Continuity and rupture between argumentation and proof in historical texts and physics textbooks on parabolic motion

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In this paper, we analyze different presentations in a historical text by Galilei and a textbook for high school of the parabolic motion of a projectile with a lens developed within Mathematics education research on argumentation and proof (cognitive unity; Mariotti et al., 1997; Pedemonte, 2005). The analysis highlights possibilities and problematic issues, with particular attention to the aspects related to continuity and rupture between argumentation and proof in textbooks and the different interdisciplinary relationships between mathematics and physics mirrored by historical sources and textbooks. We discuss how a comparison between them can be exploited to develop a discourse about interdisciplinary that can enlarge the view of the relationship between the two disciplines and didactical implications that can be inferred from this comparison.

Keywords: Interdisciplinary approach, epistemology, proof, cognitive unity, textbook evaluation.

Introduction

To introduce the topic of our contribution, we start from three different representations of the trajectory of a projectile:

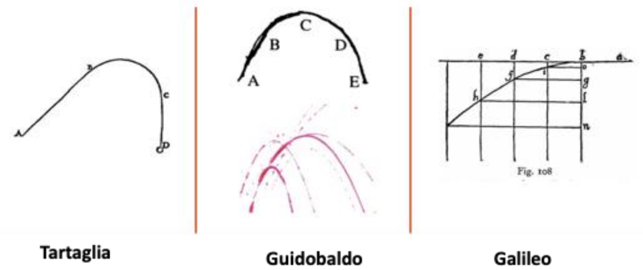


Figure 1: Three different representations of trajectory of projectile in historical texts

In Tartaglia's representation, the trajectory of a projectile consists of three parts: a straight part, followed by an arc of a circle and then ending in a straight vertical line. As stressed in Renn et al. (2000, p. 316): "in the Aristotelian tradition, projectile motion was conceived of as resulting from the contrariety of natural and violent motion, the latter according to medieval tradition acting through an impetus impressed by the mover into the moving body. According to this understanding of projectile motion, the trajectory cannot be symmetrical because the motion of the projectile is determined at the beginning and at the end by quite different causes. At the beginning it is dominated by the impetus impressed into the projectile, at the end by its natural motion towards the center of the earth.". Principles elaborated to interpret motion on the Earth were "embodied" in the form of trajectory, pursuing the aim to provide an axiomatic foundation to the analysis of projectile motion. Guidobaldo's sketch comes from an experiment. The paradigm was slowly changing in science and his transition work was crucial to challenge the medieval perception of motion. As we can see in his representation, the "symmetry" that he had experimentally found in the trajectory (it the ball will take the same path in falling as in rising, and the shape is that which, when inverted under the horizon) led him to corroborate the idea that not necessarily the different kinds of motions are consecutive. This opened the path to new hypotheses compatible with the possibility that motions can compose each other; in this frame, the trajectory could resemble a catenary or hyperbola or parabola. Galilei (1638), as we will show, in *Discourses and Mathematical Demonstration Relating to Two New Science*, completed the process of proving that the trajectory is parabolic, setting up an axiomatic system and grounding reasoning on rigorous proofs inspired to Euclidean ones. These steps were crucial in the birth of Physics and clearly show that the structure of reasonings developed mainly in Geometry, like axiomatics and deductive proofs, from the very beginning played a key role in the development of Physics (Renn et al., 2000). Udhen et al. (2012) stressed that: "the relationship between physics and mathematics has many facets, from the possibility to discover new physics within the mathematical structure to the mathematical nature of basic physical concepts. […] students should not only recognize that mathematics is a valuable tool for physics, but also that it can provide the underlying structure of a physical theory" (p. 493). These historical cases clarify why mathematics is said to play a structural role in physics.

Institutional context and differences between historical texts and textbooks

To promote students' awareness of the interdisciplinary relationships between mathematics physics and philosophy, in a historical perspective, is a goal of secondary school in the Italian Licei (Mathematics curriculum). In particular teachers are asked to pay attention to these aspects with respect to the XVII century and the birth of modern science. The books by Galilei are the primary sources to consider in order to analyse the topic from the historical-epistemological point of view. In this book the conceptions of disciplines and their relationship differs from today since it is a foundative book, one of pillars of modern scientific method, and an example of rich scientific text that intertwines explicitly many dimensions of knowledge that nowadays are codified in disciplines (mathematics, physics, engineering, philosophy). Physics textbooks for secondary school present a disciplinary didactical transposition that is consistent with the (implicit or explicit) didactical goals of the authors. The topic is not addressed in the same way as Galilei: parabolic motion is presented as a particular case of two-dimensional motion and introduced deserving a lot of space to algebraic passages and formulas, also in the proof. The main differences can be due to the targets (scientific community vs students), the goals (proposing a new theory vs teaching), the development of disciplines and their epistemologies (Euclidean geometry and study of motion vs M&P curriculum at school), interdisciplinarity (scientific discourse intertwining different dimensions vs combination of elements of knowledge taught with a disciplinary perspective).

Literature review and research questions

In this paper, we focus on a specific aspect, epistemologically relevant from the disciplinary and interdisciplinary point of view, that is the way argumentation and proof (A&P) are presented in two texts about parabolic motion: Galilei (1638) and the chapter Two-Dimensional Kinematics in the physics textbook by Walker (2017; high school edition, translated also in Italian). Among the different textbooks used in Italy, we chose that one because it is quite rich from the epistemological point of view (Bagaglini et al., 2021). A first reading of the books showed that in both cases they deal with proving/demonstrating that the trajectory of a projectile is an arc of parabola, but the meanings of the term "proving" seemed to change, as well as the way proof were presented and intertwined with other aspects of the scientific argumentation. We consider A&P key concepts to analyse the structural role of mathematical thinking in physics learning in an interdisciplinary perspective. On one hand awareness about the relationship between mathematical proofs and physical argumentation contribute to developing an authentic picture of the role of mathematics in physics. On the other hand, to trigger a reflection about the meaning of A&P in mathematics and physics (M&P) is an opportunity to investigate the epistemology of such disciplines. With respect to the literature review in mathematics education, our research aims at contributing to address some open questions proposed in the handbook by Durand-Guerrier et al. (2012) about A&P in mathematics and empirical sciences: To what extent should mathematical proofs in the empirical sciences, such as physics, figure as a theme in mathematics teaching so as to provide students with an adequate and authentic picture of the role of mathematics in the world? Could a stronger emphasis on the process of establishing hypotheses (in the empirical sciences) help students better understand the structure of a proof that proceeds from assumptions to consequences and thus the meaning of axiomatics in general? We consider the way the bridge built between mathematical and physical aspects of A&P is presented crucial to address the nature of such a relationship from the didactical point of view. The type of presentation of a proof is already under investigation in mathematics education; open questions we are interested in are: To what extent and how is the presentation of a proof (verbal, visual, formal etc.) (in)dependent on the nature of the proof? Do students perceive different types of proofs as more or less explanatory or convincing?" (Durand-Guerrier et al., 2012). We hypothesized that connecting the notions of A&P in M&P makes this aspect even more important, since the verbal, the visual and the formal aspects of proof might play a different role in explaining and convincing students when "mathematizing" observation and reasonings about empirical phenomena or experiments, and in mirroring the nature of such a kind of proof, whose complexity is evident also in the historical cases briefly resumed in the introduction.

In this paper we analyse the way knowledge belonging to M&P (objects, reasonings, assumptions, epistemological issues) is used in argumentative steps and proof in different texts. We consider the analysis of A&P in texts and the comparison with historical texts a key step to move from the historical-epistemological and cognitive analyses to the classroom practices, in particular considering teacher-students education. This issue has been investigated by papers presented in CERME10 (Stylianides et al., 2018); among the themes discussed, we contribute to highlight the role of language in teaching and writing proofs and to search for analytical frameworks for argumentation and proof in textbook expositions.

Research framework

The didactic value of inserting proof into an argumentative process that involves students in the formulation of conjectures has been highlighted by many studies as a way to move from a reproductive approach to demonstration to a productive one and to focus on proof as a process more than on proof as a product. The construct of cognitive unity has been introduced by Mariotti et al.

(1997) to encode this idea and to stress the need for didactical situations in which the construction of a proof naturally follows from the exploration of a problematic situation by students. In particular we refer to this key aspect: “some aspect of continuity, concerning the production, during the construction of the conjecture, of the elements (“arguments”) that are used later during the construction of the proof” (p. 1). This way some elements that characterize the proof (the choice of a statement to refer to, or of the semiotic representation register) are not artificially and suddenly introduced but arise naturally from the exploration, as it happens when statements are proved in research. Otherwise there is a cognitive rupture (Pedemonte, 2005). Proving that the trajectory of a projectile motion is parabolic can be considered a conjecture-proving problem, according to the characterization of Mariotti et al. (1997).

We assume that continuity should be pursued also in physics teaching to guarantee a productive approach of students to proving in this field, in particular when mathematics appears in the statements and semiotic representations of physical entities, since students need to activate resources related to their conception and experience of mathematical processes. What happens to the flow of observation and conjectures about physical phenomena when mathematics enters the discourse? If teachers have to guide a classroom discussion to help the students to include these aspects, is continuity between A&P pursued or do their interventions cause cognitive rupture? As we showed, the issue is critical from the epistemological point of view, so we think teacher-students need examples and meta reflection to guide the students properly in such classroom discussions. The cognitive unity has been developed, and is mainly used, to analyse students’ reasonings. We consider texts targeted to non-expert readers as examples of forms of presentation of reasonings, as they were teachers' speeches when they guide students who made observations and conjectures to gradually organize their reasonings. These can be prototypes of different ways the teachers scaffold students’ approach to interdisciplinary A&P in the classroom, with possible different impacts on students’ learning. We consider thus it useful to carry out analyses with the same lens used with students of the ways the texts guide the readers to move from exploration to A&P.

Methods

The books were analysed at two scales: a global analysis of the organization of the books with epistemological and linguistic lenses (Bagaglini et al., 2021), and zooming in on some excerpts where we could find relevant aspects to analyse in order to identify continuity and rupture between A&P in the texts. In this paper we focus on the second aspect. From the methodological point of view, we referred to the analysis of cognitive unity and rupture proposed by Pedemonte (2005):

– *structural analysis*: refers to the link between the structures of statements used in argumentations and in proofs. There is structural cognitive unity when statements used in the argumentation are also used in the proof. Otherwise, there is structural cognitive rupture.

– *referential analysis*: refers to the systems of reference used in argumentations and in proofs, that is, the systems of signs (drawings, calculations, algebraic expressions, etc.) and systems of knowledge (definitions, theorems, etc.) used. There is referential cognitive unity when some systems of signs or knowledge are used both in the argumentation and the proof. Otherwise, there is referential cognitive rupture. We enlarged it according to our goal (interdisciplinary analysis of prototypes of A&P

connections). We carried out a structural and referential analysis of relevant excerpts from the third and fourth day, concerning the study of local motions in Galilei (1638) and Walker (2017). We identified statements in A&P related to parabolic motion and then systems of representation and knowledge belonging to both mathematics and physics (considered as disciplines taught at school in grades 9-10 in Italy in the textbook's analysis and as historical disciplines analyzing Galilei's excerpts). We organized them on tables reporting on the left the excerpt (statements), on the right the referential analysis. By comparing the A&P steps, thanks to the structural and referential analysis, we detected unity or rupture in both texts. Because of space constraints, we report only a few excerpts to show the analysis of the proof of the statement "the trajectory of a projectile is parabolic" and the previous choices made in the argumentative part.

Main results of the analysis of unity or rupture in Galilei's and Walker's texts

By steady or uniform motion [1], I mean one in which the distances traversed by the moving particle [2] during any equal intervals of time [3], are themselves equal. [D1].	Definition of uniform motion using proportions (equal space in equal time)
A motion is said to be uniformly accelerated [4], when starting from rest, it acquires, during equal time-intervals [3], equal increments of speed.[...] the distances traversed [2] are proportional [D1] to the squares [5] of the times.	Definition of accelerated motion using proportions (equal increments of speed in equal time, space proportional to the square of time)
Imagine any particle projected along a horizontal plane without friction; if the plane is limited and elevated [6] the resulting motion which I call projection [7], is compounded of one which is uniform and horizontal [1] and of another which is vertical and naturally accelerated [4].	Definition of projectile, that incorporates the assumption of composition of motions
Theorem 1 – Proposition 1: A projectile [7] which is carried by a uniform horizontal motion [1] compounded with a naturally accelerated [4] vertical motion describes a path which is a semi-parabola [8].	Theorem formulated using previous definitions
The section of this cone [..] which is called a parabola [8] [..] the square [5] of bd is to the square [5] of fe in the same ratio [9] as the axis ad is to the portion ae.	Definition of parabola
Let us imagine an elevated [6] horizontal line or plane ab along which a body moves with uniform [1] speed from a to b. Suppose this plane to end abruptly at b [6] [..]. Draw the line be along the plane ba to represent the flow, or measure, of time; divide this line into a number of segments, bc, cd, de, representing equal intervals of time [3] [..] in proportion [D1] to the squares [5] of cb, db, eb, or, [..] in the squared ratio [9] of these same lines. [...]the distance traversed [2] by a freely falling body varies as the square [5] of the time; in like manner the space eh traversed [2] during the time be will be nine times [D1] ci; thus it is evident that the distances eh, df, ci will be to one	Proof is presented, where: - the same terms introduced before are used, as well as the same spatial representation (segments/intervals of time) - it is stressed the use of proportional reasoning, that was used to define the kinds of motions that are combined - G. recalls the assumptions about the composition of motions

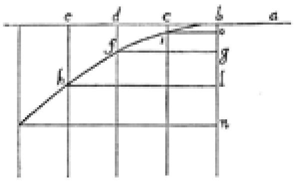
another as the squares [5] of the lines be, bd, bc. The square [5] of hl is to that of fg as the line lb is to bg [D1]; and the square [5] of fg is to that of io as gb is to bo; therefore the points i, f, h, lie on one and the same parabola [8]. 	- G. recalls the setting associated to the definition of projectile with the same words - G. intertwines the definition of parabola and the characterization of accelerated motion in order to exploit the linguistic analogies to stress that the points must lie on a parabola.

Table 1: Analysis of Galilei's excerpts

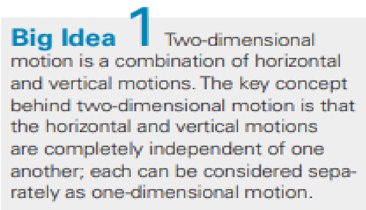
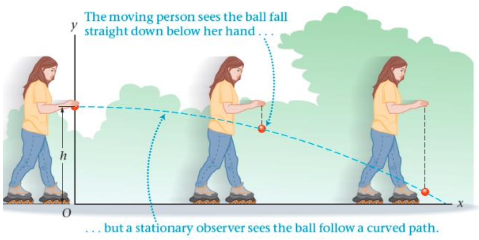
 **Big Idea 1** Two-dimensional motion is a combination of horizontal and vertical motions. The key concept behind two-dimensional motion is that the horizontal and vertical motions are completely independent of one another; each can be considered separately as one-dimensional motion.	The combination and independence of horizontal and vertical motions are initially introduced in a lateral box as Big Idea. The status of the statement in terms of elements of A&P (axiom, theorem) is not expressed.
Projectile Motion: Basic Equations We now apply the independence of horizontal and vertical motions to projectiles. Just what do we mean by a projectile? Well, a projectile is an object that is thrown, kicked, batted, or otherwise launched into motion and then allowed to follow a path determined solely by the influence of gravity.	The Big Idea is applied to projectile motion to obtain its equations and a phenomenological description of the projectile is presented.
Demonstrating Independence of Motion: A simple demonstration illustrates the independence of horizontal and vertical motions in projectile motion. [..] Notice that the ball goes straight down, lands near your feet, and returns almost to the level of your hand in about a second. [...] To you, its motion looks the same as before. The fact that you were moving in the horizontal direction the whole time had no effect on the ball's vertical motion — the motions are independent.  	A figure represents a moving person with a roller skate and a falling ball; the two combined motions are represented with a reference to real life. The motion is seen also by an external observer and the trajectory is linear and vertical in the system of person and curved in the external system, that is represented through cartesian axes put onto the real life figure. The relativity of motion in different systems is used to demonstrate independence of motions.
To an observer who sees you walking by, the ball follows a curved path, as shown. The precise shape of this curved path—a parabola—is verified in the next section.  (a)	A picture (photo with a camera to a real world phenomenon) is proposed. In the description of the figure, it is mentioned the visualization of concepts and presented as one among other"examples of principle" of independence of motions.

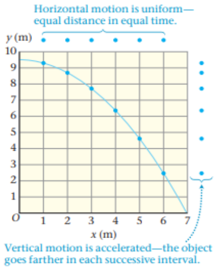
FIGURE 4-4 Visualizing Concepts - Independence of Motion (a) An athlete jumps upward from a moving skateboard.	It is anticipated that the shape is a parabola and that this will be verified later.
▸ **FIGURE 4-6 Trajectory of a projectile launched horizontally** In this plot, the projectile was launched from a height of 9.5 m with an initial speed of 5.0 m/s. The positions shown in the plot correspond to the times $t = 0.20$ s, 0.40 s, 0.60 s, ... Notice the uniform motion in the x direction, and the accelerated motion in the y direction. and $y_0 = h$ This is illustrated in Figure 4-3. The initial velocity is horizontal in this case, corresponding to $\theta = 0$ in **FIGURE 4-5**. As a result, the x component of the initial velocity is simply the initial speed: $v_{0x} = v_0 \cos 0° = v_0$ The y component of the initial velocity is zero: $v_{0y} = v_0 \sin 0° = 0$ Substituting these specific values into our fundamental equations for projectile motion (Equations 4-6) gives the following simplified results for zero launch angle ($\theta = 0$): $x = v_0 t \quad v_x = v_0 = \text{constant} \quad v_x^2 = v_0^2 = \text{constant}$ $y = h - \frac{1}{2}gt^2 \quad v_y = -gt \quad v_y^2 = -2g\Delta y$ 4-7 Snapshots of this motion at equal time intervals are shown in **FIGURE 4-6**.  Dropped and Thrown Balls **PROBLEM-SOLVING NOTE** **Identify Initial Conditions** The launch point of a projectile determines x_0 and y_0. The initial velocity of a projectile determines v_{0x} and v_{0y}.	A graph, resembling the one by Galilei but the use of x,y and units on the axes, is in a lateral box. The horizontal uniform motion is presented using proportions (equal space in equal time) without mentioning the nature of this description as definition. The same happens to vertical accelerated motion. Symbolic expressions are used for the generic case and the Galilei case is obtained substituting a value into equations for projectile motion.
Parabolic Path **RWP** Just what is the shape of the curved path followed by a projectile launched horizontally? This can be found by combining $x = v_0 t$ and $y = h - \frac{1}{2}gt^2$, which allows us to express y in terms of x. First, solve for time using the x equation. This gives $t = \frac{x}{v_0}$ Next, substitute this result into the y equation to eliminate t: $y = h - \frac{1}{2}g\left(\frac{x}{v_0}\right)^2 = h - \left(\frac{g}{2v_0^2}\right)x^2$ 4-8 It follows that y has the form $y = a + bx^2$	An algebraic version of the proof is presented (never named proof), with: - reference to a curved path: - the term “found” instead of verify - use of symbolic expression of the two motions combined, as well as the parabolic generic equation - no reference to assumptions about the combination of motions - the use of terms “substitution” and “eliminate” - no mention of the previous graph and the exemplification of principles of independence.

Table 2: Analysis of Walker’s excerpts

Discussion and conclusions

The first analysis shows that Galilei’s text is characterized by structural and referential unity: he mathematized the relationship between space and time with magnitudes and proportions and used always the same objects and properties to merge the observation of phenomena, empirical laws and geometrical properties of conic sections. The mathematization of the experimental setting allowed him to prove, deductively, that the trajectory is a semi-parabola, under the hypothesis that the motion of a projectile results from a composition of independent uniform and accelerated motion. The theory of magnitudes bridges the concrete action of measuring and the theoretical comparison between geometrical magnitudes. The graphic representation plays a crucial role, since the action itself to trace a line/curve with a motion of a point is a sort of ideal machine that draws a trajectory, hybridizing the notion ofs trajectory and geometrical curve to treat the trajectory geometrically. In this case the structural role of mathematics clearly emerges: “importing” the structure of Euclidean proof in the investigation of motion allows to refine and strengthen argumentation.

In Walker's chapter, it is visible the effort to consider the dimension of A&P: there are physical assumptions, a definition of projectile, examples that ground the assumptions about the composition of independent motions on empirical facts, stressing that they are realistic. Some referential choices are consistent: the motion of a projectile is a particular case of a more general motion, equations of evolution are used to derive new equations treating time and space as algebraic variables. However, many elements of rupture are present. both in terms of structural and referential analysis of the relationship between argumentation and proof. Indeed, the presentation of the argument concerning physical principles and entities and the proof are presented with figures and pictures related to real life, while in the derivation of the equation they switch suddenly to algebraic language and analytical reasoning (substituting variables in functions). Moreover definitions, principle, inference, proof are never mentioned. The link between empirical aspects and mathematical knowledge is hard to establish for a reader, because of the strong discontinuity in terms of use of signs and semiotic registers for the expression of the statements.

Our analysis highlighted issues that we consider crucial from the didactical point of view since they connect relevant issues of mathematics education to interdisciplinarity M&P. In particular, from such a comparison prospective teachers can gain awareness about the ruptures that can be found in textbooks and thus adapt their teaching practices to pursue cognitive unity by reflecting on the aspects we stressed with their students and compensating for the weakness of textbooks.

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