On arbitrarily regular conforming virtual element methods for elliptic partial differential equations

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Abstract

The Virtual Element Method (VEM) is a very effective framework to design numerical approximations with high global regularity to the solutions of elliptic partial differential equations. In this paper, we review the construction of such approximations for an elliptic problem of order p_1 using conforming, finite dimensional subspaces of $H^{p_2}(\Omega)$, where p_1 and p_2 are two integer numbers such that $p_2 \ge p_1 \ge 1$ and $\Omega \in \mathbb{R}^2$ is the computational domain. An abstract convergence result is presented in a suitably defined energy norm. The space formulation and major aspects such as the choice and unisolvence of the degrees of freedom are discussed, also providing specific examples corresponding to various practical cases of high global regularity. Finally, the construction of the "enhanced" formulation of the virtual element spaces is also discussed in details with a proof that the dimension of the "regular" and "enhanced" spaces is the same and that the virtual element functions in both spaces can be described by the same choice of the degrees of freedom.

Keywords: polyharmonic problem, virtual element method, polytopal mesh, high-order methods, high-regular methods

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1. Introduction

The conforming finite element method is based on the construction of a finite dimensional approximation spaces that are typically only C^0 -continuous [39] on the meshes covering the computational domain. The construction of sych approximation spaces with higher regularity is normally deemed a difficult task because it requires a set of basis functions with such global regularity. Examples in this direction can be found all along the history of finite elements from the oldest works in the sixties of the last century, e.g., [12, 26, 40] to to the most recent attempts in [44, 45, 63, 64]. Despite its intrinsic difficulty, designing approximations with global C^1 - or higher regularity is still a major research topic. Such kind of approximations have indeed a natural application in the numerical treatment of problems involving high-order differential operators.

The Virtual Element Method (VEM) [16] does not require the explicit knowledge of the basis functions spanning the approximation spaces in its formulation and implementation. The crucial idea behind the VEM is that the elemental approximation spaces, which are globally "glued" in a highly regular conforming way, are defined elementwise as the solutions of a partial differential equation. The functions that belong to such approximation spaces are dubbed as "virtual" as they are never really computed, with the noteworthy exception of a subspace of polynomials that are indeed used in the formulation of the method. The virtual element functions are uniquely characterized by a set of values, the *degrees of freedom*, that are actually solved for in the method.

The virtual element 'paradigm' thus provides a major breakthrough in obtaining highly regular Galerkin methods as it allows the construction of numerical approximation of any order of accuracy and global conforming regularity that work on unstructured two-dimensional and three-dimensional meshes with very general polytopal elements.

Roughly speaking, the VEM is a Galerkin-type projection method that generalize the finite element method, which was originally designed for simplicial and quadrilateral/hexahedral meshes, to polytopal meshes. Other important families of methods that are suited to polytopal meshes are the polygonal/polyhedral finite element method [59] the mimetic finite difference method[21] the discontinuous Galerkin method on polygonal/polyhedral grids[7, 34]; the hybrid discontinuous Galerkin method[41]; and the hybrid high–order method [42].

The conforming VEM was first developed for second-order elliptic problems in primal formulation [16, 19], and then in mixed formulation [18, 32] and nonconforming formulation [15]. Despite its relative youthness (the first paper was published in 2013), the VEM has been very successful in a wide range of scientific and engineering applications. A non-exhaustive list of applications includes, for example, the works of References [2, 5, 6, 8, 13, 14, 27, 28, 35, 37, 50, 56–58, 62]. Virtual element spaces forming de Rham complexes for the Stokes, Navier-Stokes and Maxwell equations were proposed in [17, 22, 23]. A VEM for Helmholtz problems based on non-conforming approximation spaces of Trefftz functions, i.e., functions that belong to the kernel of the Helmholtz operator, is found [51]

The first works using a C^1 -regular conforming VEM addressed the classical plate bending problems [33, 38], second-order elliptic problems [24, 25], and the nonlinear Cahn-Hilliard equation [3]. More recently, highly regular virtual element spaces were considered for the von Kármán equation modelling the deformation of very thin plates [49], geostrophic equations [53] and fourth-order subdiffusion equations [48], two-dimensional plate vibration problem of Kirchhoff plates [52], the transmission eigenvalue problems [54] the fourth-order plate buckling eigenvalue problem [55]. In [10], we proposed the highly-regular conforming VEM for the two-dimensional polyharmonic problem $(-\Delta)^{p_1}u = f, p_1 \ge 1$. The VEM is based on an approximation space that locally contains polynomials of degree $r \ge 2p_1 - 1$ and has a global H^{p_1} regularity. In [9], we extended this formulation to a virtual element space that can have arbitrary regularity $p_2 \ge p_1 \ge 1$ and contains polynomials of degree $r \ge p_2$. This VEM is a generalization of the VEMs for second- and fourth-order problems since the approximation space for $p_2 = p_1 = 1$ coincides with the conforming virtual element spaces for the Poisson equation of Reference [16] and the approximation space for $p_2 = p_1 = 2$ coincides with the conforming virtual element spaces for the and the biharmonic equation of Reference [33]. VEMs for three-dimensional problems are also available for the fourth-order linear elliptic equation [20] (see also [31]), and highly-regular conforming VEM in any dimension has been proposed in [46].

In this paper, we review the detailed construction of the virtual element spaces with arbitrary order of accuracy and regularity for the numerical approximation of two-dimensional problems involving the polyharmonic operator of degree p_1 . Such a construction follows the standard guidelines of the VEM, which we briefly summarize here. As the VEM is a conforming Galerkin variational method, its formulation requires the definition of a suitable finite dimensional approximation space, which is obtained by combining in a conforming way local (elemental) finite dimensioal spaces. The local virtual element spaces are defined in every mesh element by all the solutions of a specific polyharmonic problem of degree $p_2 \ge p_1$. The loading terms of the partial differential equations defining the elemental virtual element spaces can be all the polynomials of degree (up to) $r - 2p_1$, where the integer number $r \ge p_2$ is the order of the virtual element space. The traces of the virtual element functions and all its normal derivatives of order *j* from one to $p_2 - 1$ on the elemental edges are univariate polynomials of degree at least r - j (in some cases the polynomial degree can be a little higher than r - j). From the definition, it also follows the fundamental property that the polynomials of degree up to *r* inside all elements are a linear subspace of the virtual element space of degree *r*. Then, the elemental spaces are "glued" together to form a global space with H^{p_2} -regularity. A very careful choice of the degrees of freedom, which as usual are nodal values associated with the mesh vertices or polynomial moments associated with edges and elements, makes the elliptic projection onto the polynomials of degree r computable. An L^2 -orthogonal projection onto the polynomials of degree $r - p_1$ in every mesh element is also computable in the "modified" (or "enhanced") formulation of the virtual element method. In this work, we also present a detailed discussion of the enhanced formulation and its major properties. The enhanced formulation is obtained by extending the similar contruction for the Poisson equation presenting in the pioneering paper [1] to our case. These polynomial projection operators are finally used to construct the discrete approximation of the bilinear form and the right that are used in the virtual element approximation. An abstract convergence result holds, that can be proved by assuming only a few fundamental properties of the virtual element formulation.

The remaining part of the manuscript is organized as follows. In Section 2 we introduce the continuous polyharmonic problem and its weak formulation. In Section 3 we introduce the virtual element discretization and recall the main abstract convergence result. In Section 4 we present the formulation of the conforming virtual element approximation with higher-order regularity. Finally, in Section 5 we draw our conclusions.

2. The continuous problem

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, convex domain with polygonal boundary Γ . For any integer $p_1 \ge 1$, we consider the polyharmonic problem

$$(-\Delta)^{p_1} u = f \qquad \text{in } \Omega, \tag{1a}$$

$$\partial_n^J u = 0$$
 for $j = 0, \dots, p_1 - 1$ on Γ , (1b)

where $\partial_n u = \mathbf{n} \cdot \nabla u$ is the normal derivative of u and $\partial_n^j u$ is the normal derivative applied j times to u with the useful convention that $\partial_n^0 u = u$ for j = 0. Let

$$V := H_0^{p_1}(\Omega) = \{ v \in H^{p_1}(\Omega) : \partial_n^j v = 0 \text{ on } \Gamma, \ j = 0, \dots, p_1 - 1 \}.$$

Denoting the duality pairing between V and its dual V' by $\langle \cdot, \cdot \rangle$, the variational formulation of the polyharmonic problem (1) reads as

Find
$$u \in V$$
 such that: $a_{p_1}(u, v) = \langle f, v \rangle \quad \forall v \in V,$ (2)

where, for any nonnegative integer ℓ , the bilinear form $a_{p_1}(\cdot, \cdot) : V \times V \to \mathbb{R}$ is given by

$$a_{p_1}(u,v) := \begin{cases} \int_{\Omega} \nabla \Delta^{\ell} u \cdot \nabla \Delta^{\ell} v \, d\mathbf{x} & \text{for } p_1 = 2\ell + 1, \, \ell \ge 0, \\ \int_{\Omega} \Delta^{\ell} u \, \Delta^{\ell} v \, d\mathbf{x} & \text{for } p_1 = 2\ell, \, \ell \ge 1. \end{cases}$$
(3)

If $f \in L^2(\Omega)$ we have

$$\langle f, v \rangle := (f, v) = \int_{\Omega} f v \, d\mathbf{x},$$
(4)

where (\cdot, \cdot) denotes the L^2 -inner product. The bilinear form $a_{p_1}(\cdot, \cdot)$ is coercive and continuous with respect to $||u||_V := (a_{p_1}(u, u))^{1/2}$, which is a norm on $H_0^{p_1}(\Omega)$. The coercivity and continuity constants are respectively denoted by α and M, and their value depends on the regularity of Ω and its boundary

Γ. Coercivity and continuity implies existence and uniqueness of the solution to (2) from an application of the Lax-Milgram theorem [30, Theorem 2.7.7]. About the regularity of the solution to (2), it is worth mentioning the result in [43, Corollary 2.21]. Accordingly, if the domain boundary ∂Ω is C^k -regular for $k \ge 2p_1$ and $f \in H^{k-2p_1}(\Omega)$, then $u \in H^k(\Omega) \cap H_0^{p_1}(\Omega)$ and it holds that $||u||_k \le C||f||_{k-2p_1}$. As pointed out in [9], the regularity of *u* for domains with irregular boundaries is still an open issue. However, we know that a similar result holds for the biharmonic problem, i.e., $p_1 = 2$, if Ω is a bounded, convex, polygonal domain see [29].

3. The discrete problem and an abstract convergence result

Let *r* and p_2 be two integer numbers such that $r \ge p_2 \ge p_1 \ge 1$. The virtual element approximation to the variational problem (2) reads as

Find
$$u_h \in V_{h,r}^{p_2,p_1}$$
 such that: $a_h(u_h,v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in V_{h,r}^{p_2,p_1},$ (5)

where the virtual element space $V_{h,r}^{p_2,p_1}$ is a finite-dimensional conforming subspace of V; $a_h(\cdot, \cdot)$: $V_{h,r}^{p_2,p_1} \times V_{h,r}^{p_2,p_1} \to \mathbb{R}$ is the virtual element bilinear form that approximates the bilinear form (3); $\langle f_h, \cdot \rangle : V_{h,r}^{p_2,p_1} \to \mathbb{R}$ is the continuous linear functional that approximates (4) through an element f_h of the dual space $(V_{h,r}^{p_2,p_1})^*$ of $V_{h,r}^{p_2,p_1}$. The formal definition and properties of $V_{h,r}^{p_2,p_1}$, $a_h(\cdot, \cdot)$ and f_h are discussed in the next section.

3.1. Mesh notation, mesh regularity and some basic definitions

The virtual element method is formulated on the mesh family $\{\Omega_h\}_h$, where each mesh Ω_h is a partition of the computational domain Ω into nonoverlapping polygonal elements P and is labeled by the mesh size parameter *h* that is defined below. A polygonal element P is a compact subset of \mathbb{R}^2 with boundary ∂P , area |P|, center \mathbf{x}_P , and diameter $h_P = \sup_{\mathbf{x},\mathbf{y}\in P} |\mathbf{x} - \mathbf{y}|$. The mesh elements of Ω_h form a finite cover of Ω such that $\overline{\Omega} = \bigcup_{P \in \Omega_h} P$ and the mesh size labeling each mesh Ω_h is defined by $h = \max_{P \in \Omega_h} h_P$. A mesh edge *e* has center \mathbf{x}_e and length h_e and we denote the set of mesh edges by \mathcal{E}_h . A mesh vertex v has position vector \mathbf{x}_v and we denote the set of mesh vertices by \mathcal{V}_h . Moreover, in the definition of the degrees of freedom of the next section, we associate every vertex v with a characteristic lenght h_v . This characteristic lenght h_v can be the average of the diameters of the polygons sharing v.

For any integer number $\ell \ge 0$, we let $\mathbb{P}_{\ell}(\mathsf{P})$ and $\mathbb{P}_{\ell}(e)$ denote the space of polynomials defined on P and e, respectively, and $\mathbb{P}_{\ell}(\Omega_h)$ denotes the space of piecewise polynomials of degree ℓ on the mesh Ω_h . Accordingly, if $q \in \mathbb{P}_{\ell}(\Omega_h)$ then it holds that $q_{|\mathsf{P}} \in \mathbb{P}_{\ell}(\mathsf{P})$ for all $\mathsf{P} \in \Omega_h$. Finally, we define the (broken) seminorm of a function $v \in \prod_{\mathsf{P} \in \Omega_h} H^{p_1}(\mathsf{P})$ by

$$||v||_{h}^{2} = \sum_{\mathsf{P}\in\Omega_{h}} a_{p_{1}}^{\mathsf{P}}(v,v).$$

Throughout the paper, we use the multi-index notation, so that $v = (v_1, v_2)$ is a two-dimensional index defined by the two integer numbers $v_1, v_2 \ge 0$. Moreover, $D^v w = \partial^{|v|} w / \partial x^{v_1} \partial y^{v_2}$ denotes the partial derivative of order $|v| = v_1 + v_2 > 0$ of a given bivariate function w(x, y), and we use the conventional notation that $D^{(0,0)}w = w$ for v = (0,0). We denote the partial derivatives of w versus x and y by the shortcuts $\partial_x w, \partial_y w, \partial_{xx} w, \partial_{xy} w, \partial_{yy} w$, etc. We denote the normal and tangential derivatives with respect to a given edge and their mixed combination by $\partial_n w, \partial_t w, \partial_{tt} w, \partial_{nt} w, \partial_{nn} w$, etc, and use the shorter notation $\partial_t^j w$ and $\partial_n^j w$ for the tangential and normal derivatives of w of order j.

3.2. Abstract convergence theorem

For the mathematical formulation of the virtual element approximation (5), we require the two following assumptions on the virtual element space $V_{h,r}^{p_2,p_1}$ and the bilinear form $a_h(\cdot, \cdot)$:

- (H1). For all h > 0, the global virtual element space $V_{h,r}^{p_2,p_1}$ is a conforming, finite-dimensional subspace of $V = H_0^{p_1}(\Omega) \cap H^{p_2}(\Omega)$ such that for all elements P of all mesh partitions Ω_h it holds that
 - $V_{h,r}^{p_2,p_1}(\mathsf{P})$, the *local* (elemental) virtual element space that is defined as the restriction of $V_{h,r}^{p_2,p_1}$ to the element P is a finite-dimensional subspace of $H^{p_2}(\mathsf{P})$;
 - $\mathbb{P}_r(\mathsf{P})$, the space of polynomials of degree up to r defined on P is a subspace of $V_{h,r}^{p_2,p_1}(\mathsf{P})$.
- (H2). The bilinear form $a_h(\cdot, \cdot): V_{h,r}^{p_2,p_1} \times V_{h,r}^{p_2,p_1} \to \mathbb{R}$ admits the elementwise decomposition

$$a_h(u_h, v_h) = \sum_{\mathsf{P} \in \Omega_h} a_h^{\mathsf{P}}(u_h, v_h) \quad \forall u_h, v_h \in V_{h, r}^{p_2, p_1},$$

where for all element P the local bilinear form $a_{h}^{P}(\cdot, \cdot)$ is symmetric and such that

(*r*-Consistency): for every polynomial $q \in \mathbb{P}_r(\mathsf{P})$ and every virtual element function $v_h \in V_{h,r}^{p_2,p_1}(\mathsf{P})$ it holds that

$$a_{h}^{\mathsf{P}}(v_{h},q) = a_{p_{1}}^{\mathsf{P}}(v_{h},q); \tag{6}$$

(Stability): there exist two positive constants α_* , α^* independent of h and P such that for every $v_h \in V_{h,r}^{p_2,p_1}(\mathsf{P})$ it holds that

$$\alpha_* a_{p_1}^{\mathsf{P}}(v_h, v_h) \le a_h^{\mathsf{P}}(v_h, v_h) \le \alpha^* a_{p_1}^{\mathsf{P}}(v_h, v_h).$$
(7)

The stability constant α_* and α^* may depend on the polynomial approximation degree *r*, see, e.g., [11] for the case $p_1 = 1$.

Assumption (H2) implies that the symmetric bilinear form $a_h(\cdot, \cdot)$ is coercive and continuous, so that the existence and uniqueness of the solution u_h follows from an application of the Lax-Milgram theorem [30, Theorem 2.7.7]. Under these assumptions we can prove this abstract convergence result.

Theorem 3.1. Let $u \in V$ be the solution to the variational problem (1) and $u_h \in V_{h,r}^{p_2,p_1}$, $r \ge p_2 \ge p_1 \ge 1$, the solution to the virtual element approximation (5) under assumptions (H1)-(H2). Then, there exists a constant *C* independent of *h* such that

$$||u - u_h||_V \le C \Big(||u - u_I||_V + ||u - u_\pi||_h + ||f_h - f||_{(V_h^{p_2, p_1})^*} \Big), \tag{8}$$

for every virtual element approximation u_I in $V_{h,r}^{p_2,p_1}$ and any piecewise polynomial approximation $u_{\pi} \in \mathbb{P}_r(\Omega_h)$ of u. The constant C is proportional to $(M/\alpha)(\alpha^*/\alpha_*)$.

The proof of this theorem was first published in [10] for the case with $p_2 = p_1$ and then extended to the case for $p_2 \ge p_1$ in [9].

4. The virtual element spaces of higher-order continuity

In this section, we present the formulation of the virtual element method of Eq. (5). To this end, we first introduce the local virtual element spaces, the degrees of freedom, and the global virtual element space, which is obtained by "gluing" in a conforming way the local spaces. Then, we discuss the computability of the elliptic projection operator, and the enhancement of the local spaces, which allows us to compute the orthogonal projection operators onto the subspace of polynomials of degree up to $p_1 - 1$. Finally, we discuss the construction of the bilinear form $a_h(\cdot, \cdot)$ and the local term $\langle f_h, \cdot \rangle$.

4.1. Local space definitions

Let P be a mesh element. For $p_2 \le r \le 2p_2 - 2$, we consider the local virtual element space defined as

$$V_{h,r}^{p_2,p_1}(\mathsf{P}) = \left\{ v_h \in H^{p_2}(\mathsf{P}) : \Delta^{p_2} v_h \in \mathbb{P}_{r-2p_1}(\mathsf{P}), \ \partial_n^j v_h \in \mathbb{P}_{\alpha_j(p_2,r)}(e), \\ j = 0, \dots, p_2 - 1 \ \forall e \in \partial \mathsf{P} \right\}, \quad (9)$$

where $\alpha_j(p_2, r) = \max\{2(p_2-j)-1, r-j\}$. For $r \ge 2p_2-1$ it holds that $\alpha_j = r-j$ for all $j = 0, ..., p_2-1$ and the definition of the local virtual element space on the element P takes the simpler form

$$V_{h,r}^{p_2,p_1}(\mathsf{P}) = \left\{ v_h \in H^{p_2}(\mathsf{P}) : \Delta^{p_2} v_h \in \mathbb{P}_{r-2p_1}(\mathsf{P}), \ \partial_n^j v_h \in \mathbb{P}_{r-j}(e), \\ j = 0, \dots, p_2 - 1 \ \forall e \in \partial \mathsf{P} \right\}.$$
(10)

In both definitions (9) and (10) we use the conventional notation that $\mathbb{P}_r(\mathsf{P}) = \{0\}$ if r < 0.

Remark 4.1. The space of polynomials $\mathbb{P}_r(P)$ is a subspace of $V_{h,r}^{p_2,p_1}(P)$ for both definitions (9) and (10).

Remark 4.2. Let $\#(\mathcal{P})$ denote the cardinality of a (finite dimensional) space \mathcal{P} and $N_{\mathcal{P}}^{\mathcal{E}}$ and number of edges and vertices of element \mathcal{P} . The dimension of the local virtual element space (9) is given by

$$dim V_{h,r}^{p_2,p_1}(P) = \# (\mathbb{P}_{r-2p_1}(P)) + \sum_{e \in \partial P} \sum_{j=0}^{p_2-1} \# (\mathbb{P}_{\alpha_j(p_2,r)}(e)) - N_P^{\mathcal{V}} \frac{(p_2+1)p_2}{2}$$
$$= \frac{(r-2p_1+1)(r-2p_1+2)}{2} + N_P^{\mathcal{E}} \sum_{j=0}^{p_2-1} (\alpha_j(p_2,r)+1)$$
$$- N_P^{\mathcal{V}} \frac{(p_2+1)p_2}{2}.$$
(11)

The dimension of the local virtual element space (10) is given by

$$dim V_{h,r}^{p_2,p_1}(P) = \# (\mathbb{P}_{r-2p_1}(P)) + \sum_{e \in \partial P} \sum_{j=0}^{p_2-1} \# (\mathbb{P}_{r-j}(e)) - N_P^{\mathcal{V}} \frac{(p_2+1)p_2}{2}$$
$$= \frac{(r-2p_1+1)(r-2p_1+2)}{2} + N_P^{\mathcal{E}} \frac{p_2(2r+3-p_2)}{2}$$

$$-N_{P}^{V}\frac{(p_{2}+1)p_{2}}{2}.$$
(12)

In both equations (11) and (12), the last term of the right-hand side, i.e., $N_P^{\mathcal{W}}p_2(p_2+1)/2$, is subtracted to take into account the C^{p_2-1} -regularity of v_h at the elemental vertices.

4.2. Local degrees of freedom

Let $\beta_j = \alpha_j - \min\{2(p_2 - j) - 1, r - j\} - 1$. For $r = 2p_2 - 1 - k$ with $k = 1, \dots, p_2 - 1$, the virtual element functions in the elemental space (9) are uniquely identified by the following degrees of freedom

(D1) $h_{\mathsf{v}}^{|\mathsf{v}|} D^{\mathsf{v}} v_h(\mathsf{v}), |\mathsf{v}| \le p_2 - 1$ for any vertex v of $\partial \mathsf{P}$;

(**D2**)
$$h_e^{-1+j} \int_e q \partial_n^j v_h \, ds$$
 for any $q \in \mathbb{P}_{\beta_j}(e)$ and edge e of $\partial \mathsf{P}, j = k+1, \dots, p_2-1;$

(D3)
$$h_{\mathsf{P}}^{-2} \int_{\mathsf{P}} q v_h \, d\mathbf{x}$$
 for any $q \in \mathbb{P}_{r-2p_1}(\mathsf{P})$.

For $r \ge 2p_2 - 1$, we note that $\beta_j = r - (2p_2 - j)$ and we consider the polynomial edge moments in (**D2**) for $j = 0, ..., p_2 - 1$.

Remark 4.3. The L^2 -projection operator $\Pi_{r-2p_1}^{0,P}$ onto the polynomial space $\mathbb{P}_{r-2p_1}(P)$ is computable from the degrees of freedom (D3).

A virtual element function in $v_h \in V_{h,r}^{p_2,p_1}(\mathsf{P})$ has the regularity property that $(v_h)_{|\partial\mathsf{P}|} \in C^{p_2-1}(\partial\mathsf{P})$. This regularity is reflected by the choice of the degrees of freedom, and is, indeed, provided by the vertex degrees of freedom of (**D1**). Furthermore, the traces of v_h (j = 0) and the *j*-th normal derivatives (up to order $j = p_2 - 1$) on every edge $e \in \partial\mathsf{P}$ are univariate polynomials of degree at least r - j. The information provided by (**D1**) makes it possible to build polynomial traces of degree higher than r - j if *r* is equal to p_2 (or not "too bigger" than p_2 as shown in the following examples). In such a case, we can compute the edge traces $(v_h)_{|e}$ and $(\partial_n^j v_h)_{|e}$ by solving the interpolation problem that uses the vertex values of v_h and its partial derivatives.

In the next three examples, we discuss the trace interpolation problem for $r \ge p_2$ and j = 0, j = 1, $j \ge 2$. This process is also shown in Table 2 and Fig. 1

Example 4.4 (j = 0). We derive the higher-order tangential derivatives of v_h by repetitively applying the differential operator $\mathbf{t} \cdot \nabla = (t_x \partial_x + t_y \partial_y)$ to the univariate polynomial trace of v_h , i.e., $\partial_t^{\ell} v_h$ along every elemental edge (recall that $\partial_t^0 v_h = v_h$ for $\ell = 0$). For example, for $\ell = 1, 2, 3, 4$, the tangential derivatives $\partial_t^{\ell} v_h$ are given by

$$\partial_t v_h(\mathbf{v}_i) = t_x \partial_x (v_h) (\mathbf{v}_i) + t_y \partial_y (v_h) (\mathbf{v}_i),$$

$$\partial_t^2 v_h(\mathbf{v}_i) = (t_x \partial_x + t_y \partial_y) (\partial_t v_h) (\mathbf{v}_i)$$

$$= t_x \partial_x (\partial_t v_h) (\mathbf{v}_i) + t_y \partial_y (\partial_t v_h) (\mathbf{v}_i)$$

$$= t_x t_x \partial_{xx} v_h (\mathbf{v}_i) + 2t_x t_y \partial_{xy} v_h (\mathbf{v}_i) + t_y t_y \partial_{yy} v_h (\mathbf{v}_i),$$

$$\partial_t^3 v_h(\mathbf{v}_i) = (t_x \partial_x + t_y \partial_y) (\partial_t^2 v_h) (\mathbf{v}_i)$$

$$= t_x \partial_x (\partial_t^2 v_h) (\mathbf{v}_i) + t_y \partial_y (\partial_t^2 v_h) (\mathbf{v}_i)$$

$$= t_x t_x t_x \partial_{xxx} v_h(v_i) + 3t_x t_x t_y \partial_{xxy} v_h(v_i) + 3t_x t_y t_y \partial_{xyy} v_h(v_i)$$

+ $t_y t_y t_y \partial_{yyy} v_h(v_i)$,
 $\partial_t^4 v_h(v_i) = (t_x \partial_x + t_y \partial_y) (\partial_t^3 v_h)(v_i)$
= $t_x \partial_x (\partial_t^3 v_h)(v_i) + t_y \partial_y (\partial_t^3 v_h)(v_i)$
= $t_x t_x t_x t_x \partial_{xxxx} v_h(v_i) + 4t_x t_x t_x t_y \partial_{xxxy} v_h(v_i) + 6t_x t_x t_y t_y \partial_{xxyy} v_h(v_i)$
+ $4t_x t_y t_y t_y \partial_{xyyy} v_h(v_i) + t_y t_y t_y t_y \partial_{yyyy} v_h(v_i)$.

It is easy to recognize the pattern of the combinatorial coefficients in these expansions. According to the third column (j = 0) of Table 1, the degrees of freedom (**D1**) at vertex v_i for a given regularity index p_2 yield p_2 pieces of information, $v_h(v_i)$, $\partial_t v_h(v_i)$, $\partial_t^2 v_h(v_i)$, \dots $\partial_t^{p_2-1} v_h(v_i)$. Since each edge has two vertices, we have $2p_2$ pieces of information and we can interpolate the edge trace of v_h as a univariate polynomial of degree $2p_2 - 1$. Such polynomial degree is clearly bigger than r if we choose r such that $p_2 \leq r < 2p_2 - 1$. This fact is not in conflict with the property that the virtual element space contains the subspace of polynomials of degree r.

The polynomial degrees of the edge trace of v_h that we can interpolate from the degrees of freedom (**D1**)-(**D2**) are illustrated in Table 2 by the rows for j = 0 and different values of p_2 . In this table, the values of r such that $r = 2p_2 - 1$ are reported in bold, and the ones for $r < 2p_2 - 1$ are those preceeding the bold ones on the same row. For these values of r the trace of v_h can be interpolated from the information provided by (**D1**). However, if we increase the polynomial degree r so that $r > 2p_2 - 1$, the degrees of freedom (**D1**) are no longer enough to solve the interpolation problem. In such a case, we need the additional degrees of freedom of (**D2**), i.e., the moments of v_h against a (basis of) polynomials of degree $r - (2p_2 - 1)$ defined on e.

Example 4.5 (j = 1). As for the case j = 0, we derive the higher-order tangential derivatives of $\partial_n v_h = n_x \partial_x v_h + n_y \partial_y v_h$ by repetitively applying the differential operator $\mathbf{t} \cdot \nabla = (t_x \partial_x + t_y \partial_y)$ to the univariate polynomial trace of $\partial_n v_h$, i.e, $\partial_t^{\ell} \partial_n v_h$ along every elemental edge (recall that $\partial_t^0 \partial_n v_h = \partial_n v_h$ for $\ell = 0$). For example, for $\ell = 1, 2, 3$ we find that

$$\begin{aligned} \partial_t \partial_n v_h(\mathbf{v}_i) &= \left(t_x \partial_x + t_y \partial_y \right) \left(\partial_n v_h \right) (\mathbf{v}_i) \\ &= t_x \partial_x \left(\partial_n v_h \right) (\mathbf{v}_i) + t_y \partial_y \left(\partial_n v_h \right) (\mathbf{v}_i) \\ &= t_x n_x \partial_{xx} v_h(\mathbf{v}_i) + \left(t_x n_y + t_y n_x \right) \partial_{xy} v_h(\mathbf{v}_i) + t_y t_y \partial_{yy} v_h(\mathbf{v}_i), \\ \partial_t^2 \partial_n v_h(\mathbf{v}_i) &= \left(t_x \partial_x + t_y \partial_y \right) \left(\partial_t \partial_n v_h \right) (\mathbf{v}_i) \\ &= t_x \partial_x \left(\partial_t \partial_n v_h \right) (\mathbf{v}_i) + t_y \partial_y \left(\partial_t \partial_n v_h \right) (\mathbf{v}_i) \\ &= t_x t_x n_x \partial_{xxx} v_h(\mathbf{v}_i) + \left(t_x \left(t_x n_y + t_y n_x \right) + t_y \left(t_x n_y + t_y n_x \right) \right) \partial_{xxy} v_h(\mathbf{v}_i) \\ &+ t_x t_y \left(n_x + n_y \right) \partial_{xyy} v_h(\mathbf{v}_i) + t_y t_y n_y \partial_{yyy} v_h(\mathbf{v}_i), \\ \partial_t^3 \partial_n v_h(\mathbf{v}_i) &= \left(t_x \partial_x + t_y \partial_y \right) \left(\partial_t^2 \partial_n v_h \right) (\mathbf{v}_i) \\ &= t_x \partial_x \left(\partial_t^2 \partial_n v_h(\mathbf{v}_i) \right) (\mathbf{v}_i) + t_y \partial_y \left(\partial_t^2 \partial_n v_h(\mathbf{v}_i) \right) (\mathbf{v}_i), \\ &= \dots \end{aligned}$$

According to the fourth column (j = 1) of Table 1, the degrees of freedom (**D1**) at vertex v_i for a given p_2 yield $p_2 - 1$ pieces of information, $\partial_n v_h(v_i)$, $\partial_t \partial_n v_h(v_i)$, $\partial_t^2 \partial_n v_h(v_i)$, \dots $\partial_t^{p_2-2} \partial_n v_h(v_i)$. Since each edge has two vertices, we have $2(p_2-1)$ pieces of information that we can interpolate as a polynomial of degree $2p_2 - 3$. Such polynomial degree is clearly bigger than r - 1 if we choose r such that $p_2 \le r < 2(p_2 - 1)$.

The polynomial degrees of the edge trace of $\partial_n v_h$ that we can interpolate from the degrees of freedom (**D1**)-(**D2**) are illustrated in Table 2 by the rows for j = 1 and different values of p_2 . In this table, the values of r such that $r - 1 = 2(p_2 - 1) - 1$ are reported in bold, and the ones for $r - 1 < 2(p_2 - 1) - 1$ are those preceeding the bold ones on the same row. For these values of r the trace of $\partial_n v_h$ can be interpolated from the information provided by (**D1**). However, if we increase the polynomial degree r so that $r - 1 > 2(p_2 - 1) - 1$, the degrees of freedom (**D1**) are no longer enough to solve the interpolation problem. In such a case, we need the additional degrees of freedom of (**D2**), i.e., the moments of $\partial_n v_h$ against a (basis of) polynomials of degree $r - 2(p_2 - 1)$ defined on e.

Example 4.6 $(j \ge 2)$. As for the cases j = 0 and j = 1, we derive the higher-order tangential derivatives of $\partial_n^j v_h$ by repetitively applying the differential operator $\mathbf{t} \cdot \nabla = (t_x \partial_x + t_y \partial_y)$ to the univariate polynomial trace of $\partial_n^j v_h$, i.e., $\partial_t^\ell \partial_n^j v_h$ along every elemental edge. For example, for j = 2, since $\partial_n^2 v_h = n_x n_x \partial_{xx} v_h + 2n_x n_y \partial_{xy} v_h + t_y t_y \partial_{yy} v_h$, we find that

$$\begin{aligned} \partial_t \partial_n^2 v_h(\mathsf{v}_i) &= \left(t_x \partial_x + t_y \partial_y \right) \left(\partial_n^2 v_h \right) (\mathsf{v}_i) \\ &= t_x \partial_x \left(\partial_n^2 v_h \right) (\mathsf{v}_i) + t_y \partial_y \left(\partial_n^2 v_h \right) (\mathsf{v}_i) \\ &= t_x n_x n_x \partial_{xxx} v_h + \left(2 t_x t_x n_x + t_y n_x n_x \right) \partial_{xxy} v_h \\ &+ \left(t_x n_y n_y + 2 t_y n_x n_y \right) \partial_{xyy} v_h + t_y n_y n_y \partial_{yyy} v_h. \end{aligned}$$

For $j \ge 0$, each edge vertex v_i , i = 1, 2, provides the values of $\partial_n^j v_h$ and its first $p_2 - j - 1$ tangential derivatives. Hence, there are $2(p_2 - j)$ pieces of information available on each edge and we can interpolate the edge trace of $\partial_n^j v_h$ as a univariate polynomial of degree $2(p_2 - j) - 1$. Such polynomial degree is clearly bigger than r - j if we choose r such that $p_2 \le r < 2p_2 - j - 1$.

The polynomial degrees of the edge trace of $\partial_n^j v_h$ that we can interpolate from the degrees of freedom (**D1**)-(**D2**) are illustrated in Table 2 by the rows for $j \ge 0$ and different values of p_2 . In this table, the values of r such that $r = 2p_2 - j - 1$ are reported in bold, and the ones for $r < 2p_2 - j - 1$ are those preceeding the bold ones on the same row. For these values of r the trace of $\partial_n^j v_h$ can be interpolated from the information provided by (**D1**). However, if we increase the polynomial degree r so that $r > 2p_2 - j - 1$, the degrees of freedom (**D1**) are no longer enough to solve the interpolation problem. In such a case, we need the additional degrees of freedom of (**D2**), i.e., the moments of $\partial_n^j v_h$ against a (basis of) polynomials of degree $r - (2p_2 - j - 1)$ defined on e. It is worth noting that for increasing values of r, we need to supplement this information starting from the higher-order normal derivatives (see Fig. 1).

Lemma 4.7. The degrees of freedom (D1)-(D3) are unisolvent in the virtual element space $V_{h r}^{p_2, p_1}(P)$.

Proof. Let P be a polygonal element. First, a counting argument shows that the number of degrees of freedom (**D1**)-(**D3**) is equal to the dimension of $V_{h,r}^{p_2,p_1}(\mathsf{P})$ (see Remark 4.2). Then, we prove that a virtual element function v_h is necessarily zero if all its degrees of freedom (**D1**)-(**D3**) are zero. In particular, assuming that the degrees of freedom (**D1**) and (**D2**) are zero implies that the polynomial traces of v_h and its normal derivatives of order up to $p_2 - 1$ are identically zero on all edges of $\partial \mathsf{P}$, and so are their tangential derivatives of any order. Likewise, assuming that the degrees of freedom (**D3**) are zero implies that the elemental moments of v_h against the polynomials of degree up to $r - 2p_1$ (for $r \ge 2p_1$) are zero.

<i>p</i> ₂ – 1	Degrees of freedom (D1)	j = 0	<i>j</i> = 1	<i>j</i> = 2	<i>j</i> = 3	<i>j</i> = 4
0	$v_h(v_i)$	v _h				
1	$\partial_x v_h(v_i), \partial_y v_h(v_i)$	$\partial_t v_h$	$\partial_n v_h$			
2	$ \begin{aligned} \partial_{xx} v_h(v_i), \partial_{xy} v_h(v_i), \\ \partial_{yy} v_h(v_i) \end{aligned} $	$\partial_t^2 v_h$	$\partial_t \partial_n v_h$	$\partial_n^2 v_h$		
3	$\partial_{xxx} v_h(\mathbf{v}_i), \partial_{xxy} v_h(\mathbf{v}_i), \\ \partial_{xyy} v_h(\mathbf{v}_i), \partial_{yyy} v_h(\mathbf{v}_i)$	$\partial_t^3 v_h$	$\partial_t^2 \partial_n v_h$	$\partial_t \partial_n^2 v_h$	$\partial_n^3 v_h$	
4	$ \begin{aligned} \partial_{xxxx} v_h(v_i), \partial_{xxxy} v_h(v_i), \\ \partial_{xxyy} v_h(v_i), \partial_{xyyy} v_h(v_i), \\ \partial_{yyyy} v_h(v_i) \end{aligned} $	$\partial_t^4 v_h$	$\partial_t^3 \partial_n v_h$	$\partial_t^2 \partial_n^2 v_h$	$\partial_t \partial_n^3 v_h$	$\partial_n^4 v_h$

Table 1: Vertex degrees of freedom for the trace interpolation process on the elemental edges. The first column shows the value of max $\{|v|\} = p_2 - 1$ that we use to define the degrees of freedom (**D1**). The second column shows the degrees of freedom at vertex v_i corresponding to $p_2 - 1$ in the first column. The remaining columns shows the quantities that we can compute using the degrees of freedom listed in the second column. Recalling that $|v| \le p_2 - 1$ and $0 \le j \le p_2 - 1$, on the columns for $j = 0, \ldots, 4$ we read the pieces of information that are available for the interpolation of $\partial_n^j v_h$ (with $\partial_n^0 v_h = v_h$ for j = 0). For example, if $p_2 = 3$, we can use only the objects of the first three table rows (i.e., |v| = 0, 1, 2), and each column for j = 0, 1, 2 lists the pieces of information that are available to construct the polynomial trace of v_h , $\partial_n v_h$ and $\partial_n^2 v_h$ on each edge. In such a case, the vertex degrees of freedom allows us to interpolate the trace of v_h as a polynomial of degree 5, the trace of $\partial_n v_h$ as a polynomial of degree 3, and the trace of $\partial_n^2 v_h$ as a polynomial of degree 1. These trace interpolations are consistent with $r = p_2$ and a global virtual element space with C^{p_2-1} -regularity on Ω .

Consider separately the case of odd and even values of p_2 . If $p_2 = 2\ell + 1$ with $\ell \ge 0$, a repeated application of the integration by parts formula yields

$$\int_{\mathsf{P}} \left| \nabla \Delta^{\ell} v_{h} \right|^{2} d\mathbf{x} = -\int_{\mathsf{P}} \left(\Delta^{p_{2}} v_{h} \right) v_{h} d\mathbf{x} + \int_{\partial \mathsf{P}} \left(\partial_{n} \Delta^{\ell} v_{h} \right) \Delta^{\ell} v_{h} ds + \sum_{i=1}^{\ell} \left(\int_{\partial \mathsf{P}} \left(\partial_{n} \Delta^{p_{2}-i} v_{h} \right) \Delta^{i-1} v_{h} ds - \int_{\partial \mathsf{P}} \left(\Delta^{p_{2}-i} v_{h} \right) \partial_{n} \Delta^{i-1} v_{h} ds \right).$$
(13)

Similarly, if $p_2 = 2\ell$ with $\ell \ge 1$, we find that

$$\int_{\mathsf{P}} \left| \Delta^{\ell} v_h \right|^2 \, d\mathbf{x} = \int_{\mathsf{P}} \left(\Delta^{p_2} v_h \right) v_h \, d\mathbf{x}$$
$$- \sum_{i=1}^{\ell} \left(\int_{\partial \mathsf{P}} \left(\partial_n \Delta^{p_2 - i} v_h \right) \Delta^{i - 1} v_h \, ds - \int_{\partial \mathsf{P}} \left(\Delta^{p_2 - i} v_h \right) \partial_n \Delta^{i - 1} v_h \, ds \right). \tag{14}$$

Since $\Delta^{p_2}v_h$ is a polynomial of degree $r - 2p_1$ according to the definition of the virtual element space, the volume integral in the right-hand sides of (13) and (14) is an elemental moment of v_h . This integral must be zero since we assumed that the degrees of freedom (**D3**) of v_h are zero.

		r – j									
p ₂ = 1	$2(p_2-j)-1\\$	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7			
j = 0	1	1	2	3	4	5	6	7			
p ₂ = 2	$2(p_2-j)-1\\$	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7			
j = 0	3	-	2	3	4	5	6	7			
<i>j</i> = 1	1	-	1	2	3	4	5	6			
p ₂ = 3	$2(p_2-j)-1\\$	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7			
j = 0	5	-	-	3	4	5	6	7			
<i>j</i> = 1	3	-	-	2	3	4	5	6			
<i>j</i> = 2	1	-	-	1	2	3	4	5			
p ₂ = 4	$2(p_2-j)-1\\$	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7			
j = 0	7	-	-	_	4	5	6	7			
<i>j</i> = 1	5	-	-	_	3	4	5	6			
<i>j</i> = 2	3	-	_	_	2	3	4	5			
<i>j</i> = 3	1	-	_	-	1	2	3	4			

Table 2: Polynomial orders of the edge traces of v_h and its normal derivatives $\partial_n^j v_h$ for $p_2 = 1, 2, 3, 4$ and $r = p_2, \dots 7$ (we recall that $r \ge p_2$). The first column on the left reports the value of p_2 and $j = 0, \dots, p_2 - 1$; the second column reports the value of $2(p_2 - j) - 1$, which is a threshold value, and the remaining columns the possible values of r - j (remember that on each edge $\partial_n^j v_h \in \mathbb{P}_{\alpha_j}$ with $\alpha_j = \max\{2(p_2 - j) - 1, r - j\}$). The values of the polynomial degree r - 1 such that $r - j = 2(p_2 - j) - 1$ (or, equivalently, that $r = 2p_2 - j - 1$) are reported in bold font. The polynomial traces with degree equal or higher than this bold value, which are above it in every column ad correspond to the smaller order j of derivation, can be interpolated using only the vertex degrees of freedom (**D1**). To interpolate the remaining edge traces we need the additional information provided by (**D2**).

To prove that the edge integrals in (13) and (14) are zero, we first note that such integrals contain the edge trace of $\Delta^{\mu}v_h$ for $\mu = 0, \dots, p_2 - 1$ and its normal and tangential derivatives. Since $\Delta v_{h|e} = \partial_t^2 v_h + \partial_n^2 v_h$, it holds that

$$\Delta^{\mu} v_{h} = \left(\partial_{t}^{2} + \partial_{n}^{2}\right)^{\mu} v_{h} = \sum_{\nu=0}^{\mu} C_{\mu,\nu} \,\partial_{t}^{2(\mu-\nu)} \,\partial_{n}^{2\nu} v_{h}, \tag{15}$$

where $C_{\mu,\nu}$ denote the ν -th combinatorial coefficient of the μ -th power expansion. Therefore, all the edge integrals either contain the normal derivatives $\partial_n^{\ell} v_h$ for some integer $\ell = 0, \ldots, p_2 - 1$, or the tangential derivatives of these quantities. As noted at the beginning of this proof, all these quantities are zero since we assumed that the degrees of freedom (**D1**)-(**D2**) of v_h are zero.

Finally, we note that a function $v_h \in V_{h,r}^{p_2,p_1}(\mathsf{P})$ with all zero degrees of freedom also belongs to $H_0^{p_2}(\mathsf{P}) = \{v \in H^{p_2}(\mathsf{P}) : \partial^j v_{|\partial\mathsf{P}} = 0 \forall j = 0, \dots, p_2 - 1\}$. Since both left-hand sides of (13) and (14) are a norm on $H_0^{p_2}(\mathsf{P})$, it follows that $v_h = 0$.

4.3. Global virtual element spaces

Building upon the local virtual element spaces, the *global* conforming virtual element space $V_{h,r}^{p_2,p_1}$ is defined on Ω as

$$V_{h,r}^{p_2,p_1} = \left\{ v_h \in H_0^{p_1}(\Omega) \cap H^{p_2}(\Omega) : v_{h|\mathsf{P}} \in V_{h,r}^{p_2,p_1}(\mathsf{P}) \ \forall \mathsf{P} \in \Omega_h \right\},\tag{16}$$



Figure 1: Edge degrees of freedom of the virtual element space $V_{h,r}^{p_2,p_1}$ with regularity index $p_2 = 1$ (Laplace operator), $p_2 = 2$ (bi-harmonic operator), $p_2 = 3$ (tri-harmonic operator), $p_2 = 4$, and polynomial degree r such that $p_2 \le r \le p_2 + 4$. The (green) dots at the vertices represent the vertex values and each (red) vertex circle represents an order of derivation. The (black) dots on the edge represent the polynomial moments of the trace $v_{h|e}$; the arrows represent the polynomial moments of $\partial_n v_{h|e}$; the double arrows represent the polynomial moments of $\partial_n^2 v_{h|e}$.

where $V_{h,r}^{p_2,p_1}(\mathsf{P})$ is the local space defined in (9) if $p_2 \le r \le 2p_2 - 2$ and the local space defined in (10) if $r \ge 2p_2 - 1$.

Remark 4.8. Let $N^{\mathcal{P}}$, $N^{\mathcal{E}}$ and $N^{\mathcal{V}}$ denote the number of element, edges and vertices of Ω_h . The dimension of the global virtual element space built upon (9) is given by

$$dim V_{h,r}^{p_2,p_1} = N^{\mathcal{P}} \frac{(r-2p_1+1)(r-2p_1+2)}{2} + N^{\mathcal{E}} \sum_{j=0}^{p_2-1} \left(\alpha_j(p_2,r) + 1 \right) \\ - N^{\mathcal{V}} \frac{(p_2+1)p_2}{2}.$$

The dimension of the global virtual element space built upon (10) is given by

$$\dim V_{h,r}^{p_2,p_1} = N^{\mathcal{P}} \frac{(r-2p_1+1)(r-2p_1+2)}{2} + N^{\mathcal{E}} \frac{(p_2-1)(2(r+1)-(p_2-1))}{2} - N^{\mathcal{V}} \frac{(p_2+1)p_2}{2}.$$

The set of global degrees of freedom are inherited from the local degrees of freedom of section 4.2. Therefore, we consider

(**D1**) $h_v^{|\nu|} D^{\nu} v_h(v), |\nu| \le p_2 - 1$ for every vertex v of \mathcal{V}_h ;

(**D2**)
$$h_e^{-1+j} \int_e q \partial_n^j v_h \, ds$$
 for any $q \in \mathbb{P}_{\beta_j}(e)$ and every edge e of \mathcal{E}_h , $j = k + 1, \dots, p_2 - 1$;
(**D3**) $h_{\mathsf{P}}^{-2} \int_{\mathsf{P}} q v_h \, d\mathbf{x}$ for any $q \in \mathbb{P}_{r-2p_1}(\mathsf{P})$ and every element P of Ω_h ,

where, again, $\beta_j = \alpha_j - \min\{2(p_2 - j) - 1, r - j\} - 1$ and $\alpha_j(p_2, r) = \max\{2(p_2 - j) - 1, r - j\}, j = 0, ..., p_2 - 1$. For $r \ge 2p_2 - 1$, these degrees of freedom become

(D1)
$$h_v^{|\nu|} D^{\nu} v_h(\nu), |\nu| \le p_2 - 1$$
 for every interior vertex v of \mathcal{V}_h ;

(**D2**)
$$h_e^{-1+j} \int_e q \partial_n^j v_h \, ds$$
 for any $q \in \mathbb{P}_{r-2p_2+j}(e)$ $j = 0, \dots, p_2 - 1$ and every interior edge e of \mathcal{E}_h ;

(D3)
$$h_{\mathsf{P}}^{-2} \int_{\mathsf{P}} q v_h d\mathbf{x}$$
 for any $q \in \mathbb{P}_{r-2p_1}(\mathsf{P})$ and every element P of Ω_h

We remark that the associated global space is made of $H^{p_2}(\Omega)$ functions. Indeed, the restriction of a virtual element function v_h to each element P belongs to $H^{p_2}(P)$ and glues with C^{p_2-1} -regularity across the internal mesh faces.

Finally, the unisolvence of these degrees of freedom is an immediate consequence of the unisolvence of the elementwise degrees of freedom in any elemental space $V_{h,r}^{p_2,p_1}(\mathsf{P})$, cf. Lemma 4.7.

4.4. Elliptic projection operator

The elliptic projection operator $\Pi_r^{p_1,\mathsf{P}}: V_{h,r}^{p_2,p_1}(\mathsf{P}) \to \mathbb{P}_r(\mathsf{P})$ is such that for all $v_h \in V_{h,r}^{p_2,p_1}(\mathsf{P})$, the projection $\Pi_r^{p_1,\mathsf{P}}v_h$ is the solution of the finite dimensional variational problem

$$a_{p_1}^{\mathsf{P}}(\Pi_r^{p_1,\mathsf{P}}v_h - v_h, q) = 0 \quad \forall q \in \mathbb{P}_r(\mathsf{P}),$$
(17)

$$\int_{\partial \mathsf{P}} \left(\Pi_r^{p_1,\mathsf{P}} v_h - v_h \right) q \, ds = 0 \quad \forall q \in \mathbb{P}_{p_1 - 1}(\mathsf{P}).$$
⁽¹⁸⁾

Condition (18) allows us to fix the nontrivial kernel of $a_{p_1}^{\mathsf{P}}(\cdot, \cdot)$, which is the subspace of polynomials of degree (up to) $p_1 - 1$.

Remark 4.9. Instead of (18), we can consider the alternative condition [10]

$$\widehat{\Pi}^{P} D^{\nu} \Pi_{r}^{p_{1},P} v_{h} = \widehat{\Pi}^{P} D^{\nu} v_{h} \quad with \ |\nu| \le p_{1} - 1,$$

by using the vertex average projection $\widehat{\Pi}^{P} : C(P) \to \mathbb{P}_{0}(P)$, which is such that

$$\widehat{\Pi}^{P}\psi = \frac{1}{N^{\mathcal{P}}} \sum_{\mathbf{v} \in \partial P} \psi(\mathbf{v}), \tag{19}$$

for all continuous function ψ .

Lemma 4.10. The elliptic projection operator $\Pi_r^{p_1,P}$ is polynomial preserving in the sense that $\Pi_r^{p_1,P}q = q$ for every $q \in \mathbb{P}_r(P)$.

Proof. Let $\mathbb{P}_r(\mathsf{P}) \setminus \mathbb{P}_{p_1-1}(\mathsf{P})$ denote the linear space of polynomials of degrees *s* such that $p_1 \leq s \leq r$, and consider the decomposition

$$\mathbb{P}_{r}(\mathsf{P}) = \mathbb{P}_{r}(\mathsf{P}) \setminus \mathbb{P}_{p_{1}-1}(\mathsf{P}) \oplus \mathbb{P}_{p_{1}-1}(\mathsf{P}).$$
(20)

We expand the polynomial $q \in \mathbb{P}_r(\mathsf{P})$ and its projection $\Pi_r^{p_1,\mathsf{P}}q$ as follows

$$q = \sum_{\ell'} c_{\ell'}(q) \mu_{\ell'} + \sum_{\ell'} \widetilde{c}_{\ell'}(q) \widetilde{\mu}_{\ell'}, \qquad (21)$$

$$\Pi_{r}^{p_{1},\mathsf{P}}q = \sum_{\ell'} c_{\ell'}(\Pi_{r}^{p_{1},\mathsf{P}}q)\mu_{\ell'} + \sum_{\ell'} \widetilde{c}(\Pi_{r}^{p_{1},\mathsf{P}}q)\widetilde{\mu}_{\ell'},$$
(22)

where $\{\mu_{\ell'}\}$ is a basis of $\mathbb{P}_r(\mathsf{P}) \setminus \mathbb{P}_{p_1-1}(\mathsf{P}), \{\tilde{\mu}_{\ell'}\}$ is a basis of $\mathbb{P}_{p_1-1}(\mathsf{P})$, and $c_{\ell'}(q), \tilde{c}_{\ell'}(q), c_{\ell'}(\Pi_r^{p_1,\mathsf{P}}q)$, and $\tilde{c}(\Pi_r^{p_1,\mathsf{P}}q)$ are the coefficients of such expansions. The range of the summation index ℓ' , which is not expicitly indicated in (21) and (22), is consistent with the dimensions of $\mathbb{P}_r(\mathsf{P}) \setminus \mathbb{P}_{p_1-1}(\mathsf{P})$ and $\mathbb{P}_{p_1-1}(\mathsf{P})$. We assume that the polynomials $\mu_{\ell'}$ are orthogonal with respect to the semi-inner product $a_{p_1}^{\mathsf{P}}(\cdot,\cdot)$, which is the restriction of $a_{p_1}^{\mathsf{P}}(\cdot,\cdot)$ to a polygonal element P , so that $a_{p_1}^{\mathsf{P}}(\mu_{\ell'},\mu_{\ell}) = |\mathsf{P}| \,\delta_{\ell',\ell}$. Since the polynomials $\tilde{\mu}_{\ell'}$ belong to the kernel of $a_{p_1}^{\mathsf{P}}(\cdot,\cdot)$, we substitute the expansions (21) and (22) in (17) (with $v_h = q$ and $q = \mu_{\ell}$) and we find that

$$0 = a_{p_1}^{\mathsf{P}}(\Pi_r^{p_1,\mathsf{P}}q - q, \mu_{\ell}) = \sum_{\ell'} \left(c_{\ell'}(\Pi_r^{p_1,\mathsf{P}}q) - c_{\ell'}(q) \right) a_{p_1}^{\mathsf{P}}(\mu_{\ell}, \mu_{\ell'})$$
$$= |\mathsf{P}| \sum_{\ell'} \left(c_{\ell'}(\Pi_r^{p_1,\mathsf{P}}q) - c_{\ell'}(q) \right) \delta_{\ell,\ell'} = |\mathsf{P}| \left(c_{\ell}(\Pi_r^{p_1,\mathsf{P}}q) - c_{\ell}(q) \right),$$

which holds for all possible integers ℓ . This relation implies that

$$\Pi_r^{p_1,\mathsf{P}}q - q = \sum_{\ell'} \left(\widetilde{c}_{\ell'}(\Pi_r^{p_1,\mathsf{P}}q) - \widetilde{c}_{\ell'}(q) \right) \widetilde{\mu}_{\ell'} \in \mathbb{P}_{p_1-1}(\mathsf{P}).$$
(23)

Then, we assume that the polynomials $\tilde{\mu}_{\ell'}$ are orthogonal with respect to the inner product $(v, u)_{\partial P} = \int_{\partial P} v u \, ds$, so that $(\tilde{\mu}_{\ell'}, \tilde{\mu}_{\ell})_{\partial P} = |\partial P| \, \delta_{\ell',\ell}$, where $|\partial P|$ is the perymeter of ∂P . We substitute (23) in (18) (with $v_h = q$ and $q = \tilde{\mu}_{\ell}$) and we find that

$$0 = \int_{\partial \mathsf{P}} \left(\Pi_r^{p_1,\mathsf{P}} q - q \right) \widetilde{\mu}_{\ell} \, ds = \sum_{\ell'} \left(\widetilde{c}_{\ell'} (\Pi_r^{p_1,\mathsf{P}} q) - \widetilde{c}_{\ell'}(q) \right) \int_{\partial \mathsf{P}} \widetilde{\mu}_{\ell'} \widetilde{\mu}_{\ell} \, ds$$
$$= |\partial \mathsf{P}| \sum_{\ell'} \left(\widetilde{c}_{\ell'} (\Pi_r^{p_1,\mathsf{P}} q) - \widetilde{c}_{\ell'}(q) \right) \delta_{\ell',\ell} = |\partial \mathsf{P}| \left(\widetilde{c}_{\ell} (\Pi_r^{p_1,\mathsf{P}} q) - \widetilde{c}_{\ell}(q) \right).$$

which holds for all possible integers ℓ . This implies that $\prod_{r=1}^{p_1, \mathsf{P}} q - q = 0$, which is the assertion of the lemma.

Lemma 4.11. The polynomial projection $\Pi_r^{p_1, P} v_h$ is computable using only the degrees of freedom (D1)-(D3) of $v_h \in V_{h,r}^{p_2, p_1}(P)$.

Proof. To prove the assertion of the lemma, we only need to prove that $a_{p_1}^{\mathsf{P}}(v_h, q)$ and $(v, u)_{\partial \mathsf{P}} = \int_{\partial \mathsf{P}} v_h q \, ds$ are computable for all $v_h \in V_{h,r}^{p_2,p_1}(\mathsf{P})$ and scalar polynomial $q \in \mathbb{P}_r(\mathsf{P})$. To this end, we integrate by parts $a_{p_1}^{\mathsf{P}}(v_h, q)$. For an odd p_1 , i.e., $p_1 = 2\ell + 1$, we find that

$$a_{p_1}^{\mathsf{P}}(v_h, q) = -\int_{\mathsf{P}} \left(\Delta^{p_1} v_h \right) q \, d\mathbf{x} + \int_{\partial \mathsf{P}} \left(\partial_n \Delta^{\ell} v_h \right) \Delta^{\ell} q \, ds$$

$$+\sum_{i=1}^{\ell} \left(\int_{\partial \mathsf{P}} \left(\partial_n \Delta^{p_1 - i} v_h \right) \Delta^{i-1} q \, ds - \int_{\partial \mathsf{P}} \left(\Delta^{p_1 - i} v_h \right) \partial_n \Delta^{i-1} q \, ds \right). \tag{24}$$

For an even p_1 , i.e., $p_1 = 2\ell$, we find that

$$a_{p_{1}}^{\mathsf{P}}(v_{h},q) = \int_{\mathsf{P}} \left(\Delta^{p_{1}}v_{h}\right) q \, d\mathbf{x}$$
$$-\sum_{i=1}^{\ell} \left(\int_{\partial\mathsf{P}} \left(\partial_{n}\Delta^{p_{1}-i}v_{h}\right) \Delta^{i-1}q \, ds - \int_{\partial\mathsf{P}} \left(\Delta^{p_{1}-i}v_{h}\right) \partial_{n}\Delta^{i-1}q \, ds\right). \tag{25}$$

The first integral of the right-hand side of both formulas (24) and (25) is computable from the degrees of freedom (**D3**). In turn, all the edge integrals are computable since we can expand the trace of $\Delta^{\mu}v_h$ in terms of $\partial_t^{2(\mu-\nu)}\partial_n^{2\mu}v_h$ and use the same argument of the proof of Lemma 4.7. Since the edge traces of v_h and its normal derivatives (and all their tangential derivatives) are computable from the degrees of freedom of (**D1**)-(**D2**) through a polynomial interpolation, we deduce that all the edge integrals for both odd and even p_1 and the boundary integral $(v, u)_{\partial P}$ are computable.

4.5. Enhancement

As noted in Remark 4.3, the L^2 -projection operator $\Pi_{r-2p_1}^{0,\mathsf{P}}: V_{h,r}^{p_2,p_1}(\mathsf{P}) \to \mathbb{P}_r(\mathsf{P})$ is computable from the degrees of freedom (**D3**). Instead, to compute the orthogonal projection onto the polynomial subspace $\mathbb{P}_{r-p_1}(\mathsf{P})$, we need to modify the space definition as follows thus obtaining the so called "*enhanced*" virtual element space. Our construction follows the guidelines in [1]. First, we consider the mesh element P and the "*extended*" virtual element space for $r \ge 2p_2 - 1$ (recall that $p_2 \ge p_1$) defined as

$$\widetilde{V}_{h,r}^{p_2,p_1}(\mathsf{P}) := \left\{ v_h \in H^{p_2}(\mathsf{P}) : \Delta^{p_2} v_h \in \mathbb{P}_{r-p_1}(\mathsf{P}), \ \partial_n^j v_h \in \mathbb{P}_{r-j}(e), \\ j = 0, \dots, p_2 - 1 \ \forall e \in \partial \mathsf{P} \right\}.$$
(26)

Then, we define the enhanced virtual element space as

$$W_{h,r}^{p_2,p_1}(\mathsf{P}) := \left\{ v_h \in \widetilde{V}_{h,r}^{p_2,p_1}(\mathsf{P}) : \int_{\mathsf{P}} v_h q \, d\mathbf{x} = \int_{\mathsf{P}} \Pi_{r-p_1}^{p_1,\mathsf{P}} v_h q \, d\mathbf{x} \\ \forall q \in \mathbb{P}_{r-p_1} \setminus \mathbb{P}_{r-2p_1}(\mathsf{P}) \right\}.$$
(27)

The polynomial space $\mathbb{P}_r(\mathsf{P})$ is a subspace of $W_{h,r}^{p_2,p_1}(\mathsf{P})$ and, thus, of $\widetilde{V}_{h,r}^{p_2,p_1}(\mathsf{P})$, and the elliptic projection $\Pi_{r-p_1}^{p_1,\mathsf{P}}: \widetilde{V}_{h,r}^{p_2,p_1}(\mathsf{P}) \to \mathbb{P}_{r-p_1}(\mathsf{P})$ that is defined in (17)-(18) is still computable and only depends on the degrees of freedom (**D1**), (**D2**) and (**D3**). This assertion can easily be proved by repeating the argument of Lemma 4.10.

The virtual element functions of the space $\widetilde{V}_{h,r}^{P_2,P_1}(\mathsf{P})$ are uniquely characterized by the set of degrees of freedom (**D1**), (**D2**), (**D3**) and the set of additional degrees of freedom (**D3**)

$$(\widetilde{\mathbf{D3}}) h_{\mathsf{P}}^{-2} \int_{\mathsf{P}} q v_h \, d\mathbf{x} \text{ for any } q \in \mathbb{P}_{r-p_1}(\mathsf{P}) \setminus \mathbb{P}_{r-2p_1}(\mathsf{P}).$$

We state the unisolvence of these degrees of freedom in the following lemma. The proof is equal to the proof of Lemma 4.7 (consider the degrees of freedom $((D3), (\widetilde{D3}))$ instead of (D3)) and is omitted.

Lemma 4.12. The degrees of freedom (D1), (D2), (D3), ($\widetilde{D3}$) are unisolvent in the virtual element space $\widetilde{V}_{h,r}^{P_2,p_1}(P)$.

Remark 4.13. According to Lemma 4.12, the dimension of $\widetilde{V}_{h,r}^{p_2,p_1}(P)$ must be equal to the cardinality of the set of the degrees of freedom (D1), (D2), (D3), ($\widetilde{D3}$). This statement can also be proved by a counting argument.

Next, we want to prove that the degrees of freedom (**D1**), (**D2**) and (**D3**) are unisolvent in the enhanced space $W_{h,r}^{p_2,p_1}(\mathsf{P})$. To this end, we first need to establish a technical result. Consider the set of linear, bounded functionals $\lambda_{\ell_1}^{(\mathbf{D1})}, \lambda_{\ell_2}^{(\mathbf{D2})}, \lambda_{\ell_3}^{(\mathbf{D3})} : \tilde{V}_{h,r}^{p_2,p_1}(\mathsf{P}) \to \mathbb{R}$, which respectively return the degrees of freedom (**D1**), (**D2**) and (**D3**) when applied to a virtual element function $v_h \in \tilde{V}_{h,r}^{p_2,p_1}(\mathsf{P})$. The indices ℓ_1, ℓ_2 , and ℓ_3 run from 1 to #(**D1**), #(**D2**) and #(**D3**), respectively, where #(\mathcal{D}) denotes the cardinality of the discrete set \mathcal{D} . Renumbering ℓ_2 and ℓ_3 may require the introduction of suitable sets of basis functions for the polynomial spaces $\mathbb{P}_{r-j}(e)$ and $\mathbb{P}_{r-2p_1}(\mathsf{P})$ in (**D2**) and (**D3**), respectively. We left this aspect undefined as this technicality is not crucial in this presentation, although important in the practical implementations of the method. We introduce the additional set of linear functionals $\lambda_{\ell}^{(\overline{D3})}$ that are such that

$$\lambda_{\widetilde{\ell_3}}^{(\widetilde{\mathbf{D3}})}(v_h) = h_{\mathsf{P}}^{-2} \int_{\mathsf{P}} q_{\widetilde{\ell_3}} (\Pi_{r-p_1}^{p_1,\mathsf{P}} - \Pi_{r-p_1}^{0,\mathsf{P}}) v_h \, d\mathbf{x} \qquad \widetilde{\ell_3} = 1, \dots, \# \widetilde{D3}$$

where $\{q_{\tilde{\ell}_3}\}_{\tilde{\ell}_3}$ is a basis of $\mathbb{P}_{r-p_1}(\mathsf{P}) \setminus \mathbb{P}_{r-2p_1}(\mathsf{P})$, and the index $\tilde{\ell}_3$ runs from 1 to $\#(\widetilde{\mathbf{D3}})$, the number of degrees of freedom of $(\widetilde{\mathbf{D3}})$.

Then, we collect these different types of functionals in the functional set

$$\Lambda = \left(\lambda_{\ell}\right) = \left(\lambda_{\ell_1}^{(\mathbf{D}1)}, \lambda_{\ell_2}^{(\mathbf{D}2)}, \lambda_{\ell_3}^{(\mathbf{D}3)}, \lambda_{\widetilde{\ell_3}}^{(\overline{\mathbf{D}3})}\right).$$
(28)

We assume that the integer index ℓ is consistent with a suitable renumbering of such degrees of freedom; so that ℓ runs form 1 to $m' = m + \#(\widetilde{\mathbf{D3}}) = \dim \widetilde{V}_{h,r}^{P_2,P_1}(\mathsf{P})$ and $m = \#(\mathbf{D1}) + \#(\mathbf{D2}) + \#(\mathbf{D3})$. These functionals satisfy the property stated in the following lemma.

Lemma 4.14. The linear functionals Λ are linearly independent.

Proof. Let $v_h \in \widetilde{V}_{h,r}^{p_2,p_1}(\mathsf{P})$ such that $\lambda_{\ell}(v_h) = 0$ for all $\ell = 1, ..., m'$. Now, the degrees of freedom (**D1**), (**D2**) and (**D3**) of v_h are (obviously) zero as they are the values of the functionals $\lambda_{\ell_1}^{(\mathbf{D1})}(v_h)$, $\lambda_{\ell_2}^{(\mathbf{D2})}(v_h)$ and $\lambda_{\ell_3}^{(\mathbf{D3})}(v_h)$. Moreover, it holds that $\Pi_r^{p_1,\mathsf{P}}v_h = 0$, and, hence, $\Pi_{r-p_1}^{p_1,\mathsf{P}}v_h = 0$, as these projections only depend on the degrees of freedom (**D1**), (**D2**) and (**D3**), cf. Lemma 4.10. Then, we observe that the definition of the orthogonal projection $\Pi_{r-p_1}^{0,\mathsf{P}}v_h$ and the facts that $\lambda_{\ell_3}^{(\widetilde{D3})}(v_h) = 0$ and $\Pi_{r-p_1}^{p_1,\mathsf{P}}v_h = 0$ imply that

$$\int_{\mathsf{P}} q v_h \, d\mathbf{x} = \int_{\mathsf{P}} q \Pi_{r-p_1}^{0,\mathsf{P}} v_h \, d\mathbf{x} = \int_{\mathsf{P}} q \Pi_{r-p_1}^{p_1,\mathsf{P}} v_h \, d\mathbf{x} = 0 \tag{29}$$

for all $q \in \mathbb{P}_{r-p_1}(\mathsf{P}) \setminus \mathbb{P}_{r-2p_1}(\mathsf{P})$. Therefore, the degrees of freedom ($\widetilde{\mathbf{D3}}$) of v_h are equal to zero and finally $v_h = 0$ because the degrees of freedom (**D1**), (**D2**), (**D3**) and ($\widetilde{\mathbf{D3}}$) are unisolvent in $\widetilde{V}_{h,r}^{p_2,p_1}(\mathsf{P})$. This argument proves that the intersection of the kernels of all the linear functionals λ_ℓ contains only the virtual element function that is identically zero over P, so that these linear functionals are necessarily linearly independent.

Using the linear functionals Λ , we reformulate the definition of space $W_{h,r}^{p_2,p_1}(\mathsf{P})$ in the following equivalent way:

$$W_{h,r}^{p_2,p_1}(\mathsf{P}) := \left\{ v_h \in \widetilde{V}_{h,r}^{p_2,p_1}(\mathsf{P}) : \lambda_{\widetilde{\ell}_3}^{(\widetilde{\mathsf{D3}})}(v_h) = 0 \quad \forall \widetilde{\ell}_3 = m+1,\dots,m' \right\}.$$
(30)

In other words, $W_{h,r}^{p_2,p_1}(\mathsf{P})$ belongs to the intersection of the kernels of all the additional linear functionals λ_{ℓ} with $\ell = m + 1, \ldots, m'$. The space $W_{h,r}^{p_2,p_1}(\mathsf{P})$ has the two important properties that are stated in the following lemma.

Lemma 4.15. The virtual element space $W_{h,r}^{p_2,p_1}(P)$ has the same dimension of the "regular" space $V_{h,r}^{p_2,p_1}(P)$ and the set of degrees of freedom (**D1**), (**D2**) and (**D3**) are unisolvent in $W_{h,r}^{p_2,p_1}(P)$.

Proof. In view of Lemma 4.14 the linear functionals in Λ are linearly independent and the cardinality of Λ , i.e., $m' = \#(\Lambda)$, is equal to the dimension of $\widetilde{V}_{h,r}^{p_2,p_1}(\mathsf{P})$. Therefore, $(\mathsf{P}, \mathbb{P}_r(\mathsf{P}), \Lambda)$ is a *finite element* in the sense of Ciarlet, cf. [39]. So, there exists a set of m' dual basis functions ψ_{ℓ} such that

$$\lambda_{\ell}(\psi_{\ell'}) = \delta_{\ell,\ell'} \qquad \ell, \ell' = 1, \dots, m'$$

Now, it holds that $\lambda_{\tilde{\ell}_3}^{(\widetilde{D3})}(\psi_{\ell'}) = \lambda_{\ell}(\psi_{\ell'}) = 0$ for $\ell' = 1, \ldots, m, \ell = m + 1, \ldots, m'$ (and the corresponding values of the index $\tilde{\ell}_3$). This fact has the following consequences. The first *m* linearly independent functions $\psi_{\ell'}, \ell' = 1, \ldots, m$, belong to $W_{h,r}^{p_2,p_1}(\mathsf{P})$, cf. formulation (30), since $\lambda_{\tilde{\ell}_3}^{(\widetilde{D3})}(\psi_{\ell'}) = \lambda_{\ell}(\psi_{\ell'}) = 0$ for $\ell = m + 1, \ldots, m'$ (and corresponding indices $\tilde{\ell}_3$). This implies that dim $W_{h,r}^{p_2,p_1}(\mathsf{P}) \ge m$. Furthermore, according to the space definition (30) all virtual element functions $w_h \in W_{h,r}^{p_2,p_1}(\mathsf{P})$ are such that $\lambda_{\tilde{\ell}_3}^{(\widetilde{D3})}(w_h) = \lambda_{\ell}(w_h) = 0$ for $\ell = m + 1, \ldots, m'$. Therefore, such functions can be written as a linear combination of only the first *m* basis functions $\psi_{\ell'}, \ell' = 1, \ldots, m$, are thus identified by the values of the linear functionals $\lambda_{\ell_1}^{(D1)}v_h, \lambda_{\ell_2}^{(D2)}v_h$ and $\lambda_{\ell_3}^{(D3)}v_h$. Consequently, all virtual element functions of $W_{h,r}^{p_2,p_1}(\mathsf{P}) = m$.

In view of Lemma 4.15, the orthogonal projection operator $\Pi_{r-p_1}^{0,\mathsf{P}}$: $W_{h,r}^{p_2,p_1}(\mathsf{P}) \to \mathbb{P}_{r-p_1}(\mathsf{P})$ is computable from the degrees of freedom (**D1**), (**D2**) and (**D3**).

Finally, we collect the local virtual element spaces into a *global* conforming virtual element space $W_{hr}^{p_2,p_1}$ defined on Ω as

$$W_{h,r}^{p_2,p_1} = \left\{ w_h \in H_0^{p_1}(\Omega) \cap H^{p_2}(\Omega) : w_{h|\mathsf{P}} \in W_{h,r}^{p_2,p_1}(\mathsf{P}) \ \forall \mathsf{P} \in \Omega_h \right\},\tag{31}$$

where $W_{h,r}^{p_2,p_1}(\mathsf{P})$ is the local space defined above.

4.6. The virtual element bilinear form $a_h(\cdot, \cdot)$

We discuss the definition of the bilinear form $a_h(\cdot, \cdot)$ that approximates the bilinear form $a(\cdot, \cdot)$ in the virtual element discretization (5). This construction is the same for the "regular" virtual element spaces (9) and (10) and the "enhanced" space (27). In this section we use the symbol $V_{h,r}^{p_2,p_1}(\mathsf{P})$ to denote both choices of the spaces. However, such construction holds also for the enhanced virtual element space (31). The symmetric bilinear form $a_h : V_{h,r}^{p_2,p_1} \times V_{h,r}^{p_2,p_1} \to \mathbb{R}$, is written as the sum of local terms

$$a_h(u_h, v_h) = \sum_{\mathsf{P} \in \Omega_h} a_h^{\mathsf{P}}(u_h, v_h), \tag{32}$$

where each local term $a_h^{\mathsf{P}}: V_{h,r}^{p_2,p_1}(\mathsf{P}) \times V_{h,r}^{p_2,p_1}(\mathsf{P}) \to \mathbb{R}$ is a symmetric bilinear form. We set

$$a_{h}^{\mathsf{P}}(u_{h}, v_{h}) = a_{p_{1}}^{\mathsf{P}}(\Pi_{r}^{p_{1}, \mathsf{P}}u_{h}, \Pi_{r}^{p_{1}, \mathsf{P}}v_{h}) + S^{\mathsf{P}}(u_{h} - \Pi_{r}^{p_{1}, \mathsf{P}}u_{h}, v_{h} - \Pi_{r}^{p_{1}, \mathsf{P}}v_{h}),$$
(33)

where $S^{\mathsf{P}}: V_{h,r}^{p_2,p_1}(\mathsf{P}) \times V_{h,r}^{p_2,p_1}(\mathsf{P}) \to \mathbb{R}$ provides the stabilization term. The stabilization form $S^{\mathsf{P}}(\cdot, \cdot)$ is a symmetric, positive definite bilinear form for which there exist two positive constants σ_* and σ^* such that

$$\sigma_* a_{p_1}^{\mathsf{P}}(v_h, v_h) \le S^{\mathsf{P}}(v_h, v_h) \le \sigma^* a_{p_1}^{\mathsf{P}}(v_h, v_h) \quad \forall v_h \in V_{h,r}^{p_2, p_1}(\mathsf{P}) \text{ with } \Pi_r^{p_1, \mathsf{P}} v_h = 0.$$
(34)

The constants σ_* , σ^* are independent of *h* (and P). A possible proof of the validity of (34) for the so called "dofi-dofi" stabilization in the context of arbitrarily regular conforming VEM can be found in [46] (for the case $p_1 = 2$ see also [36, 61]). This construction has the *r*-consistency and stability properties stated in (6) and (7),

4.7. The virtual element approximation of the load term

To approximate the right-hand side term of (5) we first assume the elemental decomposition

$$\langle f_h, v_h \rangle = \sum_{\mathsf{P} \in \Omega_h} \int_{\mathsf{P}} f_h v_h \, d\mathbf{x}.$$
 (35)

In Eq. (35), the elemental term $f_{h|P}$ is defined as

$$f_{h|\mathsf{P}} = \begin{cases} \Pi_{r-2p_1}^{0,\mathsf{P}} f, & (a) \text{ if } p_2 + 2p_1 - 1 \le r, \\ \Pi_{r-p_1}^{0,\mathsf{P}} f, & (b) \text{ if } p_2 \le r \le p_2 + 2p_1 - 2. \end{cases}$$
(36)

We discuss the two definitions of f_h given above separately.

Remark 4.16. The right-hand side of (35) is fully computable by using only the degrees of freedom (D3) if $r \ge 2p_1$ and we choose f_h as the piecewise polynomial approximation of f on Ω_h in accordance with (a). In such a case, we do not need to resort to the enhanced virtual element space defined in (27).

Now, consider decomposition (35) and definition (*a*). Since $p_2 \ge p_1$, it holds that $r - 2p_1 \ge p_2 - 1$ (equivalently, $r \ge 3p_1 - 1$). Thus, using the definition of the L^2 -orthogonal projection, from (35), we find that

$$\langle f_h, v_h \rangle = \sum_{\mathsf{P} \in \Omega_h} \int_{\mathsf{P}} \Pi_{r-2p_1}^{0,\mathsf{P}} f v_h \, d\mathbf{x} = \sum_{\mathsf{P} \in \Omega_h} \int_{\mathsf{P}} \Pi_{r-2p_1}^{0,\mathsf{P}} f \Pi_{p_1-1}^{0,\mathsf{P}} v_h \, d\mathbf{x}.$$
 (37)

Applying standard approximation results to (37) and recalling that $v_h \in V_{h,r}^{p_2,p_1} \subset H_0^{p_1}(\Omega) \cap H^{p_2}(\Omega)$ yield the following estimate

$$\langle f - f_h, v_h \rangle \le Ch^{r-p_1+1} |v_h|_{p_1} |f|_{r-2p_1+1},$$

for some positive constant *C* that is independent of *h*. In particular, for $p_1 = p_2 \ge 2$ it is enough to choose $r \ge 2p_2 + 1$ (the case $p_1 = p_2 = 2$ and $r \ge 5$ has been originally treated in [33]). Note that for fixed values of p_1 , larger values of the regularity parameter p_2 ensure higher convergence rate for the approximation of the right-hand side. This is a specific attractive feature of arbitrarily regular conforming VEM (which can not be exploited, e.g., in the nonconforming setting).

Now, consider decomposition (35) and definition (b). Similarly to the previous case, using the definition of the L^2 -orthogonal projection yields

$$\left\langle f_h, v_h \right\rangle = \sum_{\mathsf{P} \in \Omega_h} \int_{\mathsf{P}} \Pi_{r-p_1}^{0,\mathsf{P}} f \, v_h \, d\mathbf{x} = \sum_{\mathsf{P} \in \Omega_h} \int_{\mathsf{P}} \Pi_{r-p_1}^{0,\mathsf{P}} f \, \Pi_0^{0,\mathsf{P}} v_h \, d\mathbf{x}. \tag{38}$$

Applying again standard approximation results to (38) we we find that

$$\langle f - f_h, v_h \rangle \le Ch^{r-p_1+2} |v_h|_{p_1} |f|_{r-p_1+1}$$

For arbitrary values of p_1 and p_2 , the use of the enhancement approach might be avoided using arguments similar to those employed in [47].

4.8. Error analysis

In this section, we briefly recall a convergence result in the energy norm [10] (see also [4, 46]) for the approximation of (1a)-(1b). In particular, employing Theorem 3.1 together with standard results of approximation (see, e.g., Reference [19, 46, 60]) and the approximation properties of the right-hand side contained in Section 4.7.

Theorem 4.17. Let $u \in H_0^{p_1}(\Omega) \cap H^{r+1}(\Omega)$ be the solution of the polyharmonic problem (1a)-(1b) and let $u_h \in V_{h,r}^{p_2,p_1}$ be the solution of the discrete problem (5). Assume that f is sufficiently regular. Then, there exists a positive constant C independent of h such that

$$||u - u_h||_V \le Ch^{r - (p_1 - 1)}.$$
(39)

Remark 4.18. Convergence estimates in lower order norms can be established provided that classical duality arguments can be used and that the polynomial approximation order r is sufficient large[4, 10, 38].

5. Conclusion

We reviewed the construction of highly regular virtual element spaces for the conforming approximations in two spatial dimensions of elliptic problems of order $p_1 \ge 1$. The resulting finite dimensional virtual spaces are subspaces of $H^{p_2}(\Omega)$, $p_2 \ge p_1$. We presented an abstract convergence result in a suitably defined energy norm. Moreover, after discussing the construction of the approximation spaces and major aspects such as the choice and unisolvence of the degrees of freedom, we provided specific examples of highly regular virtual spaces, corresponding to various practical cases. Finally, a detailed discussion of the properties of the "enhanced" formulation of the virtual element spaces is provided.

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References

- Ahmad, B., Alsaedi, A., Brezzi, F., Marini, L.D., Russo, A.: Equivalent projectors for virtual element methods. Comput. Math. Appl. 66(3), 376–391 (2013). https://doi.org/10.1016/j.camwa.2013.05.015
- [2] Aldakheel, F., Hudobivnik, B., Hussein, A., Wriggers, P.: Phase-field modeling of brittle fracture using an efficient virtual element scheme. Comput. Methods Appl. Mech. Engrg. 341, 443–466 (2018). https://doi.org/10.1016/j.cma.2018.07.008
- [3] Antonietti, P.F., Beirão da Veiga, L., Scacchi, S., Verani, M.: A C¹ virtual element method for the Cahn-Hilliard equation with polygonal meshes. SIAM J. Numer. Anal. 54(1), 34–56 (2016). https://doi.org/10.1137/15M1008117
- [4] Antonietti, P.F., Berrone, S., Borio, A., D'Auria, A., Verani, M., Weisser, S.: Anisotropic a posteriori error estimate for the virtual element method. IMA J. Numer. Anal. (02 2021). https://doi.org/10.1093/imanum/drab001
- [5] Antonietti, P.F., Bertoluzza, S., Prada, D., Verani, M.: The virtual element method for a minimal surface problem. Calcolo **57**(4), 1–21 (2020). https://doi.org/10.1007/s10092-020-00388-0
- [6] Antonietti, P.F., Bruggi, M., Scacchi, S., Verani, M.: On the virtual element method for topology optimization on polygonal meshes: A numerical study. Comput. Math. Appl. 74(5), 1091–1109 (2017). https://doi.org/10.1016/j.camwa.2017.05.025
- [7] Antonietti, P.F., Giani, S., Houston, P.: hp-version composite discontinuous Galerkin methods for elliptic problems on complicated domains. SIAM J. Sci. Comput. 35(3), A1417–A1439 (2013). https://doi.org/10.1137/120877246
- [8] Antonietti, P.F., Manzini, G., Mazzieri, I., Mourad, H., Verani, M.: The arbitrary-order virtual element method for linear elastodynamics models: Convergence, stability and dispersion-dissipation analysis. Internat. J. Numer. Methods Engrg. 122, 934–971 (2021)
- [9] Antonietti, P.F., Manzini, G., Scacchi, S., Verani, M.: A review on arbitrarily regular virtuale element methods for elliptic partial differential equations. arXiv technical report arXiv:2104.03402 (2021)
- [10] Antonietti, P.F., Manzini, G., Verani, M.: The conforming virtual element method for polyharmonic problems. Comput. Math. Appl. 79(7), 2021–2034 (2020). https://doi.org/10.1016/j.camwa.2019.09.022
- [11] Antonietti, P.F., Mascotto, L., Verani, M.: A multigrid algorithm for the *p*-version of the virtual element method. ESAIM Math. Model. Numer. Anal. 52(1), 337–364 (2018). https://doi.org/10.1051/m2an/2018007
- [12] Argyris, J.H., Fried, I., Scharpf, D.W.: The TUBA family of plate elements for the matrix displacement method. Aeronaut. J. R. Aeronaut. Soc. 72, 701–709 (1968)
- [13] Artioli, E., de Miranda, S., Lovadina, C., Patruno, L.: A stress/displacement virtual element method for plane elasticity problems. Comput. Methods Appl. Mech. Engrg. 325, 155–174 (2017). https://doi.org/10.1016/j.cma.2017.06.036
- [14] Artioli, E., de Miranda, S., Lovadina, C., Patruno, L.: A family of virtual element methods for plane elasticity problems based on the Hellinger-Reissner principle. Comput. Methods Appl. Mech. Engrg. 340, 978–999 (2018). https://doi.org/10.1016/j.cma.2018.06.020

- [15] Ayuso de Dios, B., Lipnikov, K., Manzini, G.: The non-conforming virtual element method. ESAIM Math. Model. Numer. 50(3), 879–904 (2016)
- [16] Beirão da Veiga, L., Brezzi, F., Cangiani, A., Manzini, G., Marini, L.D., Russo, A.: Basic principles of virtual element methods. Math. Models Methods Appl. Sci. 23(1), 199–214 (2013). https://doi.org/10.1142/S0218202512500492
- [17] Beirão da Veiga, L., Brezzi, F., Marini, L.D., Russo, A.: H(div) and H(curl)-conforming VEM. Numer. Math. 133(2), 303–332 (2016)
- [18] Beirão da Veiga, L., Brezzi, F., Marini, L.D., Russo, A.: Mixed virtual element methods for general second order elliptic problems on polygonal meshes. ESAIM: Mathematical Modelling and Numerical Analysis 50(3), 727–747 (2016)
- [19] Beirão da Veiga, L., Brezzi, F., Marini, L.D., Russo, A.: Virtual element methods for general second order elliptic problems on polygonal meshes. Math. Models Methods Appl. Sci. 26(4), 729–750 (2016)
- [20] Beirão da Veiga, L., Dassi, F., Russo, A.: A C¹ virtual element method on polyhedral meshes. Comput. Math. Appl. 79(7), 1936–1955 (2020). https://doi.org/10.1016/j.camwa.2019.06.019
- [21] Beirão da Veiga, L., Lipnikov, K., Manzini, G.: The Mimetic Finite Difference Method, MS&A. Modeling, Simulations and Applications, vol. 11. Springer, I edn. (2014)
- [22] Beirão da Veiga, L., Lovadina, C., Vacca, G.: Divergence free virtual elements for the Stokes problem on polygonal meshes. ESAIM: M2AN 51(2), 509–535 (2017). https://doi.org/10.1051/m2an/2016032
- [23] Beirão da Veiga, L., Lovadina, C., Vacca, G.: Virtual elements for the Navier-Stokes problem on polygonal meshes. SIAM J. Numer. Anal. 56(3), 1210–1242 (2018). https://doi.org/10.1137/17M1132811
- [24] Beirão da Veiga, L., Manzini, G.: A virtual element method with arbitrary regularity. IMA J. Numer. Anal. 34(2), 782–799 (2014)
- [25] Beirão da Veiga, L., Manzini, G.: Residual *a posteriori* error estimation for the virtual element method for elliptic problems. ESAIM Math. Model. Numer. Anal. 49(2), 577–599 (2015). https://doi.org/10.1051/m2an/2014047
- [26] Bell, K.: A refined triangular plate bending finite element. Int. J. Numer. Meth. Eng. 1(1), 101–122 (1969). https://doi.org/10.1002/nme.1620010108, cited By 156
- [27] Benedetto, M.F., Berrone, S., Borio, A.: The virtual element method for underground flow situations in fractured data. In: Advances in discretization methods, SEMA SIMAI Springer Ser., vol. 12, pp. 167–186. Springer, [Cham] (2016)
- [28] Benvenuti, E., Chiozzi, A., Manzini, G., Sukumar, N.: Extended virtual element method for the Laplace problem with singularities and discontinuities. Comput. Methods Appl. Mech. Engrg. 356, 571 – 597 (2019)
- [29] Blum, H., Rannacher, R.: On the boundary value problem of the biharmonic operator on domains with angular corners. Math. Methods Appl. Sci. 2(4), 556–581 (1980). https://doi.org/10.1002/mma.1670020416

- [30] Brenner, S.C., Scott, R.: The mathematical theory of finite element methods, vol. 15. Springer Science & Business Media (2008)
- [31] Brenner, S.C., Sung, L.Y.: Virtual enriching operators. Calcolo **56**(4), 1–25 (2019). https://doi.org/10.1007/s10092-019-0338-z
- [32] Brezzi, F., Falk, R.S., Marini, L.D.: Basic principles of mixed virtual element methods. ESAIM Math. Model. Numer. Anal. 48(4), 1227–1240 (2014)
- [33] Brezzi, F., Marini, L.D.: Virtual element methods for plate bending problems. Comput. Methods Appl. Mech. Engrg. 253, 455–462 (2013). https://doi.org/10.1016/j.cma.2012.09.012
- [34] Cangiani, A., Dong, Z., Georgoulis, E.H., Houston, P.: *hp*-version discontinuous Galerkin methods on polygonal and polyhedral meshes. SpringerBriefs in Mathematics, Springer (2017)
- [35] Certik, O., Gardini, F., Manzini, G., Vacca, G.: The virtual element method for eigenvalue problems with potential terms on polytopic meshes. Applications of Mathematics **63**(3), 333–365 (2018)
- [36] Chen, L., Huang, J.: Some error analysis on virtual element methods. Calcolo 55(1), 1–23 (2018)
- [37] Chi, H., Pereira, A., Menezes, I.F., Paulino, G.H.: Virtual element method (VEM)-based topology optimization: An integrated framework. Struct. Multidiscip. Optim. 62(3), 1089–1114 (2020). https://doi.org/10.1007/s00158-019-02268-w
- [38] Chinosi, C., Marini, L.D.: Virtual element method for fourth order problems: L²-estimates. Comput. Math. Appl. 72(8), 1959–1967 (2016). https://doi.org/10.1016/j.camwa.2016.02.001
- [39] Ciarlet, P.G.: The finite element method for elliptic problems. Classics in Applied Mathematics 40, 1–511 (2002)
- [40] Clough, R.W., Tocher, J.L. (eds.): Finite element stiffness matrices for analysis of plates in bending. Proceedings of the Conference on Matrix Methods in Structural Mechanics (1965)
- [41] Cockburn, B., Dong, B., Guzmán, J.: A superconvergent LDG-hybridizable Galerkin method for second-order elliptic problems. Math. Comp. 77(264), 1887–1916 (2008). https://doi.org/10.1090/S0025-5718-08-02123-6
- [42] Di Pietro, D.A., Droniou, J.: The Hybrid High-Order Method for Polytopal Meshes: Design, Analysis, and Applications. MS&A, Springer (2020)
- [43] Gazzola, F., Grunau, H.C., Sweers, G.: Polyharmonic boundary value problems, Lecture Notes in Mathematics, vol. 1991. Springer-Verlag, Berlin (2010). https://doi.org/10.1007/978-3-642-12245-3
- [44] Hu, J., Lin, T., Wu, Q.: A construction of C^r conforming finite element spaces in any dimension (2021), arXiv:2103.14924
- [45] Hu, J., Zhang, S.: The minimal conforming H^k finite element spaces on \mathbb{R}^n rectangular grids. Math. Comp. **84**(292), 563–579 (2015). https://doi.org/10.1090/S0025-5718-2014-02871-8
- [46] Huang, X.: H^m-conforming virtual elements in arbitrary dimension (2021), arXiv:2105.12973
- [47] Huang, X.: Nonconforming virtual element method for 2mth order partial differential equations in \mathbb{R}^n with m > n. Calcolo **57**(4), Paper No. 42, 38 (2020)

- [48] Li, M., Zhao, J., Huang, C., Chen, S.: Conforming and nonconforming vems for the fourthorder reaction–subdiffusion equation: a unified framework. IMA J. Numer. Anal. (2021). https://doi.org/10.1093/imanum/drab030
- [49] Lovadina, C., Mora, D., Velásquez, I.: A virtual element method for the von Kármán equations. ESAIM Math. Model. Numer. Anal. 55(2), 533–560 (2021). https://doi.org/10.1051/m2an/2020085
- [50] Mascotto, L., Perugia, I., Pichler, A.: A nonconforming Trefftz virtual element method for the Helmholtz problem. Math. Models Methods Appl. Sci. 29(9), 1619–1656 (2019). https://doi.org/10.1142/S0218202519500301
- [51] Mascotto, L., Perugia, I., Pichler, A.: A nonconforming Trefftz virtual element method for the Helmholtz problem: numerical aspects. Comput. Methods Appl. Mech. Engrg. 347, 445–476 (2019). https://doi.org/10.1016/j.cma.2018.12.039
- [52] Mora, D., Rivera, G., Velásquez, I.: A virtual element method for the vibration problem of Kirchhoff plates. ESAIM Math. Model. Numer. Anal. 52(4), 1437–1456 (2018). https://doi.org/10.1051/m2an/2017041
- [53] Mora, D., Silgado, A.: A C¹ virtual element method for the stationary quasi-geostrophic equations of the ocean. Comput. Math. Appl. (2021). https://doi.org/https://doi.org/10.1016/j.camwa.2021.05.022
- [54] Mora, D., Velásquez, I.: A virtual element method for the transmission eigenvalue problem. Math. Models Methods Appl. Sci. 28(14), 2803–2831 (2018). https://doi.org/10.1142/S0218202518500616
- [55] Mora, D., Velásquez, I.: Virtual element for the buckling problem of Kirchhoff-Love plates. Comput. Methods Appl. Mech. Engrg. 360, 112687, 22 (2020). https://doi.org/10.1016/j.cma.2019.112687
- [56] Park, K., Chi, H., Paulino, G.H.: On nonconvex meshes for elastodynamics using virtual element methods with explicit time integration. Comput. Methods Appl. Mech. Engrg. 356, 669–684 (2019). https://doi.org/10.1016/j.cma.2019.06.031
- [57] Park, K., Chi, H., Paulino, G.H.: Numerical recipes for elastodynamic virtual element methods with explicit time integration. Internat. J. Numer. Methods Engrg. 121(1), 1–31 (2020). https://doi.org/10.1002/nme.6173
- [58] Perugia, I., Pietra, P., Russo, A.: A plane wave virtual element method for the Helmholtz problem. ESAIM Math. Model. Num. 50(3), 783–808 (2016)
- [59] Sukumar, N., Tabarraei, A.: Conforming polygonal finite elements. Internat. J. Numer. Methods Engrg. 61(12), 2045–2066 (2004). https://doi.org/10.1002/nme.1141
- [60] Beirão da Veiga, L., Mora, D., Rivera, G.: Virtual elements for a shear-deflection formulation of Reissner-Mindlin plates. Math. Comp. 88(315), 149–178 (2019)
- [61] Beirão da Veiga, L., Lovadina, C., Russo, A.: Stability analysis for the virtual element method. Math. Models Methods Appl. Sci. 27(13), 2557–2594 (2017)
- [62] Wriggers, P., Rust, W.T., Reddy, B.D.: A virtual element method for contact. Comput. Mech. 58(6), 1039–1050 (2016). https://doi.org/10.1007/s00466-016-1331-x
- [63] Zhang, S.: A family of 3D continuously differentiable finite elements on tetrahedral grids. Appl. Numer. Math. 59(1), 219–233 (2009)
- [64] Zhang, S.: A family of differentiable finite elements on simplicial grids in four space dimensions. Math. Numer. Sin. 38(3), 309–324 (2016)