

Article

Unfolding Teachers' Interpretative Knowledge into Semiotic Interpretative Knowledge to Understand and Improve Mathematical Learning in an Inclusive Perspective

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Abstract: In this article, grounded in the concept of interpretative knowledge (IK), which is well known in the literature, we introduce and discuss the construct of semiotic interpretative knowledge (SIK). This theoretical tool unfolds the interpretation of conceptual knowledge into a broader construct that intertwines the use of semiotic representations with the manifold aspects of mathematical learning. In the first part of the article, we first introduce Duval's semio-cognitive approach, in which semiotics is the element characterizing the specific cognitive functioning of mathematics. On the basis of some classic examples from the relevant literature, we then show the necessity of introducing the semiotic component into IK to expand the interpretive power of the teacher. In the second part of the article, through the analysis of six episodes involving students with a specific learning disorder (SLD) engaged in mathematical activities, we show how the development of the teacher's SIK emerges as a necessary condition for the implementation of inclusive teaching practices. This permits us to face the research question focused on how SIK allows us to understand the student's behavior in a special needs educational context and provide effective feedback. In this sense, SIK is shown to be an intrinsic prerequisite of inclusive teaching, thus extending the concept of IK to the components of pedagogical content knowledge (PCK).

Keywords: interpretative knowledge (IK); semiotic interpretative knowledge (SIK); inclusive semiotic interpretative knowledge (inclusive SIK); teacher knowledge; semio-cognitive approach; components of mathematical learning; semiotic function



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1. Introduction

The knowledge needed by mathematics teachers to successfully fulfill their teaching tasks in mathematics classrooms is widely discussed in the literature [1–7]. The theorization of the construct of mathematical knowledge for teaching (MKT), proposed by Ball and co-authors [1], can be considered as the starting point for deepening the aspects related to the different kinds of mathematical knowledge needed for instructional scopes. Starting from research related to the conceptualization of MKT, Ribeiro and coauthors [5] introduced the notion of interpretative knowledge (IK) as the part of the mathematical knowledge “that allows teachers to give sense to pupils' non-standard answers (i.e., adequate answers that differ from the ones teachers would give or expect) or to answers containing errors” [6] (p. 9). As IK is a kind of knowledge related to problem-solving strategies and errors, it is a piece of typically conceptual knowledge and does not consider explicitly the semiotic aspects related to signs and sign use in mathematical activity. Referring to a semiotic perspective in mathematics activity, Ernest states that, going “beyond behavioral performance this viewpoint also concerns patterns of sign use and production, including individual creativity in sign use, and the underlying social rules, meanings and contexts of sign use as

internalized and deployed by individuals” [8] (p. 68). As highlighted by Duval in the early 1990s of the 20th century, in mathematics “there is no noesis without semiosis” [9] (p. 23); in other words, conceptualization cannot be accomplished without an adequate competence of what Ernest calls patterns of sign use and production. The student has to learn to use and transform signs in order to be able to conceptualize mathematical content but she does not need to acquire explicit and aware semiotic knowledge. Indeed, semiosis is at the core of mathematical activity [10], but it is not a mathematical content and there is no explicit reference to semiotic aspects in mathematics textbooks or math school curricula. A strong semiotic competence is indispensable for a cognitively meaningful mathematical activity, but semiotics is not knowledge that could be taught, compared to, e.g., algebra, geometry, or probability. It might be for this reason that the importance of semiotic competencies for teachers is often disregarded when discussing the characteristics of MKT. However, as research shows [9,11–15], interpreting a student’s oral, written, or gesture production requires a strong semiotic competence. To reserve these competencies—to researchers in mathematics, education means to deprive teachers of basic interpretative tools when they analyze and interpret students’ answers. Even if signs are used in other disciplines, sign production, and use in mathematics are specific [9,16], and require the coordination of semiotic registers, as the theory of semiotic registers [16] shows.

In this paper, we introduce the notion of semiotic interpretative knowledge (SIK), broadening the seminal notion of IK proposed in the literature [4–6]. We claim that acquiring and developing a suitable SIK are important objectives for teachers, and discuss this position in Section 2, by analyzing and interpreting some examples taken from the literature. However, we also claim that a strong SIK should become an indispensable objective when teachers have to interpret the answers of students with special educational needs. Del Zozzo and Santi [14] show that students with special educational needs often struggle during mathematical activities precisely due to difficulties related to sign use and production, even before conceptualization issues. Although this kind of difficulty is not unique to students with special educational needs, it is more frequent in these cases. If the teacher lacks the appropriate semiotic lenses needed to give sense to some of the students’ wrong or unusual answers in the mathematics class, this might contribute to undermining the right to learning for all. In this sense, in Section 2, the need for SIK specific to an inclusive mathematics classroom is discussed and the notion of inclusive SIK is introduced. In Section 3, some classroom episodes are analyzed and interpreted, adopting a lens in accordance with the notion of inclusive SIK.

In order to frame the teaching–learning processes of mathematics from an inclusive perspective, we make use of a semiotic approach that combines the principles underlying discipline-specific cognitive functioning with the potential to accommodate the differences of all students when they engage with mathematics. We present a series of examples of students engaged in mathematical tasks and show how difficulties and problem-solving strategies are interconnected with the management of semiotic transformations between the representational systems underlying mathematical thinking. Analysis of the examples shows that students’ difficulties are not intrinsic phenomena but arise and evolve in the relationship between the specific characteristics of each individual and the semiotic resources he or she employs in mathematical practice. In some cases, non-specific interpretation of the pupils’ cognitive processes risks overlooking this relationship by considering procedures that express unforeseen forms of thinking as incorrect. The semiotic perspective is, thus, proposed as a possible theoretical lens that realizes inclusive mathematical learning contexts, in the sense of differentiation for all (and each of) the pupils. This study focuses on the efficacy of SIK as a theoretical lens to interpret students’ mathematics behaviors. The research question we deal with addresses how SIK allows us to understand the student’s behavior, for this article, in a special needs educational context, and provide effective feedback.

Section 2 presents SIK built around Duval’s semio-cognitive approach. In Section 3, in light of SIK, we discuss some significant examples of students with mathematical learning

difficulties. In the last paragraph, we present some concluding remarks on SIK and its connection with mathematical learning and inclusion as differentiation for all.

2. Methodological Framework

Referring to Radford [17], we follow a methodological approach that intertwines theory and practice. In fact, in Radford's view, a theory is a triplet composed of a set of theoretical principles, a template of research questions, and a methodology.

In our study, we follow the methodological approach as outlined by Radford [17] (p. 9):

"Furthermore, as the theory produces results, theoretical principles and methods may be affected. This is why it is better to think of a theory as a system in motion and transformation. One of the consequences of such a view of theories is that methods, notwithstanding the empiricists, cannot be reduced to a series of steps to be rigorously followed. A method is something much more complex. This is why, for Vygotsky (1997a, p. 27), "Finding a method is one of the most important tasks of the researcher." For Vygotsky, the main characteristic of a method is to be inquisitional and reflective. In Vygotsky's view, a method is a truly philosophical practice. To emphasize this non-instrumentalist conception of method, in what follows, I will use the term methodology instead of method."

The aim of this article is to introduce SIK as a new tool to understand the mathematics teaching–learning process in its dialectical tension between the principles and the method of a theory. We define the main features of SIK at the crossroads of Duval's semio-cognitive approach, Ball's mathematical knowledge for teaching, and the inclusive paradigm. Within a qualitative approach that involves students with learning difficulties, we verify the interpretive efficacy of SIK and, thereby, reflect on and scrutinize the theoretical tool.

In the remainder of this section, we introduce the theoretical perspectives that drive our study, we define SIK and inclusive SIK, pose our research question, and describe the experimental setting.

2.1. The Semiotic Perspective

In the early 1990s, Raymond Duval proposed semiotics as a new theoretical lens to investigate and characterize mathematical thinking and learning. His seminal research [9,16,18] introduced a structural and functional approach to semiotics, outlining the characteristics and potential of semiotic systems, and the transformational functions that inform mathematical thinking and knowledge.

According to Duval, every mathematical concept refers to objects that do not belong to our perceptual experience. In mathematics, ostensive references are impossible, as we cannot directly access mathematical objects through our senses. Therefore, every concept inherently requires the use of semiotic (semio-cognitive) representations of mathematical objects.

The lack of ostensive references led Duval to assign a constitutive role in mathematical thinking to the use of representations belonging to specific semiotic systems. From this point of view, Duval [9] (p. 23) argues that, as stated above, there is no 'noetics without semiotics', i.e., there is no conceptual understanding without the use of signs.

The peculiar nature of mathematical objects allows us to delineate a specific cognitive functioning that characterizes the evolution and learning of mathematics. We can say that conceptualization itself, in mathematics, is identified with this complex coordination of several semiotic systems.

A semiotic system (or register) is defined by Duval [18] and Ernest [8] as:

- A set of basic signs that only have meaning when set against or in relation to other basic signs (e.g., the meaning of the digits within the decimal numbering system);
- A set of rules for the production of signs, starting from basic signs, and for their transformation.

D'Amore [19] identifies conceptualization with the following semiotic functions, specific to mathematics:

- Choice of the distinctive features of a mathematical object;
- Processing, i.e., the transformation of one representation into another representation of the same semiotic register;
- Conversion, i.e., the transformation of one representation into another representation of another semiotic register.

The very combination of these three 'actions' on a mathematical object can be considered as the 'construction of knowledge in mathematics'. However, the handling of these three actions is not spontaneous, quite the contrary. Fandiño Pinilla [20] declines the process of learning mathematics into (at least) five components:

- Conceptual learning, related to noetics and conceptual understanding;
- Strategic learning, linked to problem-solving and processes, such as conjecturing, analyzing, and understanding (a problem situation);
- Algorithmic learning, linked to calculating, operating, and performing processes and procedures;
- Communicative learning, which, considering the essentially social character of learning, takes on a very broad and extended meaning;
- Semiotic learning, linked to the specificities of semiotic functions.

This last component, for reasons well brought into focus by Duval, has a transversal role with respect to the other components of learning. However, for the same reasons, it is also a sort of envelope that, in some way, encloses and interpenetrates everything else. Indeed, as D'Amore and colleagues [21] point out, there is no conceptual, algorithmic, strategic, or communicative learning divided from semiotic learning.

On the other hand, the management of such semiotic complexity, in the structure of semiotic systems and the processing and conversion functions, comes up against Duval's famous cognitive paradox [16,22]. The semiotic functions described above are not easily handled due to an inevitable identification by the student of the mathematical object with its representations; the outcome of the cognitive paradox:

"(. . .) on the one hand, learning mathematical objects can only be conceptual learning and, on the other hand, it is only by means of semiotic representations that an activity on mathematical objects is possible. This paradox can constitute a real vicious circle for learning. How could learning subjects not confuse mathematical objects with their semiotic representations if they can only relate to semiotic representations? The impossibility of direct access to mathematical objects, outside of any semiotic representation, makes confusion almost inevitable. And, conversely, how can they master mathematical treatments, necessarily linked to semiotic representations, if they do not already have conceptual learning of the objects represented? This paradox is even stronger if one identifies mathematical activity and conceptual activity and if one considers semiotic representations as secondary or extrinsic" [16] (p. 38, the authors' translation).

In other words, we know mathematical objects through the use of symbols that represent them, but the effective use of these symbols requires the conceptualization of the objects they represent.

Mathematical objects are recognized as invariant entities that link different semiotic representations when processing and converting transformations, and as such cannot be referred to directly. As already mentioned, this is why Duval identifies the specific cognitive functioning of mathematics with the coordination of a variety of representational registers. Both the development of mathematics as a field of knowledge and its learning are realized through this specific cognitive functioning.

Duval goes beyond Frege's classic semiotic triangle, Sinn–Bedeutung–Zeichen (sense–signified–expression), and identifies meaning with the pair (sign–object), i.e., with the relationship between a sign and an object. The sign becomes a rich structure that con-

denses both expression (Zeichen, which in Duval's terminology corresponds to the semiotic representation) and meaning (Sinn), i.e., the way the semiotic representation offers the object according to the meaning underlying the structure of the semiotic system. For Duval, meaning, thus, has a dual dimension:

- *Sinn*, the way in which a semiotic representation offers a mathematical object;
- *Bedeutung*, the reference to an inaccessible mathematical object [23].

The processes of meaning construction and learning require connecting the different senses (Sinn) through possible semiotic transformations, without losing the reference (Bedeutung) to the mathematical object. The Sinn is the meaning that representations take on within the semiotic system and offers the reference to the mathematical object in a specific way. For example, the representations $(x + 1)(x - 1)$ and $x^2 - 1$ refer to the same object in a given set of numbers with different 'Sinn', the first highlights the factorial aspect and the second highlights the polynomial one. Furthermore, the signs involved in the expression ("x", "+", "-", etc.) do not have meaning per se but they do in relation to other signs within the structure of the algebraic semiotic system.

Duval's semio-cognitive perspective that we described is capable of accommodating an idea of inclusion that considers students' learning differences and, therefore, considers differentiation a tool that involves all students and all teachers. According to the semio-cognitive perspective, learning mathematics cannot and should not follow only one cognitive pathway, so that the learner achieves noetics according to his or her specific characteristics. The network of semiotic transformations available to the teacher and learner in order to achieve the learning objectives makes it possible to design creative and differentiated teaching activities that take into account the needs of each student. An obstacle that students often have to overcome is related to teaching practices that present mathematical concepts in only one semiotic register, typically the symbolic one for arithmetic and algebra, and the figurative one for geometry. This choice on the part of teachers reinforces the cognitive paradox and the difficulties it entails. In these situations, the pupil not only tends to identify the mathematical object with the semiotic representation, but does not develop the capacity to interpret the meaning of concepts consistent with the context of the mathematical practices to which he or she is exposed; interpretative capacities that are developed in the richness and depth of the sign-object pair and, faced with semiotic choices that are inconsistent with the mathematical object and inadequate to the pupil's specific needs, the sign-object pair is flattened on the side of the sign, empty of meaning.

2.2. Inclusion in Mathematics as Differentiation for All Students

Inclusive pedagogy has been conceptualized in different ways. In the international literature, we find some consensus on a general distinction between narrow and broad definitions [24–32]. Narrow definitions focus on students with disabilities, their presence in mainstream schools and classrooms, and the support needed. The broader definitions concern school systems and school communities and their commitment and capacity to accommodate all students with all of their individual differences, ensuring participation and effective learning processes.

Göransson and Nilholm [33], trying to take a more differentiated and nuanced look, systematized four different meanings of inclusive pedagogy: (a) the inclusion of pupils with disabilities in mainstream classes, (b) meeting the social and academic needs of pupils with disabilities, (c) meeting the social and academic needs of all pupils, and (d) creating communities with specific characteristics.

A broad idea of inclusion poses a great challenge to how learning processes can be supported in schools, taking into account the differences of all students while ensuring their participation in a common learning project. Differentiation has been discussed by several authors as a tool that can help to meet this challenge. However, what exactly is differentiation? Differentiation has been conceptualized by different authors in various ways. Spandagou and colleagues [34] represented it as a continuum that has on the one hand a socio-constructivist view of learning that takes into account all students' differences [35]

and on the other hand an understanding based on psychometric theories of intelligence and ability, where differentiation is understood as specific strategies provided by a teacher to a student in a class [36].

It is immediately apparent that not all ways of defining differentiation contribute to the broad idea of inclusion, simply because some definitions, such as Levy's, do not consider all students' differences but focus on only some of them. For this article, we have chosen a conceptualization of differentiation that considers students' learning differences and, thus, considers differentiation to be inclusive of all students.

The learning of mathematics plays a major role in a student's education. It requires the attainment of high cognitive standards, in terms of creativity, rationality, control of different semiotic registers, metacognition, etc. Mathematics can be a field of knowledge in which an individual's self-esteem and self-efficacy can flourish.

We can also recognize a social and political value in the learning of mathematics, as it is a fundamental tool for contemporary citizens to access the complexity of our society. Mathematics is at the heart of science and technology, shaping the world in unpredictable, unexpected, and rapidly changing scenarios. The Italian national mathematics curriculum [37] attaches considerable importance to 'mathematics for the citizen' as a guideline for mathematics teachers. Mathematics can be a tool for equity or discrimination, depending on how well (and how many) students are able to grasp its cognitive, social, and cultural-historical potential.

The educational potential of mathematics clashes against the difficulties that research in mathematics education has precisely outlined. For the sake of brevity, we will mention some of the most important of these, which are closely linked to the constituent ontological and epistemological traits of mathematics. First, mathematical knowledge refers to ideal entities that do not allow any ostensive reference. The only access to mathematical objects is through signs, structured in complex semiotic systems, which students must handle with specific cognitive competence. In this respect, the main risk that students run into is the identification of semiotic representations with the mathematical objects they refer to. This identification represents a real obstacle that hides several pitfalls in learning mathematics [19]. Second, mathematical concepts require an unnatural cognitive leap from procedural-situated thinking to highly relational-generalized thinking—as Vygotsky [38] puts it, from spontaneous concepts to scientific concepts.

2.3. *The Notion of Inclusive Semiotic Interpretative Knowledge (Inclusive SIK)*

In his seminal paper, Shulman [39] classified the knowledge needed for teaching into three categories: pedagogical knowledge (PK), pedagogical content knowledge (PCK), and subject matter content knowledge (SMCK). While PK is subject-matter independent, PCK and SMCK are related to the specificities of the discipline to be taught. In mathematics education research, the authors of [1] introduced the construct of mathematical content for teaching (MCT). According to these authors, MCT is the mathematical knowledge needed by teachers to perform the usual tasks related to teaching mathematics [1]. In the model proposed by Ball and colleagues (Figure 1), the MCT is subdivided into SMK and PCK (both considered in reference to mathematics), and both these categories have three subcategories: SMK is split into common content knowledge (CCK), horizon content knowledge (HCK), and specialized content knowledge (SCK), while the PCK is split into the knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC).

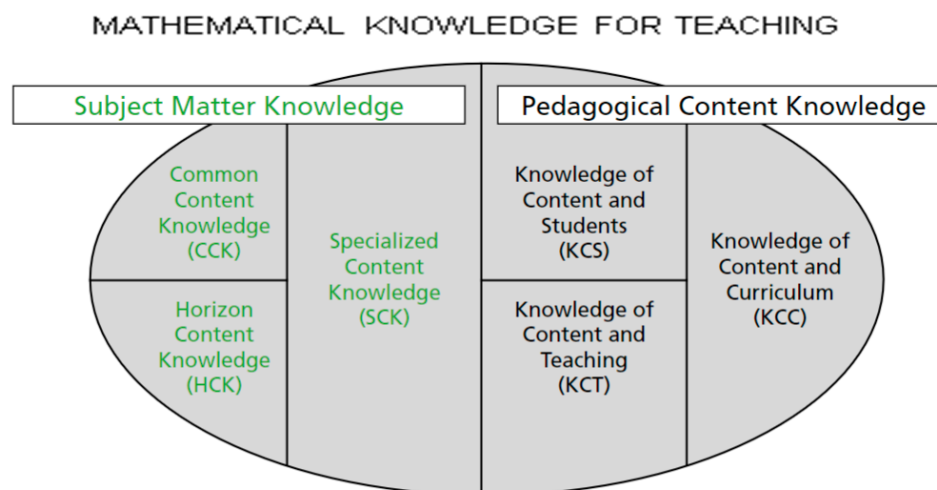


Figure 1. The elements of the mathematical knowledge for teaching (MKT) [1] quoted by [40] (p. 52).

The notion of interpretative knowledge (IK) introduced by Ribeiro and coauthors [5] is a kind of MKT that can be considered as a part of the SMK, “in the intersection between the common content knowledge and the specialized content knowledge” [3] (p. 4). These authors derive the characterization of the IK as belonging to SCK, but as strongly related to the CCK, from the conclusion that a strong CCK is necessary but not sufficient to develop a good level of IK, but at the same time, teachers with a strong CCK have difficulties in accepting unusual strategies that differ from their own [3]. In this sense, the IK is a kind of SCK that is “specialized” because it is specific to mathematics teaching, but it is also “pure” because it is not mixed with knowledge of students and pedagogy [1] (quoted by Di Martino et al. [3]).

To characterize IK, Ribeiro et al. [5] focus on the reconsideration of the notion of error: errors—but also non-standard reasoning—are learning opportunities in the sense of Borasi [41], rather than pitfalls in a learning trajectory. Di Martino et al. [3] point out that Ball and co-authors [1] distinguish between the knowledge needed to diagnose incorrect strategies or to understand correct but nonstandard ones (belonging to SCK), and the knowledge needed to be able to prevent typical errors (belonging to the PCK). The notion of IK is related to the former idea of “error” and could be located, as stated above, in the area of SCK, but on the boundary to the CCK. Indeed, the boundaries in the model represented in Figure 1 should not be considered clear-cut separations but rather blurred boundaries [42].

To introduce the notions of SIK and inclusive SIK, we first discuss from a semiotic point of view two paradigmatic examples proposed by Ribeiro et al. [6] and Galleguillos and Ribeiro [43] in grounding the need for IK for teachers. In this way, we aim to show how the notion of SIK supports and broadens the notion of IK.

The first example we want to discuss here refers to a task used by the authors [6] in prospective teachers’ courses in order to assess and develop their IK. The participants were asked to first solve a problem and then to interpret some examples of students’ solutions to the same problem. The problem at stake was the following: If we divide five chocolate bars equally among six children, what amount of chocolate would each child get? Mariana’s answer (Figure 2) could be considered a paradigmatic non-standard example that shows the need for a strong IK (e.g., [3,6]).

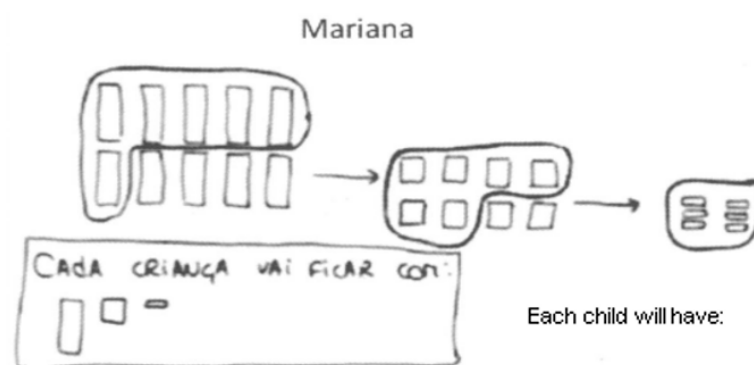


Figure 2. Mariana's solution to the task [6] quoted by [40] (p. 56).

The need for strong SIK becomes evident in considering the answer given by prospective teacher 1 (PT1) [6], as she has to interpret Mariana's solution: "Mariana's solution cannot be understood, so the first question would be, what does this representation mean? After listening to her answer, I would try to show her my own representation, in order for us to better arrive at the solution together." [6] (p. 10). PT1's answer testifies to her/his lack of SIK ("What does this representation mean?") that hinders her/his conceptual understanding ("cannot be understood"). To understand Mariana's strategy, one needs to accomplish a conversion between semiotic registers, and only after this step can the conceptual IK be used to interpret the student's strategy. We notice that the second prospective teacher (PT 2) mentioned by Ribeiro and coauthors [6] stresses that Mariana's solution is "very confusing" and that "the reasoning paths are very disorderly and lead to confusion" (p. 10). Moreover, in this case, the confusion seems to be due to the semiotic difficulty to interpret Mariana's representation and to accomplish the conversion of this representation in the arithmetic register of fractions.

Mariana's solution is very interesting from a semiotic point of view and brings to the fore two different difficulties, one lived by the student and the other lived by the teacher. The student is probably unaware that she is lacking a full understanding of the notion of fractions as expected by Duval's semio-cognitive perspective that requires the pupil to deploy the semiotic functions mentioned in Section 2.1. The solution provided by Mariana does not involve the coordination of semiotic transformations via treatments and conversions and it is confined to a series of treatments in the iconic semiotic system. Mariana seems bridled in the cognitive paradox and she is identifying the abstract and inaccessible notion of fraction with the chocolate bars represented by the icons drawn in Figure 2. Mariana's solution is, thus, restricted to treatment transformations. Looking at Fandiño Pinilla's five components—conceptual, strategic, algorithmic, communicative, and semiotic—that make up mathematical learning [20], Mariana effectively handles the strategic component of learning. Granted that the five components are distinguishable but inseparable, Mariana deploys a very effective and creative strategic thinking connected to a correct conceptualization of fractions, understood as the result of a division. Mariana's algorithmic procedure, confined to iconic representations, is deficient in terms of abstraction from the context of chocolate bars and generalization. Indeed, she does not arrive at a clear quantitative solution expressed with a rational number. Furthermore, the communicative aspects of the solution, both argumentative and explanatory, are somehow obscure. In short, Mariana's narrow semiotic approach to the problem does not allow her to overcome the cognitive paradox and reach a fully-fledged understanding.

A teacher lacking suitable SIK cannot delve into Mariana's reasoning and he/she could be perplexed by her solution. Moreover, without SIK, the teacher cannot unravel the connections between the conceptual, strategic, algorithmic, and communicative components of learning established by the underlying semiotic one. PT1 correctly singles out the semiotic issue in Mariana's solution, but she does not cast her representation into a semiotic perspective and analyze the involved components of learning. For example, the fact that

“the solution cannot be understood” is related to the communicative component related to Mariana’s semiotic choice. PT2 expresses the lack of communicative learning—both argumentative and explanatory—and deems the reasoning hindered and confusing, without interpreting what this confusion is and where it comes from. However, the reasoning paths are very ordered but difficult to recognize and improve with effective feedback without an appropriate SIK.

Mariana’s solution is very creative and breaks [44,45] the didactical contract; that is, the set of mutual interpretations, obligations, and expectations that bind the students and the teacher toward a mathematical task. In fact, Mariana does not put forward a formal solution in the symbolic arithmetical language as a teacher would usually expect. A lack of SIK on the part of the teacher could re-establish the norms of the didactical contract to direct Mariana’s solution to the one expected by the teacher, thereby hindering her meaningful and creative personal learning. Of course, the solution is incomplete and needs to be developed further (but by exploiting Mariana’s knowledge conveyed in the solution outside of the constraints and limitations of the didactical contract).

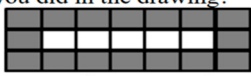
The second example that shows the need for SIK is taken from Galleguillos and Ribeiro [43].

In this case, prospective mathematics teachers were asked first to solve and then interpret students’ solutions to the task represented in Figure 3.

Imagine you have white and grey square tiles of the same size. We make a row with the white tiles and then surround the white tiles with grey tiles, as the drawing shows:



1. How many grey tiles would you need if you had 1320 white tiles and wanted to surround them in the way you did in the drawing?



2. How many grey tiles are necessary for any number of white tiles?

Figure 3. Tiles problem [46] quoted by [43] (p. 3283).

The solutions teachers had to discuss were different: two of them were wrong and were based on erroneous proportional reasoning, while one was correct.

The authors classify the teachers’ interpretations according to four different categories: (I) feedback on how to solve the problem; (II) confusing feedback; (III) counterexample as feedback; (IV) superficial feedback [43] (pp. 3284–3285). In analyzing the answer given by the teachers belonging to the second category (Figure 4), a semiotic lens seems to be able to explain why the feedback provided by these prospective teachers is confusing.

Indeed, the conversion between the representation that correctly shows the aspect that the student should become aware of to understand why proportional reasoning does not work in this case, and its representation in the register of natural language fails. Indeed, the teachers correctly represent that the six boundary tiles always remain the same and that the proportional thinking is valid only if “the edges” (that means the two external columns) are not considered, but as they try to express this aspect in the register of the natural language, they fail to create a correct correspondence between the pictorial representation (the number of “edge tiles” neither increases nor decreases when the number of white tiles varies) and the discursive representation (“the gray tiles do not increase by the same amount as the white tiles”). Thus, the teachers have chosen a suitable first semiotic representation to provide feedback to the student, but they fail to convert it into another semiotic register and this makes their feedback confusing.

An exhaustive understanding of the task requires an effective SIK that takes into account the intertwining of the semiotic functions described in Section 2.1. First of all, the immediate conversion from the pictorial semiotic system to symbolic arithmetic, in which the student carries out the proportion, could be driven by the didactical contract. They probably assume that the teacher deems acceptable only the formal solution carried out via arithmetic calculations, thereby losing the meaning of the arithmetic symbolic

representation in its reference to the mathematical object. This approach to the solution of the task disregards the important role of picking out the distinctive traits in relation to the solution of the mathematical object, offered by the pictorial representation. In fact, although the task is inscribable in the realm of arithmetic problems, it entails a strong algebraic relational nature that is the key to the solution. The relational distinctive trait is expressed by the structure of the pictorial representation that allows the student to recognize the functional relationship between the white and the gray tiles. There are two rows of five gray tiles above and beneath the row of white tiles. There are also two columns of three gray tiles on the left and on the right of the white row. The Sinn of the pictorial representation—that is, the meaning the representations take within the semiotic system—allows the student to choose the relational distinctive trait that binds the white and gray tiles. At this point, a possible solution to solve the task is to perform a series of treatments to see how the representation of the number of tiles changes for example with three, four, six, and seven white tiles, and work out the general relation between the white and gray tiles. The student can then carry out a conversion in natural language to describe and grasp the fact that the number of gray tiles is double the number of white ones plus six. A further conversion in the arithmetical symbolic language for the case required in the problem leads to $1320 \cdot 2 + 6$ that after trivial calculations (treatment) leads to the number of gray tiles corresponding to 1320 white ones. The transformations between the pictorial system of signs, natural language, and arithmetic symbolic system of signs involve Radford's [15] levels of algebraic generalizations: from the factual level related to a specific number of tiles in the pictorial system to the contextual level that is bound to a specific number but recognizes the general schema beyond the particular case (in the coordination of natural language and pictorial representations). SIK that is able to look at the possible configurations of semiotic functions would allow the teacher to delve into the students' learning components outlined by Fandiño Pinilla [20] and provide effective feedback. In this example, we can see how the semiotic component creeps in the other four components as the student faces the task, thereby allowing the teacher to both understand her/his behavior and provide suitable feedback.

d) Resolver por proporción la actividad es decir cada 5 baldosas blancas hay 16 grises

$$\frac{5}{16} = \frac{1320}{x}$$

$$x = 4224$$

lo menos de abordar el problema es erróneo ya que no se usa el pensamiento proporcional ya que la cantidad de baldosas grises no aumenta en la misma cantidad que las baldosas blancas por lo tanto no son proporcionales

Entonces si podemos afirmar que las baldosas que están arriba y abajo de las baldosas blancas están en proporción si contamos los bordes

después preguntarle ¿que pasa con los bordes la cantidad de baldosas en los bordes ¿aumenta o disminuye con las baldosas blancas?

Para llevar a que el alumno haga la conexión entre la cantidad de baldosas superior e inferior más la baldosa de los bordes, de esta manera se genera una generalización para cualquier número de baldosas blancas.

Perform the activity by proportion, that is, for every group of 5 white tiles, there are 16 grey tiles.

$$\frac{5}{16} = \frac{1320}{x}, x = 4.229.$$

The way to approach the problem is erroneous, since proportional thinking should not be used, because the number of grey tiles does not increase by the same amount as the white tiles. Therefore, they are not proportional. So, yes, we can say that the tiles that are above and below the white tiles are in a 1:2 ratio, not counting the edges.

Then ask what happens with the number of tiles at the edges when the white tiles increases or decreases? Lead the student to make the connection between the number of upper and lower tiles plus the tiles at the edges before prompting him/her to generalise for any number of white tiles.

Figure 4. Teachers' confusing feedback to the student who assumed a proportional relation between white and gray tiles [43] (p. 3285).

We conclude that in becoming aware of the semiotic constraints in mathematics, teachers would also become aware of the need for specific SIK. Indeed, the examples

we discussed here show the importance of SIK as a special kind of IK that enriches and supports the teachers' interpretative competencies.

We can characterize *SIK* as the knowledge needed by teachers in order to interpret students' answers (be they standard or non-standard), as well as students' behaviors, and give appropriate feedback to them, when conceptual knowledge is hindered and, thus, remains hidden, behind difficulties related to patterns of sign use and production, including individual creativity in sign use. In this study, although we envisage this theoretical tool as a building block of a mathematics teacher's specialized knowledge, we do not focus on how the teacher uses SIK but on its interpretative functioning from a general point of view.

SIK can be considered as underlying to the conceptual IK and, thus, it belongs to SCK and also concerns the CCK because it is related to the ontological and epistemological realm of mathematics. However, it also crosses the boundary of the PCK. Indeed, if KCT is related to "the ways of representing and formulating the subject that make it comprehensible to others" [39] (p. 9), then the choice of semiotic resources in an inclusive perspective concerns the KCT. On the other hand, if KCS is related to the knowledge of students' specificities, including those of students with special educational needs, then the awareness of possible semiotic pitfalls specific to such students concerns the KCS.

We introduce *Inclusive SIK* as the SIK needed by the teacher to provide interpretations of (and feedback to) each student's response in the context of equal opportunities for all students. SIK has an intrinsic inclusive potential for all because the semiotic perspective is characterized by a web of transformations between a broad set of semiotic registers of representation and each student moves in this web following different configurations, according to her/his way of knowing. Thus, inclusive SIK has crossed the boundaries between SCK and PCK in that it allows the interpretation of KCS and KCT.

In the next section, we show, through a few episodes involving students with special educational needs, how the difficulties and the strategies employed to overcome them are specific to each student, dependent on the way he or she manages both the semiotic resources provided by the teacher and those deployed by the student himself or herself. Then we characterize the type of inclusive SIK that the teacher would need to interpret the student's response and provide suitable feedback to her/him.

2.4. Research Question

In this phase of the study, we explore the interpretative potential of SIK in a special needs learning context in order to single out a suitable lens to understand the students' cognitive behaviors in terms of obstacles and potentials. Disregarding the semiotic aspects, we could miss some important causes and conditions behind the students' mathematical experiences and lack appropriate feedback. Our research question is, thus, the following: How does SIK allow us to understand the student's behavior when facing a mathematical task in a special needs educational context and provide effective feedback?

2.5. Experimental Setting

The data collection is based on a qualitative method in which students are exposed to mathematical tasks. We adopt a participant observation strategy [47,48] where the researcher is both an observer and participant in the activity. The research method we follow recalls the naturalistic approach [49] to document and study activity, focusing on the teachers or the learners as they cooperate in mathematics meaning-making processes. The individuals involved in the experiment are students with mathematical learning difficulties. The choice is driven by the need for significant and varied situations, in terms of both learning obstacles and divergent thinking, to test the interpretative efficacy of SIK.

The study was carried out in Italy, where education starts from 6 years of age and is divided into primary school (from the 1st to 5th grade), lower secondary school (from the 6th to 8th grade), and upper secondary school (from the 9th to 13th grade). In order to investigate the importance of SIK regardless of the school level, we felt it was significant to analyze the students' behaviors from different grades. Thus, the data involve a group of lower and

upper secondary school students with special learning disorders (SLD) attending an after-school specialized for SLD learners, and consist of notes taken by the teacher–researcher during the activity. The data collected come from the mathematical activities the students carried out with the second author of this article. The choice of collecting data from an after-school environment rather than a standard secondary school classroom is dictated by the need to expose students to an environment free of the constraints of the didactical contract, fear of assessment, and time limitations to carry out the tasks and competition. This approach allows students to act freely and in accordance with their learning style. The activities included both direct teaching of mathematics topics and mathematical problems that exposed students to standard and non-standard semiotic representations. Problems were used to trigger divergent and unexpected behaviors. The teacher–researcher worked with groups of 3–4 students. The teacher–researcher’s intervention with the student was not transmissive but mainly dialogic, creating an environment of inquiry. Furthermore, the teacher–researcher offered support to foster self-confidence and self-efficacy in students whose learning disorders could undermine their positive affective attitude toward mathematics, thereby overshadowing the true nature of their learning. In the data collection, the teacher–researcher did not bring in any interpretative process in terms of IK or SIK, to test the theoretical tool a posteriori of the student activity without adding the interpretation of the teacher as a further variable. Indeed, at this stage of our research, the focus is on the theorization of SIK as a tool needed by the researcher in mathematics education. As we aim to show the discrepancies that can arise when considering the teacher’s IK without distinguishing the semiotic aspects, our analysis of the examples is undertaken, showing the possible interpretations that could be given by the teachers with and without strong SIK. We investigate the efficacy of the theoretical lens looking at examples involving SLD students. The next step of this line of research involves teachers’ acquisition of SIK and how they are able to use it in their everyday school practice.

We selected six episodes categorized into three groups according to the semiotic features involved in their interpretation according to SIK. All the episodes concern students diagnosed with SLD, and are taken from the professional experience of the second author of this article as an operator in an after-school specialized for children with SLD. For the protection of privacy, the names of the students are fictional.

3. Results

In this section, we will describe and comment on six episodes, divided into three groups, with the aim to put into evidence the need for a solid SIK within an inclusive setting.

Group 1: semiotic aspects in the specific case of symbolic systems.

Episode 1 “... There with those p’s and q’s I make a mess ...”.

Emilio is dyslexic and is attending his first year at a scientific high school. Emilio has always been a successful student and his performance in mathematics in the previous school cycle has always been good. The school year has just started and the teacher has introduced a new topic, something Emilio has never heard of before: logical connectives and truth tables. Emilio is confused and is never able to fill out a truth table in the expected way. What is worse, however, is the sense of disorientation he feels as he realizes that he cannot establish the truth values of statements constructed through logical connectives. One afternoon, disconsolate, he says, “In these logic things I didn’t understand anything ... already there with those p’s and q’s I make a mess ...” and opens the book in which he sees the notation used for truth tables (similar to that shown in Figure 5).

Episode 2: From the blackboard to the notebook.

Carola is a student diagnosed with SLD; she is in the ninth grade. Carola has always had great difficulties at school and mathematics is one of the subjects that she struggles with the most. Carola is at school, and the teacher has just started a frontal lesson on a new topic: proportions. To introduce students to the symbolic notation of a generic proportion, the teacher writes the following expression on the blackboard: $a:b = c:d$. Carola, who

is taking notes in her notebook, transcribing what the teacher writes on the blackboard, writes: “ $a:b = c:b$.” The lesson goes on and the main properties of the proportions are also addressed, for example, the “fundamental property”, according to which, given a generic proportion $a:b = c:d$, the following equality always applies: $ad = bc$. At the end of the lesson, the teacher assigns some exercises to be done for the next lesson. In the afternoon, Carola will go home and, using her notes, set about doing the assigned exercises. However, as happened in the transcription of the generic proportion into symbolic form, the risk that there may be other exchanges between the letters b and d is present (and high).

p	q	$p \wedge q$	p	q	$p \vee q$
V	V	V	V	V	V
V	F	F	V	F	V
F	V	F	F	V	V
F	F	F	F	F	F

Figure 5. Type of notation used in Emilio’s book.

Analysis of group 1 episodes.

In the case of a diagnosis of dyslexia, some of the typical errors involve the interchange of signs that are visually the same but differently oriented in space—such as the letters in lowercase b , d , p , and q —or signs that sound similar—such as v and f [50]. Thus, Emilio’s statement “already there with those p and q I mess up” is not only not surprising but could (should?) have been predictable. If we then make a finer analysis of the notations commonly used for connectives and truth tables, we realize that the problematic signs in this type of argument are many more, for example:

- One of the conventions by which truth values allowed in classical logic are indicated involves the use of the letter “V” for the value “true” and the letter “F” for the value “false.”
- The connectives $\vee$ (OR) and $\wedge$ (AND) (themselves confusable with the letter V);
- The right ($\Rightarrow$) and left ($\Leftarrow$) implications.

A superficial analysis of the activities performed by Emilio and Carola might lead one to think that they did not understand the meaning of logical connectives and proportions, respectively. Given the semiotic perspective we presented in Section 2, it is correct to say that there was no conceptual acquisition. The lack of conceptualization can be traced back to the incorrect implementation of semiotic systems in logic and algebra rather than to factors intrinsic to the cognitive functioning of the two students. On the other hand, the teacher’s lack of solid SIK would not allow for the prediction of such difficulty. The dyslexia-related specificity of Emilio and Carola requires making a selection within the symbolic system to prevent reversal between certain pairs of symbols as shown above. The difficulty of the two students precedes semiotic transformations and concerns the management of Sinn—that is, the meaning that representations take on within the symbolic system—to understand meaning and access mathematical objects according to the structure of the semiotic system. It is clear that, from a mathematical point of view, the choice of letters is absolutely irrelevant, but in the relationship between the student’s characteristics and semiotic resources, it can lead to significant differences, in some cases compromising learning processes. So, the SIK required for appropriate interpretation of the student’s behavior in this context would be, yes, specialized knowledge (SCK), as it relates to the possible ways of conceptualizing mathematical content. However, it would also be inclusive, as it is necessary to ensure equal opportunities for all students. In this sense, it involves elements related to PCK since (1) an inclusive SIK appropriate to this context requires KCS to outline possible difficulties in handling semiotic representations related to the student’s specificities, and (2) at the same time it should single out teaching modes that reduce these difficulties as much as

possible, addressing KCT. In addition, the possible difficulties that (at least) some students might encounter with the management of the symbolic system involved could provide an opportunity for a discussion about the arbitrariness of the signifier–signified relationship when it comes to symbolic language, thereby providing learning opportunities for all.

Group 2: the calculator.

In the case of a diagnosis of a specific learning disorder, the difficulties associated with the acquisition of certain automatisms, and frequent weaknesses in short-term memory and working memory often make it necessary to resort to the use of a calculator to carry out calculations. Indeed, such a tool makes it possible to bypass the purely algorithmic aspects of calculations (which for children with specific learning disorders can require a disproportionate investment of energy in which the risk of error can be high), providing space for the more strategic and conceptual aspects. On the other hand, a calculator is an object specially programmed to perform calculations of different kinds, and its operation is in part governed by design choices that often remain implicit. The following examples will allow us to reflect on the possible consequences, in terms of learning mathematics, of not considering such implicit or seemingly invisible details.

Episode 3: “but . . . then . . . 5000 is not divisible by 2”.

Claudio is a sixth-grade student with a specific learning disorder who is completing some exercises on prime factor decomposition. Claudio is a very strategic student who tries very hard and, although he has many difficulties, his performance in mathematics is quite good. The first exercise he faces asks him to decompose the number 126 into prime factors. Claudio, relying on the divisibility criteria, carefully chooses the prime number to divide by in each step and then performs the calculation with the calculator. Arriving at the end of the exercise, Claudio is excited: he feels he has figured it out and calmly moves on to the next exercise. This time, the number to be factorized is 5000, and Claudio starts confidently because he knows what to do and how to proceed. As in the previous exercise, he evaluates the possible prime divisors of 5000, determining that he can start with 2; then, he picks up his calculator and types in “5000:2 =”. In the display on his calculator, as a result, the digits “2500” appear. Claudio enters in crisis and, bewildered says, “but . . . then . . . 5000 is not divisible by 2.” Everything that seemed to be clear to him crumbles and his confidence begins to waver.

Episode 4: how much is $(-3)^2$?

Sonia is an eighth-grade girl with a specific learning disorder. Sonia is a very strategic student who tries very hard, but mathematics is a subject in which she has particular difficulties. The teacher has recently started tackling the set of integers, and after working on algebraic sum and multiplication operations, she assigned some exercises on calculating powers. The first exercise asks to calculate $(-3)^2$. Sonia, after using the calculator, writes -6 as the result but knows something is wrong because a different value is given in the book. Sonia tries and tries again; she is also told that “she should not do the multiplication 3 by 2 but should multiply the base by itself as many times as the exponent says” but there is nothing to be done: she keeps ending up with -6 . What is going on?

Sonia, who is doing the calculations with her calculator, types “ $-3 \times -3 =$ ”, but in her calculator, the second “ $-$ ” goes in place of the multiplication sign before it. Thus, Sonia inputs into the calculator -3×-3 , but obtained $-3-3$. Once this was clarified, Sonia began to carry out the power calculation by first evaluating the sign and then carrying out the power of the absolute value of the base.

Analysis of group 2 episodes.

As anticipated in the introductory premise of this second group of examples, for students with SLD, the use of the calculator is often what enables the development of fluid reasoning, which would otherwise be at risk of being disrupted due to fragilities at the level of short-term memory and working memory.

The development of fluid reasoning and strategic aspects, as presented in Section 2, cannot be divorced from semiotic learning, with respect to which the introduction of the calculator requires the integration of an additional semiotic system. In fact, skillful and functional use of the calculator is not an obvious and taken-for-granted starting point but is a major educational goal that should receive explicit and focused attention. Examples of specific activities and reflections along these lines can be found, for example, in Fandiño Pinilla [20] and Arrigo [51].

Claudio's episode highlights how the calculator enables a sign system specific to this device that undermines Claudio's conceptual learning. In fact, he went from an attitude of confidence and control in performing the factorization in the prime of a natural number to one of insecurity and questioning of a concept he had acquired. This conflict situation was further reinforced by the didactical contract, whereby Claudio uncritically delegates to the formal representation of the calculator the real mathematical meaning he had already internalized.

On the other hand, Sonia's episode raises a question that is far from trivial; as an example, by entering the same sequence of keys as Sonia's, we show in Figure 2 the different behaviors that the calculator built into the Windows operating system has, depending on whether the standard (Figure 6a) or scientific (Figure 6b) mode is set; in Figure 7, the behavior of the online calculator at <http://www.calcolatrice.io/> (accessed on 30 December 2022) is shown (Figure 7a refers to before the "=" key is pressed; Figure 7b refers to after).

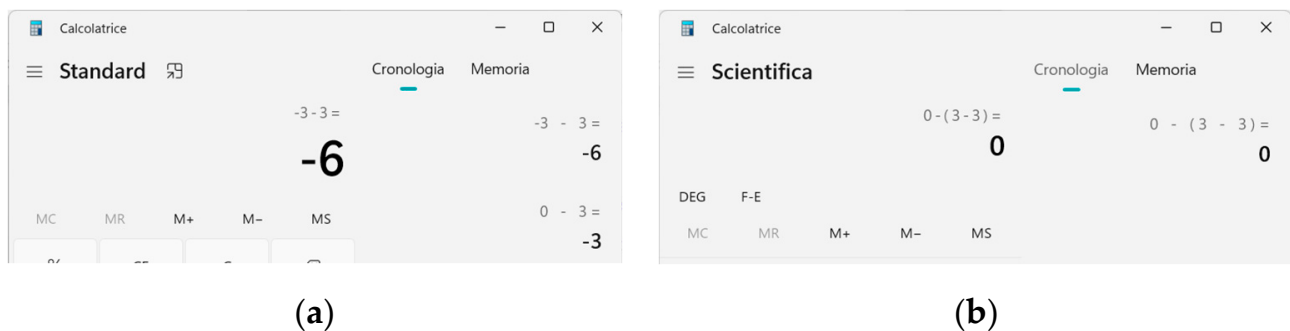


Figure 6. Output after typing " $-3 \times -3 =$ " in the calculator of the Windows operating system s (a) in standard mode, (b) in scientific mode.



Figure 7. (a) Output after typing -3×-3 in the online calculator at <http://www.calcolatrice.io/> (accessed on 20 November 2022); (b) outcome after pressing " $=$ ".

The use of the calculator introduces an additional symbolic-type semiotic system among those that the student must coordinate through (the treatment and conversion functions) and that the teacher should know how to interpret.

The first element that emerges from the analysis of the episodes related to the use of the calculator is the student's difficulty in recognizing the meanings of signs in relation to the structure of the semiotic system that identifies elementary signs and regulates the

construction of representations from these elementary signs. In using the calculator, the student should recognize the Sinn, i.e., the meaning that representations take on within the semiotic system.

In Claudio's episode, there is an interference between the meaning of the elementary sign “,” in the calculator's sign system (in which “,” is the sign used as a thousands separator and “.” as a decimal separator) and the symbolic arithmetic sign in use in Italy (in which, in contrast “,” is the sign used as a decimal separator and “.” as a thousands separator).

In Sonia's episode, the meaning of signs (Sinn) in the operations follows a specific syntax different from that of the symbolic arithmetic system used by Sonia. In fact, the representation “ -3×-3 ” takes on two different meanings in the calculator system and the arithmetic symbolic system. The situation is further complicated by the fact that this expression can take on different meanings as the type of calculator in use changes (Figures 2 and 3). In Sonia's case, “ -3×-3 ” in the arithmetic symbolic system represents repeated multiplication; instead, the calculator interprets it as subtraction. On the other hand, the adult intervention telling Sonia that “she should not do the multiplication 3 by 2 but should multiply the base by itself as many times as the exponent says”—the typical interpretation of this kind of error by teachers—proves ineffective. Indeed, their suggestion is disconnected from the semiotic activity Sonia is dealing with.

With the introduction of the calculator, Claudio and Sonia must coordinate at least two semiotic systems, i.e., of the calculator and the symbolic arithmetic system, activating a game of interpretation that recognizes the specific Sinn of each semiotic system and coordinating them while maintaining a reference to the common mathematical object. The calculator is configured, on the one hand, as an indispensable tool for students with SLD to perform calculations, and on the other hand, as a tool that enables the evolution of their semiotic learning.

The introduction of a tool that supports a student's learning requires the teacher's attention to support this evolution of semiotic learning resulting from the consequent enlargement of the representational potential it introduces.

The analysis of this second group of examples also highlights the need for the teacher to possess a solid SIK and the fact that it is inherently inclusive.

Again, we can see that SIK appropriate to manage the digital context of the calculator requires SCK. In this case, SIK is indeed connected with the conceptualization of operations with digital devices for the computation of mathematical algorithms, regardless of the specificity of the classroom context. However, it would also be inclusive, insofar as:

- It focuses on the specificities of students with specific learning disorders, in reference to the management of the semiotic system in which the digital computation is embedded, involving the KCS;
- It requires careful reflection on the introduction and use of the device in the classroom, involving the KCT. Finally, managing the links between SCK and PCK, it goes through the CCK, which allows for the recognition of arbitrariness in the signifier–signified relationship behind the choice of mathematical symbols. Such arbitrariness is assumed at the level of mathematical culture (Anglo-Saxon and Italian) in the notion of Sinn introduced by Duval as the meaning that representations take on within the sign system. SIK in its interaction with PCK and SCK also assumes (in the context of digital devices) the role of inclusive SIK.

Group 3: semiotically divergent thinking.

This group includes two episodes that provide interesting information that is in line with the strengths usually found in people with specific learning disorders, such as the ability to make unconventional connections or the ability to solve problems that require imagining possible solutions [50], which we might call “semiotically divergent” thinking.

Episode 5: 28:4-5?

Roberto is a high school student with a specific learning disorder; he is attending grade 13. Roberto is a successful, very bright student with a great passion for mathematics.

During an informal conversation about visualization, he is shown a question that is similar to the following (Figure 8):

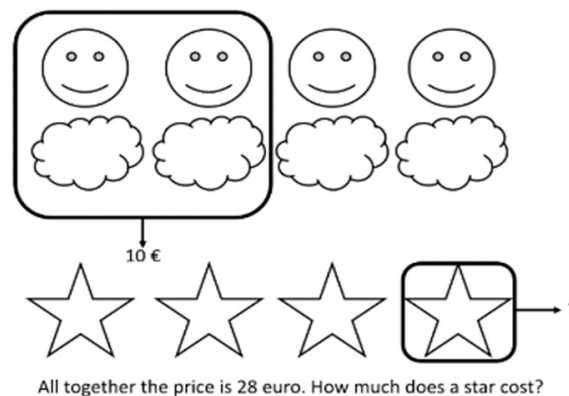


Figure 8. The task shown to Roberto.

The calculation he employs to solve the question is $28:4=5$, which, at first glance, is confusing and “seems” to have something wrong with it. In reality, Roberto’s calculation is just an expression of a particular way of perceiving the proposed situation, completely focused on the objective (which is the price of the single star). We exemplify such a way of looking at the situation in Figure 9:

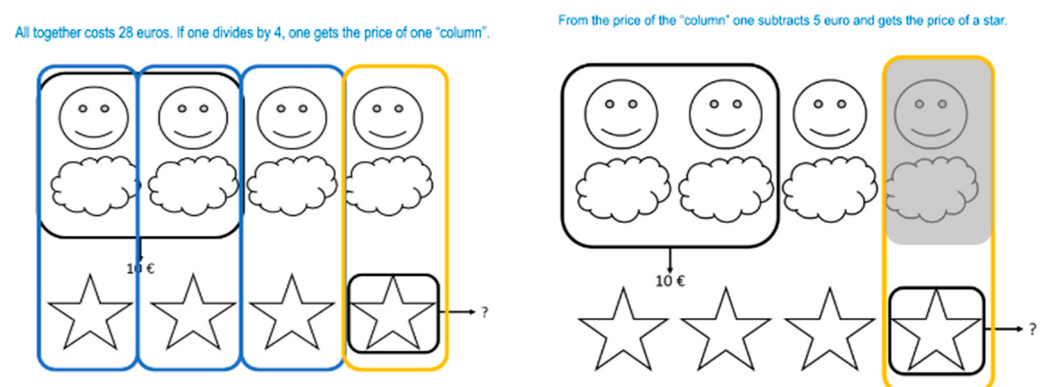


Figure 9. Roberto’s perception of the task, as unpacked by the authors. The figure is not the student’s drawing but the author’s representation of his reasoning behind the calculation.

Episode 6: the decomposed track.

The protagonist of this episode is Romeo, an eighth-grade student with a specific learning disorder. Romeo has very fast and responsive thinking and is a brilliant student but he sometimes tends to be rushed. We are heading toward the end of the school year, and Romeo, in order to train for the exam, is performing the question shown in Figure 10, taken from the National Test scheduled at the end of the lower secondary school proposed in the 2014/2015 school year by the National Institute for the Evaluation of the Education System (INVALSI). Below, we reproduce the question accompanied by the corresponding official reading guide; of course, Romeo only had the item to solve (i.e., only what is written in the left column of the table in Figure 10).

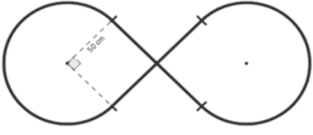
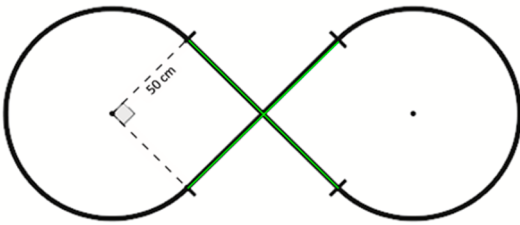
ITEM	CHARACTERISTICS	DESCRIPTION AND COMMENTS
D17. The figure represents the diagram of a track formed by: • two circumferential arcs with radius of 50 cm • two straight sections of 100 cm each, perpendicular to each other, at the midpoint.  What is the length of the track? Write down the calculations you do to find the answer and finally report the result. <		

Figure 10. Proposed question (left column) accompanied by its description (Retrieved from https://invalsi-areaprove.cineca.it/docs/attach/2015_guida_L08_DICEMBRE.pdf, p. 23, accessed on 20 November 2022, authors' translation).

In Figure 11, we show the screenshot of Romeo's solution:

D17. The figure represents the diagram of a track formed by:

- two circumferential arcs with radius of 50 cm
- two straight sections of 100 cm each, perpendicular to each other, at the midpoint.



What is the length of the track?
Write down the calculations you do to find the answer and finally report the result.

$100 + 100 = 200$
 $\pi \times \text{radius squared} = 7850$
 $7850 : 2 = 3925$ $7850 + 3925 = 11775$

Result: ca.: 11975 cm

Figure 11. Romeo's solution.

Admittedly, looking at the numerical resolution written by Romeo, we notice some problems: the units of measurement are not made explicit, the formula chosen is the one for calculating the area of the circle (instead of the one for the length of the circumference), and the value of π is approximated. However, the reasoning proposed by Romeo is perfectly in line with the (correct) expected reasoning. In fact, the choice of calculations to be performed highlights very well his solving reasoning, which we make explicit in Figure 12:

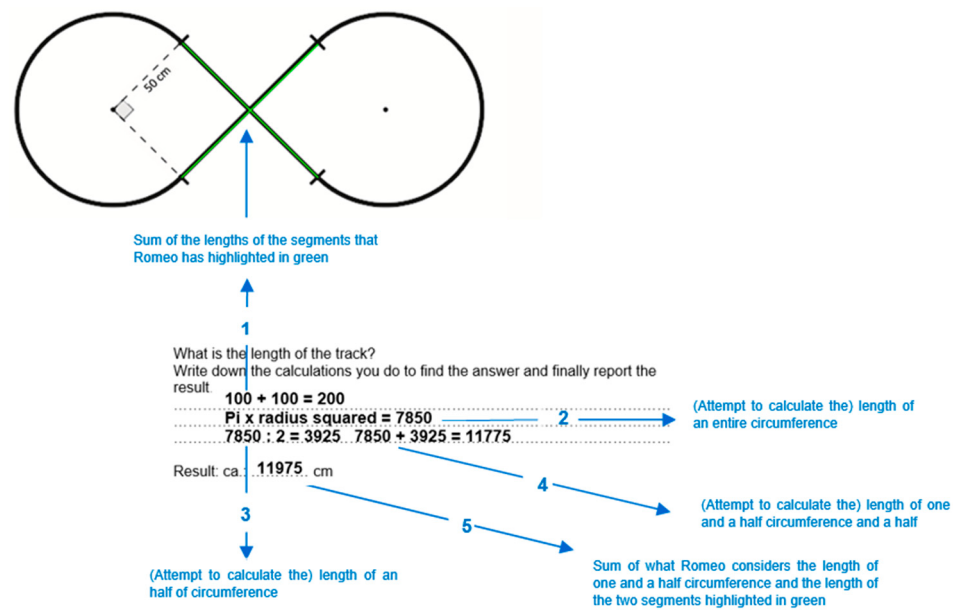


Figure 12. Authors' comments to Romeo's solution.

Analysis of group 3 episodes.

This block of episodes shows students' creative behaviors in dealing with problems. We specify that such creativity did not develop on the basis of mere abstract thinking but is intertwined with their semiotic choices in terms of distinctive features of the mathematical object, and treatment and conversion transformations. The student's behavior is in line with Duval's principle claiming that there is no noesis without semiosis. Regarding specific learning disorders, the solution strategies deployed by the students are the outcomes of the dialogue between the specificities of mathematics cognition and the specificities of students in terms of their needs and potential.

The interesting aspect common to the two episodes is the intertwining of the choice of distinctive features of mathematical knowledge at play and the subsequent semiotic transformations (treatment and/or conversion). The choice of the distinguishing features is the result of a subtle interpretative ability on the part of the student who is able to pick out the key to the solution from the Sinn offered by the initial representation.

The representation of the problem shared with Roberto, which would suggest adding up the price of the two blocks of EUR 10 each, subtracting the resulting amount from the total, and then dividing by 4, does not satisfy his way of reasoning. Roberto mentally operates a treatment so that the Sinn of the new representation immediately allows him to interpret the problem in terms of division (diagram on the left of Figure 5). At this point, he grasps the distinctive features of the division and dwells on the single column formed by the face, cloud, and star. He performs a new treatment to return to the original figure where he recognizes that the square with two smiles and two clouds costs EUR 10. He performs a conversion, and in the symbolic arithmetic language, he performs the operation $10:2$, which results in 5. He performs a further conversion to the figure with columns and knowing that the smiley face and cloud are worth 5, he subtracts this value from the entire column. Finally, after a conversion, the student arrives at the following solution: $28:4=5$. We observe that Roberto performed all of these semiotic functions mentally, without resorting to pen-and-paper representations. The key element of Roberto's solution strategy is the first treatment that turns out in a Sinn consistent with his personal meaning. Subsequent semiotic transformations support his reasoning to arrive at the solution.

The approach followed by Romeo with the track problem is similar to that followed by Roberto in the sense that Romeo identifies the distinctive traits of the mathematical object when he highlights the two straight lines in Figure 7. At this point, the Sinn of the new representation allows Romeo to interpret the representation as two arcs of a

circle, each measuring $3/4$ of the circle itself. Having identified the distinguishing features of the objects involved in the problem, Romeo performed a conversion to the symbolic arithmetic language to carry out the calculations leading to the solution. However, he makes a mistake, typical of specific learning disorders, when automatically retrieving the formula from long-term memory. Inadequate SIK might hinder the recognition of Roberto and Romeo's solution strategies by proposing explanations for the student's behavior similar to those presented in Mariana's example (paragraph 2.2). In particular, the trigger of the problem-solving strategies was the choice of effective distinguishing features that supported the subsequent treatments and conversions. Thus, SIK becomes indispensable for understanding and accommodating the processes deployed. From an inclusive perspective as differentiation for all students, a robust teacher's SIK is critical to transforming the peculiarities of students with specific learning disorders into resources and strengths for all students. In fact, the teacher's SIK can embed the configuration of semiotic operations specific to students with SLD into the broader web of transformations that encompasses the cognitive styles of all learners. In this group of examples, the choices of distinctive features play key roles.

Again, we can see how SIK appropriate to a teaching–learning environment would be SCK, but it would also be inclusive when it focuses on the specificities of students, in recognizing divergent problem-solving strategies. It engages both the KCS and the KCT when the teacher values the student's non-standard strategy as a learning resource for all.

4. Discussion and Final Remarks

In this article, we highlighted a particular aspect of teachers' interpretative knowledge. Referring to Duval's semio-cognitive approach, we identified semiotics as the defining element of mathematics-specific cognitive functioning. Thus, an underlying theoretical lens that amplifies the understanding of mathematics teaching/learning processes and the consequent feedback to be provided to students in their educational journey. Making use of Fandiño Pinilla's [20] proposal, we have shown that without a strong and appropriate SIK, the teacher's IK risks focusing on conceptual learning, without considering its intimate and intrinsic link to semiotic learning, risking leaving out important aspects in the interpretation of students' behavior when faced with mathematical tasks. We have also shown how the introduction of semiotics into IK allows the teacher's interpretive power to be expanded by also taking into consideration strategic, communicative, and algorithmic components. We showed how SIK is inherently configured as an inclusive SIK in its ability to consider the specificities (difficulties and resources) of each student in line with the broad approach of inclusion [24–32], understood as differentiation for all. We highlighted how SIK allows for a deepening of the teacher's specialized knowledge, creating a connection between SCK, KCS, and KCT, in terms of the teacher's interpretive knowledge by expanding that of Ribeiro et al. [5,6] who considered only the relation of SCK and CCK.

Specifically, in the second part of our work, we implemented SIK in its inclusive aspect, interpreting three groups of two episodes each for a total of six episodes with SLD students.

The first two groups of episodes highlight the intimate connection between semiotics and the conceptual component of learning introduced by Fandiño Pinilla [20], in that mathematics cognition relies on systems of signs where the meaning is two-fold. On the one hand, the semiotic representations we use in mathematics refer to the mathematical concept we want to objectify. On the other hand, semiotic representations do not have meaning in and of themselves but the meaning is also related to the positions they bear in the structure of the system of signs they belong to, thereby contributing to the Sinn. The episodes concerning the students' first encounters with symbolic language and the use of the calculator show that when their impairment does not allow them to recognize the two-fold meaning embedded in semiotic representations and the ensuing treatments and conversions, conceptual learning is undermined. In fact, they are not able to trigger the network of semiotic operations involving the choices of the distinguishing traits, treatment, and inclusion. From the first two sets of episodes, it emerges that students' difficulties are

not intrinsic and objective but depend on the nature, in semiotic terms, of the delivery to which they are exposed, which determines the relationship that is established between the student's specificities and the semiotic resources to which they access. In Carola's episode 2, in which she writes $a:b = c:b$, the inappropriate semiotic choice with respect to the student's needs does not allow her to acquire the conceptual component of proportions. The writing that Carola reports in her notebook is not wrong in absolute terms, but it does not represent a generic proportion. The same remark holds for Emilio whose conceptual component of learning is hindered by the inappropriate choices of the letters p and q that erases their meaning within the symbolic language of logic and the ensuing treatments and conversions.

Claudio and Sonia are struggling with the same type of difficulties when using the calculator. Their case is slightly different in that the signs used in the calculator to represent numbers and operations are fixed and not proposed by the teacher. Still, they have to understand the different meanings of “,” and “.” in the calculator with an Anglo-Saxon format and the Italian arithmetical symbolic language. The semiotic transformations they carry out are uncoupled from the mathematical object. The incorrect meanings they ascribe to such symbols hinder their conceptual knowledge.

The examples of the first two groups testify to the intertwining between semiotics and the conceptual component of learning at the point that Claudio and Carola cannot access propositional logic and proportions. In this respect, the cases involving Claudio and Sonia are even more interesting because the introduction of the calculator requires them to handle new semiotic systems; the one with the calculators impairs the previous conceptual learning of factorization and power. The first two groups of episodes show how learning disorders disconnect the interplay between semiotic operations, thus hindering and/or disrupting the students' conceptual learning. The correct interpretation and effective feedback of the student behaviors require—on the part of the teacher—appropriate SIK to delve into the articulated links between semiotic operations, conceptual learning, and the students' specific features. SIK is a pivotal professional knowledge that coordinates SCK with KCS and KCT, intrinsically ascribing SIK as an inclusive role.

The third group of episodes emphasizes the connection between semiotics and Fandiño Pinilla's strategic components of learning. In fact, it emerges that when the student is able to establish an appropriate relationship between the semiotic resources he brings into play and his learning characteristics, the mathematical practice is supported by semiotic operations functional to noetics in creative and unusual forms. The interplay between the choice of distinctive traits, treatment, and conversion is crucial as it is in the first two groups of episodes, even if from the point of view of strategic learning. Likewise, in the previous example, the students' semiotic divergent thinking unfolds from the Sinn of the semiotic representations they start from. Both Roberto and Romeo recognize in the Sinn the distinctive traits of the mathematical object resonant with the cognitive styles that trigger effective treatments and conversions, which lead to the solution of the task. The distinctive traits allow Roberto to focus on the objective of the task, i.e., the price of the star and Romeo on the perpendicularity of the straight lines and the portion of the circumference that contributes to the total track. Roberto's and Romeo's strategic learning cannot be divorced from the semiotic activity of the students. In the semio-cognitive perspective, we are advocating, the semiotic operations are not mirrors of the mathematical reasoning but the two aspects are inseparable. In order to recognize the student's divergent thinking and transform it into a resource for each student, the teacher needs a strong SIK to dig into the network of semiotic operations they deploy. The lack of an appropriate SIK could enhance expectations, routines, and interpretations typical of the didactical contract and draw the teacher's attention to local errors instead of the effectiveness of the overarching solving strategy. For example, in the case of Romeo, the teacher could focus on the wrong calculations that stem from the learning disorder that hinders him from recalling the correct formula from his long-term memory. The strong connection between the semiotic activity and the strategic component of learning confirms the need for SIK for the implementation

of an effective and meaningful environment for all students. This is possible due to the specialized knowledge inherent in SIK that blends the subject matter's knowledge and the pedagogical content knowledge via the coordination of SCK, KCS, and KCT that are embedded in semiotics.

The semiotic backing of SIK accounts for the inclusion as differentiation for all students due to the wide range of possible networks of semiotic transformations that include the specificities of all students as they learn mathematics. Thus, the realization of inclusive environments in mathematics is ensured by the teacher's specialized knowledge that includes SIK to provide the student with semiotic resources appropriate to his or her specificities in both activity designs and feedback. Interpreting a student's behavior without adequate SIK can be didactically very risky. In line with the constructive interpretations of misconceptions and errors proposed by several authors [52–55], we argue for the importance and necessity of further rethinking the view of error, building on what Ribeiro et al. [5,6] and Di Martino et al. [3] have done by looking to the way errors are considered by Borasi [41]. Indeed, we suggest looking at errors more as an expression of the delicate interweaving of Fandiño Pinilla's five components of learning in which semiotic learning plays a pivotal role.

We can answer our research question: How does SIK allow us to understand the student's behavior when facing a mathematical task in a special needs educational context and provide effective feedback? From the analysis of the six episodes, we can infer that SIK provides an effective interpretation tool, within a differentiation for an all-inclusive approach, to understand the SLD-student's cognitive functioning and offer suitable feedback. The interpretative process unfolds around the connection between Fandiño Pinilla's semiotic component of knowledge and the conceptual and strategic ones. The semiotic lens allows us to delve into the structure and functioning—made up of the syntax of the semiotic system and the three semiotic functions—that govern both conceptual and strategic learning. Thus, we can figure out the causes and conditions of the student's cognitive behavior that inform our interpretative processes and single out the appropriate feedback in terms of the teaching designs and learning tools for the students.

The objective of this study was to introduce a further unfolding of IK into SIK in order to enhance specialized knowledge that is able to embrace the connections of the five components of learning that characterize the richness and complexity of mathematics education. We showed the interpretative effectiveness of the SIK in the analysis of six episodes involving SLD students. The theoretical nature of the study requires further and structured empirical investigation that involves prospective and in-service teachers in order to identify the building blocks they need to acquire SIK, a set of suitable examples to work on, and a consistent activity design. The final aim is to build a training program that allows teachers to include SIK in their professional knowledge.

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