AN EXPLICIT CONSTRUCTION OF GENERATORS FOR $\{\mathbf{K}(R\text{–Proj})^{\perp}\}^{\perp}$

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Abstract. Let R be a ring. In two previous articles [7, 9] we studied the inclusion $j_! : \mathbf{K}(R\text{–Proj}) \longrightarrow \mathbf{K}(R\text{–Flat})$ and the orthogonal subcategory $\mathcal{S} = \mathbf{K}(R\text{–Flat})^{\perp}$. In this article we explicitly produce generators for $\mathcal{S}^{\perp}$.

Contents

0. Introduction

Let R be an associative ring with a unit. In [7, Proposition 8.1] we proved that the natural inclusion $j_! : \mathbf{K}(R\text{–Proj}) \longrightarrow \mathbf{K}(R\text{–Flat})$ has a right adjoint $j^* : \mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Proj})$. In [9, Theorem 3.1] we showed that the functor j^* has a right adjoint $j_* : \mathbf{K}(R\text{–Proj}) \longrightarrow \mathbf{K}(R\text{–Flat})$. In the two articles cited above we gave a discussion of the significance of these results and of their applications; let us not duplicate this here. The interested reader can find much more about the theory from Krause [4], Jørgensen [3], Iyengar and Krause [2], Murfet [5] as well as the survey article [8].

In [9, Theorem 3.1] the proof of the existence of $j_* : \mathbf{K}(R\text{–Proj}) \longrightarrow \mathbf{K}(R\text{–Flat})$ is not effective. It is very much a pure "existence" proof, not an explicit construction. In this article we propose to give a more down-to-earth approach. Perhaps we ought to explain what this means.

We have a fully faithful functor $j_! : \mathbf{K}(R\text{–Proj}) \longrightarrow \mathbf{K}(R\text{–Flat})$, which possesses a right adjoint j^*. Define $\mathcal{S} = \mathbf{K}(R\text{–Proj})^{\perp}$ to be the full subcategory of $\mathbf{K}(R\text{–Flat})$ whose

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objects are

$$\mathrm{Ob}(\mathcal{S}) \quad = \quad \{y \in \mathbf{K}(R\text{–Flat}) \mid \mathrm{Hom}(x,y)=0 \ \ \forall x \in \mathbf{K}(R\text{–Proj})\}\ ,$$

and let $i_* : \mathcal{S} \longrightarrow \mathbf{K}(R\text{–Flat})$ be the inclusion. Formal nonsense about triangulated categories, which the reader can find in [6, Corollary 9.1.14], tells us that the inclusion i_* must have a left adjoint i^*; diagrammatically we have

$$\mathbf{K}(R\text{–Proj}) \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathbf{K}(R\text{–Flat}) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathbf{K}(R\text{–Proj})^{\perp} \quad = \quad \mathcal{S} \quad ,$$

where each functor is left adjoint to the one below it. The same general theory tells us that the functor j^* will have a right adjoint j_* if and only if i_* has a right adjoint $i^!$. This will happen if and only if every object $y \in \mathbf{K}(R\text{–Flat})$ is part of a distinguished triangle

$$s \longrightarrow y \longrightarrow t \longrightarrow \Sigma s \qquad\qquad (*)$$

with $s \in \mathcal{S}$ and $t \in \mathcal{S}^{\perp}$. A constructive proof would produce the triangle $(*)$ above, or at the very least show that there are plenty of objects in $\mathcal{S}^{\perp}$. That is what we set out to do in this article.

Before we state the main theorem of the article we remind the reader of [9, Theorem 3.2]: it asserts that the natural inclusion $\mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Mod})$ has a right adjoint $J : \mathbf{K}(R\text{–Mod}) \longrightarrow \mathbf{K}(R\text{–Flat})$.

Facts 0.1. Let the notation be as above. In particular, let $\mathcal{S} = \mathbf{K}(R\text{–Proj})^{\perp}$ be the category of all $\mathbf{K}(R\text{–Proj})$–local objects in $\mathbf{K}(R\text{–Flat})$. Then the objects of $\mathcal{S}^{\perp} = \{\mathbf{K}(R\text{–Proj})^{\perp}\}^{\perp}$ are cogenerated by the chain complexes $J\big(\mathcal{H}om_{\mathbb{Z}}(I,\mathbb{Q}/\mathbb{Z})\big) \in \mathbf{K}(R\text{–Flat})$, where

1. The functor J is the right adjoint to the inclusion $\mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Mod})$.
2. I runs over the bounded below chain complexes of injective right R–modules, which satisfy the following two conditions:
   - (a) All but finitely many of the groups $H^i(I)$ vanish.
   - (b) For all i, $H^i(I)$ is a subquotient of a finitely generated, projective right R–module.

The proof is in Theorem 4.8.

Remark 0.2. We should remind the reader what it means for a set of objects to "cogenerate" a subcategory $\mathcal{S}^{\perp} \subset \mathbf{K}(R\text{–Flat})$. In Remark 4.2 we will see that the category $\mathbf{K}(R\text{–Flat})$ has products. The subcategory of $\mathbf{K}(R\text{–Flat})$ cogenerated by a class of objects S is the smallest triangulated subcategory containing $S \subset \mathbf{K}(R\text{–Flat})$ and closed under products. We will denote it $\rangle S \langle$. Thus Theorem 4.8 asserts that, if S is the set of all objects $J\big(\mathcal{H}om_{\mathbb{Z}}(I,\mathbb{Q}/\mathbb{Z})\big)$, then $\rangle S \langle = \mathcal{S}^{\perp}$.

Remark 0.3. In the process of proving Theorem 4.8 we discover that we also give a second proof of the existence of a right adjoint to $i_* : \mathcal{S} \longrightarrow \mathbf{K}(R\text{–Flat})$. More explicitly the argument goes as follows: it is easy to show that the objects $J\big(\mathcal{H}om_{\mathbb{Z}}(I,\mathbb{Q}/\mathbb{Z})\big)$ all lie in $\mathcal{S}^\perp$; see Remark 2.7. One immediately deduces the inclusion $\rangle S\langle \subset \mathcal{S}^\perp$. Given an object $y \in \mathbf{K}(R\text{–Flat})$ we will show how to construct a triangle

$$s \longrightarrow y \longrightarrow t \longrightarrow \Sigma s$$

with $s \in \mathcal{S}$ and $t \in \rangle S\langle$; this will automatically prove both the existence of the right adjoint $i^!$ and the fact that $\rangle S\langle = \mathcal{S}^\perp$.

Remark 0.4. As the reader can imagine it can be handy, in applications, to have a concrete construction rather than a mere existence theorem. I will say nothing about applications here; the interested reader can look at some of the recent preprints of Murfet and Salarian.

Remark 0.5. The reader should note that, for a general ring R, the proof given here of the existence of $i^!$ is not (yet) independent of the proof in [9, Theorem 3.1]. For a general ring R, the proof uses the existence of the functor J of [9, Theorem 3.2]. The existence of J was proved as a consequence of the existence of the right adjoint to i_*, thus rendering the argument circular. But in Remark 2.5 we observe that perhaps there is hope of giving a different proof of the existence of the right adjoint of J, independent of [9, Theorem 3.1].

When the ring R is right coherent, we find that the proof of Theorem 4.8 can be done without making use of the functor J. Since [9, Theorem 3.1] is an immediate consequence of Theorem 4.8 we have that, for coherent rings, the argument given here provides a second proof of [9, Theorem 3.1]. In [9]. we saw an indirect proof which appeals to the Flat Cover Conjecture, while here we will obtain a concrete, constructive proof.

If we are willing to assume, right at the start, that the ring R is right coherent, then the proof of Theorem 4.8 simplifies somewhat. Part of the reason the argument simplifies is that, in some version, the situation has already been treated in the literature. We sketch this, briefly, in §5.

Remark 0.6. In the special case where R is commutative, noetherian and of finite Krull dimension there is already a constructive argument in the literature. See Enochs and Garcia [1, Theorem 4.6]. That approach is completely different from the one we take here.

1. Tensor-phantom maps

In [9, Theorem 3.1] we proved that the inclusion $i_* : \mathbf{K}(R\text{–Proj})^\perp \longrightarrow \mathbf{K}(R\text{–Flat})$ has a right adjoint. The proof was indirect. In the remainder of the article we propose to give a more constructive approach, which has the added benefit that it yields a set of co-generators for the category $\{\mathbf{K}(R\text{–Proj})^\perp\}^\perp$. First it helps to develop a little machinery; this section is devoted to the technical background we will need.

Definition 1.1. *A* test-complex I *is a bounded below chain complex of injective right R–modules, with $H^i(I) = 0$ for all but finitely many $i \in \mathbb{Z}$. For those $i \in \mathbb{Z}$ for which $H^i(I) \neq 0$, we insist that $H^i(I)$ must be isomorphic to a subquotient of a finitely generated, projective right R–module.*

Remark 1.2. The definition is intended to ensure that, up to homotopy equivalence, there is only a set of test-complexes. Up to isomorphism, the collection of finitely generated, projective right R–modules forms a set. Hence so do all their subquotients. Therefore the triangulated subcategory $\mathcal{R}$, that these subquotients generate in $\mathbf{D}^b(R\text{–Mod})$, is essentially small. The test-complexes are injective resolutions of some of the objects in $\mathcal{R}$; there is only a set of them, up to homotopy equivalence.

Definition 1.3. *Let Y and Z be objects in $\mathbf{K}(R\text{–Flat})$. A morphism $f : Y \longrightarrow Z$ is called* tensor-phantom *if, for every test-complex I as in Definition 1.1, the map*

$$I \otimes_R Y \xrightarrow{1 \otimes f} I \otimes_R Z$$

vanishes in cohomology. That is, the induced maps $H^i(I \otimes_R Y) \longrightarrow H^i(I \otimes_R Z)$ all vanish.

Remark 1.4. The tensor-phantom maps form an ideal, in the category $\mathbf{K}(R\text{–Flat})$. We remind the reader: this means

(i) If $g, g' : Y \longrightarrow Z$ are two tensor-phantom maps, then $g+g'$ is also a tensor-phantom map.

(ii) If $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $h : Z \longrightarrow Z'$ are maps of chain complexes, and if g is tensor-phantom, then $gf : X \longrightarrow Z$ and $hg : Y \longrightarrow Z'$ are also tensor-phantom.

Lemma 1.5. *Let $f : X \longrightarrow Z$ be a tensor-phantom map. Assume X is an object in the set T of* [7, Construction 4.3]*, while Z is any chain complex of flat modules. Then, for any integer $n \in \mathbb{Z}$, the map f can be factored in $\mathbf{K}(R\text{–Flat})$ as*

$$X \longrightarrow Y \longrightarrow Z \ ,$$

with $Y \in T$, and where $Y^i = 0$ for all $i < n$.

Proof. We recall [7, Lemma 8.4]: the chain complex Z is the filtered direct limit of the category T/Z; that is, it is the filtered direct limit of the maps $\varphi : Y \longrightarrow Z$, with $Y \in T$. This means that, for any chain complex I of right R–modules, the chain complex $I \otimes_R Z$ is the filtered direct limit of all the maps $1 \otimes \varphi : I \otimes_R Y \longrightarrow I \otimes_R Z$, and furthermore that $H^i(I \otimes_R Z)$ is the filtered direct limit of $H^i(1 \otimes \varphi) : H^i(I \otimes_R Y) \longrightarrow H^i(I \otimes_R Z)$.

We are given a map $f : X \longrightarrow Z$, with $X \in T$. We know that this map is tensor-phantom. This means that, for all test-complexes I, the induced maps

$$H^i(1 \otimes f) \ : \ H^i(I \otimes_R X) \longrightarrow H^i(I \otimes_R Z)$$

vanish for all $i \in \mathbb{Z}$. By the paragraph above we conclude that, for any element $h \in H^i(I \otimes_R X)$, there exists a factorization of $f : X \longrightarrow Z$ as $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$, with $Y \in T$, and so that h is killed already by the map $H^i(1\otimes\alpha)$ in the composite

$$H^i(I\otimes_R X) \xrightarrow{H^i(1\otimes\alpha)} H^i(I\otimes_R Y) \xrightarrow{H^i(1\otimes\beta)} H^i(I\otimes_R Z)\ .$$

The proof will be based on making an intelligent choice of the test complex I, and of the element $h\in H^i(I\otimes_R X)$.

Now X belongs to T; it is a bounded below chain complex of left R–modules. This makes $X^*=\mathcal{H}\mathrm{om}_R(X,R)$ into a bounded above chain complex of right R–modules. The t–structure truncation $\{X^*\}^{>-n}$ is a bounded complex of right R–modules. Let I be an injective resolution for it. Then I is a bounded below complex of right R–modules. Since X^* is bounded above, and since $H^i(X^*)=H^i(I)$ for all $i>-n$, we conclude that $H^i(I)$ vanishes outside a finite range. Furthermore, $H^i(I)$ is isomorphic to $H^i\big(\{X^*\}^{>-n}\big)$, and $H^i\big(\{X^*\}^{>-n}\big)$ is a subquotient of the finitely generated, projective right R–module $\{X^{-i}\}^*$. That is, I is a test-complex.

We have a natural map

$$X^* \longrightarrow \{X^*\}^{>-n} \longrightarrow I\ ,$$

which we may view as an element h in H^0 of the complex $\mathcal{H}\mathrm{om}_R(X^*,I)\cong I\otimes_R X$. We wish to apply the observations of the first two paragraphs of the proof, to this particular $h\in H^0(I\otimes_R X)$.

Remark 1.6. The reader should note that the isomorphism $I\otimes_R X\cong\mathcal{H}\mathrm{om}_R(X^*,I)$ uses all the boundedness hypotheses. For each pair of integers $i,j\in\mathbb{Z}$, we have an isomorphism

$$I^j\otimes_R X^i \quad\cong\quad \mathrm{Hom}_R\big(\mathrm{Hom}_R(X^i,R)\,,\,I^j\big) \quad=\quad \mathrm{Hom}_R(\{X^i\}^*,I^j)\ ,$$

just because X^i is assumed finitely generated and projective. The chain complex $I\otimes_R X$ is formed by taking direct sums over $i+j=n$ of the groups $I^j\otimes_R X^i$, while in the complex $\mathcal{H}\mathrm{om}_R(X^*,I)$ we take the product of the groups $\mathrm{Hom}_R(\{X^i\}^*,I^j)$, over $j-(-i)=n$. The reason the direct sum agrees with the direct product is because, for each $n\in\mathbb{Z}$, there are only finitely many non-zero pairs X^i,I^j with $i+j=n$; this is because both complexes are bounded below.

Back to the proof of Lemma 1.5. Just before Remark 1.6, we defined a test-complex I and an element $h\in H^0(I\otimes_R X)$. The first couple of paragraphs, of the proof of Lemma 1.5, tell us that the map $f:X\longrightarrow Z$ must factor as $X\xrightarrow{\alpha}Y\xrightarrow{\beta}Z$, with Y belonging to T, and so that h is killed already by the map $H^i(1\otimes\alpha)$. The naturality of the isomorphism $\mathcal{H}\mathrm{om}_R(X^*,I)\cong I\otimes_R X$ tells us that the image of h under the map

$$H^0\big(\mathcal{H}\mathrm{om}_R(X^*,I)\big) \longrightarrow H^0\big(\mathcal{H}\mathrm{om}_R(Y^*,I)\big)$$

will vanish. In other words, the composite

$$Y^* \xrightarrow{\alpha^*} X^* \longrightarrow \{X^*\}^{>-n} \longrightarrow I$$

must be null homotopic. In the derived category the map $\{X^*\}^{\geq -n} \longrightarrow I$ is an isomorphism, and the composite

$$Y^* \xrightarrow{\alpha^*} X^* \longrightarrow \{X^*\}^{>-n}$$

vanishes. This means that, in the derived category, the map $\alpha^* : Y^* \longrightarrow X^*$ must factor as

$$Y^* \longrightarrow \{X^*\}^{\leq -n} \longrightarrow X^* .$$

Since Y^* is a bounded above complex of projectives, factorizations in the derived category lift to factorizations up to homotopy; the map α^* factors, up to homotopy, through the complex $\{X^*\}^{\leq -n}$, which vanishes in dimensions $> -n$. This means that the map $\alpha : X \longrightarrow Y$ is homotopic to a map $\alpha' : X \longrightarrow Y$, which vanishes in degrees $< n$.

Now we appeal to [7, Lemma 4.1], which tells us that $\alpha' : X \longrightarrow Y$ must factor as $X \longrightarrow W \longrightarrow Y$, with $W \in T$, and $W^i = 0$ for all $i < n$. □

Lemma 1.7. *As in Lemma 1.5 let $f : X \longrightarrow Z$ be a tensor-phantom map, and assume that X is an object in the set T, while Z is free to be any chain complex of flat modules. Then the map f can be factored in $\mathbf{K}(R\text{–Mod})$ as*

$$X \longrightarrow \coprod_{n=0}^{\infty} C_n \xrightarrow{\coprod_{n=0}^{\infty} f_n} \coprod_{n=0}^{\infty} Y_n \longrightarrow Z ,$$

with C_n bounded complexes, and with $Y_n \in T$.

Proof. We have that X is a bounded below chain complex

$$\cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots$$

This makes X^* the bounded above chain complex

$$\cdots \longrightarrow \{X^2\}^* \longrightarrow \{X^1\}^* \longrightarrow \{X^0\}^* \longrightarrow \{X^{-1}\}^* \longrightarrow \{X^{-2}\}^* \longrightarrow \cdots$$

As in the proof of [7, Proposition 7.14], we consider the t–structure truncations $\{X^*\}^{>-i}$; these are obtained by killing the homotopy groups in degrees $\leq -i$. For each $i \geq 0$ there exists a complex

$$\cdots \longrightarrow W_i^{i-3} \longrightarrow W_i^{i-2} \longrightarrow W_i^{i-1} \longrightarrow \{X^i\}^* \longrightarrow \{X^{i-1}\}^* \longrightarrow \cdots$$

of projective right R–modules, which is isomorphic in the derived category $\mathbf{D}(R\text{–Mod})$ to $\{X^*\}^{>-i}$. Note that we are not assuming R coherent, and therefore the W_i^j will not, in general, be finitely generated. Choose such a complex, and call it W_i. All the complexes W_i are bounded above complexes of projectives, and hence maps in the derived category correspond bijectively to homotopy equivalence classes of chain maps. We can choose chain maps $W_{i+1} \longrightarrow W_i$, lifting the canonical map in the derived category

$\{X^*\}^{>-i-1} \longrightarrow \{X^*\}^{>-i}$. That is, there is a chain map

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & W_{i+1}^{i-3} & \longrightarrow & W_{i+1}^{i-2} & \longrightarrow & \{X^{i+1}\}^* & \longrightarrow & \{X^i\}^* & \longrightarrow & \{X^{i-1}\}^* & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & W_i^{i-3} & \longrightarrow & W_i^{i-2} & \longrightarrow & W_i^{i-1} & \longrightarrow & \{X^i\}^* & \longrightarrow & \{X^{i-1}\}^* & \longrightarrow & \cdots
\end{array}$$

lifting the map of t–structure truncations. As in the proof of [7, Proposition 7.14], we leave it to the reader to check that this chain map can be chosen so that, in degrees $j \leq i$, it is just the identity map $1 : \{X^j\}^* \longrightarrow \{X^j\}^*$. That is, the diagram above becomes

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & W_{i+1}^{i-3} & \longrightarrow & W_{i+1}^{i-2} & \longrightarrow & \{X^{i+1}\}^* & \longrightarrow & \{X^i\}^* & \longrightarrow & \{X^{i-1}\}^* & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \\
\cdots & \longrightarrow & W_i^{i-3} & \longrightarrow & W_i^{i-2} & \longrightarrow & W_i^{i-1} & \longrightarrow & \{X^i\}^* & \longrightarrow & \{X^{i-1}\}^* & \longrightarrow & \cdots
\end{array}$$

Dualizing, we have a sequence of chain maps

$$W_0^* \longrightarrow W_1^* \longrightarrow W_2^* \longrightarrow W_3^* \longrightarrow \cdots$$

The reader should note that these are just chain complexes of left R–modules. We are not claiming that the modules are projective, or even flat. But what is obvious is that the colimit of the complexes W_i^* is the complex X; in this sequence, the $\{W_i^*\}^j$ stabilize to be equal to X^j after finitely many i.

Next we will show that, for each $n \geq 0$, the composite $W_n^* \longrightarrow X \longrightarrow Z$ vanishes in $\mathbf{K}(R\text{–Mod})$. Lemma 1.5 tells us that the map $X \longrightarrow Z$ factors as $X \longrightarrow Y \longrightarrow Z$, with $Y \in T$, and with $Y^i = 0$ for all $i < n$. This means that the composite

$$Y^* \longrightarrow X^* \longrightarrow W_n$$

is a map from Y^*, which vanishes in degrees $> -n$, to W_n, which is quasi-isomorphic to $\{X^*\}^{>-n}$. The composite must vanish in the derived category. Since Y is in T, Y^* is a bounded above complex of projectives. It follows that the composite $Y^* \longrightarrow W_n$ is null homotopic. Hence so is the composite $W_n^* \longrightarrow X \longrightarrow Y$, and this certainly means that the longer composite

$$W_n^* \longrightarrow X \longrightarrow Y \longrightarrow Z$$

must vanish in $\mathbf{K}(R\text{–Mod})$.

The colimit of the sequence W_n^* is X, and in each degree i the sequence $\{W_n^*\}^i$ eventually stabilizes to X^i From [7, Remark 6.3] we conclude that the chain complex X is the homotopy colimit of the complexes W_n^*. That is, there is a distinguished triangle

$$\coprod_{n=0}^{\infty} W_n^* \xrightarrow{1-shift} \coprod_{n=0}^{\infty} W_n^* \xrightarrow{u} X \xrightarrow{v} \coprod_{n=0}^{\infty} \Sigma W_n^* .$$

We have just shown that the composite

$$\coprod_{n=0}^{\infty} W_n^* \xrightarrow{u} X \xrightarrow{f} Z$$

vanishes, and hence the map $f : X \longrightarrow Z$ must factor as

$$X \xrightarrow{\ v\ } \coprod_{n=0}^{\infty} \Sigma W_n^* \longrightarrow Z \ .$$

The next point is that we know the map v; we can compute it quite explicitly.

We remind the reader: the map v comes about as follows. We have a short exact sequence of chain complexes

$$0 \longrightarrow \coprod_{n=0}^{\infty} W_n^* \xrightarrow{1-\mathit{shift}} \coprod_{n=0}^{\infty} W_n^* \xrightarrow{\ u\ } X \longrightarrow 0 \ ,$$

and, degree by degree, the sequence is split. For each i we are free to choose a splitting, that is a map $s^i : X^i \longrightarrow \oplus_{n=0}^{\infty}\{W_n^*\}^i$ splitting $u^i : \oplus_{n=0}^{\infty}\{W_n^*\}^i \longrightarrow X^i$. We are spoilt for choice; for all $n \geq i$, we have that $\{W_n^*\}^i \longrightarrow X^i$ is the identity. The splitting s we choose is:

$$\begin{array}{lll} X^i \xrightarrow{\ 1\ } \{W_{i+2}^*\}^i \xrightarrow{\text{inclusion}} \bigoplus_{n=0}^{\infty}\{W_n^*\}^i & & \text{If } n > -2 \\ X^i \xrightarrow{\ 1\ } \{W_0^*\}^i \xrightarrow{\text{inclusion}} \bigoplus_{n=0}^{\infty}\{W_n^*\}^i & & \text{If } n \leq -2 \quad . \end{array}$$

Now $s\partial - \partial s$ gives a chain map, as in the diagram below

$$\begin{array}{ccccccccc} & & & & X & & & & \\ & & & & \big\downarrow{\scriptstyle s\partial-\partial s} & & & & \\ 0 \longrightarrow & \coprod_{n=0}^{\infty} \Sigma W_n^* & \xrightarrow[1-\mathit{shift}]{} & \coprod_{n=0}^{\infty} \Sigma W_n^* & \xrightarrow[u]{} & \Sigma X & \longrightarrow & 0 \ , \end{array}$$

and the composite $u \circ \{s\partial - \partial s\}$ vanishes. Hence there is a unique factorization of $s\partial-\partial s$ as below

$$\begin{array}{ccccccc} & & & X & & & \\ & \swarrow{\scriptstyle \exists ! v} & & \big\downarrow{\scriptstyle s\partial-\partial s} & & & \\ 0 \longrightarrow \coprod_{n=0}^{\infty} \Sigma W_n^* & \xrightarrow[1-\mathit{shift}]{} & & \coprod_{n=0}^{\infty} \Sigma W_n^* & \xrightarrow[u]{} & \Sigma X \longrightarrow 0 & . \end{array}$$

and this defines the map v. Changing the choice of splitting changes v by a homotopy.

In our case we have chosen a splitting, and it becomes an exercise to compute v. The map v is a map from the chain complex X to a direct sum of chain complexes ΣW_n^*. I leave to the reader to check the computation; the map $X \longrightarrow \Sigma W_n^*$ comes down to

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & X^{n-3} & \longrightarrow & X^{n-2} & \xrightarrow{\partial^{n-2}} & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \\
 & & {\scriptstyle 0}\downarrow & & {\scriptstyle \partial^{n-2}}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & \\
\cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & \{W_n^{-n-1}\}^* & \longrightarrow & \{W_n^{-n-2}\}^* & \longrightarrow & \cdots
\end{array}$$

This chain map clearly factors as

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & X^{n-3} & \longrightarrow & X^{n-2} & \xrightarrow{\partial^{n-2}} & X^{n-1} & \longrightarrow & X^n & \longrightarrow & \cdots \\
 & & {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \partial^{n-2}} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X^{n-2} & \longrightarrow & K^{n-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & {\scriptstyle 1}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \xrightarrow{\partial^{n-1}} & X^n & \longrightarrow & \{W_n^{-n-1}\}^* & \longrightarrow & \cdots
\end{array}$$

where K^{n-1} is the kernel of $\partial^{n-1} : X^{n-1} \longrightarrow X^n$. Let C_n be the complex

$$\cdots \longrightarrow X^{n-3} \longrightarrow X^{n-2} \longrightarrow K^{n-1} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

Then C_n is a bounded complex of R–modules, and we have shown that the map v factors as

$$X \longrightarrow \coprod_{n=0}^{\infty} C_n \longrightarrow \coprod_{n=0}^{\infty} \Sigma W_n^* \ .$$

Now we remind the reader that, at the beginning of the proof, we showed that $f : X \longrightarrow Z$ factors as

$$X \xrightarrow{v} \coprod_{n=0}^{\infty} \Sigma W_n^* \longrightarrow Z \ .$$

What we now know, about the map v, allows us to factor this further as

$$X \longrightarrow \coprod_{n=0}^{\infty} C_n \longrightarrow \coprod_{n=0}^{\infty} \Sigma W_n^* \longrightarrow Z \ .$$

This is almost what we need to prove; the Lemma asserts the existence of a factorization

$$X \longrightarrow \coprod_{n=0}^{\infty} C_n \longrightarrow \coprod_{n=0}^{\infty} Y_n \longrightarrow Z \ ,$$

with Y_n in T. Our problem is that the complexes ΣW_n^* need not lie in T. They are bounded below, but the the modules $\{W_n^j\}^*$ do not need to be finitely generated or projective. To complete the proof it therefore suffices to show that, for each n, the composite $C_n \longrightarrow \Sigma W_n^* \longrightarrow Z$ can also be expressed as $C_n \longrightarrow Y_n \longrightarrow Z$, with $Y_n \in T$.

Now observe that, for every $n \geq 0$, the composite $C_n \longrightarrow \Sigma W_n^* \longrightarrow Z$ is a chain map

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & X^{n-3} & \longrightarrow & X^{n-2} & \longrightarrow & K^{n-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & {\scriptstyle 1}\big\downarrow & & {\scriptstyle 1}\big\downarrow & & {\scriptstyle i}\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & X^{n-3} & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & \{W_n^{-n-1}\}^* & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & Z^{n-2} & \longrightarrow & Z^{n-1} & \longrightarrow & Z^n & \longrightarrow & Z^{n+1} & \longrightarrow & Z^{n+2} & \longrightarrow & \cdots
\end{array}$$

and this map is unaffected by what happens in degrees $\geq n+2$. In other words, if we leave the diagram unchanged in degrees $< n+2$, then the composite above will be equal to any possible composite below

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & X^{n-2} & \longrightarrow & K^{n-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & {\scriptstyle 1}\big\downarrow & & {\scriptstyle i}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & Y^{n+2} & \longrightarrow & Y^{n+3} & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & Z^{n-1} & \longrightarrow & Z^n & \longrightarrow & Z^{n+1} & \longrightarrow & Z^{n+2} & \longrightarrow & Z^{n+3} & \longrightarrow & \cdots
\end{array}$$

To complete the proof, it suffices therefore to show that

- We can complete the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X^{n-3} & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\cdots & \longrightarrow & Z^{n-2} & \longrightarrow & Z^{n-1} & \longrightarrow & Z^n & \longrightarrow & Z^{n+1}
\end{array}$$

  to a chain map

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & Y^{n+2} & \longrightarrow & Y^{n+3} & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & Z^{n-1} & \longrightarrow & Z^n & \longrightarrow & Z^{n+1} & \longrightarrow & Z^{n+2} & \longrightarrow & Z^{n+3} & \longrightarrow & \cdots
\end{array}$$

  with the Y^j finitely generated and projective.

The remainder of the proof will establish $\bullet$.

As in the proof of [7, Lemma 4.1], we proceed inductively. Suppose we have the chain map defined as far as

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & Y^{j-3} & \longrightarrow & Y^{j-2} & \longrightarrow & Y^{j-1} & \longrightarrow & Y^j \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\cdots & \longrightarrow & Z^{j-3} & \longrightarrow & Z^{j-2} & \longrightarrow & Z^{j-1} & \longrightarrow & Z^j
\end{array} .$$

From the diagram

$$\begin{array}{ccccc} Y^{j-1} & \xrightarrow{\partial_Y^{j-1}} & Y^j & & \\ \downarrow & & \downarrow{\scriptstyle g^j} & & \\ Z^{j-1} & \longrightarrow & Z^j & \xrightarrow[\partial_Z^j]{} & Z^{j+1} \end{array}$$

we note that $\partial_Z^j g^j : Y^j \longrightarrow Z^{j+1}$ is a map from the finitely generated, projective module Y^j to the flat module Z^{j+1}, and that the composite

$$Y^{j-1} \xrightarrow{\partial_Y^{j-1}} Y^j \xrightarrow{\partial_Z^j g^j} Z^{j+1}$$

vanishes. From [7, Corollary 3.3] we deduce that the map $\partial_Z^j g^j : Y^j \longrightarrow Z^{j+1}$ factors as

$$Y^j \xrightarrow{\partial_Y^j} Y^{j+1} \xrightarrow{g^{j+1}} Z^{j+1} \ ,$$

with Y^{j+1} finitely generated and projective, and so that the composite

$$Y^{j-1} \xrightarrow{\partial_Y^{j-1}} Y^j \xrightarrow{\partial_Y^j} Y^{j+1}$$

vanishes. This precisely means that we have extended the chain map to

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & Y^{j-3} & \longrightarrow & Y^{j-2} & \longrightarrow & Y^{j-1} & \xrightarrow{\partial_Y^{j-1}} & Y^j & \xrightarrow{\partial_Y^j} & Y^{j+1} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g^j} & & \downarrow{\scriptstyle g^{j+1}} \\ \cdots & \longrightarrow & Z^{j-3} & \longrightarrow & Z^{j-2} & \longrightarrow & Z^{j-1} & \longrightarrow & Z^j & \xrightarrow[\partial_Z^j]{} & Z^{j+1} \ . \end{array}$$

This completes the inductive step. □

Lemma 1.8. *Suppose we are given two composable tensor-phantom morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \ .$$

Suppose furthermore that X belongs to the set T, of [7, Construction 4.3]*. Then the composite $gf : X \longrightarrow Z$ vanishes in $\mathbf{K}(R\text{–Flat})$.*

Proof. By Lemma 1.7 we know that f must factor as

$$X \longrightarrow \coprod_{n=0}^{\infty} C_n \xrightarrow{\coprod_{n=0}^{\infty} f_n} \coprod_{n=0}^{\infty} Y_n \longrightarrow Y \ ,$$

where the C_n are bounded complexes and the Y_n belong to T. Because C_n is bounded, we may choose integers $N(n)$, so that $C_n^i = 0$ if $i \geq N(n)$. Now the composite $Y_n \longrightarrow Y \xrightarrow{g} Z$ must be a tensor-phantom map, because g is. Being a tensor-phantom map from an object $Y_n \in T$ to the object $Z \in \mathbf{K}(R\text{–Flat})$, it must, by Lemma 1.5, factor as

$$Y_n \longrightarrow Z_n \longrightarrow Z \ ,$$

with $Z_n^i = 0$ if $i \leq N(n)$. The composite

$$C_n \longrightarrow Y_n \longrightarrow Z_n$$

is a map from the complex C_n to the complex Z_n, and $C_n^i = 0$ if $i \geq N(n)$, while $Z_n^i = 0$ if $i \leq N(n)$. The composite must therefore vanish. This means that the longer composite

$$X \longrightarrow \coprod_{n=0}^{\infty} C_n \longrightarrow \coprod_{n=0}^{\infty} Y_n \longrightarrow \coprod_{n=0}^{\infty} Z_n \longrightarrow Z$$

must also vanish, but this is $gf : X \longrightarrow Z$. □

2. The relation with $\mathcal{S}$–local objects

In §1 we studied the formal properties of tensor-phantom maps. In this section we plan to disclose why we care about them; we will give the relation between tensor-phantom maps and the right adjoint to the functor $i_* : \mathcal{S} \longrightarrow \mathbf{K}(R\text{–Flat})$. We begin by reminding the reader of bland generalities.

Reminder 2.1. Suppose we have an inclusion of triangulated categories $i_* : \mathcal{S} \longrightarrow \mathcal{T}$; in our case $\mathcal{T}$ happens to be $\mathbf{K}(R\text{–Flat})$, while $\mathcal{S} \subset \mathcal{T}$ is $\mathbf{K}(R\text{–Proj})^\perp \subset \mathbf{K}(R\text{–Flat})$. We want to produce a right adjoint to this inclusion. Now [6, Theorem 9.1.13] asserts that such an adjoint will exist if and only if, for all objects $t \in \mathcal{T}$, there exists a distinguished triangle

$$s \longrightarrow t \longrightarrow y \longrightarrow \Sigma s$$

with $s \in \mathcal{S}$ and $y \in \mathcal{S}^\perp$. This means that we have to understand how to produce many objects y, orthogonal to $\mathcal{S}$. We recall the terminology of [6, Definition 9.1.3]; such objects are usually called $\mathcal{S}$–local objects. The next lemma will produce some $\mathcal{S}$–local objects.

Lemma 2.2. *Let I be any complex of right R–modules. Then $\mathcal{H}om_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is a chain complex of left R–modules, and is $\mathcal{S}$–local.*

Proof. Let X be any complex in $\mathcal{S}$. By [7, Proposition 9.1] the chain complex $I \otimes_R X$ is acyclic. Since $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group, the complex

$$\mathcal{H}om_{\mathbb{Z}}\big(I \otimes_R X \ , \ \mathbb{Q}/\mathbb{Z}\big)$$

is also acyclic. But

$$\mathcal{H}om_R\big(X \ , \ \mathcal{H}om_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big) \quad = \quad \mathcal{H}om_{\mathbb{Z}}\big(I \otimes_R X \ , \ \mathbb{Q}/\mathbb{Z}\big) \ .$$

Therefore $\mathrm{Hom}\big(X \ , \ \mathcal{H}om_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big) = H^0\Big(\mathcal{H}om_R\big(X \ , \ \mathcal{H}om_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big)\Big)$ must vanish, and we conclude that $\mathcal{H}om_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is $\mathcal{S}$–local. □

Remark 2.3. This means that, in the category $\mathbf{K}(R\text{–Mod})$, we have no difficulty at all in producing $\mathcal{S}$–local objects. Our problem is that we are after $\mathcal{S}$–local objects in $\mathbf{K}(R\text{–Flat})$. If I is an arbitrary chain complex of right R–modules, then the modules in the chain complex $\mathcal{H}om_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ are under no obligation to be flat.

Remark 2.4. What we are about to say is an aside, which the reader can safely skip.

In Lemma 2.2 we produced a great many objects of $\mathbf{K}(R\text{–Mod})$, all of which are $\mathcal{S}$–local. It is conceivable that, using them, there could be a direct proof that the inclusion $\mathcal{S} \longrightarrow \mathbf{K}(R\text{–Mod})$ has a right adjoint. Of course, [9, Theorem 3.1] tells us that the adjoint exists; but the proof given in [9, Theorem 3.1] was indirect.

Let me note that, from the existence of an adjoint to the inclusion $\mathcal{S} \longrightarrow \mathbf{K}(R\text{–Mod})$ we formally deduced, in [9, Theorem 3.2], that the inclusion $\mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Mod})$ also has a right adjoint. And in [9, Remark 3.3] we saw how, in a few lines, one can then conclude that every R–module has a flat precover. It would be interesting to give a simple, direct proof of the existence of a right adjoint to the inclusion $\mathcal{S} \longrightarrow \mathbf{K}(R\text{–Mod})$. I do not know how to do this.

Remark 2.5. Remark 2.3 is something of a digression. In this article we are interested in constructing a right adjoint to the inclusion $i_* : \mathcal{S} \longrightarrow \mathbf{K}(R\text{–Flat})$, not the inclusion $\mathcal{S} \longrightarrow \mathbf{K}(R\text{–Mod})$. We need $\mathcal{S}$–local objects in $\mathbf{K}(R\text{–Flat})$, not $\mathbf{K}(R\text{–Mod})$.

Of course Remark 2.3 observed that, in the category $\mathbf{K}(R\text{–Mod})$, it is easy to construct $\mathcal{S}$–local objects. From [9, Theorem 3.2] we know that the inclusion $\mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Mod})$ has a right adjoint $J : \mathbf{K}(R\text{–Mod}) \longrightarrow \mathbf{K}(R\text{–Flat})$. Next we note the trivial fact:

Lemma 2.6. *Let X be an $\mathcal{S}$–local object in $\mathbf{K}(R\text{–Mod})$. Let $J : \mathbf{K}(R\text{–Mod}) \longrightarrow \mathbf{K}(R\text{–Flat})$ be a right adjoint to the inclusion $\mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Mod})$. Then JX is an $\mathcal{S}$–local object in $\mathbf{K}(R\text{–Flat})$.*

Remark 2.7. Lemma 2.2 tells us that $\mathcal{H}\mathrm{om}(I, \mathbb{Q}/\mathbb{Z})$ is an $\mathcal{S}$–local object of $\mathbf{K}(R\text{–Mod})$, while Lemma 2.6 establishes that $J\big(\mathcal{H}\mathrm{om}(I, \mathbb{Q}/\mathbb{Z})\big)$ is an $\mathcal{S}$–local object of $\mathbf{K}(R\text{–Flat})$. The idea will be to use the objects $J\big(\mathcal{H}\mathrm{om}(I, \mathbb{Q}/\mathbb{Z})\big)$ to construct the right adjoint to $i_* : \mathcal{S} \longrightarrow \mathbf{K}(R\text{–Flat})$.

Now it is high time to disclose what all of this has to do with tensor-phantom maps.

Lemma 2.8. *Let $J : \mathbf{K}(R\text{–Mod}) \longrightarrow \mathbf{K}(R\text{–Flat})$ be a right adjoint to the inclusion $\mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Mod})$. Let Z be an object of $\mathbf{K}(R\text{–Flat})$. A map $f : Y \longrightarrow Z$ in $\mathbf{K}(R\text{–Flat})$ is tensor-phantom if and only if, for every test-complex I and for every map $g : Z \longrightarrow J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big)$, the composite*

$$Y \xrightarrow{\ f\ } Z \xrightarrow{\ g\ } J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big)$$

is null homotopic.

Proof. By definition the map $f : Y \longrightarrow Z$ is tensor-phantom if and only if, for every test-complex I, the map

$$I \otimes_R Y \xrightarrow{\ 1\otimes f\ } I \otimes_R Z$$

vanishes in cohomology. If I is a test-complex, then so is any suspension $\Sigma^n I$. Hence $f : Y \longrightarrow Z$ is tensor-phantom if and only if, for every tensor-complex I, the map

$$H^0(I \otimes_R Y) \xrightarrow{H^0(1\otimes f)} H^0(I \otimes_R Z)$$

vanishes. Since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in the category of abelian groups, it is equivalent for the map

$$\mathrm{Hom}_{\mathbb{Z}}\big(H^0(I \otimes_R Z)\ ,\ \mathbb{Q}/\mathbb{Z}\big) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(H^0(I \otimes_R Y)\ ,\ \mathbb{Q}/\mathbb{Z}\big)$$

to vanish, for every test-complex I. This map identifies as

$$\mathrm{Hom}_{\mathbf{K}(\mathbb{Z})}(I \otimes_R Z\ ,\ \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbf{K}(\mathbb{Z})}(I \otimes_R Y\ ,\ \mathbb{Q}/\mathbb{Z})\ .$$

But this, in turn, identifies with the map

$$\mathrm{Hom}_{\mathbf{K}(R\text{–Mod})}\big(Z\ ,\ \mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big) \longrightarrow \mathrm{Hom}_{\mathbf{K}(R\text{–Mod})}\big(Y\ ,\ \mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big)\ ;$$

to say that all these maps vanish is to say that any chain map $Z \longrightarrow \mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ composes with $f : Y \longrightarrow Z$ to give a null homotopic map.

Finally note that Y and Z are objects in $\mathbf{K}(R\text{–Flat})$. The inclusion of $\mathbf{K}(R\text{–Flat})$ into $\mathbf{K}(R\text{–Mod})$ has a right adjoint J. The composites

$$Y \xrightarrow{\ f\ } Z \longrightarrow \mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$$

will all vanish if and only if the composites

$$Y \xrightarrow{\ f\ } Z \longrightarrow J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big)$$

are null. □

3. A reminder of right coherent rings

In Remark 2.7 we observed that the complexes $J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big)$ are $\mathcal{S}$–local objects in $\mathbf{K}(R\text{–Flat})$. In this section we will note that, as long as I is a complex of injective right R–modules over a right coherent ring R, then $\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is a complex of flat R–modules. There is no need to apply the functor J.

This section reminds the reader of the statements and proofs of a couple of basic facts about right coherent rings. I do this partly to keep the paper self-contained, but mostly to highlight where the right coherence of R is being used.

If I is a right R–module, then $\mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is a left R–module. If F is another right R–module, composition gives a map

$$F \otimes \mathrm{Hom}_R(F, I) \otimes \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}\ ,$$

which yields a natural map

$$F \otimes_R \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(F, I)\ ,\ \mathbb{Q}/\mathbb{Z}\big)\ .$$

This map is obviously an isomorphism when $F = R$ is free of rank 1. Because it commutes with finite direct sums in F, the map is an isomorphism whenever F is a free module of finite rank. Now we note

Lemma 3.1. *Let R be a right coherent ring, and let I be an injective right R–module. Then the left R–module* $\mathrm{Hom}(I,\mathbb{Q}/\mathbb{Z})$ *is flat.*

Proof. It suffices to show that, for each finitely presented right R–module M, the group $\mathrm{Tor}_1^R\big(M\ ,\ \mathrm{Hom}(I,\mathbb{Q}/\mathbb{Z})\big)$ vanishes. The ring R is assumed right coherent, and M is a finitely presented right R–module. We may choose a resolution for M

$$P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with the P_2, P_1 and P_0 finitely generated, free right R–modules. We need to show the exactness of the sequence

$$P_2 \otimes_R \mathrm{Hom}(I,\mathbb{Q}/\mathbb{Z}) \longrightarrow P_1 \otimes_R \mathrm{Hom}(I,\mathbb{Q}/\mathbb{Z}) \longrightarrow P_0 \otimes_R \mathrm{Hom}(I,\mathbb{Q}/\mathbb{Z})\ .$$

To this end observe the diagram

$$\begin{array}{ccc} P_2 \otimes_R \mathrm{Hom}(I,\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(P_2,I)\ ,\ \mathbb{Q}/\mathbb{Z}\big) \\ \downarrow & & \downarrow \\ P_1 \otimes_R \mathrm{Hom}(I,\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(P_1,I)\ ,\ \mathbb{Q}/\mathbb{Z}\big) \\ \downarrow & & \downarrow \\ P_0 \otimes_R \mathrm{Hom}(I,\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_R(P_0,I)\ ,\ \mathbb{Q}/\mathbb{Z}\big) \end{array}\ .$$

The horizontal maps are clearly isomorphisms, and we want to show the left column exact. It suffices, therefore, to show the right column exact. But now the fact that I is an injective right R–modules says that

$$\mathrm{Hom}_R(P_0,I) \longrightarrow \mathrm{Hom}_R(P_1,I) \longrightarrow \mathrm{Hom}_R(P_2,I)$$

is an exact sequence of abelian groups, while the fact that $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group gives the desired exactness of the right column. □

An immediate consequence is

Lemma 3.2. *Let R be a right coherent ring. If I is a complex of injective right R–modules, then $X = \mathcal{H}\mathrm{om}(I,\mathbb{Q}/\mathbb{Z})$ is a complex of flat left R–modules. Furthermore, X is orthogonal of $\mathcal{S}$.*

Proof. The fact that X is a chain complex of flat R–modules, that is $X \in \mathbf{K}(R\text{–Flat})$, may be found in Lemma 3.1. The orthogonality to $\mathcal{S}$, that is the fact that $X \in \mathcal{S}^{\perp}$, may be found in Lemma 2.2. □

Before we leave this section, we want to note a fact about products of flat modules.

Lemma 3.3. *Let R be a right coherent ring. Suppose that $\{F_\lambda, \lambda \in \Lambda\}$ is a set of flat left R–modules. Then the product*

$$\prod_{\lambda\in\Lambda} F_\lambda$$

is a flat left R–module.

Proof. As in the proof of Lemma 3.1 it suffices to prove that, given an exact sequence

$$P_2 \longrightarrow P_1 \longrightarrow P_0$$

of free right R–modules of finite rank, the sequence

$$P_2 \otimes_R \left\{ \prod_{\lambda\in\Lambda} F_\lambda \right\} \longrightarrow P_1 \otimes_R \left\{ \prod_{\lambda\in\Lambda} F_\lambda \right\} \longrightarrow P_0 \otimes_R \left\{ \prod_{\lambda\in\Lambda} F_\lambda \right\}$$

is also exact. Now for a free module of finite rank, the natural map

$$P \otimes_R \left\{ \prod_{\lambda\in\Lambda} F_\lambda \right\} \longrightarrow \prod_{\lambda\in\Lambda} \{P \otimes_R F_\lambda\}$$

is an isomorphism, so the sequence above becomes

$$\prod_{\lambda\in\Lambda} \{P_2 \otimes_R F_\lambda\} \longrightarrow \prod_{\lambda\in\Lambda} \{P_1 \otimes_R F_\lambda\} \longrightarrow \prod_{\lambda\in\Lambda} \{P_0 \otimes_R F_\lambda\} ,$$

which is obviously exact. □

4. Constructing $\mathcal{S}$–local objects

In this section and the next we will show how to explicitly construct $\mathcal{S}$–local objects. In the case where the ring R is right coherent the construction is elementary. In the general case it uses the functor $J : \mathbf{K}(R\text{–Mod}) \longrightarrow \mathbf{K}(R\text{–Flat})$, which is right adjoint to the inclusion $\mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Mod})$; see [9, Theorem 3.2].

Reminder 4.1. If I is any chain complex of right R–modules, then Lemma 2.2 says that the complex $\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is an $\mathcal{S}$–local object in $\mathbf{K}(R\text{–Mod})$. From Lemma 2.6 it follows that $J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big)$ is an $\mathcal{S}$–local object in $\mathbf{K}(R\text{–Flat})$. In the special case, where R happens to be right coherent and I is a complex of injective right R–modules, Lemma 3.2 tells us that $\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is a chain complex of flat modules as it stands. For right coherent rings R, There is no need to apply the functor J.

Remark 4.2. From [9, Theorem 3.2] we know that the inclusion $\mathbf{K}(R\text{–Flat}) \longrightarrow \mathbf{K}(R\text{–Mod})$ has a right adjoint $J : \mathbf{K}(R\text{–Mod}) \longrightarrow \mathbf{K}(R\text{–Flat})$. This means that the category $\mathbf{K}(R\text{–Flat})$ has products; given a collection $\{Z_\lambda, \lambda \in \Lambda\}$ of objects in $\mathbf{K}(R\text{–Flat})$, their product in $\mathbf{K}(R\text{–Flat})$ is obtained by forming the product in $\mathbf{K}(R\text{–Mod})$, and then applying the functor J. In symbols, the product in the category $\mathbf{K}(R\text{–Flat})$ is

$$J\left(\prod_{\lambda\in\Lambda} Z_\lambda \right) .$$

Note that, if R happens to be right coherent, then products of flat modules are flat by Lemma 3.3. For right coherent rings there is no need to apply J to the product.

Remark 4.3. Remark 4.2 tells us that the category $\mathbf{K}(R\text{–Flat})$ has products. What we know about the subcategory $\mathcal{S}^\perp \subset \mathbf{K}(R\text{–Flat})$ is

(i) The product of any family of objects in $\mathcal{S}^{\perp}$ belongs to $\mathcal{S}^{\perp}$.
(ii) The subcategory $\mathcal{S}^{\perp} \subset \mathbf{K}(R\text{–Flat})$ is triangulated.

Facts (i) and (ii) are standard; you can find them, for instance, in [6, Lemma 9.1.12].

Lemmas 2.2 and 2.6 tell us how to construct some $\mathcal{S}$–local objects in $\mathbf{K}(R\text{–Flat})$. Reminder 4.1 tells us that, as long as the ring R is right coherent, we can construct these objects without having to appeal to the functor $J : \mathbf{K}(R\text{–Mod}) \longrightarrow \mathbf{K}(R\text{–Flat})$. The point of (i) and (ii) above is that we can produce many more $\mathcal{S}$–local objects, using products and triangles. Now we will do this.

Construction 4.4. Let us choose a set Λ of representatives for the homotopy equivalence classes of test complexes. That is, every test-complex I is homotopy equivalent to one of $\{I_\lambda, \lambda \in \Lambda\}$. Remark 1.2 guarantees that this can be done; there is only a set of homotopy equivalence classes of test-complexes. For each $\lambda \in \Lambda$, let

$$F_\lambda \quad = \quad J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I_\lambda, \mathbb{Q}/\mathbb{Z})\big) \ .$$

Let $\mathcal{F}$ be the colocalizing subcategory generated by the objects $\{F_\lambda, \lambda \in \Lambda\}$. This means that $\mathcal{F}$ is the smallest triangulated subcategory of $\mathbf{K}(R\text{–Flat})$, containing all the F_λ and closed under products. Since $\mathcal{S}^{\perp}$ is colocalizing and contains all the objects $F_\lambda = J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I_\lambda, \mathbb{Q}/\mathbb{Z})\big)$, it follows that $\mathcal{F} \subset \mathcal{S}^{\perp}$. The main fact we will prove, in what follows, is that $\mathcal{F} = \mathcal{S}^{\perp}$.

Lemma 4.5. *Let Z be an object of $\mathbf{K}(R\text{–Flat})$. There exists a distinguished triangle in $\mathbf{K}(R\text{–Flat})$*

$$Y \xrightarrow{\ f\ } Z \longrightarrow F \longrightarrow \Sigma Y \ ,$$

so that $f : Y \longrightarrow Z$ is a tensor-phantom map, and $F \in \mathcal{F}$.

Proof. Let F be the product of F_λ, over all morphisms $Z \longrightarrow F_\lambda$, for any $\lambda \in \Lambda$. Let $Z \longrightarrow F$ be the natural map. Complete it to a triangle

$$Y \xrightarrow{\ f\ } Z \longrightarrow F \longrightarrow \Sigma Y \ .$$

Clearly $F \in \mathcal{F}$, since it is a product of F_λ's. By Construction 4.4 every test complex I is homotopy equivalent to an I_λ, and hence every $J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big)$ is homotopy equivalent to an $F_\lambda \cong J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I_\lambda, \mathbb{Q}/\mathbb{Z})\big)$. Therefore every map $Z \longrightarrow J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big) \cong F_\lambda$ must factor through $Z \longrightarrow F$. The triangle $Y \longrightarrow Z \longrightarrow F \longrightarrow \Sigma Y$ tells us that every composite

$$Y \xrightarrow{\ f\ } Z \longrightarrow J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\big)$$

must vanish. Lemma 2.8 permits us to conclude that f is tensor-phantom. □

Lemma 4.6. *Let T be the set of objects in $\mathbf{K}(R\text{–Proj})$, given in* [7, Construction 4.3]. *Every object $Z \in \mathbf{K}(R\text{–Flat})$ admits a map $Z \longrightarrow F$, where $F \in \mathcal{F}$, and so that, for all objects $t \in T$, the map*

$$\mathrm{Hom}_{\mathbf{K}(R\text{–Flat})}(t, Z) \longrightarrow \mathrm{Hom}_{\mathbf{K}(R\text{–Flat})}(t, F)$$

is injective.

Proof. By Lemma 4.5 there exists a triangle in $\mathbf{K}(R\text{–Flat})$

$$Y \xrightarrow{\ f\ } Z \longrightarrow F_1 \longrightarrow \Sigma Y ,$$

with $F_1 \in \mathcal{F}$, and with f a tensor-phantom map. Applying Lemma 4.5 again, this time to the object Y, there exists a triangle

$$X \xrightarrow{\ g\ } Y \longrightarrow F_2 \longrightarrow \Sigma X ,$$

with $F_2 \in \mathcal{F}$, and with g a tensor-phantom map. Now we build an octahedron from the two composable maps $X \xrightarrow{g} Y \xrightarrow{f} Z$. We obtain two distinguished triangles

$$X \xrightarrow{\ fg\ } Z \longrightarrow F \longrightarrow \Sigma X \quad ,$$

$$F_2 \longrightarrow F \longrightarrow F_1 \longrightarrow \Sigma F_2 \quad .$$

The second of these tells us that F must be an object of $\mathcal{F}$. In the first of these two triangles, the map $fg : X \longrightarrow Z$ is a composite of two tensor-phantom maps. Lemma 1.8 tells us that, if t is any object in T, then any composite

$$t \longrightarrow X \xrightarrow{\ g\ } Y \xrightarrow{\ f\ } Z$$

must vanish. In the exact sequence

$$\mathrm{Hom}(t, X) \xrightarrow{\mathrm{Hom}(t,fg)} \mathrm{Hom}(t, Z) \longrightarrow \mathrm{Hom}(t, F)$$

we know that the map $\mathrm{Hom}(t, fg)$ vanishes. Hence the map $\mathrm{Hom}(t, Z) \longrightarrow \mathrm{Hom}(t, F)$ must be injective. □

Lemma 4.7. *Let $X \longrightarrow F$ be a morphism in $\mathbf{K}(R\text{–Flat})$, and suppose $F \in \mathcal{F}$. Then there is a factorization*

$$X \longrightarrow F' \longrightarrow F$$

with $F' \in \mathcal{F}$ and so that, for all $t \in T$, the two maps

$$\mathrm{Hom}(t, X) \longrightarrow \mathrm{Hom}(t, F) \qquad \text{and} \qquad \mathrm{Hom}(t, F') \longrightarrow \mathrm{Hom}(t, F)$$

have the same image $I \subset \mathrm{Hom}(t, F)$.

Proof. Complete $X \longrightarrow F$ to a triangle

$$X \longrightarrow F \longrightarrow Y \longrightarrow \Sigma X .$$

By Lemma 4.6 there exists a morphism $Y \longrightarrow F''$, with $F'' \in \mathcal{F}$ and so that $\mathrm{Hom}(t, Y) \longrightarrow \mathrm{Hom}(t, F'')$ is injective. The triangle gives us an exact sequence

$$\mathrm{Hom}(t, X) \longrightarrow \mathrm{Hom}(t, F) \longrightarrow \mathrm{Hom}(t, Y) ,$$

and the injectivity of $\mathrm{Hom}(t, Y) \longrightarrow \mathrm{Hom}(t, F'')$ means that the sequence

$$\mathrm{Hom}(t, X) \longrightarrow \mathrm{Hom}(t, F) \longrightarrow \mathrm{Hom}(t, F'')$$

is also exact. Now form the distiniguished triangle

$$F' \longrightarrow F \longrightarrow F'' \longrightarrow \Sigma F' .$$

Since F, F'' lie in $\mathcal{F}$, so does F'. Because the composite $X \longrightarrow F \longrightarrow Y \longrightarrow F''$ clearly vanishes, it follows that the map $X \longrightarrow F$ must factor as $X \longrightarrow F' \longrightarrow F$. We have a commutative diagram, where the rows are exact

$$\begin{array}{ccccc} \mathrm{Hom}(t,X) & \longrightarrow & \mathrm{Hom}(t,F) & \longrightarrow & \mathrm{Hom}(t,F'') \\ \downarrow & & \| & & \| \\ \mathrm{Hom}(t,F') & \longrightarrow & \mathrm{Hom}(t,F) & \longrightarrow & \mathrm{Hom}(t,F'') \end{array}$$

from which we conclude that the images of the maps

$$\mathrm{Hom}(t,X) \longrightarrow \mathrm{Hom}(t,F) \qquad \text{and} \qquad \mathrm{Hom}(t,F') \longrightarrow \mathrm{Hom}(t,F)$$

must agree; both are equal to the kernel of $\mathrm{Hom}(t,F) \longrightarrow \mathrm{Hom}(t,F'')$. □

Now we are ready to give a second proof, valid for an arbitrary ring R, of the existence of a right adjoint to the inclusion $i_* : \mathcal{S} = \mathbf{K}(R\text{–Proj})^\perp \longrightarrow \mathbf{K}(R\text{–Flat})$. Unlike the proof given in [9, Theorem 3.1], the one here will give us explicitly that $\mathcal{S}^\perp$ is equal to $\mathcal{F}$, the colocalizing subcategory generated by the objects F_λ.

Theorem 4.8. *Let R be any ring. There is a right adjoint to the inclusion $\mathcal{S} = \mathbf{K}(R\text{–Proj})^\perp \longrightarrow \mathbf{K}(R\text{–Flat})$. Furthermore, $\mathcal{F} = \mathcal{S}^\perp = \{\mathbf{K}(R\text{–Proj})^\perp\}^\perp$.*

Proof. Let Y be any object of $\mathbf{K}(R\text{–Flat})$. Lemma 4.6 tells us that there exists a morphism $Y \longrightarrow F_0$, with $F_0 \in \mathcal{F}$, and so that, for all $t \in T$, the map $\mathrm{Hom}(t,Y) \longrightarrow \mathrm{Hom}(t,F_0)$ is injective. Now we inductively construct a sequence of morphisms in $\mathcal{F}$

$$\cdots \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \ ,$$

and a map from Y to the sequence. The construction is as in Lemma 4.7; we factor the map $Y \longrightarrow F_n$ as $Y \longrightarrow F_{n+1} \longrightarrow F_n$, so that the images of

$$\mathrm{Hom}(t,Y) \longrightarrow \mathrm{Hom}(t,F_n) \qquad \text{and} \qquad \mathrm{Hom}(t,F_{n+1}) \longrightarrow \mathrm{Hom}(t,F_n)$$

agree. For every object $t \in T$, the sequence

$$\cdots \longrightarrow \mathrm{Hom}(t,F_3) \longrightarrow \mathrm{Hom}(t,F_2) \longrightarrow \mathrm{Hom}(t,F_1)$$

identifies as the direct sum of two sequences

$$\begin{array}{ccccccc} \cdots & \xrightarrow{1} & \mathrm{Hom}(t,Y) & \xrightarrow{1} & \mathrm{Hom}(t,Y) & \xrightarrow{1} & \mathrm{Hom}(t,Y) \\ \cdots & \xrightarrow{0} & K_3 & \xrightarrow{0} & K_2 & \xrightarrow{0} & K_1 \end{array} .$$

Now let F be the homotopy limit of the sequence F_n; that is, it fits in a triangle

$$F \longrightarrow \prod_{n=1}^{\infty} F_n \xrightarrow{1-\mathit{shift}} \prod_{n=1}^{\infty} F_n \longrightarrow \Sigma F \ .$$

Since each of the F_n lies in $\mathcal{F}$, so does the homotopy limit F. The map $Y \longrightarrow \prod_{n=1}^{\infty} F_n$ factors through F, and if we apply the functor $\mathrm{Hom}(t,-)$ to the triangle, with $t \in T$, we easily see that $\mathrm{Hom}(t,Y) \longrightarrow \mathrm{Hom}(t,F)$ is an isomorphism. Now complete $Y \longrightarrow F$ to a triangle

$$S \longrightarrow Y \longrightarrow F \longrightarrow \Sigma S\ .$$

We have that $F \in \mathcal{F} \subset \mathcal{S}^\perp$. The fact that $\mathrm{Hom}(t,Y) \longrightarrow \mathrm{Hom}(t,F)$ is an isomorphism tells us that $\mathrm{Hom}(t,S)$ vanishes, for every $t \in T$. This makes S orthogonal to the category generated by T; [7, Proposition 7.4] tells us that T generates $\mathbf{K}(R\text{–Proj})$. Thus $S \in \mathbf{K}(R\text{–Proj})^\perp = \mathcal{S}$.

Given an object $Y \in \mathbf{K}(R\text{–Flat})$, we have exhibited a distinguished triangle

$$S \longrightarrow Y \longrightarrow F \longrightarrow \Sigma S\ ,$$

with $S \in \mathcal{S}$ and $F = \mathcal{S}^\perp$. From [6, Theorem 9.1.13] we deduce that the inclusion $i_* : \mathcal{S} \longrightarrow \mathbf{K}(R\text{–Flat})$ has a right adjoint. If $Y \in \mathcal{S}^\perp$, the distinguished triangle

$$S \longrightarrow Y \longrightarrow F \longrightarrow \Sigma S$$

is such that both Y and F lie in $\mathcal{S}^\perp$, and hence so does S. Therefore S belongs both to $\mathcal{S}$ and to $\mathcal{S}^\perp$, and must vanish. It follows that $Y \longrightarrow F$ is an isomorphism. Since $F \in \mathcal{F}$, we conclude that $\mathcal{F} = \mathcal{S}^\perp$. □

5. The special case of compactly generated $\mathbf{K}(R\text{–Flat})$

In Theorem 4.8 we proved, for any ring R, that the subcategory $\mathcal{S}^\perp \subset \mathbf{K}(R\text{–Flat})$ is equal to the subcategory $\mathcal{F} \subset \mathbf{K}(R\text{–Flat})$, the colocalizing subcategory cogenerated by the objects $J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I,\mathbb{Q}/\mathbb{Z})\big)$. In the process we also gave a second proof that the inclusion $i_* : \mathcal{S} \longrightarrow \mathbf{K}(R\text{–Flat})$ has a right adjoint. The second proof is independent of the first only when we do not have to use the functor J; this happens, for example, if R is right coherent.

The statement that the objects $J\big(\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I,\mathbb{Q}/\mathbb{Z})\big)$ cogenerate $\mathcal{S}^\perp$ is true for any ring R. In the general case I do not have an easier argument than the one given in the previous sections. If we are willing to restrict ourselves to the case where the ring R is right coherent, then we have already observed that the complex $\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I,\mathbb{Q}/\mathbb{Z})$ is an object of $\mathbf{K}(R\text{–Flat})$ as it stands; there is no need to apply the functor J to it. What we will note in this section is that the entire proof simplifies, because the category $\mathbf{K}(R\text{–Proj})$ is compactly generated; see [7, Proposition 7.14].

For the rest of this article R is assumed to be a right coherent ring.

Reminder 5.1. We begin by reminding the reader of formal generalities. Let $\mathcal{T}$ be a triangulated category; in the application we have in mind $\mathcal{T} = \mathbf{K}(R\text{–Flat})$. Let $\mathcal{S} \subset \mathcal{T}$ be a thick subcategory; the $\mathcal{S}$ we plan to apply this to is the category $\mathcal{S} = \mathbf{K}(R\text{–Proj})^\perp$ of [9, Notation 2.2].

Let $\mathcal{S}^\perp$ be the full subcategory of all $\mathcal{S}$–local objects in $\mathcal{T}$. Let $\pi : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ be the natural projection to the Verdier quotient. From [6, Lemma 9.1.5] we know that, if x is any object in $\mathcal{T}$ and $y \in \mathcal{S}^{\perp}$, then

$$\mathrm{Hom}_{\mathcal{T}}(x,y) \quad = \quad \mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(\pi x, \pi y) \ .$$

If we let x and y both be in the subcategory $\mathcal{S}^{\perp}$, we learn that the composite functor

$$\mathcal{S}^{\perp} \xrightarrow{\text{inclusion}} \mathcal{T} \xrightarrow{\ \pi\ } \mathcal{T}/\mathcal{S}$$

must be fully faithful. If the inclusion $i_* : \mathcal{S} \longrightarrow \mathcal{T}$ has a right adjoint, then [6, Theorem 9.1.16] asserts that the map $\mathcal{S}^{\perp} \longrightarrow \mathcal{T}/\mathcal{S}$ is an equivalence of categories. Let us note the easy observation, that the converse also holds.

Lemma 5.2. *Let $i_* : \mathcal{S} \longrightarrow \mathcal{T}$ be a fully faithful, triangulated functor of triangulated categories. Suppose that the image of i_* is equivalent to a thick subcategory of $\mathcal{T}$. Let $\mathcal{S}^{\perp} \subset \mathcal{T}$ be the full subcategory of all $\mathcal{S}$–local objects in $\mathcal{T}$. Then the following are equivalent:*

(i) *The composite functor*

$$\mathcal{S}^{\perp} \xrightarrow{\text{inclusion}} \mathcal{T} \xrightarrow{\ \pi\ } \mathcal{T}/\mathcal{S}$$

is an equivalence of categories.

(ii) *The inclusion $i_* : \mathcal{S} \longrightarrow \mathcal{T}$ has a right adjoint $i^! : \mathcal{T} \longrightarrow \mathcal{S}$.*

Proof. As mentioned in Reminder 5.1, the implication (ii)$\Longrightarrow$(i) may be found in [6, Theorem 9.1.16]. We need to prove (i)$\Longrightarrow$(ii). Let t be an object of $\mathcal{T}$. By (i), its image πt in $\mathcal{T}/\mathcal{S}$ must be isomorphic to an object in the image of the composite

$$\mathcal{S}^{\perp} \xrightarrow{\text{inclusion}} \mathcal{T} \xrightarrow{\ \pi\ } \mathcal{T}/\mathcal{S} \ ,$$

that is there exists an object $y \in \mathcal{S}^{\perp}$ and an isomorphism $\pi t \longrightarrow \pi y$. By [6, Lemma 9.1.5] we know that

$$\mathrm{Hom}_{\mathcal{T}}(t,y) \quad = \quad \mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(\pi t, \pi y) \ ,$$

and hence the isomorphism in $\mathcal{T}/\mathcal{S}$ comes from a map $f : t \longrightarrow y$, in the category $\mathcal{T}$. Complete f to a triangle in $\mathcal{T}$

$$s \longrightarrow t \xrightarrow{\ f\ } y \longrightarrow \Sigma s \ .$$

Because πf is an isomorphism, we conclude that $\pi s = 0$, which means that s is isomorphic to an object in $\mathcal{S}$. If we replace s by an isomorph, we may assume $s \in \mathcal{S}$. Now [6, Theorem 9.1.13] tells us that there is a right adjoint to the functor $i_* : \mathcal{S} \longrightarrow \mathcal{T}$. □

Remark 5.3. We know that the chain complexes $\mathcal{H}om_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$, where I are test-complexes, are $\mathcal{S}$–local objects in $\mathbf{K}(R\text{–Flat})$. That is, $\mathcal{H}om_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is an object of $\mathcal{S}^{\perp}$. It clearly suffices, by Lemma 5.2, to show that, in the category $\mathbf{K}(R\text{–Flat})/\mathcal{S}$, the objects $\mathcal{H}om_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ are cogenerators. This is what we will prove.

What makes this approach to the proof quite feasible is that we know the category $\mathbf{K}(R\text{–Flat})/\mathcal{S}$. Recall that $\mathcal{S} = \mathbf{K}(R\text{–Proj})^{\perp}$; in [7, Remark 2.12(i)] we noted that general nonsense, applied to the diagram

$$\begin{array}{ccccc} \mathbf{K}(R\text{–Proj}) & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & \mathbf{K}(R\text{–Flat}) & \underset{i_*}{\overset{i^*}{\rightleftarrows}} & \mathbf{K}(R\text{–Proj})^{\perp} \\ & & \big\downarrow \pi & & \\ & & \dfrac{\mathbf{K}(R\text{–Flat})}{\mathbf{K}(R\text{–Proj})^{\perp}} & & . \end{array}$$

tells us that $\pi j_! : \mathbf{K}(R\text{–Proj}) \longrightarrow \mathbf{K}(R\text{–Flat})/\mathcal{S}$ is an equivalence of categories. That is, the category $\mathbf{K}(R\text{–Flat})/\mathcal{S}$ is equivalent to $\mathbf{K}(R\text{–Proj})$. To establish that $i_* : \mathcal{S} \longrightarrow \mathbf{K}(R\text{–Flat})$ has a right adjoint it suffices to show that, with the various identifications we have made, the objects $\mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ cogenerate $\mathbf{K}(R\text{–Proj})$.

We need to figure out what are the $F = \mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$, viewed as objects in the category $\mathbf{K}(R\text{–Proj}) \cong \mathbf{K}(R\text{–Flat})/\mathcal{S}$, where the equivalence of $\mathbf{K}(R\text{–Proj})$ with $\mathbf{K}(R\text{–Flat})/\mathcal{S}$ is via the functor $\pi j_!$. Since F is $\mathcal{S}$–local we have that, for any object $X \in \mathbf{K}(R\text{–Proj})$,

$$\mathrm{Hom}_{\mathbf{K}(R\text{–Flat})/\mathcal{S}}(\pi j_! X, \pi F) \quad = \quad \mathrm{Hom}_{\mathbf{K}(R\text{–Flat})}(j_! X, F) \ .$$

We therefore need to understand the abelian groups $\mathrm{Hom}_{\mathbf{K}(R\text{–Flat})}(j_! X, F)$. We begin with a lemma that works for any ring R.

Lemma 5.4. *Let R be a ring, let I be a bounded below complex of right R–modules, and put $F = \mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$. Let P be a chain complex in the set T of* [7, Construction 4.3]*. Then*

$$\mathrm{Hom}_{\mathbf{K}(R\text{–Mod})}(j_! P, F) \quad = \quad \mathrm{Hom}_{K(\mathbb{Z})}\Big(\mathcal{H}\mathrm{om}_R(P^*, I) \ , \ \mathbb{Q}/\mathbb{Z}\Big) \ .$$

Proof. We should prehaps remind the reader; the objects in the set T are bounded below chain complexes of finitely generated, projective left R–modules, and the complex P^* is the complex $\mathcal{H}\mathrm{om}(P, R)$; it is a bounded above complex of finitely generated, projective right R–modules.

Now for the proof. We recall that $F = \mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$. This makes

$$\begin{aligned} \mathrm{Hom}_{\mathbf{K}(R\text{–Mod})}(j_! P, F) \quad &= \quad \mathrm{Hom}_{\mathbf{K}(R\text{–Mod})}\Big(j_! P \ , \ \mathcal{H}\mathrm{om}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})\Big) \\ &= \quad \mathrm{Hom}_{\mathbf{K}(\mathbb{Z})}\big(I \otimes_R P \ , \ \mathbb{Q}/\mathbb{Z}\big) \\ &= \quad \mathrm{Hom}_{K(\mathbb{Z})}\Big(\mathcal{H}\mathrm{om}_R(P^*, I) \ , \ \mathbb{Q}/\mathbb{Z}\Big) \ . \end{aligned}$$

□

Construction 5.5. Now let Q be a compact object in $\mathbf{K}(R\text{–Proj})$. We remind the reader that, in [7, Proposition 7.12], we classified the compact objects in $\mathbf{K}(R\text{–Proj})$. Replacing Q by an isomorph if necessary, we may assume that $Q \in T$; that is, Q is a bounded below chain complex of finitely generated, projective left R–modules. This means that Q^* is a bounded above chain complex of finitely generated, projective right R–modules. Since

Q is compact, we furthermore know that $H^i(Q^*) = 0$ if $i \ll 0$. This means that we can choose a bounded below complex of injective right R–modules, quasi-isomorphic to Q^*; choose one, and denote it $I = I(Q)$. Let us choose and fix an $I(Q)$, for every compact object Q in $\mathbf{K}(R\text{–Proj})$. Clearly $I(Q)$ is a test-complex.

Proposition 5.6. *Let R be a coherent ring. Let $\{I(Q),\ Q \in \mathbf{K}(R\text{–Proj})^c\}$ be all the test-complexes of Construction 5.5. Then the objects $F(Q) = \mathcal{H}om_{\mathbb{Z}}\big(I(Q), \mathbb{Q}/\mathbb{Z}\big)$ cogenerate the category $\mathbf{K}(R\text{–Proj}) \cong \mathbf{K}(R\text{–Flat})/\mathcal{S}$.*

Proof. First of all, because R is coherent, the objects $F(Q)$ belong to $\mathbf{K}(R\text{–Flat})$.

Let P be any object in T, and assume Q is a compact object. Both P and Q are chain complexes of finitely generated, projective right R–modules; hence

$$\mathcal{H}om_R(Q, P) \quad = \quad \mathcal{H}om_R(P^*, Q^*) \ .$$

Now P^* is a bounded above chain complex of projectives, and hence the quasi-isomorphism $Q^* \longrightarrow I(Q)$, of Construction 5.5, induces a quasi-isomorphism

$$\mathcal{H}om_R(P^*, Q^*) \ \longrightarrow \ \mathcal{H}om_R\big(P^*, I(Q)\big) \ .$$

Since the abelian group $\mathbb{Q}/\mathbb{Z}$ is injective, we have an isomorphism of abelian groups

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}\Big(\mathrm{Hom}_{\mathbf{K}(R\text{–Proj})}(Q,P)\ ,\ \mathbb{Q}/\mathbb{Z}\Big) & & \\ \Big\| & & \\ \mathrm{Hom}_{\mathbf{K}(\mathbb{Z})}\Big(\mathcal{H}om_R(Q,P)\ ,\ \mathbb{Q}/\mathbb{Z}\Big) & \longrightarrow & \mathrm{Hom}_{\mathbf{K}(\mathbb{Z})}\Big(\mathcal{H}om_R\big(P^*, I(Q)\big)\ ,\ \mathbb{Q}/\mathbb{Z}\Big) \ . \end{array}$$

Lemma 5.4 tells us that

$$\mathrm{Hom}_{\mathbf{K}(\mathbb{Z})}\Big(\mathcal{H}om_R\big(P^*, I(Q)\big)\ ,\ \mathbb{Q}/\mathbb{Z}\Big) \quad = \quad \mathrm{Hom}_{\mathbf{K}(R\text{–Flat})}\big(j_!P, F(Q)\big) \ .$$

Combining these isomorphisms, we have an isomorphism, natural in P,

$$\mathrm{Hom}_{\mathbb{Z}}\Big(\mathrm{Hom}_{\mathbf{K}(R\text{–Proj})}(Q,P)\ ,\ \mathbb{Q}/\mathbb{Z}\Big) \ \longrightarrow \ \mathrm{Hom}_{\mathbf{K}(R\text{–Flat})}\big(j_!P, F(Q)\big) \ .$$

These are isomorphisms for all $P \in T$, in particular for all compact objects P. On the subcategory $\mathbf{K}(R\text{–Proj})^c$, of the compact objects in $\mathbf{K}(R\text{–Proj})$, the functor represented by $F(Q)$ is the functor taking $(-)$ to

$$\mathrm{Hom}_{\mathbb{Z}}\Big(\mathrm{Hom}_{\mathbf{K}(R\text{–Proj})}(Q\ ,\ -)\ ,\ \mathbb{Q}/\mathbb{Z}\Big) \ .$$

This is an injective object in the abelian category $\mathcal{C}at\Big(\big\{\mathbf{K}(R\text{–Proj})^c\big\}^{\mathrm{op}}\ ,\ \mathcal{A}b\Big)$ of additive functors $\{\mathbf{K}(R\text{–Proj})^c\}^{\mathrm{op}} \longrightarrow \mathcal{A}b$. If we take the product, over all compact objects $Q \in \mathbf{K}(R\text{–Proj})$, we discover that the functor represented by

$$F \quad = \quad \prod_{Q \in \mathbf{K}(R\text{–Proj})^c} F(Q)$$

is an injective cogenerator in $\mathcal{C}at\Big(\big\{\mathbf{K}(R\text{–Proj})^c\big\}^{\mathrm{op}}\ ,\ \mathcal{A}b\Big)$. The category $\mathbf{K}(R\text{–Proj})$ is a compactly generated triangulated category because the ring R is right coherent, and by [7, Proposition 7.14]. The argument above shows that F is a Brown–Comenetz object in $\mathbf{K}(R\text{–Proj})$. By [6, Theorem 8.6.1], F cogenerates. □

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