ON Q-CONVEX HYPERSURFACES IN RIEMANNIAN MANIFOLDS

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Abstract. We prove that any closed, convex hypersurface in an $(n+1)$-dimensional Riemannian manifold with $\lceil \frac{n}{2} \rceil$-positive curvature operator is a rational homology sphere with finite fundamental group. The same conclusion holds for any $\lceil \frac{n}{2} \rceil$-convex hypersurface, provided that the mean curvature satisfies a sharp pinching condition. Both results follow from more general vanishing and estimation theorems for the Betti numbers of closed q-convex immersed hypersurfaces in $(n+1)$-dimensional Riemannian manifolds, under a lower bound on the average of the smallest $(n-p)$ eigenvalues of the curvature operator.

1. Introduction

Let M be an $(n+1)$-dimensional Riemannian manifold, and let $f\colon \Sigma \to M$ be an immersed hypersurface. In this paper, we always assume that Σ is closed, connected, oriented and f is two-sided (that is, there exists a global non-vanishing unit normal vector field ν). We say that f is q-convex (resp. strictly q-convex) for some integer $1 \leq q \leq n$ if the sum of its smallest q principal curvatures is nonnegative (resp. positive) at each point $x \in \Sigma$, with respect to ν. Observe that q-convexity implies p-convexity for all $p \geq q$ and non-negativity $H \geq 0$ of the mean curvature in the direction of ν.

The special case $q = 1$ corresponds to convexity (that is, all principal curvatures are nonnegative). In fact, when the ambient space is the Euclidean space, we know that Σ is diffeomorphic to the sphere $\mathbb{S}^n$ by the work of Hadamard [10], Chern-Lashof [8], Heijenoort [12], and Sacksteder [20] (see also do Carmo-Lima [6]). The same holds in nonflat space forms due to do Carmo and Warner [7] and also in Hadamard manifolds (that is, complete, simply connected Riemannian manifolds with nonpositive sectional curvature) due to Alexander [1]. For further discussion regarding this case, we refer the reader to the survey of Lima [14] and the references therein.

The next case of 2-convexity has also attracted considerable attention. In particular, the topological classification in Euclidean space $\mathbb{R}^{n+1}$, $n \geq 3$, was established by Huisken and Sinestrari [13], who showed that such hypersurfaces are diffeomorphic either to the sphere $\mathbb{S}^n$ or to a connected sum of $\mathbb{S}^{n-1} \times \mathbb{S}^1$. Their approach relied on the mean curvature flow with surgery together with the fact that 2-convexity is preserved along the flow. As pointed out by Brendle and Huisken [4], this may no longer be true for the mean curvature flow in general ambient Riemannian manifolds. To overcome this difficulty, they introduced a fully nonlinear flow preserving, among other things, strict 2-convexity in general ambient spaces. Their work implies that compact $(n+1)$-dimensional Riemannian manifolds, $n \geq 3$, satisfying $\bar{R}_{1313} + \bar{R}_{2323} \geq 0$, and having non-empty strictly 2-convex boundary

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are diffeomorphic to a 1-handlebody. The case $n = 2$ was also treated by Brendle and Huisken [3,5].

More generally, for q-convexity with $1 \leq q \leq n-1$, Sha [22] (see also Wu [23]) proved that every compact n-dimensional Riemannian manifold with nonnegative sectional curvature and strictly smooth q-convex boundary has the homotopy type of a CW-complex of dimension at most $q-1$. As a consequence, if $q \leq \frac{n}{2}$, then the i-th real homology group of the boundary vanishes for all $q \leq i \leq n-q$.

In the present paper, we apply the Bochner technique to provide new vanishing and estimation results for the Betti numbers of q-convex immersed hypersurfaces in general ambient Riemannian manifolds, under a lower bound on the average of the smallest $(n-p)$ eigenvalues of the curvature operator. We recall that the curvature operator of a Riemannian manifold is called ℓ-positive (resp. ℓ-nonnegative) if the sum of its smallest ℓ eigenvalues is positive (resp. nonnegative). In this sense, our results may be viewed as an extrinsic analogue of the work of Petersen and Wink [19].

Our first main theorem is the following:

Theorem A. *Let $n \geq 3$ and $1 \leq q \leq p \leq \lfloor \frac{n}{2} \rfloor$ be integers. If M is an $(n+1)$-dimensional Riemannian manifold with $(n-p)$-nonnegative curvature operator and $f\colon \Sigma \to M$ a closed, connected, oriented, q-convex immersed hypersurface, then the following hold:*

(1) *The Betti numbers of Σ satisfy $b_i(\Sigma) = b_{n-i}(\Sigma) \leq \binom{n}{i}$, for each $q \leq i \leq p$.*

(2) *If either the curvature operator of M is $(n-p)$-positive at some point $x \in f(\Sigma)$ or f is strictly q-convex at some point, then*

$$b_q(\Sigma) = \cdots = b_p(\Sigma) = 0 \quad \text{and} \quad b_{n-p}(\Sigma) = \cdots = b_{n-q}(\Sigma) = 0.$$

Moreover, if $q = 1$, then M admits a Riemannian metric of positive Ricci curvature and has finite fundamental group. In particular, if $n = 3$, then it is diffeomorphic to a spherical space form.

(3) *If $b_q(\Sigma) > 0$, then every harmonic q-form is parallel. In particular, Σ supports a nontrivial parallel q-form.*

As an immediate consequence, we get

Theorem B. *Let $n \geq 3$ and $1 \leq q \leq p \leq \lfloor \frac{n}{2} \rfloor$ be integers. If M is an $(n+1)$-dimensional Riemannian manifold with $(n-p)$-positive curvature operator and $f\colon \Sigma \to M$ a closed, connected, oriented, q-convex immersed hypersurface, then the Betti numbers of Σ satisfy*

$$b_q(\Sigma) = \cdots = b_p(\Sigma) = 0 \quad \text{and} \quad b_{n-p}(\Sigma) = \cdots = b_{n-q}(\Sigma) = 0.$$

Moreover, if $q = 1$, then Σ admits a Riemannian metric of positive Ricci curvature and has finite fundamental group. In particular, if $n = 3$, then it is diffeomorphic to a spherical space form.

For $n \geq 3$, the class of $(n+1)$-dimensional Riemannian manifolds with ℓ-positive curvature operator $(3 \leq \ell \leq n)$ is different from the class of manifolds satisfying $\overline{R}_{1313} + \overline{R}_{2323} \geq 0$ for all orthonormal sets $\{e_1, e_2, e_3\}$, that is, having nonnegative second intermediate Ricci curvature Ric_2, defined for unit vectors $v \in TM$ by

$$\mathrm{Ric}_2(v) = \inf\{\sec(v, e_1) + \sec(v, e_2) : \{e_1, e_2\} \text{ orthonormal set in } v^{\perp}\}$$

Let us present some further interesting consequences of Theorem A. For example, it yields the following "sphere theorem" for hypersurfaces.

Corollary 1. *Let M be an $(n+1)$-dimensional Riemannian manifold with $n \geq 3$ and $\lceil \frac{n}{2} \rceil$-nonnegative curvature operator. Let $f\colon \Sigma \to M$ be a closed, connected, oriented, convex immersed hypersurface. If either the curvature operator of M is $\lceil \frac{n}{2} \rceil$-positive at some point $x \in f(\Sigma)$, or f is strictly convex at some point, then Σ is a rational homology sphere, admits a Riemannian metric with positive Ricci curvature, and has finite fundamental group. In particular, if $n = 3$, then Σ is diffeomorphic to a spherical space form.*

Of course the conclusions of Theorems A-B do apply also in case the ambient manifold M satisfies the relevant curvature assumption only in a neighbourhood of $f(\Sigma)$. So, for compact Riemannian manifolds with non-empty boundary we get:

Corollary 2. *Let $n \geq 3$, and let M be a compact $(n+1)$-dimensional Riemannian manifold with non-empty smooth boundary ∂M. If the curvature operator of M is $\lceil \frac{n}{2} \rceil$-positive in a neighbourhood of ∂M and ∂M is q-convex for some $1 \leq q \leq \lfloor \frac{n}{2} \rfloor$, then each connected component of $\partial M = M_1 \sqcup \cdots \sqcup M_k$ satisfies*

$$b_q(M_j) = \cdots = b_{n-q}(M_j) = 0 \quad \textit{for all} \quad 1 \leq j \leq k.$$

In particular, if ∂M is convex then each connected component of ∂M is a rational homology sphere with finite fundamental group.

When the average of the smallest ℓ eigenvalues of the curvature operator of the ambient manifold is bounded from below by a negative constant, then we prove the following:

Theorem C. *Fix integers $n \geq 3$ and $1 \leq q \leq p \leq \lfloor \frac{n}{2} \rfloor$ and real numbers $c < 0$ and $D > 0$. Let M be an $(n+1)$-dimensional Riemannian manifold such that*

$$\frac{\lambda_1 + \cdots + \lambda_{n-p}}{n-p} \geq c,$$

where $\lambda_1 \leq \cdots \leq \lambda_{\binom{n+1}{2}}$ denote the eigenvalues of its curvature operator. Let $f\colon \Sigma \to M$ be a closed, connected, oriented, q-convex immersed hypersurface. If $\operatorname{diam}(\Sigma) \leq D$, then there exists a constant $C(n, cD^2) > 0$ such that

$$b_i(\Sigma) = b_{n-i}(\Sigma) \leq \binom{n}{i} \cdot \exp\left(C(n, cD^2) \cdot \sqrt{-cD^2 i(n-i)} \right),$$

for each $q \leq i \leq p$. Moreover, there exists $\varepsilon(n,i) > 0$ such that if $cD^2 \geq -\varepsilon(n,i)$, then $b_i(\Sigma) = b_{n-i}(\Sigma) \leq \binom{n}{i}$.

Theorems A and C naturally raise the

Question. *Can we say anything about $b_i(\Sigma)$ with $i < q \leq \frac{n}{2}$? What about when $q > \frac{n}{2}$?*

This is exactly what the next two theorems establish.

Theorem D. *Let $n \geq 3$, $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$ and $1 \leq q \leq n-1$ be integers. Let M be an $(n+1)$-dimensional Riemannian manifold with $(n-p)$-positive curvature operator, and let $f\colon \Sigma \to M$ be a closed, connected, oriented, q-convex immersed hypersurface. Set*

$$c \coloneqq \min_{x \in f(\Sigma)} \left\{ \frac{\lambda_1(x) + \cdots + \lambda_{n-p}(x)}{n-p} \right\}, \tag{1}$$

where $\lambda_1 \leq \cdots \leq \lambda_{\binom{n+1}{2}}$ denote the eigenvalues of the curvature operator of M, and

$$\beta_q \coloneqq \begin{cases} q-1, & \textit{if} \quad q \leq \frac{n}{2} \\ n-q, & \textit{if} \quad q > \frac{n}{2} \end{cases} \tag{2}$$

If for some integer $1 \leq \ell \leq \min\{p, \beta_q\}$ *the (normalized) mean curvature satisfies*

$$H \leq \frac{n-q}{n}\sqrt{\frac{\ell}{q-\ell}}c, \tag{3}$$

then the following hold:

(1) $b_i(\Sigma) = b_{n-i}(\Sigma) \leq \binom{n}{i}$, *for each* $\ell \leq i \leq \min\{p, \beta_q\}$.

(2) *If strict inequality holds at some point, then*

$$b_i(\Sigma) = b_{n-i}(\Sigma) = 0, \quad \text{for all} \quad \ell \leq i \leq \min\{p, \beta_q\}.$$

Moreover, if $\ell = 1$, *then* Σ *admits a Riemannian metric of positive Ricci curvature, and has finite fundamental group. In particular, if* $n = 3$, *then it is diffeomorphic to a spherical space form.*

(3) *If* $b_\ell(\Sigma) > 0$, *then equality holds in* (3) *everywhere on* Σ *and every harmonic* ℓ*-form is parallel. In particular,* Σ *supports a nontrivial parallel* ℓ*-form.*

We emphasize that inequality (3) is sharp; see Remarks 12(4). Observe also that Theorem D complements Theorem B. As an immediate consequence of Theorem D, we obtain the following "sphere theorem" for pinched hypersurfaces.

Corollary 3. *Let* $n \geq 3$ *and let* M *be an* $(n+1)$*-dimensional Riemannian manifold with* $\lceil \frac{n}{2} \rceil$*-positive curvature operator. If* $f\colon \Sigma \to M$ *is a closed, connected, oriented,* $\lceil \frac{n}{2} \rceil$*-convex immersed hypersurface such that*

$$H \leq \frac{n - \lceil \frac{n}{2} \rceil}{n\sqrt{\lceil \frac{n}{2} \rceil - 1}}c, \tag{4}$$

where c *is defined by* (1), *and the inequality is strict at some point, then* Σ *is a rational homology sphere, admits a Riemannian metric of positive Ricci curvature, and has finite fundamental group. In particular, if* $n = 3$, *then it is diffeomorphic to a spherical space form.*

We emphasize once again that inequality (4) is sharp; see Remarks 12(5). When the average of the smallest $(n-p)$ eigenvalues of the curvature operator of the ambient Riemannian manifold is bounded from below by a non-positive constant, we obtain the following result.

Theorem E. *Fix integers* $n \geq 3$, $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq q \leq n-1$ *and real numbers* $c \leq 0$ *and* $D > 0$. *Let* M *be an* $(n+1)$*-dimensional Riemannian manifold such that*

$$\frac{\lambda_1 + \cdots + \lambda_{n-p}}{n-p} \geq c,$$

where $\lambda_1 \leq \cdots \leq \lambda_{\binom{n+1}{2}}$ *denote the eigenvalues of its curvature operator. Let* $f\colon \Sigma \to M$ *be a closed, connected, oriented,* q*-convex immersed hypersurface with* $\operatorname{diam}(\Sigma) \leq D$. *Set* β_q *as in* (2). *Then for each* $1 \leq \ell \leq \min\{p, \beta_q\}$ *there exists a constant* $C(n, \kappa_\ell D^2) > 0$, *where*

$$\kappa_\ell = \min_{x \in \Sigma}\left(c - \frac{q-\ell}{\ell}\left(\frac{n}{n-q}\right)^2 H^2\right),$$

such that

$$b_\ell(\Sigma) = b_{n-\ell}(\Sigma) \leq \binom{n}{\ell} \cdot \exp\left(C(n, \kappa_\ell D^2) \cdot \sqrt{-\kappa_\ell D^2 \ell(n-\ell)}\right).$$

Moreover, there exists $\varepsilon(n,\ell) > 0$ *such that if* $\kappa_\ell D^2 \geq -\varepsilon(n,\ell)$*, then* $b_\ell(\Sigma) = b_{n-\ell}(\Sigma) \leq \binom{n}{\ell}$*. Furthermore, if* $q \leq p$*, then there exists a constant* $C(n, cD^2) > 0$ *such that*

$$b_i(\Sigma) = b_{n-i}(\Sigma) \leq \binom{n}{i} \cdot \exp\left(C(n, cD^2) \cdot \sqrt{-cD^2 i(n-i)}\right),$$

for each $q \leq i \leq p$*. Moreover, there exists* $\varepsilon(n,i) > 0$ *such that if* $cD^2 \geq -\varepsilon(n,i)$*, then* $b_i(\Sigma) = b_{n-i}(\Sigma) \leq \binom{n}{i}$*.*

If inequality (3) is not satisfied for some $1 \leq \ell \leq \min\{p, \beta_q\}$ in Theorem D, then one can still derive upper bounds for the corresponding Betti number in the spirit of Theorem E.

The structure of the paper is as follows. In Sections 2 and 3, we provide the necessary background material and establish several auxiliary results. The proofs of the main theorems are given in Section 4. Finally, in Section 5, we examine the special case in which the ambient space is the unit sphere.

2. An auxiliary result

We start with some algebraic preliminaries inspired by Savo [21]. Let V be a real n-dimensional vector space, $n \geq 3$, equipped with a positive definite inner product denoted by $\langle \cdot, \cdot \rangle$. We denote by $\mathrm{End}(V)$ the set of all endomorphisms of V and by $\Lambda^p V^*$ the $\binom{n}{p}$-dimensional real vector space, defined as the p-th exterior power of the dual vector space $V^* = \mathrm{Hom}(V, \mathbb{R})$ of V. To each self-adjoint $A \in \mathrm{End}(V)$, we associate the endomorphism $T_A^{[p]} \in \mathrm{End}(\Lambda^p V^*)$ defined by

$$T_A^{[p]} = (\mathrm{tr}A)A^{[p]} - A^{[p]} \circ A^{[p]},$$

where $A^{[p]} \in \mathrm{End}(\Lambda^p V^*)$ is given by

$$A^{[p]}\omega(v_1, \dots, v_p) = \sum_{i=1}^{p} \omega(v_1, \dots, Av_i, \dots, v_p),$$

with $\omega \in \Lambda^p V^*$ and $v_1, \dots, v_p \in V$. The endomorphism $T_A^{[p]}$ is self-adjoint with respect to the natural inner product $\langle \cdot, \cdot \rangle$ in $\Lambda^p V^*$. For every integer $1 \leq p \leq n$, we consider the set I_p of p-multi-indices $I_p = \{\{i_1, \dots, i_p\} : 1 \leq i_1 < \cdots < i_p \leq n\}$. Let $A \in \mathrm{End}(V)$ be self-adjoint with eigenvalues $k_1 \leq k_2 \leq \cdots \leq k_n$. Before we proceed, we recall the following elementary fact from linear algebra:

Lemma 4. *For any integer* $1 \leq p \leq n$*, the following holds:*

$$\sum_{i=1}^{p} \langle Av_i, v_i \rangle \geq k_1 + \cdots + k_p$$

for all orthonormal sets $\{v_1, \dots, v_p\} \subseteq V$*.*

For each $a = \{i_1, \ldots, i_p\} \in I_p$, the associated *p-algebraic curvature* K_a of A is the number given by $K_a = k_{i_1} + \cdots + k_{i_p}$. It follows from [21, Lemma 2] that the $\binom{n}{p}$ eigenvalues $\lambda_a(T_A^{[p]})$ of $T_A^{[p]}$ are given by

$$\lambda_a(T_A^{[p]}) = K_a K_{\star a}, \quad a = \{i_1, \ldots, i_p\} \in I_p, \tag{5}$$

where $\star a \in I_p$ is defined by $\star a = \{1, \ldots, n\} \smallsetminus \{i_1, \ldots, i_p\}$. We say that A is q-nonnegative if $k_1 + \cdots + k_q \geq 0$.

Lemma 5. *Let $A \in End(V)$ be self-adjoint and q-nonnegative for some integer $1 \leq q \leq n-1$. Then, for each integer $1 \leq p \leq \min\{q, n-q\}$ the lowest eigenvalue of $T_A^{[p]} \in End(\Lambda^p V^*)$ satisfies*

$$\min_{a \in I_p} \lambda_a(T_A^{[p]}) \geq -\frac{(n-p)(q-p)}{(n-q)^2} (\mathrm{tr} A)^2 \tag{6}$$

and inequality is strict in (6) if A is q-positive. Moreover, if $A \neq 0$ and equality holds in (6) then the eigenvalues $k_1 \leq \cdots \leq k_n$ of A satisfy

$$k_1 + \cdots + k_p = -\frac{q-p}{n-q} \mathrm{tr} A \leq 0 \qquad \text{and} \qquad k_{p+1} = \cdots = k_n = \frac{\mathrm{tr} A}{n-q} \geq 0.$$

Proof: Let $k_1 \leq \cdots \leq k_n$ denote the eigenvalues of A. We claim that

$$\min_{a \in I_p} \lambda_a(T_A^{[p]}) = \lambda_{\{1,\ldots,p\}}.$$

Indeed, the function $t \mapsto t(\mathrm{tr} A - t)$ is concave, so the minimum value of λ_a is attained when a either minimizes or maximizes K_a among all p-subsets of $\{1, \ldots, n\}$. Due to the ordering of the k_i's, this happens when $a = \{1, \ldots, p\}$ or $a = \{n-p+1, \ldots, n\}$, respectively, so we have to show that

$$\lambda_{\{n-p+1,\ldots,n\}} \geq \lambda_{\{1,\ldots,p\}}. \tag{7}$$

We compute

$$\lambda_{\{1,\ldots,p\}} = \sum_{i=1}^{p} k_i \sum_{i=p+1}^{n} k_i, \qquad \lambda_{\{n-p+1,\ldots,n\}} = \sum_{i=1}^{n-p} k_i \sum_{i=n-p+1}^{n} k_i \,.$$

Note that $p \leq \frac{n}{2}$ by assumption $p \leq \min\{q, n-q\}$. If n is even and $p = \frac{n}{2}$ then clearly $\lambda_{\{1,\ldots,p\}} = \lambda_{\{n-p+1,\ldots,n\}}$, so let us suppose that $p < \frac{n}{2}$. In this case we have $p < n-p$, so the sets $\{1, \ldots, p\}$, $\{p+1, \ldots, n-p\}$ and $\{n-p+1, \ldots, n\}$ define a partition of $\{1, \ldots, n\}$ and setting

$$B = \sum_{i=1}^{p} k_i, \qquad C = \sum_{i=p+1}^{n-p} k_i, \qquad D = \sum_{i=n-p+1}^{n} k_i$$

we have

$$\lambda_{\{1,\ldots,p\}} = B(C+D), \qquad \lambda_{\{n-p+1,\ldots,n\}} = D(B+C).$$

Note that $B \leq D$ since the sums defining them contain the same number of terms and the terms appearing in the definition of B are all less or equal than those appearing in the definition of D. Moreover, since $p \leq n-q$ we have $q \leq n-p$, so $B + C \geq 0$ by q-nonnegativity of A and therefore $C \geq 0$. (Indeed, if it were $C < 0$ then we would have

$k_{p+1} < 0$, so $k_i < 0$ for all $i \leq p$ implying $B < 0$ and then $B + C < 0$, contradiction.) In view of this, we have

$$\lambda_{\{n-p+1,\dots,n\}} - \lambda_{\{1,\dots,p\}} = C(D - B) \geq 0$$

and this proves (7).

If $\lambda_{\{1,\dots,p\}} \geq 0$ then (6) is trivial, so let us suppose that $\lambda_{\{1,\dots,p\}} < 0$. Note that

$$K_{\{1,\dots,p\}} + K_{\{p+1,\dots,n\}} = \mathrm{tr}A \geq 0$$

by q-nonnegativity of A and that $K_{\{1,\dots,p\}} = B \leq C + D = K_{\{p+1,\dots,n\}}$ by what we already observed. Since $\lambda_{\{1,\dots,p\}} = B(C + D)$, the assumption $\lambda_{\{1,\dots,p\}} < 0$ implies in particular that $B = K_{\{1,\dots,p\}} < 0$. We observe that

$$0 \leq k_q \leq \frac{\mathrm{tr}A}{n - q}\,.$$

Indeed, since A is q-non-negative we have

$$0 \leq \sum_{i=1}^{q} k_i \leq q k_q,$$

and

$$(n - q)k_q \leq \sum_{i=q+1}^{n} k_i = \mathrm{tr}A - \sum_{i=1}^{q} k_i \leq \mathrm{tr}A\,. \tag{8}$$

Using this, we estimate

$$0 \leq \sum_{i=1}^{q} k_i = \sum_{i=1}^{p} k_i + \sum_{j=p+1}^{q} k_j \leq \sum_{i=1}^{p} k_i + (q - p)k_q \leq \sum_{i=1}^{p} k_i + \frac{q-p}{n-q}\mathrm{tr}A \tag{9}$$

that is,

$$|K_{\{1,\dots,p\}}| = -\sum_{i=1}^{p} k_i \leq \frac{q-p}{n-q}\mathrm{tr}A$$

and then

$$|\lambda_{\{1,\dots,p\}}| = |K_{\{1,\dots,p\}}|(\mathrm{tr}A + |K_{\{1,\dots,p\}}|) \leq \frac{(n-p)(q-p)}{(n-q)^2}(\mathrm{tr}A)^2 \tag{10}$$

as desired. If $A \neq 0$ and equality holds in (6), then $T_A^{[p]}$ must have at least one negative eigenvalue, so $\lambda_{\{1,\dots,p\}} < 0$ indeed, and all inequalities in (8), (9) and (10) must be satisfied with equality sign. In particular, $k_1 + \cdots + k_q = 0$ and $k_{p+1} = \cdots = k_q = \cdots = k_n$ implying

$$\mathrm{tr}A = \sum_{i=q+1}^{n} k_i = (n - q)k_q \qquad \text{and} \qquad \sum_{i=1}^{p} k_i = -\sum_{i=p+1}^{q} k_i = -(q - p)k_q \tag{11}$$

as claimed.

If A is q-positive then let us first note that it must be $p < q$, since otherwise the second identity in (11) would give

$$\sum_{i=1}^{q} k_i = \sum_{i=1}^{p} k_i = 0.$$

Thus, if A is q-positive then we have strict inequality in the last of (8) and therefore, since $p < q$, also in the last of (9) and in (10). □

3. THE BOCHNER-WEITZENBÖCK OPERATOR

Let M be a closed, connected and oriented Riemannian manifold of dimension n. For each integer $1 \leq p \leq n-1$, the Hodge Laplacian acting on p-forms is defined by

$$\Delta = dd^* + d^*d : \Omega^p(M) \to \Omega^p(M),$$

where d and d^* are the differential and the co-differential operators, respectively. For every p-form ω the Bochner-Weitzenböck formula states that the Hodge Laplacian is given by

$$\Delta\omega = \nabla^*\nabla\omega + \mathcal{B}^{[p]}\omega, \tag{12}$$

where $\nabla^*\nabla$ is the connection Laplacian and

$$\mathcal{B}^{[p]} \colon \Omega^p(M) \to \Omega^p(M)$$

is a certain symmetric and self-adjoint endomorphism on the bundle of p-forms, called the *Bochner-Weitzenböck operator.*

Proposition 6. *Let M be a closed, connected and oriented n-dimensional Riemannian manifold and let $1 \leq p \leq \frac{n}{2}$. The following assertions hold:*

(1) *If $\mathcal{B}^{[p]} \geq 0$, then $b_p(M) \leq \binom{n}{p}$.*

(2) *If $\mathcal{B}^{[p]} \geq 0$, and the strict inequality holds at some point, then the de Rham cohomology satisfies*

$$H^p(M,\mathbb{R}) = H^{n-p}(M,\mathbb{R}) = 0.$$

Moreover, if $p = 1$, then M admits a Riemannian metric of positive Ricci curvature, and has finite fundamental group. In particular, if $n = 3$, then it is diffeomorphic to a spherical space form.

(3) *If $\mathcal{B}^{[p]} \geq 0$ and $H^p(M,\mathbb{R}) \neq 0$, then every harmonic p-form is parallel. In particular, M supports a nontrivial parallel p-form.*

Proof: We argue for (2) when $p = 1$. Recall that

$$\langle \mathcal{B}^{[1]}\omega, \omega\rangle = \mathrm{Ric}(\omega^\#, \omega^\#)$$

where $\omega^\#$ denotes the dual vector field corresponding to the 1-form ω. Therefore, in this case we have that the Ricci curvature is everywhere non-negative and at some point is strictly positive. Then, it follows from Aubin [2] that M carries a metric with positive Ricci curvature and the Bonnet-Myers theorem implies that the fundamental group of M is finite. In particular, if $n = 3$ then the result follows from [11]. For the rest of the proofs see [21, Proposition 3] and [16, Proposition 15]. □

The following is a special case of Theorem 1.9 in [19] adapted to our needs.

Proposition 7. *Let $n \geq 3$, $\kappa < 0$ and $D > 0$, and let M be a closed n-dimensional Riemannian manifold with $\mathrm{Ric}(M) \geq (n-1)\kappa$ and $\mathrm{diam}(M) \leq D$. Assume that for some integer $1 \leq p \leq \frac{n}{2}$ there exists $\delta > 0$ such that $\mathcal{B}^{[p]} \geq \delta\kappa$. Then, there exists a constant $C(n, \kappa D^2) > 0$ such that the dimension of the kernel of the Hodge Laplacian*

$$\ker(\Delta) = \{\omega \in \Omega^p(M) \ : \ \Delta\omega = \nabla^*\nabla\omega + \mathcal{B}^{[p]}\omega = 0\}$$

is bounded by

$$\binom{n}{p} \cdot \exp\left(C(n,\kappa D^2) \cdot \sqrt{-\kappa\delta D^2}\right).$$

Moreover, there exists $\varepsilon(n,\delta)>0$ *such that if* $\kappa D^2\geq-\varepsilon(n,\delta)$, *then* $\dim\ker(\Delta)\leq\binom{n}{p}$.

3.1. **The relation with the Curvature Operator.** Let M be an n-dimensional Riemannian manifold with Riemannian connection ∇ and curvature tensor R defined by

$$R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]},\ X,Y\in TM.$$

For each $X,Y\in TM$ we can view $R(X,Y)$ as a skew-symmetric endomorphism on TM. Next, recall that the curvature operator is the self-adjoint operator $\mathfrak{R}\colon\Lambda^2TM\to\Lambda^2TM$ defined by

$$\langle R(X,Y)Z,W\rangle=\langle\mathfrak{R}(X\wedge Y),W\wedge Z\rangle,$$

where Λ^2TM inherits naturally the inner product coming from the Riemannian metric of M. Every $X\wedge Y\in\Lambda^2TM$ acts on TM as a skew symmetric endomorphism via

$$(X\wedge Y)Z=\langle X,Z\rangle Y-\langle Y,Z\rangle X,\ Z\in TM.$$

Therefore, each element of Λ^2TM can be viewed as a skew symmetric endomorphism of TM. For this reason we write $\mathfrak{so}(TM)=\Lambda^2TM$ and endow $\mathfrak{so}(TM)$ with the inner product that comes from Λ^2TM. Now, every element of $\mathfrak{so}(TM)$ induces a derivative on p-forms in the following way: if $\omega\in\Omega^p(M)$ and $L\in\mathfrak{so}(TM)$ then

$$(L\omega)(X_1,\ldots,X_p)=-\sum_{i=1}^{p}\omega(X_1,\ldots,LX_i,\ldots,X_p).$$

To each p-form ω we assign a tensor $\hat{\omega}$ with values in Λ^2TM defined by:

$$\langle L,\hat{\omega}(X_1,\ldots,X_p)\rangle=(L\omega)(X_1,\ldots,X_p)\quad\text{for all}\quad L\in\mathfrak{so}(TM)=\Lambda^2TM.$$

It follows from [18, Lemma 9.4.9, page 354] that the Bochner-Weitzenböck operator and the curvature operator of M are related via the formula

$$\langle\mathcal{B}^{[p]}\omega,\phi\rangle=\langle\mathfrak{R}(\hat{\omega}),\hat{\phi}\rangle\quad\text{for all}\quad\omega,\phi\in\Omega^p(M).\tag{13}$$

3.2. **The splitting of the Bochner-Weitzenböck Operator.** Let $f\colon M\to\tilde{M},n\geq3$, be an isometric immersion into a Riemannian manifold $\tilde{M}$ of dimension $n+m$. The second fundamental form α_f is viewed as a section of the vector bundle $\mathrm{Hom}(TM\times TM,N_fM)$, where N_fM is the normal bundle. For each unit normal vector field $\xi\in\Gamma(N_fM)$, the associated shape operator A_ξ is given by

$$\langle A_\xi X,Y\rangle=\langle\alpha_f(X,Y),\xi\rangle,\ X,Y\in TM.$$

The (normalized) *mean curvature vector field* of f is $\mathcal{H}=(\operatorname{tr}\alpha_f)/n$, where tr means taking the trace. In particular, when $m=1$ the *mean curvature* nH is the trace of the shape operator and when M is orientable we always choose an orientation such that $H\geq0$. Let R and $\tilde{R}$ be the curvature tensors of M and $\tilde{M}$ respectively. Then from the Gauss equation we have

$$\langle R(X,Y)Z,W\rangle=\langle\tilde{R}(X,Y)Z,W\rangle+\langle R_{\text{ext}}(X,Y)Z,W\rangle\quad X,Y,Z,W\in TM,$$

where

$$\langle R_{\text{ext}}(X,Y)Z,W\rangle=\langle\alpha_f(X,Z),\alpha_f(Y,W)\rangle-\langle\alpha_f(X,W),\alpha_f(Y,Z)\rangle.$$

Accordingly, we can split the curvature operator $\mathfrak{R}$ of M as the sum of two self-adjoint operators acting on Λ^2TM by

$$\mathfrak{R}=\mathfrak{R}_{\text{res}}+\mathfrak{R}_{\text{ext}}\tag{14}$$

where $\mathfrak{R}_{\text{res}}$ and $\mathfrak{R}_{\text{ext}}$ are defined by

$$\begin{aligned}\langle \mathfrak{R}_{\text{res}}(X\wedge Y), W\wedge Z\rangle &= \langle \tilde{\mathfrak{R}}(X\wedge Y), W\wedge Z\rangle,\\ \langle \mathfrak{R}_{\text{ext}}(X\wedge Y), W\wedge Z\rangle &= \langle \alpha(X,Z),\alpha(Y,W)\rangle - \langle \alpha(X,W),\alpha(Y,Z)\rangle.\end{aligned}$$

Let $\{\xi_1,\dots,\xi_m\}$ be an orthonormal frame of the normal bundle. Then

$$\mathfrak{R}_{\text{ext}} = \sum_{\alpha=1}^{m} \mathfrak{R}_{\text{ext}}^{(\alpha)},$$

where

$$\langle \mathfrak{R}_{\text{ext}}^{(\alpha)}(X\wedge Y), W\wedge Z\rangle = \langle A_{\xi_\alpha}(X), Z\rangle\langle A_{\xi_\alpha}(Y), W\rangle - \langle A_{\xi_\alpha}(X), W\rangle\langle A_{\xi_\alpha}(Y), Z\rangle$$

It follows from (13) and (14) that the Bochner-Weitzenböck operator $\mathscr{B}^{[p]}$ of M acting on p-forms splits as

$$\mathscr{B}^{[p]} = \mathscr{B}^{[p]}_{\text{res}} + \mathscr{B}^{[p]}_{\text{ext}}, \tag{15}$$

where

$$\begin{aligned}\langle \mathscr{B}^{[p]}_{\text{res}}\omega,\phi\rangle &= \langle \mathfrak{R}_{\text{res}}(\hat{\omega}),\hat{\phi}\rangle = \langle \tilde{\mathfrak{R}}(\hat{\omega}),\hat{\phi}\rangle,\\ \langle \mathscr{B}^{[p]}_{\text{ext}}\omega,\phi\rangle &= \langle \mathfrak{R}_{\text{ext}}(\hat{\omega}),\hat{\phi}\rangle = \sum_{\alpha=1}^{m}\langle \mathfrak{R}^{(\alpha)}_{\text{ext}}(\hat{\omega}),\hat{\phi}\rangle.\end{aligned} \tag{16}$$

It is not hard to see that the splitting in (15) is in fact the one already obtained by Savo [21, Theorem 1], where he followed the approach of Petersen [17] via the formalism of Clifford multiplication. Indeed, for each unit vector field $\xi \in \Gamma(N_f M)$, we define the endomorphism

$$T^{[p]}_{A_\xi} = (\operatorname{tr}A_\xi)A^{[p]}_\xi - A^{[p]}_\xi \circ A^{[p]}_\xi,$$

where $A^{[p]}_\xi\colon \Omega^p(M)\to\Omega^p(M)$ is the self-adjoint extension of the shape operator A_ξ in the direction ξ defined by

$$A^{[p]}_\xi\omega(X_1,\dots,X_p) = \sum_{i=1}^{p}\omega(X_1,\dots,A_\xi X_i,\dots,X_p),\ X_1,\dots,X_p\in TM.$$

Then $\mathscr{B}^{[p]}_{\text{ext}}$ is given by

$$\mathscr{B}^{[p]}_{\text{ext}} = \sum_{\alpha=1}^{m} T^{[p]}_{A_{\xi_\alpha}}. \tag{17}$$

The proof of (17) follows directly from the following algebraic result.

Lemma 8. *Let A be a self-adjoint endomorphism of TM and consider the associated self-adjoint "curvature operator" $\mathfrak{R}_A$ acting on $\Lambda^2 TM$ determined uniquely by the formula*

$$\langle \mathfrak{R}_A(X\wedge Y), W\wedge Z\rangle = \langle A(X),Z\rangle\langle A(Y),W\rangle - \langle A(X),W\rangle\langle A(Y),Z\rangle$$

for all $X,Y,Z,W\in TM$. Then, the self-adjoint operator $T^{[p]}_A\colon \Omega^p(M)\to\Omega^p(M)$ defined by

$$\langle T^{[p]}_A\omega,\phi\rangle = \langle \mathfrak{R}_A(\hat{\omega}),\hat{\phi}\rangle \tag{18}$$

can be written as

$$T^{[p]}_A = (\operatorname{tr}A)A^{[p]} - A^{[p]}\circ A^{[p]}, \tag{19}$$

where $A^{[p]}$ is the self-adjoint extension of A to $\Omega^p(M)$.

Proof: Let $\{e_1,\dots,e_n\}$ be an orthonormal basis diagonalizing A with corresponding eigenvalues $k_1,\dots,k_n$. Denote by $\{\theta_1,\dots,\theta_n\}$ its dual basis and for each $a=\{i_1,\dots,i_p\}\in I_p$ consider the p-form

$$\Theta_a=\theta_{i_1}\wedge\cdots\wedge\theta_{i_p}.$$

Savo proved in [21] that Θ_a diagonalizes $T_A^{[p]}$ given by (19) with corresponding eigenvalues $K_aK_{\star a}$, where

$$K_a=\sum_{i\in a}k_i \quad \text{and} \quad K_{\star a}=\sum_{i\in\star a}k_i.$$

A direct computation shows that $T_A^{[p]}$ defined by (18) also shares the same property. Indeed,

$$\begin{aligned}\langle T_A^{[p]}\Theta_a,\Theta_a\rangle&=\langle\mathfrak{R}_A(\hat\Theta_a),\hat\Theta_a\rangle\\&=\sum_{\ell_1,\dots,\ell_p}\langle\mathfrak{R}_A(\hat\Theta_a)(e_{\ell_1},\dots,e_{\ell_p}),\hat\Theta_a(e_{\ell_1},\dots,e_{\ell_p})\rangle\\&=\frac12\sum_{\ell_1,\dots,\ell_p}\sum_{i\neq j}k_ik_j\langle\hat\Theta_a(e_{\ell_1},\dots,e_{\ell_p}),e_i\wedge e_j\rangle^2\\&=\frac12\sum_{\ell_1,\dots,\ell_p}\sum_{i\neq j}k_ik_j\Big(\big((e_i\wedge e_j)\Theta_a\big)(e_{\ell_1},\dots,e_{\ell_p})\Big)^2\\&=\frac12\sum_{\ell_1,\dots,\ell_p}\sum_{i\neq j}k_ik_j\Big(\sum_k\Theta_a(e_{\ell_1},\dots,(e_i\wedge e_j)e_{\ell_k},\dots,e_{\ell_p})\Big)^2\\&=\frac12\sum_{\ell_1,\dots,\ell_p}\sum_{i\neq j}k_ik_j\Big(\sum_k\Theta_a(e_{\ell_1},\dots,\langle e_i,e_{\ell_k}\rangle e_j-\langle e_j,e_{\ell_k}\rangle e_i,\dots,e_{\ell_p})\Big)^2\\&=\sum_{i\in a,\,j\in\star a}k_ik_j=K_aK_{\star a}.\end{aligned}$$

It is also very easy to check that $\langle T_A^{[p]}\Theta_a,\Theta_b\rangle=0$ for $a\neq b$, and this completes the proof. ☐

Next, we extend [21, Theorem 1] by applying results obtained by Petersen and Wink in [19].

Lemma 9. *Let $\tilde\lambda_1\leq\cdots\leq\tilde\lambda_{\binom{n+m}{2}}$ denote the eigenvalues of $\tilde{\mathfrak{R}}$ and assume that for some real number c and some integer $1\leq p\leq\lfloor\frac n2\rfloor$, the eigenvalues satisfy*

$$\frac{\tilde\lambda_1+\cdots+\tilde\lambda_{n-p}}{n-p}\geq c. \tag{20}$$

Then

$$\langle\mathscr{B}_{\mathrm{res}}^{[\ell]}\omega,\omega\rangle\geq c\ell(n-\ell)|\omega|^2,\quad \textit{for all}\quad 1\leq\ell\leq p$$

and non-zero ℓ-forms ω.

Proof: We only need to prove it for $\ell=p$. To this end, we apply [19, Lemma 2.1] for $\mathfrak{R}_{\mathrm{res}}$. Indeed, if $\lambda_1\leq\cdots\leq\lambda_{\binom{n}{2}}$ are its eigenvalues, then it follows from (16), (20) and

Lemma 4 that

$$\frac{\lambda_1+\cdots+\lambda_{n-p}}{n-p}\geq\frac{\tilde{\lambda}_1+\cdots+\tilde{\lambda}_{n-p}}{n-p}\geq c.$$

Since from [19, Lemma 2.2(b) and Proposition 2.5(a)] every p-form ω satisfies

$$|L\omega|^2\leq\frac{1}{n-p}|\hat{\omega}|^2|L|^2\quad\text{for all}\quad L\in\mathfrak{so}(TM)=\Lambda^2TM,$$

the proof follows. ☐

The following result plays a crucial role in the proofs of our main results.

Proposition 10. *Fix integers $n\geq 3$, $1\leq p\leq\lfloor\frac{n}{2}\rfloor$ and $1\leq q\leq n-1$. Let M be an $(n+1)$-dimensional Riemannian manifold such that*

$$\frac{\lambda_1+\cdots+\lambda_{n-p}}{n-p}\geq c,\tag{21}$$

for some real constant c, where $\lambda_1\leq\cdots\leq\lambda_{\binom{n+1}{2}}$ denote the eigenvalues of its curvature operator. If $f\colon\Sigma\to M$ is a closed, connected, oriented, q-convex immersed hypersurface, then its Bochner-Weitzenböck operator satisfies pointwise the inequality

$$\min_{\substack{\omega\in\Omega^\ell(\Sigma)\\ \|\omega\|=1}}\langle\mathcal{B}^{[\ell]}\omega,\omega\rangle\geq\ell(n-\ell)\left(c-\frac{q-\ell}{\ell}\left(\frac{n}{n-q}\right)^2H^2\right),\tag{22}$$

for all $1\leq\ell\leq\min\{p,\min\{q,n-q\}\}$. Moreover, (22) is strict if either (21) is strict or f is strictly q-convex. Furthermore, if equality holds in (22) at some point $x\in\Sigma$ and the shape operator $A(x)\neq 0$, then the principal curvatures $k_1(x)\leq\cdots\leq k_n(x)$ satisfy

$$k_1(x)+\cdots+k_\ell(x)=-\frac{q-\ell}{n-q}nH(x)\leq 0\quad\text{and}\quad k_{\ell+1}(x)=\cdots=k_n(x)=\frac{nH(x)}{n-q}\geq 0.$$

Proof: Follows from (15), (17), Lemma 5 and Lemma 9. ☐

4. Proofs of the main Theorems

Proof of Theorem A. From Proposition 10 we have $\mathcal{B}^{[i]}\geq 0$ for all integers $q\leq i\leq p$. The result follows from Proposition 6(1)-(3) and Poincaré duality. ☐

Proof of Theorem C. From Proposition 10 we have $\mathcal{B}^{[i]}\geq i(n-i)c$ for all integers $q\leq i\leq p$. The result follows from Proposition 7 and the fact that $b_i(\Sigma)$ is equal to the dimension of the kernel of the Hodge Laplacian. ☐

Proof of Theorem D. Assume $q\leq\frac{n}{2}$. From Proposition 10 and (3) we have that $\mathcal{B}^{[i]}\geq 0$ for all integers $\ell\leq i\leq\min\{p,q-1\}$. Cases (1)-(3) follow by applying Proposition 6(1)-(3). To see that equality holds everywhere in (3) in case (3), consider a non-zero ℓ-form ω that lies in the kernel of the Hodge Laplacian. Then, from (12) we obtain

$$0=|\nabla\omega|^2+\langle\mathcal{B}^{[\ell]}\omega,\omega\rangle,$$

which implies that $\langle\mathcal{B}^{[\ell]}\omega,\omega\rangle=0$. The desired equality now follows by using (22) and (3). Assume $q>\frac{n}{2}$. From Proposition 10 and (3) we have that $\mathcal{B}^{[i]}\geq 0$ for all integers $\ell\leq i\leq\min\{p,n-q\}$. The cases (1)-(3) are proved similarly as before. ☐

Proof of Theorem E. Assume $q \leq \frac{n}{2}$. It follows from Proposition 10 that $\mathscr{B}^{[\ell]} \geq \ell(n-\ell)\kappa_\ell$, for each $1 \leq \ell \leq \min\{p, q-1\}$, where

$$\kappa_\ell = \min_{x\in\Sigma}\left(c - \frac{q-\ell}{\ell}\left(\frac{n}{n-q}\right)^2 H^2\right).$$

If moreover $q \leq p$, then from Proposition 10 we have $\mathscr{B}^{[i]} \geq i(n-i)c$ for all $q \leq i \leq p$. The result follows from Proposition 7 and Poincaré duality. On the other hand, if $q > \frac{n}{2}$, then from Proposition 10 we have $\mathscr{B}^{[\ell]} \geq \ell(n-\ell)\kappa_\ell$ for each $1 \leq \ell \leq \min\{p, n-q\}$, where κ_ℓ is as before. The result again follows from Proposition 7 and Poincaré duality. ☐

5. The case of the sphere

In this section we discuss the special case where the ambient space is the unit sphere $\mathbb{S}^{n+1}$. Recall the standard immersion of a torus

$$\mathbb{T}^n_p(r) = \mathbb{S}^p(r) \times \mathbb{S}^{n-p}(\sqrt{1-r^2})$$

into $\mathbb{S}^{n+1}$, where $\mathbb{S}^p(r)$ denotes the p-dimensional sphere of radius $r < 1$. The principal curvatures are $-\sqrt{1-r^2}/r$ and $r/\sqrt{1-r^2}$ of multiplicity p and $n-p$, respectively. A direct computation gives that

$$H = \frac{nr^2 - p}{nr\sqrt{1-r^2}} \tag{23}$$

with $r^2 \geq p/n$. We note that for $1 \leq q \leq n-1$ and $1 \leq p \leq \beta_q$ (where, β_q is as in (2)), the torus $\mathbb{T}^n_p(r)$ is q-convex, when $r^2 \geq p/q$.

Theorem F. *Let $f\colon \Sigma \to \mathbb{S}^{n+1}, n \geq 3$, be a closed, connected, oriented q-convex immersed hypersurface for some integer $2 \leq q \leq n-1$. Set β_q as in (2). If for some integer $1 \leq p \leq \beta_q$ the mean curvature satisfies*

$$H \leq \frac{n-q}{n}\sqrt{\frac{p}{q-p}}, \tag{24}$$

then

1. $b_i(\Sigma) = b_{n-i}(\Sigma) \leq \binom{n}{i}$ *for each* $p \leq i \leq \beta_q$.
2. *If strict inequality holds at some point, then*

   $$b_p(\Sigma) = \cdots = b_{\beta_q}(\Sigma) = 0 \quad \textit{and} \quad b_{n-\beta_q}(\Sigma) = \cdots = b_{n-p}(\Sigma) = 0.$$

   Moreover, if $p = 1$, then Σ admits a Riemannian metric of positive Ricci curvature and has finite fundamental group.
3. *If $b_p(\Sigma) > 0$, then $f(\Sigma)$ is isometric to the torus $\mathbb{T}^n_p(\sqrt{p/q})$.*

Furthermore, if $q \leq \frac{n}{2}$, then Σ has the homotopy type of a CW-complex with no cells of dimension i for $q \leq i \leq n-q$. In particular, if $q = 2$, then the fundamental group $\pi_1(\Sigma)$ is a free group on $b_1(\Sigma;\mathbb{Z})$ generators and if moreover, $\pi_1(\Sigma)$ is finite, then Σ is homeomorphic to the sphere $\mathbb{S}^n$.

As an immediate consequence we get the following:

Corollary 11. *Let $f\colon \Sigma \to \mathbb{S}^{n+1}, n \geq 4$, be a closed, connected, oriented, 2-convex immersed hypersurface with*

$$H \leq \frac{n-2}{n}.$$

If at some point the above inequality is strict, then Σ is homeomorphic to the sphere $\mathbb{S}^n$.

Proof: Apply Theorem F for $p = 1$ and $q = 2$. □

Remarks 12.

(1) Theorem F can be viewed as an "extension" of the main Theorem of do Carmo and Warner [7].

(2) Corollary 11 is sharp since $\mathbb{T}^n_1(1/\sqrt{2})$ is 2-convex with $H = \frac{n-2}{n}$.

(3) An easy example satisfying (24) with strict inequality is $\mathbb{T}^n_{p-1}(\sqrt{(p-1)/q})$, provided that $p > 1$.

(4) Inequality (3) is sharp since $\mathbb{T}^n_\ell(\sqrt{\ell/q})$ is q-convex with $H = \frac{n-q}{n}\sqrt{\frac{\ell}{q-\ell}}$.

(5) Corollary 3 is sharp since $\mathbb{T}^n_1(1/\sqrt{\lceil \frac{n}{2} \rceil})$ is $\lceil \frac{n}{2} \rceil$-convex with $H = \frac{n-\lceil \frac{n}{2} \rceil}{n\sqrt{\lceil \frac{n}{2} \rceil - 1}}$.

Proof of Theorem F. If for some integer $1 \leq p \leq \beta_q$ inequality (24) is satisfied, then cases (1) and (2) follow directly from Theorem D(1),(2), for $c = 1$. If $b_p(\Sigma) > 0$ then from Theorem D(3) we get equality in (24) everywhere and the result follows from [21, Theorem 4]. This concludes also case (3). □

Now, assume $q \leq \frac{n}{2}$. It follows that at each point the number of negative principal curvatures is less than q. Let $v \in \mathbb{R}^{n+2}$ be a unit vector such that the height function $h\colon \Sigma \to \mathbb{R}$ defined by $h = \langle g, v\rangle$ is a Morse function, where g is the isometric immersion $g = j \circ f$, and $j\colon \mathbb{S}^{n+1} \to \mathbb{R}^{n+2}$ is the inclusion. The Hessian of h is given by

$$\operatorname{Hess} h(X, Y) = \langle \alpha_g(X, Y), v\rangle, \ X, Y \in T\Sigma.$$

It is clear that the index at each non-degenerate critical point of h is less than q or greater than $n - q$. Therefore, it follows from standard Morse theory (cf. [15, Th. 3.5] or [9, Th. 4.10]) that M has the homotopy type of a CW-complex with no cells of dimension i for $q \leq i \leq n - q$. If in addition $q = 2$, then there are no 2-cells, and thus, by the cellular approximation theorem, the inclusion of the 1-skeleton $\mathrm{X}^{(1)} \hookrightarrow \Sigma$ induces isomorphism between the fundamental groups. Therefore, $\pi_1(\Sigma)$ is a free group on $b_1(\Sigma; \mathbb{Z})$ elements. Hence, if $\pi_1(\Sigma)$ is finite, then $\pi_1(\Sigma) = 0$. Then Σ is a simply connected homology sphere over the integers and therefore a homotopy sphere. By the (generalized) Poincaré conjecture (Smale $n \geq 5$, Freedman $n = 4$, Perelman $n = 3$), Σ is homeomorphic to $\mathbb{S}^n$.

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