T_\Lambda \Psi, H_{\text{ren}}(\Lambda) T_\Lambda \Phi \rangle e^{-\Lambda_\Lambda}, \quad (4.167)$$

with  $H_{\text{ren}}(\Lambda)$  containing counterterms. The limit  $\Lambda \rightarrow \infty$  formally leads to an infinite wave function renormalization with

$$\mathbf{\Lambda} = 4! \|\omega v\|_2^2. \quad (4.168)$$

The ESS construction now allows to directly define  $\mathbf{\Lambda} \in \text{Ren}_1$ . In the simplified case  $T \propto e^{-H_0^{-1} A^\dagger}$ , we can even define the dressing transformation directly on  $\overline{\mathcal{F}}$ :

**Proposition 4.8.2.** *For  $\Psi \in \dot{\mathcal{S}}_{\mathcal{F}}^\infty$ , we have*

$$T\Psi := e^{-\Lambda/2} e^{-H_0^{-1} A^\dagger} \Psi \in \overline{\mathcal{F}}, \quad (4.169)$$

with  $e^{-\Lambda/2} \in \text{eRen}$ . In particular,  $e^{-H_0^{-1} A^\dagger}$  and  $T$  are well-defined linear operators

$$e^{-H_0^{-1} A^\dagger} : \dot{\mathcal{S}}_{\mathcal{F}}^\infty \rightarrow \dot{\mathcal{S}}_{\mathcal{F}}^\infty, \quad T : \dot{\mathcal{S}}_{\mathcal{F}}^\infty \rightarrow \overline{\mathcal{F}}.$$

*Proof.* As argued in the proof of Proposition 4.8.1,  $H_0^{-1} A^\dagger$  is well-defined on  $\dot{\mathcal{S}}_{\mathcal{F}}^\infty$  and maps Fock space vectors supported on the  $N$ -particle sector to those supported on the  $N + 1$ -particle sector.

We can write the exponential series as

$$e^{-H_0^{-1} A^\dagger} = \sum_{k=0}^{\infty} \frac{(-H_0^{-1} A^\dagger)^k}{k!} = 1 - H_0^{-1} A^\dagger + \frac{1}{2} H_0^{-1} A^\dagger H_0^{-1} A^\dagger - \dots \quad (4.170)$$

Now, each  $-H_0^{-1} A^\dagger$  maps  $\dot{\mathcal{S}}_{\mathcal{F}}^\infty$  to itself and strictly increases the sector number. So the  $N$ -sector of  $e^{-H_0^{-1} A^\dagger} \Psi$  may only depend on at most  $N + 1$  terms of the series (4.170). All terms are elements of  $\dot{\mathcal{S}}_{\mathcal{F}}^\infty \subset \overline{\mathcal{F}}$ . Hence,  $e^{-H_0^{-1} A^\dagger}$  maps  $\dot{\mathcal{S}}_{\mathcal{F}}^\infty$  into itself, as claimed. Clearly, the factor  $e^{-\Lambda/2}$  is an element of  $\text{eRen}$ , so  $T\Psi \in \overline{\mathcal{F}}$ .  $\square$

## 4.9 Proof of Lemma 4.5.2

*Proof.* We show that  $\mathcal{D}_{WS}$  is dense in  $L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ . Since  $(S_- \otimes \mathbb{1}) : L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y \rightarrow \mathcal{F}$  is surjective and bounded with norm 1, we then immediately get density of  $(S_- \otimes \mathbb{1})[\mathcal{D}_{WS}]$  within  $\mathcal{F}$ .

If  $A_1^\dagger(\varphi)$  created a boson without giving a recoil to a fermion, we would be done: In this case,  $W_{\mathcal{F},1}(\varphi)\Psi_m$  would be of the form

$$\Psi_{mx} \otimes W_y(\varphi)\Omega_y. \quad (4.171)$$

Now,  $\Psi_{mx} \in \mathcal{S}(\mathcal{Q}_x)$ , which is dense in  $L^2(\mathcal{Q}_x)$ . Further, we have that  $\text{span}\{W_y(\varphi)\Omega_y \mid \varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty\}$  is dense in  $\mathcal{F}_y$ , so

$$\text{span}\{\Psi_{mx} \otimes W_y(\varphi)\Omega_y \mid \varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty, \Psi_{mx} \in \mathcal{S}(\mathcal{Q}_x)\} \quad (4.172)$$

is dense in  $L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ .

However,  $W_{\mathcal{F},1}(\varphi)\Psi_m$  is not of the form (4.171), since  $W_{\mathcal{F},1}(\varphi)$  shifts the momentum  $\mathbf{p}_1$  by  $\sum_{\ell=1}^N \mathbf{k}_\ell$ , as in (4.36) (the first fermion gets a recoil). The same recoil occurs when applying  $A_1^\dagger(v)$ . In other words, creation and dressing “entangle” the fermion with the created boson by giving the fermion a recoil. In order to solve this problem, we introduce a **disentangling operator**  $D_1 : L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y \rightarrow L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ , which removes all recoils:

$$(D_1\Psi)(\mathbf{P}, \mathbf{K}) := \Psi\left(\mathbf{P} - \sum_{\ell=1}^N e_1 \mathbf{k}_\ell, \mathbf{K}\right). \quad (4.173)$$

Clearly,  $D_1$  is unitary. Now,

$$\begin{aligned} (W_{\mathcal{F},1}(\varphi)\Psi_m)(\mathbf{P}, \mathbf{K}) &= \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \left( \prod_{\ell=1}^N \varphi(\mathbf{k}_\ell) \right) \Psi_{mx}\left(\mathbf{P} + \sum_{\ell=1}^N e_1 \mathbf{k}_\ell\right) \\ \Rightarrow (D_1 W_{\mathcal{F},1}(\varphi)\Psi_m)(\mathbf{P}, \mathbf{K}) &= \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \left( \prod_{\ell=1}^N \varphi(\mathbf{k}_\ell) \right) \Psi_{mx}(\mathbf{P}) \\ \Leftrightarrow D_1 W_{\mathcal{F},1}(\varphi)\Psi_m &= \Psi_{mx} \otimes \sum_{N=0}^{\infty} \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \underbrace{\varphi \otimes \dots \otimes \varphi}_{N \text{ times}} \\ &= \Psi_{mx} \otimes W_y(\varphi)\Omega_y. \end{aligned} \quad (4.174)$$

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So by density of (4.172), we have that

$$\text{span}\{D_1 W_{\mathcal{F},1}(\varphi)\Psi_m \mid \varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty, \Psi_m \in \mathcal{C}_{WS}\}$$

is dense in  $L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ . And since  $D_1$  is unitary, its preimage

$$\mathcal{D}_{WS} = \text{span}\{W_{\mathcal{F},1}(\varphi)\Psi_m \mid \varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty, \Psi_m \in \mathcal{C}_{WS}\}$$

is dense in  $L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ , as well. So  $(S_- \otimes \mathbb{1})[\mathcal{D}_{WS}]$  is dense in  $\mathcal{F}$ . □

### 4.10 Proof of Lemma 4.5.3

*Proof.* For evaluating  $[A^\dagger, W]$ , we first consider the simple case where  $\Psi$  is replaced by a boson-only vector  $\Psi_y \in \mathcal{F}_y$ . A vector with one boson can be written as:

$$\phi = a^\dagger(\phi)\Omega_y, \quad (4.175)$$

with  $\phi \in \mathfrak{h} = \mathcal{F}_y^{(1)}$ . A coherent displacement can then be described using

$$W_y(\varphi)a^\dagger(\phi) = a^\dagger(\phi)W_y(\varphi) - [a^\dagger(\phi), W_y(\varphi)], \quad (4.176)$$

where  $\varphi \in \mathfrak{h} \cap \dot{\mathcal{S}}_1^\infty$ . The first expression is easily computed in momentum space:

$$\begin{aligned} (a^\dagger(\phi)\Psi_y)(\mathbf{K}) &= \frac{1}{\sqrt{N}} \sum_{\ell=1}^N \phi(\mathbf{k}_\ell)\Psi_y(\mathbf{K} \setminus \mathbf{k}_\ell) \\ \Rightarrow (a^\dagger(\phi)W_y(\varphi)\Omega_y)(\mathbf{K}) &= \frac{e^{-\frac{\|\varphi\|^2}{2}}}{\sqrt{N!}} \sum_{\ell=1}^N \phi(\mathbf{k}_\ell) \left( \prod_{\ell' \neq \ell} \varphi(\mathbf{k}_{\ell'}) \right). \end{aligned} \quad (4.177)$$

The commutator  $[a^\dagger(\phi), W_y(\varphi)]$  is computed using

$$\begin{aligned} (*) \quad & [a^\dagger(\varphi), a^\dagger(\phi)] = 0 \\ (**) \quad & [a(\varphi), a^\dagger(\phi)] = \langle \varphi, \phi \rangle. \end{aligned} \quad (4.178)$$

We have

$$\begin{aligned}
[a^\dagger(\phi), W_y(\varphi)] &= [a^\dagger(\phi), e^{a^\dagger(\varphi) - a(\varphi)}] = \sum_{k=0}^{\infty} \frac{1}{k!} [a^\dagger(\phi), (a^\dagger(\varphi) - a(\varphi))^k] \\
&\stackrel{(*)}{=} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} ([a^\dagger(\phi), a(\varphi)](a^\dagger(\varphi) - a(\varphi))^{k-1} \\
&\quad + (a^\dagger(\varphi) - a(\varphi))[a^\dagger(\phi), a(\varphi)](a^\dagger(\varphi) - a(\varphi))^{k-2} \\
&\quad + \dots + (a^\dagger(\varphi) - a(\varphi))^{k-1} [a^\dagger(\phi), a(\varphi)]) \\
&\stackrel{(**)}{=} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} k \langle \varphi, \phi \rangle (a^\dagger(\varphi) - a(\varphi))^{k-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle \varphi, \phi \rangle (a^\dagger(\varphi) - a(\varphi))^k \\
&= \langle \varphi, \phi \rangle W_y(\varphi).
\end{aligned} \tag{4.179}$$

It is easy to check that all above formulas hold as strong operator identities on the space of finite-boson states  $\mathcal{F}_{\text{fin},y}$  (defined in (4.99)). By (4.96), an application of  $k$  operators of the kind<sup>4</sup>  $a^\sharp(\varphi) \in \{a(\varphi), a^\dagger(\varphi)\}$  to  $\Psi_y \in \mathcal{F}_{\text{fin},y}$  is bounded by

$$\| (a(\varphi)^\sharp)^k \Psi_y \| \leq \sqrt{\frac{(N_{\max} + k)!}{N_{\max}!}} \|\Psi_y\| \|\varphi\|^n. \tag{4.180}$$

This allows to estimate

$$\| a^\sharp(\phi) W_y(\varphi) \Psi \| \leq \sum_{k \in \mathbb{N}_0} \frac{1}{k!} \sqrt{\frac{(N_{\max} + k + 1)!}{N_{\max}!}} \|\Psi_y\| \|2\varphi\|^n \|\phi\| < \infty, \tag{4.181}$$

and an analogous estimate shows that  $W_y(\varphi) a^\sharp(\phi)$  is well-defined.

Putting together (4.176) and (4.179), we obtain the action of  $W_y(\varphi)$  on single-boson states:

$$W_y(\varphi) a^\dagger(\phi) \Omega_y = (a^\dagger(\phi) - \langle \varphi, \phi \rangle) W_y(\varphi) \Omega_y. \tag{4.182}$$

Now, we turn to state vectors with many fermions and one boson,  $A_1^\dagger(\phi) \Psi_m \in L^2(\mathcal{Q}_x) \otimes \mathcal{F}_y$ . Further, we go over from dressings by  $W_y(\varphi)$  to  $W_{\mathcal{F},j}(\varphi)$ , which is done replacing  $a(\phi), a^\dagger(\phi)$  by  $A_j(\phi), A_j^\dagger(\phi)$ . Note that  $A_j(\phi), A_j^\dagger(\phi)$  are no longer merely creating and annihilating bosons, but they also shift a fermion's momentum. Computations in (4.179) run through in almost the same manner. We have to replace (\*\*) by the CCR (4.86). If  $j \neq j'$ , we further use that  $V_{jj'}(\varphi^* \phi)$  (which

<sup>4</sup>Here, we allow  $\sharp$  to be a different superscript in each factor. E.g.,  $(a^\sharp)^4$  would also represent  $aa^\dagger aa^\dagger$ .

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replaces  $\langle \varphi, \phi \rangle$  in (4.179)) commutes with<sup>5</sup>  $A_{j''}(\varphi)$  and  $A_{j''}^\dagger(\varphi)$ , so we can still pull it to the left.

The final result is

$$W_{\mathcal{F},j}(\varphi)A_{j'}^\dagger(\phi) = \begin{cases} (A_{j'}^\dagger(\phi) - \langle \varphi, \phi \rangle)W_{\mathcal{F},j}(\varphi) & \text{if } j = j' \\ (A_{j'}^\dagger(\phi) - V_{jj'}(\varphi^*\phi))W_{\mathcal{F},j}(\varphi) & \text{if } j \neq j' \end{cases}. \quad (4.183)$$

This is one of the four identities claimed in (4.107). The other three identities follow by summation over  $j$  or  $j'$ .

The four identities in (4.108) are obtained analogously. In place of (\*\*), we use the CCR (4.86) together with  $[A_j(\varphi), A_{j'}(\phi)] = 0$  and keep in mind that the factor  $(-1)^k$  drops out in (4.179), which yields the desired result.

It is easy to check that all identities above hold as strong operator identities on the domain  $\mathcal{F}_{\text{fin}}$  defined in (4.106). The momentum space definitions of  $A_j^\dagger(\varphi), A_j(\varphi)$  in (4.19) and (4.21) directly yield the well-known estimates

$$\|A_j^\dagger(\varphi)\Psi\| \leq \|(N+1)^{1/2}\Psi\| \|\varphi\|, \quad \|A_j(\varphi)\Psi\| \leq \|N^{1/2}\Psi\| \|\varphi\|, \quad (4.184)$$

which are analogous to (4.96) and allow for employing the same arguments as below (4.99). Thus, all expressions in (4.107) and (4.108) are well-defined on  $\mathcal{F}_{\text{fin}}$ .  $\square$

#### Remarks.

21. If we replace the form factors  $\varphi(\mathbf{k})$  and  $\phi(\mathbf{k})$  in  $A^\dagger(\phi)$  by  $\varphi(\mathbf{p}, \mathbf{k})$  and  $\phi(\mathbf{p}, \mathbf{k})$  (as they appear in more realistic QFT models), then  $[A_j^\dagger(\varphi), A_{j'}^\dagger(\phi)] = 0$  will no longer hold true. The operators  $A_j^\dagger(\phi)$  change the momentum of the fermion which is emitting a boson. So when one fermion creates two bosons with different form factors, which depend on the fermion momentum, then it can make a difference, which boson is created first. In addition,  $V_{j'j''}$  will no longer commute with  $A_j$  and  $A_j^\dagger$ . In this case, several multi-commutators of the form  $[A_{j_1}^\#, [A_{j_2}^\#, \dots, [A_{j_n}^\#, A_{j'}^\#]]]$  with  $A_j^\# \in \{A_j, A_j^\dagger\}$  appear.

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<sup>5</sup>This is shown by a similar fiber decomposition argument, as used in the proof of Lemma 4.5.8.

## 4.11 Proof of Lemma 4.6.2

*Proof.* We evaluate the pullbacks of  $A(v)$  and  $E_\infty$ , keeping in mind that by Lemma 4.5.6,  $\Psi = W(s)W_{\mathcal{F},1}(\varphi)\Psi_m = W(s)W_1(\varphi)\Psi_m$ .

- $A(v)$ : The commutator  $[W(s)W_1(\varphi), A(v)]$  is evaluated using the extended commutation relations established in Lemma 4.5.11:

$$\begin{aligned}
 A(v)\Psi &= A(v)W(s)W_1(\varphi)\Psi_m \\
 &= [A(v), W(s)W_1(\varphi)]\Psi_m + W(s)W_1(\varphi)\underbrace{A(v)\Psi_m}_{=0} \\
 &= (M\langle v, s \rangle + \langle v, \varphi \rangle + V_{\bullet 1}(v^*\varphi) + V(v^*s))\Psi.
 \end{aligned} \tag{4.185}$$

- $E_\infty$ : The self-energy operator (4.140) now exactly removes the term  $M\langle v, s \rangle$ , so we have

$$(A(v) - E_\infty)\Psi = (\langle v, \varphi \rangle + V_{\bullet 1}(v^*\varphi) + V(v^*s))\Psi. \tag{4.186}$$

Now, by Definition 4.5.9, we may commute  $\langle v, \varphi \rangle$ ,  $V$  and  $V_{\bullet 1}$  with  $W(s)$  yielding

$$\begin{aligned}
 W^{-1}(s)(A(v) - E_\infty)W(s)W_1(\varphi)\Psi_m &= (\langle v, \varphi \rangle + V_{\bullet 1}(v^*\varphi) + V(v^*s))W_1(\varphi)\Psi_m, \\
 &= (\text{res}_1(\varphi) + V)W_1(\varphi)\Psi_m \in \overline{\mathcal{F}}_{\text{ex}}.
 \end{aligned} \tag{4.187}$$

Now, Lemma 4.5.6 allows replacing  $W_1$  by  $W_{\mathcal{F},1}$ , which renders the desired result.  $\square$

### Remarks.

22. If the form factor  $v$  depended on the fermion momentum  $\mathbf{p}$ , then  $s = -\frac{v}{\omega}$  would also depend on  $\mathbf{p}$ . In that case,  $W^{-1}(s)VW(s) = V$  will no longer hold true in general, as the function  $s(\mathbf{k}, \mathbf{p})$  in the dressing operator  $W(s)$  then depends on  $\mathbf{p}$  and fermion momenta are changed by  $V$ . If  $(A(v) - E_\infty)W(s)W_{\mathcal{F},1}(\varphi)\Psi_m$  is then expressed in terms of dressed coherent states, multiple commutators appear.

## 4.12 Proof of Lemma 4.6.3

*Proof.* Again, we may use Lemmas 4.5.4 and 4.5.6, to write

$$\Psi = W(s)W_{\mathcal{F},1}(\varphi)\Psi_m = W(s)W_1(\varphi)\Psi_m \in \overline{\mathcal{F}}.$$

Following Proposition 4.4.2 we further have  $H_{0,y}\Psi, A^\dagger(v)\Psi \in \overline{\mathcal{F}}$ . We first evaluate  $(H_{0,y} + A^\dagger(v))W(s)W_1(\varphi)\Psi_m$  in momentum space and then use Definition 4.5.7 to pull  $W(s)$  to the left.

- $H_{0,y}$ : The important point is that the expression  $(H_{0,y} + A^\dagger(v))W(s)$  cancels terms in  $W(s)H_{0,y}$ . We therefore investigate the commutator expression  $[H_{0,y}, W(s)]W_1(\varphi)\Psi_m$  and compare it to  $A^\dagger(v)W(s)W_1(\varphi)\Psi_m$ .

First, we note that the application of  $H_{0,y}$  to a dressed state just changes one single photon dispersion relation. Therefore, it is equivalent to applying a creation operator to the dressed state:

$$\begin{aligned} & (H_{0,y}W_1(\varphi)\Psi_m)(\mathbf{P}, \mathbf{K}) \\ &= \frac{e^{-\frac{|\varphi|^2}{2}}}{\sqrt{N!}} \left( \sum_{\ell'=1}^N \omega(\mathbf{k}_{\ell'}) \right) \left( \prod_{\ell=1}^N \varphi(\mathbf{k}_\ell) \right) \Psi_{mx}(\mathbf{P}') \\ &= \frac{e^{-\frac{|\varphi|^2}{2}}}{\sqrt{N!}} \sum_{\ell=1}^N \omega(\mathbf{k}_\ell) \varphi(\mathbf{k}_\ell) \left( \prod_{\ell' \neq \ell}^N \varphi(\mathbf{k}_{\ell'}) \right) \Psi_{mx}(\mathbf{P}') \\ &= (A_1^\dagger(\omega\varphi)W_1(\varphi)\Psi_m)(\mathbf{P}, \mathbf{K}), \end{aligned} \tag{4.188}$$

with  $\mathbf{P}' = (\mathbf{p}_1 + \sum_{\ell'=1}^N \mathbf{k}_{\ell'}, \mathbf{p}_2, \dots, \mathbf{p}_M)$ . Replacing the dressing  $W_1(\varphi)$  by  $W(s)W_1(\varphi)$  and applying the commutation relations from Definitions 4.5.7 and 4.5.9, we obtain

$$\begin{aligned} & H_{0,y}W(s)W_1(\varphi)\Psi_m \\ &= (A^\dagger(\omega s) + A_1^\dagger(\omega\varphi))W(s)W_1(\varphi)\Psi_m \\ &= -A^\dagger(v)W(s)W_1(\varphi)\Psi_m \\ & \quad + W(s)W_1(\varphi)(A_1^\dagger(\omega\varphi) - \langle v, \varphi \rangle - V_{\bullet 1}(v^*\varphi) + \langle \varphi, \omega\varphi \rangle)\Psi_m. \end{aligned} \tag{4.189}$$

Here, we used  $s^*\omega = \omega s = -v$  and Definition 4.5.9, which allows us to pull the formal scalar products and the  $V_{\bullet 1}$ -term past the dressing operators.

By means of (4.188) and the commutation relations from Definitions 4.5.7

and 4.5.9, we also have

$$\begin{aligned} & W(s)H_{0,y}W_1(\varphi)\Psi_m \\ &= W(s)W_1(\varphi)A_1^\dagger(\omega\varphi)\Psi_m + W(s)W_1(\varphi)\langle\varphi, \omega\varphi\rangle\Psi_m. \end{aligned} \quad (4.190)$$

Combining (4.189) and (4.190), we finally obtain

$$\begin{aligned} & [H_{0,y}, W(s)]W_1(\varphi)\Psi_m \\ &= -A^\dagger(v)W(s)W_1(\varphi)\Psi_m - W(s)W_1(\varphi)(\langle v, \varphi\rangle + V_{\bullet 1}(v^*\varphi))\Psi_m. \end{aligned} \quad (4.191)$$

- $A^\dagger(v)$ : Here, we do not need to perform any calculations. The appearing term simply cancels the  $-A^\dagger(v)W(s)W_1(\varphi)\Psi_m$  from (4.191).

Now, adding both terms and using Definition 4.5.9, (4.191) amounts to

$$\begin{aligned} & ([H_{0,y}, W(s)] + A^\dagger(v)W(s))W_1(\varphi)\Psi_m \\ &= W(s)W_1(\varphi)(-\langle v, \varphi\rangle - V_{\bullet 1}(v^*\varphi))\Psi_m \\ &= W(s)(-\text{res}_1(\varphi))W_1(\varphi)\Psi_m. \end{aligned} \quad (4.192)$$

This is equivalent to

$$(H_{0,y} + A^\dagger(v))W(s)W_1(\varphi)\Psi_m = W(s)(H_{0,y} - \text{res}_1(\varphi))W_1(\varphi)\Psi_m. \quad (4.193)$$

Lemma 4.5.6 allows again to replace  $W_1$  by  $W_{\mathcal{F},1}$ , which yields the final result.

□



# 5 Implementing Bogoliubov Transformations Beyond the Shale–Stinespring Condition

## 5.1 Overview and Main Results

Bogoliubov transformations are a powerful tool in many–body quantum mechanics, as they allow simplifying Hamiltonians  $H$  and hence analyzing the dynamics of a system. Roughly speaking, a Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}$  is realized by replacing all creation and annihilation operators  $a^\dagger(f), a(f)$  inside a product  $H$  of such operators by

$$b^\dagger(f) = a^\dagger(uf) + a(v\bar{f}), \quad b(f) = a^\dagger(v\bar{f}) + a(uf),$$

with  $f$  being an element of the one–particle Hilbert space  $\mathfrak{h}$ , where  $u, v$  are linear operators on  $\mathfrak{h}$  and where  $\bar{f}$  is the complex conjugate of  $f$ . This replacement can be used to eliminate pair creation and annihilation terms from quadratic Hamiltonians  $H$ , i.e., to “diagonalize  $H$ ” [214, 215, 216, 217]. Related transformations even allow for eliminating inconvenient terms of higher order from non–quadratic operator products  $H$  [218, 219].

In particular, it is desirable to find a unitary operator  $\mathbb{U}_\mathcal{V}$  on Fock space  $\mathcal{F}$ , such that  $\mathbb{U}_\mathcal{V}$  establishes the replacement  $a^\# \mapsto b^\#$  via

$$\mathbb{U}_\mathcal{V} a^\dagger(f) \mathbb{U}_\mathcal{V}^* = b^\dagger(f), \quad \mathbb{U}_\mathcal{V} a(f) \mathbb{U}_\mathcal{V}^* = b(f). \quad (5.1)$$

In that case we say that  $\mathbb{U}_\mathcal{V}$  implements the transformation  $\mathcal{V}$  and we call  $\mathcal{V}$  “implementable” (in the regular sense). It is well–known [220, 221] that  $\mathcal{V}$  is implementable, if and only if the **Shale–Stinespring condition** holds, which asserts that  $\text{tr}(v^*v) < \infty$ .

However, there are situations in which it is desirable to modify a Hamiltonian  $H$  or to describe its dynamics by a Bogoliubov transformation  $\mathcal{V}$  that is not implementable. Such situations occur, for instance, in relativistic models [181, 222], and within many–body systems of infinite size [223, 224, 225].

## 5 Implementing Bogoliubov Transformations Beyond the Shale–Stinespring Condition

In this chapter, we therefore investigate, in which situations a Bogoliubov transformation  $\mathcal{V}$  can be implemented on either of the two Fock space extensions presented in Chapter 3. Recall that:

- The **extended state space** (ESS) framework (Section 3.2) builds around vector spaces  $\overline{\mathcal{F}} \subset \overline{\mathcal{F}}_{\text{ex}}$ . Here,  $\overline{\mathcal{F}}$  contains elements of the form  $\Psi = e^{\mathfrak{r}}\Psi_m$ , where  $\mathfrak{r}$  is a (rigorously defined) and possibly infinite renormalization constant and  $\Psi_m$  is a function on the configuration space  $\mathcal{Q}$  that may be outside  $L^2(\mathcal{Q})$ . The space  $\overline{\mathcal{F}}_{\text{ex}}$  extends  $\overline{\mathcal{F}}$  and additionally contains elements of the form  $e^{\mathfrak{r}'_1} \cdots \mathfrak{r}'_p \Psi_m$ .
- The **infinite tensor product** (ITP) framework (Section 3.1) builds on von Neumann’s ITP space [44]. Here, a Hilbert space  $\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k$  is defined, which extends the Fock space  $\mathcal{F}$ .

Both extensions are in the same spirit as Hilbert space riggings, which we discussed in Section 3.2.3, see also [204]. The construction of  $\overline{\mathcal{F}}$  may also remind about non-standard analysis, where a field extension  ${}^*\mathbb{R}$  of the real line  $\mathbb{R}$  is constructed [226]. However, our construction is not related to it, as we discuss in end of Section 5.2.2. Our implementation procedure differs from that on Fock space, so let us quickly explain it: We take a formal expression  $H$  that consists of a product of  $a^\dagger$ - and  $a$ -operators. The aim is to define an operator

$$\tilde{H} = U_{\mathcal{V}}^{-1}(H + c)U_{\mathcal{V}} \quad (5.2)$$

on a dense set  $\mathcal{D}_{\mathcal{F}} \subseteq \mathcal{F}$ , where  $\tilde{H}$  is the version of  $H$  with  $a^\dagger$  replaced by  $b^\dagger$  and normal ordering applied. The “renormalization constant”  $c$  stems from normal ordering and may be formally infinite. We define a linear (“dressing”) operator  $U_{\mathcal{V}} : \mathcal{D}_{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$  or  $U_{\mathcal{V}} : \mathcal{D}_{\mathcal{F}} \rightarrow \widehat{\mathcal{H}}$  in such a way that  $(H + c)$  maps the space  $U_{\mathcal{V}}[\mathcal{D}_{\mathcal{F}}] \subset \overline{\mathcal{F}}$  or  $U_{\mathcal{V}}[\mathcal{D}_{\mathcal{F}}] \subset \widehat{\mathcal{H}}$  into itself. So  $\tilde{H}$  is the well-defined “renormalized version” of  $H$  and the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathcal{H}} \text{ or } \overline{\mathcal{F}} \supset U_{\mathcal{V}}[\mathcal{D}_{\mathcal{F}}] & \xrightarrow{(H + c)} & U_{\mathcal{V}}[\mathcal{D}_{\mathcal{F}}] \subset \overline{\mathcal{F}} \text{ or } \widehat{\mathcal{H}} \\ \uparrow U_{\mathcal{V}} & & \downarrow U_{\mathcal{V}}^{-1} \\ \mathcal{F} \supset \mathcal{D}_{\mathcal{F}} & \xrightarrow{\tilde{H}} & \mathcal{D}_{\mathcal{F}} \subset \mathcal{F} \end{array}$$

In particular,  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$  are vector spaces without a scalar product or even a topology. Their only purpose is to allow for formal calculations involving infinities, which in the end produce an operator  $\tilde{H}$ . This  $\tilde{H}$  generates dynamics on a Hilbert space  $\mathcal{F}$ , which provides a suitable physical interpretation. Vectors in

$\overline{\mathcal{F}}, \overline{\mathcal{F}}_{\text{ex}}$  can rather be seen as “virtual particle states” that contain no immediate physical meaning.

Our **main result** is now that in the extended sense, specified in Definition 5.5.1,  $\mathcal{V}$  can indeed be implemented:

- on  $\overline{\mathcal{F}}$  in the bosonic (Theorem 5.5.6) and the fermionic case (Theorem 5.5.8),
- on  $\widehat{\mathcal{H}}$  in the bosonic (Theorem 5.5.5) and the fermionic case (Theorem 5.5.7),

if the spectrum of the operator  $v^*v$  is countable. In Theorem 5.5.8, we need the additional assumption of  $\mathcal{V}$  inducing a full particle–hole transformation on at most finitely many modes. The reason is that  $\overline{\mathcal{F}}$  can only describe a finite number of particles.

It would be highly desirable to establish a similar result in the case of a generic  $v^*v$ . As indicated in Section 3.2.5, the main difficulty is the establishment of a suitable one–particle space  $\mathcal{R}$  (replacing  $\mathcal{E}(\mathbb{N})$ ). We conjecture that a statement similar to Theorems 5.5.6 and 5.5.8 can be established for generic  $v^*v$ . By contrast, the definition of ITP spaces  $\widehat{\mathcal{H}}$  directly carries over to the case of generic  $v^*v$  and we obtain  $\widehat{\mathcal{H}} = \prod_{x \in X}^{\otimes} \mathcal{H}_x$  with  $X$  being a possibly uncountable set related to  $\sigma(v^*v)$ . As a well–defined  $a^\dagger(f), a(f)$ , we would then only be able to treat sums  $a^\# = \sum_x f(x)a_x^\#$  over countably many  $x \in X$ . So  $f$  is equal to 0 apart from finitely many points and the set of such  $f$  is not dense in  $\mathfrak{h}$ . Thus, the generalizations of Theorems 5.5.5 and 5.5.7 would be limited to a very restrictive set of form factors  $f \in \mathfrak{h}$ , possibly only containing  $f = 0$ .

One may object that the construction of  $\mathbb{U}_{\mathcal{V}}$  creates an unnecessary effort for obtaining  $\tilde{H}$ , since  $\tilde{H}$  can directly be computed from  $H$  under a replacement  $a^\# \mapsto b^\#$ . While for Bogoliubov transformations, the latter way is indeed more efficient, such a replacement may not exist for other operator transformations. These operator transformations appear in QFT and many–body systems, where a formal Hamiltonian  $H$  may not be well–defined in the infinite–volume limit [41, 63, 95].

In fact, the Fock space extension framework is intended to allow for a definition of more general operator transformations  $W$ , such that for a given formal  $H$ , the “renormalized Hamiltonian”

$$\tilde{H} = W^{-1}(H + c)W \tag{5.3}$$

is well–defined on  $\mathcal{D}_{\mathcal{F}} \subseteq \mathcal{F}$  and allows for a self–adjoint extension, which generates quantum dynamics on  $\mathcal{F}$ . Here,  $c$  is a general counterterm and not necessarily just a constant. Our consideration of  $W = \mathbb{U}_{\mathcal{V}}$  can rather be seen as a proof of

concept for this type of renormalization. In contrast to non-perturbative cutoff renormalization (see Section 1.3), this framework does not require arbitrary cutoffs that break Lorentz invariance. Another advantage is that the Hamiltonian  $\tilde{H}$  can directly be written down without involving limit processes. Further, the amount “by how much  $H$  fails to map  $W[\mathcal{D}_{\mathcal{F}}]$  into itself” may provide useful heuristics for the choice of counterterms in  $c$ . In general, the ESS framework offers a variety of mathematical tools that appear on the way of constructing  $\overline{\mathcal{F}}$  and allow for a rigorous treatment of infinite quantities that appear in formal intermediate calculations. These tools do not appear in the ITP framework. We hope that in the future, further formal transformations  $W$ , which do not exist on Fock space, can be defined in either of both frameworks.

The rest of this chapter is structured as follows: In Section 5.2, we recap some basic definitions of the second quantization language, the ESS framework, and the ITP framework.

Section 5.3 is a recap of known material about implementing Bogoliubov transformations on Fock space which, however, is necessary to establish the notation for an implementation in the extended sense and to understand which formulas are generalized therein.

The extension beyond the Shale–Stinespring condition starts in Section 5.4. Here we define the  $\mathcal{V}$ -dependent Fock space extensions  $\widehat{\mathcal{H}}$ ,  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$ , and prove that creation and annihilation operators are well-defined on them (Lemmas 5.4.7 and 5.4.8).

Section 5.5 then concerns the implementability of  $\mathcal{V}$  in the extended sense, where a precise definition is given what this implementability means. We prove that the implementer  $\mathbb{U}_{\mathcal{V}}$  is well-defined (Lemma 5.5.3), set up conditions for  $\mathbb{U}_{\mathcal{V}}$  to implement  $\mathcal{V}$  (Lemma 5.5.4) and establish these conditions within Theorems 5.5.5–5.5.8 for different cases.

These implementers  $\mathbb{U}_{\mathcal{V}}$  can then be used to diagonalize quadratic Hamiltonians, i.e., to remove the  $a^\dagger a^\dagger$ - and  $aa$ -terms. In Section 5.6, we give a precise definition of what a diagonalization in the extended sense is, and provide conditions for when it can be performed, which is in Propositions 5.6.2 (bosonic case) and 5.6.3 (fermionic case).

Section 5.7 then offers three examples for a diagonalization in the extended sense. Proposition 5.7.1 concerns a bosonic field coupled to an external classical field by a Wick square, Proposition 5.7.2 considers the fermionic BCS model and Proposition 5.7.3 treats a toy model for fermions in a strong electromagnetic field that involves pair creation and annihilation.

We provide some additional material and proofs in Sections 5.8–5.12.

## 5.2 Basic Definitions

### 5.2.1 Fock Space Notions

In this chapter, we use the general formalism of second quantization from Section 1.2.1 based on a single-particle space

$$\mathfrak{h} = L^2(X, \mu), \quad (5.4)$$

with  $(X, \mu)$  being a measure space. We assume  $X \subset \mathbb{R}^d$ , but  $\mu$  does not need to be the Lebesgue measure. Configurations are given by tuples  $q = (\mathbf{x}_1, \dots, \mathbf{x}_N)$ , so we work with an ordered configuration space (1.4):

$$\mathcal{Q}(X) := \bigsqcup_{N \in \mathbb{N}_0} \mathcal{Q}(X)^{(N)} := \bigsqcup_{N \in \mathbb{N}_0} X^N.$$

The measure  $\mu$  induces measures  $\mu_N$  on each sector  $\mathcal{Q}(X)^{(N)}$  and a measure  $\mu_{\mathcal{Q}}$  on  $\mathcal{Q}(X)$ , allowing for a definition of the ordered Fock space (1.5):

$$\mathcal{F}(X) := L^2(\mathcal{Q}(X), \mathbb{C}, \mu_{\mathcal{Q}}).$$

Bosonic/fermionic exchange symmetries will play a key role in Bogoliubov transformations. We impose them by using the symmetrization operators  $S_+, S_- : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$  (1.6). Recall:

$$(S_{\pm} \Psi)(\mathbf{x}_1, \dots, \mathbf{x}_N) := \frac{1}{N!} \sum_{\sigma \in S_N} (\pm 1)^{(1 - \text{sgn}(\sigma))/2} \Psi(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(N)}),$$

which directly leads us to the definition of the bosonic (+) and fermionic (−) Fock space (1.7):

$$\mathcal{F}_{\pm}(X) := S_{\pm}[\mathcal{F}(X)].$$

All three Fock spaces decay into sectors

$$\mathcal{F}_{\bullet} = \bigoplus_{N \in \mathbb{N}_0} \mathcal{F}_{\bullet}(X)^{(N)},$$

with  $\bullet \in \{ \cdot, +, - \}$ .

Also recall from Section 1.2.1 the definition of creation and annihilation operators  $a_{\pm}^{\dagger}, a_{\pm}$  (1.21), which implies the canonical commutation/anticommutation

relations (CCR/CAR) (1.22):

$$[a_{\pm}(f), a_{\pm}^{\dagger}(g)]_{\pm} = \langle f, g \rangle_{\mathfrak{h}}, \quad [a_{\pm}(f), a_{\pm}(g)]_{\pm} = 0 = [a_{\pm}^{\dagger}(f), a_{\pm}^{\dagger}(g)]_{\pm},$$

with  $f, g \in \mathfrak{h}$ , commutator  $[A, B]_{+} = [A, B] = AB - BA$  and anticommutator  $[A, B]_{-} = \{A, B\} = AB + BA$ . The fermionic operators  $a_{-}, a_{-}^{\dagger}$  are bounded, while the bosonic ones  $a_{+}, a_{+}^{\dagger}$  are typically not. As in Section 1.2.1 we will drop the indices  $\pm$  if there is no risk of confusion.

It may turn out advantageous to consider  $a(f), a^{\dagger}(f)$  not as operators, but as formal expressions within a  $*$ -algebra (1.23):

$$\mathcal{A} = \mathcal{A}_{\pm} \quad \text{generated by} \quad \{a_{\pm}(f), a_{\pm}^{\dagger}(f) \mid f \in \mathfrak{h}\}.$$

The involution is given by  $a(f)^{*} = a^{\dagger}(f)$  and the multiplication in  $\mathcal{A}$  is such that the CCR/CAR hold. In particular,  $\mathcal{A}_{-}$  is a  $C^{*}$ -algebra by boundedness of operators.

To simplify calculations, we will make use of a basis choice in this chapter, which identifies the separable one-particle space  $\mathfrak{h}$  with the sequence space  $\ell^2 = L^2(\mathbb{N})$ . We denote the sequence identified with  $f \in \mathfrak{h}$  by a bold symbol  $\mathbf{f} = (f_j)_{j \in \mathbb{N}} \in \ell^2$ . The same basis choice leads to an identification of the Fock space  $\mathcal{F}(X)$  with  $\mathcal{F}(\mathbb{N})$ . Just as in the previous chapters, we will drop the  $(X)$  or  $(\mathbb{N})$ , if not explicitly needed.

## 5.2.2 Extended State Space

The ESS construction follows the general scheme in Section 3.2.2. As announced in Section 3.2.5, we choose  $X = \mathbb{N}$  and define

- the **generalized one-particle space**  $\mathcal{R} = \mathcal{E}(\mathbb{N}) = \{\phi : \mathbb{N} \rightarrow \mathbb{C}\}$ .
- the **generalized  $N$ -particle space**  $\mathcal{R}^{(N)} = \mathcal{E}^{(N)}(\mathbb{N}) = \{\Psi^{(N)} : \mathbb{N}^N \rightarrow \mathbb{C}\}$ .

Note that the product of two functions from  $\mathcal{R}$  is again in  $\mathcal{R}$ , that is,  $\mathcal{R}_2 = \mathcal{R}$ . The ESS construction now provides us with:

- a space  $\mathcal{R}_{\mathcal{F}} = \mathcal{E}_{\mathcal{F}}(\mathbb{N}) := \{\Psi : \mathcal{Q}(\mathbb{N}) \rightarrow \mathbb{C}\}$  of **generalized Fock space functions**.

- a **space of renormalization factors**  $\text{Ren}_1(\mathbb{N}) := \mathcal{E}/\sim_{\text{Ren}_1}$  with  $\mathbf{r}_1 \sim_{\text{Ren}_1} \mathbf{r}_2$  iff  $(\mathbf{r}_1 - \mathbf{r}_2) \in \ell^1 = L^1(\mathbb{N})$  and  $\sum_{j \in \mathbb{N}} (r_{1,j} - r_{2,j}) = 0$ .
- a **space of renormalization polynomials**  $\text{Ren}(\mathbb{N}) = \bigcup_{P \in \mathbb{N}} \text{Ren}_P(\mathbb{N})$ , where  $\text{Ren}_P(\mathbb{N}) := \text{Pol}_P(\mathbb{N})/\sim_{\text{Ren}_P}$ . Here,  $\text{Pol}_P(\mathbb{N})$  is the vector space generated by all commutative products  $\mathfrak{R} = \mathbf{r}_1 \cdot \dots \cdot \mathbf{r}_p$ ,  $p \leq P$ ,  $\mathbf{r}_j \in \text{Ren}_1(\mathbb{N})$ , and  $\sim_{\text{Ren}_P}$  is generated by  $\mathbf{r}_1 \mathbf{r}_2 \dots \mathbf{r}_p \sim_{\text{Ren}_P} c_1 \mathbf{r}_2 \dots \mathbf{r}_p$  and  $(c_1 c_2) \mathbf{r}_1 \dots \mathbf{r}_p \sim_{\text{Ren}_P} c_1 (c_2 \mathbf{r}_1) \dots \mathbf{r}_p$ .
- a **field of wave function renormalizations**  $\text{eRen}(\mathbb{N}) := \{\mathbf{c} = a_1/a_2 \mid a_1, a_2 \in \mathbb{C}[\text{Ren}_1(\mathbb{N})]/\mathcal{I}\}$ , with group algebra  $\mathbb{C}[\text{Ren}_1(\mathbb{N})]$  and an ideal  $\mathcal{I}$  generated by  $e^c e^\mathbf{r} - e^{c+\mathbf{r}}$  with  $c \in \mathbb{C}$ ,  $\mathbf{r} \in \text{Ren}_1$ .
- the **first ESS**  $\overline{\mathcal{F}}(\mathbb{N}) := \overline{\mathcal{F}}_0(\mathbb{N})/\sim_{\text{F}}$ , where  $\overline{\mathcal{F}}_0(\mathbb{N})$  is the free<sup>1</sup> eRen–vector space over  $\mathcal{E}_{\mathcal{F}}$  and  $\sim_{\text{F}}$  is generated by  $(c\mathbf{c})\Psi_m \sim_{\text{F}} \mathbf{c}(c\Psi_m)$ .
- the **second ESS**  $\overline{\mathcal{F}}_{\text{ex}}(\mathbb{N}) := \overline{\mathcal{F}}_{\text{ex},0}(\mathbb{N})/\sim_{\text{Fex}}$ , where  $\overline{\mathcal{F}}_{\text{ex},0}(\mathbb{N})$  is the set of all countable eRen–linear combinations  $\Psi = \sum_{m \in \mathbb{N}} \mathbf{c}_m \Psi_m$  with  $\Psi_m : \mathcal{Q}(\mathbb{N}) \rightarrow \mathfrak{R}$  and  $\sim_{\text{Fex}}$  is generated by  $(c\mathbf{c})\Psi_m \sim_{\text{Fex}} \mathbf{c}(c\Psi_m)$ .

### Remarks.

23. This particular ESS construction via  $\mathcal{E}(\mathbb{N})$  may remind about the non–standard analysis construction of the extended real line  ${}^*\mathbb{R}$  [226]. The extension  ${}^*\mathbb{R}$  of  $\mathbb{R}$  is also defined as a space of equivalence classes of real–valued sequences, just as the extension  $\text{Ren}_1(\mathbb{N})$  of  $\mathbb{C}$ . However, at a closer look, both constructions are quite different:

- Two sequences are equivalent with respect to  ${}^*\mathbb{R}$ , if they agree on a sufficiently large set, and not if their difference is a null series.
- $\mathbb{R}$  is embedded into  ${}^*\mathbb{R}$  by identifying  $r \in \mathbb{R}$  with the constant sequence  $(r_j)_{j \in \mathbb{N}}$ ,  $r_j = r$ , while  $\mathbb{C}$  is embedded into  $\text{Ren}_1(\mathbb{N})$  by identifying  $r \in \mathbb{C}$  with any sequence  $(r_j)_{j \in \mathbb{N}}$  such that  $\sum_j r_j = r$ .
- Moreover,  ${}^*\mathbb{R}$  is a field, while  $\text{Ren}_1(\mathbb{N})$  is not. By contrast,  $\text{eRen}(\mathbb{N})$  is a field, but it is not defined as a set of equivalence classes of real– or complex–valued sequences.

<sup>1</sup>As in Chapter 4, we mean by “free vector space”, that  $\overline{\mathcal{F}}_0(\mathbb{N})$  contains all finite linear combinations  $\Psi = \sum_m \mathbf{c}_m \Psi_m$  with  $\mathbf{c}_m \in \text{eRen}$  and  $\Psi_m \in \mathcal{E}_{\mathcal{F}}$ .

### 5.2.3 Infinite Tensor Products

The ITP space construction in this chapter follows the same general scheme for constructing  $\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k$  as in Section 3.1.1. We use the basis approach from Section 3.1.2. The final spaces  $\widehat{\mathcal{H}}$  will be defined later in Section 5.4.2 after having found suitable bases.

Recall that ITPs were given by equivalence classes of families  $(\Psi) = (\Psi_k)_{k \in I}$ ,  $\Psi_k \in \mathcal{H}_k$ , where  $I$  is a (possibly uncountable) index set and  $\mathcal{H}_k$  are Hilbert spaces. Formally, the scalar product between two ITPs and the norm of an ITP read (3.2)

$$\langle (\Phi), (\Psi) \rangle = \prod_{k \in I} \langle \Phi_k, \Psi_k \rangle_k, \quad \|(\Psi)\| = \prod_{k \in I} \|\Psi_k\|_k.$$

To exclude infinite norms, we restrict the allowed  $(\Psi)$  to the set of so-called *C-sequences*  $(\Psi) \in \text{Cseq} : \Leftrightarrow \prod_{k \in I} \|\Psi_k\|_k < \infty$ . Now, there are still  $(\Psi) \neq 0$  with  $\langle (\Psi), (\Psi) \rangle = 0$ . We enforce positive definiteness by modding out an equivalence relation: With each  $(\Phi) \in \text{Cseq}$ , we associate the conjugate-linear functional

$$\Phi = \iota((\Phi)) : (\Psi) \mapsto \prod_{k \in I} \langle \Phi_k, \Psi_k \rangle_k. \quad (5.5)$$

Modding out the relation  $(\Phi) \sim_C (\Phi') : \Leftrightarrow \iota((\Phi)) = \iota((\Phi'))$ , we obtain a space of equivalence classes with a scalar product given by

$$\prod_{k \in I}^{\otimes} \mathcal{H}_k := \text{span}(\iota[\text{Cseq}]), \quad \langle \Phi, \Psi \rangle = \prod_{k \in I} \langle \Phi_k, \Psi_k \rangle_k, \quad (5.6)$$

where  $\Psi, \Phi$  are the equivalence classes of some  $(\Psi), (\Phi) \in \text{Cseq}$ . The scalar product induces a norm  $\|\cdot\|$ .

The **ITP space** was then defined in Section 3.1.1, (3.8), as the completion

$$\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k := \overline{\prod_{k \in I}^{\otimes} \mathcal{H}_k}^{\|\cdot\|}. \quad (5.7)$$

In this chapter, we will additionally use that  $\widehat{\mathcal{H}}$  can be divided into orthogonal subspaces. We therefore introduce some more notation from [44], which was not given in Section 3.1.1:

On the set of  $C_0$ -sequences, we define the following equivalence relations:

- equivalence:  $(\Phi) \sim (\Psi) \quad :\Leftrightarrow \quad \sum_{k \in I} |\langle \Phi_k, \Psi_k \rangle - 1| < \infty$

- weak equivalence:  $(\Phi) \sim_w (\Psi) \quad :\Leftrightarrow \quad \sum_{k \in I} ||\langle \Phi_k, \Psi_k \rangle| - 1| < \infty$

The proof that  $\sim$  and  $\sim_w$  indeed divide the  $C_0$ -sequences into equivalence classes, called  $C$  and  $C_w$ , is given in [44]. The corresponding linear spaces of an equivalence class are defined by

- $\prod_{k \in I}^{\otimes C} \mathcal{H}_k := \overline{\text{span}\{\Psi \mid \exists(\Psi) \in C : \iota((\Psi)) = \Psi\}}^{\|\cdot\|}$  for equivalence
- $\prod_{k \in I}^{\otimes C_w} \mathcal{H}_k := \overline{\text{span}\{\Psi \mid \exists(\Psi) \in C_w : \iota((\Psi)) = \Psi\}}^{\|\cdot\|}$  for weak equivalence

Now, each  $C_0$ -sequence  $(\Psi)$  in an equivalence class  $[(\Omega)] = C$  (with  $\Omega$  being interpreted as the vacuum vector) can be written in **coordinates** as follows [44, Thm. V]: Choose an orthonormal basis  $(e_{k,n})_{n \in \mathbb{N}_0}$  for each  $\mathcal{H}_k$ , such that  $\Omega_k = e_{k,0}$  (we think of  $e_{k,0}$  as mode  $k$  being in the vacuum). Then,  $(\Psi) = (\Psi_k)_{k \in I}$  is uniquely specified by stating the coordinates  $c_{k,n} := \langle e_{k,n}, \Psi_k \rangle_k \in \mathbb{C}$ . In this coordinate representation, it is true that

- $\prod_{k \in I}^{\otimes C} \mathcal{H}_k$  is the closure of the space spanned by all normalized  $C_0$ -sequences, where  $c_{k,0} = 1$  for all but finitely many  $k \in I$ .

Or heuristically speaking, “almost all  $\Psi_k$  are in the vacuum”.

By [44, Thm. V], also a generic  $\Psi \in \prod_{k \in I}^{\otimes C} \mathcal{H}_k$  can be written as

$$\Psi = \sum_{n(\cdot) \in F} a(n(\cdot)) \prod_{k \in I}^{\otimes} e_{k,n(k)}, \quad (5.8)$$

with  $F$  being the countable set of all functions  $n : I \rightarrow \mathbb{N}_0$  with  $n(k) = 0$  for almost all  $k \in I$ , and  $a(n(\cdot)) \in \mathbb{C}$  being the coordinates of  $\Psi$  with  $\sum_{n(\cdot) \in F} |a(n(\cdot))|^2 < \infty$ .

## 5.3 Bogoliubov Transformations

In this section, we introduce our notation for Bogoliubov transformations and recap some important properties. In the literature, there exist several representations of Bogoliubov transformations (which are elements of a symplectic group) as linear operators on subspaces  $W_{1,j}$ , that are isomorphic to the one-operator subspace  $W_1$  of the algebra  $\mathcal{A}$ . We present three such choices for  $W_{1,j}$  in Section 5.9.1, state the rules how to change between representations, and then finally fix the representation from Section 5.9.4, which is used in Section 5.3.1 and thereafter. In Section 5.3.2, we recap the well-known implementation process in case when the Shale–Stinespring condition is valid. Standard references on the subject are [35] and [227].

### 5.3.1 Transformation on Operators

Consider the one-operator subspace  $W_1$  of  $\mathcal{A}$ , which is spanned by all  $a_{\pm}^{\dagger}(f), a_{\pm}(g)$ . By an algebraic Bogoliubov transformation, we mean any bijective map  $\mathcal{V}_A : W_1 \rightarrow W_1$ , which sends  $a_{\pm}^{\dagger}(f), a_{\pm}(g)$  to a new set of creation and annihilation operators  $b_{\pm}^{\dagger}(f), b_{\pm}(g)$ , such that  $b_{\pm}^{\dagger}(f)$  is the adjoint of  $b_{\pm}(f)$  and the CAR/CCR are conserved under  $\mathcal{V}$ . Further, the adjoint  $\mathcal{V}^*$ , defined in Section 5.9, is also required to conserve the CAR/CCR.

The representation we use is the following: fix a basis  $(e_j)_{j \in \mathbb{N}} \subset \mathfrak{h}$ . Then, every  $f \in \mathfrak{h}$  can then be expressed by its coordinates  $f_j := \langle e_j, f \rangle$ . This way, we may identify  $f \in \mathfrak{h}$  with an equally denoted vector  $\mathbf{f} = (f_j)_{j \in \mathbb{N}} \in \ell^2$ . Further, we can identify  $a^{\dagger}(f) + a(\bar{g}) \in W_1$  with a vector  $(\mathbf{f}, \mathbf{g}) \in \ell^2 \oplus \ell^2$ , so on algebraic Bogoliubov transformations  $\mathcal{V}_A$  amounts to a linear operator  $\mathcal{V} = \mathcal{V}_3$  on  $\ell^2 \oplus \ell^2$ , which we just call “Bogoliubov transformation”.

We now encode sums of creation and annihilation operators by vector pairs  $(\mathbf{f}, \mathbf{g}) \in \ell^2 \oplus \ell^2$ . This encoding is realized by the generalized creation/annihilation operators

$$\begin{aligned} A_{\pm}^{\dagger} : \ell^2 \oplus \ell^2 &\rightarrow \mathcal{A}_{\pm}, & (\mathbf{f}_1, \mathbf{f}_2) &\mapsto a_{\pm}^{\dagger}(\mathbf{f}_1) + a_{\pm}(\overline{\mathbf{f}_2}) = \sum_j (f_{1,j} a_{\pm}^{\dagger}(e_j) + f_{2,j} a_{\pm}(e_j)) \\ A_{\pm} : \ell^2 \oplus \ell^2 &\rightarrow \mathcal{A}_{\pm}, & (\mathbf{g}_1, \mathbf{g}_2) &\mapsto a_{\pm}(\mathbf{g}_1) + a_{\pm}^{\dagger}(\overline{\mathbf{g}_2}) = \sum_j (\overline{g_{1,j}} a_{\pm}(e_j) + \overline{g_{2,j}} a_{\pm}^{\dagger}(e_j)). \end{aligned} \tag{5.9}$$

A Bogoliubov transformation can then be encoded by a  $2 \times 2$  block matrix

$$\mathcal{V} = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}, \tag{5.10}$$

with operators  $u, v : \ell^2 \rightarrow \ell^2$ . The case of unbounded  $u, v$  is treated later in Section 5.4.

The Bogoliubov transformed operators are then given by

$$\begin{aligned} b_{\pm}^{\dagger}(\mathbf{f}) &= A_{\pm}^{\dagger}(\mathcal{V}(\mathbf{f}, 0)) = a_{\pm}^{\dagger}(u\mathbf{f}) + a_{\pm}(v\overline{\mathbf{f}}) \\ b_{\pm}(\mathbf{g}) &= A_{\pm}(\mathcal{V}(\mathbf{g}, 0)) = a_{\pm}^{\dagger}(v\overline{\mathbf{g}}) + a_{\pm}(u\mathbf{g}). \end{aligned} \tag{5.11}$$

In order for  $\mathcal{V}$  to be a Bogoliubov transformation, we require that both  $\mathcal{V}$  and

$\mathcal{V}^*$  conserve the CAR/CCR, so

$$[b_{\pm}(\mathbf{f}), b_{\pm}^{\dagger}(\mathbf{g})]_{\pm} = \langle \mathbf{f}, \mathbf{g} \rangle, \quad [b_{\pm}(\mathbf{f}), b_{\pm}(\mathbf{g})]_{\pm} = 0 = [b_{\pm}^{\dagger}(\mathbf{f}), b_{\pm}^{\dagger}(\mathbf{g})]_{\pm}, \quad (5.12)$$

and the same, if in (5.11)  $\mathcal{V}$  is replaced by  $\mathcal{V}^*$ . An explicit calculation (see Section 5.9) shows that this conservation is equivalent to 4 conditions on  $u$  and  $v$ . To express them, we define the transpose, complex conjugate and adjoint as  $(u^T)_{ij} = u_{ji}$ ,  $(\bar{u})_{ij} = \overline{u_{ij}}$ ,  $(u^*)_{ij} = \overline{u_{ji}}$  and the same for  $v_{ij}$ . The 4 Bogoliubov relations then read as

$$\begin{aligned} u^*u \mp v^T\bar{v} &= 1 & u^*v \mp v^T\bar{u} &= 0 \\ uu^* \mp vv^* &= 1 & uv^T \mp vu^T &= 0. \end{aligned} \quad (5.13)$$

Since  $v^*v$  is self-adjoint, we have  $v^*v = \overline{v^*v} = v^T\bar{v}$ , so the first equation can equivalently be written as  $u^*u \mp v^*v = 1$ .

The generalized creation and annihilation operators also allow for particularly easy “generalized CAR/CCR”: Using the standard scalar product on  $\mathbf{F}, \mathbf{G} \in \ell^2 \oplus \ell^2$ :

$$\langle \mathbf{F}, \mathbf{G} \rangle = \left\langle \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix} \right\rangle = \sum_j (\overline{f_{1,j}}g_{1,j} + \overline{f_{2,j}}g_{2,j}), \quad (5.14)$$

we obtain the generalized CAR/CCR:

$$\begin{aligned} [A_{\pm}(\mathbf{F}), A_{\pm}^{\dagger}(\mathbf{G})]_{\pm} &= \langle \mathbf{F}, \mathcal{S}_{\pm} \mathbf{G} \rangle, \\ [A_{\pm}(\mathbf{F}), A_{\pm}(\mathbf{G})]_{\pm} &= [A_{\pm}^{\dagger}(\mathbf{F}), A_{\pm}^{\dagger}(\mathbf{G})]_{\pm} = 0, \end{aligned} \quad (5.15)$$

where  $\mathcal{S}_- = id$  and  $\mathcal{S}_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

### 5.3.2 Implementation on Fock Space

In the following, we will drop the index  $\pm$  for bosonic/fermionic, as both cases are separately considered. A Bogoliubov transformation is called implementable (in the regular sense), if there exists a unitary operator  $\mathbb{U}_{\mathcal{V}} : \mathcal{F} \rightarrow \mathcal{F}$ , such that

$$\mathbb{U}_{\mathcal{V}} A^{\dagger}(\mathbf{F}) \mathbb{U}_{\mathcal{V}}^* = A^{\dagger}(\mathcal{V}\mathbf{F}). \quad (5.16)$$

It is well-known that a Bogoliubov transformation is implementable, if and only if the **Shale–Stinespring condition** holds. That is,  $\text{tr}(v^*v) < \infty$ , so  $v$  is Hilbert–Schmidt [220]. We recap some of the basic steps of the implementation process presented in [35], as some of those steps will have to be carried out in a slightly modified way for an implementation in the extended sense.

## 5 Implementing Bogoliubov Transformations Beyond the Shale–Stinespring Condition

The main task within the implementation is to find the “Bogoliubov transformed vacuum”  $\Omega_{\mathcal{V}} \in \mathcal{F}$ , which is the vector annihilated by all operators  $b(\mathbf{f})$ :

$$b(\mathbf{f})\Omega_{\mathcal{V}} = (a^\dagger(u\mathbf{f}) + a(v\overline{\mathbf{f}}))\Omega_{\mathcal{V}} = 0 \quad \forall \mathbf{f} \in \ell^2. \quad (5.17)$$

If we can find such an  $\Omega_{\mathcal{V}}$ , then it is an easy task to transform any product state vector  $a^\dagger(\mathbf{f}_1) \dots a^\dagger(\mathbf{f}_n)\Omega \in \mathcal{F}$  via:

$$\begin{aligned} \mathbb{U}_{\mathcal{V}} a^\dagger(\mathbf{f}_1) \dots a^\dagger(\mathbf{f}_n)\Omega &= \underbrace{\mathbb{U}_{\mathcal{V}} a^\dagger(\mathbf{f}_1) \mathbb{U}_{\mathcal{V}}^*}_{b^\dagger(\mathbf{f}_1)} \mathbb{U}_{\mathcal{V}} \dots \mathbb{U}_{\mathcal{V}}^* \underbrace{\mathbb{U}_{\mathcal{V}} a^\dagger(\mathbf{f}_n) \mathbb{U}_{\mathcal{V}}^*}_{b^\dagger(\mathbf{f}_n)} \underbrace{\mathbb{U}_{\mathcal{V}} \Omega}_{\Omega_{\mathcal{V}}} \\ &= b^\dagger(\mathbf{f}_1) \dots b^\dagger(\mathbf{f}_n)\Omega_{\mathcal{V}}. \end{aligned} \quad (5.18)$$

The span of these state vectors (also called algebraic tensor product) is dense within  $\mathcal{F}$ , so we can transform any  $\Psi \in \mathcal{F}$  by means of (5.18).

It remains to find the Bogoliubov transformed vacuum  $\Omega_{\mathcal{V}}$ , or called *Bogoliubov vacuum*. For only one mode, i.e., with  $\ell^2$  replaced by  $\mathbb{C}$ , this would be an easier task, as  $u$  and  $v$  are just constants in that case. So we aim for “decomposing the transformation  $\mathcal{V}$  into modes”. More precisely, we seek vectors  $\mathbf{f}_j$ , such the Bogoliubov–transformed form factors  $u\mathbf{f}_j, v\overline{\mathbf{f}_j}$  are proportional to the same normalized vector  $\mathbf{g}_j \in \ell^2$ , i.e.

$$\mathcal{V} \begin{pmatrix} \mathbf{f}_j \\ 0 \end{pmatrix} = \begin{pmatrix} u\mathbf{f}_j \\ v\overline{\mathbf{f}_j} \end{pmatrix} = \begin{pmatrix} \mu_j \mathbf{g}_j \\ \nu_j \overline{\mathbf{g}_j} \end{pmatrix} \quad \Leftrightarrow \quad \mathbb{U}_{\mathcal{V}} a^\dagger(\mathbf{f}_j) \mathbb{U}_{\mathcal{V}}^* = \mu_j a^\dagger(\mathbf{g}_j) + \nu_j a(\mathbf{g}_j). \quad (5.19)$$

The constants  $\mu_j, \nu_j \in \mathbb{C}$  intuitively describe, how much of  $a^\dagger$  “stays”  $a^\dagger$  (this is  $\mu_j$ ) or “leaves”  $a^\dagger$  to become  $a$  (this is  $\nu_j$ ). If (5.19) holds, we only have to solve  $(\nu_j a^\dagger(\mathbf{g}_j) + \mu_j a(\mathbf{g}_j))\Omega_{\mathcal{V}}$  for each mode  $\mathbf{g}_j$ , separately.

### Bosonic Case

Here, (5.19) can indeed be fulfilled. Following [35], we define a suitable operator  $C$ , such that  $C\mathbf{f}_j = \lambda_j \mathbf{f}_j$ ,  $\lambda_j \in \mathbb{C}$ . We set  $\mathcal{V}(\mathbf{f}_j, 0) = (u\mathbf{f}_j, v\overline{\mathbf{f}_j}) =: (\tilde{\mathbf{g}}_j, \mathcal{J}\tilde{\mathbf{h}}_j)$  for any  $\mathbf{f}_j \in \ell^2$  with  $\mathcal{J}\tilde{\mathbf{h}}_j = \mathcal{J}^* \tilde{\mathbf{h}}_j = \overline{\tilde{\mathbf{h}}_j}$  denoting complex conjugation. Then, we can translate (5.19) into  $\tilde{\mathbf{g}}_j$  and  $\tilde{\mathbf{h}}_j$  being parallel. Now,

$$\langle \tilde{\mathbf{g}}_j, \tilde{\mathbf{h}}_{j'} \rangle = \langle u\mathbf{f}_j, v\overline{\mathbf{f}_{j'}} \rangle = \langle \mathbf{f}_j, u^* v \mathcal{J} \mathbf{f}_{j'} \rangle. \quad (5.20)$$

So the choice

$$C = u^* v \mathcal{J} \quad (5.21)$$

would lead to

$$\langle \tilde{\mathbf{g}}_j, \tilde{\mathbf{h}}_{j'} \rangle = \lambda_{j'} \langle \mathbf{f}_j, \mathbf{f}_{j'} \rangle = \lambda_j \delta_{jj'}, \quad (5.22)$$

if  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  was an eigenbasis of  $C$ . Now, by means of (5.13),  $C$  is Hermitian ( $C = u^* v \mathcal{J} = v^T \bar{u} \mathcal{J} = \mathcal{J} v^* u = C^*$ ). Since by the Shale–Stinespring condition,  $v$  is Hilbert–Schmidt and  $u$  is bounded, the operator  $C$  is also Hilbert–Schmidt, so we can indeed find an orthonormal basis of eigenvectors  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  with  $C \mathbf{f}_j = \lambda_j \mathbf{f}_j$ . In order to obtain from (5.22) that  $\tilde{\mathbf{g}}_j$  and  $\tilde{\mathbf{h}}_j$  are parallel, we still need to show that the  $\tilde{\mathbf{g}}_j$  provide an orthogonal basis of  $\ell^2$

One easily verifies  $[u^* u, C] = 0$ , so  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  can, without restrictions, be chosen to be a simultaneous eigenbasis of  $C$  and  $u^* u$ . So if  $\mu_j^2$  are the corresponding eigenvalues of  $u^* u$ , then

$$\langle \tilde{\mathbf{g}}_j, \tilde{\mathbf{g}}_{j'} \rangle = \langle \mathbf{f}_j, u^* u \mathbf{f}_{j'} \rangle = \mu_{j'}^2 \langle \mathbf{f}_j, \mathbf{f}_{j'} \rangle = \mu_j^2 \delta_{jj'}, \quad (5.23)$$

meaning that the  $\tilde{\mathbf{g}}_j$  are orthogonal and by invertibility of  $u$  (which follows from  $\text{tr}(v^* v) < \infty$ ),  $(\tilde{\mathbf{g}}_j)_{j \in \mathbb{N}}$  is a basis. Hence, (5.19) is valid. By an appropriate choice of the complex phase of  $\mathbf{g}_j$ , we can further make  $\mu_j$  real and positive.

We even have that

$$(\mathbf{g}_j)_{j \in \mathbb{N}} := \left( \frac{1}{\mu_j} \tilde{\mathbf{g}}_j \right)_{j \in \mathbb{N}} \quad (5.24)$$

is an orthonormal basis and (5.22) then implies that  $\tilde{\mathbf{h}}_j$  is indeed proportional to  $\mathbf{g}_j$ . So  $\tilde{\mathbf{h}}_j = \nu_j \mathbf{g}_j$  for some  $\nu_j \geq 0$  and with that choice of  $\mathbf{g}_j, \mu_j, \nu_j$ , indeed (5.19) holds true. Hence, for  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  being an eigenbasis of  $C$ , we have the following important formulas:

$$u \mathbf{f}_j = \mu_j \mathbf{g}_j \quad v \overline{\mathbf{f}}_j = v \mathcal{J} \mathbf{f}_j = \nu_j \mathbf{g}_j. \quad (5.25)$$

By phase rotation, we can again enforce that  $\mu_j = |\mu_j|$ . Further,

$$\lambda_j \langle \mathbf{f}_i, \mathbf{f}_j \rangle = \langle \mathbf{f}_i, C \mathbf{f}_j \rangle = \langle \mathbf{f}_i, u^* v \mathcal{J} \mathbf{f}_j \rangle = \nu_j \mu_j \langle \mathbf{g}_i, \mathbf{g}_j \rangle \quad \Rightarrow \quad \lambda_j = \mu_j \nu_j, \quad (5.26)$$

so  $\nu_j$  is real and by (5.13), we have

$$u^* u - v^* v = 1 \quad \Rightarrow \quad \mu_j^2 - \nu_j^2 = 1. \quad (5.27)$$

The Bogoliubov vacuum is determined by the condition  $(\mu_j a(\mathbf{g}_j) + \nu_j a^\dagger(\mathbf{g}_j)) \Omega_{\mathcal{V}} = 0$  for all modes  $\mathbf{g}_j, j \in \mathbb{N}$ . This condition leads to one recursion relation per mode,

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which is formally solved by

$$\Omega_{\mathcal{V}} = \left( \prod_j \left( 1 - \frac{\nu_j^2}{\mu_j^2} \right)^{1/4} \right) \exp \left( - \sum_j \frac{\nu_j}{2\mu_j} (a^\dagger(\mathbf{g}_j))^2 \right) \Omega, \quad (5.28)$$

with the prefactor coming from normalization. The Shale–Stinespring condition now indicates when  $\Omega_{\mathcal{V}}$  lies in Fock space. Investigating the two–particle sector, we obtain

$$\Omega_{\mathcal{V}}^{(2)} = - \left( \prod_j \left( 1 - \frac{\nu_j^2}{\mu_j^2} \right)^{1/4} \right) \sum_j \frac{\nu_j}{2\mu_j} \mathbf{g}_j \otimes \mathbf{g}_j. \quad (5.29)$$

The prefactor in form of an infinite product is nonequal 0, if and only if  $\sum_j \frac{\nu_j^2}{\mu_j^2} < \infty$ . And by orthogonality of  $(\mathbf{g}_j)_{j \in \mathbb{N}}$ , the sum converges in  $L^2$ –norm under the very same condition. This condition is equivalent to  $\sum_j \nu_j^2 < \infty$ :

Since  $\mu_j^2 - \nu_j^2 = 1$ , we have  $\mu_j \geq 1$ , so  $\sum_j \nu_j^2 < \infty$  implies  $\sum_j \frac{\nu_j^2}{\mu_j^2} < \infty$ .

Conversely,  $\sum_j \frac{\nu_j^2}{\mu_j^2} < \infty$  means that  $\frac{\nu_j^2}{\mu_j^2}$  is a null sequence, so it eventually drops below any  $\varepsilon > 0$ . Now, since  $\mu_j^2 \rightarrow 1$ , as  $\frac{\nu_j^2}{\mu_j^2} \rightarrow 0$ , we have  $\mu_j < 2$  for almost all  $j \in \mathbb{N}$  and hence  $\sum_j \nu_j^2 < 4 \sum_j \frac{\nu_j^2}{\mu_j^2} + \text{const.}$ , which is also convergent.

So the Shale–Stinespring condition indeed tells us, when exactly the formal expression (5.28) makes sense as a Fock space vector. Note that we exploited  $v$  being Hilbert Schmidt to arrive at (5.28) in the first place, namely when finding an orthonormal basis  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  for  $C = u^*v\mathcal{J}$ . So a proof of necessity of the Shale–Stinespring condition for implementability requires some further thoughts, see [35, Section F]

In case of implementability, the transformation is implemented by [218, (3.1)]:

$$\mathbb{U}_{\mathcal{V}} = \exp \left( - \sum_j \frac{\xi_j}{2} ((a^\dagger(\mathbf{g}_j))^2 - (a(\mathbf{g}_j))^2) \right) \mathbb{U}_{\mathbf{g}\mathbf{f}} =: \prod_{j \in \mathbb{N}} \mathbb{U}_{j,\mathcal{V}}, \quad (5.30)$$

$$\text{with} \quad \sinh \xi_j := \nu_j \quad \Rightarrow \quad \cosh \xi_j := \mu_j, \quad (5.31)$$

and where  $\mathbb{U}_{\mathbf{g}\mathbf{f}} : \mathcal{F} \rightarrow \mathcal{F}$  is the unitary transformation which changes the basis  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  to  $(\mathbf{g}_j)_{j \in \mathbb{N}}$  via

$$\mathbb{U}_{\mathbf{g}\mathbf{f}} : \mathbf{f}_{j_1} \otimes \dots \otimes \mathbf{f}_{j_n} \mapsto \mathbf{g}_{j_1} \otimes \dots \otimes \mathbf{g}_{j_n}, \quad \forall j_1, \dots, j_n \in \mathbb{N}. \quad (5.32)$$

Note that the presign of  $\pm\xi_j$  in the literature depends on whether the Bogoliubov transformation is defined as  $\mathbb{U}_\nu a^\# \mathbb{U}_\nu^*$  (as in our case) or  $\mathbb{U}_\nu^* a^\# \mathbb{U}_\nu$  with  $\# \in (\cdot, *)$ . For a more general discussion about the implementer  $\mathbb{U}_\nu$ , we refer the reader to [228, Thm. 16.47].

### Fermionic Case

In the fermionic case, (5.19) cannot be fulfilled:  $C = u^*v\mathcal{J}$  is no longer Hermitian, so we cannot expect  $C$  to have an orthonormal eigenbasis. However,  $C^*C$  is Hermitian. By (5.13),  $C^*C = v^*v - (v^*v)^2$ , so  $C^*C$  is trace class and  $[u^*u, C] = 0$  implies  $[u^*u, C^*C] = 0$ . Thus, there exists a common orthonormal eigenbasis  $(\mathbf{f}_j)_{j \in J}$  of  $C^*C$  (eigenvalues  $\lambda_j^2$ ) and  $u^*u$  (eigenvalues  $\mu_j^2$ ). As  $u^*u + v^*v = u^*u + v^T\bar{v} = 1$ , the  $\mathbf{f}_j$  are also eigenvectors of  $v^*v$  (eigenvalues  $\nu_j^2 = 1 - \mu_j^2$ ). The index set  $J$  is assumed to be countable and specified below.

In particular, we can arrange the eigenvectors  $\mathbf{f}_j$  with  $\lambda_j \neq 0$  in pairs

$$C\mathbf{f}_{2i} = \lambda_{2i}\mathbf{f}_{2i-1}, \quad C\mathbf{f}_{2i-1} = -\lambda_{2i}\mathbf{f}_{2i}, \quad (5.33)$$

where  $i$  is an element of a countable index set  $I' \subseteq \mathbb{N}$  and the eigenvector indices are in  $J' := \{j \mid j = 2i \vee j = 2i - 1, i \in I'\}$ . The eigenvectors with  $\lambda_j = 0$  will be denoted by  $\mathbf{f}_j$  with index set  $J'' \subseteq \mathbb{N}$ ,  $J' \cap J'' = \emptyset$ . The set of all used indices is then  $J = J' \cup J'' \subseteq \mathbb{N}$  and  $(\mathbf{f}_j)_{j \in J}$  is an orthonormal basis of  $\ell^2$ .

Because of the pairing, the splitting into modes (5.19) with an orthonormal basis  $(\mathbf{g}_j)_{j \in \mathbb{N}}$  can no longer be achieved. Instead, we can obtain a splitting into modes  $j \in J''$  and pairs  $i \in I'$  using an orthonormal basis  $(\boldsymbol{\eta}_j)_{j \in J}$ . Again, we define  $(\tilde{\mathbf{g}}_j, \tilde{\mathbf{h}}_j) := (u\mathbf{f}_j, v\overline{\mathbf{f}}_j)$ .

The case  $j \in J''$  still allows for a split into single modes. It consists of 2 subcases: Since by (5.13),  $C^*C = u^*u(1 - u^*u)$ , we have that  $\lambda_j^2 = \mu_j^2(1 - \mu_j^2)$ , so  $\mu_j = 1$  or  $\mu_j = 0$ . In case  $\mu_j = 0$ , we have by  $u^*u + v^T\bar{v} = 1$ , that  $\nu_j = 1$ . We denote that case by  $j \in J''_1$  and get

$$\mathcal{V} \begin{pmatrix} \mathbf{f}_j \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{\eta}_j \end{pmatrix}, \quad j \in J''_1 \quad (5.34)$$

for a suitable choice of the phase  $\boldsymbol{\eta}_j = e^{i\varphi}\tilde{\mathbf{h}}_j = e^{i\varphi}v\overline{\mathbf{f}}_j$ . The case  $\mu_j = 1$  will be

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denoted by  $j \in J_0'' = J'' \setminus J_1''$ . Here,  $\nu_j = 0$ , so

$$\mathcal{V} \begin{pmatrix} \mathbf{f}_j \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\eta}_j \\ 0 \end{pmatrix}, \quad j \in J_0'' \quad (5.35)$$

for a suitable phase choice of  $\boldsymbol{\eta}_j = e^{i\varphi} \tilde{\mathbf{g}}_j = e^{i\varphi} u \mathbf{f}_j$ .

In case  $i \in I' \Rightarrow \lambda_{2i} \neq 0, \mu_{2i} \neq 0$ , we define the normalized vectors

$$\boldsymbol{\eta}_{2i} := \alpha_i^{-1} \tilde{\mathbf{g}}_{2i}, \quad \boldsymbol{\eta}_{2i-1} := \alpha_i^{-1} \tilde{\mathbf{g}}_{2i-1}, \quad (5.36)$$

where  $\alpha_i = \mu_{2i} = \mu_{2i-1}$  can be shown within a Bogoliubov pair  $i$ . Now,  $\tilde{\mathbf{h}}_j$  is not proportional to  $\boldsymbol{\eta}_j$ , but one can show that

$$\mathcal{V} \begin{pmatrix} \mathbf{f}_{2i} \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_i \boldsymbol{\eta}_{2i} \\ \beta_i \boldsymbol{\eta}_{2i-1} \end{pmatrix}, \quad \mathcal{V} \begin{pmatrix} \mathbf{f}_{2i-1} \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_i \boldsymbol{\eta}_{2i-1} \\ -\beta_i \boldsymbol{\eta}_{2i} \end{pmatrix}, \quad i \in I', \quad (5.37)$$

where  $\beta_i \in \mathbb{R}$ ,  $\beta_i > 0$  is such that  $\alpha_i^2 + \beta_i^2 = 1$ . That means,  $\tilde{\mathbf{h}}_{2i}$  is proportional to  $\tilde{\mathbf{g}}_{2i-1}$ , which follows by an orthogonality argument, as in (5.23). The argument is based on the fact that  $(\boldsymbol{\eta}_j)_{j \in J}$  is an orthonormal basis. For proof details, see [35, (68)–(69)].

Relations (5.34), (5.35) and (5.37) now replace (5.19). Their implementation is a bit easier than in the bosonic case, since by the Pauli exclusion principle, all modes can be occupied by at most one fermion, i.e., the Fock space per mode is  $\cong \mathbb{C}^2$ .

In the case of (5.34) and (5.35), the condition  $A(\mathbf{f}_j, 0)\Omega_{\mathcal{V}} = 0$  is easily fulfilled. If  $j \in J_0''$ , then  $a(\boldsymbol{\eta}_j)\Omega_{\mathcal{V}} = 0$ , so the  $\boldsymbol{\eta}_j$ -mode is empty. If  $j \in J_1''$ , then  $a^\dagger(\boldsymbol{\eta}_j)\Omega_{\mathcal{V}} = 0$ , so the  $\boldsymbol{\eta}_j$ -mode is fully occupied.

For (5.37), the condition  $A(\mathbf{f}_i, 0)\Omega_{\mathcal{V}} = 0$  determines  $\Omega_{\mathcal{V}}$  on each two-mode subspace of  $\mathcal{F}$ . On the 4-dimensional subspace belonging to  $\boldsymbol{\eta}_{2i}, \boldsymbol{\eta}_{2i-1}$  for some  $i \in I'$ , we have a superposition of both modes being empty with amplitude  $\alpha_i$  and a ‘‘Cooper pair’’, where both modes are filled with amplitude  $-\beta_i$ :

$$\Omega_{\mathcal{V}} = \left( \prod_{j \in J_1''} a^\dagger(\boldsymbol{\eta}_j) \right) \left( \prod_{i \in I'} (\alpha_i - \beta_i a^\dagger(\boldsymbol{\eta}_{2i}) a^\dagger(\boldsymbol{\eta}_{2i-1})) \right) \Omega. \quad (5.38)$$

The corresponding implementer  $\mathbb{U}_{\mathcal{V}}$  is given by

$$\begin{aligned} \mathbb{U}_{\mathcal{V}} &= \left( \prod_{j \in J_1''} (a^\dagger(\boldsymbol{\eta}_j) + a(\boldsymbol{\eta}_j)) \right) \exp \left( - \sum_{i \in I'} \xi_i (a^\dagger(\boldsymbol{\eta}_{2i}) a^\dagger(\boldsymbol{\eta}_{2i-1}) - a(\boldsymbol{\eta}_{2i-1}) a(\boldsymbol{\eta}_{2i})) \right) \mathbb{U}_{\boldsymbol{\eta}\boldsymbol{f}} \\ &=: \left( \prod_{j \in J_0''} \mathbb{1} \right) \left( \prod_{j \in J_1''} \mathbb{U}_{j,\mathcal{V}} \right) \left( \prod_{i \in I'} \mathbb{U}_{2i,2i-1,\mathcal{V}} \right) \mathbb{U}_{\boldsymbol{\eta}\boldsymbol{f}} \\ &=: \left( \prod_{j \in J''} \mathbb{U}_{j,\mathcal{V}} \right) \left( \prod_{i \in I'} \mathbb{U}_{2i,2i-1,\mathcal{V}} \right) \mathbb{U}_{\boldsymbol{\eta}\boldsymbol{f}}, \end{aligned} \tag{5.39}$$

$$\text{with} \quad \sin \xi_i := \beta_i \quad \Rightarrow \quad \cos \xi_i := \alpha_i, \tag{5.40}$$

and where  $\mathbb{U}_{\boldsymbol{\eta}\boldsymbol{f}}$  is the unitary transformation changing the basis  $(\boldsymbol{f}_j)_{j \in J}$  to  $(\boldsymbol{\eta}_j)_{j \in J}$  via

$$\mathbb{U}_{\boldsymbol{\eta}\boldsymbol{f}} : \boldsymbol{f}_{j_1} \otimes \dots \otimes \boldsymbol{f}_{j_n} \mapsto \boldsymbol{\eta}_{j_1} \otimes \dots \otimes \boldsymbol{\eta}_{j_n}, \quad \forall j_1, \dots, j_n \in J. \tag{5.41}$$

A proof for (5.39) implementing  $\mathcal{V}$  can be found in Section 5.10. For a general discussion of the implementer  $\mathbb{U}_{\mathcal{V}}$ , we refer the reader to [228, Thm. 16.47].

## 5.4 Bogoliubov Transformations: Extended

Now, consider the case where  $v$  is no longer Hilbert–Schmidt and (in the bosonic case) possibly not even bounded. So it is only defined on some domain  $\text{dom}(v) \subseteq \ell^2$ . Our aim in Section 5.4.1 is to show that the Bogoliubov relations (5.13) also hold in this case. So it makes sense to encode a Bogoliubov transformation in a block matrix  $\mathcal{V}$ .

In Section 5.4.2, we define an extended  $*$ -algebra  $\overline{\mathcal{A}}_e$  of creation and annihilation operator products, and give the precise definitions of  $\widehat{\mathcal{H}}$ ,  $\widehat{\mathcal{F}}$  and  $\widehat{\mathcal{F}}_{\text{ex}}$  with respect to a given Bogoliubov transformation  $\mathcal{V}$ . Lemmas 5.4.7 and 5.4.8 then establish that certain elements of  $\overline{\mathcal{A}}_e$  naturally define operators on suitable subspaces of  $\widehat{\mathcal{H}}$  and  $\widehat{\mathcal{F}}$ .

### 5.4.1 Extension of the Bogoliubov Relations

Throughout the following construction, we will assume that  $v^*v$  is densely defined and self-adjoint. In that case, the spectral theorem applies, and we can define the self-adjoint operators  $C^*C := v^*v(1 \pm v^*v)$  and  $|C| = \sqrt{C^*C}$  by spectral calculus.

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It will turn out convenient to work with spectral resolutions with respect to  $|C|$ . By the spectral theorem in the form of [24, Thm. 10.9], we may then decompose the Hilbert space  $\ell^2$  as a direct integral

$$\ell^2 = \int_{\sigma(|C|)}^{\oplus} \mathbb{C}^n d\mu_1(\lambda), \quad (5.42)$$

where  $\sigma(|C|) = \sigma$  is the spectrum of  $|C|$ ,  $\mu_1$  a suitable measure on it and  $n : \sigma \rightarrow \mathbb{N} \cup \{\infty\}$  [24, Def. 7.18] is a measurable dimension function (with  $\mathbb{C}^\infty \cong \ell^2$ ). Put differently, as visualized in Figure 5.1, we can find a spectral set

$$X = \bigcup_{\lambda \in \sigma} \{\lambda\} \times Y_\lambda \subset \mathbb{R}^2, \quad (5.43)$$

with  $Y_\lambda \subseteq \mathbb{Z}$ ,  $|Y_\lambda| = n(\lambda)$  accounting for multiplicity and unitary maps<sup>2</sup>

$$U_{X\mathbf{f}} : \ell^2 \rightarrow L^2(X), \quad U_{\mathbf{f}X} = U_{X\mathbf{f}}^{-1}, \quad (5.44)$$

such that

$$|C| = U_{\mathbf{f}X} \lambda U_{X\mathbf{f}}, \quad (5.45)$$

with  $\lambda$  being the operator on  $L^2(X)$  that multiplies by  $\lambda(x)$ . In addition, we denote  $Y = \bigcup_{\lambda \in \sigma} Y_\lambda \subseteq \mathbb{Z}$  with  $|Y|$  being an upper bound for the multiplicity of any eigenvalue. Note that the  $\lambda$  here correspond to the  $\lambda_j$  in Section 5.3.2.

We also make use of the formulation [24, Thm. 10.4] of the spectral theorem, which provides us with a projection-valued measure  $P_{|C|}$ , such that

$$|C| = \int_X \lambda(x) dP_{|C|}(x) = \int_{\sigma \times Y} \lambda dP_{|C|}(\lambda, y). \quad (5.46)$$

For countable spectrum  $\sigma$ , the entire set  $X$  is countable, so we can index it by  $j \in \mathbb{N}$  and have a corresponding eigenbasis  $(\mathbf{f}_j)_{j \in \mathbb{N}} \subset \ell^2$ . This allows for resolving any  $\phi \in \ell^2$  as  $\phi = \sum_j \phi_j \mathbf{f}_j$ ,  $\phi_j \in \mathbb{C}$ . For the moment, we assume that the spectrum of  $|C|$  is arbitrary, but starting from Section 5.4.2, only countable spectra of  $|C|$  will be considered.

In the generic spectrum case, we can still choose  $Y \subseteq \mathbb{Z}$ , so  $X \subset \mathbb{R}^2$  consists of “lines” with distance 1.

The subsets of  $X$  with  $\lambda = 0$  will turn out to be critical, as the operators  $u$  or  $v$  amount to a multiplication by 0, there. We hence define the critical and regular

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<sup>2</sup>See also the formulation [24, Thm. 10.10] of the spectral theorem.

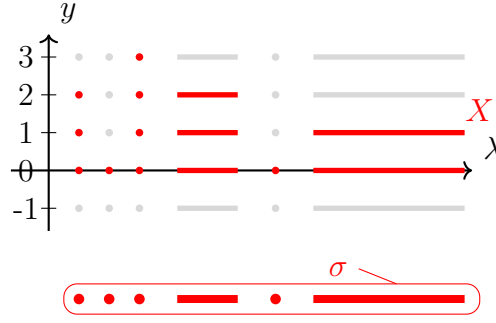


Figure 5.1: The spectral set  $X$  for a generic spectrum of  $|C|$ . Color online.

spectral sets as

$$X_{\text{crit}} := \{x = (\lambda, y) \in X \mid \lambda = 0\}, \quad X_{\text{reg}} := X \setminus X_{\text{crit}}. \quad (5.47)$$

Our (dense) space of test functions on the spectral set is then given by:

$$\mathcal{D}_X := C_c^\infty(X_{\text{crit}}) \otimes C_c^\infty(X_{\text{reg}}). \quad (5.48)$$

The corresponding test function space in  $\ell^2$  is

$$\mathcal{D}_{|C|} := U_{fX} \mathcal{D}_X. \quad (5.49)$$

For non-open  $X$ , we interpret definition (5.48) in the same way as the definition of  $\mathcal{E}(X)$ :

$$C_c^\infty(X) := C_c^\infty(\mathbb{R}^2) / \{\phi \mid \phi(x) = 0 \forall x \in X\}.$$

**Lemma 5.4.1** (Bogoliubov relations (5.13) survive the extension).

Suppose  $u$  and  $v$  are defined on a common dense domain  $\mathcal{D} \subseteq \ell^2$ , such that  $v^*v$  is densely defined and self-adjoint, and such that the linear operator

$$\mathcal{V} = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}, \quad \mathcal{V} : \mathcal{D} \oplus \mathcal{D} \rightarrow \ell^2 \oplus \ell^2 \quad (5.50)$$

defines a Bogoliubov transformation. That means, both  $\mathcal{V}$  and  $\mathcal{V}^* = \begin{pmatrix} u^* & v^T \\ v^* & u^T \end{pmatrix}$  preserve the CAR/CCR, see (5.9), (5.12) or also Section 5.9.4.

Then  $u, v, \bar{u}, \bar{v}, u^*, v^*, u^T$  and  $v^T$  are well-defined on all of  $\mathcal{D}_{|C|}$ , which was constructed above (5.49). Further, the Bogoliubov relations

$$\begin{aligned} (u^*u \mp v^*v) &= 1 & (u^*v \mp v^T\bar{u}) &= 0 \\ (u\bar{u}^* \mp v\bar{v}^*) &= 1 & (u\bar{v}^T \mp v\bar{u}^T) &= 0 \end{aligned} \quad (5.51)$$

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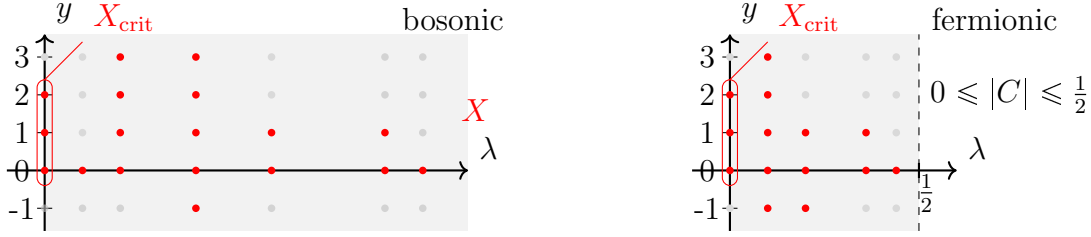


Figure 5.2: Left: Discrete spectrum of  $|C|$  in the bosonic case.

Right: In the fermionic case,  $|C| \leq \frac{1}{2}$  holds. Color online.

hold as a weak operator identity on  $\mathcal{D}_{|C|}$ . Conversely, (5.51) implies conservation of the CAR/CCR under both  $\mathcal{V}$  and  $\mathcal{V}^*$ .

*Proof.* Well-definedness of  $u, v$  on  $\mathcal{D}_{|C|}$  follows from the polar decompositions

$$v = U_v |v|, \quad u = U_u |u|, \quad (5.52)$$

with unitary operators  $U_v, U_u : L^2(X) \rightarrow \ell^2$ . The operators  $|v| = \sqrt{v^*v}$  and  $|u| = \sqrt{u^*u} = \sqrt{1 \pm v^*v}$  on  $L^2(X)$  are bounded in the fermionic case ( $-$ ), so  $u$  and  $v$  are defined on all of  $\ell^2$ . In the bosonic case ( $+$ ), they are spectral multiplications by

$$\nu(\lambda) = \sqrt{-\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda^2}} \quad \text{and} \quad \mu(\lambda) = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda^2}}, \quad (5.53)$$

which are bounded on each bounded interval  $\lambda$ . Therefore,  $|v|$  and  $|u|$  map  $\mathcal{D}_X$  onto itself, and by definition (5.49) of  $\mathcal{D}_{|C|}$ , the operators  $v$  and  $u$  map  $\mathcal{D}_{|C|} \rightarrow \ell^2$ . Well-definedness of  $\bar{u}$  and  $\bar{v}$  on  $\mathcal{D}_{|C|}$  follows by the same polar decomposition argument. And the domains of the adjoints  $u^*, v^*, u^T$  and  $v^T$  contain the domain of the respective original operators, so they all contain  $\mathcal{D}_{|C|}$ .

The CAR/CCR conservation under  $\mathcal{V}, \mathcal{V}^*$  can be equivalently translated into the two conditions  $\mathcal{V}^* \mathcal{S}_\pm \mathcal{V} = \mathcal{S}_\pm$  and  $\mathcal{V} \mathcal{S}_\pm \mathcal{V}^* = \mathcal{S}_\pm$  and a direct computation as in Section 5.9.4 shows that they are formally equivalent to (5.51) in case of bounded  $u, v$ .

For general  $u$  and  $v$  defined on  $\mathcal{D}_{|C|}$ , the first two formulas are indeed weak operator identities on  $\mathcal{D}_{|C|}$ . The last two are also weak operator identities, since  $\text{dom}(v^*) \supseteq \text{dom}(v)$  and  $\text{dom}(u^*) \supseteq \text{dom}(u)$ . □

Next, we establish that  $v^*v$  having a countable spectrum implies a countable

spectrum of  $|C|$ , as illustrated in Figures 5.2.

**Lemma 5.4.2.** *Let  $v^*v$  be self-adjoint. Then  $C = u^*v\mathcal{J}$  and  $C^*C$  are well-defined operators on  $\mathcal{D}_{|C|}$ . Here,  $\mathcal{J}$  denotes complex conjugation (see Section 5.9.1). The spectrum of  $C$  is contained in the real axis for bosons and the imaginary axis for fermions.*

*Further, if  $v^*v$  has countable spectrum, then also  $C, C^*C$  and  $|C|$  have countable spectrum.*

*Proof.* By the Bogoliubov relations,  $C^*C = v^*v \pm (v^*v)^2$  holds, wherever it is defined. Now,  $\lambda \mapsto \lambda \pm \lambda^2$  is smooth apart from the critical points 0 (bosons) or 0 and 1 (fermions). The condition  $\phi \in \mathcal{D}_{|C|}$  means that the corresponding spectral function  $\phi_X$  has compact support and is smooth apart from the critical points. This property is preserved by an application of  $C^*C$ , so  $C^*C : \mathcal{D}_{|C|} \rightarrow \mathcal{D}_{|C|}$  is well-defined. Hence, also  $|C| = \sqrt{C^*C}$  is well-defined, and by a polar decomposition also  $C = U_C|C|$  with  $U_C : \ell^2 \rightarrow \ell^2$  being a unitary operator.

In the bosonic case,  $C$  is symmetric by the Bogoliubov relations, so  $C^*C = C^2$  and further,  $\sigma(C)$  is a subset of the preimage of  $\sigma(C^2) \subseteq [0, \infty)$  under the complex map  $z \mapsto z^2$ . This preimage is contained within the real axis.

In the fermionic case, the Bogoliubov relations imply  $C^* = -C$ , so  $C^*C = -C^2$ . That means,  $\sigma(C)$  lies within the preimage of  $\sigma(C^2) \subseteq [0, \infty)$  under the map  $z \mapsto -z^2$ , which is contained within the imaginary axis.

Now, suppose  $\sigma(v^*v)$  is countable. Then,  $\sigma(|C|)$  is the image of  $\sigma(v^*v)$  under the map  $z \mapsto \sqrt{z(1 \pm z)}$ , which sends at most 2 arguments to the same value, so  $\sigma(|C|)$  is also countable.

□

**Remarks.**

- 24. For fermions,  $u^*u + v^*v = 1$  implies that  $v^*v$  is bounded, so if  $v^*v$  and also  $u^*u$  can be defined on all of  $\ell^2$ . Hence, also  $v$  and  $u$  are defined on all of  $\ell^2$ , so (5.51) holds as a strong operator identity on  $\ell^2$ .
- 25. For bosons, it is not obvious that (5.51) holds as a strong operator identity on a dense domain of  $\ell^2$ . In fact, it does not always hold as a strong operator identity on  $\mathcal{D}_{|C|}$ , as the following counter-example shows:

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For the canonical basis  $(e_j)_{j \in \mathbb{N}}$  of  $\ell^2$ , let  $v$  be the multiplication by  $j$ , i.e.,  $ve_j = je_j$ . So  $v^*v$  and  $C^*C$  are the multiplications by  $j^2$  and  $j^2(1 + j^2)$ , respectively, and  $\mathcal{D}_{|C|}$  comprises all vectors  $\phi \in \ell^2$ , where there exists some  $N \in \mathbb{N}$ , such that  $f_j = 0 \forall j \geq N$ , i.e., there is a “maximum occupied basis vector”  $e_{N-1}$ . By  $u^*u - v^*v = 1$ , we also know that  $u^*u$  is a multiplication by  $(1 + j^2)$ , and by polar decomposition, we may write

$$u = U_u|u|, \quad (5.54)$$

with  $|u|$  being the multiplication by  $\sqrt{1 + j^2}$  and  $U_u$  a unitary operator. Similarly,  $u^* = |u|U_u^*$ . Now choose  $U_u$  such that

$$U_u^*e_1 = c \sum_j j^{-1/2-\varepsilon} e_j, \quad (5.55)$$

where  $\varepsilon > 0$  guarantees that the right-hand side is in  $\ell^2$ . Here,  $c > 0$  is a normalization constant depending on  $\varepsilon$ , chosen such that  $\|U_u^*e_1\| = 1$ . Then, for  $e_1 \in \mathcal{D}_{|C|}$ , consider the formal expression

$$u^*ve_1 = |u|U_u^*e_1 = \sum_j \sqrt{1 + j^2} j^{-1/2-\varepsilon} e_j. \quad (5.56)$$

For  $\varepsilon \leq 1$ , this is obviously not in  $\ell^2$ , so  $u^*v$  is ill-defined on  $e_1 \in \mathcal{D}_{|C|}$  and the second formula in (5.51) does not hold as a strong operator identity on  $\mathcal{D}_{|C|}$ .

### 5.4.2 Extension of the Operator Algebra

In Sections 1.2.1 and 5.2.1, we defined a  $*$ -algebra (bosonic) or  $C^*$ -algebra (fermionic)  $\mathcal{A}$  generated by  $a_{\pm}^{\dagger}(f), a_{\pm}(f), f \in \mathfrak{h}$ , so  $\mathcal{A}$  contains operator products that are densely defined on Fock space. On our two Fock space extensions, we will encounter formal expressions in creation and annihilation operators that belong to a larger algebra  $\overline{\mathcal{A}}_{\mathbf{e}}$ , which is defined with respect to a basis  $\mathbf{e} = (e_j)_{j \in \mathbb{N}} \subset \ell^2$ . We introduce the shorthand notations  $a_j := a(e_j), a_j^{\dagger} := a^{\dagger}(e_j)$ , and consider the set of finite operator products

$$\Pi_{\mathbf{e}} := \{P_{\mathbf{e}} = a_{j_1}^{\#_1} \dots a_{j_m}^{\#_m} \mid j_{\ell} \in \mathbb{N}, \#_{\ell} \in \{\cdot, \dagger\}\}. \quad (5.57)$$

Then  $\overline{\mathcal{A}}_{\mathbf{e}}$  is defined as the set of all infinite sums

$$\overline{\mathcal{A}}_{\mathbf{e}} := \left\{ H = \sum_{P_{\mathbf{e}} \in \Pi_{\mathbf{e}}} H_{j_1, \#_1, \dots, j_m, \#_m} P_{\mathbf{e}} \mid H_{j_1, \#_1, \dots, j_m, \#_m} \in \mathbb{C} \right\}. \quad (5.58)$$

$\overline{\mathcal{A}}_{\mathbf{e}}$  is made a  $*$ -algebra by the involution

$$*: ca_j \mapsto \bar{c}a_j^\dagger, \quad ca_j^\dagger \mapsto \bar{c}a_j \quad \forall c \in \mathbb{C}. \quad (5.59)$$

It is easy to see that  $\overline{\mathcal{A}}_{\mathbf{e}}$  extends  $\mathcal{A}$ , as each element of  $\mathcal{A}$  is a finite sum of operator products  $a^\#(f_1) \dots a^\#(f_m)$ . Resolving each  $a^\#(f_j)$  with respect to the basis  $\mathbf{e}$ , we obtain a countable sum of the form in (5.58), that contains each term  $a_{j_1}^\# \dots a_{j_m}^\#$  at most once.

Of particular interest will be elements of  $\overline{\mathcal{A}}_{\mathbf{e}}$  corresponding to finite sums. For  $a^\#(\boldsymbol{\phi}) = \sum_{j \in \mathbb{N}} \phi_j a_j^\#$ ,  $\boldsymbol{\phi} \in \ell^2$ , this sum is finite if the form factor  $\boldsymbol{\phi}$  is an element of

$$\mathcal{D}_{\mathbf{e}} := \{ \boldsymbol{\phi} \in \ell^2 \mid \phi_j = 0 \text{ for all but finitely many } j \in \mathbb{N} \}. \quad (5.60)$$

If  $(\mathbf{e}_j)_{j \in \mathbb{N}}$  is an orthonormal eigenbasis of  $|C|$ , then  $\mathcal{D}_{\mathbf{e}} = \mathcal{D}_{|C|}$ , since both domains are spanned by finite linear combinations of eigenvectors of  $C^*C$ .

Next, we will make the Fock space extensions precise, that will be used to implement a Bogoliubov transformation  $\mathcal{V}$  and we define products of  $a^\dagger(\boldsymbol{\phi}), a(\boldsymbol{\phi})$  with  $\boldsymbol{\phi} \in \mathcal{D}_{\mathbf{e}}$  on suitable subspaces of them. Within these definitions, **it is assumed that  $v^*v$  has countable spectrum**, so the assumptions of Lemma 5.4.2 are valid and  $|C|$  also has countable spectrum.

We start with the ITP case. As argued around (5.46), the index set for the eigenvectors of  $|C|$  (i.e., modes)  $X \subseteq \sigma \times Y$  is countable and there exists an orthonormal eigenbasis  $(\mathbf{f}_j)_{j \in \mathbb{N}}$ . We may use it to construct the bases  $\mathbf{g} = (\mathbf{g}_j)_{j \in \mathbb{N}}$  (bosonic) and  $\boldsymbol{\eta} = (\boldsymbol{\eta}_j)_{j \in J}$  (fermionic), which will take the role of  $\mathbf{e}$  in  $\mathcal{D}_{\mathbf{e}}$  and  $\overline{\mathcal{A}}_{\mathbf{e}}$ . Note that the index set  $J \subseteq \mathbb{N}$  in the fermionic case is countable, so it can as well be re-indexed by  $\mathbb{N}$ .

For bosons, we follow the construction in Section 5.3.2, replacing the argument “ $C$  is Hilbert–Schmidt” below (5.22) by “ $|C|$  has countable spectrum”. This provides us with an orthonormal basis  $\mathbf{g} = (\mathbf{g}_j)_{j \in \mathbb{N}}$ . Further, it is used, that by Lemma 5.4.1, the Bogoliubov relations still hold as a weak operator identity.

Now, we consider the one-mode Fock space  $\mathcal{H}_k$  for mode  $k = j \in \mathbb{N}$  and take the ITP over all these modes:

**Definition 5.4.3.** The **bosonic infinite tensor product space** is given by

$$\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{F}(\{\mathbf{g}_k\}). \quad (5.61)$$

Note that the sequence  $(e_{k,n})_{n \in \mathbb{N}_0}$  of  $n$ -particle basis vectors is a canonical basis of each  $\mathcal{H}_k$ , and can be used to describe elements of  $\widehat{\mathcal{H}}$ .

For fermions, we obtain a similar orthonormal basis  $(\boldsymbol{\eta}_j)_{j \in J}$  with countable index set  $J \subseteq \mathbb{N}$ : Copying the construction in Section 5.3.2, while replacing “ $C^*C$  is trace class” by “ $|C|$  has countable spectrum” (which is true by Lemma 5.4.2), we have that  $(\boldsymbol{\eta}_j)_{j \in J}$  is orthonormal. The construction of the ITP space is, however, a bit more delicate in this case. The easy part are modes with a full particle–hole transformation, or no transformation at all ( $j \in J''$ ). Here we may just consider each Fock space  $\mathcal{F}(\{\boldsymbol{\eta}_j\}) \cong \mathbb{C}^2$  over the respective mode  $\boldsymbol{\eta}_j$  as a factor within the ITP. However, for Cooper pairs indexed by  $i \in I'$  (so  $j \in J'$ ), it will become necessary (see Remark 29) to introduce a separate Fock space  $\mathcal{F}(\{\boldsymbol{\eta}_{2i-1}\}) \otimes \mathcal{F}(\{\boldsymbol{\eta}_{2i}\}) \cong \mathbb{C}^4$  for each pair of modes. We index all  $j \in J''$  and  $i \in I'$  by a corresponding  $k(i)$  or  $k(j)$ , such that all  $k \in \mathbb{N}$  are used and take the tensor product over those  $k$ :

**Definition 5.4.4.** The **fermionic infinite tensor product space** is given by

$$\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k = \left( \prod_{j \in J''}^{\otimes} \mathcal{F}(\{\boldsymbol{\eta}_j\}) \right) \otimes \left( \prod_{i \in I'}^{\otimes} \mathcal{F}(\{\boldsymbol{\eta}_{2i-1}\}) \otimes \mathcal{F}(\{\boldsymbol{\eta}_{2i}\}) \right). \quad (5.62)$$

For a one–mode Fock space,  $\mathcal{H}_k := \mathcal{F}(\{\mathbf{g}_k\})$  or  $\mathcal{H}_{k(j)} := \mathcal{F}(\{\boldsymbol{\eta}_j\})$ , the sequence  $(e_{k,n})_{n \in \mathbb{N}_0}$  of  $n$ -particle state vectors (bosonic case) or the pair  $(e_{k,0}, e_{k,1})$  (fermionic case) forms a basis of each  $\mathcal{H}_k$ . For fermionic two–mode Fock spaces  $\mathcal{H}_{k(i)} := \mathcal{F}(\{\boldsymbol{\eta}_{2i-1}\}) \otimes \mathcal{F}(\{\boldsymbol{\eta}_{2i}\})$ , such a basis is given by the quadruple  $(e_{k,0,0}, e_{k,1,0}, e_{k,0,1}, e_{k,1,1})$ , where 0, 1 are the occupation numbers of the respective mode.

Our next challenge is to lift the one–mode creation and annihilation operators  $a_j^\dagger, a_j$  defined on the one– or two–mode Fock space  $\mathcal{H}_k$  to  $\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k$ .

**Lemma 5.4.5.** Consider a (possibly unbounded) operator  $A_{j,j} : \mathcal{H}_j \supset \text{dom}(A_{j,j}) \rightarrow \mathcal{H}_j$ . Then, for  $\Psi_j^{(m)} \in \text{dom}(A_{j,j})$ ,

$$A_j \Psi^{(m)} := \Psi_1^{(m)} \otimes \dots \otimes \Psi_{j-1}^{(m)} \otimes A_{j,j} \Psi_j^{(m)} \otimes \Psi_{j+1}^{(m)} \otimes \dots \quad (5.63)$$

is independent of the choice of a  $C$ -sequence  $(\Psi^{(m)}) = (\Psi_k^{(m)})_{k \in \mathbb{N}}$  representing  $\Psi^{(m)}$ , and defines an operator  $A_j$  by linearity on

$$\Psi \in \text{dom}(A_j) := \left\{ \Psi = \sum_{m \in \mathcal{M}} d_m \Psi^{(m)} \in \widehat{\mathcal{H}} \left| \left\| \sum_{m \in \mathcal{M}} d_m A_j \Psi^{(m)} \right\| < \infty \right. \right\}, \quad (5.64)$$

where  $\mathcal{M} \subseteq \mathbb{N}$ ,  $d_m \in \mathbb{C}$  and  $\Psi^{(m)}$  being such that  $A_j \Psi^{(m)}$  is well-defined by (5.63).

*Proof.* For a fixed choice of  $(\Psi^{(m)})$  representing  $\Psi^{(m)}$  such that  $\Psi_j^{(m)} \in \text{dom}(A_{j,j})$ , well-definedness of  $A_j \Psi^{(m)}$  is easy to see. By Lemma 5.8.2, we can now represent  $\Psi = \sum_{m \in \mathcal{M}} d_m \Psi^{(m)}$ . And if  $\sum_{m \in \mathcal{M}} d_m A_j \Psi^{(m)}$  converges, then it is independent of the representation, since  $A_j$  is linear. So  $\text{dom}(A_j)$  and  $A_j \Psi$  are well-defined.

It remains to be proven that  $A_k \Psi^{(m)}$  (and hence  $A_k \Psi$ ) is independent of the choice of  $(\Psi^{(m)})$  representing  $\Psi^{(m)}$ . So, for  $m \in \mathcal{M}$ , consider a second representative  $C$ -sequence  $(\tilde{\Psi}^{(m)})$  with  $\tilde{\Psi}^{(m)} = \Psi^{(m)}$ . By Proposition 5.8.1,  $\tilde{\Psi}_k^{(m)} = c_k \Psi_k^{(m)}$  for some  $c_k \in \mathbb{C}$  with  $\prod_{k \in \mathbb{N}} c_k = 1$ . By linearity,  $A_{j,j} \tilde{\Psi}_k^{(m)} = c_k A_{j,j} \Psi_k^{(m)}$ , so also  $A_j \Psi^{(m)}$  and  $A_j \tilde{\Psi}^{(m)}$  defined by (5.63) just differ by the sequence of complex factors  $(c_k)_{k \in \mathbb{N}}$  with  $\prod_{k \in \mathbb{N}} c_k = 1$ . Hence, according to Proposition 5.8.1, they correspond to the same functional  $A_j \tilde{\Psi}^{(m)} = A_j \Psi^{(m)}$ . □

Bosonic creation and annihilation operators  $a_j, a_j^\dagger$  are usually not bounded. We need to carefully choose a non-dense domain in  $\mathcal{H}_k$  in order to make them bounded. Such a choice is made possible by restricting the allowed  $\Psi$  to the following space:

**Definition 5.4.6.** In the bosonic case, the space  $\mathcal{S}^\otimes$  with sufficient decay in the particle number is defined as

$$\mathcal{S}^\otimes := \left\{ \Psi \in \bigcap_{\substack{n \in \mathbb{N} \\ k \in \mathbb{N}_0}} \text{dom}(N_k^n) \subseteq \widehat{\mathcal{H}} \left| \|N_k^n \Psi\| \leq c_{k,n} \|\Psi\| \quad \forall k \in \mathbb{N}, n \in \mathbb{N}_0 \right. \right\}, \quad (5.65)$$

where  $j = k$  (as we are in the bosonic case),  $c_{k,n} > 0$  are suitable constants for each  $n$  and  $k$ , and  $N_k$  is the number operator on  $\mathcal{H}_k$ , lifted to  $\widehat{\mathcal{H}}$ . The lift is possible by Lemma 5.4.5, which also yields a definition of  $\text{dom}(N_k^n)$ .

In the fermionic case, we simply set

$$\mathcal{S}^\otimes := \widehat{\mathcal{H}}, \quad (5.66)$$

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as the maximum particle number per mode is 1, so we always have a sufficient decay.

Within  $\mathcal{H}_k$  there may now exist one or two creation and annihilation operators  $a_j$  and  $a_j^\dagger$ , depending on whether  $\mathcal{H}_k$  describes one or two modes. We lift them to  $\widehat{\mathcal{H}}$  and formally define

$$a^\dagger(\phi) = \sum_j \phi_j a_j^\dagger, \quad a(\phi) = \sum_j \bar{\phi}_j a_j. \quad (5.67)$$

Now,  $a^\dagger(\phi)$  does only make sense for  $\phi \in \mathcal{D}_e \subseteq \ell^2$ , as will become clear in the proof of the following lemma:

**Lemma 5.4.7** (Products of  $a^\dagger, a$  are well-defined on the ITP space).

Consider the ITP space  $\widehat{\mathcal{H}} \supseteq \mathcal{S}^\otimes$  corresponding to the basis  $(\mathbf{e}_j)_{j \in \mathbb{N}}$ , which is  $(\mathbf{g}_j)_{j \in \mathbb{N}}$  (bosonic) or  $(\boldsymbol{\eta}_j)_{j \in \mathbb{N}}$  (fermionic). If  $\phi \in \mathcal{D}_\mathbf{g}$  or  $\mathcal{D}_\boldsymbol{\eta}$  (defined in (5.60)), then

$$a^\dagger(\phi) : \mathcal{S}^\otimes \rightarrow \mathcal{S}^\otimes, \quad a(\phi) : \mathcal{S}^\otimes \rightarrow \mathcal{S}^\otimes, \quad (5.68)$$

as in (5.67), are well-defined linear operators.

Note that for  $(\boldsymbol{\eta}_j)_{j \in \mathbb{N}}$ , we have re-indexed  $j \in J$  to  $j \in \mathbb{N}$ .

*Proof.* First, note that we can write

$$a^\dagger(\phi)\Psi = \sum_{j \in \mathbb{N}} \phi_j a_j^\dagger \Psi = \sum_{j: \phi_j \neq 0} \phi_j a_j^\dagger \Psi, \quad a(\phi)\Psi = \sum_{j: \phi_j \neq 0} \bar{\phi}_j a_j \Psi, \quad (5.69)$$

where the sum is finite by definition of  $\mathcal{D}_\mathbf{g}$  and  $\mathcal{D}_\boldsymbol{\eta}$ .

In the fermionic case,  $a_j, a_j^\dagger$  are bounded, so by [44, Lemma 5.1.1], they can be lifted to bounded operators on  $\mathcal{S}^\otimes = \widehat{\mathcal{H}}$ . Therefore, also the finite linear combination (5.69) is a bounded operator on  $\mathcal{S}^\otimes$ .

Within the bosonic case, where  $j = k$ , the first statement  $a^\dagger(\phi)\Psi \in \mathcal{S}^\otimes$  can be seen as follows: We start by verifying that  $a^\dagger(\phi)\Psi$  is well-defined. First, note that

$$\|a_k^\dagger \Psi\| = \|\sqrt{N_k + 1} \Psi\| \leq \|(N_k + 1) \Psi\| \leq (c_{k,1} + 1) \|\Psi\|, \quad (5.70)$$

where we used that the operator  $a_k^\dagger$  shifts all sectors up by one (keeping them orthogonal) and multiplies by  $\sqrt{N_k + 1}$ . So

$$\|a^\dagger(\phi)\Psi\| \leq \left( \sum_{k:\phi_k \neq 0} |\phi_k|(c_{k,1} + 1) \right) \|\Psi\| =: c_1 \|\Psi\|, \quad (5.71)$$

where, by definition of  $\mathcal{D}_g$  and  $\mathcal{D}_\eta$ , the sum over  $k$  contains only finitely many nonzero terms, so we may call it  $c_1 > 0$ . Hence,  $a^\dagger(\phi)\Psi \in \widehat{\mathcal{H}}$ .

It remains to establish sufficient decay in the particle number. We observe that

$$\begin{aligned} \|N_k^n a_k^\dagger \Psi\| &= \|N_k^n \sqrt{N_k + 1} \Psi\| \leq \|N_k^{n+1} \Psi\| + \|N_k^n \Psi\| \leq (c_{k,n+1} + c_{k,n}) \|\Psi\| \\ &\leq (c_{k,n+1} + c_{k,n}) \|a_k^\dagger \Psi\|, \end{aligned} \quad (5.72)$$

so by summing over  $k$ , the particle number decay condition is again satisfied and  $a^\dagger(\phi)\Psi \in \mathcal{S}^\otimes$ .

For the second statement  $a(\phi)\Psi \in \mathcal{S}^\otimes$ , the same finite-sum argument can be used. By repeating all proof steps with  $\sqrt{N_k}$  instead of  $\sqrt{N_k + 1}$ , it can be seen that  $a(\phi)\Psi \in \widehat{\mathcal{H}}$ . However, verifying the particle number decay condition needs a bit more attention, since the inequality  $\|a_k^\dagger \Psi\| \leq \|\Psi\|$  in (5.72) does not generalize to  $a_k$ , see also Remark 26. However, for all  $k$  with  $a_k \Psi \neq 0$  and  $\phi_k \neq 0$ , there is a fixed ratio  $\frac{\|\Psi\|}{\|a_k \Psi\|} =: d_k > 0$ . Denote by  $d := \max_k d_k$  the maximum over these finitely many ratios for a fixed  $\Psi$ . Then,

$$\|N_k^n a_k \Psi\| \leq (c_{k,n+1} + c_{k,n}) \|\Psi\| \leq d \cdot (c_{k,n+1} + c_{k,n}) \|a_k \Psi\|. \quad (5.73)$$

For  $a_k \Psi = 0$ , the inequality is trivially satisfied. A finite sum over  $k$  establishes  $a(\phi)\Psi \in \mathcal{S}^\otimes$  and thus finishes the proof.  $\square$

The ESS for countable spectrum of  $v^*v$  is built using the complex sequence space

$$\begin{aligned} \mathcal{E} = \mathcal{E}_g &:= \mathcal{D}'_g \cong \mathcal{E}(\mathbb{N}) && \text{(bosonic)} \\ \mathcal{E} = \mathcal{E}_\eta &:= \mathcal{D}'_\eta \cong \mathcal{E}(\mathbb{N}) && \text{(fermionic)}. \end{aligned} \quad (5.74)$$

This definition allows for a distribution pairing  $\langle \phi, \psi \rangle = \sum_j \overline{\phi_j} \psi_j$  for  $\phi \in \mathcal{D}$ ,  $\psi \in \mathcal{E}$ . In particular, both  $\mathcal{D}$  and  $\ell^2$  can be embedded into  $\mathcal{E}$ .

The spaces  $\mathcal{E}^{(N)}$ ,  $\mathcal{E}_{\mathcal{F}}$ ,  $\text{Ren}_1$ ,  $\text{Ren}$ ,  $\text{eRen}$ ,  $\overline{\mathcal{F}}$  and  $\overline{\mathcal{F}}_{\text{ex}}$  are then defined as in Sections 3.2 and 5.2.2.

Creation and annihilation operators  $a^\dagger(\phi), a(\phi)$  are defined on configuration

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space functions  $\Psi_m : \mathcal{Q}(\mathbb{N}) \rightarrow \mathbb{C}$  in similarity to (1.21). The only difference is that a configuration is no longer  $q \in \mathcal{Q}(\mathbb{R}^d)$ , but rather  $q \in \mathcal{Q}(\mathbb{N})$ , so  $q = \{j_1, \dots, j_N\}$ . Again, we have to distinguish the bosonic (+) and the fermionic (−) case. Formally,

$$\begin{aligned} (a_{\pm}^{\dagger}(\phi)\Psi_m)(q) &= \sum_{k=1}^N \frac{(\pm 1)^k}{\sqrt{N}} \phi_{j_k} \Psi_m(q \setminus j_k) \\ (a_{\pm}(\phi)\Psi_m)(q) &= \sqrt{N+1} \sum_j \overline{\phi_j} \Psi_m(q, j). \end{aligned} \tag{5.75}$$

The definition extends by linearity from  $\Psi_m \in \mathcal{E}_{\mathcal{F}}$  to any  $\Psi = \sum_m \mathbf{c}_m \Psi_m \in \overline{\mathcal{F}}$  with  $\mathbf{c}_m \in \text{eRen}$ ,  $\Psi_m \in \mathcal{E}_{\mathcal{F}}$  and with the sum over  $m$  being finite. Note that the symmetrization operators  $S_{\pm}$  from (1.6) also naturally extend to ESS vectors, as they just permute entries within a configuration  $q$ . In the following, we will again drop the index  $\pm$ , meaning that statements about  $a^{\dagger}(\phi), a(\phi)$  are both about fermionic and the bosonic operators, if not stated otherwise.

The CAR/CCR are a direct consequence of definition (5.75) and hence still valid for the operator extensions.

**Lemma 5.4.8** (Products of  $a^{\dagger}, a$  are well-defined on the ESS).

Consider the ESS  $\overline{\mathcal{F}}$  built over  $\mathcal{E}$ . Then, (5.75) uniquely defines operators  $a^{\dagger}(\phi), a(\phi)$  as follows: for  $\phi \in \mathcal{D}$ , we have

$$a^{\dagger}(\phi) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}, \quad a(\phi) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}, \tag{5.76}$$

and more generally, for  $\phi \in \mathcal{E}$ , we have

$$a^{\dagger}(\phi) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}, \quad a(\phi) : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}}_{\text{ex}}. \tag{5.77}$$

*Proof.* For  $\phi \in \mathcal{D}$ , it suffices to show that for  $\Psi_m \in \mathcal{E}_{\mathcal{F}}$ , we have  $a^{\dagger}(\phi)\Psi_m, a(\phi)\Psi_m \in \mathcal{E}_{\mathcal{F}}$ . Considering (5.75), it is easy to see that  $a^{\dagger}(\phi)\Psi_m : \mathcal{Q}(\mathbb{N}) \rightarrow \mathbb{C}$  defines a function on configuration space, as it is just the tensor product of two functions. For  $(a(\phi)\Psi_m)(q)$ , since  $\phi \in \mathcal{D}$ , each sum over  $j$  in (5.75) has a finite number of nonzero terms and is hence finite. Therefore,  $(a(\phi)\Psi_m)(q)$  is finite and  $a(\phi)\Psi_m : \mathcal{Q}(\mathbb{N}) \rightarrow \mathbb{C}$  is a well-defined function.

For  $\phi \in \mathcal{E}$ , it is again easy to see that  $a^{\dagger}(\phi)\Psi_m$  defines a function  $\mathcal{Q}(\mathbb{N}) \rightarrow \mathbb{C}$  as a tensor product of two functions. In  $a(\phi)\Psi_m$ , the sum over  $j$  may now be infinite or even divergent. However, it can be defined as a  $\text{Ren}_1$  renormalization constant:

For each  $q \in \mathcal{Q}(\mathbb{N})$ , the function  $f(j) := \overline{\phi_j} \Psi_m(q, j)$  is  $f \in \mathcal{E}(\mathbb{N})$ , so

$$\sum_j \overline{\phi_j} \Psi_m(q, j) \in \text{Ren}_1(\mathbb{N}) \quad \Rightarrow \quad (a(\phi) \Psi_m) \in \text{Ren}^{\mathcal{Q}}(\mathbb{N}).$$

Hence, for  $\Psi = \sum_m \mathbf{c}_m \Psi_m \in \overline{\mathcal{F}}$ , we have

$$a(\phi) \Psi = \sum_m \mathbf{c}_m a(\phi) \Psi_m \in \overline{\mathcal{F}}_{\text{ex}}.$$

□

### Remarks.

26. The condition  $\phi \in \mathcal{D}_f$  is indeed necessary, meaning we may not just allow any  $\phi \in \ell^2$  inside  $a^\sharp(\phi)$ , as the following counter-example shows: For the bosonic case ( $j = k$ ), consider  $\phi_k = \frac{1}{k}$ , so  $\phi \in \ell^2 \setminus \mathcal{D}_f$ . For each mode  $k$ , consider the coherent state  $\Psi_k$  defined sector-wise by

$$\Psi_k^{(N_k)} = e^{-\frac{\alpha_k}{2}} \frac{\alpha_k^{\frac{N_k}{2}}}{\sqrt{N_k!}}, \quad (5.78)$$

where all  $\alpha_k \in \mathbb{R}$  are set equal to the same  $\alpha_k = \alpha > 0$  and where  $\|\Psi_k\|_k = 1$ . Then, define the ITP  $\Psi = \prod_{k \in \mathbb{N}} \Psi_k$ . It is easy to see that  $\Psi$  satisfies the particle number decay condition (5.65), as for each  $\Psi_k$ ,  $\|\Psi_k^{(N_k)}\|_k$  decays exponentially in  $N_k$ . But still,  $(\alpha_k)_{k \in \mathbb{N}} \notin \ell^2$ , so we may think of  $\Psi$  as a ‘‘coherent state with a large displacement’’, living outside the Fock space. It follows from a well-known fact about coherent states that  $a_k \Psi = \alpha \Psi$ , so

$$\|a(\phi) \Psi\| = \left\| \sum_k \overline{\phi_k} \alpha \Psi \right\| = \alpha \sum_k \frac{1}{k} \|\Psi\| = \infty. \quad (5.79)$$

Hence,  $a(\phi)$  is ill-defined on  $\Psi$ .

The same happens with any coherent state product (5.78) and any  $\phi$ , where  $\sum_k \overline{\phi_k} \alpha_k = \infty$ . In particular, the space of allowed  $(\phi_k)_{k \in \mathbb{N}}$  is dual to the one of allowed  $(\alpha_k)_{k \in \mathbb{N}}$ .

27. The above-mentioned duality actually extends to the definition of  $\mathcal{D}$  and  $\mathcal{S}^\otimes$ : We may alter those definitions to allow for more form factors  $\phi \in \ell^2$  in  $a^\sharp(\phi)$ . The result is that fewer vectors  $\Psi \in \mathcal{S}^\otimes$  are allowed, if  $a^\sharp(\phi_1) \dots a^\sharp(\phi_N) \Psi$

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shall still be well-defined. These alternative definitions are discussed in Section 5.11.

28. It is possible to view the subspace  $\prod_{k \in \mathbb{N}}^{\otimes C} \mathcal{H}_k$  of the equivalence class  $C$  (see below (3.9)) as the *original Fock space* with respect to the vacuum  $\Omega = \prod_{k \in \mathbb{N}}^{\otimes} e_{k,0}$ :

Recall that each  $\Psi \in \prod_{k \in \mathbb{N}}^{\otimes C} \mathcal{H}_k$  can be written in coordinates as (5.8):

$$\Psi = \sum_{n(\cdot) \in F} a(n(\cdot)) \prod_{k \in \mathbb{N}}^{\otimes} e_{k,n(k)}, \quad (5.80)$$

with  $F$  containing all sequences  $(n(k))_{k \in \mathbb{N}}$ , such that  $n(k) = 0$  for almost all  $k$ . Hence, each  $\prod_{k \in \mathbb{N}}^{\otimes} e_{k,n(k)}$  is a tensor product state of finitely many particles. Since the Fock norm and the  $\widehat{\mathcal{H}}$ -norm coincide, the vector  $\prod_{k \in \mathbb{N}}^{\otimes} e_{k,n(k)}$  can be seen as a Fock space vector normalized to 1. The linear combination (5.80) with  $\sum_{n(\cdot)} |a(n(\cdot))|^2$  can hence also be interpreted a Fock space vector. Conversely, each Fock space vector can be written as a countable sequence (5.80), since the span of the above-mentioned tensor product states is dense in  $\mathcal{F}$ .

### 5.5 Implementation: Extended

Roughly speaking, implementability of  $\mathcal{V}$  on Fock space  $\mathcal{F}$  means that there exists a linear map  $\mathbb{U}_{\mathcal{V}} : \mathcal{F} \rightarrow \mathcal{F}$  that transforms  $a^{\#}$ - into  $b^{\#}$ -operators (see Section 5.3.2). In Section 5.5.1, we give a precise definition of how implementability of  $\mathcal{V}$  is to be interpreted on Fock space extensions  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$ . Lemma 5.5.3 will then establish, that  $\mathbb{U}_{\mathcal{V}}$  is indeed well-defined and 5.5.4 will give suitable conditions for when it is an implementer in the extended sense.

Within Section 5.5.2, Theorems 5.5.5 and 5.5.6 then establish these suitable conditions for countable spectrum of  $v^*v$  in the bosonic case. In Section 5.5.3, Theorems 5.5.7 and 5.5.8 do the same for the fermionic case. Note that for implementability on  $\overline{\mathcal{F}}$  in Theorem 5.5.8, there is an additional requirement that only finitely many modes with full particle–hole transformations are allowed.

### 5.5.1 Definition of Extended Implementation

The implementer  $\mathbb{U}_\mathcal{V}$  is defined on a dense subspace of Fock space  $\mathcal{D}_\mathcal{F} \subset \mathcal{F}$ , that contains a finite number of particles from the space  $\mathcal{D}_\mathbf{f}$  (defined by (5.60) with  $\mathbf{e} = \mathbf{f}$ ):

$$\mathcal{D}_\mathcal{F} := \text{span}\{a^\dagger(\phi_1) \dots a^\dagger(\phi_N)\Omega, N \in \mathbb{N}_0, \phi_\ell \in \mathcal{D}_\mathbf{f}\}. \quad (5.81)$$

Note that this is an adapted version of definition (1.28) with  $C_c^\infty$  replaced by  $\mathcal{D}_\mathbf{f}$ . The operator  $\mathbb{U}_\mathcal{V}$  now maps from  $\mathcal{D}_\mathcal{F}$  into either an ITP space  $\widehat{\mathcal{H}}$  or an ESS  $\overline{\mathcal{F}}$ . All statements provided in this subsection hold regardless of the choice of this image space.

**Definition 5.5.1.** We say that a linear operator  $\mathbb{U}_\mathcal{V} : \mathcal{D}_\mathcal{F} \rightarrow \widehat{\mathcal{H}}$  or  $\mathbb{U}_\mathcal{V} : \mathcal{D}_\mathcal{F} \rightarrow \overline{\mathcal{F}}$  **implements** a Bogoliubov transformation  $\mathcal{V}$  **in the extended sense**, if for all  $\phi \in \mathcal{D}_\mathbf{f}$ ,  $\Psi \in \mathbb{U}_\mathcal{V}[\mathcal{D}_\mathcal{F}]$ , we have that

$$\mathbb{U}_\mathcal{V}a^\dagger(\phi)\mathbb{U}_\mathcal{V}^{-1}\Psi = b^\dagger(\phi)\Psi, \quad \mathbb{U}_\mathcal{V}a(\phi)\mathbb{U}_\mathcal{V}^{-1}\Psi = b(\phi)\Psi. \quad (5.82)$$

This requires, of course, that  $\mathbb{U}_\mathcal{V}^{-1}$  is well-defined. So before establishing (5.82), we have to show that  $\mathbb{U}_\mathcal{V}$  is invertible, in order to prove that  $\mathbb{U}_\mathcal{V}$  implements  $\mathcal{V}$  in the extended sense. This will be one main difficulty within the upcoming proofs.

The implementer  $\mathbb{U}_\mathcal{V}$  is defined as follows: First we define some new vacuum vector  $\Omega_\mathcal{V} = \mathbb{U}_\mathcal{V}\Omega$  within the respective Fock space extension, such that

$$b(\phi)\Omega_\mathcal{V} = 0. \quad (5.83)$$

Then we make  $\mathbb{U}_\mathcal{V}$  change  $a^\#$ - into  $b^\#$ -operators:

**Definition 5.5.2.** Given a Bogoliubov transformed vacuum state  $\Omega_\mathcal{V} \in \mathcal{S}^\otimes$  or  $\Omega_\mathcal{V} \in \overline{\mathcal{F}}$ , the **Bogoliubov implementer**  $\mathbb{U}_\mathcal{V}$  is formally defined on  $\mathcal{D}_\mathcal{F}$  by

$$\mathbb{U}_\mathcal{V}a^\dagger(\phi_1) \dots a^\dagger(\phi_n)\Omega := b^\dagger(\phi_1) \dots b^\dagger(\phi_n)\Omega_\mathcal{V}, \quad (5.84)$$

with  $\phi_\ell \in \mathcal{D}_\mathbf{f}$  and  $b^\dagger(\mathbf{f}_j) = (a^\dagger(u\mathbf{f}_j) + a(v\overline{\mathbf{f}_j}))$  for all basis vectors  $\mathbf{f}_j$  in  $\mathbf{f}$ .

**Lemma 5.5.3** ( $\mathbb{U}_\mathcal{V}$  is well-defined).

*In the ITP case, if  $\Omega_\mathcal{V} \in \mathcal{S}^\otimes \subseteq \widehat{\mathcal{H}}$  (see (5.65)), then the formal implementer  $\mathbb{U}_\mathcal{V}$  (5.84) is a well-defined operator  $\mathbb{U}_\mathcal{V} : \mathcal{D}_\mathcal{F} \rightarrow \mathcal{S}^\otimes$ .*

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For the ESS, if  $\Omega_{\mathcal{V}} \in \overline{\mathcal{F}}$ , then the formal implementer  $\mathbb{U}_{\mathcal{V}}$  (5.84) is a well-defined operator  $\mathbb{U}_{\mathcal{V}} : \mathcal{D}_{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$ .

*Proof.* In the ITP case, by (5.25),  $u\mathbf{f}_j$  and  $v\overline{\mathbf{f}}_j$  are both proportional to the same basis vector  $\mathbf{e}_j$  (bosonic:  $\mathbf{g}_j$ , fermionic:  $\boldsymbol{\eta}_j$ ). So the right-hand side of (5.84) is a finite linear combination of vectors  $a^\sharp(\mathbf{e}_{j_1}) \dots a^\sharp(\mathbf{e}_{j_n})\Omega_{\mathcal{V}}$ . Now,  $\Omega_{\mathcal{V}} \in \mathcal{S}^\otimes$  and by Lemma 5.4.7, each application of  $a^\sharp(\mathbf{e}_j)$  leaves the vector in  $\mathcal{S}^\otimes$ . So the whole vector (5.84) is in  $\mathcal{S}^\otimes \subseteq \widehat{\mathcal{H}}$ .

For the ESS, the right-hand side of (5.84) is a finite linear combination of vectors of the kind  $a^\sharp(\mathbf{e}_{j_1}) \dots a^\sharp(\mathbf{e}_{j_n})\Omega_{\mathcal{V}}$ , where  $\mathbf{e}_j \in \{u\mathbf{f}_j, v\overline{\mathbf{f}}_j\}$  is proportional to  $\mathbf{g}_j$  or  $\boldsymbol{\eta}_j$  and hence in  $\mathcal{E}$ . We have  $\Omega_{\mathcal{V}} \in \overline{\mathcal{F}}$ , and by Lemma 5.4.8, as  $\mathbf{f}_j \in \mathcal{D}_{\mathbf{f}}$ , each application of an  $a^\sharp(\mathbf{f}_j)$  maps again into  $\overline{\mathcal{F}}$ . So (5.84) is well-defined.  $\square$

We now provide conditions, for which  $\mathbb{U}_{\mathcal{V}}$  is indeed an implementer of  $\mathcal{V}$ .

**Lemma 5.5.4** (Conditions for an implementer  $\mathbb{U}_{\mathcal{V}}$ ). *Suppose that for a Bogoliubov transformation (i.e.,  $\mathcal{V}$  satisfying (5.51)) an  $\Omega_{\mathcal{V}}$  satisfying  $b(\boldsymbol{\phi})\Omega_{\mathcal{V}} = 0$  for all  $\boldsymbol{\phi} \in \mathcal{D}_{\mathbf{f}} \subseteq \ell^2$  has been found, such that  $\mathbb{U}_{\mathcal{V}}$  in (5.84) on  $\mathcal{D}_{\mathcal{F}}$  is well-defined and has an inverse  $\mathbb{U}_{\mathcal{V}}^{-1}$  defined on  $\mathbb{U}_{\mathcal{V}}[\mathcal{D}_{\mathcal{F}}]$ .*

*Then,  $\mathbb{U}_{\mathcal{V}}$  implements  $\mathcal{V}$  in the sense of (5.82) on all  $\Psi \in \mathbb{U}_{\mathcal{V}}[\mathcal{D}_{\mathcal{F}}]$ .*

*Proof.* We write  $\Psi = \mathbb{U}_{\mathcal{V}}\Phi$  with  $\Phi \in \mathcal{D}_{\mathcal{F}}$ . By linearity, it suffices to prove the statement for  $\Phi = a^\dagger(\boldsymbol{\phi}_1) \dots a^\dagger(\boldsymbol{\phi}_n)\Omega$ , which implies by (5.84) that  $\Psi = b^\dagger(\boldsymbol{\phi}_1) \dots b^\dagger(\boldsymbol{\phi}_n)\Omega_{\mathcal{V}}$ . In that case, we have (with  $n \geq 0$ )

$$\begin{aligned} \mathbb{U}_{\mathcal{V}}a^\dagger(\boldsymbol{\phi})\mathbb{U}_{\mathcal{V}}^{-1}\Psi &= \mathbb{U}_{\mathcal{V}}a^\dagger(\boldsymbol{\phi})\mathbb{U}_{\mathcal{V}}^{-1}b^\dagger(\boldsymbol{\phi}_1) \dots b^\dagger(\boldsymbol{\phi}_n)\Omega_{\mathcal{V}} \\ &= \mathbb{U}_{\mathcal{V}}a^\dagger(\boldsymbol{\phi})a^\dagger(\boldsymbol{\phi}_1) \dots a^\dagger(\boldsymbol{\phi}_n)\Omega_{\mathcal{V}} \\ &= b^\dagger(\boldsymbol{\phi})b^\dagger(\boldsymbol{\phi}_1) \dots b^\dagger(\boldsymbol{\phi}_n)\Omega_{\mathcal{V}} = b^\dagger(\boldsymbol{\phi})\Psi, \end{aligned} \tag{5.85}$$

which is the first statement of (5.82). The second statement requires somewhat more attention, as our operator product also includes one annihilation operator. We make use of the CAR/CCR of  $a$ - and  $b$ -operators, using the combinatorial factor  $\varepsilon = (-1)$  for fermions and  $\varepsilon = 1$  for bosons. Here, the CAR/CCR are valid for  $a$ -operators by definition (5.75) and for  $b$ -operators, since by means of Lemma 5.4.1, the Bogoliubov relations survive the extension.

$$\begin{aligned}
 \mathbb{U}_\mathcal{V} a(\boldsymbol{\phi}) \mathbb{U}_\mathcal{V}^{-1} \Psi &= \mathbb{U}_\mathcal{V} a(\boldsymbol{\phi}) \mathbb{U}_\mathcal{V}^{-1} b^\dagger(\boldsymbol{\phi}_1) \dots b^\dagger(\boldsymbol{\phi}_n) \Omega_\mathcal{V} \\
 &= \mathbb{U}_\mathcal{V} a(\boldsymbol{\phi}) a^\dagger(\boldsymbol{\phi}_1) \dots a^\dagger(\boldsymbol{\phi}_n) \Omega \\
 &= \sum_{\ell=1}^n \mathbb{U}_\mathcal{V} a^\dagger(\boldsymbol{\phi}_1) \dots a^\dagger(\boldsymbol{\phi}_{\ell-1}) \varepsilon^{\ell+1} \langle \boldsymbol{\phi}, \boldsymbol{\phi}_\ell \rangle a^\dagger(\boldsymbol{\phi}_{\ell+1}) \dots a^\dagger(\boldsymbol{\phi}_n) \Omega \\
 &= \sum_{\ell=1}^n b^\dagger(\boldsymbol{\phi}_1) \dots b^\dagger(\boldsymbol{\phi}_{\ell-1}) \varepsilon^{\ell+1} \langle \boldsymbol{\phi}, \boldsymbol{\phi}_\ell \rangle b^\dagger(\boldsymbol{\phi}_{\ell+1}) \dots b^\dagger(\boldsymbol{\phi}_n) \Omega_\mathcal{V} \\
 &= b(\boldsymbol{\phi}) b^\dagger(\boldsymbol{\phi}_1) \dots b^\dagger(\boldsymbol{\phi}_n) \Omega_\mathcal{V} - \varepsilon^{n+1} b^\dagger(\boldsymbol{\phi}_1) \dots b^\dagger(\boldsymbol{\phi}_n) b(\boldsymbol{\phi}) \Omega_\mathcal{V} \\
 &\stackrel{(5.83)}{=} b(\boldsymbol{\phi}) b^\dagger(\boldsymbol{\phi}_1) \dots b^\dagger(\boldsymbol{\phi}_n) \Omega_\mathcal{V} = b(\boldsymbol{\phi}) \Psi,
 \end{aligned} \tag{5.86}$$

which is the second statement of (5.82), where we used the convention that the above sums are set to zero for  $N = 0$ .

□

### 5.5.2 Bosonic Case

We will now show that for a suitable choice of  $\Omega_\mathcal{V}$ , the operator  $\mathbb{U}_\mathcal{V}$  defined in (5.84) indeed implements the Bogoliubov transformation  $\mathcal{V}$ . The ITP case is treated in Theorem 5.5.5 and the ESS in Theorem 5.5.6.

**Theorem 5.5.5** (Implementation works, bosonic, ITPs). *Consider a bosonic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}$  with  $v^*v$  having countable spectrum. Let  $\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k$  be the ITP space (Definition 5.4.3) with respect to the basis  $(\mathbf{g}_k)_{k \in \mathbb{N}} \subset \ell^2$ . Then,  $\mathcal{V}$  is implemented in the sense of (5.82) by  $\mathbb{U}_\mathcal{V} : \mathcal{D}_\mathcal{F} \rightarrow \widehat{\mathcal{H}}$  (5.84) with the new vacuum vector*

$$\Omega_\mathcal{V} = \prod_{k \in \mathbb{N}}^{\otimes} \Omega_{k, \mathcal{V}} := \prod_{k \in \mathbb{N}}^{\otimes} \left( \left( \left( 1 - \frac{\nu_k^2}{\mu_k^2} \right)^{1/4} \right) \exp \left( -\frac{\nu_k}{2\mu_k} (a^\dagger(\mathbf{g}_k))^2 \right) \Omega_k \right). \tag{5.87}$$

In simple words, Theorem 5.5.5 says that for  $\boldsymbol{\phi} \in \mathcal{D}_\mathcal{F} \subseteq \ell^2$  (5.60), the operators  $a(\boldsymbol{\phi}), a^\dagger(\boldsymbol{\phi})$  defined on  $\mathcal{D}_\mathcal{F} \subset \mathcal{F}$  (5.81) are mapped to  $b(\boldsymbol{\phi}), b^\dagger(\boldsymbol{\phi})$ , which still satisfy the CAR/CCR.

*Proof.* By Lemma 5.5.4,  $\mathbb{U}_\mathcal{V}$  implements  $\mathcal{V}$  (5.82), if we can show the following:

1. The new vacuum  $\Omega_\mathcal{V}$  is well-defined

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2.  $\mathbb{U}_\nu$  is well-defined on  $\mathcal{D}_{\mathcal{F}}$  (Lemma 5.5.3 will be used, here)

3.  $b(\phi)\Omega_\nu = 0$

4.  $\mathbb{U}_\nu^{-1}$  exists on  $\mathbb{U}_\nu[\mathcal{D}_{\mathcal{F}}]$

1.) Well-definedness of  $\Omega_\nu$ : Expression (5.87) is an ITP of one normalized factor per space  $\mathcal{H}_k$ . Hence, it is a  $C$ -sequence, which can be identified with  $\Omega_\nu \in \widehat{\mathcal{H}}$ .

2.) Well-definedness of  $\mathbb{U}_\nu$ : This follows from Lemma 5.5.3, if we can establish  $\Omega_\nu \in \mathcal{S}^\otimes$ . By definition of  $\mathcal{S}^\otimes$ , we need to verify the particle number decay condition  $\|N_k^n \Omega_\nu\| \leq c_{k,n} \|\Omega_\nu\|$ . If it would hold, then  $\|N_k^n \Omega_\nu\| < \infty$  and we would automatically obtain  $\Omega_\nu \in \text{dom}(N_k^n)$ . Now, since all  $\Omega_{k,\nu}$  are normalized, verifying particle number decay boils down to proving

$$\|N_k^n \Omega_{k,\nu}\|_k^2 \leq c_{k,n}^2. \quad (5.88)$$

We may explicitly compute this expression:

$$\|N_k^n \Omega_{k,\nu}\|_k^2 = (1 - 4t^2)^{1/2} \sum_{N=0}^{\infty} \frac{t^{2N} (2N)!}{(N!)^2} (2N)^{2n}, \quad (5.89)$$

with  $t = \left| \frac{\nu_k}{2\mu_k} \right| \in [0, 1/2)$ . Now, the function

$$N \mapsto \frac{t^{2N} (2N)!}{(N!)^2} (2N)^{2n} \leq (2t)^{2N} (2N)^{2n} \quad (5.90)$$

is positive, bounded and decays exponentially at  $N \rightarrow \infty$  since  $0 \leq 2t < 1$ . So,

$$\sum_{N=0}^{\infty} \frac{t^{2N} (2N)!}{(N!)^2} (2N)^{2n} \leq \text{cons.} + \sum_{N=0}^{\infty} (2t)^{2N} (2N)^{2n} =: c_{k,n}^2 < \infty, \quad (5.91)$$

which establishes  $\Omega_\nu \in \mathcal{S}^\otimes$  and hence the claim.

3.)  $b(\phi)$  annihilates  $\Omega_\nu$ : This is straightforward to check. Since  $\phi \in \mathcal{D}_{\mathbf{f}}$ , the following sum over  $k$  is finite:

$$b(\phi)\Omega_\nu = \sum_k \phi_k b(\mathbf{f}_k)\Omega_\nu. \quad (5.92)$$

As in the case where the Shale–Stinespring condition holds, each  $b(\mathbf{f}_k)$  annihilates the corresponding vacuum vector  $\Omega_{k,\nu}$ , so the finite sum above is 0.

4.) Well-definedness of  $\mathbb{U}_{\mathcal{V}}^{-1}$ : The following set is a basis for  $\mathcal{D}_{\mathcal{F}}$ :

$$\{a^\dagger(\mathbf{f}_{k_1}) \dots a^\dagger(\mathbf{f}_{k_N})\Omega \mid N \in \mathbb{N}_0, k_\ell \in \mathbb{N}\}, \quad (5.93)$$

where  $\mathbf{f}_{k_\ell}$  are chosen out of the basis  $(\mathbf{f}_j)_{j \in \mathbb{N}}$  (with  $j = k$ ). This can easily be seen, since every  $a^\dagger(\boldsymbol{\phi})$ ,  $\boldsymbol{\phi} \in \mathcal{D}_{\mathbf{f}}$  can be decomposed by definition of  $\mathcal{D}_{\mathbf{f}}$  into a finite sum over operators proportional to  $a^\dagger(\mathbf{f}_k)$ . If we can show that the set

$$\{b^\dagger(\mathbf{f}_{k_1}) \dots b^\dagger(\mathbf{f}_{k_N})\Omega_{\mathcal{V}} \mid N \in \mathbb{N}_0, k_\ell \in \mathbb{N}\} \subset \widehat{\mathcal{H}}, \quad (5.94)$$

with  $b^\dagger(\mathbf{f}_k) = \mu_k a^\dagger(\mathbf{g}_k) + \nu_k a(\mathbf{g}_k)$ , is linearly independent, we are done, since then  $\ker(\mathbb{U}_{\mathcal{V}}) = \{0\}$ , so  $\mathbb{U}_{\mathcal{V}}$  is injective and hence invertible on its image.

Now, as the application of the operators  $b_k^\dagger := b^\dagger(\mathbf{f}_k)$  and  $\mathbb{U}_{\mathcal{V}}$  preserve the ITP structure, it suffices to show that on each mode  $k$ , the set

$$\{(b_k^\dagger)^N \Omega_{k,\mathcal{V}} \mid N \in \mathbb{N}_0\} \subset \mathcal{H}_k \quad (5.95)$$

is linearly independent. Now, (5.95) is just the image of the set

$$\{(a^\dagger(\mathbf{f}_k))^N \Omega_k \mid N \in \mathbb{N}_0\} \subset \mathcal{F}(\{\mathbf{f}_k\}) \quad (5.96)$$

under a one-mode Bogoliubov transformation  $\mathbb{U}_{k,\mathcal{V}} : \mathcal{F}(\{\mathbf{f}_k\}) \rightarrow \mathcal{H}_k$  (defined as  $\mathbb{U}_{j,\mathcal{V}}$  in (5.30)), where  $\mathcal{F}(\{\mathbf{f}_k\})$  is the one-mode Fock space over  $\mathbf{f}_k$ . For a finite number  $m$  of modes, Bogoliubov transformations can always be implemented by unitary operators, as then the operator  $v : \mathbb{C}^m \rightarrow \mathbb{C}^m$  is always Hilbert–Schmidt. Now, (5.96) is an orthogonal set with no vector being 0, so its image (5.95) under the unitary  $\mathbb{U}_{k,\mathcal{V}}$  is also orthogonal with no vector being 0, and hence it is linearly independent. This finishes the proof.  $\square$

**Theorem 5.5.6** (Implementation works, bosonic, ESS). *Consider a bosonic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}$  with  $v^*v$  having countable spectrum. Let  $\overline{\mathcal{F}}$  be the ESS over  $\mathcal{E}_{\mathbf{g}}$  (see (5.74)) with respect to the basis  $(\mathbf{g}_k)_{k \in \mathbb{N}} \subset \ell^2$ . Define the new vacuum vector*

$$\Omega_{\mathcal{V}} = \underbrace{\exp\left(\frac{1}{4} \sum_k \log\left(1 - \frac{\nu_k^2}{\mu_k^2}\right)\right)}_{=: e^\tau} \underbrace{\exp\left(-\sum_k \frac{\nu_k}{2\mu_k} (a^\dagger(\mathbf{g}_k))^2\right)}_{=: \Psi_{\mathcal{V}}} \Omega = e^\tau \Psi_{\mathcal{V}}. \quad (5.97)$$

Then,  $\mathcal{V}$  is implemented in the sense of (5.82) by  $\mathbb{U}_{\mathcal{V}} : \mathcal{D}_{\mathcal{F}} \rightarrow \overline{\mathcal{F}}$  (5.84).

*Proof.* By Lemma 5.5.4, it suffices to establish the four points in the proof of Theo-

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rem 5.5.5.

1.) Well–definedness of  $\Omega_{\mathcal{V}}$ : The factor  $\mathfrak{r} = \frac{1}{4} \sum_k \log \left( 1 - \frac{\nu_k^2}{\mu_k^2} \right)$  can be interpreted as an element of  $\text{Ren}_1$ , as it is the sum over elements of a complex sequence. So the wave function renormalization is indeed  $e^{\mathfrak{r}} \in \text{eRen}$ . The second factor  $\Psi_{\mathcal{V}}$  is an infinite product of exponentials

$$\exp \left( -\frac{\nu_k}{2\mu_k} (a^\dagger(\mathbf{g}_k))^2 \right) \Omega = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{(-\nu_k)^\ell}{2^\ell \mu_k^\ell} (a^\dagger(\mathbf{g}_k))^{2\ell} \Omega. \quad (5.98)$$

So the  $N$ –sector of  $\Psi_{\mathcal{V}}$  is 0 for odd  $N$ . For even  $N$ , the sector contains a sum over all choices of boson pair numbers  $(\ell_k)_{k \in \mathbb{N}}$ ,  $\ell_k \in \mathbb{N}_0$  such that  $\sum_k \ell_k = N/2$ . Each boson pair is described by a two–particle function supported on the diagonal, as shown in Figure 5.3.

Now, each point in the mode–configuration space  $q \in \mathcal{Q}(\mathbb{N})$  gets assigned at most one summand corresponding to one choice of a sequence  $(\ell_k)_{k \in \mathbb{N}}$ . This value  $\Psi_{\mathcal{V}}(q)$  can now be written as a convergent infinite product: On each mode  $\mathbf{g}_k$ , we fix the basis  $(e_{k,n})_{n \in \mathbb{N}_0} \subset \mathcal{F}(\{\mathbf{g}_k\})$ , as mentioned below (5.62), and write

$$\Psi_{k,\mathcal{V}}(n_k) := \langle e_{k,n_k}, \Psi_{k,\mathcal{V}} \rangle = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{(-\nu_k)^\ell}{2^\ell \mu_k^\ell} \langle e_{k,n_k}, (a^\dagger(\mathbf{g}_k))^{2\ell} \Omega_k \rangle, \quad (5.99)$$

where  $\Psi_{k,\mathcal{V}}$  is the sequence associated with the Bogoliubov vacuum of mode  $\mathbf{g}_k$ , meaning  $\Psi_{k,\mathcal{V}} = \exp \left( -\frac{\nu_k}{2\mu_k} (a^\dagger(\mathbf{g}_k))^2 \right) \Omega_k \in \mathcal{F}(\{\mathbf{g}_k\})$ . Then, we have

$$\Psi_{\mathcal{V}}(q) = \prod_{k \in \mathbb{N}} \Psi_{k,\mathcal{V}}(n_k), \quad (5.100)$$

where  $n_k$  counts, how often mode  $k$  is contained in configuration  $q$ , see also Figure 5.3. As there are finitely many particles in  $q$ , we have  $n_k = 0$  for all but finitely many  $k$ . And as the vacuum is normalized to 1, i.e.,  $\Psi_{k,\mathcal{V}}(0) = 1$ , and all other  $\Psi_{k,\mathcal{V}}(n_k)$  are finite, the infinite product (5.100) is finite, as well.

2.) Well–definedness of  $\mathbb{U}_{\mathcal{V}}$ : This is an immediate consequence of Lemma 5.5.3.

3.)  $b(\phi)$  annihilates  $\Omega_{\mathcal{V}}$ : This is again checked mode–by–mode, as in proof step 3.) of Theorem 5.5.5.

4.) Well–definedness of  $\mathbb{U}_{\mathcal{V}}^{-1}$ : As in proof step 4.) of Theorem 5.5.5, we have to check that  $\mathbb{U}_{\mathcal{V}}$  is injective. And as within that proof step, it suffices to prove that

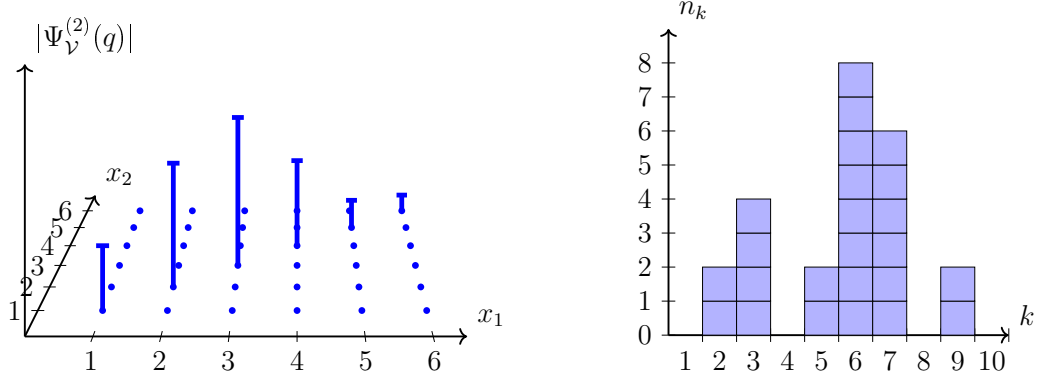


Figure 5.3: Left: The two-particle sector of  $\Psi_\nu$ , visualized.

Right: Particle numbers  $n_k$  for a typical configuration  $q$  with  $\Psi_\nu(q) \neq 0$ .  
Color online.

the set

$$\{b^\dagger(\mathbf{f}_{k_1}) \dots b^\dagger(\mathbf{f}_{k_N}) \Psi_\nu \mid N \in \mathbb{N}_0, k_\ell \in \mathbb{N}\} \subset \overline{\mathcal{F}} \quad (5.101)$$

is linearly independent. Here, the factor  $e^r$  can be removed, as it is nonzero for  $\Omega_\nu$ . Now, assume there was a (finite) nonzero linear combination

$$B = \sum_{m=1}^M b^\dagger(\mathbf{f}_{k_{m,1}}) \dots b^\dagger(\mathbf{f}_{k_{m,N_m}}), \quad \text{with } B\Psi_\nu = 0. \quad (5.102)$$

Then, there exists a maximum mode number  $K \in \mathbb{N}$ , such that  $k_{m,\ell} \leq K$  for any creation operator  $b^\dagger(\mathbf{f}_{k_{m,\ell}})$  in  $B$ . We define the Fock space of all modes  $k \leq K$  by

$$\mathcal{F}_{\leq K} := \mathcal{F}(\{1, \dots, K\}), \quad (5.103)$$

where the Bogoliubov implementer restricted to this space,

$$\mathbb{U}_{\leq K, \nu} = \prod_{k=1}^K \mathbb{U}_{k, \nu}, \quad (5.104)$$

is a unitary operator, as Bogoliubov transformations on finitely many modes are always implementable. Therefore, with  $\Omega_{\leq K} = \prod_{k \leq K}^\otimes \Omega_k$  and  $\Psi_{\leq K, \nu} = \prod_{k \leq K} \Psi_{k, \nu}$ , we have

$$\mathbb{U}_\nu a^\dagger(\mathbf{f}_{k_1}) \dots a^\dagger(\mathbf{f}_{k_N}) \Omega_{\leq K} = b^\dagger(\mathbf{f}_{k_1}) \dots b^\dagger(\mathbf{f}_{k_N}) \Psi_{\leq K, \nu} \in \mathcal{F}_{\leq K}. \quad (5.105)$$

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We can also restrict the operator  $B$  to

$$\tilde{B} : \mathcal{F}_{\leq K} \rightarrow \mathcal{F}_{\leq K} \quad \tilde{B} = \sum_{m=1}^M b^\dagger(\mathbf{f}_{k_{m,1}}) \dots b^\dagger(\mathbf{f}_{k_{m,N_m}}). \quad (5.106)$$

Now, as the vectors  $a^\dagger(\mathbf{f}_{k_1}) \dots a^\dagger(\mathbf{f}_{k_N}) \Omega_{\leq K}$  with  $k_\ell \leq K$  and  $N \in \mathbb{N}_0$  are orthogonal without a zero vector and  $\mathbb{U}_\mathcal{V}$  is unitary, also the set of  $b^\dagger(\mathbf{f}_{k_1}) \dots b^\dagger(\mathbf{f}_{k_N}) \Psi_{\leq K, \mathcal{V}}$  is orthogonal without a zero vector, and hence linearly independent. So  $\Psi_{\tilde{B}} = \tilde{B} \Psi_{\leq K, \mathcal{V}} \neq 0$ . For the associated function, that means,

$$\Psi_{\tilde{B}}(q_{\leq K}) \neq 0 \quad \text{for some } q_{\leq K} \in \mathcal{Q}(\{1, \dots, K\}). \quad (5.107)$$

Now, as  $B$  leaves all modes  $k_{m,\ell} > K$  invariant, we can write

$$\Psi_B(q) = \left( \prod_{k>K} \Psi_{k, \mathcal{V}}(n_k) \right) \Psi_{\tilde{B}}(q_{\leq K}), \quad (5.108)$$

with  $(q_{\leq K})$  containing all modes of  $q$  with  $k \leq K$  and  $n_k$  being the number of times, mode  $k$  appears in configuration  $q$ . The left bracket is 1 for  $n_k \equiv 0$  and for the right term, there exists some  $q_{\leq K}$  with  $\Psi_{\tilde{B}}(q_{\leq K}) \neq 0$  (5.107). Hence, with  $q = q_{\leq K}$ , we have  $\Psi_B(q) \neq 0$ , which contradicts  $B\Omega_\mathcal{V} = 0$  from (5.102). So we have linear independence of (5.101), leading to injectiveness of  $\mathbb{U}_\mathcal{V}$  and finishing the proof. □

### 5.5.3 Fermionic Case

**Theorem 5.5.7** (Implementation works, fermionic, ITPs). *Consider a fermionic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v^* & u \end{pmatrix}$  with  $v^*v$  having countable spectrum. Let  $\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k$  be the ITP space (Definition 5.4.4). Define the new vacuum vector*

$$\begin{aligned} \Omega_\mathcal{V} &= \prod_{j \in J''}^{\otimes} \Omega_{j, \mathcal{V}} \otimes \prod_{i \in I'}^{\otimes} \Omega_{2i, 2i-1, \mathcal{V}} \\ &:= \left( \prod_{j \in J_1''}^{\otimes} a^\dagger(\boldsymbol{\eta}_j) \Omega_j \right) \otimes \left( \prod_{j \in J_0''}^{\otimes} \Omega_j \right) \otimes \left( \prod_{i \in I'}^{\otimes} (\alpha_i - \beta_i a^\dagger(\boldsymbol{\eta}_{2i}) a^\dagger(\boldsymbol{\eta}_{2i-1})) \Omega_{2i, 2i-1} \right), \end{aligned} \quad (5.109)$$

with  $\Omega_{2i, 2i-1}, \Omega_{2i, 2i-1, \mathcal{V}} \in \mathcal{H}_{k(i)}$ . Then,  $\mathcal{V}$  is implemented in the sense of (5.82) by  $\mathbb{U}_\mathcal{V} : \mathcal{D}_\mathcal{F} \rightarrow \widehat{\mathcal{H}}$  (5.84).

*Proof.* Again, by Lemma 5.5.4, it suffices to establish the four points in the proof of Theorem 5.5.5.

1.) Well-definedness of  $\Omega_{\mathcal{V}}$ : As in Theorem 5.5.5, we have a tensor product of infinitely many normalized vectors.

2.) Well-definedness of  $\mathbb{U}_{\mathcal{V}}$ : follows from Lemma 5.5.3, as for fermions,  $\mathcal{S}^{\otimes} = \widehat{\mathcal{H}}$ .

3.)  $b(\phi)$  annihilates  $\Omega_{\mathcal{V}}$ : Follows by a similar argument, as in proof step 3.) of Theorem 5.5.5. We only have to check for  $i \in I'$ , that both  $b(\mathbf{f}_{2i})$  and  $b(\mathbf{f}_{2i-1})$  annihilate  $\Omega_{2i,2i-1,\mathcal{V}}$ , which is done exactly as in the case where the Shale–Stinespring condition holds.

4.) Well-definedness of  $\mathbb{U}_{\mathcal{V}}^{-1}$ : We proceed as in proof step 4.) in Theorem 5.5.5. That means, we have  $\ker(\mathbb{U}_{\mathcal{V}}) = \{0\}$  and hence existence of an inverse, if we can prove that the set

$$\{b^\dagger(\mathbf{f}_{j_1}) \dots b^\dagger(\mathbf{f}_{j_N})\Omega_{\mathcal{V}} \mid N \in \mathbb{N}_0, j_\ell \in J\} \subset \widehat{\mathcal{H}}, \quad (5.110)$$

with  $b^\dagger(\mathbf{f}_j) = a^\dagger(u\mathbf{f}_j) + a(v\overline{\mathbf{f}_j})$  is linearly independent. This again boils down to proving a linear independence statement on each  $\mathcal{H}_k$ . The crucial difference now is, that each tensor product factor  $\mathcal{H}_k$  may be a Fock space over either one or two modes. We abbreviate  $b_j^\# := b^\#(\mathbf{f}_j)$  and  $a_j^\# := a^\#(\boldsymbol{\eta}_j)$ . For two-mode factors indexed by  $i \in I'$ , we need to prove linear independence of the set

$$\{(b_{2i}^\dagger)^{N_1}(b_{2i-1}^\dagger)^{N_2}\Omega_{k(i),\mathcal{V}} \mid N_1, N_2 \in \{0, 1\}\} \subset \mathcal{H}_{k(i)}. \quad (5.111)$$

As in proof step 4.) of Theorem 5.5.5, this follows by the fact that the finite-mode implementer  $\mathbb{U}_{2i,2i-1,\mathcal{V}}$  (see (5.39)) is unitary and maps the orthogonal, zero-free set

$$\{(a_{2i}^\dagger)^{N_1}(a_{2i-1}^\dagger)^{N_2}\Omega_{k(i)} \mid N_1, N_2 \in \{0, 1\}\} \subset \mathcal{F}(\{\mathbf{f}_{2i}\}) \otimes \mathcal{F}(\{\mathbf{f}_{2i-1}\}) \quad (5.112)$$

onto (5.111).

For one-mode factors indexed by  $j \in J''$  we need linear independence of

$$\{(b_j^\dagger)^N\Omega_{k(j),\mathcal{V}} \mid N \in \{0, 1\}\} \subset \mathcal{H}_{k(j)}. \quad (5.113)$$

This follows again by unitarity of  $\mathbb{U}_{j,\mathcal{V}}$ , as well as orthogonality and zero-freeness of the set

$$\{(a_j^\dagger)^N\Omega_{k(j)} \mid N \in \{0, 1\}\} \subset \mathcal{F}(\{\mathbf{f}_j\}), \quad (5.114)$$

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which is mapped to (5.113). By linear independence of (5.111) and (5.113), we obtain linear independence of (5.110), which implies injectivity of  $\mathbb{U}_{\mathcal{V}}$  and finishes the proof.  $\square$

For the ESS, the situation is a bit more delicate: As in Theorem 5.5.6, we would again like to normalize the vacuum sector of each mode to  $\Psi_{j,\mathcal{V}}(0) = 1$  (compare (5.99)). However, for full particle–hole transformations  $j \in J_1''$ , we have  $\Psi_{j,\mathcal{V}}(0) = 0$ , so we cannot normalize the vacuum. As the ESS can only cover a finite number of particles within a configuration  $q \in \mathcal{Q}(\mathbb{N})$ , we also need to restrict to a **finite number of particle–hole transformations**  $|J_1''| < \infty$ .

**Theorem 5.5.8** (Implementation works, fermionic, ESS). *Consider a fermionic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v^* & u \end{pmatrix}$  with  $v^*v$  having countable spectrum. Let  $\overline{\mathcal{F}}$  be the ESS over  $\mathcal{E}_{\boldsymbol{\eta}}$  (see (5.74)) with respect to  $(\boldsymbol{\eta}_j)_{j \in J}$ , and let  $|J_1''| < \infty$ , so **the number of modes with a full particle–hole transformation is finite**. Define the new vacuum vector*

$$\Omega_{\mathcal{V}} = \underbrace{\exp\left(\sum_{i \in I'} \log \alpha_i\right)}_{=: e^{\boldsymbol{\tau}}} \underbrace{\left(\prod_{j \in J_1''} a^\dagger(\boldsymbol{\eta}_j)\right) \left(\prod_{i \in I'} \left(1 - \frac{\beta_i}{\alpha_i} a^\dagger(\boldsymbol{\eta}_{2i}) a^\dagger(\boldsymbol{\eta}_{2i-1})\right)\right)}_{=: \Psi_{\mathcal{V}}} \Omega = e^{\boldsymbol{\tau}} \Psi_{\mathcal{V}}. \quad (5.115)$$

Then,  $\mathcal{V}$  is implemented in the sense of (5.82) by  $\mathbb{U}_{\mathcal{V}} : \mathcal{D}_{\overline{\mathcal{F}}} \rightarrow \overline{\mathcal{F}}$  (5.84).

*Proof.* Also in this case, by Lemma 5.5.4, it suffices to establish the four points from the proof of Theorem 5.5.5.

1.) Well–definedness of  $\Omega_{\mathcal{V}}$ : Since  $\alpha_i \in (0, 1)$ , we have  $\log \alpha_i \in \mathbb{R}$ . By setting  $\boldsymbol{\tau}$  to be the equivalence class of the function  $r : J \rightarrow \mathbb{C}$ ,

$$r(j) = \begin{cases} \frac{1}{2} \log \alpha_j & \text{if } j \in J' \\ 0 & \text{if } j \in J'' \end{cases}, \quad (5.116)$$

we obtain  $\boldsymbol{\tau} = \sum_{i \in I'} \log \alpha_i \in \text{Ren}_1$ , so  $e^{\boldsymbol{\tau}} \in \text{eRen}$ .

The well–definedness argument for  $\Psi_{\mathcal{V}} \in \mathcal{E}_{\overline{\mathcal{F}}}$  is similar to that in proof step 1.) of Theorem 5.5.6. We can write

$$\Psi_{\mathcal{V}}(q) = \prod_{k \in \mathbb{N}} \Psi_{k,\mathcal{V}}(n_k), \quad (5.117)$$

where the occupation number  $n_k$  denotes, how often mode  $k$  appears in configuration  $q$ . We may allow for  $n_k \geq 2$ , but since we are in the fermionic case,  $\Psi_{k,\nu}(n_k) = 0$  for  $n_k \notin \{0, 1\}$ . Further, for all  $i \in I'$ , the associated vector  $\Psi_{k(i),\nu}$  describes two modes, so we have a pair of two occupation numbers  $(n_{2i}, n_{2i-1}) = n_{k(i)}$  and

$$\begin{aligned}
 \bullet \text{ for } j \in J_0'' : & \quad \Psi_{k(j),\nu}(n_{k(j)}) = 1 \text{ if } n_{k(j)} = 0 & \quad \text{and else } \Psi_{k(j),\nu}(n_{k(j)}) = 0 \\
 \bullet \text{ for } j \in J_1'' : & \quad \Psi_{k(j),\nu}(n_{k(j)}) = 1 \text{ if } n_{k(j)} = 1 & \quad \text{and else } \Psi_{k(j),\nu}(n_{k(j)}) = 0 \\
 \bullet \text{ for } i \in I' : & \quad \Psi_{k(i),\nu}(n_{k(i)}) = 1 \text{ if } n_{k(i)} = (0, 0) \\
 & \quad \Psi_{k(i),\nu}(n_{k(i)}) = -\frac{\beta_i}{\alpha_i} \text{ if } n_{k(i)} = (1, 1) & \quad \text{and else } \Psi_{k(i),\nu}(n_{k(i)}) = 0.
 \end{aligned} \tag{5.118}$$

So  $\Psi_\nu(q)$  can only be nonzero if  $q$  contains none of the modes  $j \in J_0''$ , all of the (finitely many) modes  $j \in J_1''$  and a finite subset of the modes  $2i, 2i-1$  for  $i \in I'$ . In that case,  $\Psi_\nu(q)$  is a product of infinitely many times a factor 1 and finitely many times a finite factor, so all  $\Psi_\nu(q)$  are well-defined and finite.

2.) Well-definedness of  $\mathbb{U}_\nu$ : This is an immediate consequence of Lemma 5.5.3.

3.)  $b(\phi)$  annihilates  $\Omega_\nu$ : This is proven mode-by-mode in the same way, as within proof step 3.) of Theorem 5.5.7.

4.) Well-definedness of  $\mathbb{U}_\nu^{-1}$ : The argument is also analogous to that of proof step 4.) of Theorem 5.5.6. We reduce invertibility (which means injectivity) of  $\mathbb{U}_\nu$  to linear independence of the set

$$\{b^\dagger(\mathbf{f}_{j_1}) \dots b^\dagger(\mathbf{f}_{j_N}) \Psi_\nu \mid N \in \mathbb{N}_0, j_\ell \in J\} \subset \overline{\mathcal{F}}. \tag{5.119}$$

Again, we assume there was a (finite) nonzero linear combination

$$B = \sum_{m=1}^M b^\dagger(\mathbf{f}_{j_{m,1}}) \dots b^\dagger(\mathbf{f}_{j_{m,N_m}}), \quad \text{with } \Psi_B = B\Psi_\nu = 0. \tag{5.120}$$

Then,  $B$  would possess a maximum occupied mode  $K$ , i.e., in  $B$ , there is no  $\mathbf{f}_{j_{m,N}}$  with  $k(j_{m,N}) > K$ . We denote the (finite-dimensional) fermionic Fock space and the associated Bogoliubov transformation by

$$\mathcal{F}_{\leq K} = \prod_{k \leq K}^{\otimes} \mathcal{H}_k, \quad \mathbb{U}_{\leq K, \nu} = \left( \prod_{k(j) \leq K}^{\otimes} \mathbb{U}_{j, \nu} \right) \otimes \left( \prod_{k(i) \leq K}^{\otimes} \mathbb{U}_{2i, 2i-1, \nu} \right). \tag{5.121}$$

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We restrict  $B$  to  $\tilde{B} : \mathcal{F}_{\leq K} \rightarrow \mathcal{F}_{\leq K}$ , as in (5.106), define  $\Psi_{\tilde{B}} = \tilde{B}\Psi_{\leq K, \mathcal{V}}$  and obtain

$$\Psi_B(q) = \left( \prod_{k>K} \Psi_{k, \mathcal{V}}(n_k) \right) \Psi_{\tilde{B}}(q_{\leq K}). \quad (5.122)$$

The first factor is nonzero for at least one choice of occupation numbers  $n_k$  per  $k > K$ , as  $\Psi_{\mathcal{V}} \neq 0$ . The factor  $\Psi_{\tilde{B}}$  is the image of a nonzero vector under the operator  $\mathbb{U}_{\leq K, \mathcal{V}}$ , which is unitary, as it acts on finitely many modes. So  $\Psi_{\tilde{B}}(q_{\leq K}) \neq 0$  for some  $q_{\leq K}$ , which yields  $\Psi_B \neq 0$  and hence the desired contradiction.  $\square$

29. It is crucial that the fermionic ITP space has been chosen as  $\widehat{\mathcal{H}} = \prod_{k \in \mathbb{N}}^{\otimes} \mathcal{H}_k$ , with two-mode spaces  $\mathcal{H}_k = \mathcal{F}(\{\boldsymbol{\eta}_{2i}\}) \otimes \mathcal{F}(\{\boldsymbol{\eta}_{2i-1}\})$  for Cooper pairs  $i \in I'$ . If we had just chosen a product of one-mode spaces  $\prod_{j \in J}^{\otimes} \mathcal{F}(\{\boldsymbol{\eta}_j\})$ , then  $\Omega_{\mathcal{V}}$  might not be in this space, depending on  $\mathcal{V}$ .

As a counter-example, consider a Bogoliubov transformation  $\mathcal{V}$  with countably infinitely many Cooper pairs  $i \in I'$ , such that  $\alpha_i = \beta_i = \frac{1}{\sqrt{2}}$ . Then, each Cooper pair is in the state

$$\Psi_i := \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \in \mathbb{C}^4, \quad (5.123)$$

i.e., we have a “half particle–hole transformation”. The state (5.123) cannot be written as a tensor product of two vectors in  $\mathbb{C}^2$ . So when evaluating the formal ITP

$$\Omega_{\mathcal{V}} = \prod_{i \in I'}^{\otimes} \Psi_i, \quad (5.124)$$

we obtain a sum of  $C$ –sequences: For each pair  $i$ , one has to choose either  $|0\rangle \otimes |0\rangle$  or  $|1\rangle \otimes |1\rangle$  as a contribution to  $\Omega_{\mathcal{V}}$  and sum over all choices. But now, there are uncountably many such choices, as each corresponds to a binary number of infinitely many digits. And each one gives a contribution of norm  $\prod_{i \in I'} \frac{1}{\sqrt{2}} = 0$ . So  $\Omega_{\mathcal{V}} = 0$ , making  $\mathbb{U}_{\mathcal{V}}$  non-invertible.

## 5.6 Diagonalization: Extended

By Theorems 5.5.5–5.5.8, we know that  $\mathcal{V}$  is implementable by  $\mathbb{U}_{\mathcal{V}}$  in the extended sense under fairly mild assumptions (namely  $v^*v$  having countable spectrum). Now it would be interesting to know if this  $\mathbb{U}_{\mathcal{V}}$  can be used to diagonalize quadratic Hamiltonians  $H$ .

This requires precise definitions of “quadratic Hamiltonian” and “diagonalized”, first, which we give in Section 5.6.1. The following Sections 5.6.2 and 5.6.3 provide diagonalizability criteria in Propositions 5.6.2 and 5.6.3, which are a simple consequence of combining results from [216] about when a diagonalizing  $\mathcal{V}$  exists with results from Theorems 5.5.5–5.5.8 about when  $\mathbb{U}_{\mathcal{V}}$  is implementable in the extended sense.

### 5.6.1 Definition of Extended Diagonalization

Recall that the extended operator algebra  $\overline{\mathcal{A}}_{\mathbf{e}}$ , defined in (5.58) with respect to a basis  $\mathbf{e} = (\mathbf{e}_j)_{j \in \mathbb{N}}$ , consists of all sums  $H$  that assign to each finite operator product  $P_{\mathbf{e}} = a_{j_1}^{\#1} \dots a_{j_m}^{\#m}$  a complex coefficient  $H_{j_1, \#1, \dots, j_m, \#m} \in \mathbb{C}$ . So the entire sum reads:

$$H = \sum_{P_{\mathbf{e}} \in \Pi_{\mathbf{e}}} H_{j_1, \#1, \dots, j_m, \#m} P_{\mathbf{e}}. \quad (5.125)$$

Here,  $\Pi_{\mathbf{e}}$  is the set of all finite operator products.

A **formal quadratic Hamiltonian** is an element  $H \in \overline{\mathcal{A}}_{\mathbf{e}}$ , where  $H_{j_1, \#1, \dots, j_m, \#m} \neq 0$  only appears for  $m = 2$  and where  $H^* = H$ . We will impose a normal ordering on quadratic Hamiltonians (see Remark 30), so they can be written as:

$$H = \frac{1}{2} \sum_{j, k \in \mathbb{N}} (2h_{jk} a_j^\dagger a_k \mp k_{jk} a_j^\dagger a_k^\dagger + \overline{k_{jk}} a_j a_k), \quad (5.126)$$

where for  $\mp$ , we have to take  $+$  in the bosonic and  $-$  in the fermionic case. The term “formal” stresses that  $H$  is not necessarily an operator on Fock space.

Now, to each such  $H$  we can associate a block matrix (see Section 5.12.1)

$$A_H = \begin{pmatrix} h & \mp k \\ \overline{k} & \mp \overline{h} \end{pmatrix}, \quad (5.127)$$

with  $h = (h_{jk})_{j, k \in \mathbb{N}}$ ,  $k = (k_{jk})_{j, k \in \mathbb{N}}$  being matrices of infinite size.

Consider a Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ \overline{v} & \overline{u} \end{pmatrix}$ . The corresponding algebraic Bogoliubov transformation  $\mathcal{V}_{\overline{\mathcal{A}}}$  on  $\overline{\mathcal{A}}_{\mathbf{e}}$  is defined similar to  $\mathcal{V}_{\mathcal{A}}$  on  $\mathcal{A}$  (see Section

5.9.1):

$$\begin{aligned}
 \mathcal{V}_{\overline{\mathcal{A}}_e} : \overline{\mathcal{A}}_e \ni \text{dom}(\mathcal{V}_{\overline{\mathcal{A}}_e}) &\rightarrow \overline{\mathcal{A}}_e, \\
 a_j^\dagger \mapsto b_j^\dagger &= \sum_k (u_{jk} a_k^\dagger + \overline{v_{jk}} a_k), \\
 a_j \mapsto b_j &= \sum_k (v_{jk} a_k^\dagger + \overline{u_{jk}} a_k),
 \end{aligned} \tag{5.128}$$

and a **normal ordering** is performed after the transformation. Note that this expression might not be well-defined, as the sums over  $k$  might diverge. So we needed to restrict to a subspace  $\text{dom}(\mathcal{V}_{\overline{\mathcal{A}}_e}) \subseteq \overline{\mathcal{A}}_e$ . The transformed operator and its associated block matrices are

$$\tilde{H} = \mathcal{V}_{\overline{\mathcal{A}}_e}^*(H), \quad A_{\tilde{H}} = \mathcal{V}^* A_H \mathcal{V}, \tag{5.129}$$

where the latter can easily be seen by (5.242). A diagonalization is now given by an implementable Bogoliubov transformation  $\mathcal{V}$ , which eliminates all  $a^\dagger a^\dagger$ - and  $aa$ -terms from  $H$ :

**Definition 5.6.1.** A formal quadratic Hamiltonian  $H \in \overline{\mathcal{A}}_e$  is called **diagonalizable in the extended sense** if there exists a Bogoliubov transformation  $\mathcal{V}$ , such that

$$\mathcal{V}^* A_H \mathcal{V} = \begin{pmatrix} E & 0 \\ 0 & \mp E \end{pmatrix}, \tag{5.130}$$

with  $E \geq 0$  being Hermitian,  $\mp$  being  $+$  in the bosonic and  $-$  in the fermionic case, and where  $\mathcal{V}$  is implementable in the extended sense (see Definition 5.5.1).

That means, the Hamiltonian associated with  $A_{\tilde{H}}$  is:

$$\tilde{H} = \sum_{j,k \in \mathbb{N}} E_{jk} a_j^\dagger a_k = d\Gamma(E). \tag{5.131}$$

The matrix  $E$  provides a well-defined positive semidefinite quadratic form on  $\mathcal{D}_e$ . So by Friedrichs' theorem, it has at least one self-adjoint extension on  $\text{dom}(E)$ , which we also denote by  $E$ . Following [187, Sect. VIII.10], the operator  $d\Gamma(E)$  is essentially self-adjoint on

$$\bigoplus_{n=0}^{\infty} \text{dom}(E)^{\otimes n} \subseteq \mathcal{F}, \tag{5.132}$$

so  $\tilde{H}$  defines quantum dynamics on  $\mathcal{F}$ .

30. *Normal ordering constant.* Our process of “diagonalizing” a Hamiltonian  $H$  actually consists of conjugating it with  $\mathbb{U}_{\mathcal{V}}$ , so  $a^\sharp$  is replaced by  $b^\sharp$ , plus a

subsequent normal ordering process. This process is equivalent to adding a constant to the Hamiltonian, namely

$$c = \frac{1}{2} (\operatorname{tr}(E) - \operatorname{tr}(h)) = \frac{1}{2} \sum_j (E_{jj} - h_{jj}). \quad (5.133)$$

The sum might be divergent and hence not a complex number, but we may interpret it as an infinite renormalization constant  $c \in \operatorname{Ren}_1(\mathbb{N})$  (see Sections 3.2 and 5.2.2), namely the one associated with the equally denoted function  $c \in \mathcal{E}(\mathbb{N})$ ,  $c(j) = \frac{1}{2}(E_{jj} - h_{jj})$ .

So the relation between the original Hamiltonian  $H$  and the diagonalized Hamiltonian  $\tilde{H}$  actually is

$$\tilde{H} = \mathbb{U}_{\mathcal{V}}^{-1}(H + c)\mathbb{U}_{\mathcal{V}},$$

which is in accordance with (5.2).

### 5.6.2 Bosonic Case

Conditions for the existence of a  $\mathcal{V}$ , such that  $\mathcal{V}^*A_H\mathcal{V}$  is block-diagonal, can be found in [216, Thms. 1 and 4]. We can use them to derive conditions for when a formal quadratic Hamiltonian  $H$  is diagonalizable in the extended sense:

**Proposition 5.6.2** (Extended diagonalizability, bosonic case). *Let a formal quadratic bosonic Hamiltonian  $H$  (5.126) be given such that for the associated block matrix  $A_H$  (5.127) we have  $h > 0$ , and that  $G = h^{-1/2}kh^{-1/2}$  is a bounded operator with  $\|G\| < 1$ . Following [216, Thm. 1], there exists a bosonic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}$  such that*

$$\mathcal{V}^*A_H\mathcal{V} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}. \quad (5.134)$$

Suppose further that  $v^*v$  has countable spectrum.

Then,  $H$  is diagonalizable in the extended sense, both on the ITP space  $\widehat{\mathcal{H}}$  and on the ESS  $\overline{\mathcal{F}}$ .

*Proof.* Consider Definition 5.6.1 for diagonalizability. The existence of  $\mathcal{V}$  as a block matrix associated with a bounded operator on  $\ell^2$  is a direct consequence of [216, Thm. 1], where the representation of  $\mathcal{V}$  and  $A_H$  on  $\mathfrak{h} \oplus \mathfrak{h}^*$  (see Section 5.9.3)

instead of our representation on  $\ell^2 \oplus \ell^2$  (see Section 5.9.4) has been used. The same theorem also yields  $E > 0 \Rightarrow E \geq 0$ . Note that the matrix entries  $k_{jk}$  in our representation on  $\ell^2 \oplus \ell^2$  agree with the matrix elements in the representation on  $\mathfrak{h} \oplus \mathfrak{h}^*$ , which are  $\langle \mathcal{J}e_j, ke_k \rangle = \langle e_j, \mathcal{J}^*ke_k \rangle$ . So  $\mathcal{J}^*k$  in [216] corresponds to  $k$  in our case.

If the spectrum of  $v^*v$  is countable, then implementability of  $\mathcal{V}$  in the extended sense follows from Theorems 5.5.5 (for  $\widehat{\mathcal{H}}$ ) and 5.5.6 (for  $\overline{\mathcal{F}}$ ). □

### 5.6.3 Fermionic Case

**Proposition 5.6.3** (Extended diagonalizability, fermionic case). *Let a formal quadratic fermionic Hamiltonian  $H$  (5.126) be given such that for the associated block matrix  $A_H$  (5.127),  $\dim \text{Ker}(A_H)$  is even or  $\infty$ . Following [216, Thm. 4], there exists some fermionic Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}$  such that*

$$\mathcal{V}^* A_H \mathcal{V} = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}. \quad (5.135)$$

*Suppose further that  $v^*v$  has countable spectrum.*

*Then,  $H$  is diagonalizable in the extended sense on the ITP space  $\widehat{\mathcal{H}}$ .*

*If 1 is not an eigenvalue of  $v^*v$ , or is an eigenvalue of finite multiplicity, then  $H$  is also diagonalizable in the extended sense on the ESS  $\overline{\mathcal{F}}$ .*

*Proof.* Existence of a unitary  $\mathcal{V}$  and of  $E \geq 0$  follows from [216, Thm. 4]. By unitarity,  $\mathcal{V}^* \mathcal{V} = 1 = \mathcal{V} \mathcal{V}^*$ , so  $\mathcal{V}$  is a fermionic Bogoliubov transformation.

If  $\sigma(v^*v)$  is countable, then implementability on  $\widehat{\mathcal{H}}$  follows from Theorem 5.5.7.

If further, 1 is not an eigenvalue of  $v^*v$  with infinite multiplicity, then implementability on  $\overline{\mathcal{F}}$  follows from Theorem 5.5.8. □

## 5.7 Applications

### 5.7.1 Quadratic Bosonic Interaction

Our first example for a quadratic Hamiltonian whose diagonalization requires Bogoliubov transformations “beyond the Shale–Stinespring condition” is inspired by [225]. We consider a free massive bosonic scalar field, which is interacting by a Wick square:  $\phi(\mathbf{x})^2$ ,  $\phi(\mathbf{x}) = a^\dagger(\mathbf{x}) + a(\mathbf{x})$ . We discretize the momentum by putting the system in a box  $\mathbf{x} \in [-\pi, \pi]^3$  with periodic boundary conditions. Further, the Wick

square is weighted by a real-valued external field  $\kappa \in C_c^\infty([-\pi, \pi]^3)$ ,  $\kappa(\mathbf{x}) \in \mathbb{C}$ . The Hamiltonian then reads as

$$\begin{aligned} H &= d\Gamma(\varepsilon_{\mathbf{p}}) + \frac{1}{2} \int \kappa(\mathbf{x}) : \phi(\mathbf{x})^2 : d\mathbf{x} \\ &= \frac{1}{2} \sum_{\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{Z}^3} (2\varepsilon_{\mathbf{p}_1} \delta(\mathbf{p}_1 - \mathbf{p}_2) a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2} + 2\hat{\kappa}(-\mathbf{p}_1 + \mathbf{p}_2) a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2} + \\ &\quad + \hat{\kappa}(-\mathbf{p}_1 - \mathbf{p}_2) a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger + \hat{\kappa}(\mathbf{p}_1 + \mathbf{p}_2) a_{\mathbf{p}_1} a_{\mathbf{p}_2}), \end{aligned} \quad (5.136)$$

with  $\hat{\kappa}(\mathbf{p}) = \overline{\hat{\kappa}(-\mathbf{p})}$  denoting the Fourier transform of  $\kappa(\mathbf{x})$ . For simplicity, we assume that  $\kappa(\mathbf{x}) = \text{const.}$ , so we can write  $\hat{\kappa}(\mathbf{p}) = \kappa \delta(\mathbf{p})$ ,  $\kappa \in \mathbb{R}$ .

**Proposition 5.7.1.** *For interactions  $\kappa > -\frac{m}{2}$  but  $\kappa \neq 0$ , the Hamiltonian  $H$  is diagonalizable in the extended sense both on  $\mathcal{H}$  and  $\overline{\mathcal{F}}$ . However, the transformation  $\mathcal{V}$  violates the Shale–Stinespring condition, so  $H$  is not diagonalizable on  $\mathcal{F}$ .*

*Proof.* We may directly compute  $\mathcal{V}$  and then apply Proposition 5.6.2. As demonstrated in Section 5.12.1,  $H$  can be identified with the block matrix

$$A_H = \begin{pmatrix} h & k \\ \bar{k} & \bar{h} \end{pmatrix}, \quad (5.137)$$

with entries

$$h_{\mathbf{p}_1, \mathbf{p}_2} = (\varepsilon_{\mathbf{p}_1} + \kappa) \delta(\mathbf{p}_1 - \mathbf{p}_2), \quad k_{\mathbf{p}_1, \mathbf{p}_2} = \kappa \delta(\mathbf{p}_1 + \mathbf{p}_2). \quad (5.138)$$

The diagonalization of  $H$  now translates into diagonalizing  $A_H$  by some bosonic Bogoliubov transformation  $\mathcal{V}$ , such that

$$\mathcal{V}^* A_H \mathcal{V} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \quad E_{\mathbf{p}_1, \mathbf{p}_2} = E_{\mathbf{p}_1} \delta(\mathbf{p}_1 - \mathbf{p}_2). \quad (5.139)$$

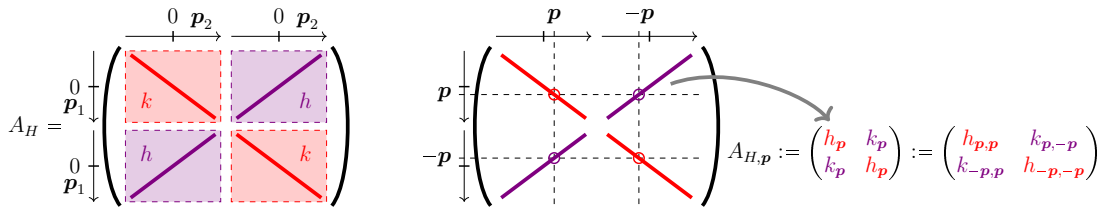


Figure 5.4: Decomposing the matrix  $A_H$  into modes. Color online.

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This can be done by decomposing the matrices  $A_H$  into modes  $\mathbf{p}$  as

$$A_H = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} A_{H,\mathbf{p}}, \quad A_{H,\mathbf{p}} = \begin{pmatrix} h_{\mathbf{p}} & k_{\mathbf{p}} \\ k_{\mathbf{p}} & h_{\mathbf{p}} \end{pmatrix} \quad (5.140)$$

(see Figure 5.4) with entries

$$h_{\mathbf{p}} = (\varepsilon_{\mathbf{p}} + \kappa), \quad k_{\mathbf{p}} = \kappa, \quad (5.141)$$

and diagonalizing all  $A_{H,\mathbf{p}} \in \mathbb{C}^{2 \times 2}$  separately via

$$\mathcal{V} = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} \mathcal{V}_{\mathbf{p}}, \quad \mathcal{V}_{\mathbf{p}}^* A_{H,\mathbf{p}} \mathcal{V}_{\mathbf{p}} = \begin{pmatrix} E_{\mathbf{p}} & 0 \\ 0 & E_{\mathbf{p}} \end{pmatrix}. \quad (5.142)$$

Following [216, Sect. 1.3], this is done for  $|k_{\mathbf{p}}| < h_{\mathbf{p}}$  by

$$\mathcal{V}_{\mathbf{p}} = \begin{pmatrix} u_{\mathbf{p}} & v_{\mathbf{p}} \\ \bar{v}_{\mathbf{p}} & \bar{u}_{\mathbf{p}} \end{pmatrix}, \quad u_{\mathbf{p}} = c_{\mathbf{p}}, \quad v_{\mathbf{p}} = c_{\mathbf{p}} \frac{-G_{\mathbf{p}}}{1 + \sqrt{1 - G_{\mathbf{p}}^2}}, \quad (5.143)$$

$$G_{\mathbf{p}} = k_{\mathbf{p}} h_{\mathbf{p}}^{-1}, \quad c_{\mathbf{p}} = \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{1 - G_{\mathbf{p}}^2}}}, \quad (5.144)$$

with eigenvalues  $E_{\mathbf{p}} = \sqrt{h^2 - k^2}$ . The condition  $|k_{\mathbf{p}}| < h_{\mathbf{p}}$  amounts to

$$|k_{\mathbf{p}}| < h_{\mathbf{p}} \quad \Leftrightarrow \quad |\kappa| < \sqrt{|\mathbf{p}|^2 + m^2} + \kappa, \quad (5.145)$$

which is satisfied for all  $\mathbf{p} \in \mathbb{Z}^3$ , if and only if  $\kappa > -\frac{m}{2}$ .  $h_{\mathbf{p}} > 0$  also holds in that case and (5.142) defines a Bogoliubov transformation  $\mathcal{V}$  diagonalizing  $A_H$ .

Out of the conditions in Proposition 5.6.2,  $h > 0$  and  $\|h^{-1/2} k h^{-1/2}\| < 1$  follow from  $h_{\mathbf{p}}$  and  $|k_{\mathbf{p}}| < h_{\mathbf{p}}$  after taking a direct sum. For the spectrum of  $v^* v$ , since  $v = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} v_{\mathbf{p}}$  can be decomposed into modes, the same holds for  $v^* v$ , which has therefore countable spectrum. So Proposition 5.6.2 applies and  $H$  is diagonalizable in the extended sense on both  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$ .

For the second claim, we have to show that

$$\mathrm{tr}(v^* v) = \sum_{\mathbf{p} \in \mathbb{Z}^3} |v_{\mathbf{p}}|^2 = \infty. \quad (5.146)$$

If  $|\mathbf{p}|$  is large enough (say,  $|\mathbf{p}| > p_{\max} > 0$ ), we have the bounds

$$G_{\mathbf{p}} = \frac{\kappa}{\kappa + \sqrt{|\mathbf{p}|^2 + m^2}} \leq \frac{\kappa}{|\mathbf{p}|}, \quad G_{\mathbf{p}} \geq \frac{\kappa d}{|\mathbf{p}|}, \quad (5.147)$$

with  $d < 1$  arbitrary, so

$$|v_{\mathbf{p}}|^2 = \frac{1 + \sqrt{1 - G_{\mathbf{p}}^2}}{2\sqrt{1 - G_{\mathbf{p}}^2}} \frac{G_{\mathbf{p}}^2}{(1 + \sqrt{1 - G_{\mathbf{p}}^2})^2} \geq \frac{\kappa^2 d^2}{4|\mathbf{p}|^2}. \quad (5.148)$$

Now, we can write the sum in (5.146) as an integral over a weighted sum of indicator functions  $\chi_{Q(\mathbf{p})}(\cdot)$  with  $Q(\mathbf{p})$  being half-open cubes with side length 1 centered at  $\mathbf{p} = (p_1, p_2, p_3)$ :

$$Q(\mathbf{p}) = [p_1 - \frac{1}{2}, p_1 + \frac{1}{2}) \times [p_2 - \frac{1}{2}, p_2 + \frac{1}{2}) \times [p_3 - \frac{1}{2}, p_3 + \frac{1}{2}), \quad (5.149)$$

$$f(\mathbf{p}') = \sum_{\mathbf{p} \in \mathbb{Z}^3} |v_{\mathbf{p}}|^2 \chi_{Q(\mathbf{p})}(\mathbf{p}'), \quad \sum_{\mathbf{p} \in \mathbb{Z}^3} |v_{\mathbf{p}}|^2 = \int_{\mathbb{R}^3} f(\mathbf{p}') d\mathbf{p}'. \quad (5.150)$$

For  $|\mathbf{p}'| > p_{\max}$ , we have

$$f(\mathbf{p}') \geq \frac{\kappa^2 d^2}{4(|\mathbf{p}'| + \frac{\sqrt{3}}{2})^2}, \quad (5.151)$$

$$\Rightarrow \sum_{\mathbf{p} \in \mathbb{Z}^3} |v_{\mathbf{p}}|^2 \geq \int_{|\mathbf{p}'| > p_{\max}} f(\mathbf{p}') d\mathbf{p}' \geq \int_{p_{\max}}^{\infty} \frac{\kappa^2 d^2}{4(|\mathbf{p}'| + \frac{\sqrt{3}}{2})^2} 4\pi |\mathbf{p}'|^2 d|\mathbf{p}'| = \infty, \quad (5.152)$$

where the integral is linearly divergent, which establishes the claim that the Shale–Stinespring condition is violated.  $\square$

31. *Infinite volume and continuous  $\mathbf{p}$ .* The original Hamiltonian  $H$  in [225] is not restricted to a finite volume. That means,  $\mathbf{p} \in \mathbb{R}^3$  instead of  $\mathbf{p} \in \mathbb{Z}^3$  is considered. In that case, a decomposition into modes is also possible and even renders a  $\mathcal{V}$  diagonalizing  $A_H$ . However, the spectrum of  $v^*v$  is then no longer countable, so Proposition 5.7.1 no longer applies. This can be seen as  $k_{\mathbf{p}}$  is the same for all  $\mathbf{p}$ , but  $h_{\mathbf{p}}$  attains all values in  $[\kappa + m, \infty)$ , i.e., uncountably many of them. So  $v_{\mathbf{p}}$  also attains uncountably many values.

In order to treat this case, it would be necessary to extend Theorems 5.5.5 and 5.5.6 to generic  $v^*v$ . If this can be done, a proof of implementability for  $\mathcal{V}$  will be straightforward.

32. *Position-dependent  $\kappa(\mathbf{x})$ .* In contrast to [225], we also assumed an interacti-

on strength  $\kappa(\mathbf{x})$  that is constant in the cube  $[-\pi, \pi]^3$ . Physically, it would be desirable to treat any  $\kappa \in C_c^\infty$ . However, in that case, we would no longer have a simple decomposition into modes  $A_H = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} A_{H, \mathbf{p}}$ , so the diagonalization can no longer be done explicitly. Further, it might occur that  $v^*v$  has uncountable spectrum, so Proposition 5.7.1 may no longer apply. However, if there was a version of Theorems 5.5.5 and 5.5.6 for generic  $v^*v$ , then we expect to obtain a diagonalizability result for small enough interactions (i.e.,  $k$  is a bounded operator with  $k < -m/2$ ).

33. *Wick square is not diagonalizable.* It may be tempting to set  $\varepsilon_{\mathbf{p}} = 0$  and to try a diagonalization of only the interaction Hamiltonian  $\frac{1}{2} \int \kappa(\mathbf{x}) : \phi(\mathbf{x}) : d\mathbf{x}$ . However, bosonic Wick squares are in general not diagonalizable by a Bogoliubov transformation, as “the interaction is too large”. For example, on one mode ( $\mathfrak{h} \cong \mathbb{C}$ ), the matrix associated with a Wick square  $H_W = 2a^\dagger a + a^\dagger a^\dagger + aa$  is

$$A_{H_W} = \begin{pmatrix} h & k \\ \bar{k} & \bar{h} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (5.153)$$

so  $\|h^{-1/2}kh^{-1/2}\| = 1$  and Proposition 5.6.2 does not apply. The same problem occurs when treating a direct sum of several independent modes. However, in the case of bosonic Wick products, an interpretation as a self-adjoint operator is possible by a suitable GNS construction [229].

### 5.7.2 BCS Model

Another example, where Bogoliubov transformations appear, which are not implementable on  $\mathcal{F}$ , is the Bardeen–Cooper–Schrieffer (BCS) model for explaining superconductivity [223, 224]. The “Hartree-like approximation” state [223, (2.16)] corresponds to a formal fermionic Bogoliubov vacuum state  $\Omega_{\mathcal{V}}$  as in (5.109). An argumentation by Haag [224] suggests that in the infinite volume limit, the BCS Hamiltonian can indeed be diagonalized by a corresponding Bogoliubov transformation, which is not implementable on Fock space.

We consider a similar model of a fermionic gas inside a box with periodic boundary conditions  $\mathbf{x} \in [-\pi, \pi]^3$ . Hence, we have discretized, arbitrarily large momenta  $\mathbf{p} \in \mathbb{Z}^3$ , as well as two spins  $s \in \{\uparrow, \downarrow\}$ , leading to a one-particle Hilbert space  $\mathfrak{h} = L^2(\mathbb{Z}^3 \times \{\uparrow, \downarrow\})$ . The corresponding Fock space is  $\mathcal{F} = \mathcal{F}(\mathbb{Z}^3 \times \{\uparrow, \downarrow\})$ . We consider the following quadratic Hamiltonian (see [224]), which provides an

approximate description for the fermionic gas:

$$H' = H_0 + H_I' = \sum_{\mathbf{p} \in \mathbb{Z}^3} \left( \varepsilon_{\mathbf{p}} a_{\mathbf{p}, \uparrow}^\dagger a_{\mathbf{p}, \uparrow} + \varepsilon_{\mathbf{p}} a_{\mathbf{p}, \downarrow}^\dagger a_{\mathbf{p}, \downarrow} - \tilde{\Delta}_{\mathbf{p}} a_{\mathbf{p}, \uparrow}^\dagger a_{\mathbf{p}, \downarrow}^\dagger + \overline{\tilde{\Delta}_{\mathbf{p}}} a_{\mathbf{p}, \uparrow} a_{\mathbf{p}, \downarrow} \right), \quad (5.154)$$

with kinetic energy  $\varepsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} - \mu \in \mathbb{R}$  and interaction strength  $\tilde{\Delta}_{\mathbf{p}} \in \mathbb{C}$ , of which we assume  $\tilde{\Delta}_{\mathbf{p}} \neq 0$ . As a basis  $(e_j)_{j \in J}$  for identifying  $\mathfrak{h}$  with  $\ell^2$ , we choose here

$$(e_{\mathbf{p}, s})_{\substack{\mathbf{p} \in \mathbb{Z}^3 \\ s \in \{\uparrow, \downarrow\}}} \subset L^2(\mathbb{Z}^3 \times \{\uparrow, \downarrow\}), \quad e_{\mathbf{p}, s}(\mathbf{p}', s') = \delta_{\mathbf{p}\mathbf{p}'} \delta_{ss'}, \quad (5.155)$$

with  $\delta$  being the Kronecker delta. The corresponding canonical basis of  $\ell^2$  is denoted  $(e_{\mathbf{p}, s})_{\substack{\mathbf{p} \in \mathbb{Z}^3 \\ s \in \{\uparrow, \downarrow\}}}$  with a countable index set. In order to obtain momentum conserva-

tion, we have to interpret  $a_{\mathbf{p}, \downarrow}^\dagger, a_{\mathbf{p}, \downarrow}$  as creating/annihilating a fermion of momentum  $-\mathbf{p}$ . The mode index  $\mathbf{p}$  is only used for an easier decomposition into modes.

**Proposition 5.7.2.** *The Hamiltonian  $H'$  (5.154) is diagonalizable in the extended sense both on  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$ .*

*Proof.* We compute  $\mathcal{V}$  directly and apply Proposition 5.6.3. Following the identification in Section 5.12.1, we can translate this Hamiltonian into a block matrix

$$A_{H'} = \begin{pmatrix} \varepsilon & -\tilde{\Delta} \\ \tilde{\Delta} & -\bar{\varepsilon} \end{pmatrix}, \quad (5.156)$$

with  $\varepsilon = \bar{\varepsilon}$  and  $\tilde{\Delta} = -\tilde{\Delta}^T$  being infinite-dimensional matrices.  $A_{H'}$  can be diagonalized by a Bogoliubov transformation  $\mathcal{V} = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}$  mode-by-mode: We use the decompositions  $A_{H'} = \bigoplus_{\mathbf{p}} A_{H', \mathbf{p}}$  and  $\mathcal{V} = \bigoplus_{\mathbf{p}} \mathcal{V}_{\mathbf{p}}$  with

$$A_{H', \mathbf{p}} = \begin{pmatrix} \varepsilon_{\mathbf{p}} & 0 & 0 & -\tilde{\Delta}_{\mathbf{p}} \\ 0 & \varepsilon_{\mathbf{p}} & \tilde{\Delta}_{\mathbf{p}} & 0 \\ 0 & \tilde{\Delta}_{\mathbf{p}} & -\varepsilon_{\mathbf{p}} & 0 \\ -\overline{\tilde{\Delta}_{\mathbf{p}}} & 0 & 0 & -\varepsilon_{\mathbf{p}} \end{pmatrix}, \quad \mathcal{V}_{\mathbf{p}} = \begin{pmatrix} u_{\mathbf{p}} & 0 & 0 & v_{\mathbf{p}} \\ 0 & u_{\mathbf{p}} & -v_{\mathbf{p}} & 0 \\ 0 & \bar{v}_{\mathbf{p}} & \bar{u}_{\mathbf{p}} & 0 \\ -\bar{v}_{\mathbf{p}} & 0 & 0 & \bar{u}_{\mathbf{p}} \end{pmatrix}, \quad (5.157)$$

with  $A_{H', \mathbf{p}}, \mathcal{V}_{\mathbf{p}, s} \in \mathbb{C}^4 \otimes \mathbb{C}^4$ . The diagonalized matrix then reads

$$\mathcal{V}_{\mathbf{p}}^* A_{H', \mathbf{p}} \mathcal{V}_{\mathbf{p}} = \begin{pmatrix} E_{\mathbf{p}} & 0 & 0 & 0 \\ 0 & E_{\mathbf{p}} & 0 & 0 \\ 0 & 0 & -E_{\mathbf{p}} & 0 \\ 0 & 0 & 0 & -E_{\mathbf{p}} \end{pmatrix}, \quad E_{\mathbf{p}} = \sqrt{\varepsilon_{\mathbf{p}}^2 + |\tilde{\Delta}_{\mathbf{p}}|^2}, \quad (5.158)$$

and the diagonalization is established by

$$u_{\mathbf{p}} = \frac{\tilde{\Delta}_{\mathbf{p}}}{\sqrt{(E_{\mathbf{p}} - \varepsilon_{\mathbf{p}})^2 + |\tilde{\Delta}_{\mathbf{p}}|^2}}, \quad v_{\mathbf{p}} = \frac{E_{\mathbf{p}} - \varepsilon_{\mathbf{p}}}{\sqrt{(E_{\mathbf{p}} - \varepsilon_{\mathbf{p}})^2 + |\tilde{\Delta}_{\mathbf{p}}|^2}}. \quad (5.159)$$

From this form, it also follows that  $\dim \text{Ker}(A_H)$  is either even or  $\infty$ . So in order to apply Proposition 5.6.3, we only need to show that the spectrum of  $v^*v$  is countable. This is the case, since  $v = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} v_{\mathbf{p}}$  decays into modes, so also  $v^*v = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} (v^*v)_{\mathbf{p}}$  decays into modes, where each  $(v^*v)_{\mathbf{p}}$  is a finite-dimensional matrix with finite spectrum. As the sum over  $\mathbf{p}$  is countable, also the spectrum of  $v^*v$  is countable, and by Proposition 5.6.3,  $H$  is diagonalizable on  $\widehat{\mathcal{H}}$ .

Now, since  $\tilde{\Delta}_{\mathbf{p}} \neq 0$ , we have only Cooper pairs labeled by  $\mathbf{p}$ , so in particular, there are no full particle–hole transformations. That means, 1 is not an eigenvalue of  $v^*v$  and by Proposition 5.6.3,  $H$  is also diagonalizable on  $\overline{\mathcal{F}}$ .  $\square$

Note that, for convenience, we can replace the pair index  $i \in I'$  by  $\mathbf{p} \in \mathbb{Z}^3$ . In particular,  $\tilde{\Delta}_{\mathbf{p}} > E_{\mathbf{p}} - \varepsilon_{\mathbf{p}}$  implies that  $u_{\mathbf{p}} > \frac{1}{\sqrt{2}}$ ,  $v_{\mathbf{p}} < \frac{1}{\sqrt{2}}$ , so we have “at most a half particle–hole transformation”.

### Remarks.

34. *Interaction-free case.* It is easy to see that, when relaxing the condition  $\tilde{\Delta}_{\mathbf{p}} \neq 0$ , then in each mode with  $\tilde{\Delta}_{\mathbf{p}} = 0$ , the matrix  $A_{H',\mathbf{p}}$  is already diagonal. So it is diagonalized by  $u_{\mathbf{p}} = 1, v_{\mathbf{p}} = 0$ , which does not generate further full particle–hole transformations. So the Bogoliubov transformation  $\mathcal{V}$  diagonalizing  $A_{H'}$  remains implementable on  $\prod_{\mathbf{p},s}^{\otimes} \mathcal{H}_{\mathbf{p},s}$  and  $\overline{\mathcal{F}}$  in this case.
35. *Infinite volume case.* The Hamiltonian  $H'$  is an approximation to  $H = H_0 + H_I$ , where  $H_I$  is an attractive quartic interaction between fermion pairs. As mentioned above, the argumentation by Haag is only valid in the infinite volume limit. For infinite volumes, i.e.,  $\mathbf{p} \in \mathbb{R}^3$ , a decomposition into modes is also possible, so (5.157)–(5.159) still yield a Bogoliubov transformation  $\mathcal{V}$ , diagonalizing  $A_{H'}$ . However, as in the previous application, the spectrum of  $v^*v$  is generally uncountable in that case, so  $\mathcal{V}$  cannot be implemented by means of Theorems 5.5.7 and 5.5.8.

### 5.7.3 External Field QED

Similar cases, where a Bogoliubov transformation is not implementable, appear in the context of Dirac fields in external electromagnetic backgrounds. Again, we restrict to  $\mathbf{p} \in \mathbb{Z}^3$ . In Section 5.12.2, we argue how to arrive at the simplified and time-dependent Hamiltonian:

$$H(t) = \sum_{\mathbf{p} \in \mathbb{Z}^3} (\varepsilon_{\mathbf{p},+}(t) a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \varepsilon_{\mathbf{p},-}(t) b_{\mathbf{p}}^\dagger b_{\mathbf{p}} - f_{\mathbf{p}}(t) a_{\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger + f_{\mathbf{p}}(t) a_{\mathbf{p}} b_{\mathbf{p}}). \quad (5.160)$$

Here,  $b_{\mathbf{p}}^\dagger, b_{\mathbf{p}}$  are operators related to positrons (not to be confused with  $b^\dagger = \mathbb{U}_{\mathcal{V}} a^\dagger \mathbb{U}_{\mathcal{V}}^{-1}$ ) and  $a_{\mathbf{p}}^\dagger, a_{\mathbf{p}}$  are related to electrons. Further,  $\varepsilon_{\mathbf{p},\pm}(t) = \overline{\varepsilon_{\mathbf{p},\pm}(t)}$  is a kinetic term and  $f_{\mathbf{p}}(t) = \overline{f_{\mathbf{p}}(t)}$  is a time-dependent interaction. We assume both interactions to be continuous, bounded functions of time for each  $\mathbf{p}$ . As in the previous example,  $b_{\mathbf{p}}^\dagger, b_{\mathbf{p}}$  have to be interpreted as creating/annihilating a positron with momentum  $-\mathbf{p}$  for momentum conservation reasons. The index  $\mathbf{p}$  is again only chosen to simplify the decomposition into modes.

**Proposition 5.7.3.** *The Hamiltonian  $H(t)$  (5.160) is diagonalizable in the extended sense both on  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$  by a Bogoliubov transformation  $\mathcal{V}(t)$ .*

*Further, the Schrödinger dynamics  $\mathcal{U}(s, t) := \exp\left(-i \int_s^t \mathcal{V}(\tau) d\tau\right)$  exist<sup>3</sup> for all  $s, t \in \mathbb{R}$  as a Bogoliubov transformation, which is implementable in the extended sense on  $\widehat{\mathcal{H}}$ .*

*Proof.* The matrix associated with  $H(t)$  is

$$A_H = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} A_{H,\mathbf{p}}, \quad A_{H,\mathbf{p}}(t) = \begin{pmatrix} \varepsilon_{\mathbf{p},+}(t) & 0 & 0 & -f_{\mathbf{p}}(t) \\ 0 & \varepsilon_{\mathbf{p},-}(t) & f_{\mathbf{p}}(t) & 0 \\ 0 & f_{\mathbf{p}}(t) & -\varepsilon_{\mathbf{p},+}(t) & 0 \\ -f_{\mathbf{p}}(t) & 0 & 0 & -\varepsilon_{\mathbf{p},-}(t) \end{pmatrix}, \quad (5.161)$$

which has the same structure as  $A_{H'}$  in Section 5.7.2, except for the existence of different kinetic terms. The diagonalization again boils down to diagonalizing the  $2 \times 2$  matrices

$$\tilde{A}_{H,\mathbf{p},\pm}(t) = \begin{pmatrix} \varepsilon_{\mathbf{p},\pm}(t) & \mp f_{\mathbf{p}}(t) \\ \mp f_{\mathbf{p}}(t) & -\varepsilon_{\mathbf{p},\pm}(t) \end{pmatrix}, \quad (5.162)$$

with eigenvalues  $E_{\mathbf{p},\pm}(t) = \sqrt{\varepsilon_{\mathbf{p},\pm}(t)^2 + f_{\mathbf{p}}(t)^2}$ .

<sup>3</sup>The exponential is here to be understood in the sense of spectral calculus.

## 5 Implementing Bogoliubov Transformations Beyond the Shale–Stinespring Condition

$A_{H,\mathbf{p}}(t)$  is diagonalized by a Bogoliubov transformation

$$\mathcal{V}(t) = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} \mathcal{V}_{\mathbf{p}}(t), \quad \mathcal{V}_{\mathbf{p}}(t) = \begin{pmatrix} u_{\mathbf{p},+}(t) & 0 & 0 & v_{\mathbf{p},+}(t) \\ 0 & u_{\mathbf{p},-}(t) & -v_{\mathbf{p},-}(t) & 0 \\ 0 & \bar{v}_{\mathbf{p},-}(t) & \bar{u}_{\mathbf{p},-}(t) & 0 \\ -\bar{v}_{\mathbf{p},+}(t) & 0 & 0 & \bar{u}_{\mathbf{p},+}(t) \end{pmatrix}, \quad (5.163)$$

with time–dependent coefficients

$$u_{\mathbf{p},\pm}(t) = \frac{f_{\mathbf{p}}(t)}{\sqrt{(E_{\mathbf{p},\pm}(t) - \varepsilon_{\mathbf{p},\pm}(t))^2 + f_{\mathbf{p}}(t)}}, \quad v_{\mathbf{p},\pm}(t) = \frac{E_{\mathbf{p},\pm}(t) - \varepsilon_{\mathbf{p},\pm}(t)}{\sqrt{(E_{\mathbf{p},\pm}(t) - \varepsilon_{\mathbf{p},\pm}(t))^2 + f_{\mathbf{p}}(t)}}. \quad (5.164)$$

It becomes again apparent from the form of  $\mathcal{V}^* A_H \mathcal{V}$ , that  $\dim \text{Ker}(A_H)$  is even or  $\infty$ , and by the same arguments as in the proof of Proposition 5.7.2, the operator  $v^*v$  has countable spectrum, which does not include 1. So by Proposition 5.6.3,  $H(t)$  is diagonalizable on both  $\widehat{\mathcal{H}}$  and  $\overline{\mathcal{F}}$ .

Concerning the second claim, it is well–known [227, 230] and has also been argued in [222, II 2.4] that the Schrödinger dynamics generated by finite–dimensional matrices of the form  $A_{H,\mathbf{p}}$  are given by a Bogoliubov transformation

$$\mathcal{U}_{\mathbf{p}}(s, t) = \begin{pmatrix} U_{\mathbf{p}}(s, t) & V_{\mathbf{p}}(s, t) \\ \bar{V}_{\mathbf{p}}(s, t) & \bar{U}_{\mathbf{p}}(s, t) \end{pmatrix} := \exp \left( -i \int_s^t A_{H,\mathbf{p}}(\tau) d\tau \right). \quad (5.165)$$

The integral exists<sup>4</sup> for finite times by the continuity and boundedness assumption on  $\varepsilon_{\mathbf{p},\pm}(t)$  and  $f_{\mathbf{p}}(t)$ . Now, the dynamics generated by  $A_H$  can easily be reconstructed from  $\mathcal{U}_{\mathbf{p}}(s, t)$  via:

$$\mathcal{U}(s, t) := \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} \mathcal{U}_{\mathbf{p}}(s, t), \quad \mathcal{U}(s, t) = \begin{pmatrix} U(s, t) & V(s, t) \\ \bar{V}(s, t) & \bar{U}(s, t) \end{pmatrix}. \quad (5.166)$$

The transformation  $\mathcal{U}_{\mathbf{p}}(s, t)$  acts on a finite number of modes (namely two) and are hence always implementable on the two–mode Fock space. As  $V = V(s, t)$  decays into countably many modes, so does  $V^*V$ . Therefore,  $V^*V$  has a countable spectrum and by Theorems 5.5.7 and 5.5.8,  $\mathcal{U}(s, t)$  is implementable on  $\widehat{\mathcal{H}}$ .  $\square$

Note that for a proof of implementability on  $\overline{\mathcal{F}}$ , we would need that there are at most finitely many particle–hole transformations, which cannot be guaranteed

<sup>4</sup>The exponential of the integral is just the exponential of a finite–dimensional matrix, which always exists.

for all time intervals  $(s, t)$ .

### Remarks.

36. *Comparison with [222].* The toy model for external field QED in [222, II 2.4] is a similar one, where the matrix  $A_H$  (5.162) plays the role of the full Hamiltonian, i.e., the system is considered before second quantization. There,  $\mathbf{p} \in \mathbb{Z}^3$  is replaced by a mode index  $n \in \mathbb{N}^3$  and  $f_{\mathbf{p}}(t)$  decays like  $\frac{f(t)}{n}$ ,  $\varepsilon_{\mathbf{p},\pm}$  is chosen independent of time.

37. *Implementability on Fock space.* To check the Shale–Stinespring condition, note that since we can decompose  $A_{H,\mathbf{p}} = \tilde{A}_{H,\mathbf{p},+} \oplus \tilde{A}_{H,\mathbf{p},-}$ , it is also possible to decompose  $\mathcal{U}_{\mathbf{p}}(s, t) := \tilde{\mathcal{U}}_{\mathbf{p},+}(s, t) \oplus \tilde{\mathcal{U}}_{\mathbf{p},-}(s, t)$ . So  $\mathcal{U}_{\mathbf{p}}(s, t)$  takes the following form:

$$\mathcal{U}_{\mathbf{p}}(s, t) = \begin{pmatrix} U_{\mathbf{p},1}(s, t) & & & V_{\mathbf{p},1}(s, t) \\ & U_{\mathbf{p},2}(s, t) & V_{\mathbf{p},2}(s, t) & \\ & \bar{V}_{\mathbf{p},1}(s, t) & \bar{U}_{\mathbf{p},1}(s, t) & \\ \bar{V}_{\mathbf{p},2}(s, t) & & & \bar{U}_{\mathbf{p},2}(s, t) \end{pmatrix}. \quad (5.167)$$

Hence, the Shale–Stinespring condition for  $\mathcal{U}(s, t)$  amounts to

$$\sum_{\mathbf{p}} (|V_{\mathbf{p},1}|^2 + |V_{\mathbf{p},2}|^2) < \infty, \quad (5.168)$$

which does not necessarily hold: We may split the diagonalizing Bogoliubov transformation  $\mathcal{V}_{\mathbf{p}} = \mathcal{V}_{\mathbf{p},+} \oplus \mathcal{V}_{\mathbf{p},-}$  as in (5.163) and get

$$\begin{aligned} \mathcal{V}_{\mathbf{p},\pm}^* \tilde{\mathcal{U}}_{\mathbf{p},\pm}(s, t) \mathcal{V}_{\mathbf{p},\pm} &= \exp \left( -i \int_s^t \mathcal{V}_{\mathbf{p},\pm}^* \tilde{A}_{H,\mathbf{p},\pm}(\tau) \mathcal{V}_{\mathbf{p},\pm} d\tau \right) \\ &= \begin{pmatrix} e^{-i(t-s)E_{\mathbf{p},\pm}} & 0 \\ 0 & e^{i(t-s)E_{\mathbf{p},\pm}} \end{pmatrix}. \end{aligned} \quad (5.169)$$

Note that neither of the matrices  $\tilde{\mathcal{U}}_{\mathbf{p},\pm}(s, t), \mathcal{V}_{\mathbf{p},\pm}$  describes a Bogoliubov transformation. Reading off  $\mathcal{V}_{\mathbf{p},\pm}$  from (5.163), we conclude that

$$V_{\mathbf{p},\pm}(s, t) = \overline{u_{\mathbf{p},\pm}} v_{\mathbf{p},\pm} (\mp 2i \sin((t-s)E_{\mathbf{p},\pm})). \quad (5.170)$$

Although there is a rapid oscillation in  $\mathbf{p}$  for small times  $(t-s)$ , convergence of the sum (5.168) depends on the values of  $E_{\mathbf{p},\pm}$  and cannot be guaranteed

in general.

## 5.8 Results on the Infinite Tensor Product Space

Our first proposition characterizes, when exactly two  $C$ -sequences correspond to the same functional  $\Psi \in \widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k$ . This allows us to define operators on  $\widehat{\mathcal{H}}$  in Section 5.4.2. Recall that any  $C$ -sequence  $(\Psi) = (\Psi_k)_{k \in I}$  gives rise to a unique functional  $\Psi \in \widehat{\mathcal{H}}$  by the embedding  $\Psi = \iota((\Psi))$ .

**Proposition 5.8.1.** *Whenever  $(\Psi), (\Psi') \in \text{Cseq}$  represent the same functional  $\Psi = \Psi'$ , then there exists a family of complex numbers  $(c_k)_{k \in I}$  such that*

$$\Psi_k = c_k \Psi'_k \quad \forall k \in I, \quad \text{and} \quad \prod_{k \in I} c_k = 1, \quad (5.171)$$

using the notion of convergence for an infinite product from Section 3.1.1.

Conversely, if  $(\Psi), (\Psi') \in \text{Cseq}$  just differ by a family  $(c_k)_{k \in I}$  as in (5.171), then they represent the same functional  $\Psi = \Psi'$ .

*Proof.* For the first statement, we must prove that  $\Psi_k$  and  $\Psi'_k$  are parallel for any  $k \in I$ . So let's fix a  $k$  and decompose  $\Psi_k = \Psi_k^{\parallel} + \Psi_k^{\perp}$  with  $\Psi_k^{\parallel} \parallel \Psi'_k$  and  $\Psi_k^{\perp} \perp \Psi'_k$ , and suppose that  $\Psi_k^{\perp} \neq 0$ . Now, choose some  $C$ -sequences  $(\Phi), (\Phi')$ , that agree on all  $k' \neq k$  and with  $\Phi_k \parallel \Psi_k$  and  $\Phi'_k \parallel \Psi'_k$ , as well as  $\|\Phi_k\|_k = \|\Phi'_k\|_k = 1$ . Then,

$$\begin{aligned} \langle \Psi, \Phi' \rangle &= \langle \Psi_k, \Phi'_k \rangle_k \prod_{k \neq k'} \langle \Psi_{k'}, \Phi'_{k'} \rangle_{k'} = \|\Psi_k^{\parallel}\|_k \prod_{k \neq k'} \langle \Psi_{k'}, \Phi_{k'} \rangle_{k'} \\ &< \|\Psi_k\|_k \prod_{k \neq k'} \langle \Psi_{k'}, \Phi_{k'} \rangle_{k'} = \langle \Psi_k, \Phi_k \rangle_k \prod_{k \neq k'} \langle \Psi_{k'}, \Phi_{k'} \rangle_{k'} = \langle \Psi, \Phi \rangle. \end{aligned} \quad (5.172)$$

By the same arguments,  $\langle \Psi', \Phi \rangle < \langle \Psi', \Phi' \rangle$ . But since  $(\Psi), (\Psi')$  correspond to the same functional  $\Psi = \Psi'$ , we can freely exchange both expressions within the scalar product:

$$\langle \Psi, \Phi' \rangle = \langle \Psi', \Phi' \rangle > \langle \Psi', \Phi \rangle = \langle \Psi, \Phi \rangle. \quad (5.173)$$

This contradicts (5.172) and thus establishes  $\Psi_k = c_k \Psi'_k$ .

Convergence of  $\prod_{k \in I} c_k$  can be seen as follows: We have

$$\|\Psi\|^2 = \langle \Psi', \Psi \rangle = \prod_{k \in I} \langle \Psi'_k, c_k \Psi'_k \rangle_k = \prod_{k \in I} c_k \|\Psi'_k\|_k^2. \quad (5.174)$$

## 5.8 Results on the Infinite Tensor Product Space

So if  $\prod_{k \in I} c_k$  was not convergent, i.e.,  $\sum_k |c_k - 1| = \infty$ , then for the product on the right-hand side, we would have

$$\begin{aligned}
 \sum_k |c_k \|\Psi'_k\|_k^2 - 1| &= \sum_k |c_k \|\Psi'_k\|_k^2 - \|\Psi'_k\|_k^2 + \|\Psi'_k\|_k^2 - 1| \\
 &\geq \sum_k (|c_k \|\Psi'_k\|_k^2 - \|\Psi'_k\|_k^2| - |\|\Psi'_k\|_k^2 - 1|) \\
 &= \underbrace{\sum_k \|\Psi'_k\|_k^2 |c_k - 1|}_{(*)} - \underbrace{\sum_k |\|\Psi'_k\|_k^2 - 1|}_{< \infty}.
 \end{aligned} \tag{5.175}$$

Now,  $\|\Psi'_k\|_k^2 > 1/2$  for all but finitely many  $k$ , so  $(*)$  and thus the first expression in (5.175) diverges. This is a contradiction to (5.174) being convergent. So  $\prod_{k \in I} c_k$  indeed yields a complex number.

But since  $\|\Psi\|^2 = \|\Psi'\|^2 = \prod_{k \in I} \|\Psi'_k\|_k^2$ , we immediately obtain  $\prod_{k \in I} c_k = 1$  from (5.174).

The converse statement can readily be seen by computing the action of the functionals  $\Psi, \Psi'$  on some  $\Phi \in \text{Cseq}$ :

$$\langle \Psi, \Phi \rangle = \prod_{k \in I} \langle \Psi_k, \Phi_k \rangle_k = \prod_{k \in I} \langle c_k \Psi'_k, \Phi_k \rangle_k = \left( \prod_{k \in I} \overline{c_k} \right) \prod_{k \in I} \langle \Psi'_k, \Phi_k \rangle_k = \langle \Psi', \Phi \rangle. \tag{5.176}$$

□

By [44, Lemma 4.1.1], all subspaces  $\prod_{k \in I}^{\otimes C} \mathcal{H}_k$  of  $\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k$  are **mutually orthogonal**. This allows for a particularly simple decomposition:

**Lemma 5.8.2.** *For any  $\Psi \in \widehat{\mathcal{H}}$ , we can write*

$$\Psi = \sum_{m \in \mathcal{M}} d_m \Psi^{(m)} = \sum_{m \in \mathcal{M}} d_m \prod_{k \in I}^{\otimes} \Psi_k^{(m)}, \tag{5.177}$$

with  $\mathcal{M}$  being a subset of  $\mathbb{N}$ ,  $\Psi^{(m)}$  defined by the mutually orthogonal  $C_0$ -sequences  $(\Psi^{(m)})$  with  $\|\Psi^{(m)}\| = 1$  and where  $\sum_m |d_m|^2 < \infty$  is a complex sequence.

Moreover, one can choose a fixed set  $Z = \{\Psi^{(a)}\}_{a \in A}$  defined by mutually orthogonal, normalized  $C_0$ -sequences  $(\Psi^{(a)})$ , such that for all  $\Psi \in \widehat{\mathcal{H}}$ , the form (5.177) can be achieved by taking only  $\Psi^{(m)} \in Z$ . The decomposition (5.177) is then unique up to the choice of the  $\Psi_k^{(m)}$  representing  $\Psi^{(m)}$ .

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So  $Z$  is an orthonormal basis of  $\widehat{\mathcal{H}}$  that might be uncountable, but the elements  $\Psi \in \widehat{\mathcal{H}}$  are all countable linear combinations with coefficient sequences in  $\ell^2$ .

*Proof.* By definition, any  $\Psi \in \widehat{\mathcal{H}}$  can be approximated by a Cauchy sequence  $(\Psi^{(r)})_{r \in \mathbb{N}} \subseteq \prod_{k \in I}^{\otimes} \mathcal{H}_k$ ,  $\Psi^{(r)} \rightarrow \Psi$ . So each  $\Psi^{(r)}$  can be written a finite linear combination of  $C$ -sequences. All  $C$ -sequences, that are no  $C_0$ -sequences, must have norm 0, so we may drop them and simply write

$$\Psi^{(r)} = \sum_{\ell=1}^{L_r} \Psi_{\ell}^{(r)}, \quad (5.178)$$

with  $(\Psi_{\ell}^{(r)})$  being  $C_0$ -sequences. Now, the  $C_0$ -sequences decay into mutually orthogonal equivalence classes  $C$ , out of which countably many are occupied by any  $\Psi^{(r)}$ . So we have

$$\Psi^{(r)} = \sum_C \sum_{\ell: (\Psi_{\ell}^{(r)}) \in C} \Psi_{\ell}^{(r)} =: \sum_C \Psi_C^{(r)}. \quad (5.179)$$

By orthogonality of the subspaces  $\Psi \in \prod_{k \in I}^{\otimes} \mathcal{H}_k$ , we have

$$\|\Psi^{(r)} - \Psi^{(s)}\|^2 = \sum_C \|\Psi_C^{(r)} - \Psi_C^{(s)}\|^2. \quad (5.180)$$

$(\Psi^{(r)})_{r \in \mathbb{N}}$  is a Cauchy sequence, so  $(\Psi_C^{(r)})_{r \in \mathbb{N}}$  is also a Cauchy sequence for all  $C$ . That means, the limit  $\Psi_C = \lim_{r \rightarrow \infty} \Psi_C^{(r)}$  exists and by orthogonality of the  $\Psi_C^{(r)}$  for each  $r$ ,

$$\lim_{r \rightarrow \infty} \sum_C \Psi_C^{(r)} = \sum_C \lim_{r \rightarrow \infty} \Psi_C^{(r)} \quad \Leftrightarrow \quad \Psi = \sum_C \Psi_C. \quad (5.181)$$

We may now write both  $\Psi_C$  and  $\Psi$  in coordinates:

$$\begin{aligned} \Psi_C &= \lim_{r \rightarrow \infty} \Psi_C^{(r)} = \lim_{r \rightarrow \infty} \sum_{\ell: (\Psi_{\ell}^{(r)}) \in C} \Psi_{\ell}^{(r)} = \sum_{n(\cdot) \in F} a_C(n(\cdot)) \prod_{k \in I}^{\otimes} e_{k, n(k)} \\ \Rightarrow \Psi &= \sum_C \lim_{r \rightarrow \infty} \Psi_C^{(r)} = \sum_C \sum_{n(\cdot) \in F} a_C(n(\cdot)) \prod_{k \in I}^{\otimes} e_{k, n(k)}. \end{aligned} \quad (5.182)$$

So  $\Psi$  can be written as a countable sum over mutually orthogonal, normalized  $C_0$ -sequences with coordinates  $a_C(n(\cdot))$ . We index the sequences and coordinates by  $(\Psi^{(m)})$  and  $d_m$ , which yields the desired form (5.177).

Square summability of the  $d_m$  can be seen by

$$\sum_m |d_m|^2 = \sum_C \sum_{n(\cdot) \in F} |a_C(n(\cdot))|^2. \quad (5.183)$$

Now, the set  $Z = \{\Psi^{(a)}\}_{a \in A}$  is exactly the union of all vectors  $\prod_{k \in I}^{\otimes} e_{k,n(k)}$  over all classes  $C$ , which is indeed a set of mutually orthogonal, normalized vectors. Uniqueness of the decomposition follows by orthogonality of the  $\Psi^{(a)}$ . □

**Remarks.**

38. Although the coefficient sequence  $a_C(n(\cdot))$  is an element of  $\ell^2$ , the space  $\widehat{\mathcal{H}} = \prod_{k \in I}^{\otimes} \mathcal{H}_k$  is generally not isomorphic to  $\ell^2$ , and hence not separable. The reason is that the set  $Z$  of eligible  $\Psi^{(m)}$ -vectors can be uncountable. For instance, already with a countable  $I$  and at least two basis vectors  $e_{k,0}, e_{k,1}$  for each  $k$ , the space  $\widehat{\mathcal{H}}$  contains the orthonormal set of vectors  $\prod_{k \in I}^{\otimes} e_{k,n}$  with  $n \in \{0, 1\}$ . This set corresponds to all families of binary digits, which is uncountable for  $I$  of infinite cardinality. For the same reason,  $Z$  is uncountable, as well as the number of equivalence classes  $C$ .

39. We may sum up all components with sequences  $(\Psi^{(m)})$  in the same equivalence class  $C$  as

$$\Psi_C := \sum_{m: (\Psi^{(m)}) \in C} d_m \Psi^{(m)} \quad \Rightarrow \quad \Psi = \sum_C \Psi_C, \quad (5.184)$$

where the sum runs only over countably many equivalence classes.

Further, by [44, Def. 6.1.1], two  $C_0$ -sequences  $(\Psi), (\Phi)$  are weakly equivalent, if and only if there exists a family  $(z_k)_{k \in I} \subseteq \mathbb{C}$  with  $(z_k \Psi_k)_{k \in I}$  being (strongly) equivalent to  $(\Phi_k)_{k \in I}$ . From that, we may conclude:

**Lemma 5.8.3.** *Let  $C_w$  be the weak equivalence class of a  $C_0$ -sequence  $(\Phi) = (\Phi_k)_{k \in I}$ , choose an orthonormal basis  $(e_{k,n})_{n \in \mathbb{N}_0}$  for each  $\mathcal{H}_k$ , such that  $\Phi_k = c_{k,0}$  and define for any  $C_0$ -sequence  $(\Psi) = (\Psi_k)_{k \in I}$  the coordinates  $c_{k,n} := \langle e_{k,n}, \Psi_k \rangle_k$ . Then,  $\prod_{k \in I}^{\otimes} C_w \mathcal{H}_k$  is exactly the closure of the span of all normalized  $C_0$ -sequences, where  $|c_{k,0}| = 1$  for all but finitely many  $k \in I$ .*

Note that the last statement means  $c_{k,n} = 0$  for  $n \geq 1$  and for those  $k$ . In simple words, Lemma 5.8.3 asserts that replacing  $C$  by  $C_w$  in the equivalence class is done

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via replacing  $c_{k,0} = 1$  by  $|c_{k,0}| = 1$ .

*Proof.* First, we prove that any normalized  $C_0$ -sequence  $(\Psi)$  with  $|c_{k,0}| = 1$  almost everywhere is weakly equivalent to  $(\Phi)$ : We can define a family of phase rotations  $|z_k| = 1$ , such that  $z_k c_{k,0} = 1$  for all  $k$  with  $|c_{k,0}| = 1$ . So  $(z_k \Psi_k)_{k \in I}$  has  $z_k c_{k,0} = 1$  almost everywhere and is hence a  $C_0$ -sequence strongly equivalent to  $(\Phi)$ . Therefore,  $(\Psi)$  is weakly equivalent to  $(\Phi)$ . So  $\Psi \in \prod_{k \in I}^{\otimes C_w} \mathcal{H}_k$  and the same holds for the span of these  $C_0$ -sequences, and their closure with respect to the Hilbert space topology on  $\widehat{\mathcal{H}}$ .

Conversely, any  $\Psi \in \prod_{k \in I}^{\otimes C_w} \mathcal{H}_k$  is within the closure of the span of normalized  $C_0$ -sequences with  $|c_{k,0}| = 1$  almost everywhere: By Lemma 5.8.2, we may write

$$\Psi = \sum_{m \in \mathcal{M}} d_m \prod_{k \in I}^{\otimes} \Psi_k^{(m)},$$

where  $\sum_m |d_m|^2 < \infty$  and the  $(\Psi^{(m)}) = (\Psi_k^{(m)})_{k \in I}$  with  $\|\Psi^{(m)}\| = 1$  are orthogonal. Further, we may choose  $(\Psi^{(m)}) \sim_w (\Phi)$ , since all  $(\Psi^{(m)})$  were constructed to come from a (strong) equivalence class  $C$  contained within  $C_w$ . So there exist families  $(z_k^{(m)})_{k \in I}$ ,  $|z_k^{(m)}| = 1$ , such that  $(z_k^{(m)} \Psi_k^{(m)})_{k \in I} \sim (\Phi)$  for all  $m \in \mathcal{M}$ . By strong equivalence, we may approximate each  $(z_k^{(m)} \Psi_k^{(m)})$  up to arbitrary precision  $\varepsilon > 0$  by a linear combination of (normalized) families  $\Psi^{(m,\varepsilon)}$ , such that, when writing these families in coordinates, we have  $c_{k,0}^{(m,\varepsilon)} = 1$  almost everywhere in  $k \in I$ . Hence, the families  $((z_k^{(m)})^{-1} \Psi_k^{(m,\varepsilon)})_{k \in I}$  approximate  $\Psi^{(m)}$  up to precision  $\varepsilon$ . They satisfy  $|(z_k^{(m)})^{-1} c_{k,0}^{(m,\varepsilon)}| = 1$  almost everywhere, so  $\Psi^{(m)}$  can be approximated up to arbitrary precision by a linear combination of  $C_0$ -sequences with the above-mentioned property. And by (5.177) and convergence of  $|d_m|^2$ , also an arbitrary approximation of  $\Psi$  is possible by linear combinations of normalized  $C_0$ -sequences with  $|c_{k,0}| = 1$  almost everywhere. □

## 5.9 Representations of Bogoliubov Transformations

Here, we compare four different ways of representing a Bogoliubov transformation: One transformation  $\mathcal{V}_A$  directly acts on the  $*$ -algebra  $\mathcal{A}$  of creation and annihilation operators. The three others  $\mathcal{V}_1, \mathcal{V}_2$  and  $\mathcal{V}_3$ , are equivalent to  $\mathcal{V}_A$  and act on different spaces  $W_{1,1}, W_{1,2}, W_{1,3}$  isomorphic to the one-operator subspace

$W_1$  of  $\mathcal{A}$ . More precisely, we consider as  $W_1$  either  $\mathfrak{h} \oplus \mathfrak{h}$ ,  $\mathfrak{h} \oplus \mathfrak{h}^*$ , or  $\ell^2 \oplus \ell^2$ . Since  $\mathcal{A}$  is the algebra generated by the one-operator subspace  $W_1$ , it suffices to define  $\mathcal{V}_j$  on  $W_{1,j}$ , in order to obtain its action on operator products. The one used in this chapter is  $\mathcal{V} = \mathcal{V}_3$ . We explicitly derive the Bogoliubov relations in all representations.

### 5.9.1 Bogoliubov Transformations on Operators

By a (bounded) algebraic Bogoliubov transformation  $\mathcal{V}_A$ , we understand any transformation mapping creation and annihilation operators  $a_{\pm}^{\dagger}(f), a_{\pm}(g)$  with  $f, g \in \mathfrak{h}$  into operators  $b_{\pm}^{\dagger}(f), b_{\pm}(g)$  via:

$$\begin{aligned} \mathcal{V}_A : \mathcal{A} &\rightarrow \mathcal{A}, & a_{\pm}^{\dagger}(f) &\mapsto b_{\pm}^{\dagger}(f) \\ b_{\pm}^{\dagger}(f) &= a_{\pm}^{\dagger}(\hat{u}f) + a_{\pm}(\hat{v}f) \\ b_{\pm}(g) &= a_{\pm}^{\dagger}(\hat{v}g) + a_{\pm}(\hat{u}g), \end{aligned} \quad (5.185)$$

with a linear operator  $\hat{u}$  and an antilinear operator  $\hat{v}$ , both of which are defined on  $\mathfrak{h}$  (i.e., bounded), such that both  $\mathcal{V}_A$  and its adjoint  $\mathcal{V}_A^*$  conserve the CAR/CCR (5.12).

In order to express  $\mathcal{V}_A$  and the conservation conditions, we define for the linear  $\hat{u}$ :

- the transpose  $\hat{u}^T$  by  $\langle f, \hat{u}^T g \rangle = \langle g, \hat{u} f \rangle$
- the complex conjugate  $\bar{\hat{u}}$  by  $\langle f, \bar{\hat{u}} g \rangle = \overline{\langle f, \hat{u} g \rangle}$
- the adjoint by  $\hat{u}^* = \bar{\hat{u}^T} = \hat{u}^{\bar{T}}$

The antilinear  $\hat{v}$  can be written as  $\hat{v} = \hat{v}_{\ell} \mathcal{J}$  with a linear operator  $\hat{v}_{\ell}$  and complex conjugation  $(\mathcal{J}f)(x) := \overline{f(x)}$ , see [231, 232]. We define the corresponding transpose by  $\hat{v}^T := \hat{v}_{\ell}^T \mathcal{J}$ , the complex conjugate by  $\bar{\hat{v}} := \overline{\hat{v}_{\ell} \mathcal{J}}$  and the adjoint by  $\hat{v}^* := \hat{v}_{\ell}^* \mathcal{J}$ . So in particular,

$$\langle f, \hat{v}^T g \rangle = \langle f, \hat{v}_{\ell}^T \mathcal{J} g \rangle = \langle \bar{\hat{v}_{\ell}} f, \mathcal{J} g \rangle = \langle \bar{\hat{v}_{\ell}} \mathcal{J} \mathcal{J} f, \mathcal{J} g \rangle = \langle \bar{\hat{v}} f, \bar{g} \rangle = \overline{\langle \hat{v} f, g \rangle}, \quad (5.186)$$

$$\Rightarrow \langle \hat{v} f, g \rangle = \overline{\langle f, \hat{v}^T g \rangle} = \langle \hat{v}^T g, f \rangle, \quad (5.187)$$

which is the “correct law for shifting antilinear operators from one side of the scalar product to the other” and replaces the familiar  $\langle f, \hat{v}_{\ell} g \rangle = \langle \hat{v}_{\ell}^* f, g \rangle$  from the linear case.

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The adjoint transformation  $\mathcal{V}_A^*$  is given by replacing  $\hat{u} \rightarrow \hat{u}^*$  and  $\hat{v}_\ell \rightarrow \hat{v}_\ell^*$ , so  $\hat{v} \rightarrow \hat{v}^T$  in  $\mathcal{V}_A$  (5.185).

Now, we investigate conservation of the CAR/CCR. From  $[a(f), a^\dagger(g)]_\pm = \langle f, g \rangle$  being preserved, we get

$$\begin{aligned}
 \langle f, g \rangle &= [b(f), b^\dagger(g)]_\pm \\
 &= [a(\hat{u}f), a^\dagger(\hat{u}g)]_\pm + [a^\dagger(\hat{v}f), a(\hat{v}g)]_\pm \\
 &= [a(\hat{u}f), a^\dagger(\hat{u}g)]_\pm \mp [a(\hat{v}g), a^\dagger(\hat{v}f)]_\pm \\
 &= \langle \hat{u}f, \hat{u}g \rangle \mp \langle \hat{v}g, \hat{v}f \rangle \stackrel{(5.187)}{=} \langle \hat{u}^* \hat{u}f, g \rangle \mp \langle \hat{v}^T \hat{v}f, g \rangle \\
 \Rightarrow 1 &= \hat{u}^* \hat{u} \mp \hat{v}^T \hat{v}.
 \end{aligned} \tag{5.188}$$

From the conservation of  $[a(f), a(g)]_\pm = 0$  we obtain

$$\begin{aligned}
 0 &= [b(f), b(g)]_\pm \\
 &= [a(\hat{u}f), a^\dagger(\hat{v}g)]_\pm + [a^\dagger(\hat{v}f), a(\hat{u}g)]_\pm \\
 &= [a(\hat{u}f), a^\dagger(\hat{v}g)]_\pm \mp [a(\hat{u}g), a^\dagger(\hat{v}f)]_\pm \\
 &= \langle \hat{u}f, \hat{v}g \rangle \mp \langle \hat{u}g, \hat{v}f \rangle \stackrel{(5.187)}{=} \langle f, \hat{u}^* \hat{v}g \rangle \mp \langle f, \hat{v}^T \hat{u}g \rangle \\
 \Rightarrow 0 &= \hat{u}^* \hat{v} \mp \hat{v}^T \hat{u}.
 \end{aligned} \tag{5.189}$$

The two conditions (5.188) and (5.189) are required also for  $\mathcal{V}_A^*$ , where we replace  $\hat{u} \rightarrow \hat{u}^*$  and  $\hat{v} \rightarrow \hat{v}^T$ . This leads to 4 conditions in total:

$$\begin{aligned}
 \hat{u}^* \hat{u} \mp \hat{v}^T \hat{v} &= 1 & \hat{u}^* \hat{v} \mp \hat{v}^T \hat{u} &= 0 \\
 \hat{u} \hat{u}^* \mp \hat{v} \hat{v}^T &= 1 & \hat{v} \hat{u}^* \mp \hat{u} \hat{v}^T &= 0.
 \end{aligned} \tag{5.190}$$

### 5.9.2 Representation by $W_{1,1} = \mathfrak{h} \oplus \mathfrak{h}$

It is most natural to encode  $a_\pm^\dagger(f_1) + a_\pm(\overline{f_2})$  by a direct sum  $F = (f_1, f_2)$ . We do this by introducing generalized creation and annihilation operators

$$\begin{aligned}
 A_{1\pm}^\dagger : \mathfrak{h} \oplus \mathfrak{h} &\rightarrow \mathcal{A}_\pm, & (f_1, f_2) &\mapsto a_\pm^\dagger(f_1) + a_\pm(\overline{f_2}) \\
 A_{1\pm} : \mathfrak{h} \oplus \mathfrak{h} &\rightarrow \mathcal{A}_\pm, & (g_1, g_2) &\mapsto a_\pm(g_1) + a_\pm^\dagger(\overline{g_2}).
 \end{aligned} \tag{5.191}$$

In a more abstract language, the representation is fixed by a bijective identification  $\iota_1 : \mathfrak{h} \oplus \mathfrak{h} \rightarrow W_1$  such that  $\iota_1(F) = A_{1\pm}^\dagger(F)$ . The operator  $A_{1\pm}(F)$  is then defined as the adjoint of  $\iota_1(F)$ .

In this representation, a Bogoliubov transformation acts by  $\mathcal{V}_1 : \mathfrak{h} \oplus \mathfrak{h} \rightarrow \mathfrak{h} \oplus \mathfrak{h}$ ,

$\mathcal{V}_1 = \iota_1^{-1} \circ \mathcal{V}_A \circ \iota_1$ , where

$$\mathcal{V}_1 \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{u}f \\ \hat{v}f \end{pmatrix} \quad \text{and} \quad \mathcal{V}_1 \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} \hat{v}\bar{g} \\ \hat{u}\bar{g} \end{pmatrix}. \quad (5.192)$$

This transformation is obviously not linear on  $\mathfrak{h} \oplus \mathfrak{h}$ . Hence, we cannot encode subsequent Bogoliubov transformations by a matrix multiplication, which makes the representation inconvenient. The nonlinearity issues can be resolved within a different representation, by complex conjugation of the second one-particle Hilbert space.

### 5.9.3 Representation by $W_{1,2} = \mathfrak{h} \oplus \mathfrak{h}^*$

We define the complex conjugation operator  $\mathcal{J} : \mathfrak{h} \rightarrow \mathfrak{h}^*$  with adjoint  $\mathcal{J}^* : \mathfrak{h}^* \rightarrow \mathfrak{h}$  and denote  $F = (f_1 \oplus \mathcal{J}f_2)$ . The generalized creation and annihilation operators are then given by

$$\begin{aligned} A_{2\pm}^\dagger : \mathfrak{h} \oplus \mathfrak{h}^* &\rightarrow \mathcal{A}_\pm, & (f_1 \oplus \mathcal{J}f_2) &\mapsto a_\pm^\dagger(f_1) + a_\pm(f_2) \\ A_{2\pm} : \mathfrak{h} \oplus \mathfrak{h}^* &\rightarrow \mathcal{A}_\pm, & (g_1 \oplus \mathcal{J}g_2) &\mapsto a_\pm(g_1) + a_\pm^\dagger(g_2). \end{aligned} \quad (5.193)$$

The corresponding identification is  $\iota_2 : \mathfrak{h} \oplus \mathfrak{h}^* \rightarrow W_1$  with  $\iota_2(F) = A_{2\pm}^\dagger(F)$ . In this representation, a Bogoliubov transformation is represented by  $\mathcal{V}_2 : \mathfrak{h} \oplus \mathfrak{h}^* \rightarrow \mathfrak{h} \oplus \mathfrak{h}^*$ ,  $\mathcal{V}_2 = \iota_2^{-1} \circ \mathcal{V}_A \circ \iota_2$ , where

$$\mathcal{V}_2 \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{u}f \\ \mathcal{J}\hat{v}f \end{pmatrix} \quad \text{and} \quad \mathcal{V}_2 \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} \hat{v}g \\ \mathcal{J}\hat{u}g \end{pmatrix}. \quad (5.194)$$

The operator  $\mathcal{V}_2$  is linear and can be written in block matrix form

$$\mathcal{V}_2 = \begin{pmatrix} U & \mathcal{J}^*V\mathcal{J}^* \\ V & \mathcal{J}U\mathcal{J}^* \end{pmatrix}, \quad (5.195)$$

with linear operators  $U : \mathfrak{h} \rightarrow \mathfrak{h}$ ,  $U = \hat{u}$  and  $V : \mathfrak{h} \rightarrow \mathfrak{h}^*$ ,  $V = \mathcal{J}\hat{v}$ . So we can write successive transformations as a matrix product. This representation is used, for instance in [216, 35].

Conservation of the CAR/CCR now translates into

$$\mathcal{V}_2^* \mathcal{S}_\pm \mathcal{V}_2 = \mathcal{V}_2 \mathcal{S}_\pm \mathcal{V}_2^* = \mathcal{S}_\pm, \quad (5.196)$$

with  $\mathcal{S}_- = id$  and  $\mathcal{S}_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  being operators on  $\mathfrak{h} \oplus \mathfrak{h}^*$ .

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In the *fermionic* case (–), the first conservation condition reads:

$$\mathcal{V}_2^* \mathcal{V}_2 = \begin{pmatrix} U^*U + V^*V & U^* \mathcal{J}^* V \mathcal{J}^* + V^* \mathcal{J} U \mathcal{J}^* \\ \mathcal{J} V^* \mathcal{J} U + \mathcal{J} U^* \mathcal{J}^* V & \mathcal{J} V^* V \mathcal{J}^* + \mathcal{J} U^* U \mathcal{J}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.197)$$

From the first line, we recover

$$U^*U + V^*V = 1 \quad U^* \mathcal{J}^* V + V^* \mathcal{J} U = 0, \quad (5.198)$$

and the second line yields exactly the same conditions.

The CAR/CCR conservation for the adjoint transformation reads

$$\mathcal{V}_2 \mathcal{V}_2^* = \begin{pmatrix} UU^* + \mathcal{J}^* V V^* \mathcal{J} & UV^* + \mathcal{J}^* V U^* \mathcal{J}^* \\ V U^* + \mathcal{J} U V^* \mathcal{J} & V V^* + \mathcal{J} U U^* \mathcal{J}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.199)$$

Again, we recover from the first line

$$UU^* + \mathcal{J}^* V V^* \mathcal{J} = 1 \quad UV^* + \mathcal{J}^* V U^* \mathcal{J}^* = 0, \quad (5.200)$$

and the second line yields the same conditions.

The calculations for *bosons* (+) are the same, except for additional minus signs. From (5.198) and (5.200), we arrive at 4 conditions in total, which are:

$$\begin{aligned} U^*U \mp V^*V &= 1 & V^* \mathcal{J} U \mp U^* \mathcal{J}^* V &= 0 \\ UU^* \mp \mathcal{J}^* V V^* \mathcal{J} &= 1 & UV^* \mp \mathcal{J}^* V U^* \mathcal{J}^* &= 0. \end{aligned} \quad (5.201)$$

### 5.9.4 Representation by $W_{1,3} = \ell^2 \oplus \ell^2$

It is also possible to write the four blocks in  $\mathcal{V}_j$  as infinite matrices with countably many complex-valued entries. This is the representation introduced in Section 5.3. Recall that we fixed a basis  $(e_j)_{j \in \mathbb{N}}$  of  $\mathfrak{h}$  and wrote  $f_j := \langle e_j, f \rangle$  for  $f \in \mathfrak{h}$ , so  $f$  could be identified with the sequence  $\mathbf{f} = (f_j)_{j \in \mathbb{N}}$ ,  $\mathbf{f} \in \ell^2$ . The generalized creation and annihilation operators are as in (5.9):

$$\begin{aligned} A_{3\pm}^\dagger : \ell^2 \oplus \ell^2 &\rightarrow \mathcal{A}_\pm, & (\mathbf{f}_1, \mathbf{f}_2) &\mapsto a_\pm^\dagger(\mathbf{f}_1) + a_\pm(\overline{\mathbf{f}_2}) = \sum_j (f_{1,j} a_\pm^\dagger(e_j) + f_{2,j} a_\pm(e_j)) \\ A_{3\pm} : \ell^2 \oplus \ell^2 &\rightarrow \mathcal{A}_\pm, & (\mathbf{g}_1, \mathbf{g}_2) &\mapsto a_\pm(\mathbf{g}_1) + a_\pm^\dagger(\overline{\mathbf{g}_2}) = \sum_j (\overline{g_{1,j}} a_\pm(e_j) + \overline{g_{2,j}} a_\pm^\dagger(e_j)). \end{aligned}$$

## 5.9 Representations of Bogoliubov Transformations

The identification is  $\iota_3 : \ell^2 \oplus \ell^2 \rightarrow W_1$  with  $\iota_3(\mathbf{F}) = A_{3\pm}^\dagger(\mathbf{F})$ . A Bogoliubov transformation can now indeed be expressed by a matrix  $\mathcal{V}_3 := \iota_3^{-1} \circ \mathcal{V}_A \circ \iota_3$  with  $\mathcal{V}_3 : \ell^2 \oplus \ell^2 \rightarrow \ell^2 \oplus \ell^2$ , which is explicitly given by

$$\mathcal{V}_3 = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix},$$

as in (5.10). Here,  $v$  and  $u$  are defined by its matrix elements  $v_{ij} = \langle e_i, \hat{v}e_j \rangle$  and  $u_{ij} = \langle e_i, \hat{u}e_j \rangle$ . In addition, transpose, complex conjugate and adjoint are given by  $(u^T)_{ij} = u_{ji}$ ,  $(\bar{u})_{ij} = \overline{u_{ij}}$ ,  $(u^*)_{ij} = \overline{u_{ji}}$  and the same for  $v_{ij}$

As in the previous representation, the CAR/CCR conservation amounts to

$$\mathcal{V}_3^* \mathcal{S}_\pm \mathcal{V}_3 = \mathcal{V}_3 \mathcal{S}_\pm \mathcal{V}_3^* = \mathcal{S}_\pm, \quad (5.202)$$

but this time,  $\mathcal{S}_- = id$  and  $\mathcal{S}_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  are operators on  $\ell^2 \oplus \ell^2$ .

For *fermions*  $(-)$ ,

$$\mathcal{V}_3^* \mathcal{V}_3 = \begin{pmatrix} u^*u + v^T \bar{v} & u^*v + v^T \bar{u} \\ v^*u + u^T \bar{v} & v^*v + u^T \bar{u} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.203)$$

Note that both the operators  $v^T \bar{v} = v^*v$  and  $u^T \bar{u} = u^*u$  are real. As for the previous representation, the second line is equivalent to the first one, which in turn yields two conditions:

$$u^*u + v^T \bar{v} = 1 \quad u^*v + v^T \bar{u} = 0. \quad (5.204)$$

The CAR/CCR conservation for the adjoint transformation is

$$\mathcal{V}_3 \mathcal{V}_3^* = \begin{pmatrix} uu^* + vv^* & uv^T + vu^T \\ \bar{v}u^* + \bar{u}v^* & \bar{v}v^T + \bar{u}u^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.205)$$

From which we recover the two conditions

$$uu^* + vv^* = 1 \quad uv^T + vu^T = 0. \quad (5.206)$$

So in total, with (5.204) and (5.206), we have 4 conditions.

For *bosons*, we get the same conditions with an additional minus signs. So in total, we have

$$\begin{aligned} u^*u \mp v^T \bar{v} &= 1 & u^*v \mp v^T \bar{u} &= 0 \\ uu^* \mp vv^* &= 1 & uv^T \mp vu^T &= 0. \end{aligned} \quad (5.207)$$

Again, the transformation can be written by a matrix multiplication, which ma-

kes this representation very convenient to handle. It is hence widely used, e.g., in [215, 225, 227, 230]. Note that depending on the resource, the definitions of  $v$  and  $\bar{v}$  might be interchanged.

## 5.10 Proof that Fermionic $\mathbb{U}_{\mathcal{V}}$ Implements $\mathcal{V}$

In this section, we prove that the fermionic Bogoliubov implementer  $\mathbb{U}_{\mathcal{V}}$  in (5.39) indeed implements the Bogoliubov transformation  $\mathcal{V}$ . The proof for the bosonic case (5.30) works similar. For shortness, we write  $a^{\sharp}(\boldsymbol{\eta}_j) = a_j^{\sharp}$ .

**Proposition 5.10.1.** *Let  $\mathcal{V} = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}$  be a fermionic Bogoliubov transformation. If  $\text{tr}(v^*v) < \infty$ , then the operator*

$$\mathbb{U}_{\mathcal{V}} = \left( \prod_{j \in J_1''} (a_j^{\dagger} + a_j) \right) \exp \left( - \sum_{i \in I'} \xi_i (a_{2i}^{\dagger} a_{2i-1}^{\dagger} - a_{2i-1} a_{2i}) \right) \mathbb{U}_{\boldsymbol{\eta}\boldsymbol{f}},$$

with  $J_1'', I'$  defined in Section 5.3.2, implements  $\mathcal{V}$ , i.e.,

$$\mathbb{U}_{\mathcal{V}} a(\boldsymbol{\phi}) \mathbb{U}_{\mathcal{V}}^* = a(u\boldsymbol{\phi}) + a^{\dagger}(v\bar{\boldsymbol{\phi}}), \quad \mathbb{U}_{\mathcal{V}} a^{\dagger}(\boldsymbol{\phi}) \mathbb{U}_{\mathcal{V}}^* = a^{\dagger}(u\boldsymbol{\phi}) + a(v\bar{\boldsymbol{\phi}}) \quad \forall \boldsymbol{\phi} \in \ell^2. \quad (5.208)$$

*Proof.* Recall that for a fermionic Bogoliubov transformation,  $u$  and  $v$  are bounded, so the above expressions are well-defined.

We first decompose  $\boldsymbol{\phi}$  according to the orthonormal basis  $(\boldsymbol{f}_j)_{j \in J}$  defined in Section 5.3.2 as an eigenbasis of the matrix  $C^*C$ . This basis is mapped under both  $u$  and  $v$  to an orthonormal basis  $(\boldsymbol{\eta}_j)_{j \in J}$ , also defined in Section 5.3.2. Since  $\mathbb{U}_{\boldsymbol{\eta}\boldsymbol{f}}$  just performs a unitary transformation, which replaces the  $\boldsymbol{f}_j$ - by  $\boldsymbol{\eta}_j$ -vectors, we may also remove the  $\mathbb{U}_{\boldsymbol{\eta}\boldsymbol{f}}$  from  $\mathbb{U}_{\mathcal{V}}$ , and replace the  $\boldsymbol{f}_j$ - with  $\boldsymbol{\eta}_j$ -vectors for all  $j \in J$ .

Recall that one may decompose  $J = J' \cup J_0'' \cup J_1''$ , where  $J'$  contains ‘‘Cooper pairs’’ of indices  $(2i, 2i - 1)$  with  $i \in I'$ ,  $J_0''$  contains all ‘‘invariant modes’’ and  $J_1''$  all modes with a full ‘‘particle–hole transformation’’. Then, it holds that

$$\begin{aligned} v\bar{\boldsymbol{\eta}}_j &= 0, & u\boldsymbol{\eta}_j &= \boldsymbol{\eta}_j & \text{for } j \in J_0'' \\ v\bar{\boldsymbol{\eta}}_j &= \boldsymbol{\eta}_j, & u\boldsymbol{\eta}_j &= 0 & \text{for } j \in J_1'' \\ v\bar{\boldsymbol{\eta}}_{2i} &= \beta_i \boldsymbol{\eta}_{2i-1}, & u\boldsymbol{\eta}_{2i} &= \alpha_i \boldsymbol{\eta}_{2i} & \text{for } i \in I' \\ v\bar{\boldsymbol{\eta}}_{2i-1} &= -\beta_i \boldsymbol{\eta}_{2i}, & u\boldsymbol{\eta}_{2i-1} &= \alpha_i \boldsymbol{\eta}_{2i-1} & \text{for } i \in I', \end{aligned} \quad (5.209)$$

## 5.10 Proof that Fermionic $\mathbb{U}_{\mathcal{V}}$ Implements $\mathcal{V}$

with  $\sin \xi_i := \beta_i$  and  $\cos \xi_i := \alpha_i$ .

(5.208) can now be checked to hold mode-by-mode.

For  $j \in J_0''$ , it is easy to see that  $\mathbb{U}_{\mathcal{V}}$  implements  $\mathcal{V}$ , as it acts as the identity.

For  $j \in J_1''$ , the implementation is also quickly verified using  $a_j a_j = a_j^\dagger a_j^\dagger = 0$ :

$$(a_j^\dagger + a_j)a_j(a_j^\dagger + a_j) = a_j^\dagger a_j a_j^\dagger \stackrel{(\text{CAR})}{=} a_j^\dagger - a_j^\dagger a_j^\dagger a_j = a_j^\dagger. \quad (5.210)$$

The statement  $(a_j^\dagger + a_j)a_j^\dagger(a_j^\dagger + a_j) = a_j$  follows by swapping the roles of  $a_j$  and  $a_j^\dagger$ .

For  $i \in I'$ , we make use of the **Lie-Schwinger formula** [233] (for a proof, see [208]):

$$e^A B a^{-A} = \sum_{n=0}^{\infty} \frac{\text{ad}^n(A)B}{n!}, \quad (5.211)$$

with  $\text{ad}^n(A)B := [A, [A, \dots [A, B] \dots]]$  being the  $n$ -fold commutator. In our case,

$$A = -\xi_i(a_{2i}^\dagger a_{2i-1}^\dagger - a_{2i-1} a_{2i}), \quad B = a_{2i}^\# \quad \text{or} \quad B = a_{2i-1}^\#. \quad (5.212)$$

The following formula will turn out useful for the calculations

$$[XY, Z] = XYZ + XZY - XZY - ZXY = X\{Y, Z\} - \{X, Z\}Y, \quad (5.213)$$

with  $\{.,.\} = [.,.]_-$  denoting the anticommutator. We start with  $B = a_{2i}$  and compute

$$\begin{aligned} \text{ad}^0(A)B &= B = a_{2i} \\ \text{ad}^1(A)B &= [A, B] = -\xi_i[a_{2i}^\dagger a_{2i-1}^\dagger, a_{2i}] + \xi_i[a_{2i-1} a_{2i}, a_{2i}] \\ &= \xi_i\{a_{2i}^\dagger, a_{2i}\}a_{2i-1}^\dagger = \xi_i a_{2i-1}^\dagger \\ \text{ad}^2(A)B &= [A, [A, B]] = -\xi_i^2[a_{2i}^\dagger a_{2i-1}^\dagger, a_{2i-1}^\dagger] + \xi_i^2[a_{2i-1} a_{2i}, a_{2i-1}^\dagger] \\ &= -\xi_i^2\{a_{2i-1}, a_{2i-1}^\dagger\}a_{2i} = -\xi_i^2 a_{2i}. \end{aligned} \quad (5.214)$$

Now,  $\text{ad}^n(A)B$ -terms of higher order repeatedly change between  $a_{2i}$  and  $a_{2i-1}^\dagger$ . More precisely,

$$\text{ad}^n(A)B = \begin{cases} (-1)^m \xi_i^{2m} a_{2i} & \text{for } n = 2m \\ (-1)^m \xi_i^{2m+1} a_{2i-1}^\dagger & \text{for } n = 2m + 1 \end{cases}. \quad (5.215)$$

## 5 Implementing Bogoliubov Transformations Beyond the Shale–Stinespring Condition

So we can split the series (5.211) into an even and an odd part:

$$\mathbb{U}_{\mathcal{V}} a_{2i} \mathbb{U}_{\mathcal{V}}^* = \sum_{m=0}^{\infty} \left( \frac{(-1)^m \xi_i^{2m}}{2m!} a_{2i} + \frac{(-1)^m \xi_i^{2m+1}}{(2m+1)!} a_{2i-1}^{\dagger} \right) = \cos(\xi_i) a_{2i} + \sin(\xi_i) a_{2i-1}^{\dagger}, \quad (5.216)$$

which is the desired result.

For  $\mathbb{U}_{\mathcal{V}} a_{2i-1} \mathbb{U}_{\mathcal{V}}^*$ , we have to interchange  $a_{2i}^{\#}$  and  $a_{2i-1}^{\#}$ , which in (5.212) leads to a change of  $\xi_i$  to  $-\xi_i$ , so we obtain

$$\mathbb{U}_{\mathcal{V}} a_{2i-1} \mathbb{U}_{\mathcal{V}}^* = \cos(\xi_i) a_{2i-1} - \sin(\xi_i) a_{2i}^{\dagger}. \quad (5.217)$$

The computations for  $a_{2i}^{\dagger}$  and  $a_{2i-1}^{\dagger}$  go through analogously. The series in (5.214) now start with  $a_{2i-1}^{\dagger}$  instead of  $a_{2i}$  and  $a_{2i}^{\dagger}$  instead of  $a_{2i-1}$ , which leads to

$$\mathbb{U}_{\mathcal{V}} a_{2i-1}^{\dagger} \mathbb{U}_{\mathcal{V}}^* = \cos(\xi_i) a_{2i-1}^{\dagger} - \sin(\xi_i) a_{2i}, \quad \mathbb{U}_{\mathcal{V}} a_{2i}^{\dagger} \mathbb{U}_{\mathcal{V}}^* = \cos(\xi_i) a_{2i}^{\dagger} + \sin(\xi_i) a_{2i-1}, \quad (5.218)$$

as desired. □

We remark that the implementation of a bosonic  $\mathcal{V}$  by  $\mathbb{U}_{\mathcal{V}}$  (5.30) can be checked similarly.

### 5.11 Alternative Definition $a^{\dagger}(\phi), a(\phi)$ on Infinite Tensor Products

In Section 5.4.2, we defined some spaces  $\mathcal{D}_e \subseteq \ell^2$  (5.60) and  $\mathcal{S}^{\otimes} \subseteq \widehat{\mathcal{H}}$  (5.65), such that for  $\phi_1, \dots, \phi_N \in \mathcal{D}_e$  and  $\Psi \in \mathcal{S}^{\otimes}$ , arbitrary operator products

$$a^{\#}(\phi_1) \dots a^{\#}(\phi_N) \Psi \in \mathcal{S}^{\otimes} \subseteq \widehat{\mathcal{H}},$$

were well-defined (Lemma 5.4.7). However, the definition of  $\mathcal{D}_e$  resembling test functions is quite restrictive. It is possible to consider larger domains for the form factor  $\phi \in \mathcal{D}_e$ . The price one has to pay is a stricter particle number decay condition in  $\mathcal{S}^{\otimes}$ . We propose two alternatives: one with a uniform particle number decay in the particle number ( $\mathcal{S}_{\text{uni}}^{\otimes}$ , Lemma 5.11.1) and a class of even smaller ones, based on Hölder sequence spaces ( $\mathcal{S}_q^{\otimes}$ , Lemma 5.11.2).

### 5.11.1 Uniform Particle Number Decay

As ITP vectors with uniform particle number decay, we define

$$\mathcal{S}_{\text{uni}}^\otimes := \left\{ \Psi \in \widehat{\mathcal{H}} \mid \|N_k^n \Psi\| \leq c_n \|\Psi\| \ \forall k \right\}. \quad (5.219)$$

In particular,  $\mathcal{S}_{\text{uni}}^\otimes \subseteq \mathcal{S}^\otimes$ , as the sequence of decay constants  $c_{k,n} = c_n$  in the definition of  $\mathcal{S}^\otimes$  (5.65) is additionally required to be in  $\ell^\infty$ . Correspondingly, the form factors  $\phi \in \ell^2$  have to be additionally in the dual space  $\ell^1$ , i.e.,

$$\sum_{k \in \mathbb{N}} |\phi_k| = \|\phi\|_{\ell^1} < \infty. \quad (5.220)$$

**Lemma 5.11.1** (Creation and annihilation operator products for  $\mathcal{S}_{\text{uni}}^\otimes$ ).

For  $\phi_1, \dots, \phi_n \in \ell^1$  and  $\Psi \in \mathcal{S}_{\text{uni}}^\otimes$ , any operator product application

$$a^\#(\phi_1) \dots a^\#(\phi_n) \Psi \in \widehat{\mathcal{H}}, \quad (5.221)$$

is well-defined.

*Proof.* By the modified particle number decay condition (5.219),

$$\|a_k^\dagger \Psi\| = \|\sqrt{N_k + 1} \Psi\| \leq \|(N_k + 1) \Psi\| \leq (c_1 + 1) \|\Psi\|, \quad (5.222)$$

where we used that on the one-mode Fock space  $\mathcal{H}_k$ , the operator  $a_k^\dagger$  shifts all sectors up by one (keeping them orthogonal) and multiplies by  $\sqrt{N_k + 1}$ . So

$$\|a^\dagger(\phi) \Psi\| \leq \sum_k |\phi_k| (c_1 + 1) \|\Psi\| = \|\phi\|_{\ell^1} (c_1 + 1) \|\Psi\|. \quad (5.223)$$

For an  $n$ -fold application of  $a^\dagger(\phi)$ , we obtain in similarity to (5.69):

$$a^\dagger(\phi_1) \dots a^\dagger(\phi_n) \Psi = \sum_{k_1, \dots, k_n} \phi_{1, k_1} \dots \phi_{n, k_n} a_{k_1}^\dagger \dots a_{k_n}^\dagger \Psi. \quad (5.224)$$

So creations are applied to at most  $n$  modes, with at most  $n$  applications, each. In the maximal application number case,

$$\|(a_k^\dagger)^n \Psi\| = \left\| \prod_{\ell=1}^n \sqrt{N_k + \ell} \Psi \right\| \leq \|(N_k + n)^n \Psi\|. \quad (5.225)$$

The operator  $(N_k + n)^n$  is a polynomial of degree  $n$  in  $N_k$ . So by the modified

particle number decay condition (5.219) we can estimate

$$\|(N_k + n)^n \Psi\| \leq \tilde{c}_n \|\Psi\|, \quad (5.226)$$

with a constant  $\tilde{c}_n$  depending on  $c_1, \dots, c_n$  and  $n$ . A similar estimate holds true for all application numbers  $l \leq n$  with a respective constant  $\tilde{c}_l$ . Denoting the maximum of those constants by  $\tilde{c}_{n,\max}$ , and keeping in mind that at most  $n$  modes can receive a creation operator, we finally obtain

$$\|a_{k_1}^\dagger \dots a_{k_n}^\dagger \Psi\| \leq \|\tilde{c}_{n,\max}^n \Psi\|. \quad (5.227)$$

Hence, when summing over all  $k$ ,

$$\|a^\dagger(\phi_1) \dots a^\dagger(\phi_n) \Psi\| \leq \left\| \sum_{k_1, \dots, k_n} |\phi_{1,k_1}| \dots |\phi_{n,k_n}| \tilde{c}_{n,\max}^n \Psi \right\| = \tilde{c}_{n,\max}^n \prod_{\ell=1}^n \left( \sum_{k_\ell} |\phi_{k_\ell}| \right) \|\Psi\|. \quad (5.228)$$

Replacing any number of creation by annihilation operators will lower the number  $\prod_{\ell=1}^n \sqrt{N_k + \ell}$  in (5.225), so the right-hand side stays an upper bound and all estimates remain valid. Hence, also  $a^\#(\phi_1) \dots a^\#(\phi_n) \Psi \in \widehat{\mathcal{H}}$ , which establishes the claim. □

### Remarks.

40. As in Lemma 5.4.7,  $\Psi \in \mathcal{S}_{\text{uni}}^\otimes$  also result in  $a^\dagger(\phi) \Psi \in \mathcal{S}_{\text{uni}}^\otimes$ , but we may have  $a(\phi) \Psi \notin \mathcal{S}_{\text{uni}}^\otimes$ .

The first statement can be seen by

$$\begin{aligned} \|N_k^n a_k^\dagger \Psi\| &= \|(N_k + 1)^n \sqrt{N_k + 1} \Psi\| \leq \|(N_k + 1)^{n+1} \Psi\| \leq \text{const.} \|\Psi\| \\ &\leq \text{const.} \|a_k^\dagger \Psi\|, \end{aligned} \quad (5.229)$$

with the const. depending on  $c_1, \dots, c_{n+1}$ . So the uniform particle number decay condition is satisfied.

This argument does not work for  $a_k$ , since  $\|\Psi\| \leq \|a_k^\dagger \Psi\|$  does not generalize to annihilation operators. A simple counter-example, where  $a(\phi) \Psi \notin \mathcal{S}_{\text{uni}}^\otimes$  is the following: Define  $\Psi = \prod_{k \in \mathbb{N}}^\otimes \Psi_k$  such that

$$\Psi_k^{(k+1)} = e^{-k}, \quad \Psi_k^{(0)} = \sqrt{1 - e^{-2k}}, \quad (5.230)$$

### 5.11 Alternative Definition $a^\dagger(\phi), a(\phi)$ on Infinite Tensor Products

and all other sectors are unoccupied ( $\Psi_k^{(N_k)} = 0$ ). Here,  $\Psi_k^{(n)} = \langle e_{k,n}, \Psi_k \rangle_k = c_{k,n}$  with respect to the basis  $(e_{k,n})_{n \in \mathbb{N}_0}$  of  $n$ -particle states in mode  $k$ . So  $\|\Psi_k\|_k = 1$  and  $\|\Psi\| = 1$ . Then, on  $\mathcal{H}_k$  we have

$$\|N_k^n \Psi_k\|_k = (k+1)^n \|\Psi_k^{(k+1)}\|_k = (k+1)^n e^{-k} = (k+1)^n e^{-k} \|\Psi_k\|_k. \quad (5.231)$$

Since the function  $x \mapsto (x+1)^n e^{-x}$  is bounded on  $[0, \infty)$ , we have a uniform particle number decay, e.g., with  $c_n = \sup_{x \geq 0} (x+1)^n e^{-x}$ . However, in  $a_k \Psi_k$ , only the  $k$ -sector is occupied with  $\|a_k \Psi_k\|_k = \sqrt{k} e^{-k}$ , so

$$\|N_k^n a_k \Psi_k\|_k = \sqrt{k} k^n e^{-k} = k^n \|a_k \Psi_k\|_k, \quad (5.232)$$

and  $k^n$  cannot be uniformly bounded in all  $k \in \mathbb{N}$ . Hence,  $a(\phi)\Psi$  does not meet the rapid decay condition if we choose  $\phi \in \ell^1$  such that  $\phi_k \neq 0$  holds for infinitely many  $k$ , for instance by  $\phi_k = \frac{1}{k^2}$ .

41. Theorem 5.5.5 *does not hold* in a similar form for  $\mathcal{S}_{\text{uni}}^\otimes$  instead of  $\mathcal{S}^\otimes$ . The Bogoliubov vacuum  $\Omega_\nu$  (5.87) is in  $\mathcal{S}_{\text{uni}}^\otimes$ , **if and only if**  $C = u^* v \mathcal{J}$  is **bounded**. This can be seen by checking the particle number decay condition. We recall (5.89):

$$\|N_k^n \Omega_{k,\nu}\|_k^2 = (1 - 4t^2)^{1/2} \sum_{N=0}^{\infty} \frac{t^{2N} (2N)!}{(N!)^2} (2N)^{2n} =: f_n(t),$$

with  $t = |\frac{\nu_k}{2\mu_k}| \in [0, 1/2)$  depending continuously and monotonically on the eigenvalues  $\lambda_k$  of  $C$  and with  $t \rightarrow 1/2$  as  $\lambda_k \rightarrow \infty$ .

If  $C$  is bounded, then there is a largest  $\lambda_k$ , meaning a largest  $t$  exists, called  $t_{\text{max}}$ . The continuous functions  $f_n$  attain a maximum on  $t \in [0, t_{\text{max}}]$ , called  $c_n$ . So  $\|N_k^n \Omega_{k,\nu}\|_k^2 \leq c_n$ , which establishes the uniform particle number decay and hence  $\Omega_\nu \in \mathcal{S}_{\text{uni}}^\otimes$ .

## 5 Implementing Bogoliubov Transformations Beyond the Shale–Stinespring Condition

However, for unbounded  $C$ , the sequence  $\|N_k \Omega_{k,\nu}\|_k$  diverges as  $\lambda_k \rightarrow \infty$ :

$$\begin{aligned}
 (1 - 4t^2)^{-1/2} \|N_k \Omega_{k,\nu}\|_k^2 &= \sum_{N=0}^{\infty} \frac{t^{2N} (2N)!}{(N!)^2} (2N)^2 \geq \sum_{N=0}^{\infty} \frac{t^{2N} (2N)!}{(N!)^2} (2N) \\
 &= t \frac{d}{dt} \left( \sum_{N=1}^{\infty} \frac{t^{2N} (2N)!}{(N!)^2} \right) + 1 \\
 &= t \frac{d}{dt} (1 - 4t^2)^{-1/2} + 1 = 4t^2 (1 - 4t^2)^{-3/2} + 1 \\
 \Leftrightarrow \|N_k \Omega_{k,\nu}\|_k^2 &\geq \underbrace{4t^2 (1 - 4t^2)^{-1}}_{\rightarrow \infty} + \underbrace{(1 - 4t^2)^{1/2}}_{\rightarrow 0} \rightarrow \infty \quad \text{as } t \rightarrow 1/2.
 \end{aligned} \tag{5.233}$$

So there is no way to set up a uniform bound for  $\|N_k \Omega_{k,\nu}\|_k^2$  or  $\|N_k \Omega_{k,\nu}\|_k$ , meaning that the uniform particle number decay condition cannot be fulfilled and  $\Omega_\nu \notin \mathcal{S}_{\text{uni}}^\otimes$ .

### 5.11.2 Hölder Condition in Decay Coefficients

We may also consider other conditions on  $\phi \in \mathcal{D}_e$ , for instance

$$\phi \in \ell^p \quad \Leftrightarrow \quad \sum_k |\phi_k|^p < \infty, \tag{5.234}$$

and define

$$\mathcal{D}_p := \ell^p, \quad 1 \leq p \leq 2. \tag{5.235}$$

The largest space on which an estimate of the kind (5.228) still goes through is the following:

$$\mathcal{S}_q^\otimes := \left\{ \Psi \in \widehat{\mathcal{H}} \left| \|(N_k + 1)^{n/2} \Psi\| \leq c_k^n \|\Psi\| \text{ with } \sum_k c_k^q < \infty \quad \forall n \in \mathbb{N} \right. \right\}, \tag{5.236}$$

where  $2 \leq q \leq \infty$  is the Hölder dual of  $p$ , i.e.,  $1/q + 1/p = 1$ . With that definition, it is easy to see that  $\mathcal{S}_2^\otimes \subseteq \dots \subseteq \mathcal{S}_q^\otimes \subseteq \dots \subseteq \mathcal{S}_\infty^\otimes$ .

**Lemma 5.11.2** (Creation and annihilation operator products for  $\mathcal{S}_q^\otimes$ ).

For  $\phi_1, \dots, \phi_n \in \mathcal{D}_p$  and  $\Psi \in \mathcal{S}_q^\otimes$ , any operator product application

$$a^\#(\phi_1) \dots a^\#(\phi_n) \Psi \in \widehat{\mathcal{H}} \tag{5.237}$$

is well-defined.

*Proof.* The arguments are almost the same as for Lemma 5.11.1, where in definition (5.236), the estimation step  $\sqrt{N_k + 1} \leq N_k$  for  $N_k \geq 1$  is dropped. An additional simplification comes from the fact that within an  $n$ -fold application of  $a_k^\dagger$ , each application gets out a maximum factor of  $c_k$ . So we don't have to find a  $\tilde{c}_{n,\max}$  anymore, but can directly estimate each application of  $a_k^\dagger$  by  $c_k$ . Thus, we get

$$\|a^\dagger(\phi_1) \dots a^\dagger(\phi_n)\Psi\| \leq \sum_{k_1, \dots, k_n} |\phi_{1,k_1}| \dots |\phi_{n,k_n}| c_{k_1} \dots c_{k_n} \|\Psi\| = \prod_{\ell=1}^n \left( \sum_{k_\ell} |\phi_{\ell,k_\ell}| c_{k_\ell} \right) \|\Psi\|. \quad (5.238)$$

The bracket can now be guaranteed to be finite, since  $(\phi_{\ell,k_\ell})_{k_\ell \in \mathbb{N}} \in \ell^p$  and  $(c_{k_\ell})_{k_\ell \in \mathbb{N}} \in \ell^q$ . So expression (5.238) is finite. The same holds true after replacing any number of  $a^\dagger$  by  $a$ -operators, which establishes the claim.  $\square$

## 5.12 Concerning Applications

### 5.12.1 Translation of Hamiltonians into Block Matrices

Consider the following formal quadratic bosonic Hamiltonian, that is only an element of  $\overline{\mathcal{A}}_e$ , (5.58) but not necessarily an operator generating dynamics on Fock space  $\mathcal{F} = \mathcal{F}(\mathbb{N})$ :

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} (2h_{jk} a_j^\dagger a_k + k_{jk} a_j^\dagger a_k^\dagger + \overline{k_{jk}} a_j a_k), \quad (5.239)$$

where  $h_{jk}, k_{jk} \in \mathbb{C}$ , so  $h = (h_{jk})_{j,k \in \mathbb{N}}, k = (k_{jk})_{j,k \in \mathbb{N}}$  are matrices of infinite size. By symmetry of  $H$ , we have  $h^* = h \Leftrightarrow h_{jk} = \overline{h_{kj}}$  and we can arrange for  $k^T = k \Leftrightarrow k_{jk} = k_{kj}$ . Now recall from Section 5.9.4, that each  $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2) \in \ell^2 \oplus \ell^2$  can be identified with a bosonic algebraic expression

$$A^\dagger(\mathbf{F}) = A_{3+}^\dagger(\mathbf{F}) = a^\dagger(\mathbf{f}_1) + a(\overline{\mathbf{f}_2}) = \sum_j (f_{1,j} a_{j+}^\dagger(e_j) + f_{2,j} a_+(e_j)), \quad (5.240)$$

or with  $A(\mathbf{F}) = (A^\dagger(\mathbf{F}))^*$ . We may now identify  $H$  with the following block matrices [215, 216, 225, 227]:

$$A_H = \begin{pmatrix} h & k \\ \overline{k} & \overline{h} \end{pmatrix}, \quad B_H = \begin{pmatrix} h & -k \\ \overline{k} & -\overline{h} \end{pmatrix} = A_H \mathcal{S}, \quad (5.241)$$

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where  $\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . With this identification,

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} \langle \mathbf{F}_j, A_H \mathbf{F}_k \rangle A^\dagger(\mathbf{F}_j) A(\mathbf{F}_k), \quad (5.242)$$

where  $\mathbf{F}_j = \begin{pmatrix} \mathbf{e}_j \\ \mathbf{e}_j \end{pmatrix}$  for a canonical basis vector  $\mathbf{e}_j$  and where by a direct calculation

$$[A(\mathbf{F}), [H, A^\dagger(\mathbf{G})]] = \langle \mathbf{F}, (\mathcal{S}B_H)\mathbf{G} \rangle_{\ell^2 \oplus \ell^2} \quad (5.243)$$

holds for all finite linear combinations

$$\mathbf{F}, \mathbf{G} \in \mathcal{D}_F, \quad \mathcal{D}_F := \text{span} \left\{ \begin{pmatrix} \mathbf{e}_\ell \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{e}_\ell \end{pmatrix}, \mid \ell \in \mathbb{N} \right\}, \quad \mathcal{D}_F \subset \ell^2 \oplus \ell^2. \quad (5.244)$$

Infinite linear combinations may or may not be well-defined.

Often  $H$  is called “second quantization of  $A_H$ ” and  $A_H$  is called “first quantization of  $H$ ”, where the first one is unique up to a normal ordering constant  $c$ , which we set to 0.

Further, the (weak) Schrödinger dynamics generated by  $-B_H$  on  $\mathbf{F}, \mathbf{G} \in \mathcal{D}_F$  via

$$\langle \mathbf{G}, i\partial_t \mathbf{F} \rangle = -\langle \mathbf{G}, B_H \mathbf{F} \rangle \quad (5.245)$$

correspond to the respective Heisenberg dynamics generated by  $H$  on expressions  $A^\dagger(\mathbf{F})$  in the following sense: If we extend the map  $A^\dagger : \ell^2 \oplus \ell^2 \rightarrow \overline{\mathcal{A}}_e$  to  $A^\dagger : \mathcal{E}(\mathbb{N}) \oplus \mathcal{E}(\mathbb{N}) \rightarrow \overline{\mathcal{A}}_e$  (which just means that we drop the assumption in  $A^\dagger(\mathbf{G})$  of  $\mathbf{G} = (\mathbf{g}_1, \mathbf{g}_2)$  containing only square-summable  $\mathbf{g}_1, \mathbf{g}_2$ ), then the following statement is true:

**Lemma 5.12.1.** *In the bosonic case, for all  $\mathbf{F} \in \mathcal{D}_F$  we have:*

$$A^\dagger(iB_H \mathbf{F}) = i[H, A^\dagger(\mathbf{F})]. \quad (5.246)$$

This lemma can also be found in a similar form in [225, (2.27)]. For completeness, we give a proof here:

*Proof.* Consider any canonical basis vector  $\mathbf{e}_\ell \in \ell^2$ . Then, for  $\mathbf{F} = \begin{pmatrix} \mathbf{e}_\ell \\ 0 \end{pmatrix}$ ,

$$B_H \mathbf{F} = \begin{pmatrix} \sum_j h_{j\ell} \mathbf{e}_j \\ \sum_j \overline{k_{j\ell}} \mathbf{e}_j \end{pmatrix} \Rightarrow A^\dagger (iB_H \mathbf{F}) = i \sum_j (h_{j\ell} a_j^\dagger + \overline{k_{j\ell}} a_j). \quad (5.247)$$

Note that the sum over  $j$  is finite due to  $\mathbf{F} \in \mathcal{D}_{\mathcal{F}}$ . On the other hand since  $k_{j\ell} = k_{\ell j}$ ,

$$i[H, A^\dagger(\mathbf{F})] = \frac{i}{2} \sum_j (2h_{j\ell} a_j^\dagger + \overline{k_{j\ell}} a_j + \overline{k_{\ell j}} a_j) = i \sum_j (h_{j\ell} a_j^\dagger + \overline{k_{j\ell}} a_j). \quad (5.248)$$

This verifies (5.246) for  $\mathbf{F} = \begin{pmatrix} \mathbf{e}_\ell \\ 0 \end{pmatrix}$ .

A similar calculation can be carried out for  $\mathbf{F}' = \begin{pmatrix} 0 \\ \mathbf{e}_\ell \end{pmatrix}$ , using  $\overline{h_{j\ell}} = h_{\ell j}$ :

$$B_H \mathbf{F}' = \begin{pmatrix} -\sum_j k_{j\ell} \mathbf{e}_j \\ -\sum_j \overline{h_{j\ell}} \mathbf{e}_j \end{pmatrix} \Rightarrow A^\dagger (iB_H \mathbf{F}') = -i \sum_j (k_{j\ell} a_j^\dagger + h_{\ell j} a_j), \quad (5.249)$$

$$i[H, A^\dagger(\mathbf{F}')] = \frac{i}{2} \sum_j (-2h_{\ell j} a_j - k_{\ell j} a_j^\dagger - k_{j\ell} a_j^\dagger) = -i \sum_j (h_{\ell j} a_j^\dagger + k_{j\ell} a_j). \quad (5.250)$$

So by taking finite linear combinations of  $\begin{pmatrix} \mathbf{e}_\ell \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ \mathbf{e}_\ell \end{pmatrix}$ , (5.246) holds for all  $\mathbf{F} \in \mathcal{D}_F$ . □

A similar statement is valid for fermions. Here, the formal quadratic Hamiltonian is of the form

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} (2h_{jk} a_j^\dagger a_k - k_{jk} a_j^\dagger a_k^\dagger + \overline{k_{jk}} a_j a_k), \quad (5.251)$$

with  $h^* = h$  and where we can arrange for  $k^T = -k$ . This time, we only associate one block matrix to  $H$ :

$$A_H = \begin{pmatrix} h & -k \\ \overline{k} & -\overline{h} \end{pmatrix}. \quad (5.252)$$

It serves for both translation conditions

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} \langle \mathbf{F}_j, A_H \mathbf{F}_k \rangle A^\dagger(\mathbf{F}_j) A(\mathbf{F}_k), \quad (5.253)$$

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and [227, (2b.38)]:

$$\{(A^\dagger(\mathbf{F}))^*, [H, A^\dagger(\mathbf{G})]\} = \langle \mathbf{F}, A_H \mathbf{G} \rangle_{\ell^2 \oplus \ell^2}, \quad \mathbf{F}, \mathbf{G} \in \mathcal{D}_F. \quad (5.254)$$

Note that the CAR imply  $k_{jj} = 0$ . For  $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2)$ , the expression  $A_{3-}^\dagger(\mathbf{F})$  is again defined as

$$A_{3-}^\dagger(\mathbf{F}) = a^\dagger(\mathbf{f}_1) + a(\overline{\mathbf{f}_2}) = \sum_j (f_{1,j} a_+^\dagger(e_j) + f_{2,j} a_+(e_j)). \quad (5.255)$$

The Schrödinger dynamics of  $(-A_H)$  can now be translated into the Heisenberg dynamics of  $H$ :

**Lemma 5.12.2.** *In the fermionic case, for all  $\mathbf{F} \in \mathcal{D}_F$ , we have:*

$$A^\dagger(iA_H \mathbf{F}) = i[H, A^\dagger(\mathbf{F})]. \quad (5.256)$$

*Proof.* The proof is similar to that of Lemma 5.12.1. We consider again a canonical basis vector  $\mathbf{e}_\ell \in \ell^2$  and  $\mathbf{F} = \begin{pmatrix} \mathbf{e}_\ell \\ 0 \end{pmatrix}$ ,  $\mathbf{F}' = \begin{pmatrix} 0 \\ \mathbf{e}_\ell \end{pmatrix}$ . Formulas (5.247) and (5.249) can be copied from the proof of Lemma 5.12.1:

$$A^\dagger(iA_H \mathbf{F}) = i \sum_j (h_{j\ell} a_j^\dagger + \overline{k_{j\ell}} a_j), \quad A^\dagger(iA_H \mathbf{F}') = -i \sum_j (k_{j\ell} a_j^\dagger + h_{\ell j} a_j), \quad (5.257)$$

with the sums in  $j$  being finite and hence convergent. For computing the commutators, we make use of the CAR and  $k_{j\ell} = -k_{\ell j}$ :

$$\begin{aligned} i[H, A^\dagger(\mathbf{F})] &= \frac{i}{2} \sum_{jk} [2h_{jk} a_j^\dagger a_k - k_{jk} a_j^\dagger a_k^\dagger + \overline{k_{jk}} a_j a_k, a_\ell^\dagger] \\ &= \frac{i}{2} \sum_{jk} (2h_{jk} \underbrace{(a_j^\dagger a_k a_\ell^\dagger + a_j^\dagger a_\ell^\dagger a_k)}_{=a_j^\dagger \delta_{k\ell}} - \underbrace{a_j^\dagger a_\ell^\dagger a_k - a_\ell^\dagger a_j^\dagger a_k}_{=0}) \\ &\quad + \overline{k_{jk}} \underbrace{(a_j a_k a_\ell^\dagger + a_j a_\ell^\dagger a_k)}_{=a_j \delta_{k\ell}} - \underbrace{a_j a_\ell^\dagger a_k - a_\ell^\dagger a_j a_k}_{=-a_k \delta_{j\ell}} \\ &= \frac{i}{2} \sum_j 2h_{j\ell} a_j^\dagger + \frac{i}{2} \sum_j \overline{k_{j\ell}} a_j - \frac{i}{2} \sum_k \overline{k_{\ell k}} a_k \\ &= i \sum_j (h_{j\ell} a_j^\dagger + \overline{k_{j\ell}} a_j), \end{aligned} \quad (5.258)$$

$$\begin{aligned}
 i[H, A^\dagger(\mathbf{F}')] &= \frac{i}{2} \sum_{jk} [2h_{jk}a_j^\dagger a_k - k_{jk}a_j^\dagger a_k^\dagger + \overline{k_{jk}}a_j a_k, a_\ell] \\
 &= \frac{i}{2} \sum_{jk} (2h_{jk}(\underbrace{a_j^\dagger a_k a_\ell + a_j^\dagger a_\ell a_k}_{=0} - \underbrace{a_j^\dagger a_\ell a_k - a_\ell a_j^\dagger a_k}_{=-a_k^\dagger \delta_{j\ell}}) \\
 &\quad - k_{jk}(\underbrace{a_j^\dagger a_k^\dagger a_\ell + a_j^\dagger a_\ell a_k^\dagger}_{=a_j^\dagger \delta_{k\ell}} - \underbrace{a_j^\dagger a_\ell a_k^\dagger - a_\ell a_j^\dagger a_k^\dagger}_{=-a_k^\dagger \delta_{j\ell}})) \\
 &= -\frac{i}{2} \sum_k 2h_{\ell k} a_k - \frac{i}{2} \sum_j k_{j\ell} a_j^\dagger + \frac{i}{2} \sum_k k_{\ell k} a_k^\dagger \\
 &= -i \sum_j (h_{\ell j} a_j + k_{j\ell} a_j^\dagger).
 \end{aligned} \tag{5.259}$$

Both results agree with (5.257). Taking finite linear combinations of  $\mathbf{F}, \mathbf{F}'$  establishes the proof.  $\square$

Of course, Lemmas 5.12.1 and 5.12.2 remain valid, if the number of modes is finite, i.e.,  $\mathbf{F} \in \mathbb{C}^N \oplus \mathbb{C}^N$ .

Both lemmas may also remain valid for further  $\mathbf{F} \in \ell^2 \oplus \ell^2$ , provided the expression  $A_H \mathbf{F} \in \ell^2 \oplus \ell^2$  exists. In that case, also an infinite linear combination of modes is well-defined.

### 5.12.2 Deriving Pair Creation in External Field QED

In Section 5.7.3, we considered a quadratic Hamiltonian which is intended to describe a simplified model for external field QED. Starting from the formal Hamiltonian of a Dirac field coupled to a time-dependent homogeneous classical electromagnetic field  $A_\mu(t) = (0, 0, 0, A_3(t)) \in \mathbb{C}^4$ , we may justify the quadratic Hamiltonian as follows:

Consider a Dirac field with discrete momentum  $\mathbf{p} \in \mathbb{Z}^3$  and spin index  $s \in \{1, 2, 3, 4\}$ , where  $s \in \{1, 2\}$  denotes an electron and  $s \in \{3, 4\}$  a positron. The formal Hamiltonian now reads:

$$\begin{aligned}
 H(t) &= H_0 + H_I(t), & H_0 &= d\Gamma(E_{\mathbf{p},0}), & E_{\mathbf{p},0} &= \sqrt{|\mathbf{p}|^2 + m^2} \\
 H_I(t) &= e \int \overline{\Psi(\mathbf{x})} \gamma^\mu \Psi(\mathbf{x}) A_\mu(t) d\mathbf{x},
 \end{aligned} \tag{5.260}$$

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where  $\gamma^\mu$  are the gamma matrices in Dirac representation with Einstein summation convention in the index  $\mu$  assumed and  $e$  is the coupling constant. The field operator-valued distributions  $\Psi(\mathbf{x})$  and  $\overline{\Psi}(\mathbf{x}) = \Psi(\mathbf{x})^* \gamma_0$  are given by

$$\Psi(\mathbf{x}) = \sum_{\mathbf{p} \in \mathbb{Z}^3} \sum_{s \in \{1,2\}} (a_{\mathbf{p},s} \mathbf{u}_{\mathbf{p},s} e^{-i\mathbf{p}\mathbf{x}} + b_{\mathbf{p},s}^* \mathbf{v}_{\mathbf{p},s} e^{i\mathbf{p}\mathbf{x}}), \quad (5.261)$$

with positron operators  $b_{\mathbf{p},1}^\# = a_{\mathbf{p},3}^\#, b_{\mathbf{p},2}^\# = a_{\mathbf{p},4}^\#$ . For the normalized Dirac spinors, there exist many conventions, out of which we adopt the following:

$$\mathbf{u}_{\mathbf{p},s} = c \begin{pmatrix} \phi_s \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \phi_s \end{pmatrix}, \quad \mathbf{v}_{\mathbf{p},s} = c \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m} \chi_s \\ \chi_s \end{pmatrix}, \quad \phi_1 = \chi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi_2 = \chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (5.262)$$

with normalization constant  $c \in \mathbb{C}$ , such that  $\|\mathbf{u}_{\mathbf{p},s}\| = \|\mathbf{v}_{\mathbf{p},s}\| = 1$  and with  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  being the Pauli matrix vector.

Evaluating  $H(t)$  now leads to a quadratic operator that can be translated into a block matrix (see Section 5.12.1):

$$\tilde{A}_H(t) = \bigoplus_{\mathbf{p} \in \mathbb{Z}^3} \tilde{A}_{H,\mathbf{p}}(t), \quad \tilde{A}_{H,\mathbf{p}}(t) = \begin{pmatrix} h(t) & -k(t) \\ k(t) & -h(t) \end{pmatrix} \in \mathbb{C}^{8 \times 8}. \quad (5.263)$$

By a direct calculation, one may verify that

$$eA_3(t) \mathbf{u}_{\mathbf{p},s}^* \gamma^0 \gamma^3 \mathbf{u}_{\mathbf{p},s'} = eA_3(t) \mathbf{u}_{\mathbf{p},s}^* \begin{pmatrix} & \sigma_3 \\ \sigma_3 & \end{pmatrix} \mathbf{u}_{\mathbf{p},s'} = (\Delta E_{\mathbf{p}}(t)) \delta_{ss'}, \quad (5.264)$$

$$\Delta E_{\mathbf{p}}(t) = \frac{2c^2 e A_3(t)}{E+m} p_3,$$

so the  $a^\dagger a$ - and  $b^\dagger b$ -terms within  $H_I(t)$  render an additional kinetic term in

$$h(t) = \begin{pmatrix} E_{\mathbf{p},0} + \Delta E_{\mathbf{p}}(t) & & & \\ & E_{\mathbf{p},0} + \Delta E_{\mathbf{p}}(t) & & \\ & & E_{\mathbf{p},0} - \Delta E_{\mathbf{p}}(t) & \\ & & & E_{\mathbf{p},0} - \Delta E_{\mathbf{p}}(t) \end{pmatrix}. \quad (5.265)$$

By charge conservation, no  $a^\dagger a^\dagger$ ,  $aa$ ,  $b^\dagger b^\dagger$ - or  $bb$ -terms will appear. Moreover, by symmetry  $u_{\mathbf{p}s} \gamma_0 \gamma_3 v_{-\mathbf{p}s'} = v_{-\mathbf{p}s'} \gamma_0 \gamma_3 u_{\mathbf{p}s}$ , and by  $a_{\mathbf{p},s}^\dagger b_{-\mathbf{p},s'}^\dagger = -b_{-\mathbf{p},s'}^\dagger a_{\mathbf{p},s}^\dagger$ , we have

$$k = \begin{pmatrix} 0 & 0 & f_{\mathbf{p},11} & f_{\mathbf{p},12} \\ 0 & 0 & f_{\mathbf{p},21} & f_{\mathbf{p},22} \\ -f_{\mathbf{p},11} & -f_{\mathbf{p},21} & 0 & 0 \\ -f_{\mathbf{p},12} & -f_{\mathbf{p},22} & 0 & 0 \end{pmatrix}, \quad (5.266)$$

with  $f_{\mathbf{p},ss'} = \overline{f_{\mathbf{p},ss'}}$ . We perform a physical simplification by setting  $f_{\mathbf{p},11} = f_{\mathbf{p},22} = 0$ , i.e., eliminating creation and annihilation of equal spins. As a result,  $A_{H,\mathbf{p}}$  decays into a direct sum of two  $\mathbb{C}^{4 \times 4}$ -matrices, one for  $s \in \{1, 4\}$  and one for  $s \in \{2, 3\}$ , of the form

$$A_{H,\mathbf{p}}(t) = \begin{pmatrix} E_{\mathbf{p},0} + \Delta E_{\mathbf{p}}(t) & 0 & 0 & -f_{\mathbf{p}}(t) \\ 0 & E_{\mathbf{p},0} - \Delta E_{\mathbf{p}}(t) & f_{\mathbf{p}}(t) & 0 \\ 0 & f_{\mathbf{p}}(t) & -E_{\mathbf{p},0} - \Delta E_{\mathbf{p}}(t) & 0 \\ -f_{\mathbf{p}}(t) & 0 & 0 & -E_{\mathbf{p},0} + \Delta E_{\mathbf{p}}(t) \end{pmatrix}, \quad (5.267)$$

with  $f_{\mathbf{p}} \in \{f_{\mathbf{p},12}, f_{\mathbf{p},21}\}$ . Setting  $\varepsilon_{\mathbf{p},+}(t) = E_{\mathbf{p},0} + \Delta E_{\mathbf{p}}(t)$  and  $\varepsilon_{\mathbf{p},-}(t) = E_{\mathbf{p},0} - \Delta E_{\mathbf{p}}(t)$ , we arrive at the form (5.161).



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