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Surface theory in the 18th and 19th centuries: the second problem of applicability

MAT/04

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Contents

Introduction

As is now well known, from its very beginnings infinitesimal calculus was gradually applied in geometry, firstly to the theory of plane curves, then to space curves and, finally, to surfaces. The aim of this thesis is to focus on some important moments in the development of the classic infinitesimal theory of surfaces in the eighteenth and nineteenth centuries that concern the second problem of applicability.

Two surfaces are applicable one upon the other when there is a continuous deformation mapping one upon the other. In slightly modernised terms, the problem can be described by making recourse to the notion of isometry between surfaces.[1] To this end, let us consider two surfaces, Σ and $\tilde{\Sigma}$, that are described in terms of local charts $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \Sigma$, $(u, v) \in U \mapsto \boldsymbol{x}(u, v)$ and $\tilde{\boldsymbol{x}} : V \subset \mathbb{R}^2 \to \tilde{\Sigma}$, $(\tilde{u}, \tilde{v}) \in V \mapsto \tilde{\boldsymbol{x}}(\tilde{u}, \tilde{v})$. Σ and $\tilde{\Sigma}$ are said to be applicable one upon the other if there exists a (regular) function $\Phi : \Sigma \to \tilde{\Sigma}$ that maps the line element—i.e., the first fundamental form—associated to Σ into the line element of $\tilde{\Sigma}$. In particular, the second problem of applicability consists in finding *all* the surfaces that are locally isometric to a given one.

The fact that two surfaces are isometric does not guarantee a priori the actual existence of a continuous deformation mapping one upon the other. Nevertheless, this distinction was usually disregarded in the nineteenth century. Actually, for sufficiently regular surfaces this distinction may not be considered. For these reasons, we adhere to nineteenth century practice regarding the term "applicable" as equivalent to "locally isometric".

Mathematicians had dealt with problems related to applicability since the early stages of surface theory. As early as 1692, for example, Johann Bernoulli used the isometric correspondence between a cone and a plane to find the geodesics on the cone. However, at the beginning the focus was only on finding conditions that guaranteed applicability on a plane. In this regard, Euler and Monge had succeeded in providing an analytical characterisation of all surfaces applicable to the plane (that are called *developable surfaces*). Moreover, developability problem, and (Monge 1780) in particular, played a fundamental role in the formulation of the definition of surfaces in terms of an infinitely thin, perfectly flexible and inextensible veil, which was first given by Gauss in the *Disquisitiones generales circa superficies curvas*, (Gauss 1828, §12).

Gauss' work, and in particular (Gauss 1828, §12-13), led to an important breakthrough by providing a general method for dealing systematically with surface theory. Gauss stated

[1] Gauss defined the notion of applicability between surfaces in (Gauss 1828, §12).

that two surfaces can be regarded as equivalent whenever they have the same line element (i.e., whenever they are applicable) and emphasised those properties called *intrinsic*, such as Gaussian curvature, which do not vary within a class of applicability. Thus, in the hands of Gauss, the subject of applicability became central to the study of surfaces.

Shortly after the publication of Gauss' *Disquisitiones*, the first to deal with problems related to applicability was Ferdinand Minding in a number of papers between 1837 and 1840. Minding succeeded in discovering that the class of ruled surfaces, in contrast to that of surfaces with the same constant curvature, was not complete, i.e., there are surfaces that are applicable to ruled surfaces that are not themselves ruled; he also succeeded in determining necessary and sufficient conditions for two surfaces to be applicable, but failed to address the second problem of applicability without requiring very restrictive conditions.

Unfortunately, this was a rather isolated work. Gauss' theory did not become a widely cultivated research theme within the realm of infinitesimal geometry until the late 1840s. Gauss' ideas found a favourable milieu especially in France, where a school of differential geometry was well established thanks to Monge, and where Liouville was committed to promoting the ideas of his German colleague.

It is thus not surprising that, on the occasion of the 1860 Grand Prix des Mathématiques, the Académie des Sciences in Paris called attention to the question of elaborating general methods to obtain *all* surfaces that are applicable to a given one. Attempts at solving this problem were made by Edmond Bour, Pierre Ossian Bonnet and Delfino Codazzi. Bour and Bonnet determined a second-order partial differential equation (PDE) of Monge-Ampère type (the *Bour's equation*), whose solutions give all the surfaces applicable to a given one. Unfortunately, it was integrable by known methods only in the case of developable surfaces. Moreover, both Bour and Codazzi succeeded in deducing the so-called Mainardi-Codazzi equations (MCE), which are a pair of first-order differential equations that establish relations between the coefficients of the first and the second fundamental forms of a surface. In 1867, Bonnet used the MCE to prove the fundamental theorem of surface theory (FT), according to which a surface is uniquely determined (up to rigid motions) by the assignment of the coefficients of its first and second fundamental forms.

The MCE and the FT were not completely new. Some years before, they were stated and partially demonstrated in works by Peterson (1853) and Mainardi (1856). However, the presence of some mistakes (especially on Mainardi's part) along with a poor exposition of the results, which did not take into account the second problem of applicability, resulted in a lack of appreciation. The MCE were not understood as the differential equations of all surfaces applicable to a given one, but simply as relations between the coefficients of the first and second fundamental forms, whose deduction was linked to the proof of the FT. It was only after the 1860 prize competition that the relevance of the MCE, and consequently of the FT for surface theory, was definitively recognised.

However, the results of Bour, Bonnet and Codazzi did not completely solve the second problem of applicability. Important contributions on this subject were offered by Julius Weingarten, who won the second Grand Prix des Mathématiques that the Académie des

Sciences in Paris awarded in 1894 on the theme of applicability. Weingarten succeeded in deducing *another* second-order PDE of Monge-Ampère type, whose solutions give all the surfaces applicable to a given one. In contrast to Bour's equation, it is integrable by known methods in all the cases in which complete classes of applicable surfaces had already been found.

This new method for applicability was the outcome of a ten-year-long process that began in 1884, when Weingarten identified some critical points that affected Bour's equation. Over the years 1884-1894, Weingarten devoted several memoirs to surface theory, namely to the second problem of applicability, and gradually started to develop a new approach to the problem. At first, his focus was mainly on the determination of new complete classes of applicable surfaces. Only later, probably on Darboux's advice, did he shift his attention to a general method for the second problem of applicability. This process is documented in numerous letters that Weingarten sent to Bianchi, which were published in (Bianchi 1959).

Weingarten's equation did not stimulate the same interest as Bour's equation and MCE had done 30 years earlier. Nevertheless, it stimulated the research of Luigi Bianchi, who employed it on several occasions from 1896 onwards. Bianchi's results provide interesting insights into the development of differential geometry in Italy between the nineteenth and twentyth centuries, which, as is well known,[2] was conditioned by the sceptical welcome for the absolute differential calculus invented by Gregorio Ricci Curbastro. In this context, the most emblematic episode is the failure to award Ricci the 1901 Royal Prize for Mathematics announced by the Accademia dei Lincei when Bianchi was head of the examining committee. On that occasion, Ricci received various criticisms, including that he had created an artificial instrument that did not lead to truly new results and overshadowed the geometric content of the tackled problems. It was a severe judgement that can only be understood in its historical context. For this purpose, the figure of Bianchi, who well represents this attitude of obstruction towards the new techniques, is relevant both because of his role in the commission of the Lincei Prize and his prominent position in the European scientific community. It emerges from an analysis of Bianchi's and Ricci's applications of Weingarten's new method that, to some extent, Bianchi and Ricci were motivated by the same intention: both wished to provide a treatment of surface theory (Bianchi) and of various branches of mathematics including surface theory (Ricci) by the methods that they considered most effective—Bianchi by means of Gauss' method, Ricci by means of his absolute differential calculus. However, while Ricci in part limited himself to a (noteworthy) revision of already-established results, Bianchi showed the effectiveness of Gauss' method by proving numerous results that also met various needs of his time.

These themes are organised in the different chapters as follows.

Chapter 1 introduces surface theory through an overview of the historical process that led to its first major results. We will mainly focus on the evolution of the concept of

[2] In this regard, see (Cogliati 2022, Chap. 11).

curvature, which is of primary importance for the description of surfaces, and on the role of *developable surfaces*. Beyond preparing the ground for the following chapters, this chapter contributes to a more accurate historical understanding of Euler's and Monge's theory of developable surfaces. Although their works[3] are often presented together, Euler's and Monge's interest in this class of surfaces seems to be motivated for different reasons. Indeed, according to some letters that Euler addressed to Lagrange, he was probably interested in furthering the study of PDEs. In this respect, investigations on the second problem of applicability were a way to solve the corresponding system of PDEs. Monge, on the other hand, arrived at developable surfaces by extending the notion of evolutes of a plane curve to space curves. It was only after reading (Euler 1772a) that Monge emphasised their constitution as a complete class of applicable surfaces in (Monge 1780).

After presenting Gauss' main ideas on surface theory contained in (Gauss 1825a) and in (Gauss 1828), Chapter 2 describes Minding's investigations on applicability.[4] This reading of Minding's work will show the difficulty of the second problem of applicability.

Chapter 3 offers[5] a detailed analysis of three works, (Bour 1862b), (Codazzi 1883) and (Bonnet 1867), which were submitted for the 1860 prize competition, as well as a later additional note appended by Bonnet in 1867 to his original manuscript, in which a thorough proof of the FT was conveyed for the first time. Previous studies have mainly focused on priority issues by emphasising the role of Peterson and Mainardi as anticipators of both the MCE and the FT, for example (Coolidge 1947), (Reich 1973) and (Phillips 1979). In contrast, through this technical analysis, we intend to show that both the MCE and the FT were essential tools in investigations related to the difficult problem of applicability. From this point of view, it is not surprising that Peterson and Mainardi's contributions went almost unnoticed and were forgotten soon afterwards. We argue that this was probably due, at least in part, to the fact that their works were not as yet inserted systematically within the general framework of applicability problems.

Chapter 4 conveys a brief biographical sketch of Julius Weingarten by integrating the available literature with the content of letters that Weingarten addressed to Bianchi. Weingarten was a first-rank mathematician on the European scene in the nineteenth century. His original talent found its expression in geometry and inspired at least two of the leading differential geometers of his time —Luigi Bianchi and Gaston Darboux. The contents of several entire chapters in Bianchi's *Lezioni di geometria differenziale* and Darboux's *Leçons sur la Théorie Générale des Surfaces*, which contributed to the education of young mathematicians for generations, originated from many of Weingarten's investigations, such as those on finite and infinitesimal deformations, $W-$ surfaces and $W-$ congruences. According to (Knobloch 1989, p. 257), Weingarten founded a classic surface theory school, whose members included Scheffers, Hessenberg, Rembs and Grotemayer, that began a tradition of studies on surface theory in Berlin. Nevertheless, there is a lack of secondary literature

[3](Euler 1772a), (Monge 1785), (Monge 1780)

[4](Minding 1837; Minding 1838a; Minding 1838b; Minding 1839; Minding 1840)

[5]Chapter 3 is mainly based on the content of (Cogliati and Rivis 2022).

dedicated to Weingarten. The few comments on his work are mainly related to the discovery of the theorems that launched the $W-$surfaces between 1861 and 1863, such as (Reich 1973) and (Coddington 1905). In this chapter, we intend to provide a broader overview of his research.

In addition to presenting the content of the awarded memoir (Weingarten 1897) in some detail, Chapter 5 offers a reconstruction of Weingarten's research in the years 1884-1894 that led him to this important result. To this end, an in-depth reading of his correspondence with Bianchi integrated the content of Weingarten's published results on applicability. This allows us to verify Bianchi's words: *"by studying his particular procedures more carefully, Weingarten succeeded in obtaining a radical transformation of the applicability equation"*.[6]

Chapter 6 is devoted to the exposition of some applications of Weingarten's method provided by Bianchi and Ricci. Besides showing the scientific value of Weingarten's method, an overview of Bianchi's applications—especially those relating to non-Euclidean geometry—will offer an interesting insight into his work. This will be useful for a twofold purpose. First, it will provide a greater awareness of the scientific collaboration between Bianchi and Weingarten, and, in particular, it will clarify the difference between Weingarten's analytical approach and Bianchi's geometrical approach. Second, it will provide a closer look at the interests and the state of the research in differential geometry at the time, which can clarify some reasons for the initial reservation with respect to absolute differential calculus. Moreover, comparing Bianchi's and Ricci's results on the same subject (Weingarten's method for applicability) will highlight the particular aspects of their respective research. This concretely shows the difference in their views and, consequently, provides further insight into the delicate issue of the initial diffidence towards Ricci's techniques.

Two appendices can be found at the end of the dissertation. Appendix A is devoted to providing a more accurate description of the erroneous interpretation of applicability as a problem of local isometry in the nineteenth century. Appendix B briefly presents how the meaning of surface also changed in the eighteenth and nineteenth centuries in relation to the state of research in analysis.

A few remarks on methodological issues are in order. As a rule, we have adhered to the original notation. However, we make the notation uniform among the different authors and translate the original notation into a more familiar one to make reading easier. As a rule, we have employed vector notation,[7] denoted the Gaussian parameters of surfaces by u, v and used the standard functions E, F, G, D, D'' of the first and second fundamental forms.[8] We believe that this choice, necessary for comparison in Chapter 3, might help to

[6] (Bianchi 1910a, p. 224)

[7] Generally, none of the authors to be taken into account in the following employed vector notation. Vectors only began to be diffusely used in differential geometry from the early 1900s. For a complete account on the history of the vector calculus, we refer the interested reader to (Crowe 1985).

[8] The coefficients E, F, G, D, D', D'' were first introduced by Gauss in his *Disquisitiones* and their widespread use was probably the result of the influence of Darboux's and Bianchi's textbooks (*Leçons sur la Théorie Générale des Surfaces* and *Lessons of Differential Geometry* respectively) in the last quarter of the 19th century.

facilitate readability with little harm to textual exactitude.

Chapter 1

The beginnings of surface theory: the role of developable surfaces

1.1 The emergence of calculus in geometry

The discovery of differential calculus by Newton and Leibniz stimulated new questions and fostered the creation of new mathematical disciplines. Among them, we should mention the systematic introduction of infinitesimal techniques into coordinate geometry for the investigation of curves and surfaces,[1] which finally led to the creation of differential geometry.[2]

This process began in the second half of the seventeenth century[3] and was rather slow and gradual: this was primarily a consequence of the poor state of analytical geometry, which can be considered as the starting point for the application of calculus to geometry since it allows for the representation of geometric entities through equations. At that time, investigations in analytic geometry were mainly directed towards the study of plane curves rather than of space geometry, although Descartes had already shown how space curves could be treated by using the coordinate method.[4] Thus, it is not surprising that the same asymmetry occurred in the field of infinitesimal geometry and that surface theory was only addressed at a later stage.

[1] See Appendix B for a brief overview of how the meaning of the term "surface" changed during the eighteenth and nineteenth century.

[2] It appears that Bianchi was the first to use the term *differential geometry* in the title of his textbook, *Lezioni di geometria differenziale*, see (Reich 2007, p. 479). Before then, the discipline had no specific name. Expressions such as "infinitesimal geometry" or "geometric applications of infinitesimal calculus" were used, as one can see in (Chasles 1870), for example. Darboux openly criticised Bianchi's choice of title for his *Lezioni*. In this regard, we refer to the footnote 23 in Section 5.3.

[3] For a more detailed description of the beginnings of infinitesimal geometry and an overall report of first contributions, see (Struik 1933a), (Taton 1951, Chap. IV) and (Kline 1972, Chap. 23).

[4] For instance, Descartes suggested studying space curves in terms of their projections on two orthogonal planes. See (Descartes 1637, La Géométrie, 368–369). Space curves are also called *curves with double curvature*. This expression derives precisely from the possibility of associating each space curve with a pair of plane curves each characterised by its own curvature.

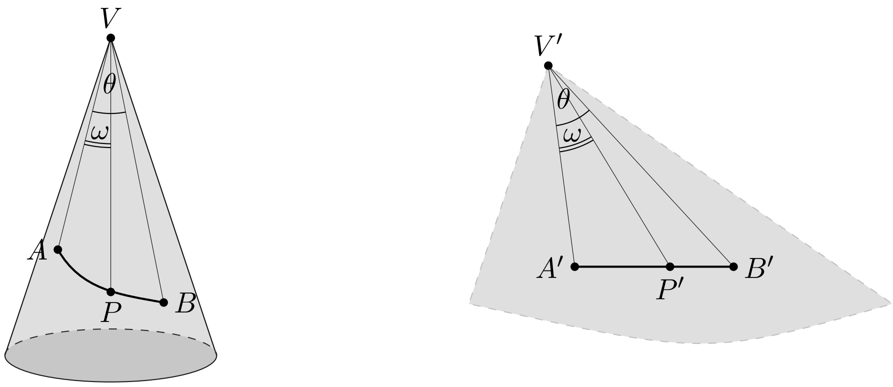

Figure 1.1: Illustration of the construction of the isometric correspondence between cone and plane proposed by Bernoulli.

According to (Kline 1972, p. 562), the first to turn his attention to the study of surfaces was Johann Bernuolli, as late as 1697. After challenging the scientific community with the determination of the curve of fastest descent, known as the *brachistochronous curve*, he posed the problem of finding the shortest arc between two points on a convex surface.[5] Partial results were obtained by Johann himself and by Jacob Bernoulli, but a complete answer only came 30 years later in 1728, when the young Leonhard Euler (1707-1783) discovered an analytic characterisation for geodesics on a surface in (Euler 1732).

While outlining the history of geodesics theory (Stäckel 1893, p. 452) highlights the following important step in the development of surface theory. In (Bernoulli 1698, pp. 265–266), Johann Bernoulli used an isometric mapping between the surface of a cone and a plane in order to establish a characterisation of geodesics upon a cone. Through a simple geometric construction he showed that it is possible to determine these by cutting and unfolding the cone on a plane and by matching the geodesics of the plane—i.e., straight lines—with those of the cone. The correspondence is established by requiring the preservation of distances and angles between particular lines chosen on the cone and on the plane. Bernoulli's argument is not a proper demonstration of the developability of a cone on a plane, but rather the exposition of a consequence of the following statement he attributed to Barrow:[6]

> I suppose here that the angles of the conical surface around the top are compa-

[5]See (Bernoulli 1697, p. 204). In general, *shortest curves* are not the same as *geodesics* between two points, though the two concepts are closely related. Initially, the term geodesics only indicated the shortest lines drawn on surfaces that approximated the Earth's surface, such as the ellipsoid of rotation, while for all other surfaces the term *shortest line* was used. It was only with Liouville that the term *geodesic* acquired a broader meaning, denoting all those curves for which at each point there exists an osculating plane orthogonal to the surface itself. Finally, Jacobi showed that geodesics are only locally the shortest distance between points. For an historical account of the solution of the problem of geodesics, one can refer to the classics (Stäckel 1893) and (Eneström 1899), and to (Reich 1973, pp. 308–309). (Nabonnand 1995) provides a historical analysis of the emergence of a global viewpoint in geodesics theory.

[6]Bernoulli did not provide a more precise reference, but he probably referred to a result by Isaac Barrow, whose main geometrical work is (Barrow 1670).

> rable, or that they may be assumed equal to the angles of the plane: how this can be done has already been explained by Barrow.[7]

The construction proposed by Bernoulli is as follows. Given two points A and B on the cone as illustrated in Figure 1.1, the two lines passing through each of them and through the vertex V form an angle θ. Bernoulli considered two lines in a plane that form the same angle θ at their point of intersection V'. The validity of this construction was guaranteed by Barrow's statement. Then, he considered two points A' and B' that are as far from V' as A and B from V. Finally, he traced the line r passing through A' and B': each point P' of r has a specific distance d from V' and the line joining it to V' forms a specific angle, ω, with the line joining V' and A'. The point P on the cone that isometrically corresponds to P' is that point on the line that forms an angle ω with the line through V and A, whose distance from V is d. The geodesic g on the cone consists of the collection of all these points P.

Bernoulli, therefore, used the possibility of switching from a surface that was difficult to treat (the cone) to another surface (the plane), which is equivalent to the first but somewhat simpler to deal with, to tackle the proposed problem. This was a first attempt to exploit the isometric mapping of one surface onto another to study its properties. As Stäckel remarked:

> This solution also proves Johann Bernoulli's extraordinary inventiveness. Imagining a curved surface as flexible was certainly not unfamiliar to him, as the problem of the sail surface, which he had investigated in 1692, shows.[8] But here, for the first time, the unfolding of curved surfaces is used to solve a problem.[9]

Hence, it is from the very origins of surface theory that the problem arose, of determining which surfaces can be considered isometrically equivalent to an assigned one, albeit not in these general terms. Furthermore, it is remarkable that a 100 years before its definitive formulation with Gauss' seminal work, *Disquisitiones Generales Circa Superficies Curvas* (Gauss 1828), this idea was already employed as a tool to study surfaces.

In the first half of the eighteenth century, more extensive studies on three dimensional geometry were carried out. The most significant one was by Alexis Clairaut (1713-1765). At the age of 16, he wrote a treatise on the theory of space curves, (Clairaut 1731), where

[7]"Suppono hic angulos superficiei conicae circa verticem comparabiles esse, vel aequales assumi posse angulis planis: quod quomodo faciendum, iam Barrowius docuit." (Bernoulli 1698, p. 266)

[8]In (Bernoulli 1692) and in (Bernoulli 1691-1692, Lectio 42-44), Bernoulli investigated the shape of sails inflated by the wind and the shape of "those curves [*curvae*] that any flexible material takes on, as if it were considered to be devoid of gravity, like a veil, cloth, thread, etc." (Bernoulli 1691-1692, p. 507). Later, he further developed the subject in (Bernoulli 1714).

[9]"Auch diese Lösung beweist Johann Bernoullis ausserordentliche Erfindungskraft. Eine krumme Oberfläche sich als biegsam vorzustellen war ihm freilich nicht fremd, wie das Problem der Segelfläche zeigt, welches er 1692 untersucht hatte. Aber hier wird zum ersten Male die Abwickelung krummer Oberflächen zur Lösung einer Aufgabe verwendet". (Stäckel 1893, p. 452)

curves are conceived as intersections of surfaces. He imagined a surface as the juxtaposition of infinite curves placed infinitely close and proved that such a surface can be described by a functional relation between three variables, $F(x, y, z) = 0$. It is worth noting that the possibility of describing surfaces by using an equation with three unknowns, such as $F(x, y, z) = 0$ or $z = z(x, y)$, was not obvious at that time. A few lines from the preface of (Clairaut 1731) might clarify the state of the art:

> I do not believe this matter [curved surfaces] to be less new than that of curves with double curvature, and what I know to be already known on this subject is only how to express curved surfaces by equations in three variables, which I learned was mentioned occasionally in a memoir by the famous M. Bernoulli inserted in the Acts of Leipzig.[10]

The idea of rolling conical and cylindrical surfaces on the plane was taken up from a geometric point of view by others besides Bernoulli, such as Clairaut himself, but also Henri Pitot (1695-1771) and Amédée François Frézier (1682-1773).[11] By relying on geometric intuition, these geometers recognised as evident the fact that curved surfaces of cones and cylinders could be unfolded onto a plane isometrically, i.e., without being deformed or torn. On the other hand, they were perfectly aware that there are surfaces that do not possess this property, such as the sphere. A clear manifestation of this awareness can be seen in the efforts made to invent methods for cartography, whose purpose was to represent the Earth's surface, which is appreciably spherical, on a plane. Given the impossibility of representing it exactly on a plane, a number of approximation methods were devised: among these, the so-called *projection by development* should be mentioned. This essentially consists in replacing a small region of a surface (generally a sphere) with certain portions of developable surfaces—the most common are conic and cylindrical surfaces—which work as intermediate surfaces that can be easily represented on a plane. Flamsteed, for example, substituted the region between two parallels with a conical frustum.[12] However, the possibility of unfolding one surface onto another was only thought of in relation to the plane: methods were therefore contrived to map on the plane even surfaces that were known (intuitively) not to be isometric to the plane, but none were sought to map non-developable surfaces isometrically.

1.2 Euler: first insight into a differential approach to surfaces

Until the mid-eighteenth century contributions to space geometry were mainly related to analytical geometry and a general infinitesimal theory of surfaces was still waiting to be

[10]"Je ne crois point cette matiere [surfaces courbes] moins neuve que celle des courbes à double courbure, et je ne sçai de connu sur ce sujet, que la façon d'exprimer les surfaces courbes par des équations à trois variables dont j'ai appris qu'il étoit fait mention par occasion dans un mémoire du celebre M. Bernoulli inséré dans les Actes de Leypsie." (Clairaut 1731, preface). According to (Le Goff 1993, p. 105), Clairaut referred to (Bernoulli 1728).

[11]For a brief overview of these contributions, see (Stäckel 1893, p. 453).

[12](Marie 1885, p. 56)

created. The first significant results, and the necessary tools for the future development of a general differential theory of surfaces, were provided by Euler.

Euler tackled problems related to space geometry throughout his long career. In addition to the aforementioned (Euler 1732), in (Euler 1744, Chap. V, Ex. VII) he provided the first non-trivial example of minimal surface, the catenoid, which is the result of the rotation of a catenary about its directrix.[13] In (Euler 1748) he dealt with analytic geometry of curves and surfaces: in particular, he classified second-degree surfaces by reducing them to a canonical form. Euler also offered an interesting investigation into the theory of space curves in (Euler 1786) by employing a parametric representation through the arc length and defining a moving frame at each point of the curve consisting of the set of tangent, normal and binormal vectors to the curve.

However, his most significant memoirs that gave solid foundations to the differential theory of surfaces were (Euler 1767) and (Euler 1772a).

The first (Euler 1767) was presented to the Academy of Sciences of Berlin on 8th September 1760 but was only published seven years later. As is evident from the title, *"Recherches sur la courbure des surfaces"*, the aim of the paper is the investigation of the curvature of a surface, which is a theme of primary importance that no-one had yet addressed.

Euler's plan was to describe the curvature of all those curves that are obtained by intersecting the surface with a plane by using the classical notion of curvature for plane curves formulated in terms of the osculating circle.[14] However, the large number of these sectional curves made this extension difficult.

Euler succeeded in providing a description of the curvature of a surface at its point P as follows: he selected, among all possible sections, the normal sections, which are obtained by intersecting the surface with planes containing the normal to the surface at that point. He then described the curvature of the surface at a given point P as the collection of curvatures of all normal sections in that point, i.e., as a function of P and $\boldsymbol{v}$, which is a

[13] Among all the infinitesimally closed surfaces that satisfy the same boundary conditions, a minimal surface is the one with the smallest area. Lagrange improved Euler's investigation in (Lagrange 1760-1761), especially in the first appendix. Lagrange's equation was efficiently solved by Bonnet in (Bonnet 1853). For a more detailed account of the main achievements related to the theory of minimal surfaces up to the 1860s, the interested reader can see (Beltrami 1867, pp. 3–24) and (Gray 2021, Chap. 23).

[14] The osculating circle at a given point P on a plane curve is defined as that circle passing through P and through a pair of additional points on the curve, which are infinitesimally closed to P. Its centre lies on the normal line to the curve, and its radius ρ defines the curvature k of the given curve at that point as $k = \frac{1}{\rho}$. The centre and the radius of the osculating circle at a given point are called, respectively, *centre of curvature* and *radius of curvature* of the curve at that point. Huygens provided a description of the osculating circle through synthetic methods in his treatise *Horologium oscillatorium: sive de motu pendulorum ad horologia aptato dimostrationes geometricae* (Huygens 1673). However, his aim was to determine the evolute of a given curve and he did not discuss the notion of curvature. The connection between curvature and osculating circle was finally established through the investigations of Newton, Leibniz and Johann and Jacob Bernoulli. The interested reader can see (Lodder 2018) for an overview of Huygens's results and (Nauenberg 1996) for an account of early contributions to the definition of curvature.

tangent vector to the surface at P.[15] In this sense, *"the question about the curvature of surfaces is not susceptible of a simple answer, but requires at the same time an infinite number of determinations"*.[16] Furthermore, Euler was able to find two special directions, the principal directions, which summarised the notion of curvature.

Indeed, by writing down the radius of curvature ρ of the generic normal section in function of the angle φ that the chosen section plane forms with a fixed one, Euler noticed that the curvature of the normal section, which is $k = \frac{1}{\rho}$, assumes maximum and minimum values along two orthogonal directions, today known as *principal directions.* More precisely, denoting the curvature radii related to principal directions with ρ_1, ρ_2, the radius of curvature ρ of every other normal section is given by

$$\rho = \frac{2\rho_1\rho_2}{\rho_1 + \rho_2 - (\rho_1 - \rho_2)\cos 2\varphi}.$$

Commenting on this outstanding result, Euler expressed his astonishment:

> And so the measurement of the curvature of surfaces, however complicated it appeared at the beginning, is reduced at each point to the knowledge of two radii of curvature, one the largest and the other the smallest, at that point; these two things entirely determine the nature of the curvature and we can determine the curvatures of all possible perpendicular sections at the given point.[17]

1.2.1 Euler's deduction of all developable surfaces

A few years later, Euler again employed the methods of analysis in the study of surface theory when, in (Euler 1772a), he provided the first analytical treatment of developable surfaces.[18] This memoir was presented to the St. Petersburg Academy on 5th March 1770 and published two years later. Here, Euler turned his attention to a new problem that he described as *"curious and noteworthy"*:[19]

> To find a general equation for all solids whose surface can be unfolded onto a plane.[20]

[15]Here, the use of vector notation is anachronistic. In general, neither Euler nor any of the authors we will consider used vector notation, but we will adopt it whenever possible to facilitate the modern reader.

[16]"La question sur la courbure des surfaces n'est pas susceptible d'une réponse simple, mais elle exige à la fois une infinité de déterminations" (Euler 1767, pp. 119–120)

[17]"Ainsi le jugement sur la courbure des surfaces, quelque compliqué qu'il ait paru au commencement, se réduit pour chaque élément à la conoissance de deux rayons osculateurs, dont l'un est le plus grand et l'autre le plus petit dans cet élément; ces deux choses déterminent entièrement la nature de la courbure en nous découvrant la courbure de toutes les sections possibles, qui sont perpendiculaires sur l'élément proposé." (Euler 1767, p. 143)

[18]The term *developable surface* to indicate particular ruled surfaces (i.e., surfaces having straight lines passing through at any point) with null Gaussian curvature is subsequent and due to the work of Monge, see (Taton 1951, p. 169).

[19]"Quaestio aeque curiosa ac notatu digna" (Euler 1772a, p. 3)

[20]"Invenire aequationem generalem pro omnibus solidis, quorum superficiem in planum explicare licet." (Euler 1772a, p. 3)

Euler used the words *"in planum explicare licet"* without giving any explanation of how to interpret the term *explicare*. Rather, he referred to notions that were common at the time and justified by unspecified results *"ex ipsa Geometria elementari"*:[21]

> The property of cylinders and cones according to which their surface can be unfolded onto the plane is well known, and this property is also extended to all cylindrical and conical solids, which have any figure as their base; on the other hand, a sphere is deprived of this property, since its surface cannot be unfolded onto a plane in any way, nor can it be overlaid with a flat surface.[22]

Since not all surfaces can be unfolded on the plane, Euler wondered whether cones and cylinders were the only solids whose surface had this property.

First Euler's approach relies on purely analytical methods and provided conditions for developability.[23] He introduced two new tools into infinitesimal geometry, whose importance would be fully appreciated in their generality only after Gauss' work some 50 years later: the parametrization of a developable surface and its line element.

Apparently for the first time, in Euler's work a surface in Euclidean space is defined by means of its parametric equations and not as $F(x, y, z) = 0$ or $z = z(x, y)$. Parametric equations are the components of a vector function

$$\begin{aligned} \boldsymbol{x} : U \subset \mathbb{R}^2 \to & \mathbb{R}^3 \\ (u, v) \mapsto & \boldsymbol{x}(u, v) = (x(u, v), y(u, v), z(u, v)) \end{aligned}$$

that assign to a couple of parameters u, v the Cartesian coordinates $x(u, v), y(u, v), z(u, v)$ describing the surface at least in a neighbourhood of a given point. Probably, the formulation of the problem itself led Euler to this representation: since the plane is described by two orthogonal coordinates, u and v, and each point of the plane corresponds to a point of the surface, it is natural to consider the coordinates x, y, z of points upon the surface as dependent in some certain way on u and v and, therefore, each can be regarded as certain functions of these parameters. The map $\boldsymbol{x} : (u, v) \to (x(u, v), y(u, v), z(u, v))$ precisely assigns to each point on the plane the point on the surface that leans against it. This is confirmed by Euler's own words:

> Then let us imagine the surface of this solid, already unfolded on a plane [...], in which that point Z falls on [*incidat*] V, whose position is defined by two

[21] (Euler 1772b, p. 5)

[22] "Notissima est proprietas cylindri et coni, qua eorum superficiem in planum explicare licet atque adeo haec proprietas ad omnia corpora cylindrica et conica extenditur, quorum bases figuram habeant quamcunque; contra vero spherae hac proprietate destituitur, quum eius superficies nullo modo in planum explicari neque superficie plana obduci queat." (Euler 1772a, p. 3)

[23] More precisely, in (Euler 1772a, pp. 4–7) Euler found a set of equations (system (1.2)) that developable surfaces must necessarily satisfy. He did not specify that the validity of these equations is also sufficient to require the developability of a surface on the plane. However, from an examination of (Euler 1772a, pp. 7–26), in which he geometrically integrated the equations (1.2), it appears that Euler was aware that (1.2) describes only developable surfaces.

orthogonal coordinates so that $OT = t$ and $TV = u$: it is then clear that the previous three coordinates x, y and z must depend in some way on the pair t, u, and the idea is that each of them can be considered as certain functions of these t and u.[24]

Interestingly, Euler also imagined the problem in terms of a sheet of paper or a foil (*"charta"*, (Euler 1772a, p. 7)) which was suitably folded to cover the solid. Thus, while continuing to refer to surfaces as the boundaries of solids, as the title of (Euler 1772a) suggests (*De solidis quorum superficiem in planum explicare licet*), he apparently conceived the idea that surfaces are two-dimensional entities.

Here, we also find the expression of the line element[25] of a (developable) surface, i.e., the square of the distance between two infinitesimally close points on it, usually denoted by ds^2. However, it must be emphasised that Euler did not give this the prominence that we are used to assigning to it now and it did not have until the appearance of Gauss' *Disquisitiones*. Indeed, Euler did not even use a dedicated symbol to indicate it.

Euler naturally achieved the expression of the line element by translating the possibility of unfolding a surface on a plane into analytical terms. To this end, he required the existence of a local isometry by requiring corresponding infinitesimal triangles to be congruent.[26] He constructed an infinitesimal triangle on the plane, whose vertices are[27] $A = (u, v), B = (u + du, v), C = (u, v + dv)$, and its image through the map $\boldsymbol{x}$, whose vertices are $A' = (x, y, z), B' = (x + x_u du, y + y_u du, z + z_u du), C' = (x + x_v dv, y + y_v dv, z + z_v dv)$ up to infinitesimals of higher order than the first. Then, in the case that the surface and the plane are isometric, the lengths of homologous edges must be preserved. In particular, the requirement $\overline{BC} = \overline{B'C'}$ gives

$$\sqrt{du^2 + dv^2} = \sqrt{\boldsymbol{x}_u \cdot \boldsymbol{x}_u du^2 + 2\boldsymbol{x}_u \cdot \boldsymbol{x}_v du dv + \boldsymbol{x}_v \cdot \boldsymbol{x}_v dv^2}.$$

In anachronistic terminology used for the first time in (Gauss 1828, §§12-13), Euler required that the line element of a (developable) surface, which is

$$ds^2 = E du^2 + 2F du dv + G dv^2$$

where $E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u$, $F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v$, $G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v$, must be equal to the line element of the plane, which is $du^2 + dv^2$. This condition splits into three second-order partial differential

[24]"Deinde concepiamus superficiem huius solidi, iam in planum explicatam [...], in qua punctum illud Z incidat in V, cuius locus per binas coordinatas orthogonales ita definiatur ut sit, $OT = t$ et $TV = u$ atque manifestum est, ternas coordinatas priores x, y et z certo quodam modo ab his binis t et u pendere debere, ideaque singulas earum tamquam certas functiones istarum t et u spectari posse." (Euler 1772a, p. 4)

[25]Today line element is more commonly known as *first fundamental form* or *metric* of the surface.

[26]In this regard, see Appendix A.

[27]Here, we used the parameters u, v instead of the original t, u in order to make the notation uniform and we made the following substitutions

$$l = x_u \qquad \lambda = x_v \qquad m = y_u \qquad \mu = y_v \qquad n = z_u \qquad \nu = z_v.$$

We also adopt a vector notation, where subscripts u or v denote differentiation with respect to u, v.

equations

$$\boldsymbol{x}_u \cdot \boldsymbol{x}_u = 1 \qquad \boldsymbol{x}_v \cdot \boldsymbol{x}_v = 1 \qquad \boldsymbol{x}_u \cdot \boldsymbol{x}_v = 0, \tag{1.1}$$

namely,

$$\begin{cases} \left(\dfrac{\partial x_1}{\partial u}\right)^2 + \left(\dfrac{\partial y_1}{\partial u}\right)^2 + \left(\dfrac{\partial z_1}{\partial u}\right)^2 = 1 \\ \left(\dfrac{\partial x_1}{\partial v}\right)^2 + \left(\dfrac{\partial y_1}{\partial v}\right)^2 + \left(\dfrac{\partial z_1}{\partial v}\right)^2 = 1 \\ \dfrac{\partial x_1}{\partial u}\dfrac{\partial x_1}{\partial v} + \dfrac{\partial y_1}{\partial u}\dfrac{\partial y_1}{\partial v} + \dfrac{\partial z_1}{\partial u}\dfrac{\partial z_1}{\partial v} = 0. \end{cases}$$

Moreover, Euler also considered other conditions related to the effective existence of a surface. Indeed, this latter depends on the existence of three functions, $x(u, v)$, $y(u, v)$ and $z(u, v)$, that can be chosen as parametric equations of the surface, $\boldsymbol{x} = (x(u, v), y(u, v), z(u, v))$, so that $d\boldsymbol{x} = \boldsymbol{x}_u du + \boldsymbol{x}_v dv$ is integrable. To guarantee the integrability of this differential, Euler required the validity of the so-called *integrability* (or *compatibility*) *conditions* that consist of the equality of mixed second-order partial derivatives of $x(u, v), y(u, v)$ and $z(u, v)$.[28]

Hence, the problem of finding developable surfaces is reduced to the analytic problem of finding six functions, which are the components of $\boldsymbol{x}_u = (x_u, y_u, z_u)$ and $\boldsymbol{x}_v = (x_v, y_v, z_v)$, such that

$$\begin{cases} \boldsymbol{x}_u \cdot \boldsymbol{x}_u = 1 \qquad \boldsymbol{x}_v \cdot \boldsymbol{x}_v = 1 \\ \boldsymbol{x}_u \cdot \boldsymbol{x}_v = 0 \qquad \boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}. \end{cases} \tag{1.2}$$

Unfortunately, this system was inaccessible to Euler. Nevertheless, he found an alternative, more geometrical way to successfully interpret the problem and integrate the system. To this purpose, the central remark, which appeared here for the first time, is a clear distinction between developable and ruled surfaces. Euler observed that it is not sufficient for a surface to contain an infinite number of lines for it to be developable, it must also be the case that two lines on it, which are infinitesimally close to each other, are incident and

[28] *Integrability conditions* are necessary and sufficient conditions for the the integrability of the differential $\alpha(u, v)du + \beta(u, v)dv$. For this purpose there must exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\frac{\partial f}{\partial u} = \alpha$ and $\frac{\partial f}{\partial v} = \beta$. Indeed, in this way, one has $df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv = \alpha(u, v)du + \beta(u, v)dv$. See, for example, (Spivak 1999, Vol. 1, p. 184). Until the end of nineteenth century, integrability conditions simply consisted of the request $\frac{\partial \alpha}{\partial v} = \frac{\partial \beta}{\partial u}$, that is, the equality of second-order partial derivatives $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. Today, it is well known that some requirements on the regularity of f must be demanded, but these corrections took a long time to be considered. Nicolaus Bernoulli had already used integrability conditions in an unpublished manuscript dating back to 1719. In 1743 he wrote to Euler that he regarded them as *"an axiom, which I thought to be obvious to anybody from the mere notion of differentials"* (Engelsman 1984, p. 106). First, but fallacious, proofs were published by Euler and Clairaut in 1740 and, from then, several mathematicians strove for a more rigorous, but still unsatisfactory, demonstration, as detailed in (Lindelöf 1933). It was not until 1873 that Schwarz succeeded in giving the first rigorous proof. As well as (Lindelöf 1933), a bibliographical account and more details on this subject can be found in (Higgins 1940), (Katz 1981) and (Gray 2021, pp. 27–29). As we will see, the lack of a rigorous proof did not prevent geometers from habitually employ these conditions in the course of the eighteenth and nineteenth centuries: their correctness was so widely accepted that their assumption was not even reported.

their intersection points constitute a space curve as a consequence of continuity.[29] Euler drew this conclusion by observing the possible reciprocal position of lines in the plane. He imagined unfolding a sheet of paper over a cone. The sheets must be folded along lines that converge to a point, which corresponds to the vertex of the cone. Similarly for cylinders, the folds draw parallel lines on the sheet, i.e., lines that meet at the point at infinity. But, there are infinite other possible configurations of straight lines in the plane, in which the lines are not all parallel nor convergent in a point: as an example, Euler provided Figure 1.2. By supposing that the lines cover the sheet of paper and folding it along them, this action gives rise to a new type of developable surfaces, now known as *tangent surfaces*, characterised by the fact that the generating lines are tangent to the same line. This line will later be called the *edge of regression*.

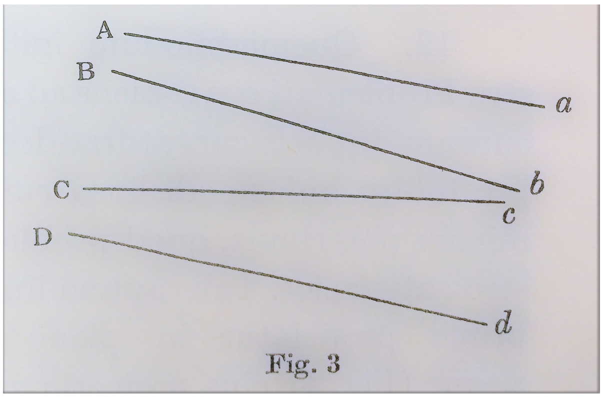

Figure 1.2: Simple drawing that illustrates a possible reciprocal position of lines in the plane. It was provided by Euler himself in (Euler 1772a). The picture is taken from Euler's *Opera omnia*, vol. 28.

This simple and acute reasoning led him to establish a bijective correspondence between developable surfaces and space curves: each developable surface can be regarded as the set of tangent lines to a space curve. Besides cones and cylinders, there exist infinitely many other developable surfaces, as many as there are space curves. Euler exploited this correspondence to write down the isometry between a generic developable surface and a plane. First, he wrote the parametric equations of a generic developable surface. To achieve this, he considered the surface as generated by the tangents of a space curve. He parametrized the edge of regression by means of its first coordinate $\tilde{x}$ as $\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y}(\tilde{x}), \tilde{z}(\tilde{x}))$ and then he parametrized its tangent lines by means of the parameter s, which represents the distance between any point of the surface and the point of tangency of the line passing through that point. In this way, the parametric equations were

$$\begin{cases} x = \tilde{x} - s\sin\theta\sin\zeta \\ y = \tilde{y} - s\sin\theta\cos\zeta \\ z = \tilde{z} - s\cos\theta, \end{cases} \tag{1.3}$$

[29]The appendix B shows that, at the end of the nineteenth century, progress in analysis led to a new definition of developable surfaces and the consequent discovery of developable surfaces that are not ruled.

where θ and ζ are angles that Euler expressed in terms of $\tilde{x}$ only.

Finally, he explicitly provided the isometry $\Phi : S \to \mathbb{R}^2$ between such a surface and the plane by requiring that the length of infinitesimal elements of the edge of regression be preserved, as well as the angle between two of its consecutive tangents.[30] Since (1.3) are all developable surfaces, Euler thus succeeded in integrating the system (1.2) via a geometric construction.

It should be noted that Euler was aware that the problem he dealt with in (Euler 1772a) was only a particular case of a more general one. An attempt at generalization can be found in (Euler 1862, pp. 494–496), which is an undated fragment of notes written by Euler's student Mikhail Evseyevich Golovin and published in 1862.[31] Here, Euler had the following purpose:

> To find two surfaces, one of which can be transformed into the other, in such a way that corresponding [homologa] points maintain the same distances from each other.[32]

For the first time, the problem of unfolding one surface upon another was no longer seen as limited to developable surfaces. However, there does not seem to be any other evidence of the circulation of these ideas among Euler's pupils and collaborators, and we have to wait until Gauss' work to find references to this subject again.

As it is evident, in (Euler 1862) the language is more abstract than that used in (Euler 1772a): here, he did not employ the notion of "surface of a solid" but only used the term "surface", nor he did refer to the act of "unfolding" a surface, but simply to that of "transforming so that distances are preserved". In addition, no comparison is made between surfaces and flexible materials such as sheets of paper. He maintained the possibility of describing a surface by means of a two-variable parametrization and, in analogy with (Euler 1772a), to realise the isometry Euler considered two infinitesimal triangles, one upon each surface, which were in correspondence and required the length of the homologous edges to be equal. In this way, Euler arrived at the equality of the line elements of the two surfaces.

By resorting to a vector notation that is not to be found in Euler's treatment, we can describe his observation as follows. He considered any two surfaces, Σ that was parametrized by means of the function $\boldsymbol{x} : U \to \mathbb{R}^3$, $\boldsymbol{x} = (x_1(u,v), x_2(u,v), x_3(u,v))$, and Σ', that was parametrized by means of the function $\boldsymbol{x'} : U \to \mathbb{R}^3$, $\boldsymbol{x'} = (x'_1(u,v), x'_2(u,v), x'_3(u,v))$. Similarly to how the system (1.1) is derived in (Euler 1772a), the isometric correspondence

[30] For a detailed description of the construction of Euler's isometry, one can see (Cogliati 2022, pp. 49–52).

[31] According to (Eneström 1913), it dated back to 1766-1783. Apparently, there is insufficient evidence to establish whether this fragment is later than (Euler 1772a) or not.

[32] "Invenire duas superficies, quarum alteram in alteram transformare liceat, ita ut in utraque singula puncta homologa easdem inter se teneant distantias" (Euler 1862, p. 494)

of two infinitesimal triangle gives

$$\begin{cases} \left(\frac{\partial x_1}{\partial u}\right)^2 + \left(\frac{\partial y_1}{\partial u}\right)^2 + \left(\frac{\partial z_1}{\partial u}\right)^2 = \left(\frac{\partial x_1'}{\partial u}\right)^2 + \left(\frac{\partial y_1'}{\partial u}\right)^2 + \left(\frac{\partial z_1'}{\partial u}\right)^2 \\ \left(\frac{\partial x_1}{\partial v}\right)^2 + \left(\frac{\partial y_1}{\partial v}\right)^2 + \left(\frac{\partial z_1}{\partial v}\right)^2 = \left(\frac{\partial x_1'}{\partial v}\right)^2 + \left(\frac{\partial y_1'}{\partial v}\right)^2 + \left(\frac{\partial z_1'}{\partial v}\right)^2 \\ \frac{\partial x_1}{\partial u}\frac{\partial x_1}{\partial v} + \frac{\partial y_1}{\partial u}\frac{\partial y_1}{\partial v} + \frac{\partial z_1}{\partial u}\frac{\partial z_1}{\partial v} = \frac{\partial x_1'}{\partial u}\frac{\partial x_1'}{\partial v} + \frac{\partial y_1'}{\partial u}\frac{\partial y_1'}{\partial v} + \frac{\partial z_1'}{\partial u}\frac{\partial z_1'}{\partial v}, \end{cases} \tag{1.4}$$

which gives the solution to the problem once solved. Later, after Gauss' *Disquisitiones*, two such surfaces came to be called *applicable* one upon the other.

However, Euler judged the system of equations (1.4) as *"maxime arduum"*[33] to be solved by known methods and he reduced it to another form by using analytical tools. This new system offered Euler the occasion for some rather enigmatic remarks. He observed that if two surfaces can be transformed into each other isometrically, then both surfaces *"must necessarily be extended to infinity, and that this extension cannot be removed by any imaginary means"* since in the system the parameters u and v can increase without limitations. *"Therefore neither a spherical figure, nor any other form that stands in a finite space, can be contained in these formulae"*.[34] He then concluded by mentioning a sort of rigidity conjecture of closed surfaces, which seemed intended as a generic impossibility to deform (*nullam amplius mutationem patitur*) closed surfaces:[35]

> As soon as the shape is everywhere closed, it can no longer be deformed. This can be understood by observing the known solid shapes that are usually called regular. Thus, insofar as the spherical surface is complete, it admits no mutation. It is therefore clear that such forms can be mutated to the extent that they are not whole or everywhere closed. Now, it is clear that the shape of the hemisphere is certainly mutable. But what kinds of mutations are possible seems to be a very difficult problem.[36]

[33](Euler 1862, p. 494)

[34]"Cum enim in utrisque formulis binae variabiles r et s in infinitum augeri quaent, facile pater utramque superficiem necessario in infinitum protendi, neque hanc extensionem per quaepiam imaginaria tolli posse. Quamobrem figura spherica neque ulla alia figura in spatio finito substitens in his formulis contenta esse potest".(Euler 1862, p. 495)

[35](Liberti and Lavor 2016) finds in these words the first formulation of a *rigidity conjecture for polyhedra*. However, this interpretation seems inadequate, at least because, in the fragment, Euler considered any surfaces (he did not use the term *surface of solids* either) and did not limit himself to polyhedra. Rather, in a letter to Lagrange from 9th November 1762, Euler explicitly discussed the impossibility of deforming polyhedra: *"Bodies have strangeness that are not found in surfaces; although all the sides of a polygon and even their order are given, the figure is still susceptible of an infinite number of determinations; but, in a polyhedron, as soon as all the edges with their order are known, the body is completely determined."* (Lagrange 1842, p. 203)

[36]"Statim enim atque figuras solida undique est clausa, nullam amplius mutationem patitur; quemadmodum ex notis illis figuris corporeis, quae corpora regularia vocari solent, intelligere licet. Unde quatenus superficies spherica est integra, nullam mutationem admittit. Hinc patet, eatenus hujusmodi figuras mutari posse, quatenus non sunt integrae seu undique clausae. Interim patet hemisphaerii figuram certe esse

Beyond the validity of the statement and its exact interpretation, it is interesting to note that Euler distinguished between global and local deformability, and did not question the possibility of locally deforming a sphere, although he was unable to prove this analytically.

Euler's interest in developable surfaces and applicable surfaces in general was probably conditioned by his interest in the theory of partial differential equations (PDEs) that was in its infancy during those years. This assumption is based on the fact that Euler in a letter addressed to Lagrange mentioned a connection between PDEs theory and a problem that was quite similar to the determination of all surfaces that correspond to the same linear element. Indeed, two years before the publication of (Euler 1772a), in (Euler 1770) he had tried to determine all the surfaces that correspond to the same *surface element.*

Euler first observed that a plane curve, which was supposed to be referred to an orthogonal coordinate system, is uniquely determined, except for translations across the y axis, by the assignment of its line element. Indeed, when a plane curve is parametrically represented through $\boldsymbol{x} = (x, f(x))$, its line element $ds^2 = dx^2 + dy^2$ can also be expressed as $ds^2 = P^2(x)dx^2$. As a consequence,

$$dy = \sqrt{P^2(x) - 1}dx$$

holds true and, when integrated, gives the equation for the required curve, save for a constant of integration. On the contrary, surfaces exhibit a different behaviour. The assignment of the surface element

$$dA = \sqrt{1 + f_u^2 + f_v^2}dudv,$$

i.e., the measure of the area of an infinitesimal parallelogram on the surface (see Figure 1.3),[37] is not sufficient to unambiguously determine it. Indeed, Euler observed that when

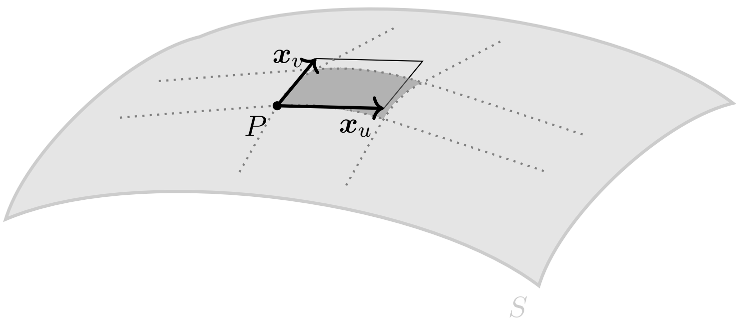

Figure 1.3: A representation of a surface element at a point P of a surface S. $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are the coordinate tangent vectors at P.

mutabilem, cuiusmodi autem mutationes recipere possit, problema videtur difficillimum." (Euler 1862, pp. 495–496)

[37]The area of the parallelogram in Figure 1.3 is obtained as $\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|$. When the parametric equations of the surface are given in the form $\boldsymbol{x} = (u, v, f(u, v))$, one has $\boldsymbol{x}_u = (du, 0, f_u(u, v)du)$ and $\boldsymbol{x}_v = (0, dv, f_v(u, v)dv)$. Hence, $\|\boldsymbol{x}_u \times \boldsymbol{x}_v\| = \sqrt{1 + f_u^2 + f_v^2}dudv$.

two surfaces S_1 and S_2 have $dA_1 = \sqrt{1 + f_{1u}^2 + f_{1v}^2}dudv$ and $dA_2 = \sqrt{1 + f_{2u}^2 + f_{2v}^2}dudv$ as surface elements respectively, then the condition $dA_1 = dA_2$, that is,

$$f_{1u}^2 + f_{1v}^2 = f_{2u}^2 + f_{1v}^2 \tag{1.5}$$

is not sufficient to guarantee that S_1 and S_2 are the same surface (up to a translation). The equation (1.5) is trivially verified when $f_{1u}^2 = f_{2u}^2$ and $f_{1v}^2 = f_{2v}^2$. In this case, S_1 and S_2 are exactly the same surface, since $df_i = f_{iu}du + f_{iv}dv$, $i = 1, 2$. But Euler showed other cases in which this equation is satisfied. For example, he chose S_1 as the elliptic paraboloid and S_2 as the hyperbolic paraboloid, whose equations are respectively

$$f_1(x, y) = \frac{x^2 + y^2}{2a} \qquad f_2(x, y) = \frac{xy}{2a}$$

where a is a constant. They are clearly distinct surfaces, but (1.5) holds true.[38] Euler called surfaces having the same surface element *congruent* and he tried to find some of these classes of surfaces in the rest of the memoir.

The fact that surfaces were not uniquely determined by the assignment of the surface element intrigued Euler so much that he referred to it as a paradoxical aspect of surfaces, as the title of (Euler 1770) suggests (*Evolutio insignis paradoxi circa aequatlitatem superficierum*). It is not surprising, therefore, that he then turned to a similar problem (the problem of isometric transformation of surfaces), which he also considered as *"curious and noteworthy"*.

Euler had already mentioned his research on congruent surfaces to Lagrange in a letter dated 9th November 1762, i.e., eight years before the submission of (Euler 1770) to the St. Petersburg Academy of Sciences:

> Bodies have strangeness that is not found in surfaces; [...] it is impossible to give two different curves, which have equal arcs for all the abscissas; but we can always find an infinite number of different surfaces where the elements $dxdy\sqrt{1 + p^2 + q^2}$ are the same. Thus, conical surfaces, whose axis is perpendicular to the basis, conform to a plane surface, and the solids expressed by these equations $az = xy$ and $2az = x^2 + y^2$ have equal surfaces, since $p^2 + q^2$ is the same on both sides;[39] but one can easily find an infinite number of other surfaces of the same nature, where one can even introduce arbitrary and discontinuous functions.[40] But it is more difficult to find such solids, whose surface agrees with that of the sphere. The task is to find such an integrable equation

[38] Indeed, one has $f_{1u} = \frac{x}{a}$, $f_{1v} = \frac{y}{a}$, $f_{2u} = \frac{y}{a}$, $f_{2v} = \frac{x}{a}$.

[39] $az = xy$ and $2az = x^2 + y^2$ are the equations of a hyperbolic paraboloid and a elliptic paraboloid, respectively.

[40] In this letter, Euler did not explain what he meant by "discontinuous functions". However, it seams that Euler generally used this expression to denote functions that are not described by a unique analytic expression. Thus, according to Euler, a hyperbola is a continuous function and a piecewise-defined function is always discontinuous, contrary to the modern interpretation of the term "discontinuous". For more detail, the interested reader can see (Yushkevich 1976, §9).

$dz = pdx + qdy$ such that $p^2 + q^2$ is equal to $\frac{x^2+y^2}{a^2-x^2-y^2}$. I can define all the possible functions for p and q, but I cannot derive any, from which the equation relating x, y and z becomes algebraic. This is still a subject that requires the new branch of Analysis that deals with functions of two or more variables, of which certain relations between their differentials are given.[41]

While Euler's interest initially seems to be motivated by curiosity about certain "strangeness" of surfaces, the last lines of the quotation above seem to suggest how strangeness shifted his interest to PDEs theory. As noted in (Gray 2021, p. 55), it was in the early 1760s that Euler published his first studies in this field, (Euler 1763) and (Euler 1764). In addition, as highlighted in (Capobianco, Enea, and Ferraro 2017), analysis occupied a prominent place in Euler's conception of mathematics and he developed a sort of programme that aimed to transform analysis into an autonomous discipline and reorganise the whole of mathematics around it.

It cannot therefore be ruled out that the study of developable surfaces, which is so similar to the study of congruent surfaces, was also motivated by an analytical interest rather than a geometric one. This hypothesis seems to find support in two other letters that Euler addressed to Lagrange, dated 16th January and 19th March 1770, in which he exposed to his colleague some passages of his research on developable surfaces. In a post scriptum to the first letter, he described the problem in purely analytical terms without resorting to any geometric interpretation:

> Some time ago, Sir, I found a complete solution to the following problem: it consists in finding three functions X, Y and Z of two variables t and u, such that, by posing $dX = Pdt + pdu$, $dY = Qdt + qdu$, $dZ = Rdt + rdu$, they satisfy the following conditions
>
> $$P^2 + Q^2 + R^2 = 1,$$
>
> $$p^2 + q^2 + r^2 = 1,$$
>
> $$pP + qQ + rR = 0.$$

[41]"Mais les corps ont des bizarreries qui ne se trouvent pas dans les surfaces; [...] on ne saurait donner deux courbes différentes qui aient pour toutes les abscisses des arcs égaux; mais on peut toujours trouver une infinité de surfaces différentes òu les éléments $dxdy\sqrt{1+p^2+q^2}$soient les mêmes. Ainsi les surfaces coniques dont l'axe est perpendiculaire à la base conviennent avec une surface plane, et les corps exprimés par ces équations $az = xy$ et $2az = x^2 + y^2$ ont leurs surfaces égales, puisque $p^2 + q^2$ est le même de part et d'autre; mais on trouve aisément une infinité d'autres surfaces de la même nature, où l'on peut même introduire des fonctions arbitraires et discontinues. Or il est plus difficile de trouver de tels corps, dont la surface convienne avec celle de la sphère. Il s'agit de trouver une telle équation intégrable $dz = pdx + qdy$ que $p^2 + q^2$ soit égal à $\frac{x^2+y^2}{a^2-x^2-y^2}$. Je puis bien définir toutes les fonctions possibles pour p et q, mais je n'en puis tirer aucune d'où l'équation entre x, y et z devient algébrique. C'est encore un sujet qui demande la nouvelle branche de l'Analyse qui roule sur les fonctions de deux ou plusieurs variables, de certains rapports entre leurs différentiels étant donnés." (Lagrange 1842, pp. 203–204)

Now the nature of differentials still requires the following conditions

$$\frac{\partial P}{\partial u} = \frac{\partial p}{\partial t} \qquad \frac{\partial Q}{\partial u} = \frac{\partial q}{\partial t} \qquad \frac{\partial R}{\partial u} = \frac{\partial r}{\partial t}.$$

As a very singular consideration has led me to the solution of this problem, which I would otherwise have judged almost impossible, I believe that this discovery may become of great importance in the new part of Integral Calculus for which Geometry is indebted to you.[42]

In the second letter he detailed his results by providing a totally analytical approach to the problem, but he stressed:

> In order that you do not think that this is a pure and sterile speculation, I have the honour to tell you that the following problem has led me to it: *To find all the solids whose surface can be unfolded (unfold, develop; explicare) onto a plane, as it happens in all cylindrical and conical bodies.*[43]

Hence, Euler viewed the system (1.2) primarily as a system of partial differential equations and only secondarily as the analytical transcription of a geometric problem. Furthermore, he saw the integration of the system (1.2) as an important step for the theory of PDEs. In this respect, the second solution presented in (Euler 1772a, pp. 7–26) can therefore be considered as an indirect and ad hoc method for solving the system of equations (1.2). In fact, Euler overcame his inability to deal with the system (1.2) by resorting to the specific geometric properties of the problem that had given rise to the equations. Hence, it can be argued that, to a certain extent, in the case of developable surfaces, Euler resorted to geometric intuition to solve an analytical problem.

1.3 Gaspard Monge and his research on developable surfaces

Euler's ideas found a particularly favourable reception in France, especially in the work of Gaspard Monge (1746-1818)[44] who significantly advanced some of Euler's results. Con-

[42]"Il y a quelque temps, Monsieur, que j'ai trouvé une solution complète du problème suivant: Il s'agit de trouver trois fonctions X, Y et Z de deux variables t et u, telles que, posant $dX = Pdt + pdu$, $dY = Qdt + qdu$, $dZ = Rdt + rdu$, on satisfasseaux conditionssuivantes $P^2 + Q^2 + R^2 = 1$, $p^2 + q^2 + r^2 = 1$, $pP + qQ + rR = 0$. Or la nature des différentielles demande encore les conditions suivantes

$$\frac{\partial P}{\partial u} = \frac{\partial p}{\partial t} \qquad \frac{\partial Q}{\partial u} = \frac{\partial q}{\partial t} \qquad \frac{\partial R}{\partial u} = \frac{\partial r}{\partial t}.$$

Comme une considération tout à fait singulière m'a conduit à la solution de ce problème, que j'aurais d'ailleurs jugé presque impossible, je crois que cette découverte pourra devenir d'une grande importance dans la nouvelle partie du Calcul intégral dont la Géométrie vous est redevable." (Lagrange 1842, pp. 217–218)

[43]"Afin que vous ne pensiez pas que c'est une pure et stérile spéculation, j'ai l'honneur de vous dire que le problème suivant m'y a conduit: *Trouver tous les solides dont la surface puisse être expliquée (Expliquer, développer; explicare) dans un plan, comme il arrive dans tous les corps cylindriques et coniques.*" (Lagrange 1842, p. 221)

[44]A detailed account of Monge's life and works is provided by (Taton 1951). For more about the subject of this section, we refer to (Taton 1951, Chap. 1 and Chap. 4).

temporarily but independently from him, Monge investigated the theory of developable surfaces from the beginning of his research career.

Around 1764, while he was working at the École Royale du Génie de Mézières, Monge was commissioned to solve an engineering problem: to determine the necessary height of an outer wall in a fortification project. Monge's solution was appreciated for its efficiency and was covered by military secrecy. Starting from these investigations, he developed a general geometric theory, which was presented to the Academy of Sciences in Paris in October 1770, but was not published until 1785 in (Monge 1785). In this wide exposition of space curves, Monge's main achievement was the extension of the theory of evolutes from plane curves to space curves by introducing the notion of *polar surface*.

For a plane curve C, its evolute is the geometric locus of all its centres of curvature. Equivalently, in view of the definition of the centre of curvature as the intersection point of two infinitesimally close normal lines to the curve, the evolute of a plane curve C can also be identified with the the envelope of the normals to the curve C (C is called *involute*). In the case of space curves, the extension of this latter definition must take into account that there are infinite normal lines passing through each point P of the curve, which together constitute a plane that is perpendicular to the tangent vector $\boldsymbol{v}$ at that point. Instead of centres of curvature, Monge thus considered the line of intersection between two normal planes to the curve that pass through two points infinitesimally close to P, which is constituted by the points of intersection of all possible couples of coplanar normals passing through infinitesimal close points of C. This line is called *axis of the osculating circle* of a point P (see Figure 1.4). By construction, it is the line that passes through the centre of the osculating circle orthogonally to the osculating plane at P.

According to Monge, the juxtaposition of the axes of osculating circles of any space curve C constitute the *polar surface*. It contains the locus of the centres of curvature of C, but also all those curves that are obtained by intersecting it with the normals of the curve that form the same angle with the osculating plane. In this way, it can be concluded that a space curve C has infinite evolutes that together constitute the polar surface.

Interestingly, Monge noted that polar surfaces have a common property that is independent from C, although the shape of the surface completely depends on that of C:

> This characteristic is that they can be developed on a plane, like any conical and cylindrical surfaces, without duplication, and without solution of continuity. [...] Therefore the surface of the poles of any double curvature curve is always a developable surface.[45]

Monge observed that the area between any two consecutive axes that constitute polar surfaces must be plane. Indeed, as illustrated in Figure 1.5, two consecutive axes, l and l', lie on the same normal plane.

[45] "Ce caractère est de pouvoir être développées sur un plan, comme les surfaces coniques et cylindriques bases quelconques, sans duplicature, et sans solution de continuité. [...] Donc la surface des pôles d'une courbe double courbure quelconque toujours une surface développable." (Monge 1785, §V)

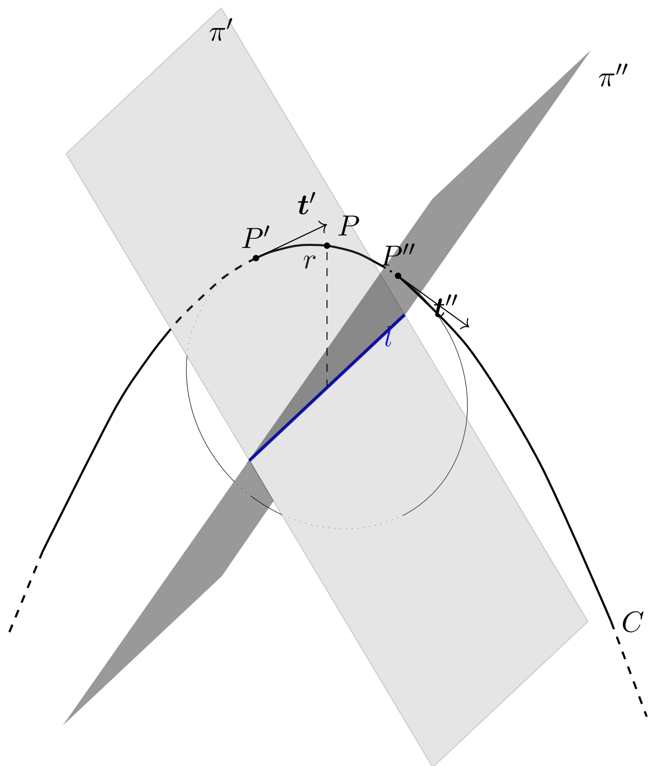

Figure 1.4: Representation of the axis of the osculating circle l at a point P of a space curve C. P' and P'' are points of the curve C infinitely close to P. π' (π'') is the plane passing through P' (P'') and orthogonal to the tangent vector $\boldsymbol{t}'$ ($\boldsymbol{t}''$). The intersection of π' and π'' is the axis of the osculating circle l at P. The centre of the osculating circle lies on the the axis l.

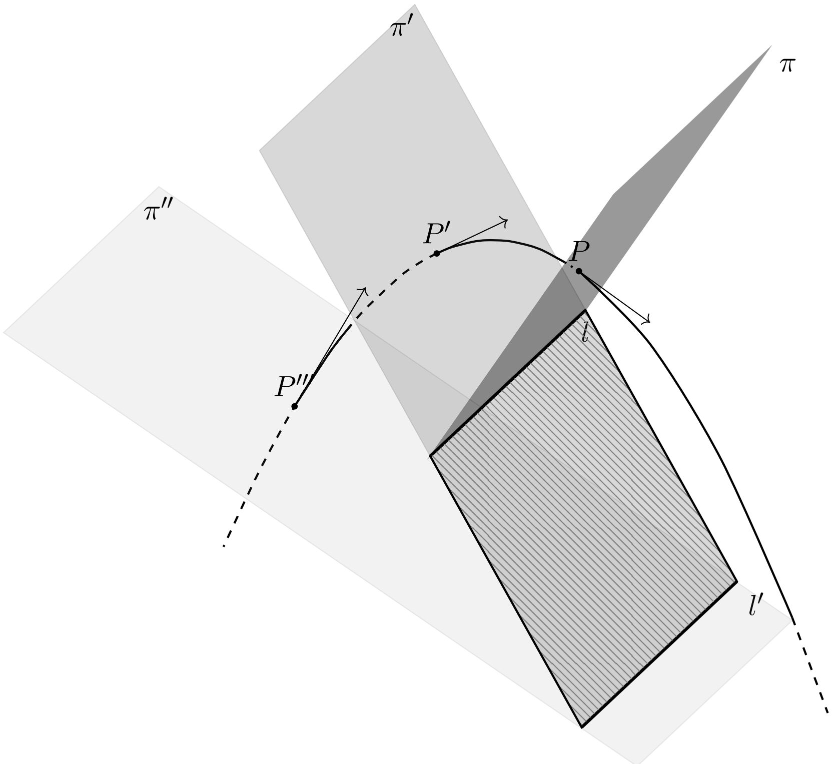

Figure 1.5: Representation of the reciprocal position of two consecutive axes of the osculating circle, l and l'. P, P' and P'' are infinitesimally close points on the curve C. π' , π' and π'' are the normal planes through P, P' and P'', respectively. The intersection of π and π' is the axis of the osculating circle l. The intersection of π' and π'' is the axis of the osculating circle l'. Both l and l' lie on the plane π'.

Conversely, for surfaces to be developable, any two consecutive axes of osculating circles must intersect at a point. These points constitute a second curve that is the *edge of regression*.[46]

Monge also deduced that infinite other developable surfaces must exist apart from cylinders and cones. He observed that the polar surface of any plane curve is a cylinder that has its evolute as base; vice versa, any cylinder can be interpreted as the polar surface of a plane curve (the involute of its base). Similarly, a curve drawn on a sphere has a cone as its polar surface, whose vertex is the centre of the sphere and whose basis depends on the curve;[47] vice versa, every cone is the polar surface of a curve on a sphere. However, Monge noted, there exist curves that are not plane or that are not drawn on a sphere; therefore, their polar surface is a developable surface that is neither conical nor cylindrical.

After these purely geometric considerations and after summarising some necessary elementary results from analytic geometry,[48] he deduced the differential equations of developable surfaces that are polar surfaces of an assigned curve C.[49]

Monge's interest in developable surfaces therefore seems to be of a different nature from Euler's. While Euler's focus appears to be on the analytical aspects of the problem, Monge was probably attracted by the way he defined them as juxtapositions of straight lines. As (Lawrence 2011, pp. 707–708) points out, geometric constructions involving a continuous motion of lines played an important role in Monge's thinking in accordance with his interpretation of geometry based on the generating principle:[50]

> developable surfaces are [...] an integral part of the Mongean treatment of space. [...] all objects imagined through the use of descriptive geometry as given by Monge are ruled surfaces (and most are developable). Monge describes all geometrical objects through generation: a point is a generatrix of a line; similarly any plane is generated by two lines. [...] every three-dimensional geometrical body can be described as a sum of planes, which are generated by some mode of motion and intersect each other. This means that the solid is not regarded as an independent entity in itself, but as a product of motion and intersection of its primary elements. [...] In this way the perception of all geometrical entities is changed from a collection of various forms, to a collection of various methods and processes, by which those forms are being generated.

Monge chose developable surfaces as a sort of primitive object, which can be adopted

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[46]See (Monge 1785, §XVI).

[47]It is a consequence of the fact that all normal planes to the curve pass through the centre of the sphere.

[48]He provided the construction of a plane perpendicular to a line, that of a line passing through a point and orthogonal to another line and that of a plane orthogonal to a space curve. Monge's choice to quote these demonstrations in spite of their (in our opinion) simplicity can offer an insight into the state of development of analytical geometry at that time.

[49](Monge 1785, p. 531)

[50]As is well known, Newton already conceived of mathematical entities as being generated by motion. For an introduction to this aspect of Newton's thought, the interested reader can see (Edwards 1979, Chap. 8). However, as far as we know, there are no explicit references to Newton's ideas in Monge's work.

to define more complex ones. A significant sample of this occurred in (Monge 1781) where the notion of *lines of curvature* of a surface was introduced for the first time. Once again, an engineering problem acted as a motivation: to optimise the transport of a mass of material composed of infinitely small particles (sand, for example). The fact that to have the required minimum it is necessary for the particles to move in a straight line led him to more deeply investigate the theory of surfaces, developable surfaces in particular, linking his results with those obtained by Euler 20 years earlier. Indeed, in (Monge 1781, pp. 685–689) he proved that at every point of a surface S the normal direction to the surface can be regarded as the intersection of two developable surfaces, which meet under a right angle, and that the intersections of these developable surfaces with the surface result in two curves, named *lines of curvature.* Therefore, lines of curvature were defined as those lines having normal vectors to the surface that generate a developable surface. The remarkable fact noted by Monge himself is that these lines have maximum and minimum normal curvature, respectively, and their tangent vectors at that point coincide with Euler's principal directions.

The construction of developable surfaces as special ruled surfaces is dominant in (Monge 1785) and the possibility of unfolding them upon a plane seems to be only a consequence, albeit a noteworthy one, of the way they were constructed.[51] This aspect assumed a more relevant role when, after reading (Euler 1772a), Monge published (Monge 1780). Here, he gave a definition of these kinds of surfaces in terms of a flexible and inextensible veil and, at least for developable surfaces, he officially removed any reference to the classical legacy that depicted surfaces as the boundary of a solid:

> A surface is developable when, supposing it to be flexible and inextensible, one may conceive of mapping it onto a plane, like those of cones and cylinders, so that the way in which it rests on the plane is without duplication or disruption of continuity, or, which is the same, having led it through any point a tangent and indefinite plane, and supposing the plane to be flexible and inextensible, it may be conceived of as folded over the surface, so that it comes into contact with it everywhere, without its having to be broken or folded up at any point.[52]

Hence, the possibility of unfolding a surface on a plane and the increasingly frequent appearance of surfaces other than quadrics gave a renewed perspective to the notion of surface and slowly led to a radical change in the definition of any surface, which was interpreted as a flexible, inextensible veil throughout the nineteenth century.

[51] In this respect, it is worth noting that Monge defined these kinds of surfaces *developable*, which is reminiscent of the term *développées* (evolute).

[52] *"Une surface est développable, lorsqu'en la supposant flexible et inextensible, on peut la concevoir appliquée sur un plan, comme celles des cônes et des cylindres, de manière qu'elle le touche sans duplicature ni solution de continuité, ou, ce qui revient au même, lorsque lui ayant mené par un point quelconque un plan tangent et indéfini, et en supposant le plan fiexible et inextenfible, on peut le concevoir plié sur la surface, de manière qu'il soir par-rout en contact avec elle, sans que pour cela il faille le briser ou replier en aucun point."* (Monge 1780, p. 383)

To introduce the object of (Monge 1780), Monge clarified its connection to (Euler 1772a):

> After returning to this subject [developable surfaces], on the occasion of a Memoir that M. Euler gave in the Volume of 1771, of the Academy of Petersburg, on developable surfaces, and in which this illustrious Geometer gives formulas for recognising whether a proposed curved surface enjoys or not the property of being able to be applied to a plane, I arrived at results that seem to me to be much simpler, and of a much easier use for the same purpose.[53]

He adopted Euler's point of view by defining developable surfaces as the envelope of a space curve and, in (Monge 1780, pp. 385–397) he developed three different ways of providing their analytical characterisation in terms of a second-order differential equation. When a surface S is represented as $z = z(x, y)$, Monge proved that it is developable if, and only if,[54]

$$z_{xx}z_{yy} - z_{xy}^2 = 0 \tag{1.6}$$

holds true. Therefore, in order to recognise whether a surface $z = z(x, y)$ is developable or not, it is sufficient to calculate its second-order partial derivatives and check whether they satisfy (1.6). Finally, in (Monge 1780, pp. 427–435), Monge also went deeper into the distinction between ruled non-developable surfaces (*surfaces gauches*) and developable surfaces by providing their general representation in terms of a third-order partial differential equation and by proving that developable surfaces are a sub-case of ruled surfaces.

These results paved the way for one of Monge's most representative ideas on differential geometry. As Morris Kline commented:

> In pursuing the correspondence between the ideas of analysis and those of geometry, Monge recognised that a family of surfaces having a common geometric property or defined by the same method of generation should satisfy a partial differential equation.[55]

Indeed, a specific feature of Monge's approach was his frequent recourse to differential equations in order to more easily describe the properties of geometric objects, as in the case of developable surfaces. In this regard, a comment by Gino Loria clearly reveals the novelty of this fusion of analysis and geometry:

> From this instant, it can be seen that in a large number of cases it is far more convenient to characterise the intimate nature of a family of surfaces by considering its differential equation than the equation in finite terms.[56]

[53] "Ayant repris cette matière, à l'occasion d'un Mémoire que M. Éuler a donné dans le Volume de 1771, de l'Académie de Pétersbourg, sur les surfaces développables, et dans lequel cet illustre Géomètre donne des formules pour reconnoître si une surface courbe proposée, jouit ou non de la propriété de pouvoir être appliquée sur un plan, je fuis parvenu à des résultats qui me semblent beaucoup plus simples, et d'un usage bien plus facile pour le même objet." (Monge 1780, p. 383)

[54] Here, subscripts x or y denote differentiation with respect to x, y.

[55] (Kline 1972, p. 566)

[56] "Da questo istante si scorge come in un gran numero di casi per caratterizzare la natura intima di una

Monge systematically applied infinitesimal calculus to the study of surfaces, by emphasising the need to harmonise geometrical and analytical treatments of a given subject. In this respect, one can say that Monge's main contribution consisted precisely in promoting a synthesis between geometry and analysis.

1.3.1 The school of differential geometry in Paris

Monge's influence on the French mathematical community operated mainly through his didactic activity both at the *École Polytechnique* and at the *École Normale*, where he taught lecture courses on both descriptive and differential geometry. The contents of his courses on differential geometry were published with some extensions in the first textbook ever devoted to the subject, which he entitled *Application de l'Analyse à la Géométrie*. Monge's commitment to education contributed to creating a favourable milieu within which the study of differential geometry acquired a special importance.

Among the students that oriented their research interests to this field were Jean-Baptiste Meusnier, Charles Dupin and Olinde Rodrigues.

Jean-Baptiste Meusnier (1754-1793)[57] was a student of Monge in 1774-75 when he taught at the École de Mézières. Meusnier turned his attention to the curvature of surfaces when his master challenged him to demonstrate some of Euler's results. Monge later remembered that episode with these words:

> The next morning, in the common rooms, he gave me a small paper containing the demonstrations, but, what was remarkable, the considerations he had employed were more direct than Euler's and the route he had followed, faster. The elegance of his solution and the little time it had cost him gave me an idea of the acuity and that exquisite sense of the nature of things evidenced in all the studies he later undertook. I showed him the volume of the Academy of Berlin, which contained Euler's memoir on the subject. He saw quickly that the means he had employed were more direct than his model's; they were also to be more fruitful, and he arrived at results that had escaped Euler.[58]

Meusnier gathered his results in the one memoir that he devoted to differential geometry, (Meusnier 1785), which was presented to the Paris Academy in 1776.

famiglia di superficie sia assai più conveniente il considerarne l'equazione differenziale che l'equazione in termini finiti" (Loria 1931, p. 136)

[57]A biographical account of Meusnier is (Darboux 1912).

[58]"Le lendemain matin, dans les salles, il me remit un petit papier qui contenait cette démonstration mais ce qu'il y avait de'remarquable, c'est que les considérations qu'il avait employées étaient plus directes, et la marche qu'il avait suivie était beaucoup plus rapide, que celles dont Euler avait fait usage. L'élégance de cette solution, et le peu de temps qu'elle lui avait coûté, me donnèrent une idée de la sagacité et de ce sentiment exquis de la nature des choses dont il a donné des preuves multipliées dans tous les travaux qu'il a entrepris depuis. Je lui indiquai alors le Volume de l'Académie de Berlin dans lequel était le Mémoire d'Euler sur cet objet il reconnut bien tôt que, les moyens qu'il avait employés étant plus directs que ceux de son modèle, ils devaient être aussi plus féconds, et il parvint à des résultats qui avaient échappé à Euler." (Darboux 1912, p. 221)

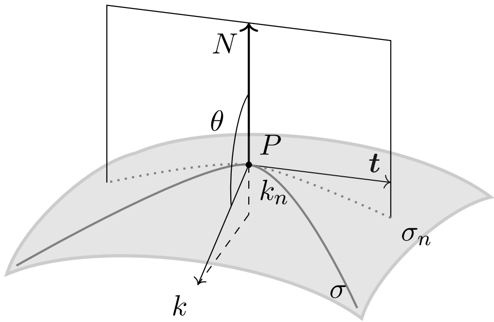

Figure 1.6: A representation of Meusnier's theorem. σ is a curve through P with $\boldsymbol{t}$ as tangent vector in P; σ_n is the normal section passing through P and having the same tangent vector $\boldsymbol{t}$ in P.

Similar to curves, for which curvature is defined in terms of the variation of the tangent line, Meusnier interpreted the curvature of a surface at one of its points P as the variation of its tangent plane from its position at P to that at a point P' infinitely close to P. Meusnier measured this variation through the infinitesimal angle ϵ that the normal directions to the surface at P and P' determine. Furthermore, just as the osculating circle best represents the curvature of a space curve at a fixed point, Meusnier noted that a surface can always be locally approximated at each of its points by two tori, which are generated by the rotation of the osculating circle of one of the principal sections of the surface around the axis of the oscillating circle of the other principal section.[59] As a consequence of this construction, he obtained Euler's results on the curvature of normal sections and proved a theorem that is known as *Meusnier's theorem*. According to it, if k_n denotes the normal curvature at a point upon a surface in a given direction and k is the curvature of a generic (not normal) section along the same direction at that same point, then $k_n = k\cos\theta$, where θ denotes the angle between the principal normal vector of the generic section and the normal to the surface (see Figure 1.6).[60]

Meusnier also improved on Euler's work by providing a first analytical characterisation of some geometric properties in terms of the principal radii of curvature, ρ_1 and ρ_2.[61] He provided noteworthy contributions, such as the analytical description of surfaces with equal principal curvatures and the characterisation of minimal surfaces as those surfaces that have zero mean curvature.[62]

He used radii of principal curvatures also to (locally) describe the concavity of a surface. If ρ_1 and ρ_2 have the same sign (and thus $\rho_1\rho_2$ is positive), then any other normal radius of curvature ρ has the same sign. In particular, no normal section has zero curvature. In this case, the surface is either concave or convex. If ρ_1 and ρ_2 have different signs (and thus $\rho_1\rho_2$ is negative), the relative sections will be one concave and the other convex and there

[59] (Meusnier 1785, p. 478)
[60] (Meusnier 1785, pp. 486–489)
[61] Meusnier denoted principal radii of curvatures with r and ρ and the radius of curvature of any normal section with R.
[62] (Meusnier 1785, pp. 500–502) and (Meusnier 1785, pp. 502–504).

are two normal sections with null curvature. In this case, the surface is concave-convex.

It is worth noting that the product $\rho_1\rho_2$ seems to be considered here for the first time. Meusnier wrote it down as[63]

$$\rho_1\rho_2 = -\frac{1}{z_{xy}^2 - z_{xx}z_{yy}} \tag{1.7}$$

where the surface is represented as $z = z(x, y)$. However, he did not place particular emphasis on this quantity, and here it is not used as a definition of the curvature of a surface.

The product of the principal radii of curvature also allowed Meusnier to deduce the differential equation of non-planar developable surfaces.[64] Following Monge, these surfaces are defined as those that are generated by the juxtaposition of infinite straight lines, which result from the movement of a line in space, in such a way that two consecutive lines are coplanar. Referring to Figure 1.7, let MN, OP and QR be three infinitely close generating

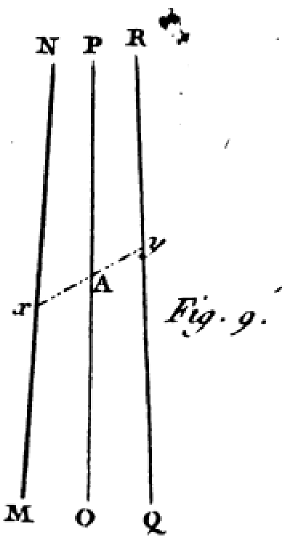

Figure 1.7: Meusnier's illustration for the demonstration of the uniqueness of the normal section having null curvature on a developable surface (Meusnier 1785).

lines of a developable surface S and A any point of S, which lies on OP. Meusnier observed that OP is the only normal section in A whose curvature is zero. Otherwise, let the straight line xAy be a second normal section through A with null curvature. Then, the two lines OP and xAy should be coplanar, causing the surface to be flat, a case excluded by Meusnier's hypotheses. The existence of a unique sectional curve with null curvature allowed Meusnier to exclude that S was concave, convex or concave-convex, and to conclude that one of the principal radii of curvature was infinite. Thus, by requiring the denominator of (1.7) to be zero, he could characterise developable surfaces as those surfaces $z = z(x, y)$ that satisfy

[63]Meusnier immediately obtained this formula by calculating the product of the two principal radii of curvature (Meusnier 1785, p. 490).

[64](Meusnier 1785, p. 509)

the following equation

$$z_{xx}z_{yy} - z_{xy}^2 = 0.$$

The idea of characterising the convexity of a surface was resumed a few years later by Charles Dupin (1784-1873). He was one of Monge's students at the École Polytechnique and his work forms a natural continuation of his master. Dupin introduced a curve now known as the *Dupin's indicatrix* to study the curvatures of the normal sections in a given point P of a surface.[65] It is a second-degree curve obtained as the limit of the intersection of a plane parallel to the tangent plane at P that is close to it and the surface itself, as the plane tends to the tangent position. Thus, the indicatrix is an ellipse for surfaces that are locally concave or convex, a hyperbola for surfaces that are locally concave-convex, and two parallel lines in the case of developable surfaces.

Starting with the indicatrix, Dupin defined other special families of curves on a surface that he called *conjugate systems* and *asymptotic curves*.[66] In (Dupin 1813), they were extensively investigated as well as curvature lines.

The investigation of particular curves on surfaces and their properties was also pursued by Olinde Rodrigues (1794-1815). He was not a direct student of Monge, but his publications in infinitesimal geometry, which are only a few clustered around 1815, were inspired by Monge's work.[67]

In the first part of (Rodrigues 1815), he dealt with the *"Analytical theory of lines of curvature and radii of curvature"*, as its title states. He characterised lines of curvature with a new system of equations, now called *Rodrigues equations*, that he derived from Monge's definition of lines of curvature in terms of developable surfaces.[68] More precisely, he exploited the fact that two consecutive normals along a line of curvature intersect at a point that is the centre of the osculating circle.

At any point $\boldsymbol{P} = (x, y, z)$ of a surface S, there is a normal unit vector $\boldsymbol{N} = (X, Y, Z)$. When one moves $\boldsymbol{P}$ to $\boldsymbol{P} + d\boldsymbol{P} = (x + dx, y + dy, z + dz)$ along a curvature line C, $\boldsymbol{N}$ changes to $\boldsymbol{N} + d\boldsymbol{N} = (X + dX, Y + dY, Z + dZ)$. Rodrigues proved that $d\boldsymbol{P}$ and $d\boldsymbol{N}$ are proportional and the constant of proportionality is exactly the length of the principal

[65] (Dupin 1813, p. 146)

[66] Any two families of curves depending on one parameter are said to form a *conjugate system* when the directions of the tangents to a curve of each family at their point of intersection have directions that coincide with the conjugate diameters of the Dupin indicatrix for that point. In a slightly modernised notation, Dupin characterised asymptotic curves as the solution of $Ddu^2 + D''dudv + D''dv^2 = 0$. In the case of hyperbolic points, i.e., where Dupin's indicatrix is a hyperbola, the asymptotic curves are tangent to the asymptotes of Dupin's indicatrix. In the case of parabolic points, i.e., where Dupin's indicatrix is a couple of parallel lines, they have as tangent vector the direction common to both lines. In the case of elliptic points, i.e., where Dupin's indicatrix is an ellipse, they are imaginary curves.

[67] Rodrigues attended courses at the Faculté des Sciences in Paris. Here, he got his doctorate, probably under the supervision of Lacroix, who was one of the Monge's first students, in 1815, when Monge was ousted from his teaching and scientific positions due to the fall of Napoleon. For a more detailed account of Rodrigues' biography, see (Altmann and Ortiz 2005).

[68] (Rodrigues 1815, pp. 162–163)

radius of curvature of C, ρ. By translating this property into coordinates, he obtained

$$dx = \rho dX \qquad dy = \rho dY \qquad dz = \rho dZ.$$

that is, if $\boldsymbol{x}(t) = (x(t), y(t), z(t))$ is a parametrization of the line of curvature and $\boldsymbol{N} = (X(t), Y(t), Z(t))$ that of the normal unit vector of the surface along it,

$$\frac{\partial x(t)}{\partial t} = \rho\frac{\partial X(t)}{\partial t} \qquad \frac{\partial y(t)}{\partial t} = \rho\frac{\partial Y(t)}{\partial t} \qquad \frac{\partial Z(t)}{\partial t} = \rho\frac{\partial Z(t)}{\partial t}. \tag{1.8}$$

Starting from these equations, he obtained a number of results and formulas that are connected with radii of principal curvatures of S, ρ_1 and ρ_2, such as an analytical expression for $\frac{1}{\rho_1} + \frac{1}{\rho_2}$ and for $\frac{1}{\rho_1}\frac{1}{\rho_2}$ that today are called mean curvature and Gaussian curvature, respectively[69]

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \qquad \frac{1}{\rho_1}\frac{1}{\rho_2} = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x},$$

in which the X, Y are still the first two components of the unit vector $\boldsymbol{N}$, but now they are functions of the variables x, y, which also parametrized S as $\boldsymbol{x}(t) = (x, y, z(x, y))$.

In the second part of (Rodrigues 1815), he investigated *"The transformation of a class of double integrals, which are directly related to the formulas of this* [curvature radii] *theory"*. The integral in question was[70]

$$\int\int \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^{3/2}} dxdy. \tag{1.9}$$

Via (1.8), Rodrigues noted that the integrand corresponds to the product $\frac{1}{\rho_1}\frac{1}{\rho_2}$ and that it can be considered as the Jacobian of the map

$$\begin{aligned} N : U \subset S &\to \mathbb{R}^3 \\ (x, y, z(x, y)) &\to (X(x, y), Y(x, y), Z(x, y)) \end{aligned} \tag{1.10}$$

that associates to each point P in a neighbourhood U of the surface a point on the unit sphere identified by the unit normal vector $\boldsymbol{N}$ at P. This allowed Rodrigues to conclude that the integral (1.9) is equivalent to the integral

$$\int\int \frac{dXdY}{\sqrt{1 - X^2 - Y^2}},$$

which expresses the area of the corresponding image via the map N.[71]

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[69] (Rodrigues 1815, pp. 167–168) They are an immediate consequence of the following quadratic relation for the principal radii of curvature derived from the Rodrigues equations

$$\left[\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}\right] R^2 - \left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right] + 1 = 0$$

[70] This integral coincide with Gauss called *curvatura integra*.

[71] (Rodrigues 1815, pp. 176–177)

Hence, Rodrigues showed that the product of the principal curvatures was equal to

$$\frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^{3/2}},$$

which was later called Gaussian curvature, and he provided a relation between an area on a surface and that of the corresponding area on a sphere via the Gauss map. All these elements can be found also in (Gauss 1828). However, while the independence of Rodrigues' and Gauss' results and the primacy of Rodrigues' discovery over Gauss does not seem fully established,[72] there is no doubt that Rodrigues' work lacks an interpretation in the frame of a general notion of curvature for surfaces.

Differential geometry in the tradition of Euler, Monge and his pupils continued to flourish in France in the nineteenth century. Elsewhere in Europe, circumstances were significantly different: apparently, nowhere else could parallel the vitality of the French *milieu* in this research field. Needless to say, a most notable exception was represented by the isolated work of Gauss and the publication of his *Disquisitiones generale circa superficies curvas*, which inaugurated a brand new phase in the development of the discipline by emphasising the importance of intrinsic properties—i.e., those properties of a surface that are independent of its immersion in Euclidean space.

[72] In this regard, see (Cogliati 2022, pp. 66–67).

Chapter 2

The establishment of the general problem of applicability

2.1 Gauss' theory for conformal maps

Gauss' main work on surface theory, *Disquisitiones generales circa superficies curvas*, (Gauss 1828), is the result of a long investigation that began as early as the end of the eighteenth century.[1] In this process, (Gauss 1825a) represents an important step. Gauss submitted the memoir on 11th December 1822 for a prize awarded by the Danish Academy that he finally won. As pointed out in the introduction, Gauss wrote this memoir quickly in order to meet the deadline for the prize and its content was only the essential part of a research he had been working on for some time and which needed to be tackled in greater depth:

> The author of this paper [...] has been induced to submit a solution found by him some considerable time since, as the lateness of the time at which he was informed of the prize question would otherwise have prevented him from sending an answer. He regrets that the latter circumstance has forced him to limit his inquiry to the essential part only, besides hinting at some obvious applications to the projection of maps and the higher branches of geodetics. Had it not been for the near approach of the term fixed by the Society, he would have followed up several inquiries, and have detailed numerous applications of the subject to geodetical operations; all which he must now reserve to himself for another moment and another place.[2]

[1] As reconstructed in detail in (Stäckel 1923) and later in (Dombrowski 1979), the introduction of the so-called Gauss map and the notion of Gaussian curvature and its expression as the product of the two principal curvatures date back to the years 1799-1813; the demonstration of invariance under isometries of the total curvature to about 1816; the deduction of the expression for Gaussian curvature in conformal coordinates and in polar geodetic coordinates to the years 1823-1825; finally, the determination of the sum of the internal angles of an infinitesimal geodetic triangle and the derivation of the expression for Gaussian curvature in arbitrary coordinates to 1825.

[2] "Der Verfasser dieser Abhandlung [...] ist dadurch aufgemuntert worden, dieser seine schon vor längerer

The topic of the prize was certainly familiar and of some interest to Gauss: he himself had proposed it to a journal as a prize question in 1816, but was rejected. It was resumed by Schumacher who proposed it to the Academy of Copenhagen for a prize that was announced initially for 1821 and then resubmitted for the following year due to lack of income.[3] The question of the prize required the mapping of two generic surfaces onto each other in such a way that the two surfaces were similar (i.e., angles and distance proportions are preserved) at an infinitesimal level. Infinitesimal level means that the similarity between surfaces exists between infinitesimal areas surrounding the corresponding points and the similarity is not necessarily uniform over the whole surface. This was a generalisation of a classic problem in cartography. It was well known that stereographic and Mercator projections responded to the posed problem when the first surface was a sphere and the second was a plane. But the Academy required the solution of the problem in the most general case.

Gauss' starting point was the possibility of parametrizing a generic surface by means of two independent variables u, v.[4] He did not provide any explanation to support the admissibility of this choice, which was officially extended to any surface after Euler had first introduced it in the case of developable surfaces:

> The nature of a curved surface is determined by an equation between the coordinates x, y, z referring to each point of it. By means of this equation, each of these three variable quantities can be regarded as a function of the other two. It is even more general to introduce two new variable quantities t, u, and to represent each of the x, y, z as a function of t and u, whereby, at least in general terms, certain values of t and u always belong to a certain point of the surface, and vice versa.[5]

Zeit gefundene Auflösung vorzulegen, wovon ihn sonst die späte von der Preisfrage erhaltene Kenntniss abgehalten haben würde. Er bedauert, dass der letztere Umstand ihn genöthigt hat, sich fast nur auf das Wesentliche und auf die Andeutung einiger näher liegenden Benutzungen für Kartenprojectionen und für die höhere Geodäsie zu beschränken, da er ohne die Nähe des Schlusstermins gern die Entwieklung einiger Nebenumstände noch weiter verfolgt, und die vielseitigen Anwendungen in der höheren Geodäsie ausführlich bearbeitet haben würde, welches er sich nun für eine andere Zeit und für einen andern Ort vorbehalten muss." (Gauss 1825a, p. 191)

[3]This reconstruction is provided by (Dombrowski 1979, p. 127). Gauss wrote to Schumacher about the prize question on 5th July 1816 (Gauss 1900, p. 370): *"I also conferred with Lindelau about a prize question that is to be published in the new journal at a price of 100 ducats. I had thought of an interesting task, namely "to project (represent) a given surface onto another (given) surface in such a way that the image resembles the original in the smallest parts". [...] this would seem to me to be a suitable prize question for a society."*

[4]To make the notation uniform, we use u, v instead of t, u. In the following, we also changed T, U in u', v' and Ω in ω'.

[5]"Die Natur einer krummen Fläche wird durch eine Gleichung zwischen den sich auf jeden Punkt derselben beziehenden Coordinaten x, y, z bestimmt. Vermöge dieser Gleichung kann jede dieser drei veränderlichen Grössen wie eine Function der beiden andern betrachtet werden. Noch allgemeiner ist es, noch zwei neue veränderliche Grössen t, u einzuführen, und jede der x, y, z als eine Function von t und u darzustellen, wodurch, wenigstens allgemein zu reden, bestimmte Werthe von t und u allemal einem bestimmten Punkte der Oberfläche angehören, und umgekehrt."(Gauss 1825a, p. 193)

Gauss translated the act of mapping or projecting (*abbilden* (Gauss 1825a, §3)) one surface upon another in mathematical terms as the establishment of a law that univocally associates each point on a surface with a point on the second one. His initial choice of two distinct families of parameters, u, v and u', v', upon each of the two surfaces facilitated the analytical description of this law:[6] if a surface $S : (u, v) \mapsto \boldsymbol{x}(u, v)$, is mapped onto a surface $S' : (u', v') \mapsto \boldsymbol{x'}(u', v')$, then there should exist a transformation of the parameters $u' = \varphi(u, v)$, $v' = \psi(u, v)$.

To ensure that this transformation was conformal, in (Gauss 1825a, §4) Gauss first required that the distance between any two infinitely close points on the first surface S, which he called *line element* and denoted with ω, was proportional to the infinitesimal distance between two corresponding point on S', ω'. Again without explanation, he stated that the line element of the first surface S, which is parametrized by means of $\boldsymbol{x} : (u, v) \to \boldsymbol{x} = (x(u, v), y(u, v), z(u, v))$, is[7]

$$\omega = \sqrt{\boldsymbol{x}_u \cdot \boldsymbol{x}_u du^2 + 2\boldsymbol{x}_u \cdot \boldsymbol{x}_v du dv + \boldsymbol{x}_v \cdot \boldsymbol{x}_v dv^2}, \tag{2.1}$$

and that of the second surface S', which is now assumed to be parametrized by the same parameters u, v of S as $\boldsymbol{x'} : (u, v) \to \boldsymbol{x'} = (x'(u, v), y'(u, v), z'(u, v))$, is

$$\omega' = \sqrt{\boldsymbol{x'}_u \cdot \boldsymbol{x'}_u du^2 + 2\boldsymbol{x'}_u \cdot \boldsymbol{x'}_v du dv + \boldsymbol{x'}_v \cdot \boldsymbol{x'}_v dv^2}. \tag{2.2}$$

Gauss concluded that ω is proportional to ω' when the coefficients of du^2, $dudv$ and dv^2 in (2.1) and (2.2) are proportional, that is

$$\boldsymbol{x}_u \cdot \boldsymbol{x}_u = \lambda \boldsymbol{x'}_u \cdot \boldsymbol{x'}_u \qquad \boldsymbol{x}_u \cdot \boldsymbol{x}_v = \lambda \boldsymbol{x'}_u \cdot \boldsymbol{x'}_v \qquad \boldsymbol{x}_v \cdot \boldsymbol{x}_v = \lambda \boldsymbol{x'}_v \cdot \boldsymbol{x'}_v \tag{2.3}$$

where λ is a function depending on the parameters u, v.[8] The fact that λ is not necessarily constant implies that the similarity ratio from one surface to the other may vary from point to point and may therefore not be uniform over the whole surface.

From (2.3), Gauss also deduced the angles invariance. In slightly modern terms, let $\boldsymbol{\gamma}_s : s \mapsto \boldsymbol{\gamma}_s(s) = (\gamma_{s1}(s), \gamma_{s2}(s), \gamma_{s3}(s))$ and $\boldsymbol{\gamma}_t : t \mapsto \boldsymbol{\gamma}_t(t) = (\gamma_{t1}(s), \gamma_{t2}(s), \gamma_{t3}(s))$ be any two curves upon S that intersect at a point P forming an angle θ. If $\boldsymbol{\gamma}_s$ and $\boldsymbol{\gamma}_t$ are chosen as coordinate lines in a neighbourhood U of S containing P, the parametric equations of S change in $\tilde{\boldsymbol{x}} = (\tilde{x}(s, t), \tilde{y}(s, t), \tilde{z}(s, t))$ and the angle θ they form at P is[9]

$$\cos\theta = \frac{\tilde{\boldsymbol{x}}_s \cdot \tilde{\boldsymbol{x}}_t}{\|\tilde{\boldsymbol{x}}_s\| \|\tilde{\boldsymbol{x}}_t\|}.$$

[6]For example, (Euler 1862) lacks an interpretation of the problem of isometric mapping of surfaces in terms of a parameter transformation.

[7]Gauss did not use vector notation and denoted $x_u, y_u, z_u, x_v, y_v, z_v$ with a, b, c, a', b', c' and $x'_u, y'_u, z'_u, x'_v, y'_v, z'_v$ with A, B, C, A', B', C'.

[8]Gauss used m instead of λ.

[9]Given two non-zero vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ the cosine of the enclosed angle θ is given by $\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}$. Gauss did not consider curves upon surfaces, but he used two different increments d and δ and wrote $\cos\theta$ as

$$\frac{(x_u du + x_v dv)(x_u \delta u + x_v \delta v) + (y_u du + y_v dv)(y_u \delta u + y_v \delta v) + (z_u du + z_v dv)(z_u \delta u + z_v \delta v)}{\sqrt{(x_u du + x_v dv)^2 + (y_u du + y_v dv)^2 + (z_u du + z_v dv)^2}\sqrt{(x_u \delta u + x_v \delta v)^2 + (y_u \delta u + y_v \delta v)^2 + (z_u \delta u + z_v \delta v)^2}}.$$

This expression and the corresponding one for S' become *"clearly equal"* if (2.3) hold true.

Analogously, if the curves that correspond to $\boldsymbol{\gamma}_s(s)$ and $\boldsymbol{\gamma}_t(t)$ through the transformation are chosen as coordinate lines upon S', the parametric equations of S' change in $\tilde{\boldsymbol{x}}' = (\tilde{x}'(s,t), \tilde{y}'(s,t), \tilde{z}'(s,t))$ and the angle θ' they form at P' is

$$\cos\theta' = \frac{\tilde{\boldsymbol{x}}'_s \cdot \tilde{\boldsymbol{x}}'_t}{\|\tilde{\boldsymbol{x}}'_s\|\|\tilde{\boldsymbol{x}}'_t\|}.$$

The result is $\theta = \theta'$ since (2.1), (2.2) and (2.3) hold true, save for changing u in s, v in t, $\boldsymbol{x}$ in $\tilde{\boldsymbol{x}}$ and $\boldsymbol{x}'$ in $\tilde{\boldsymbol{x}}'$.

Gauss thus demonstrated that requiring the punctual proportionality of the line elements of two surfaces is sufficient to guarantee that the correspondence is conformal. He also remarked:

> in the special case where m is constant, a perfect similarity will also be found in the finite parts, and if, moreover, $m = 1$, a perfect equality will take place, and the one surface can be unfolded [*abwickeln*] on the other.[10]

Then, Gauss went deeper into an analytical characterisation of the proportionality of the two line elements, $\omega = \lambda\omega'$. In (Gauss 1825a, §5), he proved that every surface S admits an isothermal coordinate system; i.e., there always exists a couple of parameters p, q such that the line element of S becomes $\omega = n(p,q)(dp^2 + dq^2)$. This implies, in particular, that every surface is locally conformal to the plane. The demonstration is based on the possibility of decomposing the trinomial $\omega = \sqrt{\boldsymbol{x}_u \cdot \boldsymbol{x}_u du^2 + 2\boldsymbol{x}_u \cdot \boldsymbol{x}_v du dv + \boldsymbol{x}_v \cdot \boldsymbol{x}_v dv^2}$ as the product of two factors that are linear with respect to du and dv, which are[11]

$$\begin{aligned} &\boldsymbol{x}_u \cdot \boldsymbol{x}_u du + \left[\boldsymbol{x}_u \cdot \boldsymbol{x}_v + i\sqrt{(\boldsymbol{x}_u \cdot \boldsymbol{x}_u)(\boldsymbol{x}_v \cdot \boldsymbol{x}_v) - (\boldsymbol{x}_u \cdot \boldsymbol{x}_v)^2}\right] dv \\ &\boldsymbol{x}_u \cdot \boldsymbol{x}_u du + \left[\boldsymbol{x}_u \cdot \boldsymbol{x}_v - i\sqrt{(\boldsymbol{x}_u \cdot \boldsymbol{x}_u)(\boldsymbol{x}_v \cdot \boldsymbol{x}_v) - (\boldsymbol{x}_u \cdot \boldsymbol{x}_v)^2}\right] dv \end{aligned} \tag{2.4}$$

where i is the imaginary unit. The integration of $\omega = 0$ can be thus decomposed into the integration of

$$\boldsymbol{x}_u \cdot \boldsymbol{x}_u du + \left[\boldsymbol{x}_u \cdot \boldsymbol{x}_v + i\sqrt{(\boldsymbol{x}_u \cdot \boldsymbol{x}_u)(\boldsymbol{x}_v \cdot \boldsymbol{x}_v) - (\boldsymbol{x}_u \cdot \boldsymbol{x}_v)^2}\right] dv \tag{2.5}$$

$$\boldsymbol{x}_u \cdot \boldsymbol{x}_u du + \left[\boldsymbol{x}_u \cdot \boldsymbol{x}_v - i\sqrt{(\boldsymbol{x}_u \cdot \boldsymbol{x}_u)(\boldsymbol{x}_v \cdot \boldsymbol{x}_v) - (\boldsymbol{x}_u \cdot \boldsymbol{x}_v)^2}\right] dv. \tag{2.6}$$

If the integral of (2.5) is $p + iq = const$, where p and q denote real functions of u and v, the integral of (2.6) is $p - iq = const$. In this way, $(dp + idq)(dp - idq)$ or, equivalently, $dp^2 + dq^2$, must be a factor of ω, that is $\omega = n(dp^2 + dq^2)$, where n is a function of u and v.

When p', q' denote the isothermal parameters of S', for which the line element assumes the form $\omega' = n'(p,q)(dp^2 + dq^2)$ Gauss stated that a necessary and sufficient condition for

[10] "In dem speciellen Falle, wo m constant ist, wird eine vollkommene Aehnlichkeit auch in den endlichen Theilen, und wenn überdies m = 1 ist, wird eine vollkommene Gleichheit Statt finden, und die eine Fläche sich auf die andere abwickeln lassen." (Gauss 1825a, §4)

[11] The product of the factors in (2.4) does not give exactly ω, but $\boldsymbol{x}_u \cdot \boldsymbol{x}_u \omega$. However, since $\omega = 0$ holds true, this fact is irrelevant.

the transformation of S into S' to be conformal is that the following conditions are verified

$$p' + iq' = F(p + iq) \qquad p' - iq' = F(p - iq),$$

where F denotes any complex function. Later, these conditions were written in an equivalent form by using a pair of differential equations, today known as the Cauchy-Riemann equations[12]

$$\frac{\partial u'}{\partial u} = \pm\frac{\partial v'}{\partial v} \qquad \frac{\partial u'}{\partial v} = \mp\frac{\partial v'}{\partial u}.$$

Finally, in the second part (Gauss 1825a, §8-13), Gauss gave examples of conformal mappings between different surfaces and the plane but also between an ellipsoid and a sphere.

2.2 Gauss' *Disquisitiones generales circa superficies curvas*

In 1827, Gauss presented to the Royal Society of Sciences of Göttingen a memoir entitled *Disquisitions generales circa superficies curvas*, which was published the following year. In spite of its brevity, only about 40 pages long, it is a complete presentation of Gauss' geometric ideas, through which he intended to renovate research in the field of surface theory from a methodological point of view:

> Although geometers have given much attention to general investigations of curved surfaces and their results cover a significant portion of the domain of higher geometry, this subject is still so far from being exhausted, that it can well be said that, up to this time, but a small portion of an exceedingly fruitful field has been cultivated. Through the solution of the problem, to find all representations of a given surface upon another in which the smallest elements remain unchanged [i.e., (Gauss 1825a)], the author sought some years ago to give a new phase to this study. The purpose of the present discussion is further to open up other new points of view and to develop some of the new truths which thus become accessible.[13]

In order to provide a good definition of curvature, Gauss borrowed the notion of spherical mapping from astronomy and established a point-to-point correspondence between a curved surface and a sphere of unit radius. With respect to Figure 2.1, every point P on a

[12]See, for example, (Bianchi 1922, §54).

[13]"Obgleich die Geometer sich viel mit allgemeinen Untersuchungen über die krummen Flächen beschäftigt haben, und ihre Resultate einen bedeutenden Theil des Gebiets der höhern Geometrie ausmachen, so ist doch dieser Gegenstand noch so weit davon entfernt, erschöpft zu sein, dass man vielmehr behaupten kann, es sei bisher nur erst ein kleiner Theil eines höchst fruchtbaren Feldes angebauet. Der Verf. hat schon vor einigen Jahren durch die Auflösung der Aufgabe, alle Darstellungen einer gegebenen Fläche auf einer andern zu finden, bei welchen die kleinsten Theile ähnlich bleiben, dieser Lehre eine neue Seite abzugewinnen gesucht: der Zweck der gegenwärtigen Abhandlung ist, abermals andere neue Gesichtspunkte zu eröffnen, und einen Theil der neuen Wahrheiten, die dadurch zugänglich werden, zu entwickeln." (Gauss 1827, p. 341)

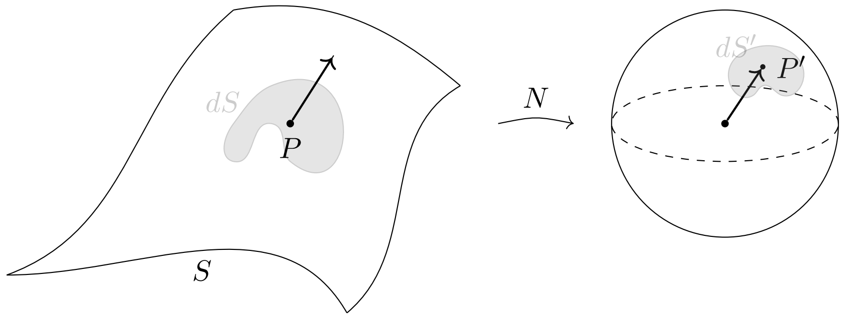

Figure 2.1: Representation of Gauss' map N of a surface S. dS' is the *curvatura integra* related to dS.

given surface S is associated with a point P' on a unit sphere that is the extremity of the radius drawn parallel to the assumed positive direction of the normal to S at P. Thus, a map $N : S \to S^2$, now called *Gauss map*, is defined at least locally. Gauss commented: *"this procedure agrees fundamentally with that which is constantly employed in astronomy, where all directions are referred to a fictitious celestial sphere of infinite radius"*.[14]

Gauss used this map to evaluate the variation of the normal vector around a point P, which, in his view, describes the curvature of S. First, he defined the *curvatura integra* of a small region dS on S containing P as the area dS' of the image through N of dS. The fact that the curvature at a point P of a surface is related to the area of the image through the Gauss map of a neighbourhood of P is clearly explained by Gauss himself, who remarked that the less dS differs from a plane, the smaller is the corresponding part on the sphere.[15] Then, he defined what today we call *Gaussian curvature* at a point P in dS as the limit of the ratio between the curvatura integra dS' and the area of dS when the diameter ϵ of dS containing P tends to zero, namely

$$|k(P)| = \lim_{\epsilon \to 0} \frac{dS'}{dS}. \tag{2.7}$$

The sign of $k(P)$ is determined according to the sign of the principal curvatures at the point. This definition of curvature was already mentioned in a Gauss' manuscript, (Gauss 1825b, §2), which was published in 1900. Here, the definition (2.7) was proposed together with a similar definition of curvature for curves, which Euler had already provided in (Euler 1786, §7). Euler imagined that a unit sphere was defined at every point on a space curve C and at each point he considered the radius that is parallel to the positive direction of the tangent vector to the curve; he thus defined the radius of curvature of C at a point P as (the limit of) the ratio between the line element of the curve and the measure of the arc that the tangent directions to the curve identify on the unit sphere.

While the concept of curvature was extensively discussed by Gauss, there is no precise definition of surface. Gauss restricted his investigations to regular surfaces, or portions of surfaces that exclude singular points where the tangent plane is not well defined (such as

[14](Gauss 1827, p. 342)
[15](Gauss 1827, p. 342)

the vertex of a cone). He represented a surface S in three different ways: as the locus of the zeros of a certain function $F(x, y, z)$, as the graph of a two-variable function $z = z(x, y)$ or, finally, as the set of three functions x, y, z depending on two parameters p, q.

By exploiting the Gauss map and the representation of S as the graph of a function $z = z(x, y)$, Gauss derived the following expression for the Gaussian curvature (2.7)

$$k(P) = \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2}. \tag{2.8}$$

He also established an important link between his definition of curvature (2.7) and Euler's results by showing that at every point P on S one has

$$k(P) = \frac{1}{\rho_1(P)\rho_2(P)},$$

where ρ_1 and ρ_2 denote the principal radii of curvature.

Gauss had already found an analogous expression for the Gaussian curvature in (Gauss 1822, p. 381), a manuscript written two days after (Gauss 1825a) was sent to the Copenhagen Academy. Gauss supposed that the surface S was parametrized by means of isothermal coordinates, i.e., S has $ds^2 = \lambda(u, v)(du^2 + dv^2)$ as line element, and he showed that

$$k(P) = -\frac{1}{\lambda^2}\left(\frac{\partial^2 \log \lambda}{\partial u^2} + \frac{\partial^2 \log \lambda}{\partial v^2}\right).$$

In that circumstance, Gauss also pointed out that $k(P)$ depended only on $\lambda(u, v)$ and its derivatives and, in addition, he deduced the invariance of the Gaussian curvature under isometries, since an isometry does not change an isothermal coordinate system. This is essentially the content of the so-called *theorema egregium* (hereafter TE), which he had known since 1816. Gauss proved TE in various ways that culminated in the analytical demonstration valid for generic coordinates contained in the *Disquisitiones*. He considered a surface $S : (u, v) \mapsto \boldsymbol{x} : U \to \mathbb{R}^3$, $\boldsymbol{x} = (x(u, v), y(u, v), z(u, v))$ and he wrote his line element in the form that is now classic[16]

$$ds^2 = E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2.$$

Gauss arrived at a rather complicated formula for the Gaussian curvature k that involves

[16] In (Gauss 1828, §17), Gauss also paused on the geometrical interpretation of the coefficient E, F, G:

> Any point whatever on the surface can be regarded as the intersection of a line of the first system with a line of the second [the coordinate lines]; and then the element of the first line adjacent to this point and corresponding to a variation dp will be equal to $\sqrt{E}dp$, and the element of the second line corresponding to the variation dq will be equal to $\sqrt{G}dq$. Finally, denoting by ω the angle between these elements, it is easily seen that we shall have $\cos\omega = \frac{F}{\sqrt{EG}}$.

as many as 15 quantities, all of which are related to the coefficients of the line element

$$\begin{aligned}4(EG-F^2)k(u,v) = E\left(\frac{\partial E}{\partial v}\frac{\partial G}{\partial v} - 2\frac{\partial F}{\partial u}\frac{\partial G}{\partial v} + \left(\frac{\partial G}{\partial u}\right)^2\right)\\ +F\left(\frac{\partial E}{\partial u}\frac{\partial G}{\partial v} - \frac{\partial E}{\partial v}\frac{\partial G}{\partial u} - 2\frac{\partial E}{\partial v}\frac{\partial F}{\partial v} + 4\frac{\partial F}{\partial u}\frac{\partial F}{\partial v} - 2\frac{\partial F}{\partial p}\frac{\partial G}{\partial p}\right) \qquad (2.9)\\ +G\left(\frac{\partial E}{\partial u}\frac{\partial G}{\partial u} - 2\frac{\partial E}{\partial u}\frac{\partial F}{\partial v} + \left(\frac{\partial E}{\partial v}\right)^2\right) - 2(EG-F^2)\left(\frac{\partial^2 E}{\partial v^2} - 2\frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 G}{\partial u^2}\right).\end{aligned}$$

When written in these terms, the Gaussian curvature $k(P)$ is independent of the functions $x(u,v), y(u,v), z(u,v)$ that determine the immersion of the surface in Euclidean space and hence of the particular parametrization that is chosen. The Gaussian curvature depends only on the coefficients of the line element and their derivatives:

> The analysis developed in the preceding article thus shows us that for finding the measure of curvature there is no need of finite formulae, which express the coordinates x, y, z as functions of the indeterminates p, q $[u, v]$; but that the general expression for the magnitude of any line element is sufficient.[17]

It must be emphasised that Gauss always considered surfaces as embedded in Euclidean space, although here he seems to set the finite equations aside. The three dimensional space is necessary for him to provide the line element of a surface, which is not considered abstractly as a generic positive quadratic differential form but as defined by $ds^2 = \boldsymbol{x}_u \cdot \boldsymbol{x}_u du^2 + 2\boldsymbol{x}_u \cdot \boldsymbol{x}_v dudv + \boldsymbol{x}_v \cdot \boldsymbol{x}_v dv^2$.

Then, he turned to the problem of surfaces that admit an equal line element and defined the problem of applicability for any surface. As already pointed out in (Gauss 1825a), a surface S, which is parametrized by means of $\boldsymbol{x} : U \to \mathbb{R}^3$, $\boldsymbol{x} = (x(u,v), y(u,v), z(u,v))$ and whose line element is $ds^2 = E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2$, is applicable[18]over a surface S', which is parametrized by means of $\boldsymbol{x'} : U \to \mathbb{R}^3$, $\boldsymbol{x'} = (x'(u,v), y'(u,v), z'(u,v))$ and whose line element is $ds^2 = E'(u,v)du^2 + 2F'(u,v)dudv + G'(u,v)dv^2$, if

$$E(u,v) = E'(u,v) \qquad F(u,v) = F'(u,v) \qquad G(u,v) = G'(u,v).$$

The fact that, unless there is a change of parameters, the coefficients of the line element must be equal for applicable surfaces allowed Gauss to explicit the geometrical meaning of equation (2.9), and to deduce the TE:

> If a curved surface, or a part of it, can be developed upon another surface, the measure of curvature at every point remains unchanged.[19]

[17]"Docet itaque analysis in art. preec. explicata, ad inveniendam mensuram curvaturae haud opus esse formulis finitis, quae coordinatas x, y, z tamquam functiones indeterminatarum p, q exhibeant, sed sufficere expressionem generalem pro magnitudine cuiusvis elementi linearis." (Gauss 1828, §12)

[18]Gauss used the words *explicari posse*, (Gauss 1828, §12).

[19]"Wenn eine krumme Fläche, oder ein Stück derselben auf eine andere Fläche abgewickelt werden kann, so bleibt nach der Abwickelung das Krümmungsmaass in jedem Punkt ungeändert".(Gauss 1827, p. 344)

Starting from the TE, Gauss derived at once the characteristic equation of developable surfaces. He did not refer to their geometric construction as particular ruled surfaces, as Monge and Meusnier did before him, but he used their definition as the class of surfaces that are applicable to the plane. Indeed, since the Gaussian curvature is invariant by isometry and a plane has null Gaussian curvature, then it must also be null for all the developable surfaces. In virtue of the equation (2.8), the equation $z = z(x, y)$ of a developable surface must satisfy the following equation

$$z_{xx}z_{yy} - z_{xy}^2.$$

These important results led Gauss to the following remark that summarises his programme for a "new and fertile" direction in surface theory:

> What we have explained in the preceding article is connected with a particular method of studying surfaces, a very worthy method which may be thoroughly developed by geometers. When a surface is regarded, not as the boundary of a solid, but as a flexible, though not extensible solid, one dimension of which is supposed to vanish, then the properties of the surface depend in part upon the form to which we can suppose it reduced, and in part are absolute and remain invariable, whatever may be the form into which the surface is bent. To these latter properties, the study of which opens to geometry a new and fertile field, belong the measure of curvature and the integral curvature, in the sense which we have given to these expressions. To these belong also the theory of shortest lines, and a great part of what we reserve to be treated later. From this point of view, a plane surface and a surface developable on a plane, e.g., cylindrical surfaces, conical surfaces, etc., are to be regarded as essentially identical; and the generic method of defining in a general manner the nature of the surfaces thus considered is always based upon the formula $\sqrt{Edp^2 + 2Fdpdq + Gdq^2}$, which connects the linear element with the two indeterminates p, q.[20]

Only here, Monge's intuition of a surface thought as a thin veil is generalised to *any* surface. A surface was thus regarded as deformable and surfaces that were obtained from each other by deformation were considered "essentially identical". Hence, a deformable surface is a class of equivalence of all locally isometric surfaces. This led Gauss to emphasise those properties

[20] "Quae in art. praec. exposuimus, cohaerent cum modo peculiari superficies considerandi, summopere digno, qui a geometris diligenter excolatur. Scilicet quatenus superficies consideratur non tamquam limes solidi, sed tamquam solidum, cuius dimensio una pro evanescente habetur, flexile quidem, sed non extensibile, qualitates superficiei partim a forma pendent, in quam illa reducta concipitur, partim absolutae sunt, atque invariatae manent, in quamcunque formam illa flectatur. Ad has posteriores, quarum investigatio campum geometriae novum fertilemque aperit, referendae sunt mensura curvaturae atque curvatura integra o sensu, quo hae expresiones a nobis accipiuntur; porro huc pertinet doctrina de lineis brevissimis, pluraque alia, de quibus in posterum agere nobis reservamus. In hoc considerationis modo superficies plana atque superfcies in planum explicabilis, e. g. cylindrica, conica etc. tamquam essentialiter identicae spectantur, modusque genuinus indolem superfciei ita consideretae generaliter exprimendi semper innititur formulae $\sqrt{Edp^2 + 2Fdpdq + Gdq^2}$, quae nexum elementi cum duabus indeterminatis p, q sistit."(Gauss 1828, §13)

of a surface that do not depend on the immersion of the surface in Euclidean space and that remain constant within an equivalence class (today they are called *intrinsic*). Whereas, for example, the principal curvatures of a surface at a point depend on the immersion of the surface, their product, that is the Gaussian curvature of a surface, depends only on the line element according to TE.

This shows how the problem of applicability acquired a dominant role in investigations of surface theory after the publication of (Gauss 1828). However, an important terminological aspect needs to be clarified. In order to translate the problem of applicability—i.e., the possibility of deforming one surface until it coincides with another one—into mathematical terms, Gauss interpreted it as a problem of isometry, i.e., by requiring the existence of a bijective correspondence between the points of the surfaces S and S' that maps the first fundamental form of S onto the first fundamental form of S'. During the nineteenth century, these two notions (applicability and local isometry) were frequently used in a loose and misleading sense, which led to continuously identify them. However, it should be borne in mind that these two points of view are not equivalent. The notion of applicability clearly implies the existence of an isometry, but the assignment of an isometry is not sufficient to guarantee the existence of a continuous motion that leaves the first fundamental form unchanged at every instant.[21] To adhere to nineteenth century practice, we will disregard any difference between the requirement of applicability and that of (local) isometry. Thus, the term "applicable" will be regarded as equivalent to "locally isometric".

In subsequent years, two main problems related to applicability emerged: the first concerned the study of applicability between two given surfaces, while the second concerned the determination of all the surfaces applicable to a given one. Bianchi in (Bianchi 1922, pp. 326, 355) named them the *first* and *second problem of applicability*, respectively. Voss, meanwhile, coined the expressions *Minding's problem* and *Bour's problem* respectively in (Voss 1903, pp. 389, 395).

2.3 Reception of Gauss' ideas

The issue of the reception of Gauss' *Disquisitiones* is a widely debated topic and has been dealt with extensively in the secondary literature.[22] Over a period of 30 years, the systematic introduction of both line elements and intrinsic properties led to a significant expansion of surface theory. However, this did not happen immediately after the publication of (Gauss 1828). Despite the programmatic nature of Gauss' work and the numerous inputs that suggested the possibility of further investigations, neither Gauss[23] nor most of his colleagues

[21] See Appendix A for more details.

[22] In addition to (Reich 1973) that offers a general overview of the development of surface theory, the interested reader may consult (Reich 1996) and (Reich 2010) for the reception of Gauss' work in France and Russia respectively and (Cogliati 2022, Chap. 4) for an investigation of the reception of the *Disquisitiones* in France, Italy and Germany.

[23] (Gauss 1828) were Gauss' last work on differential geometry. After them, he turned his research mainly to geodesic and physic problems. As reported in (Reich 2006, p. 84), Gauss taught surface theory in the

immediately followed his route. Until the late 1840s, when the French school finally made (Gauss 1828) the focus of common interest, only a few—albeit important—results in the direction indicated by Gauss had been achieved in Germany.

2.3.1 Germany: Minding's first developments of problems related to applicability

Carl Jacobi (1804-1851) and Ferdinand Minding (1806-1885) were the first to explore the routes indicated by Gauss' *Disquisitiones.* They dealt with Gaussian surface theory at almost the same period: Jacobi from 1836 to 1843, and Minding between 1838 and 1840, after initial work in 1830. Despite this, as far as we know, there was no connection between the two nor does any evidence of scientific collaboration on surface theory between them and Gauss seem to exist.

While Jacobi was mainly interested in investigating the geometric basis of the TE,[24] Minding[25] dealt with problems related to applicability and intrinsic properties. His short, but dense contributions were so relevant that, according to Kneser, *"Minding may be described as the first successor of Gauss, who, though moving along the master's path, has gone in essential points beyond Gauss"*.[26]

The first "essential point" that Minding tackled was the issue of intrinsic properties. In (Minding 1830a), he showed that a quantity, later called *geodesic curvature* by Liouville, was a bending invariant. Minding had introduced it in the previous paper (Minding 1830b) as an auxiliary tool to find the shortest curve joining two points A and B that encloses a portion of the surface with a fixed area together with a given curve between the same points A and B. In (Minding 1830a), geodesic curvature became the subject of Minding's investigation, which was clearly inspired by Gauss:

> [geodesic curvature] can be expressed in general terms in a similar way as the measure of *(mensura curvaturae)* is presented in Gauss' *Disquisitiones generales circa superficies curvas.*[27]

years 1827-1832, but, unlike Monge, he seemed not to be interested in creating a scientific community to continue his research.

[24]In (Jacobi 1842), Jacobi generalised Gauss' theorem on the angular defect of geodesic triangles on a surface and deduced an interesting corollary to the effect that the spherical image of the principal normals of a closed space curve divides the surface of the sphere into two equal parts. In this regard, see (McCleary 1994). For an overview on the prehistory of Gauss' theorem, the interested reader can see (Brummelen 2009). Besides this collection of papers, it should be noted that Jacobi regularly taught curve and surface theory at Königsberg, whose manuscripts *"show a vivid interest in the production of the French school and Gauss, and they present, with Jacobi's own work, much of this material in an original way"* (Struik 1933b, p. 166).

[25]For a biographic and scientific account of Minding, as well as a list of his publications, see (Kneser 1900).

[26](Kneser 1900, pp. 113–114)

[27]"Krümmungsmass lässt sich auf ähnliche Weise allgemein ausdrücken, wie das Maass der (mensura eurvaturae) in den *Disquisitiones generales circa superficies curvas* von Gauss dargestellt wird." (Minding 1830a, p. 159)

With reference to Figure 2.2 and using vector notation that is extraneous to Minding's treatment, Minding considered a curve σ on a surface S with line element $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. Let $\boldsymbol{k}$ be the curvature vector of σ at any of its point P, whose magnitude is k. Geodesic curvature of σ, which we denote with k_g, is equal to the magnitude of the curvature vector of the plane curve σ' that is obtained by (orthogonally) projecting σ onto the tangent plane to the surface at P. By referring to Meusnier's theorem[28] and denoting the angle between $\boldsymbol{k}$ and the tangent plane with ω, geodesic curvature is defined as $k_g = k \cos\omega$. That is to say, k_g is the magnitude of the orthogonal projection of $\boldsymbol{k}$ onto the tangent plane to the surface at P.[29] Minding showed that this quantity, which apparently depends on the embedding of S into Euclidean space, is actually only dependent on the coefficients E, F, G and their derivatives and is therefore an intrinsic property.

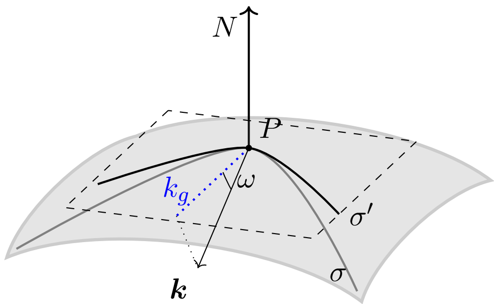

Figure 2.2: A representation of geodesic curvature k_g. σ is a generic curve on S passing through P and σ' is its orthogonal projection onto the tangent plane of S at P. k_g is the magnitude of the orthogonal projection of the curvature $\boldsymbol{k}$ of σ upon the tangent plane.

However, Minding's name is mainly associated with problems related to applicability. Within the space of four years, between 1837 and 1840, five closely interconnected articles with a total of just 35 pages appeared in one of the most influential mathematical journals, the *Journal für die reine und angewandte Mathematik*. The first is a short memoir, (Minding 1837), in which Minding connected his previous results on geodesic curvature with the theory of developable surfaces. If one considers a curve C drawn on a developable surface and the curve C', which results by the development of the surface on a plane, then, Minding proved, the ratio between their radii of curvature at corresponding points is equal to the cosine of the angle that the osculating plane of C forms with the plane tangent to the surface.

In the second paper of the series (Minding 1838a), Minding turned to ruled surfaces and successfully addressed the problem of determining by quadratures all ruled surfaces with a given line element, including the special case of developable surfaces. He provided

[28]It is sufficient to note that σ' is a normal section of the cylinder that is generated by the normal lines along σ and that σ is another curve on the cylinder, which is tangent to σ' at P. Then, Meusnier's theorem states that $k_g = k\cos\omega$.

[29]For the sake of completeness, by substituting ω with its complementary angle θ (i.e., the angle that the normal to the surface makes with the positive direction of the principal normal to σ), one has $k_g = k\sin\theta$, and thus $k^2 = k_g^2 + k_n^2$.

an intuitive description of deformation in terms of movements along the generators, which is somewhat similar to Euler's description of folding a sheet of papers in (Euler 1772a):

> Up to now, flexibility has been demonstrated only in the case of developable surfaces; here it will be shown that it takes place in the case of all surfaces that are generated by the movement of a straight line; this can be easily verified in the following way. Let $a, a', a'', ...$ be the straight lines infinitely close to one another, which form a piece (s) of such a surface, then, leaving the first strip between a and a' unmoved, one can rotate the remaining part of s about a', as a fixed axis, of an infinitesimal amount; then, leaving the strips aa' and $a'a''$ unmoved, rotate the remaining part of s about a'', etc., whereby the surface is bent without changing the length of any line on it. This bending, as can be seen, occurs in quite the same way as with developable surfaces; the previous restriction to these is therefore quite insignificant.[30]

The possibility of interpreting the deformation of ruled surfaces in these terms led Minding to consider the distinction between developable and ruled surfaces as not so significant and allowed him to generalise the treatment of developable surfaces to ruled surfaces. As Euler wrote the parametric equations of all the developable surfaces starting from a curve on the surface that is *tangent* to all the generators (i.e., the edge of regression), in the same way Minding parametrized a ruled surface starting from a generic curve C on the surface that *intersects* all the generators (its *directrix*).

Minding chose the parameter v of a ruled surface $S : (u, v) \mapsto \boldsymbol{x}(u, v)$ as the parameter of the curve $C : v \mapsto \tilde{\boldsymbol{x}}(v) = (\tilde{x}(v), \tilde{y}(v), \tilde{z}(v))$. Every generator passes through a point of C and is parametrized by means of u that represents the distance (measured along the generator) of any of its points from the point where it meets C. Hence, the position of any point $\boldsymbol{x}(u, v)$ on S is uniquely determined through the following parametric equations

$$\boldsymbol{x}(u, v) = \tilde{\boldsymbol{x}}(v) + \boldsymbol{w}(v)u$$

where $\boldsymbol{w}(v) = (w_1(v), w_2(v), w_3(v))$ is a unit vector that identifies the direction of the generator passing through $\tilde{\boldsymbol{x}}(v)$.[31] In order to obtain the corresponding line element, it is

[30] "Bisher ist die Biegsamkeit nur bei den abwickelbaren Flächen nachgewiesen worden; hier soll gezeigt werden, dass sie Statt findet bei allen Flächen, welche durch Bewegung einer geraden Linie entstehen; wovon man sich leicht auf folgende Weise überzeugt. Es seien $a, a', a'', ...$ die unendlich nahe auf einander folgenden Geraden, welche ein Stück (s) einer solchen Fläche bilden, so kann man, den ersten Streifen zwischen a und a' unbewegt lassend, den übrigen Theil von s um a', als feste Axe, unendlich wenig drehen; hierauf die Streifen aa' und $a'a''$ unbewegt lassend, den noch übrigen Theil von s um a'' drehen u. s. f., wodurch die Fläche gebogen wird, ohne die Länge irgend einer auf ihr befindlichen Linie zu ändern. Diese Biegung geschieht, wie man sieht, ganz auf dieselbe Weise, wie bei abwickelbaren Flächen; die bisherige Beschränkung auf diese ist mithin durchaus unwesentlich." (Minding 1838a, p. 297)

[31] Minding, who did not use vector notation, chose the parameters p, q instead of u, v and indicated the components of $\tilde{\boldsymbol{x}}(p) = (\tilde{x}(v), \tilde{y}(v), \tilde{z}(v))$ with $u(p), v(p), w(p)$ and those of $\boldsymbol{w}(v) = (w_1(v), w_2(v), w_3(v))$ with u', v', w'.

sufficient to consider the following derivatives[32]

$$\boldsymbol{x}_u(u,v)=\boldsymbol{w}(v) \qquad \boldsymbol{x}_v(u,v)=\tilde{\boldsymbol{x}}_v+\boldsymbol{w}_v(v)u$$

which give

$$\begin{aligned} E &= \|\boldsymbol{w}(v)\|^2 = 1 \\ F &= \boldsymbol{w}(v)\cdot\boldsymbol{x}_v = \cos\theta \\ G &= \|\tilde{\boldsymbol{x}}_v+\boldsymbol{w}_v(v)u\|^2 = \alpha+2\beta u+\gamma u^2 \end{aligned}$$

where

$$\alpha := \|\tilde{\boldsymbol{x}}_v\|^2 \qquad \beta=\tilde{\boldsymbol{x}}_v\cdot\boldsymbol{w}_v(v) \qquad \gamma=\|\boldsymbol{w}_v(v)\|^2$$

and $\theta(v)$ is the angle between the tangent to C and the generator through $\tilde{\boldsymbol{x}}(v)$. Thus, the line element of S is

$$ds^2 = du^2+2\cos\theta dudv+(\alpha+2\beta u+\gamma u^2)dv^2. \tag{2.10}$$

Then, Minding researched all the ruled surfaces $S:(u,v)\mapsto\boldsymbol{x}(u,v)$ that are applicable to a given ruled surface S_1, $\boldsymbol{x}_1(u_1,v_1)=(x_1(u_1,v_1),y_1(u_1,v_1),z_1(u_1,v_1))$, whose line element, according to the previous results, can assume the form

$$ds_1^2=du_1^2+2\cos\theta du_1dv_1+(\alpha+2\beta u_1+\gamma u_1^2)dv_1^2,$$

where $\alpha,\beta,\gamma,\theta$ are known functions. His aim is to determine six functions of v, $\tilde{\boldsymbol{x}}(v)=(\tilde{x}(v),\tilde{y}(v),\tilde{z}(v))$ and $\boldsymbol{w}=(w_1(v),w_2(v),w_3(v))$, in such a way that

$$\begin{gathered} \|\boldsymbol{w}(v)\|^2=1 \qquad \|\boldsymbol{w}_v(v)\|^2=\gamma \\ \|\tilde{\boldsymbol{x}}_v\|^2=\alpha \qquad \tilde{\boldsymbol{x}}_v\cdot\boldsymbol{w}_v(v)=\beta \qquad \boldsymbol{w}(v)\cdot\boldsymbol{x}_v=\cos\theta \end{gathered} \tag{2.11}$$

hold true.

By recurring to a spherical representation for $\boldsymbol{w}$

$$w_1=\cos\omega\cos\varphi \qquad w_2=\cos\omega\sin\varphi \qquad w_3=\sin\varphi,$$

the condition $\|\boldsymbol{w}_v(v)\|^2=\gamma$ becomes

$$\omega'^2+\cos^2\omega\varphi'^2=\gamma, \tag{2.12}$$

where apices stand for derivatives with respect to v. (2.12) provides φ by quadrature as a function of ω and thus $\boldsymbol{w}$ is determined as a function of ω. The angle ω remains arbitrary in the final solution as a consequence of the fact that (2.3.1) is a system in six unknowns with only five equations. With regard to $\tilde{\boldsymbol{x}}$, it is obtained by integrating the equations (2.3.1), when they are written with respect to $\tilde{\boldsymbol{x}}_v$.

As a consequence of the indeterminacy of the angle ω, the solution that Minding found is not a single surface but a family of deformations of S. Then, Minding observed that, just

[32]Subscripts u or v denote differentiation with respect to u, v, respectively.

as the plane constitutes the simplest representation among developable surfaces, among the infinite representations of a given ruled surface the preferred one has the generators parallel to the same plane.[33]

Finally, Minding applied his method to determine the ruled surfaces applicable to the hyperboloid and the ruled helicoid. This last example probably gave him occasion to consider more closely the characteristics of the surface applicable to a ruled surface. In fact, later but still in 1838, Minding sent a note to the journal to be appended to (Minding 1838a), which was instead published as an independent article in the same volume in Minding 1838b because of the delay in submission. Therein, Minding showed that there exist surfaces applicable to a ruled surface that are not themselves ruled, i.e., the class of surfaces applicable to a given ruled surface he had found in (Minding 1838a) is not complete. In this respect, the distinction of developable and ruled surfaces was not so secondary: only the deformations of a plane can *always* be generated by straight lines; in the case of ruled surfaces, some deformations may be not ruled. This is the case of the following family of surfaces of revolution:

$$\begin{cases} x = \dfrac{1}{a}\sqrt{n^2+v^2}\cos au \\ y = \dfrac{1}{a}\sqrt{n^2+v^2}\sin au \\ z = \dfrac{1}{a}\displaystyle\int_0^v \sqrt{\frac{a^2n^2+(a^2-1)v^2}{n^2+v^2}}dv \end{cases} \tag{2.13}$$

where a is a non null arbitrary constant. All of them can be developed on the right helicoid,[34] since their line element is $ds^2 = (n^2+v^2)du^2+dv^2$, but, as Minding stated, none of them contains straight lines.[35] This curious fact probably led Minding to approach the problem of determining the *complete* class of surfaces applicable to a given one, S. Indeed, in the remaining part of the paper, Minding proposed a first attempt to this problem that he judged to be a *"very fascinating, but at the same time extremely challenging, task"*.[36]

To overcome the difficulties of the problem and achieve some results, Minding restricted the set of surfaces S by requiring a specific form of their line element $ds^2 = Edu^2+2Fdudv+Gdv^2$. This restriction allowed him to easily write both the differentials of the equations the searched surfaces and their integrability conditions. Precisely, he required the Gaussian

[33]The deformation of ruled surfaces was also addressed in (Bonnet 1848, §7) and very extensively in (Bour 1862b, Chap. 5). In particular, the latter generalised Minding's remark by stating that a ruled surface can be deformed in such a way that the director cone assumes any arbitrary shape. Beltrami also studied the deformations of a ruled surface that leave it a ruled surface in (Beltrami 1865).

[34]The parametric equations of a right helicoid are $\boldsymbol{x}(u,v) = (v\cos u, v\sin u, nu)$. hence, $\boldsymbol{x}_u = (-v\sin u, v\cos u, n)$, $\boldsymbol{x}_v = (\cos u, \sin u, 0)$ and $E = n^2+v^2$, $F = 0$ and $G = 1$. The same values for E, F and G are obtained in the case of (2.13), since $\boldsymbol{x}_u = \left(-\sqrt{n^2+v^2}\sin au, \sqrt{n^2+v^2}\cos au, 0\right)$ and $\boldsymbol{x}_v = \left(\frac{v}{2\sqrt{n^2+v^2}}\cos au, \frac{v}{2\sqrt{n^2+v^2}}\sin au, \frac{1}{a}\sqrt{\frac{a^2n^2+(a^2-1)v^2}{n^2+v^2}}\right)$.

[35]It is noteworthy that Minding implicitly remarked that a catenoid (but he did not use this term) can be developed on a ruled helicoid by pointing out that the equations (2.13) represent a catenoid in the case $a = 1$.

[36]"Sehr anziehende, zugliech aber äußerst schwierige Aufgabe" (Minding 1838b, p. 366)

parameters u, v to be orthogonal (i.e., $F = 0$) and the Cartesian coordinate z of the surface to be a function of v only (i.e., $dz = V(v)dv$). Under these conditions, one has[37]

$$dx^2 + dy^2 = Edu^2 + (G - V^2)dv^2. \tag{2.14}$$

This latter equation can be easily decomposed as follows by multiplying it by $1 = \cos^2\psi + \sin^2\psi$, where ψ is an auxiliary function of u and v:

$$dx^2 + dy^2 = (\sqrt{E}\cos\psi du + \sqrt{G - V^2}\sin\psi dv)^2 + (-\sqrt{E}\sin\psi du + \sqrt{G - V^2}\cos\psi dv)^2$$

namely,

$$\begin{cases} dx = \sqrt{E}\cos\psi du + \sqrt{G - V^2}\sin\psi dv \\ dy = -\sqrt{E}\sin\psi du + \sqrt{G - V^2}\cos\psi dv. \end{cases} \tag{2.15}$$

The existence of such functions x, y is guaranteed by the requirement of the following integrability conditions

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{\partial^2 x}{\partial v \partial u} \qquad \frac{\partial^2 y}{\partial u \partial v} = \frac{\partial^2 y}{\partial v \partial u}$$

that give

$$\frac{\partial\sqrt{E}}{\partial v} = \sqrt{G - V^2}\frac{\partial\psi}{\partial u} \qquad \frac{\partial\sqrt{G - V^2}}{\partial u} = -\sqrt{E}\frac{\partial\psi}{\partial v}. \tag{2.16}$$

Finally, the existence of a function ψ is ensured by the condition

$$\frac{\partial^2\psi}{\partial u \partial v} = \frac{\partial^2\psi}{\partial v \partial u}. \tag{2.17}$$

By deriving (2.16), (2.17) is equivalent to

$$\frac{\partial\sqrt{E}}{\partial v}V(v)\frac{\partial V(v)}{\partial v} = MQ^2 + N$$

where

$$M = \frac{\partial^2\sqrt{E}}{\partial v^2} + \frac{1}{2\sqrt{E}}\frac{\partial^2 G}{\partial u^2} - \frac{1}{2E}\frac{\partial\sqrt{E}}{\partial u}\frac{\partial G}{\partial u}$$
$$N = \frac{1}{4\sqrt{E}}\left(\frac{\partial G}{\partial u}\right)^2 + \frac{1}{2}\frac{\partial\sqrt{E}}{\partial v}\frac{\partial G}{\partial v} - GM.$$

If $M\left(\frac{\partial\sqrt{E}}{\partial v}\right)^{-1}$ and $N\left(\frac{\partial\sqrt{E}}{\partial v}\right)^{-1}$ are function of v alone, Minding observed that

$$V(v)\frac{\partial V(v)}{\partial v} = \left(\frac{\partial\sqrt{E}}{\partial v}\right)^{-1}(MQ^2 + N)$$

can be solved and ψ can be determined. Once ψ is found and (2.16) are also satisfied, the complete class of surfaces having (2.14) as line element is obtained through the integration of (2.15).

[37] It immediately follows from $dx^2 + dy^2 + dz^2 = Edu^2 + 2Fdudv + Gdv^2$.

Minding was probably led to consider surfaces with (2.14) as a line element in view of a generalisation of the case of surfaces of revolution. Indeed, when a surface of revolution is parametrized as

$$\begin{cases} x = \Phi(v)\cos u \\ y = \Phi(v)\sin u \\ z = V(v) \end{cases}$$

where $\Phi(v)$ and $V(v)$ are function of v only, its line element is

$$ds^2 = \Phi^2(v)du^2 + \left(\left(\Phi'(v)\right)^2 + \left(V'(v)\right)^2\right)dv^2,$$

which naturally satisfies the initial requirements— i.e., $F = 0$ and $z = V(v)$. In this case, the integrability conditions (2.16) give

$$\frac{d\psi}{dv} = 0 \qquad \frac{d\psi}{du}\sqrt{G - V^2} = \Phi'$$

and it is immediately verifiable that $\psi = \dfrac{u}{a}$ is a solution, where a is a non null arbitrary constant. In addition, $G - V^2 = a^2\Phi'^2(v)$. By integrating (2.15),[38] one has

$$\begin{cases} x = a\Phi(v)\cos\left(\frac{u}{a}\right) \\ y = a\Phi(v)\sin\left(\frac{u}{a}\right) \\ z = \int\sqrt{V'^2 + (1 - a^2)\Phi'^2}dv \end{cases}$$

that gives all the surfaces applicable to a surface of revolution that respect the initial restrictions $F = 0$ and $z = V(v)$.

Minding also specialised this latter system to the case of spheres by choosing $\Phi(v) = \cos v$ and $V(v) = \sin v$. The equations

$$\begin{cases} x = a\cos v\cos\left(\frac{u}{a}\right) \\ y = a\cos v\sin\left(\frac{u}{a}\right) \\ z = \int\sqrt{1 - a^2\sin^2 v}dv \end{cases}$$

give a unit sphere for $a = 1$ and deformation of it for other values of a. Here, Minding also emphasised that this deformation cannot be interpreted globally. Indeed, if a sphere were deformed as a whole, its deformation would no longer be closed or the sphere would cover the surface of its deformation more than once:[39]

> In order to prevent misunderstandings, I would like to add that it is well known that a convex surface that is closed in itself is unbendable as an intact whole. Therefore, when speaking of its bending, one must think of the relation as being

[38]To integrate (2.15), one must take into account that $\sqrt{E} = \Phi(v)$, $\psi = \dfrac{1}{a}p$, $G - V^2 = a^2\Phi'^2(v)$ and $dz = V(v)dv$, where $V(v) = \sqrt{V'^2 + (1 - a^2)\Phi'^2}$.

[39]A more detailed description of this topic is provided in (Bianchi 1922, §130).

> interrupted in a certain extension. If, for example, we assume that a is a real break in the preceding formulae, it becomes evident that some parts of the original spherical surface fall on top of each other in the curved surface. For, in order to obtain the whole curved surface, one need only assume values from 0 to $2a\pi$ as far as p is concerned, while for the sphere p continues from 0 to 2π; the remaining strip of the sphere extending from $p = 2a\pi$ to therefore covers, after the bending, the strip of the curved surface contained between $p = 0$ and $p = 2\pi(1 - a)$. This, as one can see, results from the formulae by itself, if one only notes that the same values of p and q belong to the same point of the surface, before and after the bending.[40]

In the following year, in (Minding 1839) Minding obtained his most important results tackling the problem of applicability, but trying to avoid integration. As the title suggests (*"How to decide whether or not two given curved surfaces can be unfolded upon each other; together with observations on surfaces with constant curvature"*), its main topic[41] is the *first* problem of applicability.

Minding showed that for two surfaces S and S' to be applicable, the condition $\Delta_1(K) = \Delta_1(K'_1)$ must be required in addition to the equality of their Gaussian curves, i.e., $K(P) = K'(P)$.[42] Minding highlighted:

> From the general formulae developed above [$K(P) = K'(P)$ and $\Delta_1(K) = \Delta_1(K'_1)$], it follows that the question of whether it is possible to unfold two surfaces, whose equations are known in coordinates, onto each other, can always be answered without the aid of the integral calculus; the operations of differentiation and elimination are sufficient for their decision by calculation.[43]

[40]"Um Missverständnissen vorzubeugen, bemerke ich noch, dass bekanntlich eine in sich geschlossene convexe Fläche als unversehrtes Ganzes, unbiegsam ist. Man muss sich daher, wenn von ihrer Biegung gesprochen wird, den Zusammenhang in einer gewissen Ausdehnung unterbrochen denken. Lässt man z. B. in den hier zunächst vorhergehenden Formeln a einen ächten Bruch bedeuten, so zeigt sich, dass in der gebogenen Fläche einige Theile der ursprünglichen Kugelfläche auf einander fallen. Denn um die ganze gebogene Fläche zu erhalten, braucht man, was p betrifft, nur Werthe von 0 bis $2a\pi$ anzunehmen, während für die Kugel p von 0 bis 2π fortgeht; der übrige von $p = 2a\pi$ bis reichende Kugelstreifen deckt mithin, nach der Biegung, den zwischen $p = 0$ und $p = 2\pi(1 - a)$ enthaltenen Streifen der gebogenen Fläche. Dies ergiebt sich, wie man sieht, aus den Formeln von selbst, wenn man nur festhält, das zu demselben Puncte der Fläche, vor und nach der Biegung, dieselben Werthe von p und q geh"oron." (Minding 1838b, p. 368)

[41]"Haupt-Aufgabe", (Minding 1839, p. 380).

[42]A discussion of this fact is omitted here. The interested reader can find a complete exposition in (Cogliati 2022, pp. 109–112). The equation $\Delta_1(K) = \Delta_1(K'_1)$ corresponds to equation (Minding 1839, 381, eq. 25). $\Delta_1(f)$ is the first differential parameter of the function f, that is, $\Delta_1(f) = \frac{En^2 - 2Fmn + Gm^2}{EG - F^2}$ where $df = mdu + ndv$. Here, it is used in an anachronistic way since it was introduced only after, by Eugenio Beltrami in (Beltrami 1864-1865).

[43]"Aus den in dieser Absicht oben entwickelten allgemeinen Formeln ergiebt sich, dass die Frage, ob es möglich ist, zwei Flächen, deren Gleichungen in Coordinaten man kennt, auf einander abzuwickeln, sich immer ohne Hülfe der Integral-Rechnung beantworten lässt; es reichen nämlich die Operationen des Differentiirens und Eliminirens zu ihrer Entscheidung durch Rechnung hin"(Minding 1839, p. 387)

Despite the fact that Minding's solution is not complete,[44] it is so noteworthy that the problem is named *Minding's problem* after him.

This paper also contains the first treatment of surfaces with constant curvature. Minding proved that for such surfaces, and for them only, TE is a sufficient as well as necessary condition for applicability; that is, surfaces with the same constant curvature constitute a complete class of applicable surfaces. In this case, in fact, the map that realises the isometry between two surfaces S and S' with the same Gaussian curvature can be easily obtained by considering a system of geodesic coordinates upon each surface that is centred at corresponding points. The central remark is that for surfaces with constant curvature, whose line element is $ds^2 = Edr^2 + 2Fdrd\theta + Gd\theta^2$, the coefficient G is independent to θ. Indeed, in (Minding 1830a), he had integrated Gauss' equation in the case of a geodesic system of parameters on a surface with constant curvature K, that is,

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial r^2}$$

and obtained the expression for the corresponding line element

$$ds^2 = dr^2 + \frac{\sin^2(\sqrt{K}r)}{K}d\theta^2.$$

In this case, the condition $ds^2 = ds'^2$ that expresses the local isometry of S and S' reduces to

$$dr^2 + \frac{\sin^2(\sqrt{K}r)}{K}d\theta^2 = dr'^2 + \frac{\sin^2(\sqrt{K}r')}{K}d\theta'^2,$$

that is, $r = r'$ and $\frac{\sin^2(\sqrt{K}r)}{K}d\theta^2 = \frac{\sin^2(\sqrt{K}r')}{K}d\theta'^2$. The latter reduces to $d\theta = d\theta'$ since $r = r'$ and, thus, one has $\theta = \pm\theta' + cost$. While $r = r'$ indicates that any point on the first surface can be made to correspond to any point on the second surface, the second equation indicates that any geodesic on the first surface that passes through a point can be made to correspond to any geodesic on the second surface that passes through the corresponding point:

> Consequently, two surfaces of the same invariable curvature can be unfolded on each other in an infinite number of ways, since any two points of one can be made to correspond to any two points of the other, if the lines of the shortest lengths on the surfaces between the two pairs of points are the same.[45]

Vice versa, the map

$$r = r' \qquad \theta = \pm\theta' + cost$$

obviously preserves the line element.

[44] His demonstration is incomplete as he did not consider some singular cases. For a complete proof, the interested reader can refer to (Bianchi 1922, §122).

[45] "Folglich lassen sich überhaupt zwei Flächen von gleichem unveränderlichem Krümmungsmaasse auf einander abwickeln, und zwar auf unendlich viele Arten, indem man zwei beliebige Puncte der einen zweien beliebigen der anderen entsprechend setzen kann, wenn nur die Längen kürzester Linien auf den Flächen, zwischen beiden Paaren von Puncten, einander gleich sind." (Minding 1839, p. 375)

This remarkable achievement encouraged Minding to further investigate zero-curvature surfaces. When they are supposed to be expressed in the form $z = f(x, y)$, their differential equation is

$$z_{uu}z_{vv} - z_{uv}^2 = K(1 + z_u^2 + z_v^2).$$

Solving this equation would have meant determining a second complete class of applicable surfaces in addition to the developable surfaces. Its complicated form, however, dissuaded Minding from such an effort, and he limited his investigation to the case of helicoidal surfaces or surfaces of revolution by requiring that $x = r\cos\psi$ and $y = r\sin\psi$ and $\frac{\partial z}{\partial \psi} = h$, $h = const.$[46] The case $K < 0$ and $h = 0$, which corresponds to surfaces of revolution with constant negative curvature, is particularly interesting. In this case, he found that the curve $\gamma_a : r \mapsto z(r)$ that generates the surface through its revolution around an axis is a solution of the following equation

$$\frac{\partial z}{\partial r} = \pm\sqrt{\frac{1}{a + r^2} - 1} \tag{2.18}$$

where a is a constant. He identified three different curves γ_a that are solutions of (2.18) based on the value of the parameter a, as illustrated in Figure 2.3. In particular, for $a = 0$ he obtained the surface that Beltrami called the *pseudosphere*.[47] This classification was further investigated by Beltrami and Dini, who showed that, despite the property of being applicable to each other, these three classes of surfaces are essentially distinct, as it is not possible to find an isometric correspondence that makes the meridians of surfaces of two distinct classes coincide.[48]

Finally, Minding devoted (Minding 1840) to the theme of surfaces of revolution. Here, he proved that, if two surfaces with non-constant curvature can be deformed into each other in an infinite number of ways, then their class of applicability always contains surfaces of revolution.

2.3.2 France: Liouville's commitment to disseminating Gauss' ideas

In France, the geometrical tradition inaugurated by Monge had prepared a fertile ground for the reception of Gauss' ideas: here, a wider number of mathematicians were working on issues related to infinitesimal geometry.[49]

Liouville was the main person responsible for the dissemination of Gauss' methods in France.[50] Interest first arose as a result of the publication of (Liouville 1847), where Liouville provided an alternative demonstration of the TE based on the introduction of

[46](Minding 1839, pp. 376–380)

[47](Beltrami 1868, p. 290)

[48]During the years 1865-1866, Dini and Beltrami devoted a number of papers to the investigation of surfaces with constant curvature. The main results are in (Beltrami 1864) and (Dini 1865b). For a detailed review of their contributions, see (Coddington 1905, pp. 17–23).

[49]For a description of the relation between Gauss and the French community, see (Reich 1996).

[50]For a detailed discussion on Liouville's role in the dissemination of Gauss' ideas, see (Lützen 1990, pp. 739–755).

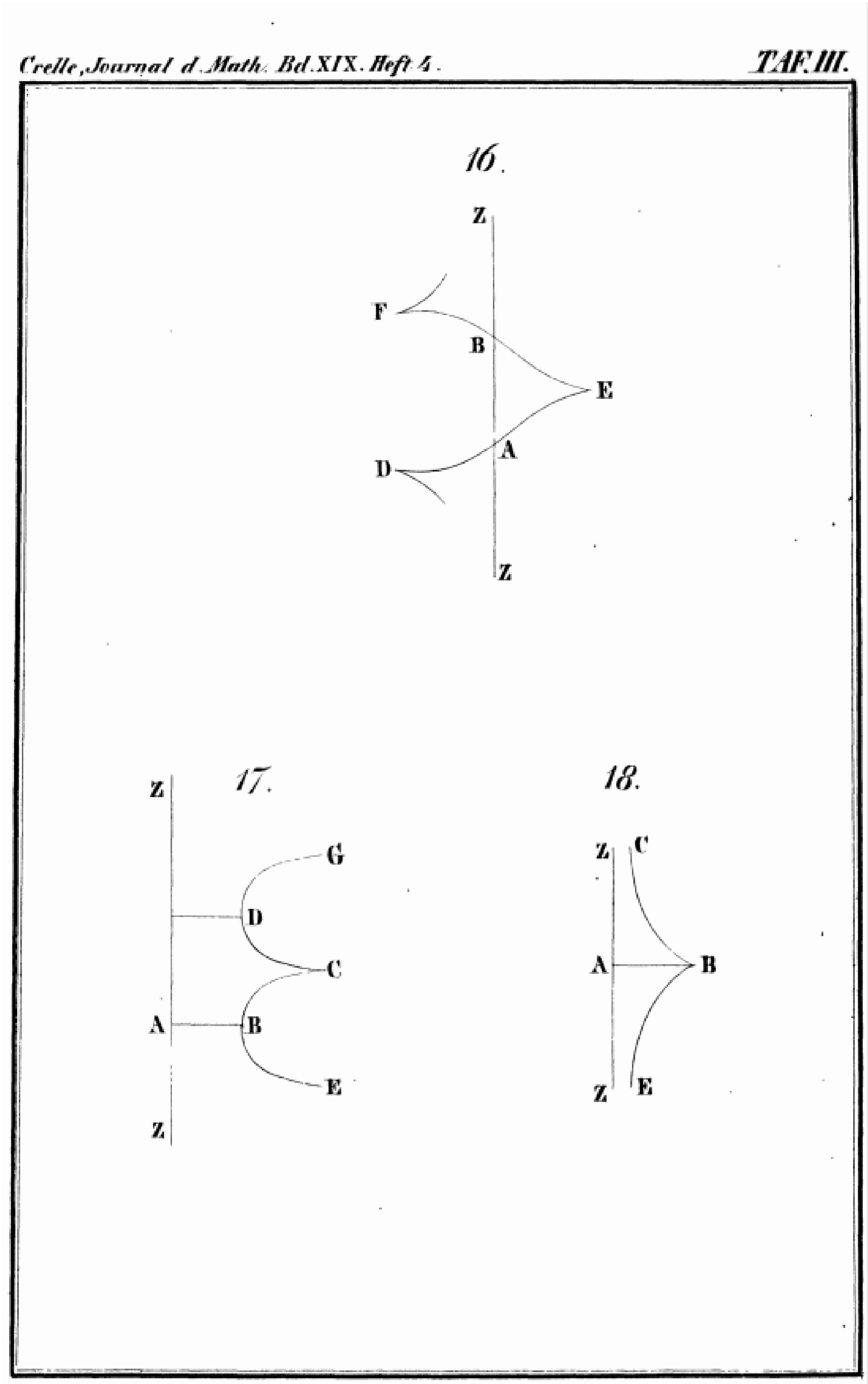

Figure 2.3: Minding's drawings that illustrate the profile of surfaces with constant negative curvature. Figure 16 corresponds to the profile of an elliptical type pseudospherical surface of revolution; Figure 17 corresponds to the profile of a hyperbolic type pseudospherical surface of revolution; Figure 18 corresponds to the profile of a pseudosphere. For a detailed description of this classification, see (Bianchi 1922, §127).

an isothermal coordinate system. This work stimulated several alternative approaches, which were proposed by Bertrand, Diguet and Puiseux with a twofold purpose: to make its geometric meaning more evident, and to simplify its demonstration so that it could be included in university curricula. Until then, only Bonnet, who was Liouville's pupil at the École Polythecnique, had mentioned the *Disquisitiones*, praising their content in (Bonnet 1844).

The event that, in some ways, represents the realisation of the value of the *Disquisitiones* was the publication of the fifth edition of Monge's treatise, *Application de l'analyse à la géométrie.* It was identical to the previous edition, except for a long addition, through which Liouville intended to draw attention to the need to update Monge's classical exposition in light of the new methods set forth by Gauss. He therefore provided a reprint of the entire *Disquisitiones*, to which he added seven notes expanding on several themes that Gauss had introduced. In (Monge 1850, p. 582) he clearly explained his aim:

> the notes that we append naturally deal with those points [...] for which Mr. Gauss has opened new ways; moreover, our aim is to indicate to young people the sources where they can learn rather than giving them regular lectures.[51]

Geodesic curvature is treated in the second and third notes.[52] After (Minding 1830a), geodesic curvature had been treated (apparently independently) by Bonnet in (Bonnet 1848). According to Lützen, Liouville's definition had the advantage of being completely intrinsic:

> By insisting on an a priori intrinsic definition, Liouville seems to have grasped better than his predecessors the opportunity of a purely intrinsic formulation, which had been characterised by Gauss as *"a very valid method that can be thoroughly developed by geometers"*.[53]

In note IV, Liouville discussed some alternative demonstrations of Gauss' theorem on the invariance of the curvature of a surface under isometries, and Minding's problem. Also in this case, Lützen notes that it is arduous to establish if Liouville knew Minding's preceding results, as well as those due to Bonnet, who had dealt with the same problem two years before in (Bonnet 1848).[54] In notes V and VI he dealt with the problem of conformal mapping and gave an alternative approach to the problem of deciding whether or not two surfaces are applicable.

Liouville's invitation was warmly welcomed by the French scientific community. In the 1850s, there was a proliferation of works on the Gaussian theory of surfaces by Liouville

[51]"Les Notes que nous ajoutons portent naturellement sur les points [...] pour lesquels M. Gauss a ouvert des voies nouvelles: nous avons d'ailleurs bien plus pour objet d'indiquer aux jeunes gens les sources où ils doivent s'instruire que de leur donner des leçons en règle." (Monge 1850, p. 582)

[52]The first note is devoted to curve theory, but geodesic curvature is defined in (Monge 1850, p. 568).

[53](Lützen 1990, p. 750)

[54](Lützen 1990, p. 743)

himself, Bonnet, and Bertrand, to name the principal actors.[55] France, thus, assumed a position at the forefront of research in differential geometry.

Circumstances were still significantly different elsewhere in Europe. Minding had left Germany in 1843 and moved to the University of Dorpat, today Tartu (Estonia), at that time under the Russian Empire. His research activity shifted mainly to analysis, but he supervised Carl Peterson during his thesis on the deformation of surfaces, which remained completely forgotten throughout the nineteenth century. In it, the Mainardi-Codazzi equations and an early attempt at the proof of the fundamental theorem of surfaces appeared for the first time. As (Reich 1973) notes, however, the Russian environment was rather closed and ideas struggled to circulate outside it.

The developments in differential geometry in France were closely followed in Italy, where interest in Gauss' theory was mainly sustained by Domenico Chelini. From 1848, he devoted a series of papers to surface theory[56] in which he expounded a proof of the TE in more familiar geometrical terms. In Pavia, we find Mainardi and his students Brioschi and Codazzi. In particular, in 1856 Mainardi wrote (Mainardi 1856), where he proposed his own version of the Mainardi-Codazzi equations and provided a heuristic argument that can be regarded as an attempt to prove the fundamental theorem of surfaces. However, (Mainardi 1856), although published in a relatively well-known journal, the *Giornale dell'Istituto Lombardo*, did not attract much attention.[57]

[55]For a list and a brief description of works dating back to this period, see (Chasles 1870).
[56](Chelini 1848a), (Chelini 1848b).
[57]A detailed description is postponed to Section 3.6.

Chapter 3

Paris, 1860: the origins of the fundamental theorem of surface theory

3.1 The Grand Prix des Mathématiques

The Copenhagen Prize that Gauss won in 1825 was part of a complex system of prizes organised by a number of institutions, including the various National Academies of Sciences, from the eighteenth century.[1]

Prizes were divided into two categories: *retrospective prizes*, which evaluated a selection of memoirs that competitors freely presented, and *prospective prizes*, which asked for answers to precise questions with new results. The latter certainly played a key role in guiding the research. Indeed, they essentially had a threefold purpose: to identify the most relevant aspects to improve, to draw the attention of the international scientific community to precise promising issues, and to motivate mathematicians to provide important contributions.

The Académie des Sciences in Paris, which was a model for the scientific community throughout the nineteenth century, invested significantly in prizes.[2] The most prestigious, the Grand Prix, on problems in mathematics and physics, was announced annually every other year starting in 1810. A committee of experts proposed a specific question on one of the major open problems of the time. Answers had to be submitted to the Academy within a certain time period, usually a couple of years, but the deadline could be extended, as was often the case, if no worthy memoirs were presented. Due to the extreme difficulty of successfully solving the prize questions, only a few works were regularly submitted. To ensure the impartiality of the committee in charge of evaluating them, anonymity was maintained through a system of mottos that identified authors. Mottos were placed on the

[1] For an historical account of prize competitions, see (Gray 2006) and (Crosland and Gálvez 1989).

[2] On the establishment of prizes by the Paris Academy and their winners, see (Maindron 1881), (Gauja 1917) and (Jaisson 2003).

memoir and on a closed envelope containing the name of the corresponding mathematician. At the award ceremony, only the winner's envelope was opened after the winning memoir was announced. The monetary prize was a rather large 3,000 francs and the winning memoirs were published in the *Comptes rendus*.

The offer of money clearly acted as an incentive, but in the case of the Grand Prix winning such a prestigious prize could also provide visibility in international environments. Only outstanding works won and, in case no meritorious memoirs were submitted, the prize was not awarded and was usually re-announced. To avoid this inconvenience, topics were often chosen from those on which it was known that someone was already working and making important progress. As we will see in Chapter 5, this was probably the case in the 1894 Grand Prix, but also for the Danish prize of 1825 in which Schumacher, who was in close correspondence with Gauss, chose a topic (conformal mapping of surfaces) that he knew to be the focus of Gauss' research at the time.

3.1.1 1860 Grand Prix

An explicit manifestation of the interest of the French community on the theme of applicability came in 1859 when the Académie des Sciences de Paris decided to devote a "Grand Prix des Mathématiques" to surface theory, and specifically to the second problem of applicability. The original statement of the prize read as follows:

> To form the differential equation or equations of surfaces applicable to a given surface, and then treat the problem in some particular cases, either by looking for all the surfaces applicable to a given surface, or by finding only those which fulfil, in addition, a second condition chosen so as to simplify the solution.[3]

The examining commission consisted of Liouville, Chasles, Lamé, Hermite, and Bertrand as rapporteur. In its final report (Comptes rendus 1861) Serret replaced Lamé. In all, five papers were submitted. Three memoirs authored by Pierre Ossian Bonnet (1819-1892), Delfino Codazzi (1824-873) and Edmond Bour (1832-1866) were particularly appreciated so much so that the commission declared: "[. . .] if each of them had been presented individually, it would certainly have won the prize" (Comptes rendus 1861, p. 554).[4] Bour's paper stood out due to the fact that he could integrate the relevant equations for surfaces of revolution. Among other outstanding results, we recall *Bour's theorem*, which assures the existence of a helicoidal surface locally isometric to a given surface of revolution.[5]

[3]"Former l'équation ou les équations différentiélles des surfaces applicables sur une surface donnée; traiter le problème dans quelques cas particuliers, soit en cherchant toutes les surfaces applicables sur une surface donnée, soit en trouvant seulement celles qui remplissent, en outre, une seconde condition choisie de manière à simplifier la solution." (Comptes rendus 1859, p. 521)

[4]According to the manuscripts at the Archives de l'Académie des sciences, the other two were submitted by Augus Commines de Marsilly and Felice Casorati.

[5]This result represented a source of inspiration for Ulisse Dini and his discovery of a helicoidal surface (later to be known as *Dini's surface*) that is locally isometric to the tractroid (i.e., the surface of revolution obtained from a tractrix that rotates about its axis).

The commission motivated the attribution of the prize to Bour with the following words (Comptes rendus 1861, p. 554):

> [...] Memoir n° 1 [Bour's memoir] also contains a very remarkable chapter, the analogue of which is not found in the other two, and which determined the unanimous choice of the Commission in its favour. The author has, in fact, proposed nothing less than the complete integration of the equations of the problem in the case where the given surface is one of revolution. The ordinary methods of integral calculus not seeming applicable here, he took advantage of a cursory indication by Lagrange in one of his Memoirs and to the application of which the illustrious geometer himself pointed out serious difficulties. This method consists in first forming a complete solution of the second-order differential equation in which there are five arbitrary constants and in deducing the general solution from it by the variation of these constants. The difficulties which Lagrange had perceived and pointed out were very skilfully and very happily overcome in Memoir n° 1. The Commission hopes that the author will generalise his fine analysis and that the integral calculus will thereby receive a notable improvement. It will be fair to bring back to Lagrange the glory of having opened this new path, but the current Competition will nevertheless occupy an important place in the history of its development.[6]

In addition, the commission decided to acknowledge a *mention honorable* to Codazzi and Bonnet for their valuable contributions.

The three manuscripts by Bour, Bonnet and Codazzi were published only after some years. According to (Chasles 1870, p. 326), Bour's memoir (Bour 1862b) was published without a section that had originally been included in the submission for the prize, which was unfortunately lost. (Darboux 1908, p. 113) specified that the memoir had been taken by Joseph Bertrand and that it was burnt during the Commune. Actually, a manuscript, which is identifiable to Bour's memoir by the motto *"Je plie et ne romps pas"* in accordance with the announcement of the winner's prize (Comptes rendus 1861, p. 555), is stored at the Archives de l'Académie des Sciences in Paris. It contains six chapters that essentially

[6] "[...] le Mémoire n° 1 contient en outre un chapitre très-remarquable, dont l'analogue ne se trouve pas dans les deux autres et qui a déterminé en sa faveur le choix unanime de la Commission. L'auteur ne s'est, en effet, proposé rien moins que l'intégration complète des équations du problème dans le cas où la surface donnée est de révolution. Les méthodes ordinaires du calcul intégral ne semblant pas ici applicables, il a mis à profit une indication rapide jetée comme en passant par Lagrange dans l'un de ses Mémoires et à l'application de laquelle l'illustre géomètre signalait lui-même de graves difficultés. Cette méthode consiste à former d'abord une solution complète de l'équation différentielle du second ordre dans laquelle figurent cinq constantes arbitraires et à en déduire la solution générale par la variation de ces constantes. Les difficultés que Lagrange avait aperçues et signalées ont été très-habilement et très-heureusement surmontées dans le Mémoire n° I. La Commission espère que le savant auteur généralisera sa belle analyse et que le calcul intégral recevra par là un perfectionnement notable. Il sera juste de rapporter à Lagrange la gloire d'avoir ouvert cette voie nouvelle, mais le Concours actuel occupera néanmoins une place importante dans l'histoire de son développement."

correspond to the first six chapters that appeared in (Bour 1862b), as well as a chapter entitled, *"Extrait du chapitre X. Surfaces applicables sur les surfaces de revolution"*, which was probably the "noteworthy chapter" referred to by the Commission in the above quotation, and two appendices, *"Théorie des caustiques"* and *"Intégration des équations des lignes géodésiques"*, respectively. These latter three parts are not present in (Bour 1862b).[7] Some insights into Bour's integration method can be inferred by analysing his proof of the theorem on helicoid surfaces, which is discussed in Section 3.4.

Bonnet's manuscript appeared in two parts (Bonnet 1865) and (Bonnet 1867), the latter containing an extensive addition in which the FT made its first appearance. Finally, (Codazzi 1883) was published only after Codazzi's death. There is no simple explanation for this delay in publication. Both Bonnet and Codazzi must have felt the need to improve the theory that they had developed on the occasion of the prize. For example (Bonnet 1863), which was devoted to the problem of deformation of non-developable ruled surfaces, offered an ameliorated deduction of *"les formules de M. Codazzi"*. On his part, in (Codazzi 1868), Codazzi included a more general deduction of the MCE that was valid for an arbitrary coordinate system. Finally, the fact that the Paris Academy published Codazzi's paper more than 20 years after it was written can be read as a tribute to Codazzi himself, to whom the name of the MCE had been associated in the meantime.

3.2 Different paths toward the prize competition: Bonnet and Bour

When Bonnet and Bour decided to commit themselves to the topic of applicability theory in 1859, their backgrounds, both personal and scientific, were quite divergent. Bonnet was a mature mathematician who had provided significant contributions to infinitesimal geometry, and to surface theory in particular. In contrast, Bour was a rising star within the French mathematical community who had recently published noteworthy memoirs dealing with analytical mechanics and the integration theory of first-order PDEs.

Looking back to Bonnet's research interests,[8] one has the impression that his participation in the prize competition was a natural step in his scientific career. Probably, the most

[7] On the contrary, paragraphs §§31-39 (which probably correspond to the end of chapter 6 and chapters 7-9 of (Bour 1862b)) are lacking in the manuscript. An approximate description of the difference between the original and the printed version is provided by Bour himself in (Bour 1862b, pp. 147–148): *"The above Memoir is not identical with the one that was awarded by the Academy of Sciences in its extraordinary session of 25th March 1861. In the latter, due to the lack of time necessary for the writing, the entire chapters VI and IX, as well as part of chapters V and VII, were only represented by a very brief analysis, which indicates, without demonstration, the results that I have established today. On the other hand, the work submitted to the Academy included an extract from the analytical research mentioned in the introduction (§6), research which relates to the general integration of equation [III] [Bour's equation] in certain cases where this equation simplifies; it also included the analysis of two appendices, to which I have had occasion to refer the reader several times in the course of the present Memoir."*

[8] A short scientific biography of Bonnet can be found in (Appell 1893) and (Struik 2008). For a detailed description of his early publications, see (Chasles 1870, pp. 199–214).

appropriate description of Bonnet's research programme can be conveyed using Lamé's words, which were quoted by Bonnet himself in (Bonnet 1844).

> If mathematical analysis, said M. Lamé, discovers new properties in the science of extension, it is important that pure geometry assimilates these properties and verifies them by methods that are its own. It is by perfecting themselves through similar trials that geometrical methods will be able to acquire all the generality and all the certainty necessary to be able to tackle the difficult questions that analysis alone has explored until now.[9]

These three elements (i.e., the elaboration and refinement of techniques and notions specific to geometry, the revision and improvement of existing results from a geometric point of view, and the search for new ones) are indeed characteristic of Bonnet's achievements.

One important result was the introduction of the notion of *geodesic curvature*, which Bonnet discovered independently of Minding in 1848.[10] He defined this as the projection upon the tangent plane of the radius of curvature of a given curve drawn on a surface. He proved its intrinsic character and employed it to generalise Gauss' theorem on the angular defect/excess of infinitesimal triangles to the case of any (i.e not necessarily geodesic) infinitesimal triangles.

Bonnet also had strong competence in advanced analysis, which he exploited to solve geometric questions. This is the case with (Bonnet 1860), where he discussed the subject of minimal surfaces and the determination of other classes of surfaces.

Edmond Bour,[11] who was 13 years younger than Bonnet, graduated from the *École Polytechnique* and then continued his education at the *École des Mines de Saint-Étienne*. Apparently, his education was profoundly influenced by Bertrand. In a letter sent to his father in 1855, he wrote: "Certainly, if I am going to receive a doctorate one day, I should thank, after God, M. Bertrand".[12] As a result of Bertrand's impulse, Bour's research interests turned to the theory of partial differential equations and analytical mechanics. He devoted his efforts to studying the work of Hamilton and Jacobi, in which he saw a way to develop a general theory of first-order PDEs.

In one of his first publications,[13] Bour discovered a method for reducing the number of variables involved in Hamilton's equations by exploiting the knowledge of some of their integrals and the elementary properties of Poisson brackets. As was soon to be discovered, the central idea to this approach had already been explored by Jacobi himself, who had expounded his so-called new method (*nova methodus*) in a long memoir that was posthumously published by Alfred Clebsch as (Jacobi 1862).

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[9] (Bonnet 1844, p. 980)

[10] (Bonnet 1848, pp. 43–44).

[11] For information on Bour's biography, see (Anonymus 1867) and (Taton 2008). An additional source can be found in (Bour 1905), which is a volume gathering hundreds of letters addressed by Bour to his family.

[12] (Bour 1905, p. 49)

[13] (Bour 1855)

The research programme that was initiated in 1855 was further pursued in (Bour 1862a), which appeared at the same time as (Bour 1862b), as a sort of analytical appendix to it. Bour extended his results on first-order differential equations to the case of systems of two or more such equations and also tried to explore possible applications for the case of second-order equations, such as those of Monge-Ampére type.[14] The reception of this work among the members of the Parisian mathematical community could not have been more favourable.[15] Upon commenting on the contents of this memoir, for example, Liouville wrote:

> Every word is an idea. [...] From now on, Mr. Bour has his rank fixed among the masters. He is no longer a promising young man, but a great geometer who has kept the brilliant promises of his youth.[16]

A further appreciation of Bour's work came in 1868, when, after his premature death, he was awarded the 1868 Grand Prix for his relevant contribution to the theory of second-order PDEs contained in (Bour 1862a).[17]

In contrast to Bonnet's geometrical background, Bour's decision to join the 1858 prize competition does not appear to have been supported by a solid experience in the study of surface theory. Once again, it was his mentor Bertrand who spurred him to work on the theme selected by the Academy.[18] A vivid description of the difficulties that Bour had to confront was provided by a fellow student of his at the École Polytechnique, Hilaire de Chardonnet (1839-1924) who recounted:

> I heard him say that he had found in a few hours the way to solve the question proposed for the Grand Prix, but that the calculations had cost him two years of application, and that he was still working on it the last night before the deadline.[19]

In view of these facts, the outcome of the competition appears to be predictable. The support of both Liouville and Bertrand must have been decisive in attributing the prize to Bour. Their decision probably reflected the high appreciation for Bour's work on the integration theory of PDEs.

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[14] In this regard, Bour commented in (Bour 1862a, p. 186): *La tâche est difficile, et je n'ai encore pu réunir qu'un bien petit nombre de matériaux capables de figurer dans le plan général que je viens d'esquisser.*

[15] In an undated letter to Houël (Gispert 1987, p. 143), on the contrary, Darboux wrote about a mystification of Bour's figure:

> Bour was a polytechnician, that is to say pretentious, he thought he had invented Peru and he was mistaken, so it seems that he got the prize for a point that appeared questionable since then. He was overestimated, that is the general opinion, and if he did not publish the integration that he had announced, it is said that he had not done it.

[16] (Liouville 1862).

[17] (Comptes rendus 1868, p. 923). The Academie had received no response to the prize question.

[18] (Bour 1905, p. 113).

[19] (Chardonnet 1866, p. 345).

In 1862, Bour and Bonnet were once again opposed to each other, this time competing for a position at the Paris Academy left vacant after Biot's death. Bonnet was elected even though Bour had the advantage of both the prize result and the support of Liouville[20], probably because of his longer experience.

3.3 Bonnet's solution and Bour's first method

Bonnet answered the question of the prize by facing the second problem of applicability in the most direct way. Curiously, this approach proved to be pretty much the same as that investigated by Bour in the first section of his memoir. By starting with the definition of applicability, he found a second-order differential equation of Monge-Ampére type that describes the entire class of surfaces applicable to a given one. However, this equation has a complicated form that makes it difficult to find its general integrals.[21]

Bonnet's paper was published in two parts, (Bonnet 1865) and (Bonnet 1867). The first part opened with a new proof of the TE that, in the author's view,[22] was to be preferred to Gauss' original proof on account of its geometrical character. In (Bonnet 1865, 221, §9), by recognising his previous mistake,[23] Bonnet made it clear that TE provides only a necessary condition for applicability. Finally, in (Bonnet 1865, §§ 10-15), one finds a treatment of the first problem of applicability that explicitly followed Minding's. The second part (Bonnet 1867) includes an answer to the problem that had been posed by the Academy, together with a long *Addition* that was absent in the original submission. We postpone our analysis of the *Addition* to Section 3.5. We now turn to (Bonnet 1867, pp. 1–31).

Both Bonnet and Bour started by writing down the equations for applicability in a particular coordinate system (symmetric coordinates), for which both the coefficients E, G are vanishing. This choice clearly required some previous knowledge of a particular system of curves upon a given surface. However, neither Bonnet nor Bour regarded this issue as central to their treatment. Bour, for example, stated that he wanted to "put aside all that was not closely related to the question under consideration".[24] Bonnet, on his part, only acknowledged the limitations that this choice implied in his *Addition* (Bonnet 1867, p. 31).

As a consequence of this choice of parametrization, the line element of a surface $\tilde{\Sigma}$,

[20] In this respect, see (Lützen 1990, pp. 223–224).

[21] A few years later, a delicate aspect of this equation was singled out (Weingarten 1884): it may admit real solutions that do not describe real surfaces. For this reason, in (Weingarten 1897) Weingarten proposed an alternative equation to describe those surfaces that are applicable to a given one. Unlike Bour and Bonnet's analytical characterisation, Weingarten's equation appeared to fit better with the geometrical problem that it describes. This achievement earned him the *Grand Prix de Mathématique* awarded by the Paris Academy in 1894.

[22] (Bonnet 1865, p. 210).

[23] See (Bonnet 1844, p. 981) and (Bonnet 1848, p. 81).

[24] (Bour 1862b, p. 5).

$\tilde{\boldsymbol{x}} : U \to \mathbb{R}^3$ can be written as[25]

$$d\tilde{s}^2 = 4\lambda(u,v)dudv \tag{3.1}$$

where $\lambda(u,v)$ denotes a sufficiently regular function of the variables u, v.

For a surface $\tilde{\Sigma}$ with line element (3.1) being given, one wants to provide a characterisation of those surfaces Σ that are locally isometric to $\tilde{\Sigma}$. To this end, Bonnet considered a surface Σ that was parametrized by means of the function $\boldsymbol{x} : U \to \mathbb{R}^3$, $\boldsymbol{x} = (x(u,v), y(u,v), z(u,v))$,[26] whose line element is:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

Because u, v can be regarded also as local coordinates for Σ, it is clear that $\tilde{\Sigma}$ and Σ are applicable if and only if the conditions $E = G = 0$ and $F = 2\lambda$ hold true; that is, if and only if the three functions $x(u,v), y(u,v), z(u,v)$ satisfy the equations:

$$\begin{cases} \left(\dfrac{\partial x}{\partial u}\right)^2 + \left(\dfrac{\partial y}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial u}\right)^2 = 0 \\ \left(\dfrac{\partial x}{\partial v}\right)^2 + \left(\dfrac{\partial y}{\partial v}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2 = 0 \\ \dfrac{\partial x}{\partial u}\dfrac{\partial x}{\partial v} + \dfrac{\partial y}{\partial u}\dfrac{\partial y}{\partial v} + \dfrac{\partial z}{\partial u}\dfrac{\partial z}{\partial v} = 2\lambda. \end{cases} \tag{3.2}$$

Thus, the entire problem of determining all surfaces applicable to Σ boils down to the resolution of (3.2). Bonnet and Bour provided a reformulation of (3.2) by using two different but essentially equivalent substitutions. In this respect, we will follow Bonnet's treatment. He introduced functions $m, m', n, n' : U \to \mathbb{C}$, which are defined by the following relations:

$$\begin{array}{lll} \dfrac{\partial x}{\partial u} = i(m^2 + n^2) & \dfrac{\partial y}{\partial u} = m^2 - n^2 & \dfrac{\partial z}{\partial u} = 2mn \\ \dfrac{\partial x}{\partial v} = i(m'^2 + n'^2) & \dfrac{\partial y}{\partial v} = m'^2 - n'^2 & \dfrac{\partial z}{\partial v} = 2m'n'. \end{array} \tag{3.3}$$

Clearly, the first two equations of (3.2) are identically satisfied as a consequence of (3.3); while the third of (3.2) implies:

$$mn' - nm' = i\sqrt{\lambda}. \tag{3.4}$$

In addition, some integrability conditions are required to guarantee the existence of a surface that is described by $\boldsymbol{x}$ satisfying (3.3). Upon derivation of the right-hand sides of

[25] To draw a comparison between Bonnet's and Bour's solutions we replace $\varphi^2(u,v)$, the notation employed by Bonnet, by $\lambda(u,v)$.

[26] The coordinates u, v can be regarded as local coordinates of both $\tilde{\Sigma}$ and Σ. To this end, it is sufficient to choose $\boldsymbol{x} = \Phi \circ \tilde{\boldsymbol{x}}$, where $\Phi : \tilde{\Sigma} \to \Sigma$ denotes a local isometry between $\tilde{\Sigma}$ and Σ.

(3.3), after a few calculations that take into account (3.4), Bonnet could obtain:

$$m\frac{\partial m}{\partial v} = m'\frac{\partial m'}{\partial u} \tag{3.5a}$$

$$\frac{\partial}{\partial u}\left(\frac{n}{m}\right) = -\frac{i}{m'}\frac{\partial(\sqrt{\lambda}/m)}{\partial u} \tag{3.5b}$$

$$\frac{\partial}{\partial v}\left(\frac{n}{m}\right) = -\frac{i\sqrt{\lambda}}{m'}\frac{\partial(1/m)}{\partial v}. \tag{3.5c}$$

As a consequence of (3.5a) m^2 and m'^2 can be regarded as the partial derivatives of a function $z = z(u, v)$. Thereby, Bonnet could introduce the following definitions:

$$m^2 = p := \frac{\partial z}{\partial u} \qquad m'^2 = q := \frac{\partial z}{\partial v}$$

When p, q are substituted into (3.5b) and (3.5c) one obtains

$$\frac{\partial}{\partial u}\left(\frac{n}{m}\right) = -\frac{i\sqrt{\lambda}}{\sqrt{q}}\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{p}}\right) - \frac{i}{\sqrt{q}\sqrt{p}}\frac{\partial\sqrt{\lambda}}{\partial u}$$

$$\frac{\partial}{\partial v}\left(\frac{n}{m}\right) = -\frac{i\sqrt{\lambda}}{\sqrt{q}}\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{p}}\right).$$

Since one must have:

$$\frac{\partial^2}{\partial v\partial u}\left(\frac{n}{m}\right) = \frac{\partial^2}{\partial u\partial v}\left(\frac{n}{m}\right),$$

Bonnet finally arrived at:[27]

$$\lambda(rt - s^2) - \frac{\partial\lambda}{\partial v}qr - \frac{\partial\lambda}{\partial u}pt + 2pq\frac{\partial^2\lambda}{\partial u\partial v} - \frac{pq}{\lambda}\frac{\partial\lambda}{\partial u}\frac{\partial\lambda}{\partial v} = 0 \tag{3.6}$$

where $r := \frac{\partial p}{\partial u}$, $t := \frac{\partial q}{\partial v}$ and $s := \frac{\partial p}{\partial v} = \frac{\partial q}{\partial u}$, according to classical notation.

In this way, the initial problem could be traced back to the integration of (3.6), which is a nonlinear second-order partial differential equation of Monge–Ampère type. When a solution $z(u, v)$ to (3.6) can actually be exhibited, this can be employed to determine both m and m', as a consequence of the definition of p, q, and also both n and n' by means of (3.5b), (3.5c) and (3.4).

Once the functions m, m', n, n' have been obtained, a surface Σ applicable to $\tilde{\Sigma}$ is given by quadratures upon integration of the system:

$$\begin{cases} dx = i(m^2 + n^2)du + i(m'^2 + n'^2)dv \\ dy = (m^2 - n^2)du + (m'^2 - n'^2)dv \\ dx = 2mndu + 2m'n'dv. \end{cases}$$

For the possibility of integrating equation (3.6), Bonnet remarked:[28]

[27] This equation coincides with Bonnet's equation (9) in (Bonnet 1867, p. 3), provided one substitutes λ with φ^2.

[28] A general discussion of the integrability of (3.6) by means of the methods due to Monge and Ampère is provided in (Forsyth 1920, §220)

> This integration is unaffordable in the general case; but a few simple cases, for which the result is easy to predict, allow the method to be verified and at the same time show how the calculations are completed.[29]

Indeed, in accordance with the question posed by the Academy, he turned to discussing particular cases, such as developable surfaces[30] (Bonnet 1867, pp. 4–7) and minimal surfaces (Bonnet 1867, pp. 7–9).

Finally, in (Bonnet 1867, pp. 29–31), after determining the lines of curvature, the asymptotic lines and the principal radii of curvature in terms of $z(u, v)$ and $\lambda(u, v)$, he demonstrated a theorem, later known as *Bonnet's theorem*, according to which it is impossible to deform a surface Σ in such a way that ∞^1 asymptotic lines remain asymptotic under deformation, unless Σ is a ruled surface whose asymptotic lines are rectilinear generators.

As for Bour's treatment, after deducing equation (3.6), he remarked significantly that:

> It is curious that one can obtain a second-order partial differential equation endowed with such a great generality. This equation has, moreover, an extremely simple form; it offers the solution to questions relating to the curvature of surfaces, as we will show by some applications.[31]

Since Bour's memoir was published first, the equation (3.6) was commonly named *Bour's equation*.

Before dealing with specific applications of equation (3.6), Bour expounded an alternative treatment of the problem. As will be seen later, a deduction of the MCE was proposed in the course of this exposition.

3.4 The fundamental equations

Bour's second approach to the applicability problem was based on the introduction of three geometrical quantities and the deduction of as many equations, which Bour named *fundamental equations*. They may today be interpreted as the MCE associated with the two fundamental forms of a given surface. At the same time, Codazzi—and Mainardi[32] before him—also came up with essentially equivalent relations in (Codazzi 1883, p. 6). As will be seen, Codazzi wrote his equations in a slightly more general coordinate system than that adopted by Bour. Therefore, this section is devoted to a thorough analysis of (Bour 1862b, pp. 17–24) and (Codazzi 1883, pp. 2–9).

[29] "Cette intégration est inabordable dans le cas général; mais quelques cas simples, pour lesquels le résultat est facile a prévoir, permettent de vérifier la méthode et montrent en meme temps comment les calculs s'achèvent." (Bonnet 1865, p. 211)

[30] In this case, (3.6) reduces to the well-known form $rt - s^2 = 0$.

[31] "il est même curieux que l'on puisse obtenir une équation aux différentielles partielles du second ordre, douée d'une aussi grande généralité. Cette équation a d'ailleurs une forme extrêmement simple; elle se prete très-bien à la solution des questions relatives la courbure des surfaces, comme nous le montrerons par quelques applications." (Bour 1862b, p. 6)

[32] (Mainardi 1856, p. 395).

3.4.1 Bour's second solution

Bour emphasised the importance of his second solution by proudly claiming that it could be regarded as completing Gauss' theory. In his words:

> They [his fundamental equations] express analytically, in the simplest form, all the properties that concern the curvature of any surface and the variation of the different elements of this curvature. They will replace advantageously, in a large number of studies, the ordinary equations of the theory of surfaces. [...] I can say that these equations, which I appropriately dubbed *fundamental*, encompass all the theoretical part of my memoir; it is from these relations that all my results, both analytical and geometrical, are drawn.[33]

A key role in Bour's treatment was played by a system of geodesic polar coordinates (r, θ). As Gauss proved in the *Disquisitiones* en route to the proof of his version of the Gauss-Bonnet theorem, in this coordinate system the line element of a surface Σ can be written as follows:

$$ds^2 = dr^2 + g^2(r,\theta)d\theta^2. \tag{3.7}$$

At each point of Σ, Bour considered the tangent unit coordinate vectors and the normal unit vector, which are given, respectively, by:

$$\boldsymbol{x}_r = (\cos\lambda_1, \cos\lambda_2, \cos\lambda_3) \quad \frac{1}{g}\boldsymbol{x}_\theta = (\cos\mu_1, \cos\mu_2, \cos\mu_3) \quad \boldsymbol{N} = (\cos N_1, \cos N_2, \cos N_3).$$

Bour's method consisted in deducing a system of equations for the first-order derivatives of the functions $\boldsymbol{x}_r, \boldsymbol{x}_\theta, \boldsymbol{N}$, which turned out to be expressible in terms of three auxiliary functions only: $T(r,\theta), H(r,\theta), H_1(r,\theta)$. The system can be written for $i = 1, 2, 3$ as follows:

$$\begin{cases} \dfrac{\partial \cos\mu_i}{\partial r} = -T\cos N_i & \dfrac{1}{g}\dfrac{\partial\cos\mu_i}{\partial\theta} = H\cos N_i - \dfrac{g_r}{g}\cos\lambda_i \\ \dfrac{\partial\cos\lambda_i}{\partial r} = H_1\cos N_i & \dfrac{1}{g}\dfrac{\partial\cos\lambda_i}{\partial\theta} = -T\cos N_i + \dfrac{g_r}{g}\cos\mu_i \\ \dfrac{\partial\cos N_i}{\partial r} = T\cos\mu_i - H_1\cos\lambda_i & \dfrac{1}{g}\dfrac{\partial\cos N_i}{\partial\theta} = -H\cos\mu_i + T\cos\lambda_i \end{cases} \tag{3.8}$$

Conversely, if functions $g(r,\theta), T(r,\theta), H(r,\theta), H_1(r,\theta)$ are appropriately given, then the system (3.8) can be thought of as providing the defining equation of a surface Σ' whose line element coincides with (3.7). In this respect, a crucial point was represented by the

[33]"Elles [his fundamental equations] expriment analytiquement, sous la forme la plus simple, toutes les propriétés qui sont relatives à la courbure des surfaces quelconques, et à la variation des différents éléments de cette courbure. Elles remplaceront avantageusement, dans un grand nombre de recherches, les équations ordinaires de la théorie des surfaces. [...] Je puis dire que ces équations, que j'ai nommèes à bon escient *fondamentales*, résument toute la partie théorique de mon Mémoire; c'est de ces relations que sont tirés tous mes résultats, tant analytiques que géométriques." (Bour 1862b, pp. 7–8).

discovery of three differential relations connecting $g(r, \theta), T(r, \theta), H(r, \theta), H_1(r, \theta)$, which can be interpreted as the MCE written in geodesic coordinates.

We now turn to Bour's deduction of the system (3.8). First, it should be pointed out that the commutativity property of cross-derivatives $\boldsymbol{x}_{r\theta} = \boldsymbol{x}_{\theta r}$ implies the following conditions:

$$\begin{cases} \dfrac{1}{g}\dfrac{\partial \cos\lambda_1}{\partial\theta} = \dfrac{\partial \cos\mu_1}{\partial r} + \dfrac{g_r}{g}\cos\mu_1 \\ \dfrac{1}{g}\dfrac{\partial \cos\lambda_2}{\partial\theta} = \dfrac{\partial \cos\mu_2}{\partial r} + \dfrac{g_r}{g}\cos\mu_2 \\ \dfrac{1}{g}\dfrac{\partial \cos\lambda_3}{\partial\theta} = \dfrac{\partial \cos\mu_3}{\partial r} + \dfrac{g_r}{g}\cos\mu_3 \end{cases}$$

where $g_r = \dfrac{\partial g}{\partial r}$. By employing vector notation, which Bour did not use, this system can be written in a more transparent form as:

$$\frac{1}{g}\frac{\partial \boldsymbol{x}_r}{\partial\theta} = \frac{\partial}{\partial r}\left(\frac{\boldsymbol{x}_\theta}{g}\right) + \frac{g_r}{g^2}\boldsymbol{x}_\theta \tag{3.9}$$

Following Bour, we introduce the function T, which is defined as follows:

$$T^2 := \left\| \frac{\partial}{\partial r}\left(\frac{1}{g}\boldsymbol{x}_\theta\right)\right\|^2 = \sum_{i=1}^{3}\left(\frac{\partial \cos\mu_i}{\partial r}\right)^2. \tag{3.10}$$

We have:[34]

$$T\boldsymbol{x}_r = \left(\frac{1}{g}\boldsymbol{x}_\theta\right) \wedge \frac{\partial}{\partial r}\left(\frac{1}{g}\boldsymbol{x}_\theta\right). \tag{3.11}$$

Bour wrote this relation in a more explicit form as:

$$\begin{cases} T\cos\lambda_1 = \cos\mu_2\dfrac{\partial\cos\mu_3}{\partial r} - \cos\mu_3\dfrac{\partial\cos\mu_2}{\partial r} \\ T\cos\lambda_2 = \cos\mu_3\dfrac{\partial\cos\mu_1}{\partial r} - \cos\mu_1\dfrac{\partial\cos\mu_3}{\partial r} \\ T\cos\lambda_3 = \cos\mu_1\dfrac{\partial\cos\mu_2}{\partial r} - \cos\mu_2\dfrac{\partial\cos\mu_1}{\partial r}. \end{cases} \tag{3.12}$$

Starting again from (3.9), upon scalar multiplication by $\dfrac{1}{g}\boldsymbol{x}_\theta$, we get:

$$\frac{1}{g^2}\frac{\partial \boldsymbol{x}_r}{\partial\theta}\cdot\boldsymbol{x}_\theta = \frac{g_r}{g}.$$

[34]Upon scalar multiplication of (3.9) by $\boldsymbol{x}_r$, since $\boldsymbol{x}_r \cdot \boldsymbol{x}_\theta = \boldsymbol{x}_r \cdot \boldsymbol{x}_{r\theta}$=0, it is easily seen that:

$$\boldsymbol{x}_r \cdot \frac{\partial}{\partial r}\left(\frac{1}{g}\boldsymbol{x}_\theta\right) = 0.$$

This means that $\boldsymbol{x}_r \perp \boldsymbol{x}_\theta$ and $\boldsymbol{x}_r \perp \dfrac{\partial}{\partial r}\left(\dfrac{1}{g}\boldsymbol{x}_\theta\right)$. Consequently, because $\|\boldsymbol{x}_r\| = 1$, T can be expressed in terms of the vector product of $\dfrac{1}{g}\boldsymbol{x}_\theta$ and $\dfrac{\partial}{\partial r}\left(\dfrac{1}{g}\boldsymbol{x}_\theta\right)$. Indeed we have:

$$T\boldsymbol{x}_r = \left(\frac{1}{g}\boldsymbol{x}_\theta\right) \wedge \frac{\partial}{\partial r}\left(\frac{1}{g}\boldsymbol{x}_\theta\right).$$

.

Consequently, if (3.11) is taken into account, then one obtains:[35]

$$\det\left[\frac{1}{g}\boldsymbol{x}_\theta \quad \frac{1}{g}\frac{\partial}{\partial\theta}\left(\frac{1}{g}\boldsymbol{x}_\theta\right) \quad \frac{\partial}{\partial r}\left(\frac{1}{g}\boldsymbol{x}_\theta\right)\right] = T\frac{g_r}{g}, \tag{3.13}$$

which is equation (5) of (Bour 1862b, p. 20).

Bour then introduced spherical coordinates ϕ, ψ that allowed him to express the components of the unit vector $\frac{\boldsymbol{x}_\theta}{g}$ as:

$$\frac{\boldsymbol{x}_\theta}{g} = (\cos\phi\sin\psi, \sin\phi\sin\psi, \cos\psi).$$

Accordingly, (3.10) becomes

$$T^2 = \left(\frac{\partial\psi}{\partial r}\right)^2 + \sin^2\psi\left(\frac{\partial\phi}{\partial r}\right)^2.$$

Furthermore, this relation suggests the introduction of a new auxiliary variable ϖ, such that

$$\sin\psi\frac{\partial\phi}{\partial r} = T\cos\varpi \qquad \frac{\partial\psi}{\partial r} = T\sin\varpi.$$

If the new variables ϕ, ψ, ϖ are inserted in (3.12) and (3.13), then one obtains

$$\boldsymbol{x}_r = \begin{bmatrix} -\sin\phi\sin\varpi - \cos\phi\cos\psi\cos\varpi \\ \cos\phi\sin\varpi - \sin\phi\cos\psi\cos\varpi \\ \sin\psi\cos\varpi \end{bmatrix}$$

$$\cos\varpi\frac{\partial\psi}{\partial\theta} - \sin\psi\sin\varpi\frac{\partial\phi}{\partial\theta} = g_r$$

Finally, in light of these positions, Bour could express both the components of the normal unit vector and the function T in terms of the variables ϕ, ψ, ϖ:

$$\boldsymbol{N} = \begin{bmatrix} -\cos\phi\cos\psi\sin\varpi + \sin\phi\cos\varpi \\ -\sin\phi\cos\psi\sin\varpi - \cos\phi\cos\varpi \\ \sin\psi\sin\varpi. \end{bmatrix}$$

[35] A detailed verification of this equation goes as follows:

$$\begin{aligned}
\det\left[\frac{1}{g}\boldsymbol{x}_\theta \quad \frac{1}{g}\frac{\partial}{\partial\theta}\left(\frac{1}{g}\boldsymbol{x}_\theta\right) \quad \frac{\partial}{\partial r}\left(\frac{1}{g}\boldsymbol{x}_\theta\right)\right] &= -\frac{1}{g}\frac{\partial}{\partial\theta}\left(\frac{1}{g}\boldsymbol{x}_\theta\right)\cdot\left[\frac{1}{g}\boldsymbol{x}_\theta \wedge \frac{\partial}{\partial r}\left(\frac{1}{g}\boldsymbol{x}_\theta\right)\right] \\
&= -\frac{1}{g}\frac{\partial}{\partial\theta}\left(\frac{1}{g}\boldsymbol{x}_\theta\right)\cdot(T\boldsymbol{x}_r) \\
&= -\frac{T}{g}\frac{\partial}{\partial\theta}\left(\frac{1}{g}\boldsymbol{x}_\theta\right)\cdot\boldsymbol{x}_r \\
&= -\frac{T}{g}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{g}\underbrace{\boldsymbol{x}_\theta\cdot\boldsymbol{x}_r}_{=0}\right) - \frac{1}{g}\boldsymbol{x}_\theta\cdot\frac{\partial\boldsymbol{x}_r}{\partial\theta}\right] \\
&= T\frac{1}{g^2}\boldsymbol{x}_\theta\cdot\frac{\partial\boldsymbol{x}_r}{\partial\theta} \\
&= T\frac{g_r}{g}.
\end{aligned}$$

$$\frac{1}{g}\frac{\partial \varpi}{\partial \theta} - \cos\psi \frac{1}{g}\frac{\partial \phi}{\partial \theta} = T$$

We can now introduce the two auxiliary quantities H and H_1 that intervene in system (3.8). Bour defined them by means of the following relations:

$$\begin{aligned} &\sin\varpi \frac{1}{g}\frac{\partial \psi}{\partial \theta} + \cos\varpi \sin\psi \frac{1}{g}\frac{\partial \phi}{\partial \theta} = -H \\ &\frac{1}{g}\frac{\partial g_r}{\partial r} = T^2 - HH_1. \end{aligned} \tag{3.15}$$

Apparently, they were introduced for purely analytical purposes. Bour only provided a discussion of their geometrical meaning at a later stage. We now consider this geometrical interpretation, which Bour himself discussed in (Bour 1862b, pp. 24–27). By taking into account Rodrigues' formulas for lines of curvature, Bour deduced the following quadratic equation:

$$\left(\frac{1}{\rho}\right)^2 - (H + H_1)\frac{1}{\rho} + HH_1 - T^2 = 0,$$

whose solutions $\kappa_1 = \dfrac{1}{\rho_1}, \kappa_2 = \dfrac{1}{\rho_2}$ coincide with the principal curvatures of Σ. Consequently, it is straightforward to see that $HH_1 - T^2$ is equal to the Gaussian curvature of Σ. In addition, Bour also pointed out that the angle ω between the coordinate lines $r =$ const. and either of the two system of the lines of curvature can be expressed by means of $\tan 2\omega = \frac{2T}{H_1 - H}$. This is sufficient to identify H and H_1 with the normal curvatures of the coordinate lines and T with the torsion of geodesic lines (θ)—i.e., $\theta =$ const.

In view of the preceding positions and after straightforward computations, he finally arrived at the following equations for ϕ, ψ, ϖ,

$$\begin{cases} \sin\psi \dfrac{\partial \phi}{\partial r} = T\cos\varpi & \sin\psi \dfrac{1}{g}\dfrac{\partial \phi}{\partial \theta} = -H\cos\varpi - \dfrac{g_r}{g}\sin\varpi \\ \dfrac{\partial \psi}{\partial r} = T\sin\varpi & \dfrac{1}{g}\dfrac{\partial \psi}{\partial \theta} = -H\sin\varpi + \dfrac{g_r}{g}\cos\varpi \\ \dfrac{\partial \varpi}{\partial r} = -H_1 + T\cos\varpi\cot\psi & \dfrac{1}{g}\dfrac{\partial \varpi}{\partial \theta} = T - (H\cos\varpi + \dfrac{g_r}{g}\sin\varpi)\cot\psi \end{cases} \tag{3.16}$$

and at system (3.8), which the functions $\boldsymbol{x}(r, \theta)$ must satisfy for $i = 1, 2, 3$.

If the functions g, H, H_1, T are assigned, then every solution to this system (whenever it exists) defines a surface whose line element is given by: $ds^2 = dr^2 + g^2(r, \theta)d\theta^2$. Bour observed that, to guarantee the existence of solutions to (3.8), one requires a set of integrability conditions that are obtained by equality of cross-derivatives.

In all, there are 12 integrability conditions of (3.16) and (3.8); only two are independent. If we add equation (3.15), which defines H_1, then we finally obtain Bour's *fundamental equations*:

$$
\begin{cases}
\dfrac{g_{rr}}{g} = T^2 - HH_1 \\
\dfrac{1}{g}\dfrac{\partial T}{\partial \theta} + \dfrac{\partial H}{\partial r} + \dfrac{g_r}{g}(H - H_1) = 0 \\
\dfrac{1}{g}\dfrac{\partial H_1}{\partial \theta} + \dfrac{\partial T}{\partial r} + 2\dfrac{g_r}{g}T = 0.
\end{cases}
\tag{3.17}
$$

System (3.17) actually provides an answer to the problem posed by the Academy. Indeed, as Bour asserted, to each set of solutions (H, H_1, T)[36] there correspond surfaces in Euclidean space that are applicable to a surface with line element $ds^2 = dr^2 + g^2(r, \theta)d\theta^2$. Nonetheless, it is important to emphasise that the result according to which the assignment of the functions $g(r, \theta), H, H_1, T$ is sufficient for the determination of a surface, remained implicit. Neither did Bour seem to be interested in investigating uniqueness issues for solutions of the system (3.8). Probably, these results were simply taken for granted. The following passage from (Bour 1862b, pp. 23–24) helps us evaluate the scope of his analysis and research priority:

> It is essential to notice beforehand that all will not be done when one has integrated my fundamental equations. Other calculations will still be necessary to obtain the finite equations of the sought after surfaces; for example, the integration of the system (13) [(3.16)], or that of the system (14) [(3.8)] as the case may be. But, as I said and I cannot repeat it enough, this classification of difficulties into separate groups that can be tackled one after another is precisely the essence of the method I am explaining. As soon as we have found for H, H_1, T, a system of values satisfying equations (12) and (16) [they correspond to the system (3.17)], we can consider the problem as solved, since we will have the analytical and geometric definition of a surface or of a series of surfaces that can be developed on the proposed surface. As to the problem consisting of writing down the equations of these surfaces in the usual way, this is a minor question.[37]

[36] H, H_1, T can be easily written in terms of the coefficients of the second fundamental form D, D', D''. When a surface S is parametrised by means of $\boldsymbol{x} = \boldsymbol{x}(u, v)$ and $\boldsymbol{X} = \boldsymbol{X}(u, v)$ is its normal vector field, D, D', D'' are defined as

$$
D = \boldsymbol{X} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial u^2} \qquad D' = \boldsymbol{X} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \qquad D'' = \boldsymbol{X} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial v^2}.
$$

One can easily verify that $H = \dfrac{D''}{g^2}, H_1 = D, T = \dfrac{D'}{g}$.

[37] "Il est bien essentiel auparavant de remarquer que tout ne sera pas fait quand on aura intégré mes équations fondamentales. D'autres calculs seront encore nécessaires pour avoir les équations finies des surfaces cherchées; par exemple, l'intégration du système (13), ou celle du système (14) selon le cas.

Mais, je l'ai dit et je ne saurais trop le répéter, cette classification des difficultés en groupes séparés qu'on peut attaquer l'un après l'autre, constitue précisément l'essence de la méthode que j'expose. Dès qu'on aura trouvé pour H, H_1, T, un système de valeurs satisfaisant aux équations (12) et (16), on pourra considérer le problème comme résolu, puisqu'on aura la définition analytique et géométrique d'une surface ou d'une série de surfaces développables sur la surface proposée. Quant à mettre les équations de ces

An explicit determination of the finite equations for surfaces that are applicable to a given one (i.e., a determination of functions $(u, v) \mapsto \boldsymbol{x}(u, v)$) was carried out in some particular cases. A noteworthy example was offered by Bour's treatment of the second applicability problem for surfaces of revolution. Interestingly, en route to it, he provided a proof of the following result, which assures the existence of a whole family of helicoids applicable to a given rotation surface:

Theorem 3.1. *By quadratures, one can always find helicoids* $\tilde{\Sigma}_{n,h}$*:*

$$\begin{cases} x(u,v) = \rho \cos \Omega, \\ y(u,v) = \rho \sin \Omega, \\ z(u,v) = \dfrac{h}{2\pi}\Omega + w; \end{cases} \tag{3.18}$$

where $\Omega := n(v+\omega)$*, such that* $\tilde{\Sigma}_{n,h}$ *is applicable to a given rotation surface* $\Sigma : ds^2 = du^2 + G(u)dv^2$*. Here,* n *and* h *are constant parameters, and* $\rho(u), \omega(u), w(u)$ *denote functions of the variable* u *only.*

As Bour himself explained, he was led to this result through a direct application of his second method to the following problem: to determine all surfaces applicable to a given rotation surface, such that their functions H, H_1, T depend upon the variable u only. This additional condition introduced a noteworthy simplification in equations (3.17) that could thereby promptly be integrated. However, because the integration of systems (3.8) appeared to be too laborious, Bour preferred to convey an alternative, more viable proof of Theorem 3.1 that started with the equality

$$dx^2 + dy^2 + dz^2 = du^2 + G(u)dv^2,$$

where the functions $\boldsymbol{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ are defined by (3.18), and showed that the differential system thereby obtained admits solutions. Interestingly, Bour proved that the degree of generality of these solutions was sufficiently high to provide a *complete* solution to the problem. By this, he implicitly referred to a notion of generality for solutions to PDEs dating back to Lagrange that consisted in attributing a privileged role to solutions depending on arbitrary parameters. In the case of second-order equations, such as the present one, a complete integral depends upon five arbitrary parameters. The solutions obtained by Bour belong precisely to this category. Indeed, besides the parameters n and h, whose existence is guaranteed by the above theorem, Bour explicitly remarked that one should take into account three additional parameters corresponding to the different orientations of the surface in Euclidean space (clearly, one has to consider also three further parameters corresponding to translations, but these were regarded as not affecting the generality of the solutions).

The complete integral that is thus obtained represented the starting point of a much more elaborate procedure (which was highly appreciated by the prize commission in the

surfaces sous la forme ordinaire, ce n'est plus là évidemment qu'une question purement accessoire [...]." (Bour 1862b, pp. 23–24) previously quoted report) by means of which Bour could obtain a general solution (i.e., a solution depending on arbitrary functions) to the problem of applicability for revolution theory. Unfortunately, as we know, the details of Bour's methods, although contained in the original submission, were not incorporated in (Bour 1862b).

3.4.2 Codazzi's fundamental equations

Codazzi's choice to join the prize competition on the theme of applicability theory appears as a natural one if one considers his research interests in the late-1850s. Codazzi was at that time a professor of mathematics at the "Imperial Regio Ginnasio Liceale" in Pavia. A few years before, he had published a number of memoirs in the Italian journal *Annali di Scienze Matematiche e Fisiche* that specifically dealt with surface theory, and applicability problems in particular. Among these papers, which appeared to be profoundly influenced by the works of French mathematicians such as Monge, Dupin and Liouville, (Codazzi 1856) occupies a prominent position. Therein, Codazzi tackled the problem of determining which types of surfaces admit isometric deformations that preserve lines of curvature: it turned out that any such surface is either a developable surface or a surface for which all lines of curvature lie on parallel planes.

Another noteworthy contribution to surface theory that dates to the same period can be found in (Codazzi 1857). With hindsight, it might be considered a step forward in the reception process of non-Euclidean geometry. Indeed, it consisted of a study of trigonometric formulas for geodetic triangles that are drawn upon a surface with negative constant curvature. Beltrami would explicitly refer to it in his celebrated *Saggio* (1868) when discussing the trigonometric formulas in Lobačevskij geometry and its realisation on a pseudosphere.

The solution to the prize competition that Codazzi sent to the Academy of Sciences in Paris stood out among those of Bour and Bonnet because of the generality of the coordinate system that he adopted.[38] In (Codazzi 1883, pp. 2–7) he wrote a set of equations that are essentially equivalent to Bour's (3.17). Nonetheless, unlike Bour, Codazzi required no other condition except for orthogonality (i.e., $F = 0$) of the coordinate lines. In addition, equations that are also valid for a completely general coordinate system were discussed in a short note added at the end of the manuscript submitted for the prize competition.

Crucial to Codazzi's treatment was his use of the Frenet formulas for the coordinate lines of a surface, allowing him to introduce auxiliary functions that are essentially equivalent to Bour's functions: H, H_1, T.

Codazzi based his solution on the consideration of an orthogonal system of coordinate lines upon a surface Σ with finite equations $\boldsymbol{x} = (x(u,v), y(u,v), z(u,v))$. Accordingly, he wrote the line element as

$$ds^2 = Edu^2 + Gdv^2.$$

[38] Codazzi did not explicitly emphasise the generality of his contribution because he did not have his colleagues' results as a benchmark. Bonnet was the first to appreciate this aspect, which prompted him to revise his own work.

First, he introduced two *differently oriented* Frenet frames for curves $v = \text{cost}, u = \text{cost}$ (Figure 3.1)—we will refer to these curves as $(v)-$curves and $(u)-$curves, respectively.[39] If $\boldsymbol{t}_{(v)}, \boldsymbol{n}_{(v)}, \boldsymbol{b}_{(v)}$ and $\boldsymbol{t}_{(u)}, \boldsymbol{n}_{(u)}, \boldsymbol{b}_{(u)}$ denote the tangent, normal and binormal unit vectors to $(v)-$curves and $(u)-$curves respectively[40], then one has, by definition:

$$\boldsymbol{t}_{(v)} = \frac{\boldsymbol{x}_u}{||\boldsymbol{x}_u||} = \frac{\boldsymbol{x}_u}{\sqrt{E}}, \qquad \boldsymbol{n}_{(v)} = \frac{\boldsymbol{x}_{uu}}{||\boldsymbol{x}_{uu}||}, \qquad \boldsymbol{b}_{(v)} = -\boldsymbol{t}_{(v)} \wedge \boldsymbol{n}_{(v)};$$
$$\boldsymbol{t}_{(u)} = \frac{\boldsymbol{x}_v}{||\boldsymbol{x}_v||} = \frac{\boldsymbol{x}_v}{\sqrt{G}}, \qquad \boldsymbol{n}_{(u)} = \frac{\boldsymbol{x}_{vv}}{||\boldsymbol{x}_{vv}||}, \qquad \boldsymbol{b}_{(u)} = \boldsymbol{t}_{(u)} \wedge \boldsymbol{n}_{(u)}.$$

To write down Frenet equations, Codazzi resorted to the notion of the angle of contingence and torsion. We adopt a more familiar notation, thus denoting with $1/\rho_{(u)}$ and $1/\rho_{(v)}$ the curvatures of (u)-curves and (v)-curves, respectively. Similarly, we denote their torsions with $1/T_{(u)}$ and $1/T_{(v)}$. Consequently, the Frenet equations are written as follows:

$$\frac{\partial \boldsymbol{t}_{(v)}}{\partial u} = \boldsymbol{n}_{(v)}\frac{\sqrt{E}}{\rho_{(v)}} \qquad \frac{\partial \boldsymbol{n}_{(v)}}{\partial u} = -\boldsymbol{t}_{(v)}\frac{\sqrt{E}}{\rho_{(v)}} + \boldsymbol{b}_{(v)}\frac{\sqrt{E}}{T_{(v)}} \qquad \frac{\partial \boldsymbol{b}_{(v)}}{\partial u} = -\boldsymbol{n}_{(v)}\frac{\sqrt{E}}{T_{(v)}}$$
$$\frac{\partial \boldsymbol{t}_{(u)}}{\partial v} = \boldsymbol{n}_{(u)}\frac{\sqrt{G}}{\rho_{(u)}} \qquad \frac{\partial \boldsymbol{n}_{(u)}}{\partial v} = -\boldsymbol{t}_{(u)}\frac{\sqrt{G}}{\rho_{(u)}} - \boldsymbol{b}_{(u)}\frac{\sqrt{G}}{T_{(u)}} \qquad \frac{\partial \boldsymbol{b}_{(u)}}{\partial v} = +\boldsymbol{n}_{(u)}\frac{\sqrt{G}}{T_{(u)}}.$$

If $\boldsymbol{N} = \boldsymbol{t}_{(v)} \wedge \boldsymbol{t}_{(u)} = (N_1, N_2, N_3)$ is the normal unit vector to the surface and ω, Ω denote the angles between the normal to the surface and the principal normal of the $(v)-$curves and $(u)-$curves respectively, then we have:

$$\begin{aligned} \boldsymbol{t}_{(v)} &= \boldsymbol{b}_{(u)} \cos\Omega - \boldsymbol{n}_{(u)} \sin\Omega & \qquad \boldsymbol{t}_{(u)} &= \boldsymbol{b}_{(v)} \cos\omega - \boldsymbol{n}_{(v)} \sin\omega \\ \boldsymbol{n}_{(v)} &= \boldsymbol{N} \cos\omega - \boldsymbol{t}_{(u)} \sin\omega & \boldsymbol{n}_{(u)} &= \boldsymbol{N} \cos\Omega - \boldsymbol{t}_{(v)} \sin\Omega \\ \boldsymbol{b}_{(v)} &= \boldsymbol{N} \sin\omega + \boldsymbol{t}_{(u)} \cos\omega & \boldsymbol{b}_{(u)} &= \boldsymbol{N} \sin\Omega + \boldsymbol{t}_{(v)} \cos\Omega \\ \boldsymbol{N} &= \boldsymbol{b}_{(v)} \sin\omega + \boldsymbol{n}_{(v)} \cos\omega & \boldsymbol{N} &= \boldsymbol{b}_{(u)} \sin\Omega + \boldsymbol{n}_{(u)} \cos\Omega. \end{aligned}$$

At this point, Codazzi introduced six auxiliary quantities, j, J, k, K, w, W[41] by means of which he could characterise the immersion of a surface in Euclidean space. They were defined as follows:

$$\begin{aligned} j &:= \frac{\sqrt{E}}{\rho_{(v)}} \cos\omega, \quad k := \frac{\sqrt{E}}{\rho_{(v)}} \sin\omega, \quad w := -\frac{\sqrt{E}}{T_v} - \frac{\partial \omega}{\partial u}, \\ J &:= \frac{\sqrt{G}}{\rho_{(u)}} \cos\Omega, \quad K := \frac{\sqrt{G}}{\rho_{(u)}} \sin\Omega, \quad W := +\frac{\sqrt{G}}{T_u} - \frac{\partial \Omega}{\partial v}, \end{aligned} \tag{3.19}$$

Codazzi did not explicitly discuss the geometrical meaning of these functions. However, one can easily see that they are related to well-known geometrical quantities. Clearly, j, J

[39] It should be pointed out that this choice for orientation was not explicitly stated by Codazzi. We inferred it by requiring consistency in his formulas.

[40] We adopt a vector notation, which Codazzi did not employ, for the sake of simplicity. Subscripts u or v denote differentiation with respect to u, v.

[41] For the sake of clarity, we have modified Codazzi's notation (Codazzi 1883, p. 4) by posing $u = j, U = J, v = k, V = K$.

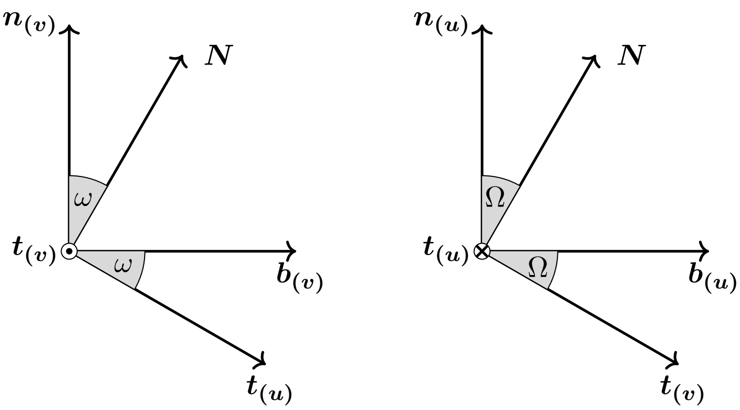

Figure 3.1: The vector $\boldsymbol{t}_{(v)}$ points out of the front of the diagram, toward the viewer. The vector $\boldsymbol{t}_{(u)}$ points in the opposite direction.

and k, K are connected to the normal and the geodesic curvatures. Furthermore, w and W can be interpreted in terms of the geodesic torsions of $(v)-$curves and $(u)-$curves, respectively.

Upon derivation of $\boldsymbol{N}, \boldsymbol{t}_{(v)}, \boldsymbol{t}_{(u)}$ with respect to u and v and insertion of (3.19), he obtained

$$\begin{cases} \boldsymbol{N}_u = -j\boldsymbol{t}_{(v)} - w\boldsymbol{t}_{(u)} & \dfrac{\partial \boldsymbol{t}_{(v)}}{\partial v} = K\boldsymbol{t}_{(u)} + W\boldsymbol{N} & \dfrac{\partial \boldsymbol{t}_{(u)}}{\partial v} = J\boldsymbol{N} - K\boldsymbol{t}_{(v)} \quad (3.20) \\ \boldsymbol{N}_v = -J\boldsymbol{t}_{(u)} - W\boldsymbol{t}_{(v)} & \dfrac{\partial \boldsymbol{t}_{(v)}}{\partial u} = j\boldsymbol{N} - k\boldsymbol{t}_{(u)} & \dfrac{\partial \boldsymbol{t}_{(u)}}{\partial u} = k\boldsymbol{t}_{(v)} + w\boldsymbol{N}. \quad (3.21) \end{cases}$$

The commutative property of the cross-derivatives

$$\boldsymbol{N}_{uv} = \boldsymbol{N}_{vu} \qquad \frac{\partial^2 \boldsymbol{t}_{(v)}}{\partial u \partial v} = \frac{\partial^2 \boldsymbol{t}_{(v)}}{\partial v \partial u} \qquad \frac{\partial^2 \boldsymbol{t}_{(u)}}{\partial u \partial v} = \frac{\partial^2 \boldsymbol{t}_{(u)}}{\partial v \partial u}$$

gives:

$$\boldsymbol{N}\left(\frac{\partial W}{\partial u} - \frac{\partial j}{\partial v} + kJ + wK\right) + \boldsymbol{t}_{(u)}\left(\frac{\partial K}{\partial u} + \frac{\partial k}{\partial v} + jJ - wW\right) = 0$$
$$\boldsymbol{N}\left(\frac{\partial w}{\partial v} - \frac{\partial J}{\partial u} + Kj + Wk\right) + \boldsymbol{t}_{(v)}\left(\frac{\partial K}{\partial u} + \frac{\partial k}{\partial v} + jJ - wW\right) = 0.$$

This in turn implies:[42]

$$\begin{cases} \dfrac{\partial k}{\partial v} + \dfrac{\partial K}{\partial u} + jJ - wW = 0 \\ \dfrac{\partial j}{\partial v} = \dfrac{\partial W}{\partial u} + wK + kJ \\ \dfrac{\partial J}{\partial u} = \dfrac{\partial w}{\partial v} + Kj + Wk. \end{cases} \tag{3.22}$$

To simplify (3.22), Codazzi took into account the equation $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$, where $\boldsymbol{x}_v = \boldsymbol{t}_{(u)}\sqrt{G} \quad \boldsymbol{x}_u = \boldsymbol{t}_{(v)}\sqrt{E}$. Easy computations give

$$\sqrt{E}W = \sqrt{G}w \qquad k = \frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v} \qquad K = \frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}. \tag{3.23}$$

[42]This is a consequence of the fact that $\boldsymbol{N}, \boldsymbol{t}_{(u)}$ and $\boldsymbol{N}, \boldsymbol{t}_{(v)}$ are linearly independent vectors.

We can easily see that the first relation simply states that the geodesic torsions of the coordinate lines are equal and opposite in sign. Finally, in light of these positions, Codazzi arrived at the following set of three equations:

$$\begin{cases} wW = \dfrac{\partial}{\partial v}\left(\dfrac{1}{\sqrt{G}}\dfrac{\partial\sqrt{E}}{\partial v}\right) + \dfrac{\partial}{\partial u}\left(\dfrac{1}{\sqrt{E}}\dfrac{\partial\sqrt{G}}{\partial u}\right) + jJ \\ \dfrac{\partial j}{\partial v} = \dfrac{\partial W}{\partial u} + \dfrac{J}{\sqrt{G}}\dfrac{\partial\sqrt{E}}{\partial v} + \dfrac{w}{\sqrt{E}}\dfrac{\partial\sqrt{G}}{\partial u} \\ \dfrac{\partial J}{\partial u} = \dfrac{\partial w}{\partial v} + \dfrac{j}{\sqrt{E}}\dfrac{\partial\sqrt{G}}{\partial u} + \dfrac{W}{\sqrt{G}}\dfrac{\partial\sqrt{E}}{\partial v}. \end{cases} \tag{3.24}$$

The first equation is a restatement of Gauss' equation for curvature

$$\mathcal{K} = \frac{DD'' - D'^2}{EG} = -\frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)\right\}.$$

The remaining relations are a slightly more general version of Bour's fundamental equations. Indeed, they can be expressed in terms of the coefficients E, G, D, D', D'' as follows:[43]

$$\begin{cases} \dfrac{\partial}{\partial v}\left(\dfrac{D}{\sqrt{E}}\right) - \dfrac{\partial}{\partial u}\left(\dfrac{D'}{\sqrt{E}}\right) = \dfrac{D''}{G}\dfrac{\partial\sqrt{E}}{\partial v} + \dfrac{D'}{\sqrt{EG}}\dfrac{\partial\sqrt{G}}{\partial u} \\ \dfrac{\partial}{\partial u}\left(\dfrac{D''}{\sqrt{G}}\right) - \dfrac{\partial}{\partial v}\left(\dfrac{D'}{\sqrt{G}}\right) = \dfrac{D}{E}\dfrac{\partial\sqrt{G}}{\partial u} + \dfrac{D'}{\sqrt{EG}}\dfrac{\partial\sqrt{E}}{\partial v}. \end{cases}$$

Codazzi himself described a particularisation of (3.24) to geodesic polar lines (as Bour did) and curvature lines that he exploited for studying applicability on special types of surfaces. He provided a detailed treatment of the applicability problem for ruled surfaces and for those surfaces whose lines of curvature lie upon parallel planes. This latter topic, as we know, had been discussed by Codazzi in a previous paper with a more elementary, almost rudimentary, approach that was now superseded by the general standpoint adopted in (Codazzi 1883).

At the time of the submission of his memoir to the Academy of Sciences, a short note[44] was added in which Codazzi explained how equations of type (3.24) could be deduced without any condition on the orthogonality of coordinate lines. By introducing the angle between coordinate curves, he obtained a system of equations that is perfectly equivalent to what we nowadays refer to as the MCE.

3.5 Bonnet 1867: Codazzi's equations and the fundamental theorem

After providing a solution to the problem posed by the Académie, Bonnet reconsidered his work in light of the solutions proposed by Bour and Codazzi. He probably had access to

[43] It can be easily proven that the following relations hold true: $j = \frac{D}{\sqrt{E}}$, $J = \frac{D''}{\sqrt{G}}$, $w = \frac{D'}{\sqrt{G}}$, $W = \frac{D'}{\sqrt{E}}$.

[44] See (Codazzi 1883, pp. 44–45).

the latter's unpublished manuscript.[45] Bonnet's aim was twofold: first, to fill a *véritable lacune* of his previous treatment that could be imputed to his resort to a geodesic coordinate system; and second, to provide a thorough study of the system of equations (3.8) that Bour had regarded as a "minor question". With all the evidence, Bonnet was of a completely different opinion.

The result of these reflections was published in a long additional note appended to (Bonnet 1867).

In this section, we will analyse of (Bonnet 1867, pp. 31–45) where a rigorous proof of the fundamental theorem of surface theory was given for the first time. As will be seen, Codazzi's equations turned out to play an essential role.

3.5.1 An alternative deduction of the MCE

En route to his own derivation of Codazzi's equations, Bonnet introduced a set of auxiliary quantities that are essentially equivalent to j, J, k, K, w, W. At the same time, he introduced a noteworthy modification with respect to Codazzi's treatment by replacing the Frenet frames of the coordinate lines with a moving trihedron (*principal trihedron*) whose axes are given by the coordinate tangent vectors and the normal unit vector.

Similarly to Codazzi, Bonnet represented a given surface by means of an orthogonal parametrization $\Sigma : (u, v) \to \boldsymbol{x}(u, v)$, whose line element is given by $ds^2 = Edu^2 + Gdv^2$. We now introduce some notation to describe his treatment. Let (v) and (u) denote the coordinate curves $\gamma_{v^*} : u \to \boldsymbol{x}(u, v^*)$, $\gamma_{u^*} : v \to \boldsymbol{x}(u^*, v)$; furthermore, let $\boldsymbol{\mathcal{X}}_1 = (\mathcal{X}_1, \mathcal{Y}_1, \mathcal{Z}_1)$, $\boldsymbol{\mathcal{X}}_2 = (\mathcal{X}_2, \mathcal{Y}_2, \mathcal{Z}_2)$, $\boldsymbol{\mathcal{X}}_3 = (\mathcal{X}_3, \mathcal{Y}_3, \mathcal{Z}_3)$ be unit vector fields defined by:

$$\boldsymbol{\mathcal{X}}_1 = \frac{\boldsymbol{x}_u}{\sqrt{E}}, \quad \boldsymbol{\mathcal{X}}_2 = \frac{\boldsymbol{x}_v}{\sqrt{G}}, \quad \boldsymbol{\mathcal{X}}_3 = \frac{\boldsymbol{x}_u}{\sqrt{E}} \wedge \frac{\boldsymbol{x}_v}{\sqrt{G}}. \tag{3.25}$$

At each point of Σ a moving trihedron consisting of the three axes $(\boldsymbol{\mathcal{X}}_1, \boldsymbol{\mathcal{X}}_2, \boldsymbol{\mathcal{X}}_3)$ was considered. To obtain differential relations connecting the components of this frame, Bonnet introduced two infinitely close points $P = \boldsymbol{x}(u, v), P' = \boldsymbol{x}(u + du, v)$ lying upon a curve γ_v. He then defined the quantities[46] ω_{ij} that describe the relative position of the frame $(\boldsymbol{\mathcal{X}}_1(u + du, v), \boldsymbol{\mathcal{X}}_2(u + du, v), \boldsymbol{\mathcal{X}}_3(u + du, v))$ at P' with respect to the frame

[45] This is all the more likely in view of the fact that in 1862 Bonnet had become a member of the Académie. See (Appell 1893, p. 1015).

[46] Here we changed Bonnet's notation by adopting a matrix formalism in an anachronistic way. Bonnet simply defined a, a', a'' ($\omega_{11}, \omega_{12}, \omega_{13}$ in our notation) as the cosines of the angles that $\boldsymbol{\mathcal{X}}_1(u+du, v)$ forms with $\boldsymbol{\mathcal{X}}_1(u, v), \boldsymbol{\mathcal{X}}_2(u, v), \boldsymbol{\mathcal{X}}_3(u, v)$; b, b', b'' ($\omega_{21}, \omega_{22}, \omega_{23}$) as the cosines of the angles that $\boldsymbol{\mathcal{X}}_2(u+du, v)$ forms with $\boldsymbol{\mathcal{X}}_1(u, v), \boldsymbol{\mathcal{X}}_2(u, v), \boldsymbol{\mathcal{X}}_3(u, v)$; and c, c', c'' ($\omega_{31}, \omega_{32}, \omega_{33}$) as the cosines of the angles that $\boldsymbol{\mathcal{X}}_3(u+du, v)$ forms with $\boldsymbol{\mathcal{X}}_1(u, v), \boldsymbol{\mathcal{X}}_2(u, v), \boldsymbol{\mathcal{X}}_3(u, v)$. We gathered the coefficients ω_{ij} in a matrix Ω as in (3.26). As Bonnet proved a few pages later, Ω is the first-order approximation of the rotation mapping a trihedron based at P into another based at P'. In this respect, see Section 3.5.3.

$(\boldsymbol{\mathcal{X}}_1(u,v), \boldsymbol{\mathcal{X}}_2(u,v), \boldsymbol{\mathcal{X}}_3(u,v))$ at P:

$$\boldsymbol{\Omega} = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} := \tag{3.26}$$

$$:= \begin{pmatrix} \boldsymbol{\mathcal{X}}_1(u+du,v)\cdot\boldsymbol{\mathcal{X}}_1(u,v) & \boldsymbol{\mathcal{X}}_1(u+du,v)\cdot\boldsymbol{\mathcal{X}}_2(u,v) & \boldsymbol{\mathcal{X}}_1(u+du,v)\cdot\boldsymbol{\mathcal{X}}_3(u,v) \\ \boldsymbol{\mathcal{X}}_2(u+du,v)\cdot\boldsymbol{\mathcal{X}}_1(u,v) & \boldsymbol{\mathcal{X}}_2(u+du,v)\cdot\boldsymbol{\mathcal{X}}_2(u,v) & \boldsymbol{\mathcal{X}}_2(u+du,v)\cdot\boldsymbol{\mathcal{X}}_3(u,v) \\ \boldsymbol{\mathcal{X}}_3(u+du,v)\cdot\boldsymbol{\mathcal{X}}_1(u,v) & \boldsymbol{\mathcal{X}}_3(u+du,v)\cdot\boldsymbol{\mathcal{X}}_2(u,v) & \boldsymbol{\mathcal{X}}_3(u+du,v)\cdot\boldsymbol{\mathcal{X}}_3(u,v) \end{pmatrix}$$

Without providing any further detail,[47] Bonnet simply stated that the following formulas for $\omega_{ij}, i,j = 1,2,3$, hold true[48]:

$$\begin{aligned} &\omega_{11} = 1 && \omega_{12} = Mdu, && \omega_{13} = Pdu, \\ &\omega_{21} = -Mdu && \omega_{22} = 1, && \omega_{23} = -Rdu, \\ &\omega_{31} = -Pdu && \omega_{32} = Rdu, && \omega_{33} = 1, \end{aligned} \tag{3.27}$$

[47]At that time, the use of moving trihedrons as a tool to investigate differential geometry of surfaces was rather well-established in France and these results were probably common knowledge. This could justify, at least in part, Bonnet's omission of the details.

[48]For the sake of brevity, we verify these formulas for $\omega_{11}, \omega_{12}, \omega_{13}$, only. The remaining relations can be proven in a similar way. Upon consideration of the Taylor expansion of ω_{11} truncated at the first order in du, we obtain:

$$\begin{aligned} \omega_{11} = \boldsymbol{\mathcal{X}}_1(u+du,v)\cdot\boldsymbol{\mathcal{X}}_1(u,v) &= \left[\boldsymbol{\mathcal{X}}_1(u,v) + \frac{\partial\boldsymbol{\mathcal{X}}_1(u,v)}{\partial u}du + o(du)\right]\cdot\boldsymbol{\mathcal{X}}_1(u,v) \\ &= \boldsymbol{\mathcal{X}}_1(u,v)\cdot\boldsymbol{\mathcal{X}}_1(u,v) + \frac{\partial\boldsymbol{\mathcal{X}}_1(u,v)}{\partial u}du\cdot\boldsymbol{\mathcal{X}}_1(u,v) + \boldsymbol{\mathcal{X}}_1(u,v)\cdot o(du) = 1. \end{aligned}$$

Similarly, one gets:

$$\begin{aligned} \omega_{12} = \boldsymbol{\mathcal{X}}_1(u+du,v)\cdot\boldsymbol{\mathcal{X}}_2(u,v) &= \left[\boldsymbol{\mathcal{X}}_1(u,v) + \frac{\partial\boldsymbol{\mathcal{X}}_1(u,v)}{\partial u}du + o(du)\right]\cdot\boldsymbol{\mathcal{X}}_2(u,v) \\ &= \boldsymbol{\mathcal{X}}_1(u,v)\cdot\boldsymbol{\mathcal{X}}_2(u,v) + \frac{\partial\boldsymbol{\mathcal{X}}_1(u,v)}{\partial u}du\cdot\boldsymbol{\mathcal{X}}_2(u,v) + \boldsymbol{\mathcal{X}}_2(u,v)\cdot o(du) = \frac{\partial\boldsymbol{\mathcal{X}}_1(u,v)}{\partial u}du\cdot\boldsymbol{\mathcal{X}}_2(u,v), \end{aligned}$$

because $\boldsymbol{\mathcal{X}}_1(u,v) \perp \boldsymbol{\mathcal{X}}_2(u,v)$. Besides, from Frenet's equations for (v)-curves, we have $\frac{\partial\boldsymbol{\mathcal{X}}_1(u,v)}{\partial u} = \sqrt{E}\frac{\boldsymbol{n}_v}{\rho_v}$ where $\boldsymbol{n}_v$ and $\frac{1}{\rho_v}$ denote the principal normal and the curvature of v-curves, respectively. The angle between $\boldsymbol{n}_v$ and $\boldsymbol{\mathcal{X}}_2(u,v)$ and the angle between $\boldsymbol{n}_v$ and $\boldsymbol{\mathcal{X}}_3(u,v)$ are complementary. Thus, by introducing the geodesic curvature, we get:

$$\omega_{12} = \frac{\sqrt{E}}{\rho_{t,v}}du.$$

Finally, we consider ω_{13}. From (3.26), we have:

$$\begin{aligned} \omega_{13} = \boldsymbol{\mathcal{X}}_1(u+du,v)\cdot\boldsymbol{\mathcal{X}}_3(u,v) &= \left[\boldsymbol{\mathcal{X}}_1(u,v) + \frac{\partial\boldsymbol{\mathcal{X}}_1(u,v)}{\partial u}du + o(du)\right]\cdot\boldsymbol{\mathcal{X}}_3(u,v) \\ &= \boldsymbol{\mathcal{X}}_1(u,v)\cdot\boldsymbol{\mathcal{X}}_3(u,v) + \frac{\partial\boldsymbol{\mathcal{X}}_1(u,v)}{\partial u}du\cdot\boldsymbol{\mathcal{X}}_3(u,v) + \boldsymbol{\mathcal{X}}_3(u,v)\cdot o(du) = \frac{\partial\boldsymbol{\mathcal{X}}_1(u,v)}{\partial u}du\cdot\boldsymbol{\mathcal{X}}_3(u,v), \end{aligned}$$

where we have taken the orthogonality between $\boldsymbol{\mathcal{X}}_1(u,v)$ and $\boldsymbol{\mathcal{X}}_3(u,v)$ into account. In light of Frenet's equations for (v)-curves, we conclude:

$$\omega_{13} = \frac{\sqrt{E}}{\rho_v}\boldsymbol{n}\cdot\boldsymbol{\mathcal{X}}_3(u,v)du = \frac{\sqrt{E}}{\rho_{n,v}}du.$$

where $M = \dfrac{\sqrt{E}}{\rho_{t,v}}, P = \dfrac{\sqrt{E}}{\rho_{n,v}}, R = \dfrac{\sqrt{E}}{T_v}$. Here $\dfrac{1}{\rho_{t,v}}, \dfrac{1}{\rho_{n,v}}, \dfrac{1}{T_v}$ denote the geodesic curvature, the normal curvature and the geodesic torsion of (v)-curves.[49]

As a consequence of (3.27), if infinitesimals of an order higher than the first are disregarded, then one can easily obtain:

$$\begin{aligned}
\mathcal{X}_1 + \frac{\partial \mathcal{X}_1}{\partial u}du &= \omega_{11}\mathcal{X}_1 + \omega_{12}\mathcal{X}_2 + \omega_{13}\mathcal{X}_3 = \mathcal{X}_1 + (M\mathcal{X}_2 + P\mathcal{X}_3)du\\
\mathcal{X}_2 + \frac{\partial \mathcal{X}_2}{\partial u}du &= \omega_{21}\mathcal{X}_1 + \omega_{22}\mathcal{X}_2 + \omega_{23}\mathcal{X}_3 = \mathcal{X}_2 - (M\mathcal{X}_1 + R\mathcal{X}_3)du\\
\mathcal{X}_3 + \frac{\partial \mathcal{X}_3}{\partial u}du &= \omega_{31}\mathcal{X}_1 + \omega_{32}\mathcal{X}_2 + \omega_{33}\mathcal{X}_3 = \mathcal{X}_3 + (-P\mathcal{X}_1 + R\mathcal{X}_2)du
\end{aligned}$$

Identical relations hold true for the remaining components $(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3), (\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3)$ of the moving frame. Upon iteration of the same procedure applied to (u)-curves, similar equations involving the first-order partial derivatives of $\boldsymbol{\mathcal{X}}$ with respect to v were obtained. Thereby, Bonnet could prove that the components $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ satisfy the following two systems of first order linear differential equations:[50]

$$\begin{cases}
\dfrac{\partial \mathcal{X}_1}{\partial u} = M\mathcal{X}_2 + P\mathcal{X}_3\\
\dfrac{\partial \mathcal{X}_2}{\partial u} = -M\mathcal{X}_1 - R\mathcal{X}_3\\
\dfrac{\partial \mathcal{X}_3}{\partial u} = -P\mathcal{X}_1 + R\mathcal{X}_2
\end{cases} \quad (\mathrm{I}) \qquad
\begin{cases}
\dfrac{\partial \mathcal{X}_1}{\partial v} = N\mathcal{X}_2 + S\mathcal{X}_3\\
\dfrac{\partial \mathcal{X}_2}{\partial v} = -N\mathcal{X}_1 + Q\mathcal{X}_3\\
\dfrac{\partial \mathcal{X}_3}{\partial v} = -S\mathcal{X}_1 - Q\mathcal{X}_2.
\end{cases} \quad (\mathrm{II})$$

Thus, if the system of functions $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ (together with $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$, $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$) corresponding to a surface $\Sigma : (u, v) \mapsto \boldsymbol{x}(u, v)$ is defined by means of (3.25), then $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ are solutions of both (I), (II). Consequently, the following integrability conditions hold true:

$$\frac{\partial}{\partial u}\frac{\partial \mathcal{X}_i}{\partial v} = \frac{\partial}{\partial v}\frac{\partial \mathcal{X}_i}{\partial u} \qquad i = 1, 2, 3.$$

These are easily seen to imply the following relations:

$$\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u} + PQ + RS \tag{3.28}$$

$$\frac{\partial P}{\partial v} = \frac{\partial S}{\partial u} - (MQ + NR) \tag{3.29}$$

$$\frac{\partial R}{\partial v} = -\frac{\partial Q}{\partial u} + NP - MS. \tag{3.30}$$

[49]It is straightforward to verify that $P = \dfrac{D}{\sqrt{E}}, R = -\dfrac{D'}{\sqrt{G}}$.

[50]By analogy with the preceding positions, the functions N, S, Q are defined as follows: $N = \dfrac{\sqrt{G}}{\rho_{t,u}}, Q = \dfrac{\sqrt{G}}{\rho_{n,u}}, S = \dfrac{\sqrt{G}}{T_u}$. Here, $\dfrac{1}{\rho_{t,u}}, \dfrac{1}{\rho_{n,u}}, \dfrac{1}{T_u}$ denote the tangent curvature, the normal curvature and the geodesic torsion of (u)-curves, respectively. Finally, it is easy to see that $S = \dfrac{D'}{\sqrt{E}}, Q = \dfrac{D''}{\sqrt{G}}$.

Furthermore, by taking into account the conditions $\dfrac{\partial \boldsymbol{x}_u}{\partial v} = \dfrac{\partial \boldsymbol{x}_v}{\partial u}$, Bonnet obtained the additional relations:

$$M = -\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v} \qquad N = \frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u} \qquad \frac{R}{\sqrt{E}} + \frac{S}{\sqrt{G}} = 0, \tag{3.31}$$

which can also be seen as a direct consequence of the geometrical meaning of M, N, R, S in terms of the geodesic curvatures and the geodesic torsions of the coordinate lines.

3.5.2 Existence theorem

After providing his own deduction of equations (3.28)-(3.31), Bonnet turned to discussing the following problem: if functions $\tilde{E}, \tilde{G} > 0$ and $\tilde{M}, \tilde{N}, \tilde{P}, \tilde{Q}, \tilde{R}, \tilde{S}$, defined upon an open set $U \subset \mathbb{R}^2$, are assigned so that they satisfy the conditions (3.28)-(3.31), does there exist a surface Σ, $(u, v) \mapsto \boldsymbol{x}(u, v)$, with corresponding functions E, G, M, N, P, Q such that $E = \tilde{E}, G = \tilde{G}$, $M = \tilde{M}, N = \tilde{N}, P = \tilde{P}$, $Q = \tilde{Q}, R = \tilde{R}, S = \tilde{S}$?

To answer in the affirmative, Bonnet relied on an existence theorem[51] for differential systems (I)-(II), which can be formulated as follows:

Theorem 3.2. *If conditions* (3.28)-(3.31) *are satisfied, then systems* (I)-(II) *admit an infinite set of solutions. Furthermore, if* $(\mathcal{X}_1^{(1)}, \mathcal{X}_2^{(1)}, \mathcal{X}_3^{(1)}), (\mathcal{X}_1^{(2)}, \mathcal{X}_2^{(2)}, \mathcal{X}_3^{(2)}), (\mathcal{X}_1^{(3)}, \mathcal{X}_2^{(3)}, \mathcal{X}_3^{(3)})$ *are three solutions of* (I)-(II) *such that*

$$\det \begin{pmatrix} \mathcal{X}_1^{(1)} & \mathcal{X}_1^{(2)} & \mathcal{X}_1^{(3)} \\ \mathcal{X}_2^{(1)} & \mathcal{X}_2^{(2)} & \mathcal{X}_2^{(3)} \\ \mathcal{X}_3^{(1)} & \mathcal{X}_3^{(2)} & \mathcal{X}_3^{(3)} \end{pmatrix} \neq 0,$$

then any other solution $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3)$ *can be expressed in terms of a linear combination of* $(\mathcal{X}_1^{(1)}, \mathcal{X}_2^{(1)}, \mathcal{X}_3^{(1)})$, $(\mathcal{X}_1^{(2)}, \mathcal{X}_2^{(2)}, \mathcal{X}_3^{(2)})$, $(\mathcal{X}_1^{(3)}, \mathcal{X}_2^{(3)}, \mathcal{X}_3^{(3)})$*:*

$$\begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \mathcal{X}_3 \end{pmatrix} = C_1 \begin{pmatrix} \mathcal{X}_1^{(1)} \\ \mathcal{X}_2^{(1)} \\ \mathcal{X}_3^{(1)} \end{pmatrix} + C_2 \begin{pmatrix} \mathcal{X}_1^{(2)} \\ \mathcal{X}_2^{(2)} \\ \mathcal{X}_3^{(2)} \end{pmatrix} + C_3 \begin{pmatrix} \mathcal{X}_1^{(3)} \\ \mathcal{X}_2^{(3)} \\ \mathcal{X}_3^{(3)} \end{pmatrix}.$$

The next step of Bonnet's analysis consisted in proving that there exist constants a_{ij}, $i, j = 1, 2, 3$ such that the vectors:

$$\boldsymbol{\Psi}_1 = \begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \mathcal{X}_3 \end{pmatrix} = a_{11} \begin{pmatrix} \mathcal{X}_1^{(1)} \\ \mathcal{X}_2^{(1)} \\ \mathcal{X}_3^{(1)} \end{pmatrix} + a_{12} \begin{pmatrix} \mathcal{X}_1^{(2)} \\ \mathcal{X}_2^{(2)} \\ \mathcal{X}_3^{(2)} \end{pmatrix} + a_{13} \begin{pmatrix} \mathcal{X}_1^{(3)} \\ \mathcal{X}_2^{(3)} \\ \mathcal{X}_3^{(3)} \end{pmatrix},$$

$$\boldsymbol{\Psi}_2 = \begin{pmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \\ \mathcal{Y}_3 \end{pmatrix} = a_{21} \begin{pmatrix} \mathcal{X}_1^{(1)} \\ \mathcal{X}_2^{(1)} \\ \mathcal{X}_3^{(1)} \end{pmatrix} + a_{22} \begin{pmatrix} \mathcal{X}_1^{(2)} \\ \mathcal{X}_2^{(2)} \\ \mathcal{X}_3^{(2)} \end{pmatrix} + a_{23} \begin{pmatrix} \mathcal{X}_1^{(3)} \\ \mathcal{X}_2^{(3)} \\ \mathcal{X}_3^{(3)} \end{pmatrix}, \tag{3.32}$$

[51] Bonnet did not provide any reference to sources of this theorem.

$$\mathbf{\Psi}_3 = \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \\ \mathcal{Z}_3 \end{pmatrix} = a_{31} \begin{pmatrix} \mathcal{X}_1^{(1)} \\ \mathcal{X}_2^{(1)} \\ \mathcal{X}_3^{(1)} \end{pmatrix} + a_{32} \begin{pmatrix} \mathcal{X}_1^{(2)} \\ \mathcal{X}_2^{(2)} \\ \mathcal{X}_3^{(2)} \end{pmatrix} + a_{33} \begin{pmatrix} \mathcal{X}_1^{(3)} \\ \mathcal{X}_2^{(3)} \\ \mathcal{X}_3^{(3)} \end{pmatrix},$$

are an orthonormal frame of $\mathbb{R}^3$, i.e., $\mathbf{\Psi}_i \cdot \mathbf{\Psi}_k = \delta_{ik}, i, k = 1, 2, 3$. Indeed, if this is the case, then the function $\boldsymbol{x} : (u, v) \to \boldsymbol{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ defined by:

$$\begin{cases} x(u,v) := \int \sqrt{E}\mathcal{X}_1 du + \sqrt{G}\mathcal{X}_2 dv \\ \\ y(u,v) := \int \sqrt{E}\mathcal{Y}_1 du + \sqrt{G}\mathcal{Y}_2 dv \\ \\ z(u,v) := \int \sqrt{E}\mathcal{Z}_1 du + \sqrt{G}\mathcal{Z}_2 dv, \end{cases} \tag{3.33}$$

is easily seen to define a surface Σ whose coefficients E, G, M, N, Q, R, S (i.e., E, G, D, D', D'') coincide with those required.

To prove the existence of such constants a_{ij}, $i, j = 1, 2, 3$, Bonnet resorted to an ingenious geometrical argument, to which we now turn.

He defined the following quantities:

$$A_i = \sum_{k=1}^{3} \left[\mathcal{X}_k^{(i)}\right]^2 \qquad i = 1, 2, 3$$

$$B_1 = \sum_{k=1}^{3} \mathcal{X}_k^{(2)}\mathcal{X}_k^{(3)}, \qquad B_2 = \sum_{k=1}^{3} \mathcal{X}_k^{(1)}\mathcal{X}_k^{(3)}, \qquad B_3 = \sum_{k=1}^{3} \mathcal{X}_k^{(1)}\mathcal{X}_k^{(2)}.$$

As a consequence of $\mathbf{\Psi}_i \cdot \mathbf{\Psi}_k = \delta_{ik}, i, k = 1, 2, 3$, one has:

$$\begin{cases} (a_{i1}, a_{i2}, a_{i3}, 1)\boldsymbol{A}(a_{i1}, a_{i2}, a_{i3}, 1)^t = 1 & i = 1, 2, 3 \\ (a_{i1}, a_{i2}, a_{i3}, 0)\boldsymbol{A}(a_{j1}, a_{j2}, a_{j3}, 0)^t = 0 & i, j = 1, 2, 3 \quad i \neq j \end{cases} \tag{3.34}$$

where $\boldsymbol{A}$ denotes the symmetric matrix

$$\begin{pmatrix} A_1 & B_3 & B_2 & 0 \\ B_3 & A_2 & B_1 & 0 \\ B_2 & B_1 & A_3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

As Bonnet observed, $\boldsymbol{A}$ is the matrix corresponding to an ellipsoid

$$\mathcal{E}: \quad A_1x^2 + A_2y^2 + A_3z^2 + 2B_1yz + 2B_2xz + 2B_3xy = 1$$

with centre at the origin $\boldsymbol{O} = (0, 0, 0)$, because[52]

$$\begin{vmatrix} A_1 & B_3 & B_2 \\ B_3 & A_2 & B_1 \\ B_2 & B_1 & A_3 \end{vmatrix} = \begin{vmatrix} \mathcal{X}_1^{(1)} & \mathcal{X}_2^{(1)} & \mathcal{X}_3^{(1)} \\ \mathcal{X}_1^{(2)} & \mathcal{X}_2^{(2)} & \mathcal{X}_3^{(2)} \\ \mathcal{X}_1^{(3)} & \mathcal{X}_2^{(3)} & \mathcal{X}_3^{(3)} \end{vmatrix} \cdot \begin{vmatrix} \mathcal{X}_1^{(1)} & \mathcal{X}_1^{(2)} & \mathcal{X}_1^{(3)} \\ \mathcal{X}_2^{(1)} & \mathcal{X}_2^{(2)} & \mathcal{X}_2^{(3)} \\ \mathcal{X}_3^{(1)} & \mathcal{X}_3^{(2)} & \mathcal{X}_3^{(3)} \end{vmatrix} = \begin{vmatrix} \mathcal{X}_1^{(1)} & \mathcal{X}_1^{(2)} & \mathcal{X}_1^{(3)} \\ \mathcal{X}_2^{(1)} & \mathcal{X}_2^{(2)} & \mathcal{X}_2^{(3)} \\ \mathcal{X}_3^{(1)} & \mathcal{X}_3^{(2)} & \mathcal{X}_3^{(3)} \end{vmatrix}^2 > 0. \tag{3.35}$$

[52] $|A|$ denotes the value of the determinant of the matrix A.

Clearly, the first equations of (3.34) assert that the points $\boldsymbol{a}_1 = (a_{11}, a_{12}, a_{13})$, $\boldsymbol{a}_2 = (a_{21}, a_{22}, a_{23})$, $\boldsymbol{a}_3 = (a_{31}, a_{32}, a_{33})$ belong to the quadric, whereas the second equations can be interpreted geometrically by saying that the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ determine three conjugate diameters of $\mathcal{E}$. The very fact that $\mathcal{E}$ is an ellipsoid (this is a consequence of (3.35)) is sufficient to establish the existence of constants $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ such that $\boldsymbol{\Psi}_i \cdot \boldsymbol{\Psi}_k = \delta_{ik}, i, k = 1, 2, 3$. Actually, the preceding argument offered a constructive procedure that could be used to determine them.

3.5.3 Uniqueness

As for the uniqueness of the surface associated to the coefficients E, G, M, N, P, Q, R, S, Bonnet proved that the vector fields $(\mathcal{X}_i', \mathcal{Y}_i', \mathcal{Z}_i')^t$ and $(\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_i)^t$, $i = 1, 2, 3$, corresponding to any two set of solutions $\boldsymbol{\Phi}_i$ and $\boldsymbol{\Psi}_i$, $i = 1, 2, 3$ of (I)-(II) such that $\boldsymbol{\Phi}_i \cdot \boldsymbol{\Phi}_k = \delta_{ik}$ and $\boldsymbol{\Psi}_i \cdot \boldsymbol{\Psi}_k = \delta_{ik}$, differ by a rotation of $\mathbb{R}^3$.

To this end, he considered a system of solutions $\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \boldsymbol{\Psi}_3$ corresponding to the conjugate diameters $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ and made the following positions: $\boldsymbol{a}_i = l_i(\theta_{i1}, \theta_{i2}, \theta_{i3}), i = 1, 2, 3$ where l_i denotes the length of the semidiameter $\boldsymbol{a}_i$ and $\boldsymbol{\theta}_i := (\theta_{i1}, \theta_{i2}, \theta_{i3})$ denotes the unit vector corresponding to the conjugate diameters $\boldsymbol{a}_i$, $i = 1, 2, 3$. Furthermore, he denoted with $\boldsymbol{a}_1' = (a_{11}', a_{12}', a_{13}'), \boldsymbol{a}_2' = (a_{21}', a_{22}', a_{23}'), \boldsymbol{a}_3' = (a_{31}', a_{32}', a_{33}')$, the points of $\mathcal{E}$ that correspond to a new set of conjugate diameters of $\mathcal{E}$.

Bonnet then introduced a new reference system $\boldsymbol{Ox'y'z'}$ whose axes coincide with the directions of the conjugate diameters corresponding to $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$. It can be checked through a direct application of conditions (3.34), that $\mathcal{E}$ is written in $\boldsymbol{Ox'y'z'}$, as follows:

$$\mathcal{E} : \frac{x'^2}{l_1^2} + \frac{y'^2}{l_2^2} + \frac{z'^2}{l_3^2} = 1.$$

Clearly, any system of conjugate diameters of $\mathcal{E}$ consists of three vectors whose components with respect to $\boldsymbol{Ox'y'z'}$ can be written as:

$$\tilde{\boldsymbol{a}}_i = (l_1\omega_{i1}, l_2\omega_{i2}, l_3\omega_{i3}) \qquad i = 1, 2, 3.$$

As a consequence of (3.34), which is now written with respect to the new reference system $\boldsymbol{Ox'y'z'}$, Bonnet could obtain the following set of equations:

$$\sum_{j=1}^{3} \omega_{ij}^2 = 1 \quad i = 1, 2, 3 \qquad \sum_{j=1}^{3} \omega_{1j}\omega_{2j} = \sum_{j=1}^{3} \omega_{2j}\omega_{3j} = \sum_{j=1}^{3} \omega_{3j}\omega_{1j} = 0. \tag{3.36}$$

These conditions imply that the matrix $\boldsymbol{\Omega} = [\omega_{ij}]$ represents a rotation of $\mathbb{R}^3$.[53] Now, the components of $\tilde{\boldsymbol{a}}_i, i = 1, 2, 3$ with respect to the system $\boldsymbol{Oxyz}$ are given by $\boldsymbol{a}_i' = \boldsymbol{\Theta}\tilde{\boldsymbol{a}}_i$, $i = 1, 2, 3$, where $\boldsymbol{\Theta}$ denotes the following matrix:

$$\boldsymbol{\Theta} = \begin{pmatrix} \theta_{11} & \theta_{21} & \theta_{31} \\ \theta_{12} & \theta_{22} & \theta_{32} \\ \theta_{13} & \theta_{23} & \theta_{33} \end{pmatrix}.$$

[53] In fact, conditions (3.36) entail that $\boldsymbol{\Omega} \in O(n)$. More precisely, $\Omega \in SO(n)$ since it maps positively oriented trihedra into positively oriented trihedra.

A straightforward computation gives

$$a'_{tk} = \sum_{s=1}^{3} \theta_{sk} l_s \omega_{ts} = \sum a_{sk}\omega_{ts}, \qquad t,k = 1,2,3.$$

Now, if we introduce new functions $\mathcal{X}'_i, \mathcal{Y}'_i, \mathcal{Z}'_i$ corresponding to a new triple of conjugate diameters $\boldsymbol{a}'_i$, $i = 1,2,3$, we have (in view of (3.32)):

$$\mathcal{X}'_i = \sum_{k=1}^{2} a'_{1k}\mathcal{X}_i^{(k)}, \qquad \mathcal{Y}'_i = \sum_{k=1}^{2} a'_{2k}\mathcal{Y}_i^{(k)}, \qquad \mathcal{Z}'_i = \sum_{k=1}^{2} a'_{3k}\mathcal{Z}_i^{(k)}, \qquad i = 1,2,3.$$

It is easily verified that the vectors $(\mathcal{X}'_i, \mathcal{Y}'_i, \mathcal{Z}'_i)^t$ and $(\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_i)^t$, $i = 1,2,3$, can be mapped one into the other by the matrix $\boldsymbol{\Omega}$. Indeed:

$$\begin{cases} \mathcal{X}'_i = \sum_{k=1}^{3} a'_{1k}\mathcal{X}_i^{(k)} = \sum_{s,k=1}^{3} a_{sk}\omega_{1s}\mathcal{X}_i^{(k)} = \omega_{11}\mathcal{X}_i + \omega_{12}\mathcal{Y}_i + \omega_{13}\mathcal{Z}_i, \\ \mathcal{Y}'_i = \sum_{k=1}^{3} a'_{2k}\mathcal{X}_i^{(k)} = \sum_{s,k=1}^{3} a_{sk}\omega_{2s}\mathcal{X}_i^{(k)} = \omega_{21}\mathcal{X}_i + \omega_{22}\mathcal{Y}_i + \omega_{23}\mathcal{Z}_i, \qquad i = 1,2,3. \\ \mathcal{Z}'_i = \sum_{k=1}^{3} a'_{3k}\mathcal{X}_i^{(k)} = \sum_{s,k=1}^{3} a_{sk}\omega_{3s}\mathcal{X}_i^{(k)} = \omega_{31}\mathcal{X}_i + \omega_{32}\mathcal{Y}_i + \omega_{33}\mathcal{Z}_i, \end{cases}$$

This entails that any two solutions $\boldsymbol{x}(u,v)$ and $\boldsymbol{x}'(u,v)$ of (3.33)—corresponding to constants $\boldsymbol{a}_i$ and $\boldsymbol{a}'_i$, $i = 1,2,3$, respectively—can be mapped one into the other by an element $\boldsymbol{\Omega} \in SO(n)$ and a translation (if additive integration constants are taken into account).

Thereby, Bonnet could conclude that an immersed surface is univocally determined by the assignment of (appropriate) functions E, G, D, D', D'', save for its position in Euclidean space.[54]

3.5.4 An application of the fundamental theorem

Bonnet did not limit himself to providing a satisfactory proof of the existence and uniqueness result, he was also well aware of its significance for treating specific problems. In this respect, a most interesting example is provided by the theorem mentioned in Section 3.3. For the sake of clarity, we recall its statement:[55]

Theorem 3.3 (Bonnet, 1867)**.** *Let us consider two surfaces Σ and $\tilde{\Sigma}$. Suppose that Σ and $\tilde{\Sigma}$ are applicable one upon the other in such a way that a system of asymptotic curves on Σ is mapped onto a system of asymptotic curves on $\tilde{\Sigma}$, then Σ and $\tilde{\Sigma}$ coincide up to a rigid motion in Euclidean space. An exception is represented by the case in which Σ is a ruled surface and the system of asymptotic lines is a system of generatrices of Σ.*

The proof proposed in (Bonnet 1867, p. 44) was far simpler than that contained in the manuscript submitted for the prize competition. Furthermore, this new approach

[54] The fact that $(\mathcal{X}'_i, \mathcal{Y}'_i, \mathcal{Z}'_i)^t$ and $(\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_i)^t$, for $i = 1,2,3$, give rise to the same functions D, D', D'', implies that $\det \boldsymbol{\Omega} = 1$. However, this remark is nowhere to be found in (Bonnet 1867).

[55] This is a slightly modified version of (Bonnet 1867, p. 44).

made it clear that the original statement contained a superfluous hypothesis; that is, the requirement that *both* systems of asymptotic curves on Σ be preserved under local isometry.

To start with, Bonnet supposed that the two surfaces are parametrized in such a way that the coordinate lines $v =$ const. and $u =$ const. are asymptotic lines and their corresponding orthogonal trajectories, respectively. More explicitly, let $\boldsymbol{x} : (u, v) \mapsto \boldsymbol{x}(u, v)$ and $\tilde{\boldsymbol{x}} : (u, v) \mapsto \tilde{\boldsymbol{x}}(u, v)$ be the parametrization of Σ and $\tilde{\Sigma}$, respectively. Their line elements (first fundamental forms) are given by $ds^2 = d\tilde{s}^2 = E(u, v)du^2 + G(u, v)dv^2$. Furthermore, the curves $\gamma_{v^*} : u \mapsto \boldsymbol{x}(u, v^*)$, $\tilde{\gamma}_{v^*} : u \mapsto \tilde{\boldsymbol{x}}(u, v^*)$ are asymptotic curves on Σ and $\tilde{\Sigma}$, respectively. As a consequence of the definition of asymptotic lines,[56] we have $D = \tilde{D} = 0$. Thus, in terms of Bonnet's functions, $P = \tilde{P} = 0$. Because the two surfaces are locally isometric, one has $E = \tilde{E}, G = \tilde{G}$. Furthermore, we remark that $M = \tilde{M}, N = \tilde{N}$, as a consequence of (3.31). In view of (3.28) and the last of (3.31), this implies that $S = \pm\tilde{S}$ and consequently $R = \pm\tilde{R}$. Finally, if $M \neq 0$, then (3.29) implies also $Q = \pm\tilde{Q}$.

Now it is clear that the triplet $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3)$ where $\mathcal{X}_1 := \dfrac{x_u}{\sqrt{E}}$, $\mathcal{X}_2 := \dfrac{x_v}{\sqrt{G}}$, $\mathcal{X}_3 := N_x$ (along with the remaining triplets $(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3), (\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3)$)[57] is a solution of the systems:

$$\begin{cases} \dfrac{\partial \mathcal{X}_1}{\partial u} = M\mathcal{X}_2 \\ \dfrac{\partial \mathcal{X}_2}{\partial u} = -M\mathcal{X}_1 - R\mathcal{X}_3 \\ \dfrac{\partial \mathcal{X}_3}{\partial u} = R\mathcal{X}_2 \end{cases} \quad (\mathrm{I}) \qquad \begin{cases} \dfrac{\partial \mathcal{X}_1}{\partial v} = N\mathcal{X}_2 + S\mathcal{X}_3 \\ \dfrac{\partial \mathcal{X}_2}{\partial v} = -N\mathcal{X}_1 + Q\mathcal{X}_3 \\ \dfrac{\partial \mathcal{X}_3}{\partial v} = -S\mathcal{X}_1 - Q\mathcal{X}_2. \end{cases} \quad (\mathrm{II})$$

Analogously, the triplet $(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2, \tilde{\mathcal{X}}_3)$, where $\tilde{\mathcal{X}}_1 := \dfrac{\tilde{x}_u}{\sqrt{E}}$, $\tilde{\mathcal{X}}_2 := \dfrac{\tilde{x}_v}{\sqrt{G}}$, $\tilde{\mathcal{X}}_3 := \tilde{N}_x$ is a solution of the systems:

$$\begin{cases} \dfrac{\partial \tilde{\mathcal{X}}_1}{\partial u} = \tilde{M}\tilde{\mathcal{X}}_2 \\ \dfrac{\partial \tilde{\mathcal{X}}_2}{\partial u} = -\tilde{M}\tilde{\mathcal{X}}_1 - \tilde{R}\tilde{\mathcal{X}}_3 \\ \dfrac{\partial \tilde{\mathcal{X}}_3}{\partial u} = \tilde{R}\tilde{\mathcal{X}}_2 \end{cases} \quad (\tilde{I}) \qquad \begin{cases} \dfrac{\partial \tilde{\mathcal{X}}_1}{\partial v} = \tilde{N}\tilde{\mathcal{X}}_2 + \tilde{S}\tilde{\mathcal{X}}_3 \\ \dfrac{\partial \tilde{\mathcal{X}}_2}{\partial v} = -\tilde{N}\tilde{\mathcal{X}}_1 + \tilde{Q}\tilde{\mathcal{X}}_3 \\ \dfrac{\partial \tilde{\mathcal{X}}_3}{\partial v} = -\tilde{S}\tilde{\mathcal{X}}_1 - \tilde{Q}\tilde{\mathcal{X}}_2. \end{cases} \quad (\tilde{II})$$

If $\tilde{S} = S, \tilde{R} = R, \tilde{Q} = Q$ then clearly $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3)$, $(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2, \tilde{\mathcal{X}}_3)$ define the same surface; that is, Σ and $\tilde{\Sigma}$ coincide except for a Euclidean motion. The same holds true also if $\tilde{S} = -S, \tilde{R} = -R, \tilde{Q} = -Q$; indeed, in this case, the triplet $(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2, \tilde{\mathcal{X}}_3)$ and the remaining triplets $(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2, \tilde{\mathcal{Y}}_3)$, $(\tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_2, \tilde{\mathcal{Z}}_3)$, which are solutions of $(\tilde{I})$ and $(\tilde{II})$, are such that $(\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2, -\tilde{\mathcal{X}}_3)$, $(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2, -\tilde{\mathcal{Y}}_3)$, $(\tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_2, -\tilde{\mathcal{Z}}_3)$ are solutions of (I) and (II). As a consequence of Bonnet's fundamental theorem and of (3.33), Bonnet concluded that Σ and $\tilde{\Sigma}$ coincide except for a Euclidean motion. Finally, if $M = 0$—that is, if γ_{v^*} and $\tilde{\gamma}_{v^*}$ are straight lines, a case that is excluded by the preceding argument—, then it is clear that both Σ and $\tilde{\Sigma}$

[56]It should be recalled that asymptotic lines are solutions to the following differential condition:

$$Ddu^2 + 2D'dudv + D''dv^2 = 0.$$

[57]We denoted the components of $\boldsymbol{\mathcal{X}}_3 = \dfrac{\boldsymbol{x}_u}{\sqrt{E}} \wedge \dfrac{\boldsymbol{x}_v}{\sqrt{G}}$ with N_x, N_y and N_z.

are ruled surfaces. Thereby Theorem 3.3 was proven.

3.6 Remarks on Mainardi's memoir

The MCE and FT were first stated and partially demonstrated in the works of Peterson (1853) and Mainardi (1856). While Peterson's memoir has been commented on extensively in (Phillips 1979) and (Reich 1973), secondary literature on Mainardi and his work is rather scarce.[58]

3.6.1 Mainardi's deduction of MCE

Although certainly conditioned by personal views,[59] Francesco Brioschi's[60] words in a letter to Angelo Genocchi dated 29th December 1857 offers an interesting review of (Mainardi 1856):

> The other memoir you tell me about *"on the general theory of surfaces"* (note that he publishes nothing but general theories) is a chief work of dishonesty, and ignorance. And I prove it to you. *He begins by still offering some essential formulas etc. ... and others known*; now, which of the formulas (1) (2) ... (18) is not known: and by citing only three authors, Chelini, Bonnet and Gauss, for the formulas (8) (7) (18) does not he suggest that a large number of the others are his? And, for example, the formulas (11) ... (15) are not already found and demonstrated less barbarously in my note, *"About some properties of a line drawn over a surface"*, Annals of Tortolini 1854? But let us turn to what the author believes to be of most interest and has the courage to compare to Gauss' formulas for the product of radii of curvature. He believes he has made a discovery by having made the ridiculous observation that just as Gauss found a relation between the six quantities E, G, F, D, D', D'' functions of the first and second derivatives of x, y, z with respect to u and v; so will it be possible to obtain relations between other quantities that are functions of the third derivatives. And why not of the fourth derivatives? And so on. There would be much more to say about this shaky idea, but observe the details. For example, in order to introduce, in the first determinants that arise after elimination, the quantities $P, Q, \ldots$ instead of multiplying all the

[58]In addition to (Belgiojoso 1879), (Millan Gasca 2006) and Mainardi's published memoirs, we refer to the correspondence between Francesco Brioschi and Angelo Genocchi (Carbone et al. 2006) and the correspondence between Mainardi himself and Felice Casorati, which is stored in the *Fondo Gabba* in Pavia and has been included in (Moglia 1992).

[59]Mainardi and Brioschi were certainly not on friendly terms. Indeed, Casorati alluded to the *"little understanding that had existed for many years between you* [Mainardi] *and Bordoni and Brioschi"*. (Moglia 1992, p. 165)

[60]Brioschi, who was both a student and a colleague of Mainardi, is considered one of the main exponents of the Renaissance of mathematical sciences in Italy. For a careful description of his commitment to Italian civic and cultural life, the interested reader can refer to (Bottazzini 1998).

> terms by an appropriate determinant; he squares and extracts the root, so leaving aside the errors that can be made by this operation, the formulas get more and more complicated. Mainardi's application of his formulas on the penultimate page clearly demonstrates his ignorance of geometrical matters, and the value $d_u\left(\frac{d_uG}{\sqrt{EG}}\frac{dG}{dr}=0\right)$ leads to the result that the surfaces having the lines of curvature of a geodesic system are those of revolut.[ion] and the developable ones; now other families of surfaces having this property are already known, e.g., those of Monge having the lines of curvature of a system situated in parallel planes. The erroneous consequence depends on the upper equation being by an error of calculation, erroneous, which is reduced instead to an identical equat.[ion]. Hence its formulas lead to no result.[61]

(Mainardi 1856) contains many more errors than those reported by Brioschi: in (Moglia 1992, pp. 55–59), 34 misprints, six miscalculations and 11 more serious miscalculations were counted. The readability of the memoir is also compromised by numerous long deductions of formulas whose contents are rarely commented on and by the large number of quantities considered.[62] These facts certainly do not help the reader appreciate the value of the last five pages, which contain the deduction of the MCE and FT.

In the incipit of (Mainardi 1856), Mainardi seemed to take the content of the FT for granted. Indeed, he hypothesised that surfaces embedded into Euclidean space are uniquely determined by the assignment of the six coefficients of the first and second fundamental

[61] L'altra memoria di cui mi parlate *"sulla teoria generale delle superficie"* (notate bene che egli non pubblica che teorie generali) è un capo d'opera di mala fede, e d'ignoranza. E ve lo provo. *Incomincia dall'offrire ancora alcune formole essenziali etc. ... ed altre conosciute*; ora quale tra le formole (1) (2) ... (18) non è conosciuta: e citando soli tre autori il Chelini il Bonnet ed il Gauss per le formole (8) (7) (18) non lascia supporre che buon numero delle altre siano sue? E per esempio le formole (11) ... (15) non sono già trovate e dimostrate meno barbaramente nella mia nota *"Intorno ad alcune proprietà di una linea tracciata sopra una superficie"* Annali del Tortolini 1854? Ma passiamo a quel che l'autore crede di maggior interesse ed ha il coraggio di paragonare alle formole di Gauss pel prodotto dei raggi di curvatura. Egli crede di aver fatto una scoperta avendo fatto la ridicola osservazione che come Gauss ha trovata una relazione fra le sei quantità E, G, F, D, D', D'' funzioni delle derivate prima e delle derivate seconde di x, y, z rispetto ad u e v; così si potranno ottenere relazioni fra altre quantità che siano funzioni delle derivate terze. E perché non delle derivate quarte? E così via. Vi sarebbe molte altre cose a dire su questa malferma idea, ma osservate i dettagli. Per esempio, per introdurre, nei primi determinanti che si presentano dopo l'eliminazione, le quantità $P, Q, \ldots$ in luogo di moltiplicare tutti i termini per un determinante opportuno; fa il quadrato ed estrae la radice, per cui lasciando da parte gli errori che ponno commettersi per questa operazioni, le formole vanno sempre più complicandosi. La applicazione che il Mainardi fa delle sue formole nella penultima pagina dimostra all'evidenza la ignoranza sua in fatto di questioni geometriche, ed il valore $d_u\left(\frac{d_uG}{\sqrt{EG}}\frac{dG}{dr}=0\right)$ conduce al risultato che le superfici aventi le linee di curvatura di un sistema geodetiche sono quelle di rivoluz.[ione] e le sviluppabili; ora sono già note altre famiglie di superficie aventi questa proprietà p. e. quelle di Monge che hanno le linee di curvatura di un sistema situate in piani paralleli. L'erronea conseguenza dipende dall'essere per uno sbaglio di calcolo, erronea la equazione superiore, la quale riducesi invece ad una equaz.[ione] identica. Per cui le sue formole non conducono a nessun risultato.(Carbone et al. 2006, pp. 289–290)

[62] Coolidge commented *"The difficulty of reading the paper, at least to me, consists in remembering the meanings of forty-eight new symbols introduced in ten pages" (Coolidge 1947, p. 360).*

forms. This probably indicates a certain widespread awareness of the validity of the FT Bonnet's rigorous proof and even before Mainardi's draft proof. From the remark that a surface is determined by the assignment of the line element, Mainardi deduced that there must exist some relations ("Essenziali dipendenze" (Mainardi 1856, p. 385)) between the six coefficients, so that D, D', D'' could be derived as a function of E, F, G. Gauss' equation for total curvature is one of these essential dependencies. The determination of the others is the purpose of memoir.

The first ten pages, however, are devoted to the exposition of those well-known formulas of surface theory that Brioschi complained about in the letter to Genocchi quoted earlier. The MCE deduction is clearly announced in (Mainardi 1856, p. 394).

Mainardi represented a given surface by means of the parametrization $S : (u, v) \to \boldsymbol{x}(u, v)$, $\boldsymbol{x} = (x(u, v), y(u, v), z(u, v))$, whose line element is given by $ds^2 = Edu^2 + Fdudv + Gdv^2$. Then, he required the validity of the following integrability conditions[63]

$$\frac{\partial x_{uu}}{\partial v} = \frac{\partial x_{uv}}{\partial u} \quad \frac{\partial y_{uu}}{\partial v} = \frac{\partial y_{uv}}{\partial u} \quad \frac{\partial z_{uu}}{\partial v} = \frac{\partial z_{uv}}{\partial u}$$
$$\frac{\partial x_{uv}}{\partial v} = \frac{\partial x_{vv}}{\partial u} \quad \frac{\partial y_{uv}}{\partial v} = \frac{\partial y_{vv}}{\partial u} \quad \frac{\partial z_{uv}}{\partial v} = \frac{\partial z_{vv}}{\partial u}.$$

Mainardi finally arrived at four rather complicated equations, which are his version of the MCE, for the unknowns D, D', D'':

$$\left(\frac{\partial m}{\partial u} - P\right)\sqrt{\begin{vmatrix} G & n & n' \\ n & P & S \\ n' & S & Q \end{vmatrix}} + \left(\frac{\partial n}{\partial u} - S\right)\sqrt{\begin{vmatrix} E & m & m' \\ m & P & S \\ m' & S & Q \end{vmatrix}} + \frac{1}{2}\frac{\partial P}{\partial u}D' - \left(\frac{\partial S}{\partial u} - \frac{1}{2}\frac{\partial P}{\partial v}\right) D = 0$$

$$\left(\frac{\partial m}{\partial v} - S\right)\sqrt{\begin{vmatrix} G & n & n'' \\ n & P & T \\ n'' & Y & R \end{vmatrix}} + \left(\frac{\partial n}{\partial v} - T\right)\sqrt{\begin{vmatrix} E & m & m'' \\ m & P & T \\ m'' & T & R \end{vmatrix}} + \frac{1}{2}\frac{\partial P}{\partial v}D'' - \left(\frac{\partial U}{\partial u} - \frac{1}{2}\frac{\partial Q}{\partial v}\right) D = 0$$

$$\left(\frac{\partial m''}{\partial u} - U\right)\sqrt{\begin{vmatrix} G & n' & n'' \\ n' & Q & U \\ n'' & U & R \end{vmatrix}} + \left(\frac{\partial n''}{\partial v} - R\right)\sqrt{\begin{vmatrix} E & m' & m'' \\ m' & Q & U \\ m'' & U & R \end{vmatrix}} + \frac{1}{2}\frac{\partial R}{\partial v}D' - \left(\frac{\partial U}{\partial v} - \frac{1}{2}\frac{\partial R}{\partial u}\right) D'' = 0$$

$$\left(\frac{\partial m''}{\partial u} - T\right)\sqrt{\begin{vmatrix} G & n & n'' \\ n & P & T \\ n'' & T & R \end{vmatrix}} + \left(\frac{\partial n}{\partial v} - U\right)\sqrt{\begin{vmatrix} E & m' & m'' \\ m & P & T \\ m'' & T & R \end{vmatrix}} + \frac{1}{2}\frac{\partial R}{\partial u}D - \left(\frac{\partial S}{\partial v} - \frac{1}{2}\frac{\partial Q}{\partial u}\right) D'' = 0$$

[63] With respect to Mainardi's original notation, we made the following changes

$$\alpha = x_{uu} \quad \alpha' = x_{uv} \quad \beta = y_{uu} \quad \beta' = y_{uv} \quad \gamma = z_{uu} \quad \gamma' = z_{uv}.$$

where

$$
\begin{array}{llll}
m = \boldsymbol{x}_u \cdot \boldsymbol{x}_{uu} & n = \boldsymbol{x}_v \cdot \boldsymbol{x}_{uu} & P = \|\boldsymbol{x}_{uu}\|^2 & S = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{uv} \\
m' = \boldsymbol{x}_u \cdot \boldsymbol{x}_{uv} & n' = \boldsymbol{x}_v \cdot \boldsymbol{x}_{uv} & Q = \|\boldsymbol{x}_{uv}\|^2 & T = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} \\
m'' = \boldsymbol{x}_u \cdot \boldsymbol{x}_{vv} & n'' = \boldsymbol{x}_v \cdot \boldsymbol{x}_{vv} & R = \|\boldsymbol{x}_{vv}\|^2 & U = \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{vv}
\end{array}
$$

Mainardi provided no explanation to support his choice to consider the integrability conditions of the second derivatives of surface's parametric equations. Brioschi seems justly to wonder why to consider the third derivatives. An answer seems possible only by taking into account a system such as (3.8) and (3.20) that Bour and Codazzi specifically constructed for the deductions of their MCE starting with the second applicability problem.

Finally, Mainardi explained how an immersed surface could be constructed from the assignment of the functions E, F, G and D, D', D'' through the following argument.

> If we want to construct the surface we will begin, for example, by drawing a curve s_v corresponding to $u = 0$. The initial position of the tangent and of the osculating plane are arbitrary. The equation $\dfrac{ds_v}{dv} = \sqrt{G}$ and (6), (8), (12), (13), (15) [Mainardi referred to some equations obtained in the first part of (Mainardi 1856)] where $u = 0$, will provide the values of the successive sides and angles of the first and second curvature of the curve, attributing to the variable v infinitesimal increasing values. After having drawn the line s_v, we will be able from any of its points to draw the corresponding line s_u. The direction of the initial tangent is determined from the formulas that give $\cos\omega, \cos(\rho_v s_u)$. Then we will proceed by using formulas $\dfrac{ds_u}{du} = \sqrt{E}$ and (6), (8), … in which the corresponding constant value is attributed to v.[64]

The main idea at the basis of this constructive approach can be described in slightly modernised terminology as follows.

First, it is to be observed that the position of the Frenet frame at the initial point of a curve $\boldsymbol{\gamma}_0 : v \mapsto \boldsymbol{\gamma}_0(v)$ can be arbitrarily chosen, which is what Mainardi meant by saying "the initial position of both the tangent and the osculating plane are arbitrary" . The formulas to which Mainardi referred in the text above can be used to assign the value of curvature and the torsion at each point of $\boldsymbol{\gamma}_0$. More precisely, through a combination of formula (6), which gives the geodesic curvature of a curve drawn upon a surface, together with formula (8), which provides the expression of the normal curvature, one easily obtains

[64]"Quando avessimo a costruire la superficie incominceremo, per esempio, a delineare una curva s_v corrispondente ad $u = 0$. La posizione iniziale della tangente e del piano osculatore sono arbitrarie. La equazione $\dfrac{ds_v}{dv} = \sqrt{G}$ e le (6), (8), (12), (13), (15) in cui si faccia $u = 0$, forniranno i valori dei lati e degli angoli successivi di prima e seconda flessione della curva, attribuendo alla variabile v piccolissimi valori crescenti. Descritta quella linea s_v, potremo da qualunque suo punto condurre la linea s_u che vi corrisponde. La direzione della tangente iniziale si determina colle formole che danno $\cos\omega, \cos(\rho_v s_u)$ e quindi procederemo impiegando le formule $\dfrac{ds_u}{du} = \sqrt{E}$ e le (6), (8), … in cui si attribuisca a v il valore costante che vi corrisponde." (Mainardi 1856, p. 396).

the expression for the absolute curvature of $\boldsymbol{\gamma}_0$ in terms of the functions E, F, G, D, D', D'' evaluated at points $(u, v) = (0, v)$. These data are sufficient to determine the curve γ_0. Indeed, although Mainardi did not state it, we can interpret his assertion in the light of the fundamental theorem of the local theory of curves in Euclidean space.

Similarly, the assignment of the functions E, F, G, D, D', D'' allows us to univocally define at each point $\gamma_0(v)$ of $\boldsymbol{\gamma}_0$, a curve $\boldsymbol{\gamma}_v : u \mapsto \boldsymbol{\gamma}_v(u)$ such that $\boldsymbol{\gamma}_v(0) = \boldsymbol{\gamma}_0(v)$. This happens because the position of the tangent $\dot{\boldsymbol{\gamma}}_v(0)$ with respect to $\dot{\boldsymbol{\gamma}}_0(v)$ is prescribed by the functions $\cos\omega = \frac{F}{\sqrt{EG}}$ and by formula (3) in (Mainardi 1856, p. 386), which gives the angle between $\dot{\boldsymbol{\gamma}}_v(0)$ and the normal vector of $\gamma_0(v)$. Likewise, the normal vector of $\boldsymbol{\gamma}_v$ at $u = 0$ is determined by taking into account formulas (6) and (8), which can be used to determine the normal and the geodesic curvature of $\boldsymbol{\gamma}_v$ at $u = 0$.

Albeit not fully rigorous, Mainardi evidently regarded these remarks as a satisfactory proof of the fact that the assignment of functions E, F, G, D, D', D'' uniquely determines a surface defined by $\boldsymbol{x}(u, v) := \boldsymbol{\gamma}_v(u)$, save for its position in space.

3.6.2 Mainardi's claim to authorship of Codazzi's equations

According to (Moglia 1992), Mainardi only read Bour and Bonnet's memoirs containing MCE but with no mention of his results in 1869.

In general, he was conscious of the poor diffusion of many of his works, not only of (Mainardi 1856). As a claim about his own merits, namely *"to have found many truths and methods which have been appropriated by illustrious mathematicians"*,[65] he attempted to prove his priority in the discovery of various results, including the MCE, first in (Mainardi 1870), and then, when this did not produce the desired effect, in (Mainardi 1872).

We do not know why Mainardi did not participate in the Paris prize, but he considered himself its moral winner. Indeed, he wrote to Casorati: *"The prize of Bour, Bonnet and Codazzi is Mainardi!!! O glory!!"*;[66] and also: *"my cited memoirs* [he referred to (Mainardi 1856) and (Mainardi 1870)]*, to which a geometer of genius, Bour, would (pardon the hilarity) confer nothing less than the grand prix of the French Academy…"*.[67]

The only fault he imputed to himself was laziness: with his *"usual apathy for details"*[68] he did not prove that the four equations he had arrived at in (Mainardi 1856) could be reduced to three, which are simpler than those he derived in 1857 and more general than those derived by Bour because they are compatible with any coordinate system, as he showed in (Mainardi 1870). This judgement seems to be too simplistic. In addition to having published a rather obscure article, Mainardi seemed unaware of the real significance of MCE and their connection with the long-standing problem of applicability, to which he

[65]"Ho trovate moltissime verità e metodi che i Matematici distinti si sono appropriati" (Moglia 1992, p. 208)

[66]"Il premio di Bour, Bonnet, Codazzi cioè Mainardi!!! O gloria!!" (Moglia 1992, p. 225)

[67]"Le mie memorie succitate, alle quali un geometra di genio, il Bour, avrebbe (mi si perdoni l'ilarità) nientemeno che conferito il gran premio dell'Accademia Francese..." (Mainardi 1872, p. 78)

[68]"Io però non ho mai dimostrato che le equaz. (19, 20, 21, 22) sono due per esem.o le (21) (22): colla mia solita apatia ai dettagli" (Moglia 1992, p. 199)

never referred. The MCE were not understood as the differential equations of all surfaces applicable to a given one, but simply as relations between the coefficients of the first and second fundamental forms, whose deduction was expedient to the proof of the FT. The relevance of MCE was instead completely explained by Bour, who wrote: "[the MCE] *elegantly complete Gauss' theorem, by making known* all *the conditions which limit the number of possible deformations of a given surface"*.[69]

3.7 Reception of the Fundamental theorem of surface theory

Peterson, and Mainardi after him, were the first to provide proofs for the MCE along with a statement of the FT. However, it was only with Bonnet that a fully rigorous and complete proof of this existence and uniqueness result was actually given. Indeed, previous attempts were limited to proposing heuristic considerations of a geometrical kind that did not discuss the analytical properties of the solutions of differential systems, such as (I)-(II).

The look at (Mainardi 1856) should have provided an idea of the novelty introduced by Bonnet's analytical formulation. A similar approach was adopted in Peterson's doctoral dissertation. Peterson, too, did not provide any discussion of the existence and uniqueness result from the point of view of the solutions of a system of partial differential equations. (Phillips 1979, p. 155) explains this by observing that Peterson would certainly have furnished such a detailed analysis, *"were he not working under the pressure of a deadline"*. More convincingly, one can argue that the need for a systematic investigation on this subject could be appreciated only after the MCE were recognised as an essential tool for tackling various problems concerning applicability between surfaces, such as that discussed in 3.5.4.

From this perspective, we can regard the FT as the result of a collective work to which Bour, Bonnet and Codazzi contributed in different ways. As shown earlier, the Prize competition offered a common ground for comparison that turned out to be essential in promoting Bonnet's fundamental achievements.

For the reception of the FT in the early years after the publication of (Bonnet 1867), it is interesting to observe that Bianchi in the first edition of his *Lezioni* devoted no special attention to either the MCE or to the FT. Indeed, he just presented a brief discussion of the FT in an appendix to Chapter 4, in which he made use of the notion of Jacobian systems associated to the equations (I)-(II). Curiously, no mention was made of (Bonnet 1867). Instead, Bianchi referred to (Lipschitz 1883), where Lipschitz, probably unaware of Bonnet's investigations, had offered an alternative proof of the FT. Similarly to Bonnet, Lipschitz's main motivation behind the composition of (Lipschitz 1883) was the need to complete Bour's second method for dealing with the second applicability problem.

An explicit recognition of Bonnet's priority was given, probably for the first time, in (Stahl and Kommerell 1893, §8), where the FT was referred to as the "Satz von Bonnet".

[69]"quell'elles complètent élégamment le théorème de Gauss, en faisant connaître *toutes* les conditions qui limitent le nombre des déformations possibles d'une surface donnée." (Bour 1862b, p. 7)

The attribution was then reinforced and definitely established, as it were, in Felix Klein's influential *Encyklopädie der Mathematischen Wissenschaften*.[70]

[70] See (Lilienthal 1902, p. 158).

Chapter 4

Weingarten through his letters to Bianchi

4.1 The correspondence between Bianchi and Weingarten

We now pause the study of the second applicability problem, which we will resume in Chapter 5, to focus on Julius Weingarten (1836-1910), who will be the protagonist of that chapter. An original description of his figure is offered by his dense correspondence with Luigi Bianchi (1856-1928).[1]

A considerable part of Bianchi's scientific correspondence, which was published in (Bianchi 1959), consists of letters sent to him by his friend Julius Weingarten.[2] These cover a long period of time: the first letter dates back to 28th October 1884, when Bianchi, just 28 years old, was already highly esteemed in international academic circles, and they stopped in 1905-1908,[3] when Weingarten, who was twenty years older than Bianchi, retired.

It is unclear how contact between the two began. Although Blaschke claimed they had met during Bianchi's stay in Munich and Göttingen in 1879-1881,[4] a letter dated back to October 1885 disproves this reconstruction, as Weingarten expresses surprise at the sight of Bianchi's image in a photograph that his colleague had sent him:

> I received your picture, which you were so kind as to send me, with great pleasure and heartfelt interest. The idea I had of you in my mind was different from that given by the photograph itself. I imagined you, as we North Germans are accustomed to here, with a more southern, sharper facial expression.[5]

[1]For a biographical and scientific account of Luigi Bianchi, the interested reader can refer to (Scorza 1930) and (Fubini 1928).

[2]As far as we know, the letters from Bianchi to Weingarten are lost.

[3]In the last letter, which is undated, Weingarten wrote that he was teaching in Freiburg and was wondering whether to join the International Congress of Mathematicians 1908 in Rome.

[4](Blaschke 1954, p. 44)

[5]"Mit grosser Freude und innigstem Interesse habe ich Ihr Bild, das Sie so freundlich waren, mir zu schicken, entgegengenommen. Die Vorstellung die ich mir in Gedanken von Ihnen machte, war eine andere, als die durch die Photographie selbst gegeben. Ich habe mir Sie, wie wir Norddeutsche hier gewohnt sind, mit südlicherem, scharferem Gesichtsausdruck vorgestellt." (Bianchi 1959, p. 169)

Moreover, in the very first lines of (Bianchi 1911), which contains, with the family's permission, some formulas that Weingarten had communicated to him, he stated that the correspondence took place from 1884 onwards.[6]

In these 73 letters, the two scientists discussed their most recent investigations and the guidelines of their research, and soon developed a close scientific collaboration. These letters can be divided into distinct groups according to their subjects, each of which constitutes a discussion on different topics. To provide a rough indication of their content, we summarise the main ones here, but the dialogue between the two was much more varied than this list reveals:

- the letters that cover the period October 1884-December 1885 are mainly devoted to triple orthogonal systems of surfaces with constant curvature
- the letters between December 1885-April 1887 are mainly related to Weingarten's investigations on infinitesimal deformations, later reported in (Weingarten 1886) and (Weingarten 1887b)
- the letters in October-November 1887 contain Weingarten's notes on surfaces with isothermal lines of curvature, later published in (Bianchi 1887b)
- the letters that cover the period June 1888-February 1889 mainly expound the research in hydrodynamics later published in (Weingarten 1890)
- the letters between February 1889-February 1896 mainly deal with the search for complete classes of applicable surfaces and Weingarten's new method for applicability
- from the letter written in Rome during Weingarten's trip to Italy in April 1895 to the last letter, the mathematical content is sparser.

Alongside this professional affinity, in reading the correspondence between Bianchi and Weingarten one can also appreciate the consolidation of what Bianchi called a "sincere friendship",[7] emerging especially in the last letters where Weingarten indulged in more personal confessions.

4.2 Brief portrait of Julius Weingarten

Little is know about Weingarten's family of origin.[8] His father Joseph[9] was a Polish Jew, who converted to Anglicanism and emigrated to Berlin. He was a weaver who married the

[6] From the content of the first letter in (Bianchi 1959), one can also deduce that it is not the first letter they actually exchanged.

[7] (Bianchi 1910a, p. 217)

[8] Besides the official commemorative speeches (Lüroth 1910), which also contains the list of Weingarten's publications, (Bianchi 1910a) and (Jolles 1911), other sources are a short curriculum vitae at the end of (Weingarten 1864a) and the biography of his brother, Hermann Weingarten, (Jülicher 1910).

[9] Sources do not agree on the exact date of his death. According to Weingarten's words in (Bianchi 1959, p. 260), however, it is probable that he died in 1868, although, in the same sentence, he incorrectly stated that he was 43 years old when his mother died.

daughter of one of his Bavarian colleagues, Dorothea Ebner. She was a strict adherent of the Pietistic current, as were her family, and brought up her two sons, Hermann (1834-1892) and Julius (1836-1910) according to these principles. Hermann remembered his recently deceased mother in the introduction of his book (Weingarten 1877a) emphasising the decisive role of faith and Pietist approach in his and his brother's upbringing:

> I associate with this work the memory of my dear, recently deceased mother, whose youth in the congregation and community of Jänicke and Gossner provided a strong, firm direction for life and her often difficult days, whose strength of self-sacrifice helped my brother and me to prepare the foundations of our careers, whom I thank for the love, care and loyalty without measure with which she stood by my side in Berlin, Marburg and Breslau.[10]

In spite of economic difficulties, Joseph and Dorothea Weingarten provided their sons with a comprehensive education. While Hermann's studies were mostly directed towards theology, which led to him becoming a theologian and a church historian, Julius' were more technical. He first attended the *Gymnasium Zum Grauen Kloster* and then the *Berliner Gewerbeschule*, where he graduated in 1852. His exceptional talents in the fields of mathematics and science attracted the attention of the school's director, Karl Friedrich von Klöden, and the mathematician Röber, who influenced Julius' career choice by introducing him to higher studies.

Hence, Julius enrolled at the University of Berlin in 1853, which required ministerial authorisation as he did not have a gymnasium diploma. In particular, he remembered the lectures given by Magnus, Dirichlet, Weierstrass and Pohlke. In addition, according to quite a common practice at that time, he combined theoretical studies with more practical subjects by studying chemistry at the Gewerbeinstitut in Berlin, where he was also assistant for mathematics between 1855 and 1857.

After graduating in 1857, the two brothers obtained doctorates and attempted academic careers. In the meantime, in order to compensate for the family's financial problems, they both taught in schools in Berlin. Julius taught from 1858, when he obtained an exemption to compensate for the absence of a gymnasial title, to 1864, when he obtained his doctorate at the University of Halle on 6th February. According to Leo Koenigsberger, who frequented Weingarten during his time in Berlin between 1857 and 1864, however, teaching in high school did not fit well with his personality.[11]

[10]"Mit dieser Arbeit das Gedächtnis verbinde meiner lieben, vor kurzem entschlafenen Mutter, deren Jugend noch in der Gemeinde und Gemeinschaft Jänickes und Gossners die starke feste Richtung für das Leben und ihre oft schweren Tage gewonnen hat, deren Kraft aufopferndster Selbstverläugnung meinem Bruder und mir die Grundlagen unserer Laufbahn hat bereiten helfen, der ich danke für Liebe, Sorge und Treue ohne Mass, mit der sie in Berlin, Marburg und Breslau mir zur Seite gestanden." Hermann never married and lived with his mother until her death on 13th April 1877.

[11]See (Koenigsberger 1919, p. 21). Koenigsberger also provided further fragments of Julius's life during the 1860s in the first section of his autobiography (Koenigsberger 1919). For example, he wrote that he had sometimes met Weingarten and, among others, Heinrich Bertram, Jochmann, Hermes, Natani, Meyer,

During this difficult period, in which his occupations were divided between research and school teaching, Weingarten wrote two papers of remarkable quality, (Weingarten 1861) and (Weingarten 1863), which made great contributions to surface theory and which made his name known in the European mathematical milieu within a relatively short time.[12]

Despite international recognition for his research, his academic career was not so brilliant. In order to support himself, he had to take more unpretentious positions than he aspired to for most of his life. In 1864 Weingarten became *private lecturer* at the Bauakademie in Berlin and finally became a *permanent professor* of mechanics in 1874. As reconstructed in (Kändler 2009, p. 216), Weingarten addressed the minister with quite sharp words at the beginning of January 1874 to obtain this position. He had been promised a regular job for the previous year, and had in the meantime reached an age *"at which the insecurity of an official position tends to be tainted with the stigma of incompetence"*; moreover, the failure to acquire the position was even more serious, since a younger colleague had recently been permanently hired. Weingarten succeeded in mid-January.

In 1879 he moved to the Berlin Technische Hochschule, which was founded through the merger of the Bauakademie and the Vocational Training Academy to become the leading Prussian technological training institution. Since both the Bauakademie and the Technische Hochschule were institutes aiming at technical education, mathematics was considered as an auxiliary discipline and there was no professorship in mathematics. Thus, in his time at Berlin Technische Hochschule, Weingarten never held a mathematics professorship and never taught geometry, only mechanics, elasticity theory and hydrodynamics.[13]

Regarding his teaching activity, according to (Bianchi 1910a, p. 218), Weingarten was surrounded by a considerable number of students who were attracted by the fame of his lectures, especially in his first decade of teaching. Together with Kronecker, he was doctoral tutor to Paul Stäckel, who received the title in 1885.[14] *"Then, unfortunately, the poor state of his health and painful events in his family forced him to gradually reduce his teaching activities"*.[15] In (Koenigsberger 1919, p. 21), there is an even more enigmatic statement:

> I saw him several times in Switzerland with his wife and children and I was later extremely saddened by the tragic end of his family fortune, even though I did not find such a conclusion to his life surprising given the unsteady character of this excellent man.[16]

Hamburger and Paul du Bois-Reymond over a beer to spend a few hours in stimulating and fruitful scientific conversations.

[12] These papers were cited in (Dini 1865a), (Beltrami 1864-1865), (Bonnet 1867), to name a few.

[13] On this subject, see (Knobloch 1989, p. 257).

[14] In the introduction of his thesis, in (Stäckel 1885, p. 3), Stäckel wrote: *"Before moving on to the subject, I would like to express my most respectful thanks to my esteemed teachers Professor L. Kronecker and Professor J. Weingarten for their kind advice and for the participation they have shown in the progress of my work"*.

[15] (Bianchi 1910a, p. 218)

[16] "Ich habe ihn im Laufe der Jahre öfter in der Schweiz im Kreise seiner Frau und Kinder gesehen und war später durch das tragische Ende seines Familienglücks überaus traurig berührt worden, wenn ich auch bei dem unsteten Charakter dieses ausgezeichneten Mannes einen solchen Lebensabschluss nicht überraschend

Unfortunately, there is little information about Weingarten's private life. We know that he was married to Auguste Aber in August 1873 and that they had two daughters born around 1875 and 1879, Rose and Dora.[17] During the autumn-winter 1888-1889 several familiar troubles kept him away from work for almost six months. He commented:

> Many family matters have kept me from work for almost half a year. Nothing has progressed and I do not yet see when I will again find time or, better still, peace and quiet to my liking.[18]

We know that in September he had to take care of his brother, who had had a stroke.[19] Some unspecified health problems also affected Weingarten: in August 1889 he wrote to Bianchi that the doctor had prescribed a rest cure.[20]

In the meantime, his eldest daughter Rose became increasingly close to her father. She accompanied him to the first International Congress of Mathematicians in Zurich in 1897, where she met and became close to Vito Volterra, to whom her father had been bound by a great esteem and friendship since 1886 when they had collaborated on infinitesimal deformations. Incidentally, on this occasion Weingarten also met Ricci Curbastro, Brioschi and Pincherle. In 1895 Rose also went with her father to Italy for a holiday in Naples, Venice, Pisa, Bologna and Rome, during which they met Bianchi and his family. From then on, Weingarten never failed to send warm greetings from him and his daughter to Bianchi's family in his letters.

A passage of a letter dated 26th November 1905 is particularly interesting both for Weingarten's personal tone that confirms his familiarity with Bianchi and for the mention of his daughter Rose that illuminates the father-daughter relationship:

> It was with great sadness that my daughter Rose and I read the sad news that your lovely letter of the 24th of this month brought us. We spent the whole evening talking about you and your dear family, and about the unforeseen and incomprehensible fates of people. In my life, which has now spanned almost seven decades, I too have experienced many profoundly painful situations and have been able to put myself in your shoes. What makes life still worthwhile and desirable under such circumstances is the thought of the children to whom one came to be something and must be something. And the only way to get over such hurdles in life is to work, which is what you have chosen.[21]

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fand." (Koenigsberger 1919, p. 21)

[17] The correspondence with Bianchi also provides some images of their everyday life, such as Weingarten's descriptions of summer holidays.

[18] "Mannichfache Familienangelegenheiten haben mich seit fast einem halben Jahr vom Arbeiten sehr zurückgehalten. Es ging Nichts vorwärts, und ich übersehe noch nicht, wann ich wieder so recht nach meinem Geschmack werde Zeit oder besser Ruhe finden".(Bianchi 1959, p. 216)

[19] After a brief recovery in 1889, his condition worsened and he had to be transferred to the sanatorium in Pöpelwitz as a hopeless nervous patient in September 1890 and died there on 25th April 1892.

[20] (Bianchi 1959, p. 233)

[21] "Mit grosser Betrübniss habe ich und meine Tochter Rose die traurigen Nachrichten gelesen, die Ihr

The period of fatigue culminated in 1902, when Weingarten requested a sabbatical year. Then, he retired to Freiburg-im-Breisgau, moving away from his hometown in Berlin. His project was to spend his last working years in a quiet town in *"an enchanting location"*[22] at the foot of the Black Forest. Rose was also living there before moving to Paris.

Here, Weingarten could continue working at the local university. The place was provided for him by Jacob Lüroth, who was an *"intelligent and friendly"*[23] man he became attached to during his stay in Freiburg. Among his colleagues, besides Lüroth, Weingarten mentioned only Ludwig Stickelberger and Alfred Loewy. *"Nevertheless"*, he wrote to Bianchi, *"there is not much activity here in Freiburg in the field of mathematics, and the stock of literature is also quite poor. There is no library like yours in Pisa, and even our Berlin library was behind yours"*.[24] The level of education was also generally quite low and *"only future gymnasium teachers, not future mathematicians, study here"*.[25]

He was appointed honorary ordinary professor in 1905. When, on the occasion of sending New Year's greetings in 1908, he communicated the news to Koenigsberger, he wrote:

> When I became *professor honorarius* in Freiburg through the kind interest that Lüroth took in me, I remembered the good old Fuchs; he once said (40 years ago, when he was still a teacher at Gallenkamp): "Yes, Weingarten, nothing will come of us; one day I will still be a professor, but at the age of 70 I will be an associate professor in Tübingen"; at that time Tübingen was the last place among the universities for him. Now, I have become a university professor at the age of 70; he did not live to see it, he would have been amused.[26]

Between autumn 1905 and summer 1908 he was also asked to teach part-time courses in mechanics, with an introduction to the rudiments of surface theory that he had never taught before. However,

lieber Brief vom 24. d. M. uns brachte. Wir haben den ganzen Abend von Ihnen und Ihrer lieben Familie gesprochen, und über die unvorhergesehenen unbegreiflichen Menschenschicksale. Auch ich habe in meinem, jetzt fast schon sieben Jahrzehnte umfassenden Leben manches tief Traurige erfahren, und mich in Ihre Lage hineinversetzen können. Was unter solchen Umstanden das Leben noch werth und wünschenswerth macht ist der Gedanke an die Kinder, denen man etwas sein kann und sein muss. Und der einzige Weg über solche Lebensklippen hinwegzukommen ist der zu arbeiten, den Sie ja gewählt haben." (Bianchi 1959, p. 288)

[22](Bianchi 1959, p. 285)

[23](Bianchi 1959, p. 287)

[24](Bianchi 1959, p. 289). Weingarten sometimes complained to Bianchi about the condition of scarce resources in the libraries in Berlin and wrote to him to gain information about the state of art in the topic of his researches.

[25](Bianchi 1959, p. 290)

[26]"Als ich duch das liebenswürdige Interesse, das Lüroth an mir nahm *professor honorarius* in Freiburg wurde, erinnerte ich mich des guten alten Fuchs; er sagte einmal (vor 40 Jahren, als er noch Lehrer bei Gallenkamp war): "Ja Weingarten, aus uns wird nichts; ich werde einmal noch Professor werden, aber mit 70 Jahren außerordentlicher Professor in Tübingen"; Tübingen nahm damals für ihn die letzte Stelle unter den Universitäten ein. Nun bin ich mit 70 Jahren doch Universitätsprofessor geworden; er hat es nicht mehr erlebt, er hätte sich darüber amüsiert." (Koenigsberger 1919, p. 22)

> Here in Freiburg, only very few students study pure mathematics. But the few who attend my course are intelligent and hard-working. There are two in total, the third is irregular.[27]

His election to professor was not the first manifestation of esteem from the Academy milieu. As is well known,[28] in 1891 the *Mathematical Society of Berlin* was founded in an attempt to organise the mathematical life in Berlin. It was mainly stimulated by professors connected with the Technische Hochschule, which was a more dynamic environment than that of the University of Berlin. Weingarten supported this initiative and was chosen as the Society's first president. In 1896 he was elected as corresponding member of the Göttingen Academy of Sciences; in 1890 he was appointed member of the Leopoldina; and in 1899 foreign member of the Italian Accademia dei Lincei. In the meantime, in 1894 he also won a Grand Prix des Mathématiques on the theme of applicability, awarded by the Académie des Sciences in Paris.

Weingarten was also an invited speaker at the International Congress of Mathematicians, which took place in Heidelberg in 1904. He talked about "A simple example of the stationary and rotation-free movement of a heavy liquid capable of forming drops with a free boundary".[29] Bianchi also invited him to the 1908 International Congress of Mathematics in Rome, but he could not accept due to his health condition. For the same reason, he had to suspend his course in summer 1908.

Weingarten had long been lamenting the advance of old age. As he wrote to Bianchi in some letters dating back to the first decade of the twentieth century, he found it increasingly difficult to quickly understand the ideas of others and to develop his own thoughts in a coherent manner. As early as 1903 he wrote to Bianchi:

> I could be satisfied with my health, if age did not cause many problems. I am not too far from my seventieth year. Long walks in the mountains and, unfortunately, also in the realm of thought, no longer go as well as they used to. The elasticity of the body and also of the mind is already decreasing. It is no longer easy for me to find my way into new thoughts and to settle in.[30]

A 1908 letter to Koenigsberger highlights how this condition humiliated him:

> Images from the ancient past have surfaced in my memory. But also the nagging feeling of having grown old. What one could absorb back then! When I read

[27]"Hier in Freiburg studiren nur sehr wenig Studenten reine Mathematik. Die wenigen aber, die mein Colleg hören sind intelligent und fleissig. Es sind im ganzen zwei, der dritte ist unregelmässig." (Bianchi 1959, p. 289)

[28](Knobloch 1989, p. 266)

[29]"Ein einfaches Beispiel einer stationären und rotationslosen Bewegung einer trupfbaren schweren Flüssigkeit mit freier Begrenzung", (Weingarten 1904)

[30]"Mit meiner Gesundheit konnte ich wohl zufrieden sein, wenn nicht das Alter vieles Lästige hervorriefe. Ich bin ja nicht allzuferne dem siebzigsten Jahr. Grössere Spaziergange im Gebiete der Berge und leider auch im Gebiete der Gedanken, wollen nicht mehr so gut gehen wie ehemals. Die Elasticität des Körpers und auch des Geistes nimmt schon ab. Es wird mir nicht mehr leicht, mich in neue Gedanken hineinzufinden und einzuleben. " (Bianchi 1959, p. 286)

new things today, I have the feeling that I lack the dexterity to use these new things, that I will not be able to use them for my own reasonings, and that I am too old to learn, however beautiful what I have read may be. That is an embarrassing feeling.[31]

On 6th March 1910 he suffered a stroke from which he could not recover. He died on 16th June. At the funeral, Luroth synthesized his friend's personality as follow:

> Weingarten was a likeable, kind and noble man, unpretentious and full of spirit, a good companion in society; he attached himself to his friends and was grateful for all the courtesies he received. He also possessed great acuity in judging men and great finesse in knowing them.[32]

Ambition was certainly not part of his personality and this probably influenced his academic career along with the aforementioned personal problems. His humility and sense of gratitude led him to extend warm thanks for the tokens of esteem he received. He was aware of the role of prestigious figures such as Bianchi and Darboux in the success and dissemination of his work:

> In my scientific career, I have had undeserved luck with some things, and that is because important people like you [Bianchi] and Darboux have focused their attention on them. This has benefited me and given me great pleasure.[33]

4.2.1 An overture to his major research

The list of Weingarten's works includes around 60 publications, generally short and concise. His research activity essentially covers three areas: mathematical physics, geodesy and surface theory.

With regard to the first, Weingarten worked on problems related to theoretical physics from time to time throughout his career, inspired by Dirichlet's teachings on potential theory, which *"lively excited his interest and inspired in him new ideas"*, as his collegue at the Technische Hochschule, Stanislas Jolles, wrote in (Jolles 1911, p. 143). The first papers were written early in his career: in 1855 he dealt with some properties of the Laplace equation and the dynamics of pendulums, while in 1864 he corrected a mistake in (Roch 1863)

[31]"Bilder aus alter Vergangenheit sind in meinem Gedächtnis aufgetaucht. Aber auch wieder das leidige Gefühl, altgeworden zu sein. Was konnte man damals alles in sich aufnehmen! Wenn ich heute Neues lese, so habe ich das Gefühl, daßmir die Fingerfertigkeit abgeht, das Neue zu verwerten, daßich es nicht werde gebrauchen können für einen eignen Gedankengang, und daßich zu alt bin, um zu lernen, wie schön auch immer das Gelesene sein mag. Das ist ein peinliches Gefühl." (Koenigsberger 1919, p. 21)

[32]"Weingarten era un uomo simpatico, buono e nobile, senza pretese e pieno di spirito, un ottimo compagno in società; si attaccava agli amici ed era riconoscente per tutte le cortesie che riceveva. Egli possedeva inoltre una grande acutezza nel giudicare gli uomini e una grande finezza nel conoscerli. (Lüroth 1910, p. 66)

[33]"In meiner wissenschaftlichen Laufbahn habe ich mit einigen Dingen unverdientes Glück gehabt, und zwar dadurch, dass so bedeutende Leute wie Sie und Darboux ihre Aufmerksamkeit darauf gerichtet liaben. Das hat mir genutzt und grosse Freude bereitet." (Bianchi 1959, p. 291)

about an application of potential theory to the movement of electricity inside a conductor.[34] Weingarten returned to physics in 1890 when he published a treatise on a problem of hydrodynamics, (Weingarten 1890). To study particular stationary movements in a homogeneous and incompressible liquid, he investigated functions $V(x, y, z)$ with vanishing Hessian determinant that satisfy the Laplace equation

$$\Delta_2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{4.1}$$

The differential of such a function V is given by

$$dV = \xi(x, y, z)dx + \eta(x, y, z)dy + \zeta(x, y, z)dz,$$

whose components ξ, η, ζ can be regarded as angular coordinates of the points of a surface S and thus represented as functions of two independent variables u, v. In this way, the parametric equations of S are $(u, v) \mapsto (\xi(u, v), \eta(u, v), \zeta(u, v))$. To Weingarten's great pleasure and astonishment, he succeeded in leading the study of this kind of motions back to that of minimal surfaces trough the following result.[35] He showed that a function V with vanishing Hessian determinant satisfies the Laplace equation (4.1) if, and only if,

$$\frac{1}{\rho} + \frac{1}{\rho'} = 0 \qquad \Delta_2 \theta = 0$$

holds true, where ρ and ρ' are the principal radii of curvature of S and $\theta(u, v)$ is a function such that

$$\theta = \xi x + \eta y + \zeta z - V$$
$$\frac{\partial \theta}{\partial u} = x\frac{\partial \xi}{\partial u} + y\frac{\partial \eta}{\partial u} + z\frac{\partial \zeta}{\partial u} \qquad \frac{\partial \theta}{\partial v} = x\frac{\partial \xi}{\partial v} + y\frac{\partial \eta}{\partial v} + z\frac{\partial \zeta}{\partial v}.$$

In the early years of the twentieth century, in the short work (Weingarten 1901), which was published in Italian in the journal *Rendiconti della Reale Accademia Lincei*, as suggested by Volterra,[36] he devoted himself to the description of elastic bodies through surfaces of discontinuity. He noted the possibility of the existence of elastic deformations and internal tensions in continuous elastic bodies, even in absence of external forces, when material is inserted (e.g. a not entirely closed ring whose ends are welded with a thin layer of material). He investigated this case by studying some surfaces, which these bodies necessarily contain, where displacements are discontinuous. Also on this occasion Weingarten could apply his previous geometrical studies—specifically those related to the problem of infinitesimal deformations of surfaces—to describe physical phenomena. Volterra then elaborated in great detail the content of (Weingarten 1901) in a series of papers that include (Volterra 1907). These works were the beginning of the theory of dislocations.[37]

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[34](Weingarten 1855b), (Weingarten 1855a), (Weingarten 1864b)

[35]See (Weingarten 1890, p. 318). With respect to the original notation, we changed the Gaussian parameters p, q with u, v.

[36]It is a kind of homage to the Accademia dei Lincei that had elected him as its member.

[37]The interested reader can refer to (Sluckin 2018) for a brief description of Weingarten's and Volterra's work on elasticity.

From then on, throughout his time in Freiburg, although now rather elderly, which was *"A quality that is not conducive to work"*,[38] he devoted himself exclusively to problems related to physics; hydrodynamics, the theory of elastic bodies, and the theory of vortices.

It is not clear, however, which circumstances determined his interest in differential geometry. Early traces of his works on this subject date back to 1857, when he presented (Weingarten 1864a) on the occasion of a competition organised by the University of Berlin, for which the determination of curvature lines for new classes of surfaces was required. The committee attributed the prize to the 21-year-old Weingarten *"for the seriousness of the issues and for the subtlety of their reasoning"*.[39] His paper was appreciated, especially for the general approach to the problem. It also attested to his knowledge of modern results as well as classical ones, which allowed him to give new and elegant demonstrations of already-known theorems, such as Joachimsthal's theorem, and deeply analyse the case of developable surfaces and minimal surfaces. This work also earned him his doctoral title in 1864.

Before obtaining his doctorate, he also wrote (Weingarten 1861) and (Weingarten 1863), two papers of remarkable quality that made great contributions to surface theory. In the former, he introduced the so-called $W-$surfaces, which are surfaces whose radii of curvature are bounded by a functional relation. Therein, he also proved *"two famous theorems, which would not spoil the great Memoir of Gauss"*, as Darboux stated on the occasion of his speech at the 1908 International Congress.[40] These outstanding results essentially show the equivalence of the theory of evolutes with the difficult problem of finding surfaces locally isometric to surfaces of revolution. In the latter, by using $W-$surfaces, he succeeded in finding the first two examples of non-trivial complete classes of applicable surfaces.[41] These results made $W-$surfaces *"one of the more attractive chapters of geometry"*.[42]

In the same period, a lesser-known work, (Weingarten 1862),[43] gives further details about Weingarten's education. It is included in a book by Johann Jacob Baeyer, who was a German lieutenant-general in the Royal Prussian Army. As a geodesist, he was one of the driving forces establishing the Royal Prussian Geodetic Institute, as well as its first director until his death. Baeyer promoted the establishment of a comprehensive, approved method for the measurement of the curvature of the Central Europe meridian, which led to the creation of the International Association of Geodesy. In presenting the content of a chapter in his book *Das Messen auf der sphäroidischen Erdoberfläche*, Baeyer wrote: *"Mr Weingarten, a talented young mathematician I tried to interest in geodesy some time*

[38]"eine Eigenschaft die dem Arbeiten nicht günstig ist." (Bianchi 1959, p. 284)

[39](Anonymous 1858, p. 7)

[40](Darboux 1908, p. 111)

[41]For a more detailed description of the results, the reader is referred to Section 5.3.1.

[42](Darboux 1894, p. 317)

[43]Here, Weingarten generalised one of Baeyer's results on the rotational spheroid to the case of a generic surface and showed that McLaurin's development of the difference in the distance between two points, measured along the geodesic line and along the normal section passing through the two points, only begins with the fifth power for any surfaces, and is therefore generally imperceptible.

ago...".[44] Weingarten does not explicitly mention him as one of his masters, however he dedicated (Weingarten 1864a) to him, *"viro illustrissimo et excellentissimo".* At Baeyer's suggestion, Weingarten devoted only two other papers to geodesy, (Weingarten 1869) and (Weingarten 1870), in which he tackled the problem of the Earth's shape in a theoretical way. He was also a member of the Prussian commission for the measurement of the central Europe meridian at the time, as attested to by the meeting protocols.

From 1877, when he dealt with the theory of triple orthogonal systems in (Weingarten 1877b), to the end of the century, his research topics were almost exclusively related to differential geometry. After tackling problems related to the existence of isothermal surfaces (Weingarten 1881), the content of (Christoffel 1868) on the movement of geodesic triangles on a surface (Weingarten 1882), and the determination of surfaces with isothermal curvature lines (Weingarten 1883b), Weingarten returned to the subject of applicability and deformations. He contextualised the study of surfaces and the problem of applicability in the framework of quadratic differential forms and deformation invariants by treating surfaces with constant curvature in (Weingarten 1883a) first, and by dealing with surface theory more generally in (Weingarten 1884). In particular, (Weingarten 1884) offered an interesting and original deduction of the MCE. Weingarten imagined studying a surface $S : (x_1, x_2) \mapsto \boldsymbol{y}(x_1, x_2)$ that is embedded in the Euclidean space through the study of two quadratic differential forms—the first and the second fundamental forms—that vary simultaneously. In a slightly different notation due to Bianchi that makes use of Ricci's absolute differential calculus,[45] two quadratic differential forms

$$f = \sum_{r,s=1}^{2} a_{rs} dx_r dx_s \qquad \varphi = \sum_{r,s=1}^{2} b_{rs} dx_r dx_s \tag{4.2}$$

vary simultaneously if φ is covariant with respect to f. In the case (4.2) are the first and the second fundamental forms of S, respectively, Weingarten defined the components of a covariant tensor as follows[46]

$$b_{112} = \frac{\partial b_{11}}{\partial x_2} - \frac{\partial b_{12}}{\partial x_1} + \sum_{\mu} \Gamma^{\mu}_{11} b_{2\mu} - \sum_{\mu} \Gamma^{\mu}_{1l} b_{2\mu} \tag{4.3}$$

$$b_{221} = \frac{\partial b_{22}}{\partial x_1} - \frac{\partial b_{21}}{\partial x_2} + \sum_{\mu} \Gamma^{\mu}_{22} b_{1\mu} - \sum_{\mu} \Gamma^{\mu}_{1l} b_{2\mu}, \tag{4.4}$$

[44]"Herr Weingarten, ein junger talentvoller Mathematiker, den ich vor einiger Zeit für die Geodäsie zu interessiren suchte..." (Weingarten 1862)

[45]Bianchi explained Weingarten's theory of simultaneous quadratic forms in (Bianchi 1922, §40-42).

[46]Weingarten denoted b_{112} and b_{221} with $c_a(u)$ and $c_a(v)$, respectively, where a is the determinant of the first fundamental form. See (Weingarten 1884, pp. 19–20). In the multidimensional case, Bianchi showed that the components of the corresponding covariant tensor are

$$b_{ikl} = \frac{\partial b_{ik}}{\partial x_l} - \frac{\partial b_{il}}{\partial x_k} + \sum_{\mu} \Gamma^{\mu}_{ik} b_{l\mu} - \sum_{\mu} \Gamma^{\mu}_{il} b_{k\mu}, \qquad i, k, l = 1, \ldots, n.$$

Since $b_{ikl} + b_{ilk} = 0$ and $b_{ikl} + b_{kli} + b_{lik} = 0$ hold true, in the case of surfaces the independent components of b_{ikl} reduce to two—b_{112} and b_{221}.

where Christoffel's symbols[47] Γ^l_{ik} are computed with respect to f. By requiring b_{112} and b_{221} to vanish, Weingarten obtained two conditions that correspond precisely to the MCE.[48]

In (Weingarten 1886) and (Weingarten 1887b), he proposed a new approach for infinitesimal deformations. An *infinitesimal deformation* of a given surface $S : (u, v) \mapsto (x(u, v), y(u, v), z(u, v))$ is a surface $\tilde{S} : (u, v) \mapsto (\tilde{x}(u, v), \tilde{y}(u, v), \tilde{z}(u, v))$ that is applicable to S and such that

$$\begin{cases} \tilde{x} = x(u, v) + \epsilon x_1(u, v) \\ \tilde{y} = y(u, v) + \epsilon y_1(u, v) \\ \tilde{z} = z(u, v) + \epsilon z_1(u, v) \end{cases}$$

where ϵ is a small constant and x_1, y_1, z_1 are generic functions of u and v.[49] Complaining of the lack of symmetry in the use of Cartesian coordinates when describing a surface by $z = z(x, y)$, in which z has a privileged role with respect to x and y, he developed an elegant theory on the basis of differential invariants. In particular, Weingarten supposed that a given surface $S : (u, v) \mapsto \boldsymbol{x}(u, v) = (x(u, v), y(u, v), z(u, v))$, whose normal vector field is $\boldsymbol{X} = (X_1, X_2, X_3)$, was infinitesimally deformed into $S' : (u, v) \mapsto \boldsymbol{x'}(u, v) = (x'(u, v), y'(u, v), z'(u, v))$ in such a way that S and S' have the same line element. He then defined a surface $S_0 : (u, v) \mapsto \tilde{\boldsymbol{w}}(u, v) = (\tilde{w}_1(u, v), \tilde{w}_2(u, v), \tilde{w}_3(u, v)) = (\frac{x-x'}{2}, \frac{y-y'}{2}, \frac{z-z'}{2})$. The search for the infinitesimal deformation S' is finally reduced to the search of a function φ, called *characteristic function* by Bianchi, that is solution of the following differential equation[50]

$$\frac{\partial}{\partial u}\left(\frac{D''\frac{\partial\varphi}{\partial u} - D'\frac{\partial\varphi}{\partial v}}{k\sqrt{a}}\right) + \frac{\partial}{\partial v}\left(\frac{D\frac{\partial\varphi}{\partial v} - D'\frac{\partial\varphi}{\partial u}}{k\sqrt{a}}\right) = \sum_{i=1}^{3} \frac{\partial X_i}{\partial v}\frac{\partial w_i}{\partial u} - \sum_{i=1}^{3} \frac{\partial X_i}{\partial u}\frac{\partial w_i}{\partial v}$$

where k and a are the Gaussian curvature and the determinant of the line element of S, respectively. Along with Volterra, who was working on similar problems, this was also favourably welcomed by Darboux and Bianchi, who both improved Weingarten's results and included them in their textbooks.

Apart from (Weingarten 1888), which shows that the geodesics of a $W-$surface can always be determined by quadratures, publications after 1887 became more sporadic and related to the second problem of applicability. This period culminated with the winning of the second Grand Prix des Mathematiques awarded by the Academie des Sciences in Paris

[47] Christoffel's symbols of a quadratic form $f = \sum_{r,s=1}^{2} a_{rs} dx_r dx_s$ are defined as

$$\Gamma^l_{ik} = \frac{1}{2} a^{lj} \left(\frac{\partial a_{ji}}{\partial x^k} + \frac{\partial a_{jk}}{\partial x^i} - \frac{\partial a_{ik}}{\partial x^j}\right).$$

[48] See (Weingarten 1884, p. 26).

[49] The interested reader can refer to (Bianchi 1903, §223) and (Darboux 1896, Chap. 1).

[50] See (Weingarten 1887b, p. 302) and (Bianchi 1892). With respect to Weingarten's notation, we have changed u, v and w to $\tilde{w}_1$, $\tilde{w}_2$ and $\tilde{w}_3$, respectively. Weingarten also denoted the coefficient of the second fundamental form D, D', D'' with c_{11}, c_{12}, c_{22}. It should be noted that Weingarten, like many of his contemporaries, denoted with $\sum \frac{\partial X}{\partial v}\frac{\partial U}{\partial u}$, where X is the first component of the vector field normal to the surface (X, Y, Z) and u is the first component of the parametrisation of the surface S_0, what we today denote by $\sum_{i=1}^{3} \frac{\partial X_i}{\partial v}\frac{\partial w_i}{\partial u}$.

on this theme for 1894. Here, he submitted (Weingarten 1897), where a new method to tackle the second problem of applicability based on a second-order PDE of Monge-Ampère type was presented.[51]

Probably because of his expertise in the problem of applicability, Weingarten was chosen by Stäckel to comment on one of Gauss' handwritten notes entitled *Allgemeinste auflösung des Problems der Abwickelung der Flächen* (Gauss 1900, pp. 447–449). Here, Gauss briefly treated the problem of applicability beginning with the existence of a certain function, which was guaranteed by the exactness of a specific differential, $udt + UdT$. Weingarten was not able to explain Gauss' argument in its entirety, but he explained the existence of this function, which he denoted with φ, by using the spherical representation of applicable surfaces. Weingarten recalled this episode to Bianchi, expressing his confidence that the material would be as fruitful:

> What has particularly interested me, however, and will continue to do so, is your continuation of the path taken by Gauss to treat the general theory of deformation. Several years ago, Professor Stäckel wrote to me and informed me of the note in Gauss' diary, asking me whether I understood it. It was quite strange and obscure. I must confess that when I read this note, I lost all comprehension of it at the moment, and stood headless in the face of it. However, I soon remembered some formulas from old studies that touched on the Gaussian foundations, and so my comment has originated. At first, its treatment was completely different from the one given, which was only communicated to Stäckel in the second line. But so far I have no idea in what way Gauss' view can be used for the deformation theory, as I have also indicated. I am therefore very curious about your further contributions, and I am pleased that you have succeeded in getting to grips with the matter, like so many others, which would be beyond my powers. It will certainly be very fruitful, like everything that Gauss has inspired, once it has been truly understood.[52]

Weingarten wrote this passage in December 1903, probably after having seen a proof of (Bianchi 1904b). Here, following (Gauss 1900, pp. 447–449), Bianchi studied the problem

[51] A careful description of the content of these memoirs is in Chapter 5.

[52] "Was mich aber ganz besonders interessirt hat und interessiren wird, ist Ihr Weitergehen auf dem von Gauss eingeschlagenen Wege die allgemeine Theorie der Abwickelung zu behandeln. Vor mehreren Jahren schrieb Professor Stäckel an mich, und machte mir Mittheilung von der Note im Gauss'schen Tagebuch, mit der Anfrage ob ich dieselbe verstände. Sie wäre ganz sonderbar und undurchsichtig. Ich muss gestehen, dass als ich diese Note las, mir im Augenblick jedes Verständniss abging, und ich derselben kopflos gegenüberstand. Ich erinnerte mich aber bald einiger Formeln aus alten Studien, die mit der Gauss'schen Grundlage sich berührten, und so entstand die Note von mir. Ihre Behandlung war zuerst eine ganz andere, wie die gegeben, die erst in 2ter Linie an Stäckel mitgetheilt wurde. Aber ich habe bis jetzt keine Ahnung, in welcher Weise die Anschauung von Gauss für die Abwickelungstheorie verwerthet werden kann, wie ich das auch angedeutet habe. Ich bin daher sehr gespannt auf Ihre weiteren Mittheilungen, und freue mich, dass Ihnen das Eindriugen in die Sache gelungen ist, wie ja so Vieles, was meine Kräfte übersteigen würde. Es wird gewiss sehr fruchtbar werden, wie alles was Gauss angeregt hat, sobald es erst wirklich verstanden wurde." (Bianchi 1959, pp. 286–287)

of finite deformations by exploiting the remarkable property (highlighted in (Gauss 1900, p. 372)) that two applicable surfaces have spherical images with the same area.[53]

4.3 Weingarten's working method

Reading Weingarten's letters to Bianchi has led us to highlight three characteristic aspects of Weingarten's research: Weingarten as a careful writer and reader, his natural inclination to an analytical approach to geometry, and his predilection for Gauss' methods.

4.3.1 A careful writer and reader

Weingarten was accurate and acute in his research, clear and meticulous in the exposition. His works were the result of a slow writing process that often led him to postpone publication, as he confessed to Bianchi on May 1890:

> It is very difficult for me to write down a long work in an orderly manner; I tear up almost every sheet of manuscript because I dislike the way it is presented, and after a while I tear up everything I have written down again. So it happens that the things remain lying around for years and grow more and more. I should finish them more quickly.[54]

Therefore, his pleasure is understandable when, a few months after this letter, Darboux proposed publishing part of two letters in which Weingarten had shared some of his most recent progress concerning the problem of applicability, which was probably unfinished, though remarkable in his opinion. They appeared in (Weingarten 1891a; Weingarten 1891b) and Weingarten remarked in a letter to Bianchi: *"I was pleased that this exchange of letters had once again forced me to publish, which is* always *very difficult for me"*.[55]

In this regard, Horace's epigram *"saepe stilum vertas"* that Weingarten chose as a motto to identify his answer to the 1894 Grand Prix des Mathématiques, seems to adequately described this aspect of his personality.[56]

His predilection for publishing only definitive results in a clear and perfect form certainly condemned some of his research to oblivion. Other achievements, however, were published by Bianchi, who received them during the course of their correspondence and always cited Weingarten as their first author. In this respect, with an irony that reveals the nature of the relationship between the two, Blaschke stated:

[53]Bianchi provided a twofold treatment of the subject: in (Bianchi 1904b) he used tools from elliptic geometry, and in (Bianchi 1904c) the same results are obtained in a direct way without using non-Euclidean geometry.

[54]"Es wird mir sehr schwer eine lange Arbeit geordnet niederzuschreiben, ich zerreisse fast jeden Bogen Manusscript wieder, weil mir die Darstellung missfällt, und nach einiger Zeit wieder alles Aufgeschriebene. So kommt es, dass die Sachen Jahre laug liegen bleiben, und immer mehr anwachsen. Ich sollte sie schneller erledigen." (Bianchi 1959, p. 235)

[55](Bianchi 1959, p. 246). The letters were published in (Weingarten 1891a) and (Weingarten 1891b).

[56]It literally means *"turn often the stylus"* and advocates a continuous revision of one's writings.

> While scientists sometimes argue about priority, these two friends competed in attributing their discoveries to each other.[57]

It is therefore possible to reconstruct Weingarten's interests beyond those traceable in his publications by also perusing Bianchi's works.

The first outstanding example is the introduction of $W-$systems. A $W-$system is an orthogonal triple system of surfaces,[58] in which one of the families is made of surfaces with the same constant curvature.

None of Weingarten's publications deals with this subject. Yet he did succeed in finding a way to construct these systems by means of infinitesimal considerations that give the transition from an initial surface to an infinitely close surface with the same constant curvature, so that the infinitesimal normal distance between the two surfaces satisfies Cayley's equation.[59] Bianchi was informed of this result in the first letter in (Bianchi 1959) and he developed the theory in more detail. Besides citing Weingarten as the true author of the discovery in (Bianchi 1885a) and (Bianchi 1885b), Bianchi honoured Weingarten by calling these particular systems $W-$systems. Four years later, in 1888, Bianchi shared with Weingarten his recent studies of minimal surfaces embedded in space with constant curvature that he proved to be characterised by having isothermal lines of curvature. Bianchi concluded the memoir containing these results (Bianchi 1887b) with a note Weingarten had communicated to him in (Bianchi 1959, pp. 193–196), where he provided an interesting way of verifying that such surfaces have isothermal lines of curvature by employing a criterion he had shown in (Weingarten 1881). In 1889 it was the turn of a result on systems of linear, homogeneous and completely integrable partial differential equations of the second order about which Bianchi did not hesitate to write, *"For the case of only two independent variables, this theorem had already been long known by Mr Weingarten"*.[60] After Weingarten's death, in (Bianchi 1911) Bianchi also published a series of formulas that his friend had confided to him in (Bianchi 1959, pp. 155–157). They express, in invariant form, the variations of the coefficients of the two fundamental forms of a surface when each point of the surface is subjected to an infinitesimal displacement along the normal direction.

At the same time, Weingarten was an equally scrupulous reader: by requiring the same

[57]"Mentre gli scienziati qualche volta litigano sulla priorità, questi due amici competevano nell'attribuire le loro scoperte l'uno all'altro." (Blaschke 1954, p. 44)

[58]An *orthogonal triple system* is the set of three families of surfaces, each of which depends on one parameter, such that at any point three representatives of each family passing through that point have tangent planes that are pairwise orthogonal.

[59]When a surface S is referred to its line of curvature and its line element is $ds^2 = Edx^2 + Gdy^2$, Cayley equation is

$$s = \frac{\partial \log \sqrt{E}}{\partial y} p + \frac{\partial \log \sqrt{G}}{\partial x} q,$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$, with z a function of x and y. The solutions $z(x, y)$ of this equation are related to triple orthogonal systems as follows. If one takes an infinitesimal segment $\rho = \epsilon z$, where ϵ is an infinitesimal constant, on each normal of S, then the surface S' that is the locus of the extremities of the segments ρ is the surface that follows S in a triple orthogonal system. In this regard, see (Bianchi 1888a).

[60](Bianchi 1959, p. 227), (Bianchi 1887b, p. 119)

precision in what he read, he identified several errors or shortcomings in the treatises of his colleagues. Weingarten discussed with Bianchi the validity of some results: besides remarks about Backlund's questionable methods, he observed that Jellet's proof on infinitesimal deformations of convex surfaces could be invalidated by discontinuities in the considered domain and noted that Helmholtz and Kirchhoff's justification on a hydrodynamics result was erroneous, since *"they fail to see the discontinuity which does not permit their conclusions. Strangely enough, their final results are not erroneous"*.[61]

As we have seen, some of his papers are also a revision of the work of others: (Weingarten 1864b), (Weingarten 1882), but even his new method for the second problem of applicability was the result of a critical reading and revision of Bour's equation.

4.3.2 An analytical character

Weingarten was aware of his penchant for analytical methods. He considered himself a *"bad geometer who uses analytical methods more than geometry"*.[62] Complaining about the scarce presence of geometric intuition in his research, he was also aware of the limits of his perspective:

> My way of thinking is never geometrical but purely analytical, and therefore I do not progress with things as far as you, who think so distinctly geometrically.[63]

Whenever the opportunity arose, he praised Bianchi's capability to extract important results even from simple statements. Commenting on Bianchi's successful development of $W-$systems, he wrote:

> It is therefore a special honour for me if you include me in one of your works [...] and at the same time a great pleasure that a remark that has been unfruitful for me for so long, of which I did not own more than I shared with you, gains greater value in your hands.[64]

Several among Weingarten's colleagues lamented the non-geometrical approach of his research. Sometimes his results were seen as the result of a magical and obscure combination of remarks and reasoning. Such is, for example, the judgement expressed by Renato Calapso on his theory of infinitesimal deformations:

> Weingarten, who has the merit of having first and completely solved the aforementioned problem, made the resolution of it dependent on the determination

[61]"sie übersehen die Unstetigkeit welche ihre Schlüsse nicht zu lasst. Merkwürdigerweise sind ihre Endresultate nicht fehlerhaft." (Bianchi 1959, p. 210), (Bianchi 1959, pp. 189–191), (Bianchi 1959, p. 210).

[62](Bianchi 1959, p. 153)

[63]*"Meine Art der Vorstellung ist nie geometrisch sondern rein analytisch, und daher komme ich nicht so weit mit den Dingen wie Sie, der so ausgepragt geometrisch denkt." (Bianchi 1959, p. 274)*

[64]*"Es ist mir daher eine besondere Ehre, wenn Sie mich in einer Ihrer Arbeiten [...] und auch gleichzeitig eine grosse Freude, dass eine fur mich so lange unfruchtbare Bemerkung, von der mir nicht mehr gehort, als ich Ihnen mitgetheilt habe, in Ihren Handen einen grosseren Werth gewinnt.* "(Bianchi 1959, p. 154)

> of the so-called characteristic function, which he introduced as an irrational invariant of certain differential forms. We will call this characteristic function Weingarten's function. But while recognising that the introduction of the aforementioned function made it possible to solve the problem of infinitesimal deformation in a truly brilliant manner, things nevertheless remained buried under such a tangle of formulas that Weingarten's results appeared miraculous, as if they were the result of divination rather than systematic scientific investigation. [...] However, the need for more geometric light, which would be a sure foothold for further research, was felt because only a clearer geometric vision could advance studies on the subject, as was in fact the case. And here, Bianchi's note was a revelation, both because of the great simplicity of the result and because the notion of the associated surface also shed new light on the resolution of numerous other problems that could hardly have been tackled in any other way.[65]

Bianchi's geometrical vision made Weingarten's method appreciated in its entirety by highlighting the natural geometrical structure underlying Weingarten's artificial theory.

Even the new method for the second problem of applicability was affected by the same shortcoming, as it is clear from Goursat's words in 1927:

> Weingarten is to be credited with a "singular method" for the determination of surfaces admitting a given line element [he referred to (Weingarten 1891a; Weingarten 1891b)] [...] Weingarten did not indicate the ideas that guided him, and on this point we cannot but conjecture. In exposing the results obtained by this new method, G. Darboux indicated a geometrical interpretation where isotropic lines intervene. A few years later, in a Memoir crowned by the Academy of Sciences in 1894, Weingarten made known, still without any indication of the continuation of his ideas, a method seemingly much more general than the first. In the last chapter of his great work, G. Darboux explained the geometrical meaning of the principle of the method and of the transformation on which it is based. In spite of G. Darboux's remarks, this method still had a somewhat mysterious side, which I have tried to clarify.[66]

[65]"Il Weingarten, cui spetta il merito di aver risolto per primo e in modo completo il suddetto problema, fece dipendere la risoluzione di esso dalla determinazione della cosiddetta funzione caratteristica, che egli introdusse come invariante irrazionale di certe forme differenziali. Diremo funzione di Weingarten questa funzione caratteristica. Ma pur riconoscendosi come l'introduzione della predetta funzione permetteva di risolvere il problema della deformazione infinitesima in maniera veramente brillante, tuttavia le cose rimanevano sepolte sotto un tale groviglio di formole, che i risultati del Weingarten apparivano miracolosi, quasi fossero frutto di divinazione, piuttosto che di sistematica indagine scientifica. [...] Ma pur si sentiva il bisogno di una maggior luce geometrica, che fosse un sicuro punto di appoggio per ulteriori ricerche, perché, come col fatto avvenne, solo una più chiara visione geometrica poteva far progredire gli studi sull'argomento. E qui, la nota del Bianchi fu una rivelazione, sia per la grande semplicità, del risultato, sia perchè la nozione di superficie associate portò altresì nuova luce nella risoluzione di numerosi altri problemi che ben difficilmente si sarebbero potuti affrontare per altra via." (Bianchi 1952-1959, vol.II, pp.12-13)

[66]"On doit à Weingarten une "méthode singulière" pour la détermination des surfaces admettant un

Darboux was also of the same opinion.[67] In this regard, it is significant that both Darboux and Bianchi welcomed Weingarten's applicability theory and devoted entire chapters of their textbooks to it, but reworked Weingarten's treatment in order to include a geometrical interpretation that would shed light on the more hermetic aspects.

On the other hand, it was probably this analytical perspective on problems that were geometric in nature that allowed Weingarten to have an original approach to the discipline and to provide interesting ideas to geometers such as Bianchi and Darboux, whose inventive capacity was instead guided by a marked geometric intuition. Especially with respect to Bianchi, sharing a basic ideal for geometry made scientific collaboration between two such different minds possible and extremely fruitful. In (Bianchi 1910a, p. 219) Bianchi wrote:

> In all his works that brilliance in research and that perfection in form that satisfies reader's minds stand out, allowing at the same time a glimpse of new and wider horizons.[68]

4.3.3 Weingarten's preference for Gauss' method

As seen in Section 2.2, Gauss identified the basis of an effective way to study surfaces in three elements: the systematic use of the line element $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ to describe a surface, the description of its immersion in the Euclidean space through the coefficients D, D', D'' and the distinction between intrinsic and extrinsic properties of a surface. This corpus has been given the name *Gauss' method*. With reference to the examples already seen, Mainardi adhered to Gauss' method, as did the Italian school generally, along with the German-speaking geometers, and Weingarten.

Starting from the *Disquisitiones*, geometers refined Gauss' method. Gradually, the coefficients D, D', D'' were recognised as the coefficients of a quadratic differential form

$$dS^2 = Ddu^2 + 2D'dudv + D''dv^2$$

that, alongside with the line element, was sufficient to describe a theory of surfaces, as Weingarten emphasised:

élément linéaire donné [...] Weingarten n'a pas indiqué les idées qui l'avaient guidé, et l'on en est réduit sur ce point aux conjectures. En exposant les résultats obtenus par cette nouvelle méthode, G. Darboux a indiqué une interprétation géométrique où interviennent les droites isotropes. Quelques années plus tard, dans un Mémoire couronné par l'Académie des Sciences en 1894 Weingarten faisait connaître, toujours sans aucune indication sur la suite de ses idées, une méthode beaucoup plus générale en apparence que la première. En exposant cette méthode dans le dernier Chapitre de son grand Ouvragé, G. Darboux a donné la signification géométrique du principe de la méthode et de la transformation qui lui sert de base. Malgré les remarques de G. Darboux, cette méthode présentait encore un côté un peu mystérieux, que j'ai essayé d'éclaircir." (Goursat 1927, p. 5)

[67] In (Darboux 1896, p. 309), Darboux complained that Weingarten did not provide explanations in support of the radical transformation of the equation for applicability.

[68] "In tutti i suoi lavori spicca quella genialità di ricerca e quella perfezione di forma che appaga la mente del lettore, lasciando insieme intravedere nuovi e più vasti orizzonti".

> This idea, so simple and fertile, informs all of his works, and the methods that derive from it provide a complete system of formulas, which, skilfully handled, have demonstrated that they are sufficient up to now for the treatment of any problem of infinitesimal geometry.[69]

In this regard, (Weingarten 1884) is interesting. It is a dense and comprehensive article, in which Weingarten framed the study of surfaces in the general context of differential quadratic forms, by offering a renewed perspective on the general aspects of the theory of surfaces embedded in the Euclidean space. According to Weingarten, it *"will shed more light on the path hitherto taken in investigation in the theory of curvature of surfaces"*.[70] In this respect, it is plausible that Christoffel's work on the theory of differential invariants with respect to the first fundamental form of a generic manifold influenced Weingarten's thought. Weingerten highly esteemed Christoffel's geometric work, as can be guessed from a short passage in (Bianchi 1959, p. 180):

> Two years ago Christoffel wrote to me that he intended to publish his research and his lectures on infinitesimal geometry. But since then I have not heard anything. His work, too, I think, had to be good.[71]

French geometers, on the other hand, had a more geometric and kinematic approach, as can be seen from the writings of Bour and Bonnet. Indeed, both Bour and Bonnet described a surface by imagining a trihedron that moves along the surface and is composed of two orthogonal vectors that are tangent to the surface and of the normal vector to the surface. This point of view was systematised by Darboux and proved to be an alternative to Gauss'. The general ideas at the basis of Darboux's approach to infinitesimal geometry can be briefly described as follows.[72]

Let a surface S be parametrised by $\boldsymbol{x}(u,v) = (x(u,v), y(u,v), z(u,v))$. Three mutually orthogonal unit vector fields on the surface, $\boldsymbol{X}_1 = (X_1, Y_1, Z_1), \boldsymbol{X}_2 = (X_2, Y_2, Z_2), \boldsymbol{X}_3 = (X_3, Y_3, Z_3)$, of which the first two are tangent to the surface, constitute a *Darboux frame*. These vector fields are *invariably* linked to the surface, in the sense that, when S is deformed, the vector fields vary in the same way. Once a given surface S is equipped with such a frame, the three vector fields must satisfy the following systems, which were naturally

[69]"Questa idea, cosi semplice e feconda, informa tutti i lavori del Nostro, ed i metodi che ne derivano forniscono un sistema completo di formole, le quali, abilmente inaneggiate, hanno dimostrato di bastare fin qui alla trattazione di qualunque problema di geometria infinitesimale." (Bianchi 1910a, p. 221)

[70](Weingarten 1884, p. 11)

[71]"Vor zwei Jahren hat mir Christoffel geschrieben, dass er eine Herausgabe seiner infinitesimalgeometrischen Untersuchungen und Vorlesungen beabsichtigt. Er hat aber nichts wieder hören lassen. Auch sein Werk, denke ich, musste ein gutes werden".

[72]According to (Darboux 1887, p. 49), it seems that Darboux expounded at least part of this theory in a course he taught in the Academic Year 1866-1867. In the same period, Combescure obtained some similar results in (Combescure 1867). For a detailed introduction to Darboux frame, the interested reader can see (Bianchi 1922) or (Darboux 1887).

derived from the above characterization of the trihedron:[73]

$$\begin{cases} \dfrac{\partial \boldsymbol{X}_1}{\partial u} = r\boldsymbol{X}_2 - q\boldsymbol{X}_3 & \dfrac{\partial \boldsymbol{X}_1}{\partial v} = r_1\boldsymbol{X}_2 - q_1\boldsymbol{X}_3 \\ \dfrac{\partial \boldsymbol{X}_2}{\partial u} = p\boldsymbol{X}_3 - r\boldsymbol{X}_1 & \dfrac{\partial \boldsymbol{X}_2}{\partial v} = p_1\boldsymbol{X}_3 - r_1\boldsymbol{X}_1 \\ \dfrac{\partial \boldsymbol{X}_3}{\partial u} = q\boldsymbol{X}_1 - p\boldsymbol{X}_2 & \dfrac{\partial \boldsymbol{X}_3}{\partial v} = q_1\boldsymbol{X}_1 - p_1\boldsymbol{X}_2, \end{cases} \tag{4.5}$$

where p, q, r, p_1, q_1, r_1 are functions of u and v and are called *rotations*. In addition, the coordinate tangent vectors, $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, must satisfy the following equations[74]

$$\begin{cases} \dfrac{\partial x}{\partial u} = aX_1 + bX_2 & \dfrac{\partial x}{\partial v} = \alpha X_1 + \beta X_2 \\ \dfrac{\partial y}{\partial u} = aY_1 + bY_2 & \dfrac{\partial y}{\partial v} = \alpha Y_1 + \beta Y_2 \\ \dfrac{\partial z}{\partial u} = aZ_1 + bZ_2 & \dfrac{\partial z}{\partial v} = \alpha Z_1 + \beta Z_2 \end{cases} \tag{4.6}$$

where a, b, α, β are functions of u and v, and are called *translations*. By means of this latter system, the line element of the surface can be expressed as

$$ds^2 = (a^2 + b^2)du^2 + 2(a\alpha + b\beta)dudv + (b^2 + \beta^2)dv^2.$$

Vice versa, in the case (4.5) and (4.6) are integrable, that is, the functions p, q, r, p_1, q_1, r_1, a, b, α, β satisfy the integrability conditions[75]

$$\begin{cases} \dfrac{\partial p}{\partial v} - \dfrac{\partial p_1}{\partial u} = qr_1 - rq_1 \\ \dfrac{\partial q}{\partial v} - \dfrac{\partial q_1}{\partial u} = rp_1 - pr_1 \\ \dfrac{\partial r}{\partial v} - \dfrac{\partial r_1}{\partial u} = pq_1 - qp_1 \end{cases} \tag{4.7}$$

$$\begin{cases} \dfrac{\partial a}{\partial v} - \dfrac{\partial \alpha}{\partial u} = br_1 - \beta r \\ \dfrac{\partial b}{\partial v} - \dfrac{\partial \beta}{\partial u} = \alpha r - ar_1 \\ bp_1 - aq_1 + \alpha q - \beta p = 0, \end{cases} \tag{4.8}$$

then systems (4.5) and (4.6) define a unique surface, up to a rigid motion.

Moreover, if φ denotes the angle that $\boldsymbol{X}_1$ makes with the tangent vector $\frac{\partial \boldsymbol{x}}{\partial u}$ and ω the angle between the coordinate lines,[76] the translations a, b, α, β can be rewritten as

$$a = \sqrt{E}\sin(\varphi + \omega) \qquad \alpha = \sqrt{G}\sin\varphi \qquad b = \sqrt{E}\cos(\varphi + \omega) \qquad \beta = \sqrt{G}\cos\varphi. \tag{4.9}$$

[73](4.5) is a consequence of the orthogonality of $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ and $\boldsymbol{X}_3$.

[74]For the linear dependence of $\frac{\partial \boldsymbol{x}}{\partial u}$ and $\frac{\partial \boldsymbol{x}}{\partial v}$ with the tangent vectors $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, one has (4.6).

[75]As usual, the integrability conditions are given by $\frac{\partial}{\partial v}\frac{\partial \boldsymbol{X}_i}{\partial u} = \frac{\partial}{\partial u}\frac{\partial \boldsymbol{X}_i}{\partial v}$, $i = 1, 2, 3$ for (4.5), and by $\frac{\partial}{\partial v}\frac{\partial \boldsymbol{x}}{\partial u} = \frac{\partial}{\partial u}\frac{\partial \boldsymbol{x}}{\partial v}$, for (4.6).

[76]φ determines the orientation of Darboux's trihedron.

In particular, they imply that also the rotations r and r_1 are known once having the line element and φ fixed. Indeed, by substituting (4.9) in the first two equations in (4.8), they can be written as

$$\frac{\partial \varphi}{\partial u} = \frac{\sqrt{EG-F^2}}{E}\Gamma_{11}^2 - r \qquad \frac{\partial \varphi}{\partial v} = \frac{\sqrt{EG-F^2}}{E}\Gamma_{12}^2 - r_1. \tag{4.10}$$

Hence, the integrability conditions (4.7) reduce to

$$\begin{cases} \dfrac{\partial p}{\partial v} - \dfrac{\partial p_1}{\partial u} = qr_1 - rq_1 \\ \dfrac{\partial q}{\partial v} - \dfrac{\partial q_1}{\partial u} = rp_1 - pr_1 \\ K\sqrt{EG-F^2} = pq_1 - qp_1, \end{cases} \tag{4.11}$$

and (4.8) to

$$bp_1 - aq_1 + \alpha q - \beta p = 0. \tag{4.12}$$

Darboux's equations (4.10), (4.9), (4.11), (4.12) can supersede for the Gauss' method, which is based on the first and the second fundamental forms.

Weingarten positively judged Darboux's *Leçons*. He considered it a highly significant work, which would be authoritative for study throughout the next century.[77] At the same time, however, he was quite sceptical about the French fashion of Darboux's frame. This is how Weingarten commented on the publication of the first volume of Darboux's *Leçons* in November 1887:

> A few days ago I received Darboux's new Leçons sur la théorie des surfaces. They were very interesting for me, but as far as the *general* aspects are concerned, I found *your* Lectures more pleasant and more sympathetic. The introduction of the motion of a trihedron and of rotation, as elegant as it appears at first glance, is by no means fundamental, and is linked to orthogonal *curvilinear* coordinates; on the other hand, with the omission of this condition, it is quite useless and complicated. [...] Forgive these general critical remarks, which I do *not* intend to make in a critical sense against Darboux's excellent book, since they have occupied me for years. I was only struck by the fact that Darboux has placed so much emphasis on the introduction of trihedron, which entangles the *analytical* relationships.[78]

[77](Bianchi 1959, p. 222)

[78]"Vor einigen Tagen habe ich Darboux's neue Leçons sur la théorie des surfaces in die Hände bekommen. Sie sind sehr interessant für mich gewesen, aber was das *Allgemeine* anbetrifft, so sind mir *Ihre* Vorlesungen angenehmer und sympatischer erschienen. Die Einführung der Bewegung eines Trieders und der Rotation, ist, so elegant sie auch auf den ersten Blick erscheint, keineswegs fundamental, und an orthogonale *krummlinige* Coordinaten geknüpft, dagegen mit Weglassung dieser Bedingung ziemlich unbrauchbar und complicirt. [...] Verzeihen Sie diese allgemeinen critischen Bemerkungen, die ich allerdings *nicht* in critischem Sinne gegen das ausgezeichnete Darboux'sche Buch nächstens auszuführen gedenke, da sie mich schon seit Jahren beschäftigt haben. Es fiel mir eben nur auf, dass Darboux einen so besonderen Werth auf Triedereinfübrung gelegt hat, welche die analytischen Beziebungen verwickelt." (Bianchi 1959, pp. 196, 199)

Weingarten, on the other hand, preferred Bianchi's *Lezioni* because they made exclusive use of Gauss' method, which was more popular in Germany than Darboux's method. For his part, Bianchi appropriated the idea of constructing the theory of surfaces on the definition of two quadratic forms, as can be clearly read in the first printed edition of his *Lezioni*, where the two quadratic forms are defined as fundamental. Commenting on this edition, Weingarten wrote in June 1894:

> Kindly accept my sincere thanks for sending me the second part of your Infinitesimal Geometry. The dear gift is very valuable, especially with regard to the content. The first part is the most excellent handbook that one could wish for, its methods elegant, and for us Germans more pleasant than the method of trihedron, because they only presuppose the work of Gauss. So my heartfelt thanks![79]

It is thus not surprising that, in 1892 Weingarten strongly supported the publication of a German translation of Bianchi's *Lezioni* when his opinion was requested by the publisher:

> Yesterday I received a letter from the B. G. Teubner bookshop in Leipzig, I don't know who sent it. The bookshop asked me whether a German edition of your Lezioni was needed to us. They offered a translation by a Mr Lukat, recommended by Hilbert and Lindemann. I immediately replied that I was in a position to know your works as well as anyone could through my own studies. My answer was that I am convinced that in a few years every mathematician, who is able to buy even the smallest detail of the best handbooks of his science, will be in possession of your work. The owners of Darboux's extensive work would also see yours as a desirable and necessary addition, since of the two paths leading to the theory in question, you had taken one and Darboux the other. Your work is beyond my praise. I do not know whether you already know anything about the translation in question, which I hope will appear very soon, but I would ask you not to speak of my own communication, since such a request is confidential and I would not like to be considered talkative. However, I did not think it necessary to remain silent with you.[80]

[79]"Nehmen Sie meinen herzlichen Dank fur die freundliche Zusendung des zweiten Theils Ihrer Infinitesimalgeometrie, freundlich an. Die liebe Gabe ist sehr werthvoll, ganz besonders aber in Beziehung auf den Inhalt. Der 1te Theil ist das vortrefflichste Handbuch, das man sich wunschen kann, seine Methoden elegant, und fur uns Deutsche angenehmer wie die Methode des Trieders, weil sie nur die Arbeit von Gauss voraussetzen. Also herzlichsten Dank!" (Bianchi 1959, p. 264)

[80]"Ich erhielt nämlich gestern einen Brief von der Buchhandlung von B. G. Teubner in Leipzig, ich weiss nicht durch wen veranlasst. Die Buchhandlung erkundigte sich bei mir darüber ob eine deutsche Ausgabe Ihrer Lezioni ein Bedürfniss für uns sei. Es wäre ihr eine Uebersetzung von einem Herrn Lukat, empfohlen durch Hilbert und Lindemann angeboten. Ich habe gleich geantwortet, dass ich in der Lage wäre durch eignes Studium Ihre Lezioni so gut zu kennen, wie es nur irgend jemand im Stande sei. Meine Antwort ging dahin, dass ich die Ueberzeugung habe, dass in wenigen Jahren jeder Mathematiker der im Stande ist sich auch nur die kleinste Angabe der besten Handbücher seiner Wissenschaft zu kaufen, im Besitze Ihres Werkes sein würde. Auch die Besitzer des umfangreichen Darboux'schen Werkes würden in dem Ihrigen

At the same time, however, Weingarten did not totally disdain Darboux's trihedron, which had proven to be an effective method for dealing with certain problems. Although he preferred Gauss' method, he appreciated having two tools to choose from, as he wrote in a letter addressed to Bianchi in June 1892:

> The fact that you have revised your Lectures on infinitesimal geometry is of great interest to me, and I hope that after the printing you will give me another copy as a present. I completely agree with you that the Gaussian methods are the easier ones to go forward with, in contrast to the use of rotations that Darboux starts from. For me, finding the rotations from tables is always something very inconvenient. But this does not mean that Darboux's methods are quite successful, and that one can be satisfied that two suitable methods run side by side. One supports the other.[81]

In this sense, Bianchi's position is much more restrictive.

eine erwünschte und nothwendige Ergänzung sehen, da von den zwei Wegen, die in die betreffende Theorie führen Sie den einen, Darboux den anderen betreten hatten. Ihr Werk stände über meinem Lobe. Ich weiss nicht, ob Ihnen von der betreffenden Ubersetzung, von der ich hoffe, dass sie sehr bald erscheine, schon etwas bekannt ist, mochte Sie aber bitten von meiner eigenen Mittheilung nicht zu sprechen, da solche Anfrage vertraulich ist, und ich nicht für schwatzhaft gelten mochte. Ihnen gegenüber glaubte ich aber nicht schweigen zu brauchen." (Bianchi 1959, pp. 265–266)

[81]"Dass Sie Ihre Vorlesungen über Infinitesimalgeometrie umgearbeitet haben, ist mir von grossem Interesse, und hoffe ich, dass Sie nach dem Druck mir wieder ein Exemplar zum Geschenk machen. Ich bin ganz Ihrer Meinung, dass die Gauss'schen Methoden die leichter vorwärts führenden sind, im Gegensatz zu der Benutzung von Drehungen von denen Darboux ausgeht. Mir ist das Aufsuchen der Drehungen aus Tabellen immer etwas sehr unbequemes. Aber es ist auch nicht zu leugnen, dass Darboux mit seinen Methoden schone Erfolge erzielt, und, dass man zufrieden sein kann, dass zwei geeignete Methoden neben einander herlaufen.Eine unterstützt die andere." (Bianchi 1959, p. 250)

Chapter 5

Weingarten's new method for applicability

5.1 Bour's equation in the 1860s-80s

The 1860 Grand Prix des Mathématiques marked an important stage in the development of surface theory and the solution of the second problem of applicability with Bour, Bonnet and Codazzi's results. However, the subject of applicability was certainly not complete at that time. The announcement of a second Grand Prix on the same topic in 1894 proves the continued keen interest in the subject. Various mathematicians in France, Germany and Italy tackled the subject from different points of view in the years between the two prizes.

The publication of Riemann's Habilitationsvortrag *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, (Riemann 1867), led to an extension of the notion of applicability to surfaces embedded in the non-Euclidean spaces. As an example, in his dissertation, (Bianchi 1878), Luigi Bianchi dealt with the problem of applicability between surfaces in spaces with constant curvature. He also provided an appropriate definition of surfaces of revolution embedded in these spaces and proved that such surfaces are always applicable on the Euclidean surfaces of revolution, according to his definition of applicability.[1] In particular, he proved that spheres of spaces with constant curvature are always applicable upon ordinary spheres, and that the surfaces of the non-Euclidean space that remain unchanged by a motion are applicable in the Euclidean space.

Regarding the realm of classic surface theory, in (Ribaucour 1870) the problem of applicability was interpreted from a kinematic viewpoint and gave rise to a new line of research that gathered the so-called *rolling problems*, to which Darboux and Bianchi contributed significantly.[2]

[1] According to (Bianchi 1878, p. 37), two surfaces of the same space, or of two different spaces with constant curvature, are said to be applicable when their line elements, which are calculated with respect to the metric of the space in which they are embedded, are equal or can be made equal by a convenient change of co-ordinates.

[2] Given two applicable surfaces S, with $\boldsymbol{x}(u, v)$, and S', with $\boldsymbol{x}'(u, v)$, suppose that S is fixed while S' is

Gradually, especially in Germany and Italy, as a consequence of the FT, surfaces were described by assigning two quadratic forms, the first and second fundamental forms. This established a link between surface theory and the emerging theory of quadratic forms.

In addition, MCE and Bour's equation were also improved. With regard to MCE, after Bonnet had published his version in 1863 and 1867, Codazzi included them in (Codazzi 1868) in the more general form that was valid for any coordinates. After Codazzi, in (Lipschitz 1883) the MCE were generalised to the case of any coordinate system by using the same quantities H, H_1 and T employed by Bour. Weingarten's aforementioned result published in (Weingarten 1884) is also an interesting and original deduction of MCE based on the simultaneous variation of the first and second fundamental forms.[3]

The revision of *Bour's equation*[4] is more interesting for the follow-up. A young Ulisse Dini (1845-1918) lamented that a specific coordinate system on the surface was required to arrive at Bour's equations. In an excerpt from his thesis written under Betti's guidance (Dini 1864), he substituted Bour's equation (3.6) with a new equation that was valid for any coordinate system.

For a surface $\tilde{\Sigma}$ with line element $d\tilde{s}^2 = \tilde{E}du^2 + 2\tilde{F}dudv + \tilde{G}dv^2$ being given, Dini's aim was to determine the differential equation of all surfaces Σ that are parametrised by means of the function $\boldsymbol{x} : U \to \mathbb{R}^3$, $\boldsymbol{x} = (x(u,v), y(u,v), z(u,v))$ and that are locally isometric to $\tilde{\Sigma}$. His approach essentially coincided with Bour's and Bonnet's first method presented in Section 3.3. By comparing the line elements of Σ and $\tilde{\Sigma}$, he considered the system

$$\begin{cases} \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = E \\ \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = F \\ \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = G \end{cases} \tag{5.1}$$

and, by eliminating x and y in these equations, he arrived at the following Monge-Ampère

rigid. Suppose that at each moment (u_t, v_t) the point $\boldsymbol{x}(u_t, v_t)$ on S coincides with $\boldsymbol{x}'(u_t, v_t)$ on S' and the tangent plane of S at $\boldsymbol{x}(u_t, v_t)$ coincides with the tangent plane of S' at $\boldsymbol{x}'(u_t, v_t)$. This results in general in a sort of two-parameter continuous motion of S', which is called *rolling.* (Darboux 1896, chap. 6) and (Bianchi 1952-1959, Vol. 7) are devoted to *rolling problems.*

[3]See Section 4.2.1.

[4]See Section 3.3.

equation for the unknown $z(u,v)$:[5]

$$
\begin{aligned}
&4(EG-F^2)(rt-s^2)++2[(GG_u+FG_v-2GF_v)p-(EG_v+FG_u-2FF_v)q]r+\\
&+4(E_vGp+EG_uq-FG_up-FE_vq)s+\\
&+2[(EE_v+E_uF-2FF_u)q-(E_uG+E_vF-2FF_u)p]t+\\
&+(-2E_{vv}G-2GG_{uu}+4GF_{uv}+E_vG_v+G_u^2-2G_vF_u)p^2+\\
&+(4FG_{uu}+4E_{vv}F-8FF_{uv}+E_uG_v-E_vG_u+4F_uF_v-2E_vF_v-2F_uG_u)pq+\\
&+(-2EG_{uu}-2EE_{vv}+4EF_{uv}+E_uG_u+E_v^2-2E_uF_v)q^2+\\
&+2(E_{vv}+G_{uu}-2F_{uv})(EG-F_v)+\\
&-E_v^2G-EG_u^2-EE_vG_u-E_uGG_u+2E_uF_vG+2EF_uG_v+\\
&+2E_vFF_v+2FF_uG_u-E_uF_uG_v+E_vFG_u-4FF_uF_v=0
\end{aligned}
\tag{5.2}
$$

where $p=\frac{\partial z}{\partial u}$, $q=\frac{\partial z}{\partial v}$, $r=\frac{\partial^2 z}{\partial u^2}$, $s=\frac{\partial^2 z}{\partial u \partial v}$, $t=\frac{\partial^2 z}{\partial v^2}$. In the case $E=G=0$, (5.2) corresponds to Bour's equation.

In (Darboux 1873, pp. 17–18),[6] Darboux found the same equation as Dini (5.2), by following a different route. When a surface $S:(u,v)\to \boldsymbol{x}=(x(u,v),y(u,v),z(u,v))$ has $ds^2=Edu^2+2Fdudv+Gdv^2$ as line element, by substituting $dz=\dfrac{\partial z}{\partial u}du+\dfrac{\partial z}{\partial v}dv$ in ds^2, one has

$$
dx^2+dy^2=\left[E-\left(\frac{\partial z}{\partial u}\right)^2\right]du^2+2\left[F-\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right]dudv+\left[G-\left(\frac{\partial z}{\partial v}\right)^2\right],
$$

which represents the line element of a plane. By requiring the Gaussian curvature to vanish[7] Darboux found that Dini's equation (5.2) must hold for z. On the contrary, if z satisfies Dini's equation, Darboux showed that it determines a surface $S:(u,v)\to \boldsymbol{x}=(x(u,v),y(u,v),z(u,v))$, where $x(u,v)$ and $y(u,v)$ are obtained by quadratures from $z(u,v)$.

In (Darboux 1873, pp. 181–182), Darboux deduced another equation for the second problem of applicability in a simple invariant form.[8] For every surface S with $\boldsymbol{x}(u,v)=(x(u,v),y(u,v),z(u,v))$ as parametrization and $ds^2=Edu^2+2Fdudv+Gdv^2$ as line element, the function $z(u,v)$ must satisfy the following equation[9]

$$
\Delta_{22}z=K(1-\Delta_1 z). \tag{D}
$$

[5]Subscripts u and v indicate derivatives with respect to u and v.

[6]A more detailed exposition is in (Darboux 1894, pp. 253–255).

[7]Darboux used the following equation for Gaussian curvature

$$
K\left(EG-F^2\right)^2=\begin{vmatrix}-\frac{1}{2}E_{vv}+F_{uv}-\frac{1}{2}G_{uu} & F_v-\frac{1}{2}G_u & \frac{1}{2}G_v\\ \frac{1}{2}E_u & E & F\\ F_u-\frac{1}{2}E_v & F & G\end{vmatrix}-\begin{vmatrix}0 & \frac{1}{2}E_v & \frac{1}{2}G_u\\ \frac{1}{2}E_v & E & F\\ \frac{1}{2}G_u & F & G\end{vmatrix}
$$

that Brioschi deduced in (Brioschi 1852, p. 12).

[8]For a detailed description of the method, the interest reader can refer to (Darboux 1894, pp. 259–260).

[9] With respect to $ds^2=Edu^2+2Fdudv+Gdv^2$, the differential parameters $\Delta_1 f$, $\Delta(f,g)$, $\Delta_2 f$ and

Conversely, every solution $z(u,v)$ of Darboux's equation (D) determines a unique surface S, which is parametrised by means of $\boldsymbol{x}(u,v) = (x(u,v), y(u,v), z(u,v))$, where $x(u,v)$ and $y(u,v)$ can be found by quadratures.

However, (D) is still a Monge-Ampère equation that reduces to Bour's equation. Hence, albeit generalised, both Dini and Darboux found essentially Bour's equation, which cannot be integrated by known methods except in the trivial case of developable surfaces, therefore it cannot be used to determine complete classes of applicable surfaces.

5.2 Weingarten, 1884: a revision of Bour's equation

In the introduction to (Weingarten 1884), Weingarten provided an interesting critical review of the deduction of Bour's equation, apparently with the aim of emphasising the importance of MCE. While Bour considered the first two methods that he devised for the problem of applicability as equivalent (that based on the second-order differential equation of the Monge-Ampère type and that based on the MCE, respectively), in Weingarten's opinion the first suffered from a serious shortcoming: real solutions of Bour's equation do not necessarily correspond to real surfaces. In the case of Bour's first method, Weingarten observed that, if $z(u,v)$ satisfies Bour's equation, by symmetry the same equation must be satisfied by $x(u,v)$ and $y(u,v)$. However, it must be verified that the set of the three equations is compatible. This fact led Weingarten to state:

> However, we cannot agree with Bour's view that this equation [Bour's equation], or equation (*d.*) [which is an equivalent form of Bour's equation that Weingarten deduced in (Weingarten 1884, p. 3)] in general represents the differential equation of the coordinates of those surfaces that can be developed on the given surface with the line element $\sqrt{Edp^2 + 2Fdpdq + Gdq^2}$. In fact, not every function x of real variables p, q that satisfies the partial differential equation (*d.*) is to be considered as a function that, combined with two other functions y, z of the same nature, is able to satisfy the fundamental equation
>
> $$dx^2 + dy^2 + dz^2 = Edp^2 + 2Fdpdq + Gdq^2 \tag{5.3}$$
>
> as it should be if the opinion we do not share were well-founded.[10]

$\Delta_{22} f$ of generic functions $f(u,v), g(u,v)$ are given by

$$\Delta_1 f := \frac{E\left(\frac{\partial f}{\partial v}\right)^2 - 2F\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} + G\left(\frac{\partial f}{\partial u}\right)^2}{\Delta}$$

$$\Delta_1(f,g) := \frac{E\frac{\partial f}{\partial v}\frac{\partial g}{\partial v} - F\left(\frac{\partial f}{\partial u}\frac{\partial g}{\partial v} + \frac{\partial f}{\partial v}\frac{\partial g}{\partial u}\right) + G\frac{\partial f}{\partial u}\frac{\partial g}{\partial u}}{\Delta}$$

$$\Delta_2(f) := \frac{1}{\sqrt{\Delta}}\left[\frac{\partial}{\partial u}\left(\frac{G\frac{\partial f}{\partial u} - F\frac{\partial f}{\partial v}}{\sqrt{\Delta}}\right) + \frac{\partial}{\partial v}\left(\frac{E\frac{\partial f}{\partial v} - F\frac{\partial f}{\partial u}}{\sqrt{\Delta}}\right)\right]$$

$$\Delta_{22} f = \frac{2\Delta_2 f \Delta_1(f, \Delta_1 f) - \Delta_1\Delta_1 f}{4\Delta_1 f}.$$

where $\Delta = EG - F^2$. It was precisely on this occasion that the differential parameter $\Delta_{22} f$ was defined.

[10] Allein der Meinung Bour's, dass diese Gleichung, oder allgemeiner die Gleichung (*d.*), die Differential-

The crucial point is that Weingarten not only required that $x(u,v), y(u,v)$ and $z(u,v)$ be real, but he also explicitly demanded that the applicable surfaces were real. Weingarten did not justify this demand, which was not widely shared. Darboux, for example, considered it to be a minor problem.[11] However, one must consider that at the time, geometers were generally still somewhat diffident about tackling problems in the complex field rather than in the real field. (Cajori 1929, p. 435) described Scheffers' discovery in 1901 that the imaginary cylinder is developable on a plane, as *"a result which would have astonished most of the eighteenth century geometers"*. In this regard, for example, Renato Calapso also commented:

> It should be noted that a certain obstinacy of the geometers of the time in wanting everything to take place in the real field, and their evident reluctance to discuss questions in the complex field, undoubtedly delayed the progress of geometry, at least in this area [Calapso was commenting the emerging of a transformation theory for surfaces with constant positive curvature]. Because the studies in question, like those of algebraic geometry, had precisely the natural seat of their development in the complex field.[12]

Weingarten's position on this issue can be inferred by considering the position of Bianchi, with whom he shared the main principles of his research. Scorza placed Bianchi in the middle of these two positions: he did not disdain the use of imaginary quantities, which he often resorted to, but always accompanied their use with a real interpretation of the results he obtained.[13]

It is also interesting to note that the same criticism of a lack of symmetry in the treat-

gleichung der Coordinaten derjenigen Flächen darstelle, welche auf die gegebene Fläche, vom Linienelement $\sqrt{Edp^2 + 2Fdpdq + Gdq^2}$, abwickelbar sind, können wir uns nicht anschliessen. In der That ist nicht jede reellwerthige Function x der Variablen p, q welche der partiellen Differentialgleichung (d.) Genüge leistet, als eine Function zu betrachten, die mit zwei anderen Functionen y, z derselben Natur combinirt, geeignet sei die Fundamentalgleichung

$$dx^2 + dy^2 + dz^2 = Edp^2 + 2Fdpdq + Gdq^2$$

zu erfüllen, wie es doch sein musste, wenn die von uns nicht getheilte Meinung eine begründete wäre.(Weingarten 1884, p. 3)

[11]See (Darboux 1894, pp. 255–256). In general, it seems that Darboux is more confident with the use of the complex field. On the occasion of his speech to the 1908 International Congress of Mathematics, he focused on the need to introduce imageries into geometry: *"In infinitesimal geometry, as in Poncelet geometry, the use of imaginaries would be no less fruitful than in Analysis. But it took much longer to win. Why? I do not know. Is it because, according to some, Geometry must always have in view real objects and lead to constructions? Perhaps. In any case there is much progress to be made in this regard."* (Darboux 1908, p. 111).

[12]"Bisogna osservare che una certa ostinazione dei geometri del tempo a voler che tutto si svolgesse nel campo reale e la loro evidente ripugnanza a discutere le questioni nel campo complesso, fecero senza dubbio ritardare, almeno in questo settore, il progresso della geometria. Perché gli studi in discorso, come quelli della geometria algebrica, avevano proprio nel campo complesso la sede naturale del loro sviluppo." (Bianchi 1952-1959, Vol. IV.1, 5)

[13](Scorza 1930, p. 26)

ment of x, y, z was also fruitful two years later when Weingarten improved the infinitesimal deformations theory. In a letter to Bianchi, Weingarten wrote:

> I would like to thank Mr. Vito Volterra for sending me his publication and to kindly inform him that I have occupied myself with the investigations on small deformations of surfaces in a *theoretical form* and that I intend to publish something about them, especially with regard to surfaces of constant curvature. His interesting investigations have prompted me to treat the question in a *symmetric* way, without considering z as a function of x and y, as Jellet does. This does not make it more complicated, but simpler.[14]

For both finite and infinitesimal deformations, this insight was decisive for a renewed and more effective perspective on the problem.

To demonstrate the necessity of some corrections to Bour's first solution, Weingarten was inspired by Darboux's method for deriving Dini's equation. Weingarten supposed that[15] a function $x(u, v)$ satisfies equation (D) and wrote (5.3) as

$$dy^2 + dz^2 = \left[E - \left(\frac{\partial x}{\partial u}\right)^2\right] du^2 + 2\left[F - \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\right] dudv + \left[G - \left(\frac{\partial x}{\partial v}\right)^2\right]. \tag{5.4}$$

Its right side, as Darboux also noted, must be the line element of a surface with null curvature and thus there exists a coordinate system (symmetric parameters) such that the coefficients of the line element can be chosen as $E = G = 0$ and $F = 1$. Then, there must exist two functions $\alpha(u, v)$ and $\beta(u, v)$ such that

$$dy^2 + dz^2 = d\alpha d\beta, \tag{5.5}$$

where

$$d\alpha = e^{\phi}(adu + bdv) \qquad d\beta = e^{-\phi}(a'du + b'dv)$$

and ϕ is an auxiliary function that can be determined by quadratures.[16] However, as was well known, only when the symmetric parameters α, β assume conjugate complex values

[14] "Herrn Vito Volterra bitte ich meinen Dank für die Uebersendung seiner Publication zu sagen, und ihm freundlichst mitzutheilen, dass ich mich *formentheoretisch* mit den Untersuchungen über kleine Deformationen der Flächen beschäftigt habe, und über dieselbe, besonders in Rücksicht auf Flächen constanter Krümmung, einiges zu publiciren gedenke. Seine interessanten Untersuchungen haben mich veranlasst die Frage in *symmetrischer* Weise zu behandeln, ohne wie Jellet es thut z als Function von x und y zu betrachten. Sie wird dadurch nicht complicirter, sondern einfacher." (Bianchi 1959, p. 171)

[15] To make the notation uniform and to facilitate readability, we changed Weingarten's parameters p, q with u, v.

[16] By decomposing the left-hand side of (5.5) as $(adu + bdv)(a'du + b'dv)$, where a, b, a', b' are function of u and v, one has

$$d\alpha = e^{\phi}(adu + bdv) \qquad d\beta = e^{-\phi}(a'du + b'dv)$$

where φ is determined by quadratures through the integrability conditions

$$\frac{\partial e^{\phi}a}{\partial v} = \frac{\partial e^{-\phi}b}{\partial u} \qquad \frac{\partial e^{\phi}a'}{\partial v} = \frac{\partial e^{-\phi}b'}{\partial u}.$$

For more details, see (Darboux 1894, §684).

$y + iz$, $y - iz$, is the corresponding surface real. Indeed, only in this case, the function x that satisfies equation (D) has the property that two real-valued functions y and z exist, together with which x makes the equation

$$dx^2 + dy^2 + dz^2 = Edu^2 + 2Fdudv + Gdv^2$$

an identity.

Weingarten also noted that α, β assume conjugate complex values only when the function x, besides satisfying equation (D), makes the right-hand side of (5.3) a positive definite quadratic form, that is, only when x also satisfy the inequality

$$\left[E - \left(\frac{\partial x}{\partial u}\right)^2\right]\left[G - \left(\frac{\partial x}{\partial v}\right)^2\right] - \left[F - \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\right]^2 > 0. \tag{5.6}$$

Weingarten briefly commented on this condition:

> [it] expresses a simple geometric property of those functions x that are suitable to be considered as the coordinate x of a curved surface of the line element $\sqrt{Edp^2 + 2Fdpdq + Gdq^2}$.[17]

On the other hand, in the case that this inequality does not hold true, i.e., when the functions α, β are real, the product $d\alpha d\beta$ can only assume the form $dy^2 - dz^2$, and then the function x, together with the functions y and z, satisfies the equation

$$dx^2 + dy^2 - dz^2 = Edu^2 + 2Fdudv + Gdv^2,$$

"which is extraneous to the problem of applicability of the curved surfaces".[18]

In the 1886 lithographed edition of his *Lezioni*, Bianchi had already reported the need to add the request $\Delta_1 x > 1$ to equation (D) and emphasised the impossibility of having a single equation that represented the second problem of applicability.[19] A geometric interpretation of the case $\Delta_1 x < 1$ occurred only in the second edition, in (Bianchi 1903, §271). Here, Bianchi clearly distinguished two problems that Bour's equation represented, i.e.,

$$\Delta_{22} x = K(1 - x) \qquad \Delta_1 x < 1$$
$$\Delta_{22} x = K(1 - \Delta_1 x) \qquad \Delta_1 x > 1.$$

The former couple of equations corresponds to the exact transcription of the second problem of applicability in the Euclidean space $ds^2 = dx^2 + dy^2 + dz^2$; the latter is interpreted as a problem of applicability in the indefinite space $ds^2 = dx^2 + dy^2 - dz^2$.[20]

[17] "eine einfache geometrische Eigenschaft derjenigen Functionen x ausdruckt, welche geeignet sind, als Coordinate x einer krummen Fläche vom Linienelement $\sqrt{Edp^2 + 2Fdpdq + Gdq^2}$ betrachtet zu werden." (Weingarten 1884, p. 5)

[18] " welche dem problem der abwickelbarkeit der krummen Flächen auf einander, fremd ist." (Weingarten 1884, p. 5)

[19] According to the definition of $\Delta_1 f$ in Note 9, the condition (5.6) can be easily written as $\Delta_1 x > 1$.

[20] This was not the first time that Bianchi considered the indefinite space. In (Bianchi 1888b) he investigated indefinite quadratic differential forms as the line element of spaces with constant curvature.

5.3 Complete classes of applicable surfaces

Weingarten's interest in applicability had arisen about 20 years before (Weingarten 1884), when, early in his career, he published (Weingarten 1861) and (Weingarten 1863). Here, he deduced the first non-trivial complete classes of applicable surfaces in finite terms, that is to say, the finite equations (up to quadratures) of all the surfaces applicable to a given one.

Such a problem should not be surprising. We have already seen in Section 3.1.1 how the Paris Prize required not only the deduction of equations representing the second problem of applicability, but also their use to deduce particular classes of surfaces. We have also seen how Bour's memoir was preferred to those of Bonnet and Codazzi, which were also meritorious of the prize, precisely because of the determination of a complete integral of Bour's equation in the special case of surfaces of revolution.

It should also be considered that searching for finite equations of geometric entities in general is not only related to the problem of applicability, but was a typical interest in the nineteenth century. To name a few of the main problems, there is the determination in finite terms of geodesic lines, of minimal surfaces, or of surfaces that admit an isothermal system of lines of curvature.[21]

Although describing properties in terms of differential equations had become customary, starting with Monge, the determination of finite equations of geometric entities still remained a central problem. In this respect, Enea Bortolotti's comments on the early geometrical works of Ulisse Dini may illuminate the point:

> The general theory of differential and partial differential equations, which is the substantial basis of differential geometry and, moreover, finds in it a natural field of exercise and experience, required a wealth of concrete models of the general facts discovered or intuited: hence the stimulus to seek, perhaps through laborious developments and with the aid of the most varied artifices, the determination in finite terms of more or less extensive classes—possibly of the larger one that constitutes the general integral—of geometrical entities satisfying prescribed differential conditions, that is, solutions of equations or systems assigned. Gradually, this search for concrete solutions has been replaced by the observation that the same imposed differential conditions often succeed in being simpler and more manageable, in order to progress in the knowledge of the geometrical properties of the entities and their representation in finite terms.[22]

[21] An overview of these issues is provided by (Reich 1973) and (Bianchi 1887b).

[22] *"La teoria generale delle equazioni differenziali e a derivate parziali, che è la base sostanziale della geometria differenziale e d'altra parte vi trova un naturale campo d'esercitazione e d'esperienza, richiedeva copia di modelli concreti dei fatti generali scoperti o intuiti: di qui lo stimolo a ricercare, magari attraverso laboriosi svolgimenti e col sussidio dei più svariati artefici, le determinazione in termini finiti di classi più o meno estese—possibilmente di quella più ampia che costituisce l'integrale generale—di enti geometrici soddisfacenti a prescritte condizioni differenziali, cioè soluzioni di equazioni o sistemi assegnati. A poco a*

The fundamental role of integration in differential geometry was also emphasised by Darboux in a review of the first German edition of Bianchi's *Lezioni*, (Bianchi 1899c), when he complained about the title choice:

> Everything is to be praised in these Lessons, except their title: the French geometers had included under the name of *Infinitesimal Geometry* the whole of the studies exposed by M. L. Bianchi; why have they weakened this denomination in a way, by reducing it to that of Differential Geometry? The risk is to let people believe that Geometry finds its whole domain in the operations of differentiation, whereas it has to its credit so many beautiful integrations inspired or obtained by the methods which constitute it. Moreover, some people are given a pretext for neglecting it, by including it entirely in Differential Calculus, which is considered to be entirely complete and even to be reduced to purely mechanical operations.[23]

The role of geometry in problems concerning the integration of differential equations was also not secondary for Weingarten. As an example, commenting on Bianchi's results on the equation of surfaces with constant positive curvature, he remarked:

> It [Weingarten probably referred to (Bianchi 1886b)] casts new light on the peculiar properties of the partial differential equations which appear in the theory of surfaces of constant curvature, which are still almost unknown. Geometry will have to show the way to Analysis in order to solve the partial differential equations of this kind. General investigations of such equations have yielded nothing.[24]

5.3.1 $W-$surfaces for investigating applicability

Bour's equation is not very satisfactory with respect to the integrability. Bour himself noted that it was integrable by known methods—essentially the Monge-Ampère method—only in the trivial case of developable surfaces, and thus fails to provide the finite equations

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poco a questa ansia di ricerca di concrete soluzioni ha subentrato la constatazione che le stesse condizioni differenziali imposte spesso riescono più semplici e maneggevoli, allo scopo di progredire nella conoscenza delle proprietà geometriche degli enti la rappresentazione di questi in termini finiti"(Bortolotti 1953)

[23]"Tout est à louer dans ces Leçons, sauf leur titre: les géomètres français avaient compris sous le nom de *Géométrie infinitésimale* l'ensemble des études exposées par M. L. Bianchi; pourquoi avoir affaibli en quelque sorte cette dénomination, en la réduisant celle de Géométrie différentielle? On risque ainsi de laisser croire que la Géométrie trouve tout son domaine dans les opérations de différentiation, alors qu'elle a à son actif tant de belles intégrations inspirées ou obtenues par les méthodes qui la constituent. On fournit en outre à certains un prétexte pour la négliger, en la faisant figurer tout entière dans le Calcul différentiel, qui est considéré comme entièrement achevé et même comme se réduisant à des opérations purement mécaniques." (Darboux 1899, p. 323)

[24]"Sie wirft ein neues Licht auf die merkwürdigen Eigenschaften der partiellen Differentialgleichungen, die in der Theorie der Flächen von constantem Krummungsmass auftreten, die ja bisher noch fast unergründet sind. Die Geometrie wird der Analysis die Wege weisen müssen, um die partiellen Differentialgleichungen dieser Art zu ergründen. Allgemeine Untersuchungen über solche haben nichts ergeben." (Bianchi 1959, pp. 179–180)

of complete classes of applicable surfaces except in the case already known to Euler and Monge. Weingarten succeeded in overcoming this impasse by exploiting, among the possible *"varied artifices"*, $W-$surfaces.

$W-$surfaces were first studied in (Weingarten 1861), which is devoted to Monge's promising theory of evolutes that, according to Weingarten, had not garnered its deserved attention until then. Analogously to curves, the evolute of a surface is defined as the geometric locus of the centres of the circles that osculate the lines of curvature. Generally, an evolute has two distinct connected components, which are called *nappes* of the evolute, since there are two lines of curvature at every point of the given surface.

Weingarten considered the case in which the radii of curvature, $\rho_1(u, v)$ and $\rho_2(u, v)$, of a given surface are bounded together by a functional relation $\rho_2(u, v) = F(\rho_1(u, v))$ at every point of the domain of definition of the surface. Such a surface is called $W-$*surface* in his honour.

This restriction allowed Weingarten to demonstrate that each of the nappes of the evolute of any $W-$surface is applicable to a surface of revolution whose line element depends on the functional relation between the principal radii of the involute. Generally, the two nappes are not applicable on the same surface of revolution, unless the relation between the principal radii is symmetrical with respect to them. Thus, depending on which line of curvature is considered, Weingarten found that the two nappes are applicable to surfaces of revolution having as line element, respectively:

$$ds_1^2 = d\rho_1^2 + e^{2\int \frac{d\rho_1}{\rho_1-\rho_2}} du^2 \qquad ds_2^2 = d\rho_2^2 + e^{2\int \frac{d\rho_2}{\rho_2-\rho_1}} dv^2. \tag{5.7}$$

Moreover, the evolutes of *every* surface in the same $W-$class are applicable to the same surfaces of revolution (5.7). He also demonstrated the converse of this theorem: any surface applicable to a surface of revolution can be regarded as the evolute of a $W-$surface.[25] Hence, the discovery of these theorems, which we denote with $W1$, proved the equivalence of two distinct problems: the research of surfaces applicable to surfaces of revolution, and the determination of $W-$surfaces.

Two years later, in (Weingarten 1863), Weingarten proved that the determination of $W-$surfaces can also be reduced to the research of orthogonal systems on a sphere for which the coefficients of the line element are bounded by a specific relation according to the following results, which we denote with $W2$. When a surface S is given and $\boldsymbol{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ is its parametrization, its *spherical representation* is its image through the Gauss map

$$\begin{aligned} &N : S \to S^2 \\ &(u, v) \mapsto \boldsymbol{X} = (X(u, v), Y(u, v), Z(u, v)) \end{aligned}$$

[25]In (Beltrami 1864-1865, p. 166) Beltrami showed that ruled helicoidal surfaces constitute an exceptional case, since these surfaces are applicable upon a surface of rotation but cannot be considered as a nappe of a surface of curvature centres.

where $\boldsymbol{X} = (X(u,v), Y(u,v), Z(u,v))$ is a normal unit vector field upon S. The line element of the spherical representation is given by the following quadratic differential form

$$d\mathcal{S}^2 = dX^2 + dY^2 + dZ^2 = \mathcal{E}du^2 + 2\mathcal{F}dudv + \mathcal{G}dv^2.$$

Weingarten proved that the line element of the spherical representation of a $W-$surface referred to its lines of curvature as coordinate lines can always assume the form

$$d\mathcal{S}^2 = \frac{du^2}{\alpha} + \frac{dv^2}{\Theta'^2(\alpha)}, \tag{5.8}$$

where α and Θ are functions of u, v and Θ' denotes the derivative of Θ with respect to α. Moreover, the curvature radii ρ_1 and ρ_2 are defined by means of α and Θ through the following equations

$$\rho_1 = \Theta(\alpha) \qquad \rho_2 = \Theta(\alpha) - \alpha\Theta'(\alpha). \tag{5.9}$$

By substituting these values of ρ_1 and ρ_2, the line elements of the two nappes of the evolutes of a $W-$surface (5.7) become

$$ds_1^2 = \Theta'^2(\alpha)d\alpha^2 + \alpha^2 du^2 \qquad ds_2^2 = \alpha^2\Theta''^2(\alpha)d\alpha^2 + \Theta'^2 dv^2. \tag{5.10}$$

Conversely, if (5.8) is a line element of a unit sphere, then it can be interpreted as a spherical representation of a $W-$surface. Moreover, the knowledge of all parametrizations on a unit sphere, for which the coefficients $\mathcal{E}$ and $\mathcal{G}$ are bounded together as in (5.8) for a fixed function Θ, allows the determination of all surfaces of the same $W-$class.

Weingarten also noted that the finite equations of this spherical representation, which are $\boldsymbol{X} = (X(u,v), Y(u,v), Z(u,v))$, enable determination of the finite equations of the associated $W-$surface by quadratures[26] and then those of its evolutes

$$\begin{cases} \xi = -\displaystyle\int \Theta'(\alpha)(Xd\alpha + \alpha\frac{\partial X}{\partial u}du) \\ \eta = -\displaystyle\int \Theta'(\alpha)(Yd\alpha + \alpha\frac{\partial Y}{\partial u}du) \\ \zeta = -\displaystyle\int \Theta'(\alpha)(Zd\alpha + \alpha\frac{\partial Z}{\partial u}du), \end{cases}$$

in the case the nappe is associated to ρ_1, and through similar ones in the case of ρ_2, whose line elements are (5.10), respectively. In particular, when *all* the orthogonal systems on a sphere, for which the coefficients $\mathcal{E}$ and $\mathcal{G}$ are bounded as in (5.8) for the same function Θ, are known in finite terms, then both the *complete* class of the associated $W-$surfaces and that of their evolutes, which constitute a *complete* class of applicable surfaces, are known in finite terms (see Figure 5.1).

Unfortunately, however, the spherical representation was proved to assume the form (5.8) only in two cases—canal surfaces,[27] which correspond to $\Theta(\alpha) = \alpha + a$, where a is a constant, and minimal surfaces, which are associated with $\Theta(\alpha) = \frac{\alpha^2}{2}$.

[26] They are equations (4) in (Weingarten 1863, p. 162).

[27] Canal surfaces are envelopes of a family of spheres with the same radius. They have one of the radii of curvature that is constant. See (Eisenhart 1909, §29)

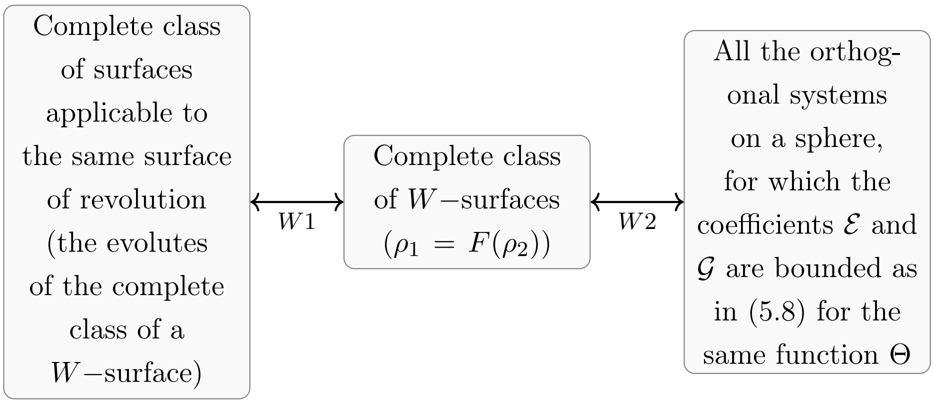


Figure 5.1: Diagram of Weingarten's theorems, $W1$ and $W2$. Each $W-$surface corresponds to a surface applicable to a surface of revolution (as a consequence of $W1$) and to a particular orthogonal system on a unit sphere (as a consequence of $W2$). By considering a new $W-$surface in the same $W-$class, another surface applicable to the same surface of revolution and another orthogonal system of the same type are obtained. Weingarten's deduction of new complete classes of applicable surfaces is based on the following remark: when *all* orthogonal systems on the unit sphere such that (5.8) holds true are known, both the *complete* class of $W-$surfaces and the *complete* class of surfaces applicable to a surface of revolution are known.

In the case of canal surfaces, $\rho_1 = \alpha + a$ and $\rho_2 = a$ and the two nappes of the evolute are trivial since (5.10) reduce to

$$ds_1^2 = d\alpha^2 + \alpha^2 du^2 \qquad ds_2^2 = dv^2,$$

where ds_1 is the line element of a developable surface[28] and ds_2^2 that of a curve.

In the case of minimal surfaces, $\rho_1 = \frac{\alpha^2}{2}$ and $\rho_2 = -\frac{\alpha^2}{2}$ and (5.10) reduce to

$$ds_1^2 = \alpha^2(d\alpha^2 + du^2) \qquad ds_2^2 = \alpha^2(d\alpha^2 + dv^2).$$

Hence, the two nappes are applicable one upon the other. Moreover, the parameters that give this form to the spherical line element are isothermal and thus the orthogonal systems on a sphere of the second theorem of Weingarten are all well known. Hence, Weingarten succeeded in determining both the finite equations of minimal surfaces with respect to their lines of curvature and those of their evolutes. These latter constituted the first non-trivial complete class of applicable surfaces ever found. Geometers often referred to this as the class of the evolute of the catenoid, since the catenoid is the minimal surface of revolution. Its representative line element is

$$ds^2 = u^2 du^2 + u^2 dv^2 \tag{5.11}$$

In 1863, Weingarten succeeded in also deducing another complete class of $W-$surfaces that admits (5.8) as a possible spherical representation. He supposed $\Theta(\alpha) = \frac{1}{2}(\arcsin k +$

[28] One can easily be convinced of this by computing the Gaussian curvature of ds_1^2.

$k\sqrt{1-k^2}$). According to (Weingarten 1863, §3), this case occurs when the position of a point on a surface is determined by its geodetic distance from two curves that are arbitrarily drawn on it. In (Weingarten 1863, §4), Weingarten found the finite equations of the associated $W-$surfaces and showed that they are defined through the strange and complicated equation $2(\rho' - \rho) = \sin 2(\rho + \rho')$. By means of his theorems, he also deduced the finite equations of their evolutes, which constitute the complete class of surfaces, whose line element can assume the form[29]

$$ds^2 = (1 - u^2)du^2 + u^2dv^2. \tag{5.12}$$

To give a description of this new class, Weingarten explicitly specified one of its representatives, that is

$$\begin{cases} x = \alpha r \cos \dfrac{v}{\alpha} \\ y = \alpha r \sin \dfrac{v}{\alpha} \\ z = \displaystyle\int_0^u \sqrt{1 - \alpha^2 - u^2} du \end{cases} \tag{5.13}$$

where α is a constant. Later, in (Darboux 1894, pp. 332–333), Darboux noted that, in the case $\alpha = 1$ (5.13) corresponds to the imaginary paraboloid $x^2 + y^2 + 2iz = 0$. He also demonstrated that Weingarten's theorems could be used for the deduction of the complete class of the real paraboloid $x^2 + y^2 + 2z = 0$, whose line element is

$$ds^2 = (1 + u^2)du^2 + u^2dv^2. \tag{5.14}$$

Indeed, this paraboloid can be interpreted as the evolute of a $W-$surface that has radii of curvature linked by the relation $2(\rho' - \rho) = \sinh 2(\rho + \rho')$, whose spherical line element (5.10) was known.

5.3.2 The discovery of a new complete class

In 1887, Weingarten published another remarkable short paper (Weingarten 1887a), on the determination in finite terms of a complete class of applicable surfaces. Its incipit highlights the novelty of his results:

> The number of cases in which it has so far been possible to fully represent the totality of surfaces that can be developed on a given one is, if we are not mistaken, three. So far, this representation has been given in finite terms only for surfaces that can be developed on a plane and for two classes that can be developed on special surfaces of revolution [(5.11) and (5.12)]. With this small number of examples, the inclusion of a new class of surfaces that can be developed on each other, among which there are no surfaces of revolution, may

[29] Since $\sin 2(\rho + \rho')$ is symmetric with respect to the radii of curvatures, the two nappes are applicable one upon the other also in this case.

perhaps be of some interest. The square of the line element of the surfaces belonging to this new class can be brought to the form

$$ds^2 = (\alpha^{\frac{2}{3}} + \beta^{\frac{2}{3}})(d\alpha^2 + d\beta^2), \tag{5.15}$$

from which one recognises, according to a well-known theorem by Liouville,[30] that the geodesic lines for each surface belonging to this class can be determined by quadratures, a property which also applies to the three previously known classes of surfaces which can be developed on each other.[31]

Weingarten's idea was to establish a link between a generic minimal surface, whose finite equations $\Sigma : (u, v) \to \boldsymbol{x} = (x(u, v), y(u, v), z(u, v))$ are known, and a complete class of applicable surfaces that he showed to be described by the line element (5.15).

A system of equations, today known as *Weingarten's equations* since it was first deduced at the beginning of (Weingarten 1861), turns out to be fundamental. If the first and second fundamental forms of Σ are $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ and $dS^2 = Ddu^2 + 2D'dudv + D''dv^2$, respectively, the tangent vectors $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ to the coordinate lines can be expressed in terms of the derivatives of the normal vector $\boldsymbol{X} = (X, Y, Z)$ to Σ as follows

$$\boldsymbol{x}_u = A\boldsymbol{X}_u + A'\boldsymbol{X}_v \quad \boldsymbol{x}_v = B\boldsymbol{X}_u + B'\boldsymbol{X}_v \tag{5.16}$$

where

$$A = \frac{FD' - GD}{EG - F^2} \quad B = \frac{FD'' - GD'}{EG - F^2} \quad A' = \frac{FD - ED'}{EG - F^2} \quad B' = \frac{FD' - ED''}{EG - F^2}.$$

By referring to Rodrigues' equations, Weingarten also showed that A, A', B, B' are related to the principal radii of curvature ρ_1 and ρ_2 of Σ through the following relations[32]

$$A + B' = \rho_1 + \rho_2 \qquad AB' - BA' = \rho_1\rho_2.$$

[30] In (Liouville 1846) Liouville demonstrated that geodesic lines of surfaces with $ds^2 = (U^2(u) + V^2(v))(du^2 + dv^2)$, where U and V are functions of u and v only, respectively, can be found by quadratures.

[31] "Die Anzahl der Fälle, in denen es bisher gelungen ist, die Gesammtheit aller Flächen, welche auf eine gegebene Fläche abwickelbar sind, vollständig darzustellen, ist, wenn wir nicht irren, auf drei beschränkt. Nur für die auf eine Ebene abwickelbaren Fläche und für zwei Classen von Flächen, welche auf specialle Rotationsfläche abwickelbar sind, ist diese Darstellung bisher durch endliche Gleichungen gegeben worden. Bei dieser kleinen Zahl von Beispielen dürfte die Mittheilung einer neuen Classe von auf einander abwickelbaren Flächen, unter deren Individuen sich Rotationsflächen nicht vorfinden, vielleicht einiges Interesse darbieten. Das Quadrat des Linienelementes der dieser neuen Classe angehörenden Flächen kann auf die Form

$$ds^2 = (\alpha^{\frac{2}{3}} + \beta^{\frac{2}{3}})(d\alpha^2 + d\beta^2)$$

gebracht werden, aus welcher man nach einem bekannten Liouvilleschen Theorem erkennt, dass die geodätischen Linien für jede dieser Classe angehörende Fläche durch Quadraturen bestimmbar sind, eine Eigenschaft, welche in gleicher Weise den drei bis jetzt bekannt gewesenen Classen von auf einander abwickelbaren Flächen zukommt. " (Weingarten 1887a, p. 28)

[32] Weingarten considered two points, $\boldsymbol{P} = (x, y, z)$ and $\boldsymbol{P} + d\boldsymbol{P} = (x + dx, y + dy, z + dz)$ along the same line of curvature. From Weingarten's equations, one has $d\boldsymbol{x} = \boldsymbol{x}_u du + \boldsymbol{x}_v dv = \boldsymbol{X}_u(Adu + Bdv) + \boldsymbol{X}_v(A'du + B'dv)$. On the other hand, as a consequence of Rodrigues' equations $d\boldsymbol{x} = \rho d\boldsymbol{X}$, one has

In (Weingarten 1887a), Weingarten's equations were simplified by choosing $u = xX + yY + zZ$ and $v^2 = x^2 + y^2 + z^2$. As a consequence, (5.16) became[33]

$$\boldsymbol{x}_u = -\frac{\rho_1\rho_2}{v}\boldsymbol{X}_v \qquad \boldsymbol{x}_v = v\boldsymbol{X}_u + (\rho_1 + \rho_2)v\boldsymbol{X}_v \tag{5.18}$$

These conditions were further simplified by supposing Σ be a minimal surface. Indeed, in this case, one has $\rho_1 + \rho_2 = 0$ and

$$\boldsymbol{x}_u = -\frac{\rho_1\rho_2}{v}\boldsymbol{X}_v \qquad \boldsymbol{x}_v = v\boldsymbol{X}_u. \tag{5.19}$$

Weingarten noted that the validity of the equations (5.19) ensures that the following differentials are exact[34]

$$\begin{cases} d\xi = xdu + vXdv \\ d\eta = ydu + vYdv \\ d\zeta = zdu + vZdv \end{cases} \tag{5.20}$$

and thus define by quadratures three functions $\xi(u, v), \eta(u, v), \zeta(u, v)$ that determine a surface S with line element

$$d\sigma^2 = d\xi^2 + d\eta^2 + d\zeta^2 = v^2du^2 + 2uvdudv + v^2dv^2. \tag{5.21}$$

By posing $v + u = \sqrt{3}\alpha^{\frac{2}{3}}$ and $v - u = \sqrt{3}\beta^{\frac{2}{3}}$, (5.21) reduces to the form (5.15). Clearly, as the minimal surface Σ varies, different surfaces S are obtained by integrating (5.20), but their line element is always the same.

Vice versa, every surface whose line element can be reduced to the form (5.21), leads to a minimal surface and thus can be obtained through this procedure.

$\boldsymbol{x}_u du + \boldsymbol{x}_v dv = \rho\boldsymbol{X}_u du + \rho\boldsymbol{X}_v dv$. The comparison of these equations leads to

$$\begin{cases} Adu + Bdv = \rho du \\ A'du + B'dv = \rho dv, \end{cases}$$

that is,

$$\begin{bmatrix} A - \rho & B \\ A' & B' - \rho \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix} = 0. \tag{5.17}$$

Finally, the assertion is obtained by requiring that the determinant of the matrix in (5.17), which is $\rho^2 - (A + B')\rho + (AB' - A'B)$, is zero.

[33] In slightly modern terms, Weingarten defined $u = \boldsymbol{x} \cdot \boldsymbol{X}$ and $v^2 = \|\boldsymbol{x}\|^2$. By deriving them with respect to u and v, one has

$$1 = \boldsymbol{X}_u \cdot \boldsymbol{x} \qquad 0 = \boldsymbol{X}_v \cdot \boldsymbol{x} \qquad 0 = \boldsymbol{x} \cdot \boldsymbol{x}_u \qquad v = \boldsymbol{x} \cdot \boldsymbol{x}_v.$$

By substituting (5.16) in the latter two of these equations and by taking into account the first two, one has

$$0 = \boldsymbol{x} \cdot \boldsymbol{x}_u = A\boldsymbol{x} \cdot \boldsymbol{X}_u + A'\boldsymbol{x} \cdot \boldsymbol{X}_v = A$$
$$v = \boldsymbol{x} \cdot \boldsymbol{x}_v = B\boldsymbol{x} \cdot \boldsymbol{X}_u + B'\boldsymbol{x} \cdot \boldsymbol{X}_v = B.$$

Combining them with $A + B' = \rho_1 + \rho_2$ and $AB' - BA' = \rho_1\rho_2$ one finally has $A = 0$, $A' = -\frac{1}{\rho_1\rho_2}$, $B = v$ and $B' = \rho_1 + \rho_2$.

[34] The integrability conditions are $\frac{\partial x}{\partial v} = \frac{\partial vX}{\partial u}$, $\frac{\partial y}{\partial v} = \frac{\partial vY}{\partial u}$, $\frac{\partial z}{\partial v} = \frac{\partial vZ}{\partial u}$ are immediately satisfied by referring to (5.19).

Weingarten supposed that a surface S, whose line element is (5.21), is given and he defined six functions $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{X} = (X, Y, Z)$ by means of $\xi(u, v), \eta(u, v), \zeta(u, v)$ as follows

$$x := \frac{\partial \xi}{\partial u} \qquad y := \frac{\partial \eta}{\partial u} \qquad z := \frac{\partial \zeta}{\partial u}$$
$$X := \frac{1}{v}\frac{\partial \xi}{\partial v} \qquad Y := \frac{1}{v}\frac{\partial \eta}{\partial v} \qquad Z := \frac{1}{v}\frac{\partial \zeta}{\partial v}.$$

While the first three equations define the parametric equations of a surface Σ, the remaining three equations guarantee that $\boldsymbol{X}$ is its spherical representation.[35] Moreover, these equations guarantee that[36]

$$Xx + Yy + Zz = u \qquad x^2 + y^2 + z^2 = v^2 \qquad \rho_1 + \rho_2 = 0.$$

Hence, Σ is a minimal surface linked to S through (5.20). *All* the surfaces with (5.21) as line element are therefore deduced by quadratures from the minimal surfaces.

In Weingarten's letters to Bianchi, we do not find any reference to the content of this memoir, but we read that Weingarten sent a copy to his friend in August of that year.[37] Bianchi, however, had already read the memoir, as some handwritten notes dated before July 1887 attest. They are entitled *Observation on Weingarten's note "Eine neue Classe auf einander abwickelbarer Flächen"* (Osservazioni sulla nota "Eine neue Classe auf einander abwickelbarer Flächen". See Figures 5.2 and 5.3). Herein, Bianchi adapted Weingarten's procedure to the case in which Σ has constant (positive or negative) curvature.

> [The associated surface S] is thus applicable to the surface of translation[38] with equations
>
> $$\xi' = \int \sqrt{1 - k^2u^2}du \qquad \eta' = \frac{ku^2}{2} \mp \frac{v^2}{2k} \qquad \zeta' = \int \sqrt{v^2 - \frac{1}{k^2}}vdv,$$
>
> whose generating curves are in the plane $\xi'\eta'$ a cycloid of radius $\frac{1}{4k}$ tangent to the axis ξ at the vertex and in the plane η' a semicubic parabola having the axis η' and facing the positive or negative side depending on whether $\rho\rho' = \mp 1$. Conversely, every surface applicable to this translation surface has a surface with constant curvature".[39]

[35] Indeed, they imply $Xdx + Ydy + Zdz = 0$ and $X^2 + Y^2 + Z^2 = 1$, since $\frac{\partial^2 \xi}{\partial v^2} + \frac{\partial^2 \eta}{\partial v^2} + \frac{\partial^2 \zeta}{\partial v^2} = v$ by definition of the line element of (5.21).

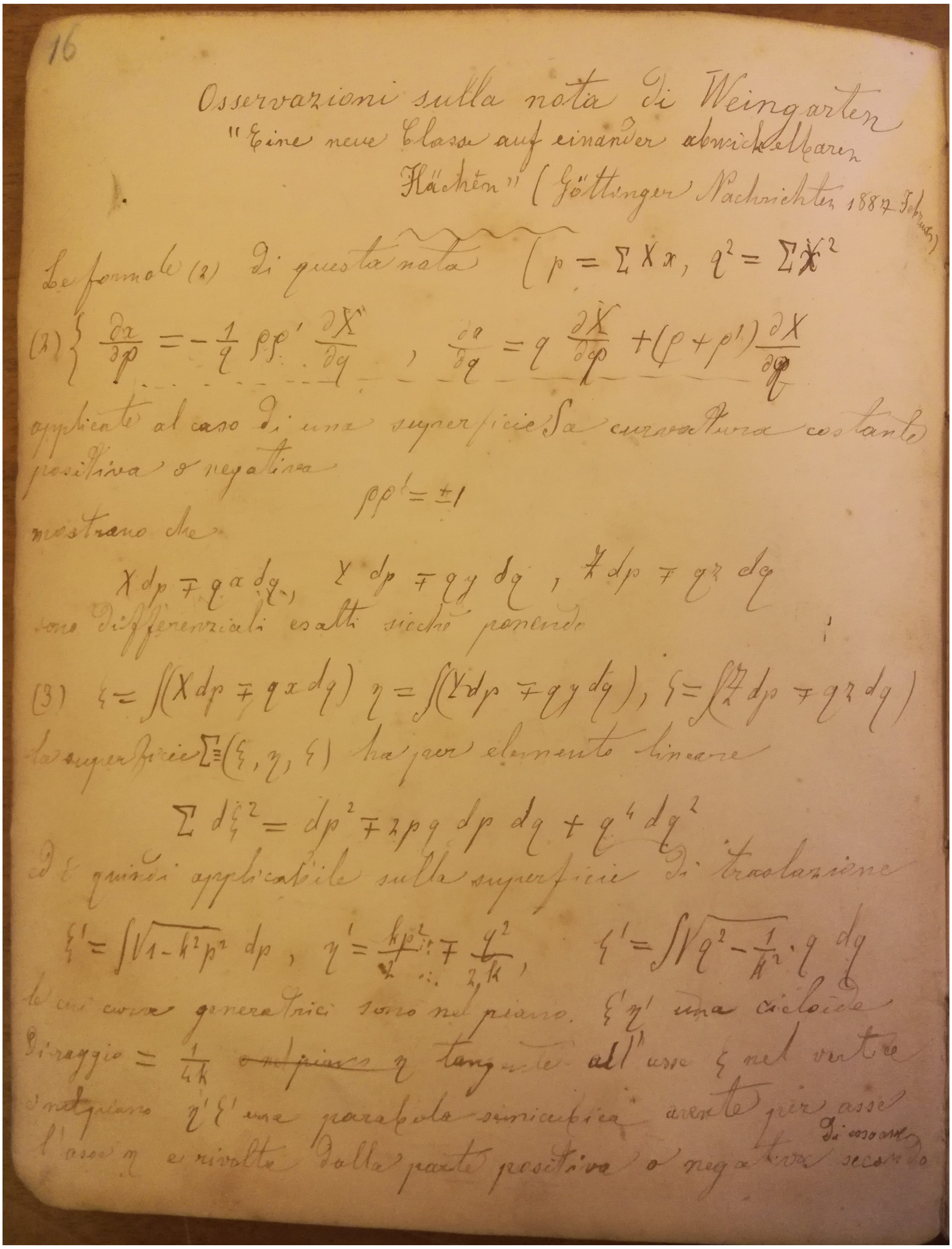
[36] The first and the second equations are a consequence of the definition of the coefficient (E and F, respectively) of the line element of S as written in (5.21). The third equation, instead, is due to the comparison between (5.18) with $\frac{\partial x}{\partial v} = v\frac{\partial X}{\partial u}$, which is obtained from the definitions of $\boldsymbol{x}$ and $\boldsymbol{X}$.

[37] (Bianchi 1959, p. 192).

[38] A surface of translation is a surface generated by translating a curve Γ in such a way that each of its points describes a curve that is congruent with a curve C. See (Eisenhart 1909, §81).

[39] "È quindi applicabile alla superficie di traslazione

$$\xi' = \int \sqrt{1 - k^2u^2}du \qquad \eta' = \frac{ku^2}{2} \mp \frac{v^2}{2k} \qquad \zeta' = \int \sqrt{v^2 - \frac{1}{k^2}}vdv,$$



16

Osservazioni sulla nota di Weingarten
"Eine neue Classe auf einander abwickelbaren Flächen" (Göttinger Nachrichten 1887 Februar)

Le formole (2) di questa nota [$p = \Sigma X x$, $q^2 = \Sigma x^2$]

(2) $\frac{\partial x}{\partial p} = -\frac{1}{q}\rho\rho'\frac{\partial X}{\partial q}$, $\frac{\partial x}{\partial q} = q\frac{\partial X}{\partial p} + (\rho+\rho')\frac{\partial X}{\partial q}$

applicate al caso di una superficie S a curvatura costante positiva o negativa

$$\rho\rho' = \pm 1$$

mostrano che

$$X dp \mp q x\, dq, \quad Y dp \mp q y\, dq, \quad Z dp \mp q z\, dq$$

sono differenziali esatti sicchè ponendo

(3) $\xi = \int (X dp \mp q x\, dq)$, $\eta = \int (Y dp \mp q y\, dq)$, $\zeta = \int (Z dp \mp q z\, dq)$

la superficie $\Sigma \equiv (\xi, \eta, \zeta)$ ha per elemento lineare

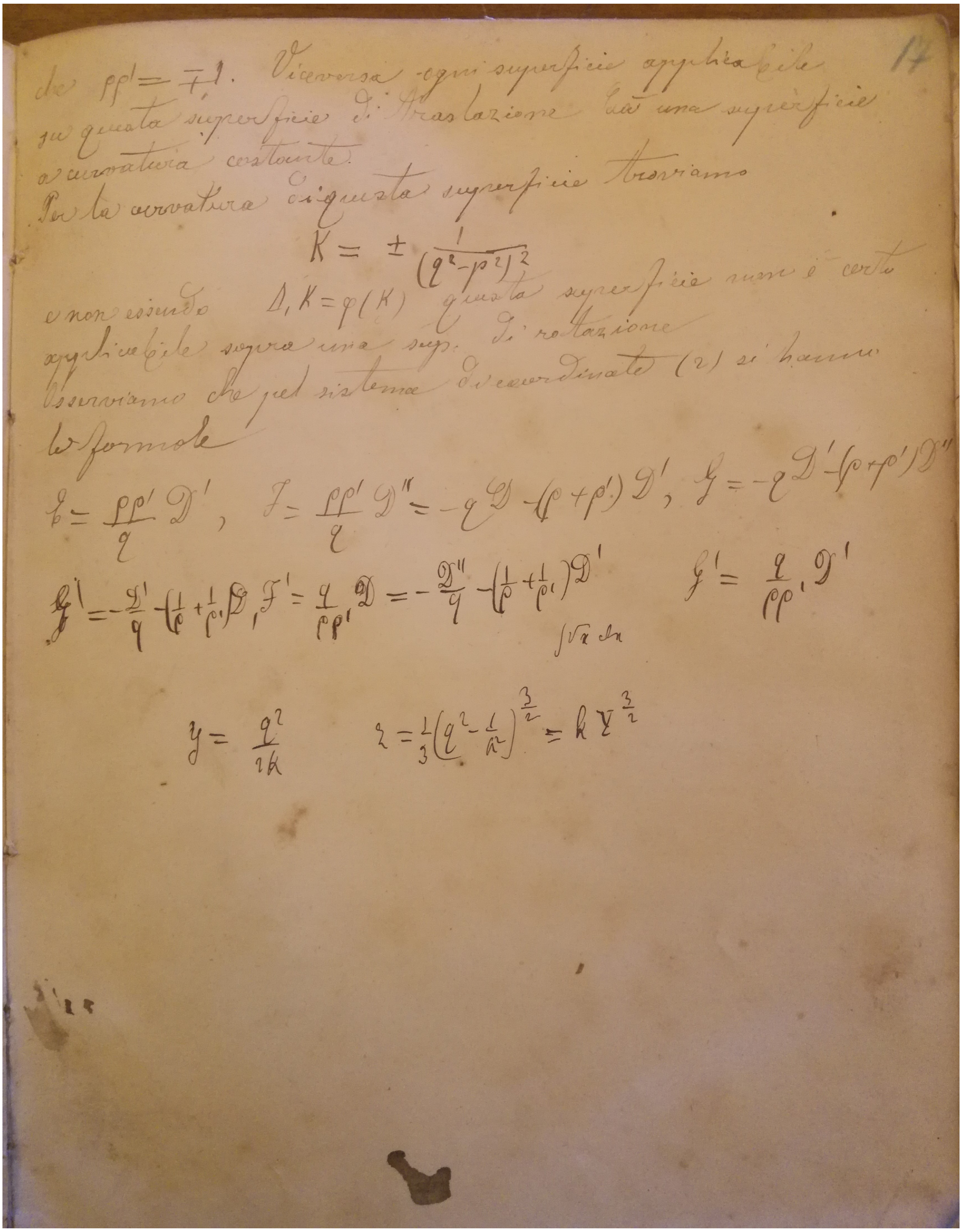
$$\Sigma d\xi^2 = dp^2 \mp 2pq\, dp\, dq + q^4 dq^2$$

ed è quindi applicabile sulla superficie di traslazione

$\xi' = \int \sqrt{1-k^2p^2}\, dp$, $\eta' = \frac{kp^2}{2} \mp \frac{q^2}{2k}$, $\zeta' = \int \sqrt{q^2 - \frac{1}{k^2}} \cdot q\, dq$

le cui curve generatrici sono nel piano $\xi'\eta'$ una cicloide di raggio $= \frac{1}{4k}$ ~~col piano~~ η tangente all'asse ξ nel vertice e nel piano $\eta'\zeta'$ una parabola semicubica avente per asse l'asse η e rivolta dalla parte positiva o negativa secondo

Figure 5.2: Picture taken from Bianchi's notebook "Fascicolo 6, 1886-87-88", Fondo Luigi Bianchi, Archivio della Scuola Normale Superiore di Pisa. Here and in the following picture, we can read Bianchi's note on (Weingarten 1887a).



17

che $\rho\rho' = \mp 1$. Viceversa ogni superficie applicabile
su questa superficie di traslazione è una superficie
a curvatura costante.
Per la curvatura di questa superficie troviamo

$$K = \pm \frac{1}{(q^2-p^2)^2}$$

e non essendo $\Delta_1 K = \varphi(K)$ questa superficie non è certo
applicabile sopra una sup. di rotazione
Osserviamo che pel sistema di coordinate (2) si hanno
le formole

$$\mathcal{E} = \frac{\rho\rho'}{q}\mathcal{D}', \quad \mathcal{F} = \frac{\rho\rho'}{q}\mathcal{D}'' = -q\mathcal{D} - (\rho+\rho')\mathcal{D}', \quad \mathcal{G} = -q\mathcal{D}' - (\rho+\rho')\mathcal{D}''$$

$$\mathcal{E}' = -\frac{\mathcal{D}'}{q} - \left(\frac{1}{\rho}+\frac{1}{\rho'}\right)\mathcal{D}, \quad \mathcal{F}' = \frac{q}{\rho\rho'}\mathcal{D} = -\frac{\mathcal{D}''}{q} - \left(\frac{1}{\rho}+\frac{1}{\rho'}\right)\mathcal{D}' \qquad \mathcal{G}' = \frac{q}{\rho\rho'}\mathcal{D}'$$

$\int \sqrt{x}\, dx$

$$y = \frac{q^2}{2k} \qquad z = \frac{1}{3}\left(q^2 - \frac{1}{k^2}\right)^{\frac{3}{2}} = k Y^{\frac{3}{2}}$$

Figure 5.3: Picture taken from Bianchi's notebook "Fascicolo 6, 1886-87-88", Fondo Luigi Bianchi, Archivio della Scuola Normale Superiore di Pisa.

We cannot exclude the possibility that Bianchi shared this observation—that is perfectly in line with Weingarten's later research for extending the method to other applicable surface classes, as we shall see in the next Section—with his colleague and friend, Weingarten.

5.4 A reconstruction of Weingarten's research on applicability in the period 1888-1893

(Weingarten 1884) and (Weingarten 1887a) were Weingarten's first and unconscious steps towards a new method for applicability. While (Weingarten 1884) led him to emphasise the necessity of a different approach to the second problem of applicability, (Weingarten 1887a) indicated to him a possible way for finding new complete classes of applicable surfaces at least in specific cases, which was alternative to the theory of $W-$ surfaces. In the following years, Weingarten mainly developed this latter aspect.

Between the end of the 1880s and the early 1890s, Weingarten was unable to continuously and profitably work because of, at least in part, the personal vicissitudes highlighted in Section 4.2 and only three publications dealt with the problem of applicability—(Weingarten 1891a) and (Weingarten 1891b), which are actually parts of letters he sent to Darboux and not proper memoirs, and (Weingarten 1893). However, it is possible to gain a more detailed picture of Weingarten's research activity by looking at his correspondence with Bianchi. This shows that Weingarten considered the subject of applicability, and in particular the determination of new complete classes of applicable surfaces, more or less intensively, throughout the entire period 1888-1893. The letters also show that the announcement of a prize—probably the Bordin Prize of 1892—as well as the exchange of letters with Darboux certainly played an important role as stimulus for new research that led him to elaborate a new method for applicability.

5.4.1 22th February 1888

Weingarten himself clearly identified the origins of his new method in a letter dated 14th May 1890.[40] While updating Bianchi on his recent and satisfactory research, Weingarten explained how he had managed to obtain the complete class of surfaces with line element (5.14):[41]

> The method that leads to this is fundamentally different from the former one [probably a reference to the theory of $W-$surfaces]. However, it is exactly the

le cui curve generatrici sono nel piano $\xi'\eta'$ una cicloide di raggio $\frac{1}{4k}$ tangente all'asse ξ nel vertice e nel piano η' una parabola semicubica avente per asse l'asse η' e rivolta dalla parte positiva o negativa secondo che $\rho\rho' = \mp 1$. Viceversa, ogni superficie applicabile su questa superficie di traslazione ha una superficie a curvatura costante" (Bianchi 1884-1927, Fascicolo 6, 16–17)

[40] (Bianchi 1959, pp. 234–240)

[41] For the sake of clarity, we note that Weingarten's deduction of the complete class of surfaces having (5.14) as line element in this letter predates the publication of the same result in (Darboux 1894, pp. 232–233).

same as the one used for the isothermal mapping of the lines of curvature.[42]

Surprisingly, there are no publications on isothermal mapping of lines of curvature. Weingarten probably referred to an investigation on the isothermal spherical mapping of curvature lines that he said he intended to publish on several occasions from 1888 onwards. According to a letter dated 22th February 1888,[43] he had begun to work on it following the reading of some research, later published in (Bianchi 1887b), that Bianchi had shared with him, in which he identified a new class of surfaces with isothermal lines of curvature:

> Your discovery of surfaces with isothermal lines of curvature prompted me to consider the analogous question of surfaces with isothermal spherical representations of the lines of curvature (Representation spherique by Darboux), for which the elements had been in my mind for years and which, as Darboux has shown, offers interesting details.[44]

In particular, Weingarten found a complete class of surfaces whose lines of curvature have an isothermal system as image via the Gauss map. This class of surfaces $S : (u, v) \mapsto \boldsymbol{x}(u, v) = (x(u, v), y(u, v), z(u, v))$, whose normal unit vector field is $\boldsymbol{X} = (X, Y, Z)$, is described through the following second-order differential equation

$$\rho_1 + \rho_2 = \frac{x^2 + y^2 + z^2}{xX + yY + zZ}, \tag{5.22}$$

where ρ_1 and ρ_2 denote the principal radii of curvature.[45] Weingarten transformed (5.22) into[46]

$$\Delta_2 u + 2u = \frac{u^2 + \Delta_1(u)}{u}, \tag{5.23}$$

by using the following remarkable system of equations[47]

$$\begin{gathered} x = uX + \Delta(X, u) \qquad y = uY + \Delta(Y, u) \qquad z = uZ + \Delta(Z, u) \\ \rho_1 + \rho_2 = \Delta_2(u) + 2u, \end{gathered} \tag{5.24}$$

where $u = xX + yY + zZ$. Finally, Weingarten solved (5.23) by requiring the spherical representation of S be isothermal and by computing the differential parameters with respect to it.

[42] "Die Methode, die dazu führt ist von der ehemaligen grundverschieden. Sie hängt aber mit derjenigen, die für die Flächen von isothermer Abbildung der Krümmungslinien verfolgte, genau zusammen" (Bianchi 1959, p. 236)

[43] (Bianchi 1959, pp. 199–202)

[44] "Ihre Auffindung von Flächengattungen mit isothermen Krümmungslinien, veranlasste mich die analoge Frage nach Flächen mit isothermen sphärischen Abbildungen der Krümmungslinien (Representation spherique von Darboux) aufzunehmen, für die mir die Elemente seit Jahren vorschwebten, und die wie Darboux gezeigt bat, interessante Einzelnheiten darbietet." (Bianchi 1959, p. 199)

[45] Weingarten denoted the principal radii of curvature with ρ and ρ'.

[46] By multiplying by x, y, z the first three in (5.24), respectively, and summing, one has $x^2 + y^2 + z^2 = u^2 + \Delta_1(u)$, from which (5.23) immediately follows by considering the fourth in (5.24).

[47] With respect to Weingarten's original notation, we change P in u. Weingarten had deduced (5.24) in (Weingarten 1884, p. 42). On that occasion, they enabled him to prove that a surface is determined up to a rigid motion by the assignment of its spherical representation and $u = xX + yY + zZ$.

At the very end of the letter in question, Weingarten briefly mentioned the problem of applicability without providing many details. He noted that the family of surfaces for which $\rho_1 + \rho_2 = 2u$ allowed him to obtain, through an unspecified method, the class of surfaces applicable to the evolute of the catenoid, which he had already achieved by using the theory of $W-$surfaces in the case $\rho_1 + \rho_2 = 0$:

> The surfaces $\rho + \rho' = 2P$ as well as similar ones also lead to developable (among each other) surface classes, just like $\rho + \rho' = 0$, e.g., the family $\rho + \rho' = 2P$ leads to surfaces of revolution with line element $dp^2 + pdq^2$, i.e., to the evolutes of a catenoid, to which the surfaces of the centres of curvature of the surfaces $\rho + \rho' = 0$ also lead.[48]

Unfortunately, Weingarten's project to publish his results in Crelle's Journal did not go as planned. First, he had to wait for Bianchi's publication of (Bianchi 1887b), which was read to the Accademia dei Lincei only in March 1888. Consequently, he temporarily set it aside and began a new work on hydrodynamics, which was not published until two years later in 1890. The writing of this work also encountered difficulties due to his family situation, but finally he was motivated to publish (Weingarten 1890) when he discovered that Klein was working on some aspects related to his investigations and worried that his research would lose part of its value.[49]

5.4.2 19th January and 14th August 1889

The following year, on 19th January 1889,[50] Weingarten communicated further considerations on the search for new families of applicable surfaces. He noted that the method he had used in (Weingarten 1887a) could also be extended to the case in which the sum of the principal radii of curvature is any function f of the Gaussian parameter u. Unfortunately, his method required knowledge of the integrals of $\rho_1 + \rho_2 = f(u)$:

> But only a part of these surfaces [i.e. surfaces with $\rho_1 + \rho_2 = f(u)$] is known, that is, those for which
>
> $$\rho + \rho' = KP$$
>
> when $K = n^2 + n + 2$ and n is an integer, (I write this by heart!),[51] or those which are determined by Euler's equation
>
> $$\frac{\partial^2 z}{\partial x \partial y} = \frac{m(m+1)z}{(x-y)^2}.$$

[48]"Die Flächen $\rho + \rho' = 2P$ ebenso wie ahnliche führen auch auf abwickelbare (unter einander) Flächenclassen, gerade wie $\rho + \rho' = 0$, z. B. führt die Familie $\rho + \rho' = 2P$ auf Rotationsflähen vom Linienelement $dp^2 + pdq^2$, dh. auf die Kettenlinienevoluten, auf die auch die Krummungsmittelpunktsflächen der Flächen $\rho + \rho' = 0$ führen." (Bianchi 1959, p. 202)

[49]See (Bianchi 1959, p. 239). It is not clear what Klein's and Weingarten's research had in common. Weingarten only wrote that Schwarz told him that Klein had found some integrals of (4.1) by using surface theory.

[50](Bianchi 1959, pp. 216–218)

[51]According to (Baroni 1890, p. 373), surfaces for which $\rho + \rho' = KP$ are known when $K = -n^2 - n + 2$, where n is an integer.

> [...] The Appell type $\rho + \rho' = 2P$ belongs to these.[52] If one wants to use this latter type for the surfaces that can be developed on one another, one obtains a class, but a well-known one, namely that of the surfaces that can be developed on the surface of rotation of the *evolute of a catenoid*, which I have already indicated a long time ago. The other surfaces are too complicated for me, not even having the Liouville element. I wanted the result to be as elegant as possible.[53]

During the summer holiday in the same year, on 14th August 1889,[54] Weingarten updated Bianchi about the status of his research with a brief communication. Besides connecting his research on isothermal spherical images of the lines of curvature with his current studying on hydrodynamics, he realised that the family of surfaces that was described by the relation (5.22) originated a *new* class of surfaces by using the same method he used in (Weingarten 1887a):

> According to the method with which I found the surfaces of the line element
>
> $$ds^2 = (\alpha^{\frac{2}{3}} + \beta^{\frac{2}{3}})(d\alpha^2 + d\beta^2),$$
>
> the surfaces 1) [he referred to (5.22)] lead to a new class of surfaces that can be developed on each other, an individual of which, however, can be be developed on a surface of revolution, as you will easily see.[55]

[52] Appell investigated surfaces for which $\rho + \rho' = 2P$, according to Weingarten's notation, in (Appell 1888). These studies were later extended to surfaces for which the distance of a fixed point from the tangent plane is proportional to the sum of the radii of curvature ($\rho + \rho' = 2P$) in (Goursat 1888) and (Baroni 1890).

[53] "Aber von diesen Flächen kennt man nur einen Theil derjenigen für welche

$$\rho + \rho' = KP$$

wenn nämlich $K = n^2 + n + 2$ und n eine ganze Zahl ist,(Dies schreibe ich aus dem Gedächtniss!), oder diejenigen welche durch die Eulersche Gleichung

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{m(m+1)z}{(x-y)^2}$$

bestimmbar sind. [...] Zu ihnen gehört die Appell'sche Gattung $\rho + \rho' = 2P$. Wenn man diese letztere Gattung für die auf einander abwickelbaren Flächen verwerthen will, so erhält man zwar eine Classe, aber eine bekannte, nämlich diejenige der auf die Rotationsfläche der *Kettenlinienevolute* abwickelbaren, welche ich schon vor langer Zeit angegeben habe. Die übrigen Flächen werden mir zu complicirt, haben auch nicht das Liouville'sche Element. Es lag mir an möglichst elegantem Resultat." (Bianchi 1959, p. 217)

[54] (Bianchi 1959, pp. 233–234)

[55] "Die Flächen 1) führen nach der Methode, mit der ich die Flächen vom Linienelemente

$$ds^2 = (\alpha^{\frac{2}{3}} + \beta^{\frac{2}{3}})(d\alpha^2 + d\beta^2)$$

gefunden habe, auf eine *neue* Classe von auf einander abwickelbaren Flächen, von denen aber ein Individuum auf eine *Rotationsfläche* abwickelbar ist, wie Sie leicht sehen werden".(Bianchi 1959, p. 234)

5.4.3 14th May 1890

These hints finally found a more elaborate exposition in the already-mentioned letter of 14th May 1890.[56] Here, the actual implementation of a generalisation of the method he employed in (Weingarten 1887a) was explicitly shown in some detail.

Weingarten assumed that the parametric equations $\boldsymbol{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ of a surface S were known, and assumed that the parameters were[57]

$$u = xX + yY + zZ \qquad v = \frac{x^2 + y^2 + z^2}{2},$$

By means of an auxiliary function φ, Weingarten wrote the following differentials

$$\begin{cases} d\xi = xd\frac{\partial \varphi}{\partial v} + Xd\frac{\partial \varphi}{\partial u} = (x\frac{\partial^2 \varphi}{\partial v \partial u} + X\frac{\partial^2 \varphi}{\partial u^2})du + (x\frac{\partial^2 \varphi}{\partial v^2} + X\frac{\partial^2 \varphi}{\partial u \partial v})dv \\ d\eta = yd\frac{\partial \varphi}{\partial v} + Yd\frac{\partial \varphi}{\partial u} = (y\frac{\partial^2 \varphi}{\partial v \partial u} + Y\frac{\partial^2 \varphi}{\partial u^2})du + (y\frac{\partial^2 \varphi}{\partial v^2} + Y\frac{\partial^2 \varphi}{\partial 1 \partial 2})dv \\ d\zeta = zd\frac{\partial \varphi}{\partial v} + Zd\frac{\partial \varphi}{\partial u} = (z\frac{\partial^2 \varphi}{\partial v \partial u} + Z\frac{\partial^2 \varphi}{\partial u^2})du + (z\frac{\partial^2 \varphi}{\partial v^2} + Z\frac{\partial^2 \varphi}{\partial u \partial v})dv, \end{cases} \tag{5.25}$$

which define the finite equations of *another* surface Σ when their exactness is required. Weingarten thus obtained the following equation as their integrability condition[58]

$$\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial u \partial v}(\rho_1 + \rho_2) + \frac{\partial^2 \varphi}{\partial v^2}\rho_1 \rho_2 = 0, \tag{W_0}$$

where ρ_1, ρ_2 are the principal radii of curvature of S. If one knows all the integrals of the equation (W_0) that are obtained by assigning a particular (but not unique) value to the function φ, i.e., if one knows all the surfaces S whose radii of curvature satisfy (W_0) for a certain value of φ, then Weingarten proved that one can obtain all the surfaces Σ that have

$$ds^2 = 2v\left(d\frac{\partial \varphi}{\partial v}\right)^2 + 2u\left(d\frac{\partial \varphi}{\partial v}\right)\left(d\frac{\partial \varphi}{\partial u}\right) + \left(d\frac{\partial \varphi}{\partial u}\right)^2, \tag{5.26}$$

as line element by integrating the equations in (5.25).

Weingarten did not comment on the equation (W_0) nor on the generality of this method. Rather, the important aspect of his new research seems once again to be the explicit determination of complete classes of applicability. Indeed, in the remaining part of the letter, he focused on specific applications. He considered the case $\varphi = vP(u) + R(u)$, where P and R are generic functions of u only, and he was able to obtain previous results as sub-cases by substituting particular values of P and R. In particular, by choosing

[56] See Section 5.4.1.

[57] With respect to Weingarten notation, we changed P, Q in u, v.

[58] The integrability conditions are

$$\frac{\partial(x\frac{\partial^2 \varphi}{\partial v \partial u} + X\frac{\partial^2 \varphi}{\partial^2 u})}{\partial v} = \frac{\partial(x\frac{\partial^2 \varphi}{\partial^2 v} + X\frac{\partial^2 \varphi}{\partial u \partial v})}{\partial u},$$

and similarly for y, z. By calculating the derivatives and substituting Weingarten's equations (5.18), one obtains (W_0).

$\varphi = uv$, the method coincides with that was exposed in (Weingarten 1887a) and in the case $\varphi = v + \frac{u^2}{2}$ the method reduces to that which led him to the class (5.22).

Finally, Weingarten commented:

> The other families are not so interesting, since the Liouville form does not occur. There is also only a very limited class of families. It is strange, however, that the difficult problem of applicability can be completely solved in some cases, albeit by complicated quadratures.[59]

5.4.4 March-April 1891

The following year, between the end of March and the beginning of April, Weingarten conducted a remarkable correspondence with Darboux, part of which was published (to Weingarten's delight) on the *Comptes rendus* of the Paris Academy in (Weingarten 1891a) and in (Weingarten 1891b). This exchange appears decisive for the orientation of Weingarten's subsequent research. In the meanwhile, he also wrote two letters to Bianchi, where he updated him on the communications he had had with their French colleague.[60]

As is clear from the first lines in (Weingarten 1891a), Weingarten wrote to Darboux after reading the first fascicle of the third volume of Darboux's *Leçons*:[61]

> The study of your Lessons on the theory of surfaces leads me to present to you a remark I made, some years ago, concerning surfaces, the square of whose line element has the form
> $$ds^2 = du^2 + (u + \alpha v)dv^2$$
> which are contained in equations (27), page 234 of Part III of the Leçons.[62]

Weingarten referred to the search for ruled surfaces that can be applied upon surfaces of revolution, which Darboux discussed in (Darboux 1894, pp. 230–235). Here, Darboux

[59]"Die anderen Familien sind nicht so interessant, da die Liouville'sche Form sind nicht wieder auftritt. Man kommt auch nur auf eine sehr beschrankte Classe von Familien. Es ist aber doch merkwurdig, dass sich das so schwierige Problem der Abwickelung in einigen Fallen vollstandig erledigen lasst, wenngleich durch complicate Quadraturen." (Bianchi 1959, p. 238)

[60](Bianchi 1959, pp. 240–246)

[61](Darboux 1894) was initially published in three steps: *première fascicule* (1890), *deuxième fascicule* (1891), and *troisième fascicule* (1895). In (Bianchi 1959, p. 245), Weingarten referred to the second fascicle (*"Die Bemerkung von Darboux über das Rotationsparaboloid finden Sie im 2ten Fasciculum der 3ten partie seiner léçons sur la théorie générale des surfaces, von dem ich glaubte, dass Sie ihn schon in Händen hatten. In diesem Theil bereitet er auch schon auf Ihre schonen Untersuchungen vor, die sich auf die orthogonalflächen mit einer Schaar von constanter Krümmung beziehen."*), but, according to the content of footnote 67 and by considering the date of publication of the fascicle, he probably meant the first fascicle.

[62]"L'étude de vos Leçons sur la théorie des surfaces me conduit à vous présenter une remarque que j'ai faite, il y a quelques années, concernant les surfaces, dont le carré de l'élément linéaire possède la forme

$$ds^2 = du^2 + (u + \alpha v)dv^2$$

contenue dans les équations (27), page 234 de la III Partie des Leçons." (Weingarten 1891a, p. 607). See also the footnote 63.

identified four classes of surfaces

$$
\begin{aligned}
ds^2 &= du^2 + udv^2 \\
ds^2 &= du^2 + (u^2 + a)dv^2 \\
ds^2 &= du^2 + [(u + av)^2 + b^2]dv^2 \\
ds^2 &= du^2 + (u + av)dv^2 \qquad (5.27)
\end{aligned}
$$

where a is a constant, that correspond to a tractroid, to a catenoid, to an ellipsoid or hyperboloid of revolution, and to a paraboloid of revolution, respectively. This result so impressed Weingarten that he wrote to Bianchi:

> I would like to tell you how much I was struck and delighted by Darboux's determination of the surfaces that can be developed on the paraboloid of revolution. His way of looking at things is and remains ingenious. I am getting more and more curious about his continuations, which have been lacking for far too long.[63]

In addition to presenting the deduction of the equation (W_0) and the method that derived from it, as he had done in the letter to Bianchi on 14th May 1890,[64] in (Weingarten 1891a) Weingarten also exposed to Darboux its application to the case $\varphi = uv - u^2\frac{\beta}{2} - \frac{p^3}{3}$, i.e., in the case in which Σ has principal radii of curvature such that $\rho + \rho' = 2u + \gamma$, with $\gamma = const.$ If $\gamma = 0$, then the class of surfaces correspond to the well-known class of the evolute of a catenoid; in all the other cases, γ can be assumed to be 1 and Weingarten found precisely the paraboloids found by Darboux $ds^2 = du^2 + (u + av)dv^2$, (5.27). Moreover, he noted:

> As a consequence, the family of surfaces applicable upon the surface of revolution
>
> $$X = \alpha t \cos\frac{r}{\alpha} \qquad Y = \alpha t \sin\frac{r}{\alpha} \qquad Z = \int \sqrt{t^2 - \alpha^2 - 1}dt$$
>
> can be determined by quadratures. By a theorem which I gave thirty years ago in Crelle's Journal, it is easy to see that one will also determine by quadratures the surfaces verifying the equation
>
> $$2(\rho' - \rho) = S(2\rho' + 2\rho),$$
>
> where $S(G)$ denotes the quantity $\frac{e^G - e^{-G}}{2}$.[65]

[63]"Zunachst mochte ich Ihnen aussprechen wie sehr mich Darboux's Bestimmung der Flächen die auf dem Rotationsparaboloid abwickelbar sind frappirt und gefreut hat. Seine Art die Dinge zu betrachten ist und bleibt genial. Ich werde immer neugieriger auf seine Fortsetzungen, die mir viel zu lange ausbleiben." (Bianchi 1959, p. 241)

[64]See Section 5.4.3.

[65]"En consequence, la famille de surfaces applicable sur la surface de révolution

$$X = \alpha t \cos\frac{r}{\alpha} \qquad Y = \alpha t \sin\frac{r}{\alpha} \qquad Z = \int \sqrt{t^2 - \alpha^2 - 1}dt$$

Hence, Weingarten found the complete class of applicable surfaces with (5.14) as line element. As is evident from the following passage from a letter to Bianchi in which Weingarten himself reported of what happened after his first message to Darboux, Weingarten was unaware that the same result was previously obtained by Darboux in (Darboux 1894, pp. 332–333):

> Darboux informed the Academy of the remark that had been made on the basis of the formulas in page 234, Part III of the Leçons, and wrote to me that he attached more importance to the general theorem than to the specific application; he also drew my attention to the second fascicule, which I now know. In a reply, I informed him that the application he had made had become trivial due to his developments,[66] which I did not yet know, but I gave him the application to surfaces with the Liouville element, which I have also described to you. Now he wrote back to me today that he had also submitted this to the Academy, because Goursat had given him a note on the theorem in question (for the Academy), and he believes that my earlier application must now also be made public.[67]

While Goursat[68] had connected the integration of $\rho_1 + \rho_2 = 2\alpha u$, which corresponds to $\varphi = uv - \alpha\frac{p^3}{3}$, to one of his previous results, the further application Weingarten referred to concerns the case $\varphi = P' + p^2$, where P is a function of the variable u only, i.e., $\rho + \rho' + P'' = 0$: while searching for those admitting a line element of the Liouville form $ds^2 = (A - B)(d\alpha^2 + d\beta^2)$, Weingarten found by quadratures a *new* class of applicable surfaces whose line element is

$$ds^2 = (\alpha - \beta)\left[\frac{d\alpha^2}{\alpha^2}(\alpha - 2) - \frac{d\beta^2}{\beta^2}(\beta - 2)\right],$$

which reduces to surfaces of the class he found in 1887 (with (5.15) as line element) in a sub-case.

---

est déterminable par quadratures. Par un théorème que j'ai donné, il y a trente ans, dans le Journal de Crelle, il est aisé de voir que l'on déterminera aussi par des quadratures les surfaces vérifiant l'équation

$$2(\rho' - \rho) = S(2\rho' + 2\rho),$$

where $S(G)$ désignant la quantité $\frac{e^G - e^{-G}}{2}$." (Weingarten 1891a, p. 609)

[66] He probably referred to (Darboux 1894, pp. 332–333).

[67] "Darboux theilte die Bemerkung, die auf Grund der Formeln Seite 234, III partie der leçons gemacht waren, der Academie mit, und schrieb mir, dass er mehr Werth auf das *allgemeine* Theorem lege, wie auf die *specielle* Anwendung; machte mich auch auf das 2te fascicule, das ich nun schon kannte, aufmerksam. In meiner Antwort theilte ich ihm mit, dass die gemachte Anwendung durch seine Entwickelungen, die ich noch nicht kannte, trivial geworden sei, gab ihm aber dafür die Anwendung auf Flächen mit dem Liouville'schen Element, die ich Ihnen auch geschrieben habe. Nun schrieb er mir heute zurück, dass er auch dies der Academie vorgelegt habe, weil Goursat ihm schon on eine Note über das betreffende Theorem vergelegt habe (für die Academie), und er glaube dass nun auch meine frühere Anwendung publicirt werden müsse." (Bianchi 1959, p. 245)

[68] (Goursat 1891)

However, the most noteworthy aspect of the above-quoted passage in the letter is Darboux's interest in his colleague's research. Indeed, he encouraged Weingarten to work more on the development of a general theory than delving into particular examples. As far as can be seen from the letters, Weingarten followed this valuable advice, since he no longer mentioned applications of the equation (W_0) to particular families of surfaces. Furthermore, in two letters dated 6th and 12th November 1891, Weingarten expounded some thoughts that developed more theoretical aspects of his research on the equation W_0 concerning the generation of surfaces with constant positive curvature from an assigned one.[69]

Before Darboux's advice, Weingarten's research was stimulated by a prize awarded by the Paris Academy, as a few lines in the correspondence with Bianchi clearly reveal. He wrote:

> You will notice, my dear friend, that the prize question of the Paris Academy has also had some effect on me, for these results are closely connected with it. I will not apply, because an old man will probably no longer win over the young forces. Thirty-three years ago, such a prize question (Deformation des surfaces)[70] came a little too early for me. Today it is too late. But the remark of the result gave me pleasure, that one can find all the surfaces of the above line element by quadratures alone.[71]

The commentators of (Bianchi 1959) identify the mentioned prize with the Grand Prix of the Paris Academy awarded in 1894, in which Weingarten actually participated. However, this does not seem to be plausible. The 1894 Grand Prix was not announced until 1892, i.e., more than a year after the letter to Bianchi. It is more likely that Weingarten was referring to another Paris Academy prize, the Bordin Prize, which was first announced in 1888 for 1890 and then re-announced in 1890 for 1892: it required the study of surfaces with a line element $ds^2 = (f(u) - \varphi(v))(du^2 + dv^2)$.[72] In the light of this hypothesis, one could also understand the content of a passage in the letter dated 19th January 1888 already mentioned in Section 5.4.2. He wrote:

> Knoblauch told me about the new prize competition of the Academy in Paris. I do not know the right approach for a more general solution, but I will certainly think about it a lot."[73]

[69](Bianchi 1959, pp. 246–250)

[70]Weingarten referred to the Grand Prix won by Bour.

[71]Sie bemerken verehrter Freund, dass die Preisfrage der Pariser Academie auch auf mich einige Bewegung ausgeübt hat, denn diese Resultate hängen mit ihr nahe zusammen. Bewerben werde ich mich nicht, denn ein alter Mann wird über die jungen Kräfte wohl nicht mehr siegen. Vor 33 Jahren kam mir eine solche Preisfrage (Deformation des surfaces) etwas zu *früh*. Heute zu *spät*. Aber die Bemerkung des Resultats hat mir Vergnügen gemacht, dass man nämlich durch *Quadraturen* allein *alle* Flächen vom obigen Linienelement finden kann." (Bianchi 1959, p. 244)

[72]See (Comptes rendus 1888) and (Comptes rendus 1890, p. 1025).

[73]"Von der neuen Preisaufgabe der Pariser Academie hat mir Knoblauch erzählt. Ich weiss keinen rechten Zugang zu einer allgemeineren Beantwortung, werde aber wohl vielfach darüber nachdenken." (Bianchi 1959, p. 216)

Immediately after this essential statement, Weingarten expounded his research on classes of applicable surfaces, admitting a Liouville form for their line element that we mentioned in Section 5.4.2. Moreover, as we have seen, Weingarten made several references to Liouville's surfaces also in the subsequent letters.

5.4.5 12th February and 29th December 1893

The most remarkable result in the direction of a theoretical characterisation of the equation (W_0) was achieved in 1893. On 12th February 1893, Weingarten wrote:[74]

> In order that our correspondence, which is so dear to me, does not fall asleep, I would like to tell you briefly about a work that I have completed in the last few days and which has cost me a lot of effort and patience over the last six months, even though its result can almost be considered negative.[75]

Later published in (Weingarten 1893), Weingarten first showed that (W_0) was a Monge-Ampère equation for u by substituting $\rho_1 + \rho_2$, $\rho_1\rho_2$ and v with their values with respect to u, and then he studied the cases in which the Monge-Ampère method can be applied.[76] Finally, Weingarten proved that the equation (W_0) is integrable by general and known methods only when the surface S has a line element of the form $(\alpha + \beta u^2)du^2 + u^2dv^2$, where α and β, $\beta \neq 0$, are arbitrary constants. This line element essentially corresponds to four cases

$$
\begin{aligned}
ds^2 &= u^2du^2 + u^2dv^2 \\
ds^2 &= (1 + u^2)du^2 + u^2dv^2 \\
ds^2 &= (1 - u^2)du^2 + u^2dv^2 \\
ds^2 &= (u^2 - 1)du^2 + u^2dv^2
\end{aligned}
$$

for which the complete classes of applicable surfaces were already found, as Darboux proved in (Darboux 1894, Chap. IX).[77]

Commenting on this, Weingarten clarified:

> Therefore, unless new methods of integrating the second-order differential equations are found, a reduction of the previously discovered number of classes, which include all surfaces of a given line element, seems to be hopeless. In this

[74](Bianchi 1959, pp. 253–255)

[75]"Damit unsere mir so liebe Correspondenz nicht einschläft, möchte ich Ihnen eine kurze Mittheilung über eine Arbeit machen, die ich in den letzten Tagen vollendet habe, und die mir wohl ein halbes Jahr viele Mühe und Geduld gekostet hat, wenngleich ihr Resultat beinahe ein negatives genannt werden kann." (Bianchi 1959, p. 253)

[76]For details, see (Weingarten 1893, p. 495). Weingarten omitted the demonstration of the integrability because of its extension.

[77]Here, Darboux obtained the finite equations (Darboux 1894, 370, eq. (45)) of all the surfaces that are applicable to the paraboloid of revolution $x^2 + y^2 = 8hiz$, where h is an arbitrary constant. The evolute of the catenoid ($ds^2 = u^2du^2 + u^2dv^2$) corresponds to the case $h = 0$. See (Darboux 1894, §770).

> respect, the result is negative. [...] I would have liked to find new results, but truths do not depend on our wishes.[78]

As Weingarten confided to Bianchi some time later, on 20th Febrary 1896, this result played an important clarifying role in his mind:

> The fact that there are many functions $\varphi(u, v)$ that derive from surfaces with the same line element, even if transformed, was at first a great trouble to me, because I always thought I had found a new class. Only an investigation of the integrability through characteristics gave me clarification. In the end, all special cases belonged to the same genus.[79]

Moreover, the fact that the same line element can be obtained by means of many functions $\varphi(u, v)$ was appreciated by both Bianchi and Darboux in light of the problem of integration. The fact that the same complete class of applicable surfaces correspond to many functions $\varphi(u, v)$ means that Weingarten's method allows the integration of as many distinct equations as the number of functions $\varphi(u, v)$ that can be matched to a given class.

Finally, on 29th December 1893, Weingarten told Bianchi he was studying the applicability of *real* surfaces: the method, he said, was very different from the one presented to the Crelle's Journal, but he had no time to explain it to him. He returned to the subject a few months later when, on 2nd February and 5th and 17th June 1894, he gave precise indications of his studies.

5.5 1894 Grand Prix

In 1892, the Paris Academy announced a second prize on the theme of applicability.[80] The question was rather vague and read as follows: *"Perfectionner en un point important la théorie de la déformation des surfaces"*.[81] Contributions in anonymous form had to be received by 1st October 1894.

On the basis of the dossier related to the Prix that is stored at the Archives de l'Académie des sciences in Paris, the participants were Julius Weingarten, Paul Adam,

[78]"Wenn daher nicht neue Methoden der Integration der Differentialgleichungen 2ter Ordnung aufgefunden werden erscheint eine *Vergrösserung* der bisher entdeckten Angabe von Classen, welche *alle Flächen* von einem gegebenen Linienelement umfassen, aussichtslos. In dieser Beziehung ist das Resultat negativ. Ich hätte gern neue Resultate herausgefunden, aber die Wahrheiten richten sich nicht nach unseren Wünschen." (Bianchi 1959, p. 255)

[79]"Der Umstand, dass es sehr viele Functionen $\varphi(u, v)$ giebt, die aus Flächen von demselben wenn auch *transformirten* Linienelement herrühren, hat mich zuerst sehr viel geplagt, weil ich immer glaubte eine *neue* Classe gefunden zu haben. Erst eine Untersuchung der Integrirbarkeit durch Characteristiken hat mir darin Aufklärung gegeben. Alle Specialfälle gehörten schliesslich derselben Gattung an." (Bianchi 1959, p. 273)

[80]It cannot be excluded that this topic was proposed by Darboux, who was aware of Weingarten's research.

[81](Comptes rendus 1892)

Claude Guichard, Xavier Stouff and Emile Borel.[82] The committee that judged their contributions was composed of Picard, Poincaré, Jordan, Hermite, and Darboux as rapporteur. They particularly appreciated the memoirs by Weingarten and Guichard.

While Guichard received a *mention honorable* for his investigations using four-dimensional geometry on the properties of conjugate lines that are preserved during deformation, Weingarten was unanimously designated the winner for the following motivation:

> The author of Memoir N. 1 recalls that the determination of surfaces admitting a given line element is reduced to the integration of a partial differential equation of the second order, the direct study of which has not hitherto made it possible to obtain a result of any importance in the theory of deformation. He therefore proposes to look for a new way of reducing the problem to the integration of a partial differential equation of the second order, by means of a method which seems to us to be extremely ingenious, although the principles on which it rests are not fully revealed by the exposition. After having formed this equation with partial derivatives, the author of the Memoir investigates in which cases it admits intermediate integrals of the first order. He thus finds the known case of the surfaces applicable to the paraboloid of revolution and of the evolute of minimal surfaces. But at least he achieves this by a regular method and the application of known theories relating to the integration of partial differential equations.[83]

On Mittag-Leffer's advice,[84] Weingarten decided to publish the memoir in the Acta Mathematica, as well as in the *Comptes Rendus* of the Paris Academy, as was usually the case for prize-winning memoirs. However, this caused a delay in publication, so that it did not appear until 1897, in (Weingarten 1897).

Weingarten's memoir is essentially divided into two parts: the first is devoted to the deduction of the *"radical transformation"*[85] of Bour's equation, the second to the study of its integrability.

[82]From a comparison with the manuscripts in the Archives de l'Académie des sciences in Paris, among which Weingarten's manuscript is missing, suggests that at least part of Adam's and Guichard's memoirs were published in (Adam 1895a), (Adam 1895b), (Guichard 1896).

[83]"L'auteur du Mémoire n° 1 rappelle que la détermination des surfaces admettant un élément linéaire donné se ramène à l'intégration d'une équation aux dérivées partielles du second ordre, dont l'étude directe n'a pas permis jusqu'ici d'obtenir un résultat de quelque importance dans la théorie de la déformation. Il se propose donc de chercher par une voie différente de toutes celles qui sont connues, une réduction nouvelle du problème à l'intégration d'une équation aux dérivées partielles du second ordre, et il atteint le but qu'il s'est proposé à l'aide d'une méthode qui nous a paru extrêmement ingénieuse, bien que les principes sur lesquels elle repose ne soient pas complètement mis en évidence par l'exposition. Après avoir formé cette équation aux dérivées partielles, l'auteur du Mémoire cherche dans quels cas elle admet des intégrales intermédiaires du premier ordre. Il retrouve ainsi le cas connu des surfaces applicables sur le paraboloïde de révolution et des développées des surfaces minima. Mais du moins il y parvient par une méthode régulière et l'application des théories connues relatives à l'intégration des équations aux dérivées partielles." (Comptes rendus 1894, pp. 1050–1051)

[84](Bianchi 1959, p. 269)

[85](Darboux 1896, p. 319)

5.5.1 Weingarten's new method for applicability in his letters to Bianchi

The aspects highlighted by the prize report, as well as by the introduction to (Weingarten 1897), are detailed in three letters that Weingarten sent to Bianchi on 2nd February, 5th June, and 9th September 1894.

Firstly, Weingarten clarified the limits of Bour's equation that he intended to overcome: not only the fact that Bour's equation is only integrable in the (trivial) case of developable surfaces, but also, as he emphasised in (Weingarten 1884), the fact that the problem was treated asymmetrically and that real solutions did not necessarily correspond to real surfaces:

> But this equation [Bour's equation] has several inconveniences.
>
> 1. There is no line element at all (the developable surfaces excluded) for which this equation admits first integrals, thus becoming integrable according to Monge-Ampère. It is also not possible by other known methods to extract from it a class of surfaces of a given element.
> 2. For the task of finding all real surfaces that can be developed on a given one, it does not give an immediate decision through its real integrals. One must still make a distinction among these. It only has significance for the general problem as posed by Darboux, in which all variables must be complex variables.
> 3. The treatment of the problem is asymmetrical in that it favours one coordinate, and from it finds the other two through quadratures.[86]

In order to solve these critical issues, Weingarten took a completely new route that led him to a new equation, of which (W_0) was just a special case. He followed a path prescribed by the results contained in (Weingarten 1887a), as he wrote to Bianchi in June 1894:

> Personally, I am working on my newest theory of applicability, which takes a completely different path than the previous works. The path, however, was prescribed by the results I had noticed for the surfaces of the element
>
> $$ds^2 = (\alpha^{\frac{2}{3}} + \beta^{\frac{2}{3}})(d\alpha^2 + d\beta^2)$$

[86]"Aber diese Gleichung hat mehrere Unbequemlichkeiten.

(a) Es giebt überhaupt *kein* Linienelement (die developpablen Flächen ausgeschlossen) für welche diese Gleichung erste Integrale zulässt, also nach Monge-Ampère integrabel wird. Man kann auch durch andre bekannte Methoden ihr keine Flächenclasse von gegebnem Element abgewinnen.

(b) Für die Aufgabe alle *reellen* Flächen zu finden die auf eine *gegebne* abwickelbar sind giebt sie keine unmittelbare Entscheidung durch ihre *reellen* Integrale. Man muss unter diesen noch eine Unterscheidung machen. Sie hat nur Bedeutung für das allgemeine Problem wie es Darboux stellt, in dem alle Variable complexe Grössen sein dürfen.

(c) Die Behandlung des Problems ist *unsymmetrisch* indem sie eine *Coordinate* bevorzugt, und aus ihr die beiden anderen durch Quadraturen findet." (Bianchi 1959, p. 267)

> and the extended type. The formula
>
> $$\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial u\partial v}(\rho_1 + \rho_2) + \frac{\partial^2\varphi}{\partial v^2}\rho_1\rho_2 = 0$$
>
> is a special case of my general formulae, and I believe that insight into the general theory is fostered by my work. I have not found out any specific new classes of surfaces, which contain all the surfaces with the same line element. If it is of any use to you, I will gladly share with you some of the investigations that I have completed, or rather that I now want to exchange with others.[87]

In these last lines, we also read Weingarten's satisfaction with the progress of his research. Despite his reluctance to publish,[88] Weingarten was now ready to share his personal work with others. Weingarten's words also implicitly indicate that Weingarten did not seem to be thinking about the Paris Prize, as is also attested to by a few words written in December 1893, where Weingarten spoke of a work on the deformation of surfaces to be published in Crelle's Journal:

> I wanted to write you a longer letter some time ago, but an attack of influenza prevented me from writing it, and to share with you a method of investigating the *real* deformations of a real surface, which is very different from the previous one and which I am now working out for Crelle's Journal. But I am so busy these days with non-mathematical business that I will only be able to do it in the next few years.[89]

Weingarten also emphasised the advantages of his solution:

> My new treatment is free of these objections [he referred to the numbered list mentioned above]. It leads to a second-order differential equation of Ampère form, which in certain cases admits two general first integrals. I give these cases, and there are no others, completely. They occur for Darboux's rotational

[87]"Ich selbst sitze bei der Ausarbeitung meiner neuesten Theorie der Abwickelbarkeit, die einen ganz andern Weg einschlägt, wie die bisherigen Arbeiten. Der Weg aber war vorgeschrieben durch die Resultate, die ich für die Flächen vom Element

$$ds^2 = (\alpha^{\frac{2}{3}} + \beta^{\frac{2}{3}})(d\alpha^2 + d\beta^2)$$

und der erweiterten Gattung bemerkt hatte. Die Formel ist ein besonderer Fall meiner allgemeinen Formeln, und glaube ich, dass der Einblick in die allgemeine Theorie durch meine Arbeit gefördert wird. Specielle neue Classen von Flächen die *alle* Flächen desselben Linienelements enthalten habe ich nicht heraus bekommen. Wenn es Ihnen Wergnügen machen kann, so theile ich Ihnen gern Einiges von der Untersuchung mit, die ich beendet habe, oder vielmehr die ich nun mit anderen vertauschen will." (Bianchi 1959, p. 265)

[88]See Section 4.3.1.

[89]"Ich wollte Ihnen schon seit einiger Zeit einen längeren Brief schreiben, an dessen Abfassung mich ein Anfall der Influenza verhindert hat, und Ihnen eine von der bisherigen sehr verschiedene Methode der Untersuchung der *reellen* Deformationen einer reellen Fläche mittheilen, die ich jetzt gerade für das Crelle'sche Journal ausarbeite. Aber ich bin in diesen Tagen so durch nicht mathematische Geschäfte beansprucht, dass ich erst im nächsten Jahre dazu kommen werde." (Bianchi 1959, p. 262)

surfaces

$$ds^2 = u^2 dv^2 + (\alpha u^2 + \beta) du^2$$

and for all line elements arbitrarily transformed from them, but only for these. Moreover, this equation becomes integrable for other surfaces [other than developable surfaces], and gives all known classes of surfaces which can be unfolded on each other from one and the same source.

2) *All real* integrals of this equation[90] also give all real surfaces of the given line element by simple quadratures.

3) The treatment is absolutely symmetrical in relation to the variables x, y, z. I have not succeeded in gaining new, at least essentially new, integrals from the basic equation, which contain two arbitrary functions.[91]

Weingarten also presented Bianchi with the guidelines of his new method. To overcome the third trouble—the asymmetric treatment of the variables—Weingarten's method *"also gives an Ampère's equation, but for one of the two variables p or q, by which the surface line element is expressed"*.[92] This equation is obtained through invariants theory. Until that moment, the second problem of applicability had been thought of as the investigation of conditions for a *generic* surface in order to have, up to a change of coordinates, the same line element of a *given* surface. Weingarten's approach overcame the necessity to consider other surfaces in addition to the assigned one by taking advantage of its intrinsic properties. In particular, from the finite equations of the given surface Weingarten deduced its intrinsic equations, which are (by definition) the equations of all the surfaces applicable to the given one. In order to guarantee the existence of solutions of the intrinsic equations, the validity of their integrability condition is required: this equation, which is a second-order differential equation of the Monge-Ampère type, replaces Bour's equation and, once integrated, allows the identification of *every* surface belonging to the same class as the given surface.

---

[90]The characteristics of which are also the asymptotic lines of the surfaces to be found. (This footnote is due to Weingarten, who wrote: "die Characteristiken derselben sind auch die asymptotischen Linien der gesuchten Flächen.")

[91]"Meine neue Behandlung ist von diesen Vorwürfen frei. Sie führt auf eine Differentialgleichung zweiter Ordnung von der Ampère'schen Form, welche in gewissen Fallen *zwei allgemeine* erste Integrale zulässt. Ich gebe diese Fälle, und es giebt keine anderen, vollständig an. Sie treten für die Darboux'schen Rotationsflächen

$$ds^2 = u^2 dv^2 + (\alpha u^2 + \beta) du^2$$

und für alle aus ihnen beliebig transformirten Linienelemente, ein, aber *nur* für diese. Ausserdem wird diese Gleichung in anderen Fällen integrabel, und giebt alle bisher bekannten Classen von auf einander abwickelbaren Flächen aus einer und derselben Quelle. 2) *Alle reellen* Integrale dieser Gleichung geben auch alle reellen Flächen von dem gegebenen Linienelement durch einfache Quadraturen. 3) Die Behandlung ist in Beziehung auf die Variablen x, y, z absolut symmetrisch. Der Grundgleichung neue, wenigstens wesentlich neue Integrale abzugewinnen, welche zwei willkürliche Functionen enthalten ist mir nicht gelungen." (Bianchi 1959, pp. 267–268)

[92]"Meine neue Methode giebt auch eine Ampère'sche Gleichung, aber für eine der beiden Variablen p oder q, durch die das Flächenlinienelement ausgedrückt wird." (Bianchi 1959, p. 263)

More precisely, Weingarten's treatment was based on differential parameters that were computed with respect to a tangent vector field:

> If XYZ are the coordinates of an arbitrary point of a sphere
>
> $$X^2 + Y^2 + Z^2 = 1$$
>
> of radius one, expressed by two independent variables u, v, then the differential equation for p
>
> $$F\left(\frac{\partial^2 p}{\partial u^2}, \frac{\partial^2 p}{\partial u \partial v}, \frac{\partial^2 p}{\partial v^2}, \frac{\partial p}{\partial u}, \frac{\partial p}{\partial v}\right)$$
>
> is an invariant form in relation to the spherical line element
>
> $$dX^2 + dY^2 + dZ^2 = b_{11}du^2 + 2b_{12}dudv + b_{22}dv^2,$$
>
> from the differential parameters
>
> $$\Delta(p) \qquad \Delta(p, \Delta(p)) \qquad \Delta(\Delta(p), \Delta(p)) \qquad \Delta_2(p)$$
>
> and p itself. This differential equation determines p. The other variable q is then always equal to $\Delta(p)$. After integrating them, one obtains ξ, η, ζ by quadratures. The differential equation itself represents the common integrability condition of these three quadratures. But this differential equation admits under given conditions first integrals of a general kind. Secondly, its real solutions lead to real surfaces of the given line element.[93]

Weingarten also specified that he reached the equation by using Gaussian representation, but composed the answer for the prize using Darboux's trihedron, *"which is an easy task when one knows the result"*. This choice is likely due to the fact that the manuscript

[93]"Sind XYZ die Coordinaten eines beliebigen Punctes einer Kugel

$$X^2 + Y^2 + Z^2 = 1$$

vom Radius Eins, ausgedrückt durch zwei Unabhangige u, v so wird die Differentialgleichung für p

$$F\left(\frac{\partial^2 p}{\partial u^2}, \frac{\partial^2 p}{\partial u \partial v}, \frac{\partial^2 p}{\partial v^2}, \frac{\partial p}{\partial u}, \frac{\partial p}{\partial v}\right)$$

eine in Beziehung auf das Kugellinienelement

$$dX^2 + dY^2 + dZ^2 = b_{11}du^2 + 2b_{12}dudv + b_{22}dv^2$$

invariante Form, aus den Differentialparametern

$$\Delta(p) \qquad \Delta(p, \Delta(p)) \qquad \Delta(\Delta(p), \Delta(p)) \qquad \Delta_2(p)$$

und p selbst zusammengesetzt. Diese Differentialgleichung bestimmt p. Die andere Variable q ist dann stets gleich $\Delta(p)$.

Nach ihrer Integration erhalt man ξ, η, ζ durch Quadraturen. Die Differentialgleichung selbst stellt die gemeinsame Integrabilitatsbedingung dieser 3 Quadraturen dar. Aber diese Differentialgleichung lasst unter angebbaren Bedingungen 1te Integrale allgemeiner Art zu. Zweitens führen ihre sämmtlichen reellen Losungen auf sämmtliche reellen Flächen des gegebenen Linienelements."

was submitted to a committee of French mathematicians, led by Darboux, who were more familiar with the trihedron method and probably preferred it. Traces of what may have been the original method can be found in the letters to Bianchi cited in the previous sections. We have seen in Section 5.4.1, for example, how Weingarten had managed to translate the differential equation of surfaces (5.22) into terms of the Gaussian parameter u and the differential parameters calculated with respect to a spherical representation for the differential parameter u, as in (5.23).

5.5.2 Weingarten's new method for applicability

In (Weingarten 1897, pp. 161–182) Weingarten constructed an equation that we denoted with (W) as a necessary condition for the existence of the finite equations of the given surface S by proving the following theorem, which also shows that the first vector field of a Darboux trihedron is related to the deformation of the surface:

Theorem 5.1. *Let S be a surface, whose parametric equations $\boldsymbol{x} = (x(z,\omega), y(z,\omega), z(z,\omega))$ are supposed to be known. Let z, σ be a new pair of Gaussian parameters that brings S to its* reduced form, *that is, two Gaussian parameters so that $K\sqrt{\Delta} = \frac{1}{2\sqrt{\sigma^3}}$, where K is the Gaussian curvature of S, $ds^2 = Edz^2 + 2Fdzd\sigma + Gd\sigma^2$ is the line element of S and $\Delta = \sqrt{EG - F^2}$.*

Furthermore, let $\boldsymbol{X}_1 = (X_1(u,v), Y_1(u,v), Z_1(u,v))$ be a vector field, whose equations with respect to generic parameters u, v are known and which reduces to the first tangent vector field of a Darboux trihedron on S when u, v are chosen as z, σ. Let $\Psi : dX_1^2 + dY_1^2 + dZ_1^2 = e_{11}du^2 + 2e_{12}dudv + e_{22}dv^2$ be the line element of the spherical representation of $\boldsymbol{X}_1$. Then, the Gaussian parameter z, which is thought of as a function of u and v, is a solution of

$$a\Delta_1 z - b\sqrt{\Delta_1 z}\Delta_2 z - \left(\alpha + \frac{b}{2\sqrt{\Delta_1 z}}\right)\Delta_1(z, \Delta_1 z) - 2\beta\sqrt{\Delta_1 z}\Delta_{22} z = 0, \qquad (W)$$

where differential parameters are computed with respect to Ψ and $a(u,v)$, $b(u,v)$, $\alpha(u,v)$, $\beta(u,v)$ are the rotation of the Darboux trihedron of S. Moreover, $\sigma(u,v) = \Delta_1 z$.

Then, by exploiting the intrinsic character of the above equations Weingarten also stated (without explicit proof) the converse:

Theorem 5.2. *Let S be a surface, whose parametric equations $\boldsymbol{x} = (x(z,\sigma), y(z,\sigma), z(z,\sigma))$ are supposed to be known and whose line element is in the reduced form, that is, if*

$$ds^2 = (a^2 + b^2)dz^2 + 2(a\alpha + b\beta)dzd\sigma + (\alpha^2 + \beta^2)d\sigma^2,$$

where a, b, α, β are functions of z, σ, then one has

$$\frac{\partial a}{\partial v} - \frac{\partial \alpha}{\partial u} = -\frac{\beta}{\sigma} \qquad \frac{\partial b}{\partial v} - \frac{\partial \beta}{\partial u} = -\frac{\alpha}{\sigma}.$$

Furthermore, let $\boldsymbol{X}_1 = (X_1(u,v), Y_1(u,v), Z_1(u,v))$ be a vector field, whose equations with respect to generic parameters u, v are known and which reduces to the first tangent vector *field of a Darboux trihedron on S when u, v are chosen as z, σ. Let $\Psi : dX_1^2 + dY_1^2 + dZ_1^2 = e_{11}du^2 + 2e_{12}dudv + e_{22}dv^2$ be the line element of the spherical representation of $\boldsymbol{X}_1$. Then, any integral $\tilde{u}$ of (W)*

$$a\Delta_1 u - b\sqrt{\Delta_1 u}\Delta_2 u - \left(\alpha + \frac{b}{2\sqrt{\Delta_1 u}}\right)\Delta_1(u, \Delta_1 u) - 2\beta\sqrt{\Delta_1 u}\Delta_{22}u = 0,$$

renders the following differential

$$\begin{aligned}
dx &= \left[a(\tilde{u},\tilde{v})X_1(\tilde{u},\tilde{v}) + b(\tilde{u},\tilde{v})\frac{\Delta_1(X_1,\tilde{u})}{\sqrt{\Delta_1\tilde{u}}}\right]d\tilde{u} &+ \left[\alpha(\tilde{u},\tilde{v})X_1(\tilde{u},\tilde{v}) + \beta(\tilde{u},\tilde{v})\frac{\Delta_1(X_1,\tilde{u})}{\sqrt{\Delta_1\tilde{u}}}\right]d\tilde{v} \\
dy &= \left[a(\tilde{u},\tilde{v})Y_1(\tilde{u},\tilde{v}) + b(\tilde{u},\tilde{v})\frac{\Delta_1(Y_1,\tilde{u})}{\sqrt{\Delta_1\tilde{u}}}\right]d\tilde{u} &+ \left[\alpha(\tilde{u},\tilde{v})Y_1(\tilde{u},\tilde{v}) + \beta(\tilde{u},\tilde{v})\frac{\Delta_1(Y_1,\tilde{u})}{\sqrt{\Delta_1\tilde{u}}}\right]d\tilde{v} \\
dz &= \left[a(\tilde{u},\tilde{v})Z_1(\tilde{u},\tilde{v}) + b(\tilde{u},\tilde{v})\frac{\Delta_1(Z_1,\tilde{u})}{\sqrt{\Delta_1\tilde{u}}}\right]d\tilde{u} &+ \left[\alpha(\tilde{u},\tilde{v})Z_1(\tilde{u},\tilde{v}) + \beta(\tilde{u},\tilde{v})\frac{\Delta_1(Z_1,\tilde{u})}{\sqrt{\Delta_1\tilde{u}}}\right]d\tilde{v},
\end{aligned}$$

where $\tilde{v} = \Delta_1\tilde{u}$ and differential parameters are formed with respect to Ψ, exact and it determines a surface applicable to S.

These results were later complemented by (Weingarten 1899), in which Weingarten further investigated the content of (Hessenberg 1900), where singularities of Weingarten's method, such as the case in which Ψ reduces to the line element of a curve, were first pointed out.

Combining these results, Weingarten essentially proved that

Theorem 5.3. *All real integrals of the* fundamental equation (W)*, which is formed with respect to a given real surface S, give all real surfaces that are applicable to S by quadratures.*

5.5.3 Detailed exposition of Weingarten's proof

Weingarten considered[94] a non-developable surface $S : (z, \omega) \in U \mapsto \boldsymbol{x}(z, \omega)$, whose finite equations $\boldsymbol{x} = (x(z,\omega), y(z,\omega), z(z,\omega))$ and line element $ds^2 = \bar{E}dz^2 + 2\bar{F}dzd\omega + \bar{G}d\omega^2$ were known. He also supposed that z, ω are real and $\bar{E}$, $\bar{F}$, $\bar{G}$ continuous and finite.

First, Weingarten changed the Gaussian parameter ω with another function σ that gave a convenient form to the line element of S, which he called *reduced form.* Up to appropriately restricting the domain U so that neither K nor $\Delta = \bar{E}G - \bar{F}^2$ vanishes, he considered the transformation $(z, \omega) \to \left(z, \left(\int_{\omega_0}^{\omega} K\sqrt{\bar{E}G - \bar{F}^2}d\omega\right)^{-2}\right)$. It should be noted that Weingarten's decision is only apparently a backward step compared with the generality of the results obtained up to then. Unlike Bour's solution, which was difficult to use in practice since it required the resolution of the geodesic line equation, Weingarten's change of coordinates can be realised in infinite manners through a quadrature.

According to Weingarten, a line element $ds^2 = Edz^2 + 2Fd\sigma dz + Gd\sigma^2$ is said to be *in reduced form* when

$$K\sqrt{\Delta} = \frac{1}{2\sqrt{\sigma^3}} \tag{5.28}$$

[94] With respect to the original, we have made minor changes in the notation: we replaced Weingarten's Euclidean coordinates ξ, η, ζ with $\boldsymbol{x} = (x, y, z)$; Darboux's rotations Q, Q_1, P, P_1 with q, q_1, p, p_1; and Gaussian parameters p, q with u, v.

holds true with respect to the fixed parametrization.

Then, Weingarten referred S to a Darboux trihedron.[95] By comparing (5.28) with Gauss equation

$$K\sqrt{\Delta} = \frac{\partial}{\partial z}\left[\Gamma^1_{22}\frac{\sqrt{\Delta}}{G}\right] - \frac{\partial}{\partial \sigma}\left[\Gamma^1_{12}\frac{\sqrt{\Delta}}{G}\right],$$

one has

$$\frac{\partial}{\partial z}\left[\Gamma^1_{22}\frac{\sqrt{\Delta}}{G}\right] = \frac{\partial}{\partial \sigma}\left[\Gamma^1_{12}\frac{\sqrt{\Delta}}{G} - \frac{1}{\sqrt{\sigma}}\right].$$

This latter guarantees the existence of a function φ such that

$$\frac{\partial \varphi}{\partial \sigma} = \Gamma^1_{22}\frac{\sqrt{\Delta}}{G} \qquad \frac{\partial \varphi}{\partial z} = \Gamma^1_{12}\frac{\sqrt{\Delta}}{G} - \frac{1}{\sqrt{\sigma}}. \tag{5.29}$$

The comparison of these equations with the equations (4.10) of the Darboux trihedron's general theory proves that φ is the angle that fixes the orientation of the Darboux trihedron and that $r = -\frac{1}{\sqrt{\sigma}}$ and $r_1 = 0$. Vice versa, when $r = -\frac{1}{\sqrt{\sigma}}$ and $r_1 = 0$, or equivalently (5.29), hold true, S is in the reduced form.[96]

The angle φ allows Weingarten to deduce Darboux's translations a, b, α, β as

$$a = \sqrt{E}\sin(\varphi + \omega) \qquad \alpha = \sqrt{G}\sin\varphi \qquad b = \sqrt{E}\cos(\varphi + \omega) \qquad \beta = \sqrt{G}\cos\varphi.$$

where ω is the angle between the coordinate lines. Finally, Weingarten obtained the vector fields of the Darboux trihedon $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{X}_3$ by inverting the equations[97]

$$\begin{cases} \dfrac{\partial x}{\partial z} = aX_1 + bX_2 & \qquad \dfrac{\partial x}{\partial \sigma} = \alpha X_1 + \beta X_2 \\ \dfrac{\partial y}{\partial z} = aY_1 + bY_2 & \qquad \dfrac{\partial y}{\partial \sigma} = \alpha Y_1 + \beta Y_2 \\ \dfrac{\partial z}{\partial z} = aZ_1 + bZ_2 & \qquad \dfrac{\partial z}{\partial \sigma} = \alpha Z_1 + \beta Z_2. \end{cases} \tag{5.30}$$

By now, we have seen how the Gauss map and the associated spherical representation were usually adopted to describe the embedding of a surface in Euclidean space. This gave a privileged role to the normal vector field to the surface. On the other hand, Weingarten noted that Darboux trihedron defined *three* vector fields, each of which has a "Gauss map" and an associated "spherical representation". More precisely, for $i = 1, 2, 3$, $\boldsymbol{X}_i$ defines a map

$$\begin{aligned} N_i &: S \to S^2 \\ (u, v) &\mapsto \boldsymbol{X}_i = (X_i(u, v), Y_i(u, v), Z_i(u, v)) \end{aligned}$$

and the following quadratic differential form

$$d\mathcal{S}_i^2 = dX_i^2 + dY_i^2 + dZ_i^2 = \mathcal{E}_i du^2 + 2\mathcal{F}_i du dv + \mathcal{G}_i dv^2,$$

[95]See section 4.3.3.

[96]For more details, the interested reader can see (Weingarten 1897, pp. 166–167).

[97]See system (4.6) in section 4.3.3.

which is the line element of the spherical representation of $\boldsymbol{X}_i$.

Moreover, Weingarten supposed that each of the two tangent vector fields, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, is *"intimately"* linked to the deformation of the surface:[98]

> We can therefore conceive three spherical representations of our surface, by looking at the quantities X, Y, Z or the others X', Y', Z' or finally X'', Y'', Z'' as the coordinates of the image of the corresponding point of the surface on a sphere whose radius is the unit.
>
> It will be our next aim to show how closely the two representations determined respectively by the coordinates (X, Y, Z) and (X', Y', Z') are intimately linked to the deformation of this surface.[99]

In addition to the choice of the reduced form, this remark is the key point of Weingarten's new method. Weingarten, therefore, chose $\boldsymbol{X}_1$ and its spherical representation $d\mathcal{S}_i^2$ that he denoted with ψ, but he also showed how considering $\boldsymbol{X}_2$ yields quite similar results.[100]

When the line element is in the reduced form, ψ is obtained by substituting (4.5) with $r = 0, r_1 = -\frac{1}{\sqrt{\sigma}}$ into $dX_1^2 + dY_1^2 + dZ_1^2$. In particular,

$$\psi = \left(\frac{1}{\sigma} + q^2\right) dz^2 + 2qq_1 dz d\sigma + q_1^2 d\sigma^2.$$

where p, q, p_1, q_1 are Darboux's rotations.

Weingarten's purpose was to substitute Bour's equation with the integrability condition of (**??**), which he aimed to write in terms of differential parameters computed with respect to ψ, and thus valid for all the surfaces applicable to S.

The computation of the first differential parameters with respect to ψ gives[101]

$$\Delta_1 z = \sigma \tag{5.31}$$

$$\Delta_1(z, X_1) = -\sqrt{\sigma} X_2 \qquad \Delta_1(z, Y_1) = -\sqrt{\sigma} Y_2 \qquad \Delta_1(z, Z_1) = -\sqrt{\sigma} Z_2. \tag{5.32}$$

Since differential parameters are invariant under changes of coordinates, (5.31) and (5.32) are still valid when they are formed with respect to Ψ instead of ψ, where Ψ is obtained from $\psi(z(u, v), \sigma(u, v))$ through a change of parameters. In particular, when Δ_1 is computed with respect to Ψ, $\Delta_1 z(u, v) = \sigma(u, v)$ holds true, as required in Theorem 5.1.

It also follows from (5.32) that the other tangent vector of Darboux trihedron is

$$\boldsymbol{X}_2 = \left(-\frac{\Delta(z, X_1)}{\sqrt{\sigma}}, -\frac{\Delta(z, Y_1)}{\sqrt{\sigma}}, -\frac{\Delta(z, Z_1)}{\sqrt{\sigma}}\right), \tag{5.33}$$

[98] The proof of this statement essentially consists in the demonstration of theorem 5.1.

[99] "On peut donc concevoir *trois* représentations sphéeriques de notre surface, en regardant les quantités X, Y, Z ou les autres X', Y', Z' ou enfin X'', Y'', Z'' comme les coordonnées de l'image du point correspondant de la surface sur une sphère dont le rayon est l'unité.

Ce sera notre but le plus prochain de montrer combien les deux représentations déterminées respectivement par le coordonées (X, Y, Z) et (X', Y', Z') sont liées intimement à la déformation de cette surface." (Weingarten 1897, p. 172)

[100] (Weingarten 1897, §3)

[101] We refer to footnote 9 for the formulae of the differential parameters. They can be easily determined by noting that $\frac{\partial z}{\partial z} = \frac{\partial \sigma}{\partial \sigma} = 1$ and $\frac{\partial z}{\partial \sigma} = \frac{\partial \sigma}{\partial z} = 0$.

and, consequently, the differential equations of S (5.30) become

$$
\begin{aligned}
dx &= \left[aX_1(z,\sigma) - b\frac{\Delta_1(X_1,z)}{\sqrt{\Delta_1 z}}\right] dz &&+ \left[\alpha X_1(z,\sigma) - \beta\frac{\Delta_1(X_1,z)}{\sqrt{\Delta_1 z}}\right] d\sigma \\
dy &= \left[aY_1(z,\sigma) - b\frac{\Delta_1(Y_1,z)}{\sqrt{\Delta_1 z}}\right] dz &&+ \left[\alpha Y_1(z,\sigma) - \beta\frac{\Delta_1(Y_1,z)}{\sqrt{\Delta_1 z}}\right] d\sigma \\
dz &= \left[aZ_1(z,\sigma) - b\frac{\Delta_1(Z_1,z)}{\sqrt{\Delta_1 z}}\right] dz &&+ \left[\alpha Z_1(z,\sigma) - \beta\frac{\Delta_1(Z_1,z)}{\sqrt{\Delta_1 z}}\right] d\sigma.
\end{aligned}
\tag{5.34}
$$

Weingarten commented:

> If one remembers that the four quantities a, b, α, β are *known* functions of the variables z and $\sigma = \Delta(z)$, one will notice that the important formulas (13) [(5.34)] give the values of the differentials $d\xi, d\eta, d\zeta$ [dx, dy, dz for us] of the coordinates ξ, η, ζ [x, y, z for us] of any point of the given surface in an **invariant** form, that is to say that the values of these differentials remain the same, whatever the independent variables u and v used.[102]

Hence, (5.34) gives all the surfaces applicable to S when z, σ are substituted with generic parameters u, v. However, their integrability conditions, or those of the equivalent system (**??**), must be satisfied. Darboux's theory states that[103] they are

$$
\begin{cases}
\dfrac{\partial a}{\partial v} - \dfrac{\partial \alpha}{\partial u} = br_1 - \beta r \\
\dfrac{\partial b}{\partial v} - \dfrac{\partial \beta}{\partial u} = \alpha r - ar_1 \\
bp_1 - aq_1 + \alpha q - \beta p = 0.
\end{cases}
\tag{5.35}
$$

Since $r = -\frac{1}{\sqrt{\Delta_1 z}}, r_1 = 0$ holds true independently of the parametrization, the first two correspond to (5.29) that are satisfied as S is supposed to be given in the reduced form.

In order to obtain the intrinsic version of 5.35 Weingarten computed the following differential parameters with respect to ψ[104]

$$
\begin{aligned}
\Delta_1\Delta_1 z &= \frac{1}{q_1^2} + \frac{q^2\sigma}{q_1^2} &\qquad \Delta_1(z, \Delta_1 z) &= -\frac{q\Delta_1 z}{q_1} \\
\Delta_2 z &= \frac{\sqrt{\sigma}p_1}{q_1} - \frac{q}{2q_1} &\qquad \Delta_{22} &= -\frac{p\sqrt{\sigma}}{2q_1},
\end{aligned}
\tag{5.36}
$$

[102]"Si l'on se souvient que les quatre quantités a, b, α, β sont des fonctions *connues* des variables z et $\sigma = \Delta(z)$, on remarquera bien que les formules importantes (13) donnent les valeurs des différentielles $d\xi, d\eta, d\zeta$ des coordonndées ξ, η, ζ d'un point quelconque de la surface donnée sous une forme **invariante**, c'est à dire que les valeurs de ces differentielles restent les mémes, quelles que soient les variables indépendantes u et v dont on a fair usage." (Weingarten 1897, p. 175)

[103]See (4.8) in section 4.3.3.

[104]Weingarten used Θ to denote Δ_{22} and

$$
J(f) = \frac{\left(\frac{\partial f}{\partial u}\right)^2 \frac{\partial^2 f}{\partial v^2} - 2\frac{\partial f}{\partial u}\frac{\partial f}{\partial v}\frac{\partial^2 f}{\partial u \partial v} + \left(\frac{\partial f}{\partial v}\right)^2 \frac{\partial^2 f}{\partial u^2}}{EG - F^2},
$$

which we substitute with $\Delta_1 z \Delta_2 z + \frac{1}{2}\Delta_1(z, \Delta_1 z)$ for better readability. We point out that there is a misprint in Weingarten's expression for $J(z)$: it should be replaced with $+\sqrt{\sigma^3}\frac{p_1}{q_1}$.

which give

$$\frac{q}{q_1} = -\frac{\Delta_1(z,\Delta_1 z)}{\Delta_1 z} \qquad \frac{p}{q_1} = -\frac{2\Delta_{22}z}{\sqrt{\Delta_1 z}} \qquad \frac{p_1}{q_1} = \frac{1}{\sqrt{\Delta_1 z}}\left(\Delta_2 z - \frac{\Delta_1(z,\Delta_1 z)}{2\Delta_1 z}\right).$$

By dividing[105] (5.35) by q_1 and substituting $\frac{q}{q_1}, \frac{p}{q_1}, \frac{p_1}{q_1}$ with the respective expressions in terms of the differential parameters obtained from (5.36), the integrability condition gives the equation (W)

$$a\Delta_1 z - b\sqrt{\Delta_1 z}\Delta_2 z - \left(\alpha + \frac{b}{2\sqrt{\Delta_1 z}}\right)\Delta_1(z,\Delta_1 z) - 2\beta\sqrt{\Delta_1 z}\Delta_{22}z = 0, \qquad (W)$$

where a, b, α, β are now functions of z and Δz. Hence, this equation involves only z and differential parameters of z formed with respect to ψ. On account of the invariant character of these differential parameters the surface S may be expressed in terms of any parameters u, v and (W) remain the same when they are calculated with respect to Ψ and $z(u, v)$ must be its solution.

Weingarten named the equation (W) *fundamental equation.* It is a second-order differential equation of the Monge-Ampère type, which, thanks to Theorem 5.2, replaces Bour's equation. Indeed, as Weingarten remarked, each function z of the variables u and v satisfying equation (W), will make the differentials of the general equations (5.34) integrable and the corresponding surfaces all have the same line element

$$ds^2 = (a^2 + b^2)du^2 + 2(a\alpha + b\beta)dudv + (\alpha^2 + \beta^2)dv^2.$$

In more detail than Weingarten did, let us consider a vector field $\boldsymbol{X}_1$ and its quadratic form $\psi = Edu^2 + 2Fdudv + Gdv^2$ with respect to a generic parametrization u, v. Then, we assume that $\tilde{u}(u, v)$ is a solution of (W) and define $\tilde{v} = \Delta_1\tilde{u}$, the differential parameter being computed with respect to ψ. Supposing that $\tilde{u}(u, v), \tilde{v}(u, v)$ are independent regular functions, they can be chosen as new Gaussian parameters and ψ becomes $\tilde{\psi} = \tilde{E}d\tilde{u}^2 + 2 + 2Fd\tilde{u}d\tilde{v} + \tilde{G}d\tilde{v}^2$. By defining $\boldsymbol{X}_2 := \left(-\frac{\Delta(z,X_1)}{\sqrt{\sigma}}, -\frac{\Delta(z,Y_1)}{\sqrt{\sigma}}, -\frac{\Delta(z,Z_1)}{\sqrt{\sigma}}\right)$, where differential parameters are computed with respect to $\tilde{\psi}$ and $\boldsymbol{X}_3$ as the orthogonal vector field to both $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, rotations in Darboux's system (4.5) satisfy its integrability conditions (4.7).[106] Moreover, the integrability conditions (4.8) are also satisfied, being the first two equations valid by hypothesis and the third equivalent to (W). Thus, (5.34) is integrable and the corresponding surfaces have

$$ds^2 = (a^2 + b^2)du^2 + 2(a\alpha + b\beta)dudv + (\alpha^2 + \beta^2)dv^2$$

as line element, concluding that every solution of (W) leads to applicable surfaces.

[105] Since S is a non-developable surface, we have $q_1 \neq 0$.

[106] From the definition of system (4.5) $\boldsymbol{X}_2$ one has

$$r = \boldsymbol{X}_2 \cdot \frac{\partial \boldsymbol{X}_1}{\partial u} \qquad p = -\boldsymbol{X}_2 \cdot \frac{\partial \boldsymbol{X}_3}{\partial u} \qquad q = -\boldsymbol{X}_1 \cdot \frac{\partial \boldsymbol{X}_3}{\partial u}$$
$$r_1 = \boldsymbol{X}_2 \cdot \frac{\partial \boldsymbol{X}_1}{\partial v} \qquad 1 = -\boldsymbol{X}_2 \cdot \frac{\partial \boldsymbol{X}_3}{\partial v} \qquad q_1 = -\boldsymbol{X}_1 \cdot \frac{\partial \boldsymbol{X}_3}{\partial v}$$

The definition of $\boldsymbol{X}_2$ also gives that $r = -\frac{1}{\sqrt{\sigma}}$ and $r_1 = 0$.

5.5.4 Discussion on the integrability of equation (W)

In the second part of (Weingarten 1897, pp. 183–200), Weingarten investigated the integrability of the equation (W). Weingarten commented:

> We will show that this fundamental equation must be regarded as the source of the discoveries of all the general classes of surfaces applicable to each other that are known today. [...] Our efforts to derive from our fundamental equation a class containing all surfaces which admit an essentially new line element, failed up to now.[107]

He achieved a result completely identical to (Weingarten 1893): the equation (W) is integrable by general methods only in the same cases, already known, in which the equation (W_0) is also integrable.[108] According to the report of the Paris prize, (Comptes rendus 1894), this remarkable fact earned Weingarten the Grand Prix. The importance of this result was also stressed by Bianchi in (Bianchi 1923):

> [Bour's equation] with the exception of the case $K = 0$ of the developable surfaces, does not admit intermediate integrals of the first order, and does not belong to any of the other known cases of integrability; it did not explain the successes previously obtained with geometry, which had led to the complete determination of other applicable classes of surfaces. Weingarten's transformation, on the other hand, possesses intermediate integrals in these cases, which analytically explains the reason for the success of the geometric methods.[109]

First, Weingarten rewrote (W) in a form that allowed him to apply an analytical result to the search for intermediate integrals. According to it, if a second-order PDE has the following form

$$(r - A_{11})(t - A_{22}) - (s - A_{12})(s - A_{21}) = 0$$

where $A_{ij},\ i, j = 1, 2$, are known functions of x, y, z, p, q and p, q, r, s, t follow Monge's classic notation

$$p = \frac{\partial z}{\partial x} \qquad q = \frac{\partial z}{\partial y} \qquad r = \frac{\partial p}{\partial x} \qquad s = \frac{\partial p}{\partial y} \qquad t = \frac{\partial q}{\partial y},$$

[107]"Nous ferons voir que cette équation fondamentale doit être regardée comme la source des découvertes de toutes les classes générales de surfaces applicables l'une sur l'autre qu'on connait aujourd'hui.

[...] Nos efforts pour tirer de notre équation fondamentale une classe contenant foutes les surfaces qui admettent un élément linéaire essentielle ment nouveau, ont échoué jusqu'a ce moment." (Weingarten 1897, p. 160)

[108]The problem of the integration of the equation (W) was later addressed also in (Gau 1925) and (Gosse 1928).

[109]"[L'equazione di Bour] salvo nel caso $K = 0$ delle superficie sviluppabili, non ammette integrali intermediari del primo ordine, e non rientra in alcuno degli altri casi noti d'integrabilità; essa non rendeva per ciò ragione dei successi già prima ottenuti colla geometria, che aveva condotto alla determinazione completa di altre classi di superficie applicabili. La trasformata di Weingarten possiede invece appunto in questi casi integrali intermediari, ciò che spiega analiticamente la ragione del successo dei metodi geometrici." (Bianchi 1923, p. 254)

then the search for its intermediate integrals can be reduced to the search for the solutions of one of the following two systems

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} + p\dfrac{\partial \varphi}{\partial z} + A_{11}\dfrac{\partial \varphi}{\partial p} + A_{12}\dfrac{\partial \varphi}{\partial q} = 0 \\ \dfrac{\partial \varphi}{\partial y} + q\dfrac{\partial \varphi}{\partial z} + A_{21}\dfrac{\partial \varphi}{\partial p} + A_{22}\dfrac{\partial \varphi}{\partial q} = 0 \end{cases} \tag{5.37}$$

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} + p\dfrac{\partial \varphi}{\partial z} + A_{11}\dfrac{\partial \varphi}{\partial p} + A_{21}\dfrac{\partial \varphi}{\partial q} = 0 \\ \dfrac{\partial \varphi}{\partial y} + q\dfrac{\partial \varphi}{\partial z} + A_{12}\dfrac{\partial \varphi}{\partial p} + A_{22}\dfrac{\partial \varphi}{\partial q} = 0. \end{cases} \tag{5.38}$$

Weingarten thus considered S as the graph of a function, $\boldsymbol{x}(x, y) = (x, y, z(x, y))$ so that the line element of the spherical representation of $\boldsymbol{X}1$ acquires the simple form

$$dX_1^2 + dY_1^2 + dZ_1^2 = \frac{2dxdy}{(1+xy)^2}.$$

Then, one has

$$\begin{gathered} \boldsymbol{X}_1 = \left(\frac{x+y}{1+xy}, \frac{1}{i}\frac{x-y}{1+xy}, \frac{1-xy}{1+xy}\right) \\ \frac{\partial^2 z}{\partial x^2} = r + \frac{2y}{1+xy}p \qquad \frac{\partial^2 z}{\partial y^2} = t + \frac{2x}{1+xy}q \\ \frac{\partial^2 z}{\partial x \partial y} = s \end{gathered} \tag{5.39}$$

By substituting these values, (W) becomes[110]

$$\left(r + \frac{2y}{1+xy}p - \rho p^2\right)\left(t + \frac{2x}{1+xy}q - \rho q^2\right) - \left(s - \frac{\tau}{\sigma}pq\right)\left(s - \frac{\tau_*}{\sigma}pq\right) = 0 \tag{5.40}$$

where $\sigma = \Delta(z) = (1+xy)^2 pq$ and

$$\begin{aligned} \rho &:= -\frac{b + 2a\sqrt{\sigma}}{2\beta\sigma} & W^2 &:= -\frac{1}{4K\sigma\beta^2} \\ \tau &:= \rho\sigma + 2(\lambda + Wi) & \tau_* &:= \rho\sigma + 2(\lambda - Wi). \end{aligned}$$

In this context, systems (5.37)-(5.38) become, respectively,

$$\begin{cases} \mathfrak{U}(\varphi) = \dfrac{\partial \varphi}{\partial x} + p\dfrac{\partial \varphi}{\partial z} + \left(\rho p^2 - \dfrac{2yp}{1+xy}\right)\dfrac{\partial \varphi}{\partial p} + \dfrac{\tau}{\sigma}pq\dfrac{\partial \varphi}{\partial q} = 0 \\ \mathfrak{V}(\varphi) = \dfrac{\partial \varphi}{\partial y} + q\dfrac{\partial \varphi}{\partial z} + \left(\rho q^2 - \dfrac{2xq}{1+xy}\right)\dfrac{\partial \varphi}{\partial q} + \dfrac{\tau_*}{\sigma}pq\dfrac{\partial \varphi}{\partial p} = 0 \end{cases} \tag{5.41}$$

$$\begin{cases} \mathfrak{U}'(\varphi) = \dfrac{\partial \varphi}{\partial x} + p\dfrac{\partial \varphi}{\partial z} + \left(\rho p^2 - \dfrac{2yp}{1+xy}\right)\dfrac{\partial \varphi}{\partial p} + \dfrac{\tau_*}{\sigma}pq\dfrac{\partial \varphi}{\partial q} = 0 \\ \mathfrak{V}'(\varphi) = \dfrac{\partial \varphi}{\partial y} + q\dfrac{\partial \varphi}{\partial z} + \left(\rho q^2 - \dfrac{2xq}{1+xy}\right)\dfrac{\partial \varphi}{\partial q} + \dfrac{\tau}{\sigma}pq\dfrac{\partial \varphi}{\partial p} = 0. \end{cases} \tag{5.42}$$

[110] For more details, see (Weingarten 1897, pp. 183–185). (5.40) also shows that Weingarten's equation is of the Monge-Ampère type.

Weingarten noted that the existence of an intermediate integral φ of (5.40) guarantees the existence of another intermediate integral φ_* of (5.40) and these integrals are always distinct.[111] Indeed, if φ satisfies (5.41), then another function φ_* must necessarily exist that satisfies (5.42). This is a consequence of the fact that (5.42) is transformed into (5.41) by permuting p with q and x with y, and by leaving z and σ unchanged (consequently τ, τ_* and ρ also remain unchanged) and therefore $\varphi_*(x, y, z, p, q) = \varphi(y, x, z, q, p)$ is a solution of (5.42).

Thus, Weingarten could refer to Bour and Jacobi's method and a "laborious and tedious calculation"[112] led him to rewrite Jacobian condition

$$\mathfrak{U}(\mathfrak{V}(\varphi)) = \mathfrak{V}(\mathfrak{U}(\varphi))$$

as

$$2W\frac{\partial\varphi}{\partial z} - (Q + Pi)\frac{\partial\varphi}{\partial p}p - (Q - Pi)\frac{\partial\varphi}{\partial q}q = 0 \tag{5.43}$$

where

$$P = W^2\frac{\partial \log\frac{\sqrt{\sigma}}{\sqrt{\Delta}}}{\partial\sigma} \qquad Q = \frac{W}{2}\left(\frac{b}{\beta}\frac{\partial \log\frac{\sqrt{\sigma}}{\sqrt{\Delta}}}{\partial\sigma} - \frac{\partial \log\frac{\sqrt{\sigma}}{\sqrt{\Delta}}}{\partial z}\right)$$

Weingarten distinguished two cases, depending on whether (5.43) is identically satisfied by φ or not. In any case, he showed that:

Theorem 5.4. *The equation* (W) *does not admit two general intermediate integrals of the first order unless the line element of the given surface* S *is of the form*

$$ds^2 = v^2du^2 + (lv^2 + m)dv^2 \tag{5.44}$$

where l *and* m *are arbitrary constants* ($l \neq 0$)*. Conversely, all surfaces that admit* (5.44) *as line element lead to an equation* (W) *that has two general intermediate integrals of the first order.*

Hence, (W) is integrable in the same well-known cases of equation (W_0):

> Thus the integration of all the fundamental equations of our theory, which can be integrated by the method of characteristics, is complete, geometry having preceded analysis.
>
> Nevertheless, our fundamental equations that can be integrated by the method of Ampère provide the teacher with an unlimited series of examples, more or less complicated, of partial differential equations of the second order, allowing the

[111]Due to the symmetry of (5.41) and (5.42), φ can coincide with φ_* only in the case $\tau = \tau_*$. However, this case is impossible since W cannot be null.

[112]"Nous jugeons superflu d'écrire ici le calcul fatigant et prolixe" (Weingarten 1897, p. 188)

various calculations required for integration by the method of characteristics to be carried out. And these examples are very rare at the moment.[113]

In the end, Weingarten also rediscovered the other known cases of applicable surfaces by specifically choosing some values for Darboux's rotations. By posing

$$a = -2f(z) \qquad b = -2\sqrt{\sigma} \qquad \alpha = 1 \qquad \beta = 0$$

(W) becomes $\Delta_2(z) = f(z)$, that is, $\rho_1 + \rho_2 = 2z + f(z)$. In the case $f(z) = k(1-k)z$, $k = const$, it reduces to Euler's equation

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{k(1-k)z}{(1+xy)^2}$$

whose solution was extensively discussed by Darboux in (Darboux 1896, pp. 322–329). In particular, if $k = 0$ or $k = 1$, i.e., $f(z) = 0$, the corresponding complete class of surfaces is that of the evolute of a catenoid (5.11), as Weingarten had already observed in his letter of 19th January 1889. In the case $k = 2$, i.e., $f(z) = -2z$, Weingarten returned to the case he discovered in (Weingarten 1887a), whose line element can be also written down as $ds^2 = du^2 + 2(u - v^2)dv^2$. Bianchi commented: *"This result, obtained by Weingarten in 1887, was the starting point for his further research that led him to the new method".*[114]

The case $f(z) = k$, $k \neq 0$, gives the complete class (5.14) that Weingarten discussed in the letters during the years 1889-1890. The results obtained in (Baroni 1890) and (Goursat 1891) correspond to the case $f(z) = hz$.

In the case $f(z) = e^z + 2$, (W) reduces to Liouville's equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \pm e^{2\theta},$$

whose integrals correspond to the well-known orthogonal isothermal coordinate system on the sphere. Weingarten had discussed the discovery of this new class of applicable surfaces, whose line element can assume also the form

$$ds^2 = du^2 + (2u^2 + 2v + e^{2v})dv^2, \tag{5.45}$$

in (Weingarten 1891b) and in (Bianchi 1959, pp. 242–244). This latter class of applicable surfaces, which is presented in detail in (Darboux 1896, pp. 330–337), gave Darboux the opportunity to emphasise the great progress made in the field of deformation theory and, indirectly, also the importance of studies on complex surfaces. Indeed, he proved that the surfaces with (5.45) as line element are applicable on an imaginary paraboloid whose generator is tangent to the circle at infinity. He concluded:

[113]"Donc l'intégration de toutes les équations fondamentales de notre théorie intégrables par la méthode des caractéristiques est achevée, la géométrie ayant devancé l'analyse.

Néanmoins nos équations fondamentales intégrables par la méthode d'Ampère donnent à l'enseignement une série illimitée d'exemples, plus ou moins compliqués, d'équations aux dérivées partielles du second ordre, permettant de mener à bonne fin les différents calculs qu'exige l'intégration par la méthode des caractéristiques. Et ces exemples sont bien rares en ce moment." (Weingarten 1897, p. 195)

[114](Bianchi 1903, p. 214)

Although the paraboloids considered in the previous sections are imaginary, we thought it useful to introduce them in order to specify the degree of generality of the new results obtained in the theory of deformation and to show how far we are from a somewhat extended solution of the problem.[115]

Finally, in the case

$$a = \frac{\partial^2 \varphi}{\partial^2 z^2} - 2\frac{\partial \varphi}{\partial \sigma} \qquad b = -2\sqrt{\sigma}\frac{\partial^2 \varphi}{\partial z \partial \sigma}$$

$$\alpha = \frac{\partial^2 \varphi}{\partial z \partial \sigma} \qquad \beta = -2\sqrt{\sigma}\frac{\partial^2 \varphi}{\partial z^2 \sigma^2},$$

where φ is a generic function of z and σ, Weingarten showed that (W) reduced to (W_0).

Weingarten did not find any new cases but his equation analytically justified and summarised all the results obtained over the previous 30 years on the determination of complete classes of applicable surfaces in a unified theory:

> Weingarten, by studying his particular procedures more closely, succeeded in obtaining a radical transformation of the applicability equation into another that is still of Ampère's form, but on which the application of the general theory leads precisely to finding all the cases of complete integration already obtained.[116]

According to Weingarten, his result also had an important implication for PDE theory:

> But our analysis has brought to light for the first time the fact, worthy of further research, that a partial differential equation of the second order, which resists integration by the known regular methods, can be reduced to another partial differential equation of the same order, which is not so rebellious.[117]

[115] "Bien que les paraboloïdes considérés dans les numéros précédents soient imaginaires, il nous a paru utile de les introduire pour préciser le degré de généralité des résultats nouveaux qui ont été obtenus dans la théorie de la déformation et pour bien montrer combien on est encore éloigné d'une solution quelque peu étendue du problème." (Darboux 1896, p. 337)

[116] "Il Weingarten, studiando più attentamente i suoi procedimenti particolari, riuscì ad ottenere una trasformazione radicale dell'equazione della applicabilità in un'altra che è ancora della forma d'Ampère, ma sulla quale l'applicazione della teoria generale conduce appunto a ritrovare tutti i casi d'integrazione completa già ottenuti." (Bianchi 1910a, p. 224)

[117] "Mais notre analyse a mis en évidence pour la première fois ce fait bien digne de recherches ultérieures, qu'une équation aux dérivées partielles du second ordre, qui résiste à l'intégration par les méthodes régulaires connues, peut étre réduite à une autre équation aux dérivées partielles du méme ordre, qui ne se montre pas si rebelle." (Weingarten 1897, pp. 199–200)

Chapter 6

Two perspectives on Weingarten's method

6.1 Bianchi's interpretation of Weingarten's method

Bianchi's applications[1] of Weingarten's new method for applicability allow for a closer look at some of his little-known results that can help clarify some fundamental aspects of his research. In this regard, the first question that may provide interesting insights is: what did Bianchi mean by "Weingarten's method"?

A comparison between the editions of his *Lezioni* may provide to be interesting in this regard. Bianchi's commitment to writing manuals for student use reflected his great commitment to teaching and disseminating the most modern scientific results.[2] Hence, the three Italian and two German editions[3] present significant differences, which reflect Bianchi's personal process of organising the theory as the theory progressively developed.

Bianchi's expositions of Weingarten's method in the second and third editions are quite different. In the second edition, Bianchi devoted two chapters to it. In the first, (Bianchi 1903, Chap. XIX), he explained it by following (Weingarten 1897) and also included Darboux's geometrical interpretation of Weingarten's reduced system.[4] In the second, (Bianchi 1903, Chap. XX), Bianchi proved that equation (W) solved the problem

[1] Weingarten's new method for applicability was also investigated by Goursat and Darboux. See in particular (Goursat 1927), (Darboux 1896) and (Gambier 1927, p. 10) for an overview on the main results.

[2] To improve the exposition of the most recent achievements in differential geometry, Bianchi demonstrated important results, such as the famous *Bianchi's identities*, which are one of the few discoveries named after Bianchi. Indeed, they are proved in a lemma that allowed him to provide a more direct demonstration of Schur's theorem (See (Bianchi 1902b)). These results were immediately included in his *Lezioni*, (Bianchi 1902a, §161).

[3] The *Lezioni* appeared first in a lithographed version in (Bianchi 1886a), then printed in (Bianchi 1894); the second edition appeared in two volumes, (Bianchi 1902a) and (Bianchi 1903); the third had three volumes, (Bianchi 1922), (Bianchi 1923), (Bianchi 1924). The two German editions were (Bianchi 1899c) and (Bianchi 1910b).

[4] In (Darboux 1896, Chap. XIV), Darboux had clarified the geometric meaning of Weingarten's reduced system by interpreting it as a coordinate system u, v of Σ, whose $u-$lines consist of the second lines of striction of the surface generated by the first vector X_1 of Darboux's trihedron of σ.

of applicability as well as equation (W_0) and, thus, the method employed in (Weingarten 1897) was "equivalent" to the method employed in (Weingarten 1891a). This second chapter also contains a structured, albeit partial, exposition of Bianchi's personal research on the equation (W_0), mainly published in (Bianchi 1896b), (Bianchi 1896a), (Bianchi 1899a) and (Bianchi 1902c).

In the third edition, Bianchi gave a more concise exposition, probably because he preferred to expand on the theory of deformation of quadrics that had characterised the early 1920s.[5] (Bianchi 1923, pp. 253–265) presented Bianchi's deduction of the equation (W_0), which was taken as the new equation for applicability. Its demonstration, provides a better understanding of the first version of Weingarten's method by making explicit the link between the surface Σ, of which the complete class of applicable surfaces is to be determined, and the surfaces that are integrals of the equation (W_0).

Bianchi first showed how to associate the equation (W_0) with a given surface Σ, whose complete class of applicable surfaces is to be determined.

His starting point was the existence of a cyclic system lying on the tangent planes of Σ. Cyclic systems are congruences (i.e., families of curves depending on two parameters) of circles which admit a family of orthogonal surfaces depending on one parameter. They were first studied in (Ribaucour 1891). The connection between deformations of a surface and the search for a cyclic system on its tangent planes had been pointed out by the early investigations of Ribaucour and Darboux, who had showed that the search of such a cyclic system depends on the solution of Darboux's equation (D).[6] By considering the case of real surfaces, Bianchi had also shown that[7] a cyclic system on the tangent planes of a surface Σ, whose line element is $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, determines three functions ξ, η, ζ such that

$$d\xi^2 + d\eta^2 - d\zeta^2 = Edu^2 + 2Fdudv + Gdv^2. \tag{6.1}$$

Hence, the search for these particular cyclic systems reduces to the determination of surfaces embedded in the indefinite space, whose first fundamental form coincides with that of Σ, i.e. to the integration of a Darboux's equation $\Delta_{22}\theta = K(1 - \Delta_1\theta)$, where $\Delta_1\theta > 1$.

To deduce the equation (W_0), Bianchi substituted $\xi = \alpha$, $\beta = \eta - \zeta$, $2\psi = \eta + \zeta$ into (6.1). The line element of Σ becomes[8]

$$ds^2 = d\alpha^2 + 2\frac{\partial\psi}{\partial\alpha}d\alpha d\beta + 2\frac{\partial\psi}{\partial\beta}d\beta^2 \tag{6.2}$$

and $\boldsymbol{x} = (x(\alpha,\beta), y(\alpha,\beta), z(\alpha,\beta))$ are the corresponding parametric equations.

[5]For a more detailed exposition, Bianchi referred to (Bianchi 1903, Chap. XIX-XX), as well as to (Darboux 1896) and (Weingarten 1897).

[6]For more details on the theory of cyclic system, one can refer to (Bianchi 1923, Chap. XXI) and (Eisenhart 1909, Chap. 13). In particular, (Bianchi 1923, §365) explains how to construct a cyclic system on the tangent planes of Σ.

[7]See (Bianchi 1923, §367).

[8]To this end, Bianchi wrote (6.1) as $d\xi^2 + d(\eta - \zeta)d(\eta + \zeta)$.

Then, Bianchi defined an auxiliary surface $\bar{S}$, which he called *Weingarten's derived surface of* Σ, through the equations

$$\bar{\boldsymbol{x}}(\alpha,\beta) = (\bar{x}(\alpha,\beta), \bar{y}(\alpha,\beta), \bar{z}(\alpha,\beta)) = (\frac{\partial x}{\partial \beta}(\alpha,\beta), \frac{\partial y}{\partial \beta}(\alpha,\beta), \frac{\partial z}{\partial \beta}(\alpha,\beta)).$$

Its normal vector field is[9] $\bar{\boldsymbol{X}}(\alpha,\beta) = \boldsymbol{x}_\alpha(\alpha,\beta)$ and its curvature radii, r_1 and r_2, must be the roots of the following equation[10]

$$\left(D\frac{\partial^2\psi}{\partial\alpha\partial\beta} - D'\frac{\partial^2\psi}{\partial\alpha^2}\right) r^2 + r\left(D\frac{\partial^2\psi}{\partial\beta^2} - D''\frac{\partial^2\psi}{\partial\alpha^2}\right) + D'\frac{\partial^2\psi}{\partial\beta^2}. - D''\frac{\partial^2\psi}{\partial\alpha\partial\beta} = 0.$$

Thus, $r_1 r_2$ and $r_1 + r_2$ must satisfy the following equation

$$\frac{\partial^2\psi}{\partial\alpha^2} r_1 r_2 - \frac{\partial^2\psi}{\partial\alpha\partial\beta}(r_1 + r_2) + \frac{\partial^2\psi}{\partial\beta^2} = 0.$$

Finally, through a Legendre transformation of α, β, ψ into u, v, φ such that $p = \frac{\partial\psi}{\partial\alpha}$, $p = \frac{\partial\psi}{\partial\beta}$ and $\varphi = \alpha\frac{\partial\psi}{\partial\alpha} + \beta\frac{\partial\psi}{\partial\beta} - \psi$,[11] the preceding equation became

$$\frac{\partial^2\varphi}{\partial u^2} r_1 r_2 + \frac{\partial^2\varphi}{\partial u\partial v}(r_1 + r_2) + \frac{\partial^2\varphi}{\partial v^2} = 0, \qquad (W_0)$$

which corresponds to equation (W_0), and the line element (6.2) of Σ with respect to the new parameters u, v becomes

$$ds^2 = 2v\left(d\frac{\partial\varphi}{\partial v}\right)^2 + 2u\left(d\frac{\partial\varphi}{\partial v}\right)\left(d\frac{\partial\varphi}{\partial u}\right) + \left(d\frac{\partial\varphi}{\partial u}\right)^2, \qquad (6.3)$$

which corresponds to Weingarten's line element (5.26) in Section 5.4.3.

Vice versa, Bianchi proved that any integral of (W_0) is a Weingarten's derived surface of a surface Σ, whose line element is (6.3).

Let $\bar{S}$ be a generic integral surface of (W_0) with parametric equations $\bar{\boldsymbol{x}}(u,v) = (\bar{x}(u,v), \bar{y}(u,v), \bar{z}(u,v))$, where u, v are Weingarten's parameters, that is, $u = \bar{\boldsymbol{x}} \cdot \bar{\boldsymbol{X}}$, $2v = \|\bar{\boldsymbol{x}}\|^2$, where $\bar{\boldsymbol{X}}$ is the normal unit vector field of $\bar{S}$. Under these hypotheses, Weingarten's equations

$$\begin{cases} \dfrac{\partial\bar{\boldsymbol{x}}}{\partial u} = -r_1 r_2 \dfrac{\partial\bar{\boldsymbol{X}}}{\partial v} \\ \dfrac{\partial\bar{\boldsymbol{x}}}{\partial v} = \dfrac{\partial\bar{\boldsymbol{X}}}{\partial u} + (r_1 + r_2)\dfrac{\partial\bar{\boldsymbol{X}}}{\partial v}, \end{cases}$$

[9] The definition of Σ gives

$$\|\boldsymbol{x}_\alpha\|^2 = 1 \qquad \|\boldsymbol{x}_\beta\|^2 = 2\frac{\partial\psi}{\partial\beta} \qquad \boldsymbol{x}_\alpha \cdot \boldsymbol{x}_\beta = \frac{\partial\psi}{\partial\alpha}.$$

By deriving the first with respect to β, one has $\boldsymbol{x}_{\alpha\beta} \cdot \boldsymbol{x}_\alpha = 0$. By deriving the second with respect to β one has $\boldsymbol{x}_{\alpha\beta} \cdot \boldsymbol{x}_\beta + \boldsymbol{x}_\alpha \cdot \boldsymbol{x}_{\beta\beta} = \frac{\partial^2\psi}{\partial\alpha\partial\beta}$, which gives $\boldsymbol{x}_\alpha \cdot \boldsymbol{x}_{\beta\beta} = 0$ since one has $\boldsymbol{x}_{\alpha\beta} \cdot \boldsymbol{x}_\beta = \frac{\partial^2\psi}{\partial\alpha\partial\beta}$ by deriving the third. This proves that $\boldsymbol{x}_\alpha$ is orthogonal to both the coordinate tangent vectors of $\bar{S}$, $\boldsymbol{x}_{\alpha\beta}$ and $\boldsymbol{x}_{\beta\beta}$.

[10] It can be easily obtained by adapting to $\bar{S}$ the general procedure contained in (Bianchi 1922, §71).

[11] This choice is due to the fact that in this case u and v are given by $u = \bar{\boldsymbol{x}} \cdot \bar{\boldsymbol{X}}$ and $v = \frac{\|\bar{\boldsymbol{x}}\|^2}{2}$, in accordance with Weingarten's notation.

together with (W_0), guarantee that

$$\bar{\boldsymbol{X}} d\frac{\partial \varphi}{\partial u} + \bar{\boldsymbol{x}} d\frac{\partial \varphi}{\partial v}$$

or, equivalently,

$$\left(\bar{\boldsymbol{x}}\frac{\partial^2 \varphi}{\partial u \partial v} + \bar{\boldsymbol{X}}\frac{\partial^2 \varphi}{\partial u^2}\right) du + \left(\bar{\boldsymbol{x}}\frac{\partial^2 \varphi}{\partial^2 v} + \bar{\boldsymbol{X}}\frac{\partial^2 \varphi}{\partial u \partial v}\right) dv$$

is exact, that is, there exists a function $\boldsymbol{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ such that

$$d\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial u} du + \frac{\partial \boldsymbol{x}}{\partial v} dv = \left(\bar{\boldsymbol{x}}\frac{\partial^2 \varphi}{\partial u \partial v} + \bar{\boldsymbol{X}}\frac{\partial^2 \varphi}{\partial u^2}\right) du + \left(\bar{\boldsymbol{x}}\frac{\partial^2 \varphi}{\partial^2 v} + \bar{\boldsymbol{X}}\frac{\partial^2 \varphi}{\partial u \partial v}\right) dv.$$

By referring to the preceding Legendre transformation,[12] this equation requires that $\bar{\boldsymbol{x}} = \frac{\partial x}{\partial \beta}$ and $\bar{\boldsymbol{X}} = \frac{\partial x}{\partial \alpha}$. Hence, $\bar{S}$ is precisely a Weingarten's derived surface of a surface Σ.

This demonstration allowed Bianchi to provide a comprehensive interpretation of the two separate problems described by Darboux's equation for applicability: the knowledge of a cyclic system, i.e., a solution of $\Delta_{22}\theta = K(1 - \Delta_1\theta)$, where $\Delta_1\theta > 1$, enables the transformation of the solutions of $\Delta_{22}\theta = K(1 - \Delta_1\theta - 1)$, where $\Delta_{22}\theta < 1$, into the solution of the unique equation (W_0).

6.1.1 Bianchi's preference for Gauss' method

Bianchi probably returned to the first version of Weingarten's equation also for his preference for Gauss' method, which he had taken as the one basis of his research in differential geometry. Serge Finikoff (1883-1964) provided an interesting comparison of Bianchi's and Darboux's approach to geometry. In his opinion, Darboux was *"an exuberant source of ideas"*, who had an ad hoc approach to problems, each of which is solved by the most suitable method that was tailored on it.[13] On the other hand,

> Bianchi is extraordinarily modest in the tools he uses; essentially, it is only one method. To determine a surface, he almost always resorts to the differential quadratic forms of Gauss, making use of Christoffel's symbols... He chooses the straightest path, without artifice, I would say almost elementary.[14]

Bianchi's position can be considered to be a limiting one. In this respect, a passage from a letter from Weingarten to Bianchi focused on a noteworthy aspect. On 30th January 1889, Weingarten wrote:

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[12] Since $dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$, one also has $\frac{\partial x}{\partial u} = \bar{\boldsymbol{x}}\frac{\partial^2 \varphi}{\partial u \partial v} + \bar{\boldsymbol{X}}\frac{\partial^2 \varphi}{\partial u^2}$. According to the Legendre transformation, $\alpha = \frac{\partial \varphi}{\partial u}$ and $\beta = \frac{\partial \varphi}{\partial v}$ and the preceding equation becomes $\frac{\partial x}{\partial u} = \bar{\boldsymbol{x}}\frac{\partial \beta}{\partial u} + \bar{\boldsymbol{X}}\frac{\partial \alpha}{\partial v}$ that gives $\bar{\boldsymbol{x}} = \frac{\partial x}{\partial \beta}$ and $\bar{\boldsymbol{X}} = \frac{\partial x}{\partial \alpha}$.

[13] (Blaschke 1954, p. 45)

[14] *"Il Bianchi è straordinariamente modesto nei mezzi usati; essenzialmente non si tratta che d'un metodo solo. Per determinazione d'una superficie egli ricorre quasi sempre alle forme quadratiche differenziali di Gauss, facendo uso dei simboli di Christoffel... Egli sceglie la via più dritta, senza artifici, direi quasi elementare"* (Blaschke 1954, p. 45).

> It seems that you have now also accepted the use of Γ^a_{bc}, a way of looking at things that Gauss introduced [...].[15]

Thus, Bianchi's severity in deciding which tools were admissible or not initially also affected Christoffel symbols, which Weingarten, on the other hand, regarded as having been introduced by Gauss himself.

In Bianchi's view, the choice of privileging Gauss' method had a twofold motivation: first, it had proven its usefulness to a large extent by leading to a remarkable and rapid development of surface theory; second, he preferred a "minimalist" exposition based on only a few techniques that could better highlight the geometric implications of the treatment. This choice seems to be a convincing one: Bianchi's *Lezioni* had a success that did not find a parallel in Darboux's *Leçons*, as attested to by the numerous Italian and German editions. On the occasion of the obituary of his friend Weingarten, he defended Gauss' methods from the unjust accusation of being inadequate and remarked that it had proved to be sufficient for the treatment of *any* problem in infinitesimal geometry. Then, he added:

> the importance of a research in the realm of infinitesimal geometry depends much more on the depth of the geometrical thought that characterises it, than on the analytical guise of which the author is pleased to dress it, according to his taste.[16]

In that circumstance, Bianchi explicitly referred to Darboux's mobile trihedron, but this judgement can certainly be extended to invariant theory and Ricci's absolute differential calculus, as is evident in his *Lezioni*. Indeed, as he briefly discussed the mobile trihedron,[17] in the same way he concisely presented the fundamental aspects of invariant theory and quadratic forms in (Bianchi 1902a, Chap.2) and (Bianchi 1922, Chap.2). Bianchi was aware that the use of these notions was now indispensable for a systematic development of differential geometry. However, as explicitly stated in the first lines of (Bianchi 1902a, Chap.2) or (Bianchi 1922, Chap.2), he limited himself to introducing only what was *"strictly necessary"*. Therefore, Bianchi did not completely welcome these notions or, consequently, Ricci's studies, which were based precisely on the concept of invariance.

All these aspects suggest that Bianchi preferred Weingarten's method based on equation (W_0) to that based on equation (W), as it did not make use of either the Darboux's trihedron or the invariant theory.

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[15] "Sie haben wie es scheint, jetzt auch mehr den Gebrauch der Γ^a_{bc} acceptirt, eine Betrachtungsweise, die Gauss eingefuhrt hat [...]" (Bianchi 1959, p. 221) We changed the original notation $\left\{ {bc \atop a} \right\}$ into the modern one Γ^a_{bc}.

[16] "L'importanza di una ricerca di geometria infinitesimale dipende ben più dalla profondità del pensiero geometrico che la informa, che non dalla veste analitica di cui l'autore, a seconda delle sue preferenze, si compiace di rivestirlo." (Bianchi 1910a, p. 221)

[17] A summary of Darboux's mobile trihedron theory works as a preface for Weingarten's new method of applicability in (Bianchi 1903, pp. 176–183) and constitutes a dedicated appendix in (Bianchi 1922).

6.1.2 Bianchi's applications of equation (W_0)

Instead of new tools, Bianchi introduced new connections between the different chapters of differential geometry, providing a unitary aspect to his research. This led him to numerous discoveries and to multiple demonstrations of the same theorem using different geometric theories. The interdependence existing between Bianchi's memoirs clearly emerges on the occasion of the publication of the ten volumes of his *Opere*, each of which is dedicated to a different theme. Commentators on the single volumes often lamented the impossibility of clearly separating the publications by subject.

The search for connections, however, was not Bianchi's ultimate goal, but a means to achieve other results:

> Bianchi had succeeded in establishing such a dense network of lines of communication between the various chapters of differential geometry that any progress in one of them was reflected in improvements in all the others; and theorems flourished in his hands with such abundant ease that even the results of a single memoir would often require a too long enumeration to be remembered.[18]

Bianchi's ability to connect notions from different fields of geometry is also evident in his applications of Weingarten's method, most of which date back to the period 1896-1902. The first memoirs,[19] (Bianchi 1896b) and (Bianchi 1896a), were inspired by (Darboux 1896, pp. 322–323), where Darboux applied Weingarten's method to the case of surfaces with constant curvature. Here, Bianchi connected the equation (W_0) for surfaces that are applicable to surfaces of revolution with his complementary transformation. In particular, if a surface is obtained by applying the complementary transformation to a pseudosphere, the surface that corresponds to it via Weingarten's method has different properties, which depend on the congruence chosen on the surface to calculate the transformation.

In (Bianchi 1899b), Weingarten's method was linked to Guichard's theorems on the deformation of quadrics of revolution. This was the beginning of an intense work that led him to deal with the general theory of deformation of quadrics, which was defined as a *"heroic phase of differential geometry"*.[20]

The origin of the demonstration that was presented at the beginning of Section 6.1 can be found in (Bianchi 1902c), which contains Bianchi's in-depth study of cyclic systems and their connection to the problem of applicability.

Finally, we draw attention to (Bianchi 1899a), where Weingarten's method is extended to non-Euclidean geometry. This is discussed in the next section.

[18]"Tra i singoli capitoli della geometria differenziale il Bianchi era riuscito a stabilire un così fitto reticolato di linee di comunicazione, che ogni progresso in uno di essi si rifletteva in perfezionamenti in tutti gli altri; e i teoremi gli fiorivano fra le mani con sì abbondevole facilità che spesso anche i risultati di una sola memoria ad esser ricordati richiederebbero una troppo lunga enumerazione." (Scorza 1930, p. 28)

[19]Their content was also discussed by Weingarten himself in (Bianchi 1959, pp. 273–274).

[20](Bianchi 1952-1959, vol. IV.1, p.1)

6.1.3 Weingarten's method in non-Euclidean geometry

By looking at Bianchi's original work, a modern reader will probably be surprised by an abundance of investigations concerning surfaces with special properties (such as pseudo-spherical, minimal or isothermal surfaces), rather than more general studies. In 1958, when the eighth volume of Bianchi's *Opere* entitled *Special classes of surfaces* appeared, geometry was already quite unfamiliar with such specific studies, as Mario Villa's[21] words attests:

> In a period such as ours, in which mathematics is increasingly asserting its abstract position—so wittily defined by Russell—always towards wider horizons, this classical Euclidean metric geometry, of which Bianchi was certainly one of the greatest exponents, brings us back, so to speak, to the concrete problems of physical geometry, which respond to our immediate intuition.[22]

Geometric intuition is fundamental to Bianchi's process of discovery:

> Bianchi never tackled a calculation, even the simple one, if he was not guided by geometric intuition.[23]

This attitude led him to a multitude of results, sometimes quite specific but often progressively generalised. The extremely detailed and concrete study is therefore not secondary, but rather one of the strengths of Bianchi's research that helped his contemporaries to familiarise themselves with the contents of the discipline. Indeed, it must also be considered that Bianchi was one of the pioneers of the discipline: his work, which often brought mathematics closer to a sort of experimental science, allowed him to explore new and still little-known fields. Bianchi's taste for specific examples was undoubtedly part of the needs of the time and was shared with Darboux, among others. In addition, Mario Villa, who edited the preface to the eighth volume of Bianchi's works, noted: *"In algebraic geometry, in a parallel phase of development, the same taste for particular cases can be found"*.[24] In this regard, the following words from a Guido Castelnuovo speech on the History of Algebraic Geometry may be clarifying:

[21] Mario Villa (1907-1973) graduated in mathematics from the University of Pavia in 1930. He spent his academic career mainly in Bologna where he taught from 1939. He worked on projective geometry of algebraic entities, geometry in the complex field and projective-differential geometry. See (Anonymous 1974).

[22] "In un periodo come il nostro in cui la matematica va sempre più affermando la sua posizione astratta - così argutamente definita da Russell - sempre verso più larghi orizzonti, questa geometria metrica euclidea classica, di cui il Bianchi fu certamente uno dei maggiori esponenti, ci riporta, per così dire, ai problemi concreti della geometria fisica, di quella che risponde alla nostra immediata intuizione."(Bianchi 1952-1959, vol. 8, p. 3)

[23] "Bianchi non affrontava mai un calcolo, anche semplice, se non era guidato dalla intuizione geometrica" (Fubini 1928, p. 48)

[24] "Nella geometria algebrica, in una fase di sviluppo parallela, si riscontra lo stesso gusto del particolare." (Bianchi 1952-1959, vol. 8, p. 1)

> The study of algebraic entities progressed thanks to the alternation of research on general entities and on particular entities, which provided examples on which to test or divine properties that could be extended to other entities.[25]

Moreover, on that occasion, Castelnuovo also urged against the abandonment of geometric intuition since it was an effective tool for guiding research through the *"dark wood"*:

> The problem of solving an equation by means of rational functions of parameters, of which I spoke earlier, or by means of pre-established irrationalities, is suggested by the grand views of Galois. Certainly the search presents arduous difficulties. It may be that the most suitable way to carry it out has not yet been discovered. But to renounce geometric intuition, the only way that has so far allowed us to find our way through this intricate territory, would be to extinguish the tenuous light that can guide us through the dark wood.[26]

However, it must be emphasised that, while the use of intuition became a serious methodological problem for the Italian school of algebraic geometry, Bianchi's research, and that of the Italian school of differential geometry, was not exposed to such criticism. Bianchi always rigorously justified his results through precise analytic exposition.

Focusing on practical examples also enabled dealing with specific PDEs and finding strategies to solve them, which was a quite urgent problem at the time. Bianchi's own career was launched on the international scene when, at the age of just 23, he discovered the *complementary transformation*, which can be considered as a method to integrate the equation of surfaces with constant negative curvature, since it generates an infinite family of its solutions. By combining his analytical skills and and his geometric intuition, he also achieved solutions of other differential equations during his career.

Applications of the equation (W_0) in non-Euclidean geometry also address this type of problem by noting that the search for certain complete classes of Euclidean applicable surfaces can be carried over to that of non-Euclidean surfaces, whose radii of curvature satisfy an equation (W_0). Bianchi did not consider non-Euclidean spaces as a mere generalisation of Euclidean ones. For example, he often created a parallel between Euclidean and non-Euclidean geometry and exploited it to obtain new results for Euclidean geometry or non-Euclidean geometry. In this respect, here is a clear example of how Bianchi's research aimed *"to compose the whole in a unitary body of doctrines that would be increasingly rich*

[25]"Lo studio degli enti algebrici è progredito grazie all'alternarsi di ricerche sopra enti generali e sopra enti particolari, i quali fornivano esempi su cui saggiare o divinare proprietà estendibili ad altri enti."(Castelnuovo 1929, p. 194)

[26]"Il problema di risolvere una equazione mediante funzioni razionali di parametri, del quale poc'anzi vi parlavo, o mediante irrazionalità prestabilite, è suggerito dalle grandiose vedute di Galois. Certo la ricerca presenta ardue difficoltà. Può darsi che la via più adatta per eseguirla non sia ancora scoperta. Ma rinunziare all'intuizione geometrica, la sola che abbia permesso sinora di orientarsi in questo territorio intricato, vorrebbe dire spegnere la tenue fiammella che può guidarci nell'oscura foresta."(Castelnuovo 1929, p. 201)

in mutual comparisons and reflections between the individual parts".[27]

Bianchi had dealt with non-Euclidean geometry since his first publication (Bianchi 1878), where he extended the problem of applicability to surfaces embedded into spaces with constant curvature.

As Bianchi's handwritten notes in the Archive of the Scuola Normale Superiore in Pisa attest,[28] Bianchi began working on applications of the (W_0) equation, which were later published in (Bianchi 1899a) in 1896. In the 1890s, he was fervently working on non-Euclidean geometries. During his university lectures in the academic year 1894/1895 he had expounded the Riemannian theory of constant curvature manifolds of dimension n, which were finally included in the last two chapters of (Bianchi 1899c) at first and in (Bianchi 1903) subsequently. In particular, among Bianchi's original results, which also attest to his constant attention to questions concerning applicability, there is a new proof for the applicability of two manifolds with the same constant curvature, and the deduction of the Gauss equation and the MCE for manifolds from Christoffel's formulae for the equivalence of quadratic differential forms.[29] In 1896, he dealt with surfaces with null absolute curvature in spaces of constant curvature[30] and published (Bianchi 1894-1895), which was followed by (Bianchi 1896c). This was also the year in which he used Weingarten's method in the aforementioned memoirs (Bianchi 1896b) and (Bianchi 1896a). Probably, the conjunction of these studies led Bianchi to apply Weingarten's method firstly to surfaces with null absolute curvature in spaces of constant curvature.

The handwritten notes on the contents of (Bianchi 1899a) are quite detailed and provide an in-depth account of both the time development of Bianchi's research and his way of working. Therefore, we offer here an analysis of (Bianchi 1899a) enriched by some details that emerge from the manuscripts.

Bianchi considered a space with constant curvature $K = \pm\frac{1}{R^2}$. Without loss of generality, K can be assumed to be $+1$ in the case of elliptic geometry and -1 in the case of hyperbolic geometry. The line element of the space is

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{\left(x^2 + y^2 + z^2 \pm \frac{1}{4}\right)^2}, \tag{6.4}$$

where the upper sign refers to elliptical space and the lower sign to hyperbolic space.[31] If x, y, z (*Riemannian coordinates*) are regarded as orthogonal Euclidean coordinates, the line elements (6.4) establish a conformal representation of non-Euclidean space on Euclidean space.[32] The non-Euclidean space can be also equipped with *Weierstrass coordi-*

[27]"per comporre il tutto in un corpo unitario di dottrine che fosse sempre più ricco di riscontri e di riflessi reciproci tra le singole parti." (Scorza 1930, p. 28)

[28]See (Bianchi 1884-1927, Fascicolo XI, p.45 ff)

[29](Bianchi 1899c, §§321, 322, 331)

[30]The *absolute curvature* is the curvature of a surface with respect to the metric of the space in which it is embedded.

[31]In this section, the upper sign always refers to the elliptical case and the lower sign to the hyperbolic case.

[32](Bianchi 1887a, p. 117)

nates x_0, x_1, x_2, x_3, such that

$$x_0 = \frac{\frac{1}{4} \mp \left(x^2+y^2+z^2\right)}{\pm\frac{1}{4}+x^2+y^2+z^2} \qquad x_1 = \frac{x}{\pm\frac{1}{4}+x^2+y^2+z^2}$$
$$x_2 = \frac{y}{\pm\frac{1}{4}+x^2+y^2+z^2} \qquad x_3 = \frac{z}{\pm\frac{1}{4}+x^2+y^2+z^2}.$$

Bianchi also considered a surface S embedded into a curved space, whose unit normal vector field is $(\xi_0, \xi_1, \xi_2, \xi_3)$, with respect to Weierstrass coordinates. He then assumed S parametrized by means of u and v, where $u = x_0$ and $v = \xi_0$. He also supposed that its reduced principal radii of curvature,[33] r_1 and r_2, satisfied Weingarten's equation

$$\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial u\partial v}\left(\frac{1}{r_1}+\frac{1}{r_2}\right) + \frac{\partial^2\varphi}{\partial v^2}\frac{1}{r_1r_2} = 0 \qquad (W_0)$$

for a fixed function $\varphi(u, v)$. Then, Weingarten's method guarantees that the following differentials

$$\begin{cases} dy_1 = x_1 d\left(\frac{\partial\varphi}{\partial u}\right) + \xi_1 d\left(\frac{\partial\varphi}{\partial u}\right) \\ dy_2 = x_2 d\left(\frac{\partial\varphi}{\partial u}\right) + \xi_2 d\left(\frac{\partial\varphi}{\partial u}\right) \\ dy_3 = x_3 d\left(\frac{\partial\varphi}{\partial u}\right) + \xi_3 d\left(\frac{\partial\varphi}{\partial u}\right) \end{cases} \qquad (6.5)$$

are exact and describe a complete family of surfaces $\boldsymbol{y} = (y_1(u, v), y_2(u, v), y_3(u, v))$ in the Euclidean space with (6.3) as line element.

Bianchi provided a geometric interpretation of this procedure. Together with S, he considered its conformal image $\bar{S}$ in Euclidean space. By applying Weingarten's method to $\bar{S}$ and S, he proved that they both determine the same complete class of applicable surfaces. On this point, Bianchi commented:[34]

> Weingarten's method, when applied to elliptic or hyperbolic geometry, yields nothing new in terms of its actual content, as we have shown above, but the usefulness of the transformation thus achieved will be evident from the applications that we now propose to carry out. In these, we will see the identity of apparently very different problems, or we will establish new results of non-Euclidean geometry.[35]

[33] *Reduced radii of curvature* are non-Euclidean generalisations for principal radii of curvature. The interested reader can find their detailed definition and geometric interpretation in (Bianchi 1887a, §§3-4).

[34] This quotation also highlights the following aspect of Bianchi's research:

> the simplest analytical facts receive the most unexpected geometrical applications; problems that seem insurmountable are solved with extreme ease because geometrical observation transforms and simplifies them in unexpected ways. (Fubini 1928, p. 38)

[35] "Il metodo di Weingarten, applicato alla geometria ellittica od iperbolica, non dà nulla di nuovo quanto all'effettivo contenuto, come sopra abbiamo dimostrato; l'utilità della trasformazione così conseguita risulterà però evidente dalle applicazioni che ora ci proponiamo di svolgere. In queste, o vedremo mostrarsi l'identità di problemi in apparenza ben diversi, o stabiliremo nuovi risultati di geometria non euclidea." (Bianchi 1899a, §5)

In the case of surfaces with absolute null curvature, one has $\frac{1}{r_1 r_2} = \mp 1$.[36] This condition corresponds to the equation (W_0) by posing[37]

$$\varphi(u, v) = \frac{u^2 \pm v^2}{2}.$$

For these values of $\varphi(u, v)$ and by choosing polar coordinates $u = \rho \cos\theta, v = \rho \sin\theta$, (6.3) became

$$ds^2 = (1 - \rho^2)d\rho^2 + \rho^2 d\theta^2 \qquad ds^2 = (1 + \rho^2)d\rho^2 + \rho^2 d\theta^2, \tag{6.6}$$

for the elliptic and hyperbolic case, respectively. Hence, (6.6) correspond, respectively, to the imaginary paraboloid (5.12) and the real paraboloid (5.14) that were found in Section 5.3.1. Thus, they define two complete classes of applicable surfaces, whose finite equations were completely determined.

Vice versa, any Euclidean surface whose line element can assume a form in (6.6) can be associated with a surface of zero absolute curvature in a space of constant curvature. Indeed, when $\Sigma : (\rho, \theta) \mapsto \boldsymbol{y}(\rho, \theta) = (y_1(\rho, \theta), y_2(\rho, \theta), y_3(\rho, \theta))$ is a surface with line element[38] $ds^2 = (1 - \rho^2)d\rho^2 + \rho^2 d\theta^2$, the corresponding non-Euclidean surface is defined through the equations

$$\begin{cases} x_0 = \rho \cos\theta \\ \xi_0 = \rho \sin\theta \\ x_i = \cos\theta \dfrac{\partial y_i}{\partial \rho} - \sin\theta \dfrac{\partial y_i}{\partial \theta} \\ \xi_i = \sin\theta \dfrac{\partial y_i}{\partial \rho} + \cos\theta \dfrac{\partial y_i}{\partial \theta} \end{cases}$$

since (6.5) holds true. In addition, Rodrigues' formulas for reduced radii of curvature prove that the corresponding surface has zero curvature.[39]

This result was not new. Bianchi had already shown how surfaces with absolute null curvature in spaces with constant curvature could all be obtained in finite terms,[40] *"but here their determination appears under a new aspect by connecting it to the already solved problem of finding the ordinary [Euclidean] linear element surface (24) or (24*)* [for us, (6.6)]*"*.[41]

Probably stimulated by this result, Bianchi repeated the procedure with minimal surfaces in elliptic and hyperbolic spaces, for which he had proven that $\frac{1}{r_1} + \frac{1}{r_2} = 0$ holds.[42]

[36] In general, one has $\frac{1}{r_1 r_2} = k \mp \frac{1}{R^2}$, where k is the absolute curvature.

[37] Indeed, in this case one has

$$\frac{\partial^2 \varphi}{\partial u^2} = \frac{\partial^2 \varphi}{\partial v^2} = 1 \qquad \frac{\partial^2 \varphi}{\partial u \partial v} = 0.$$

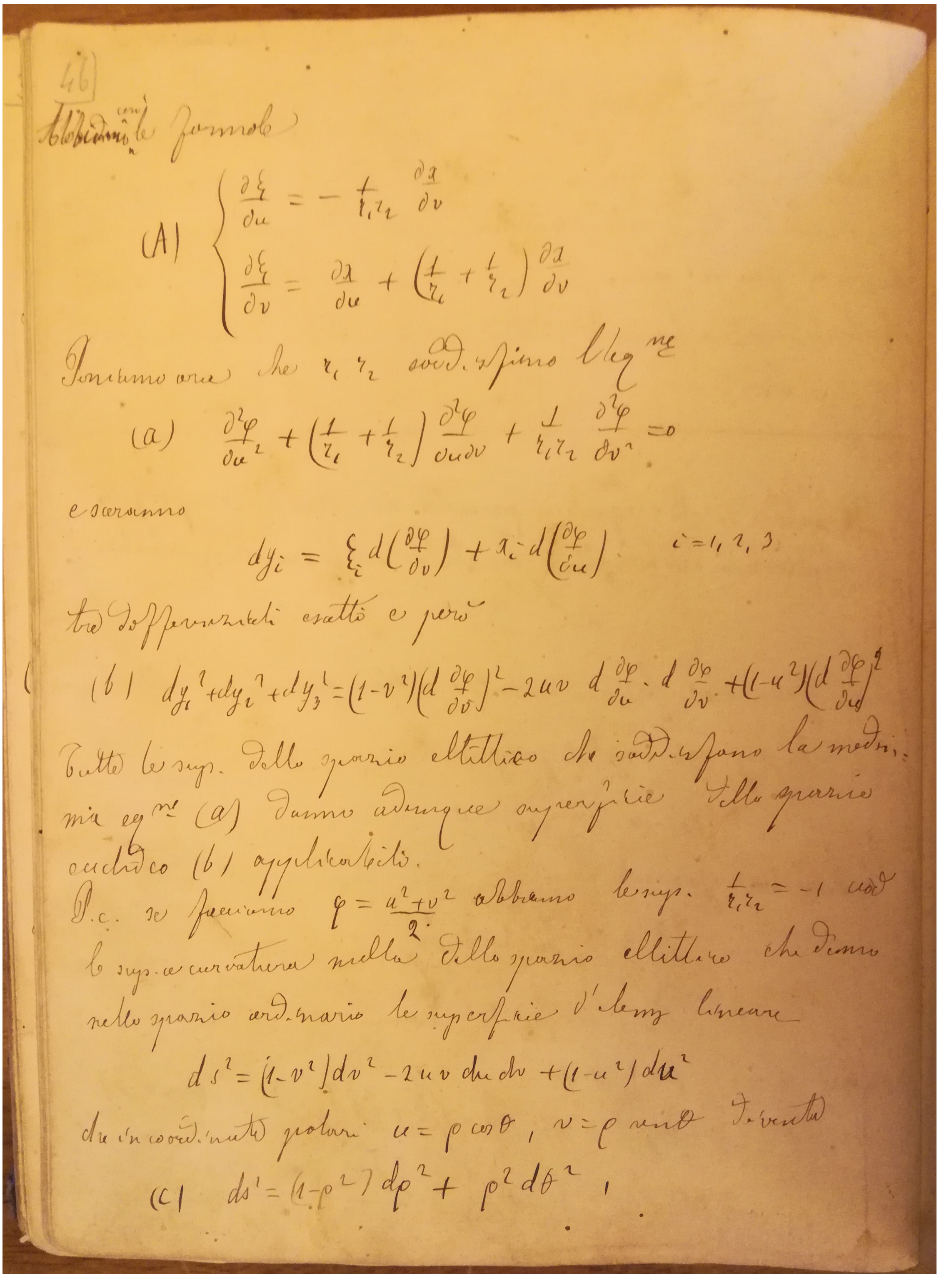
[38] The procedure is similar for $ds^2 = (1 + \rho^2)d\rho^2 + \rho^2 d\theta^2$.

[39] For a detailed exposition, see Figures 6.3 and 6.4.

[40] See (Bianchi 1894-1895) and (Bianchi 1896c).

[41] "Ma qui la loro determinazione appare sotto nuovo aspetto collegandosi al problema già risoluto di trovare le ordinarie superficie d'elemento lineare (24) o (24*)." (Bianchi 1899a, p. 124)

[42] (Bianchi 1887b, pp. 222–223)



46

Abbiamo le formole

$$(A)\quad \begin{cases} \dfrac{\partial \xi}{\partial u} = -\dfrac{1}{r_1 r_2}\dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial \xi}{\partial v} = \dfrac{\partial x}{\partial u} + \left(\dfrac{1}{r_1}+\dfrac{1}{r_2}\right)\dfrac{\partial x}{\partial v} \end{cases}$$

Poniamo ora che $r_1\, r_2$ soddisfino l'eq.ne

$$(a)\quad \frac{\partial^2\varphi}{\partial u^2} + \left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{\partial^2\varphi}{\partial u\,\partial v} + \frac{1}{r_1 r_2}\frac{\partial^2\varphi}{\partial v^2} = 0$$

e saranno

$$dy_i = \xi_i\, d\left(\frac{\partial\varphi}{\partial v}\right) + x_i\, d\left(\frac{\partial\varphi}{\partial u}\right) \qquad i=1,2,3$$

tre differenziali esatti e però

$$(b)\quad dy_1^2+dy_2^2+dy_3^2 = (1-v^2)\left(d\frac{\partial\varphi}{\partial v}\right)^2 - 2uv\, d\frac{\partial\varphi}{\partial u}\cdot d\frac{\partial\varphi}{\partial v} + (1-u^2)\left(d\frac{\partial\varphi}{\partial u}\right)^2$$

Tutte le sup. dello spazio ellittico che soddisfano la medesima eq.ne (a) danno adunque superficie dello spazio euclideo (b) applicabili.

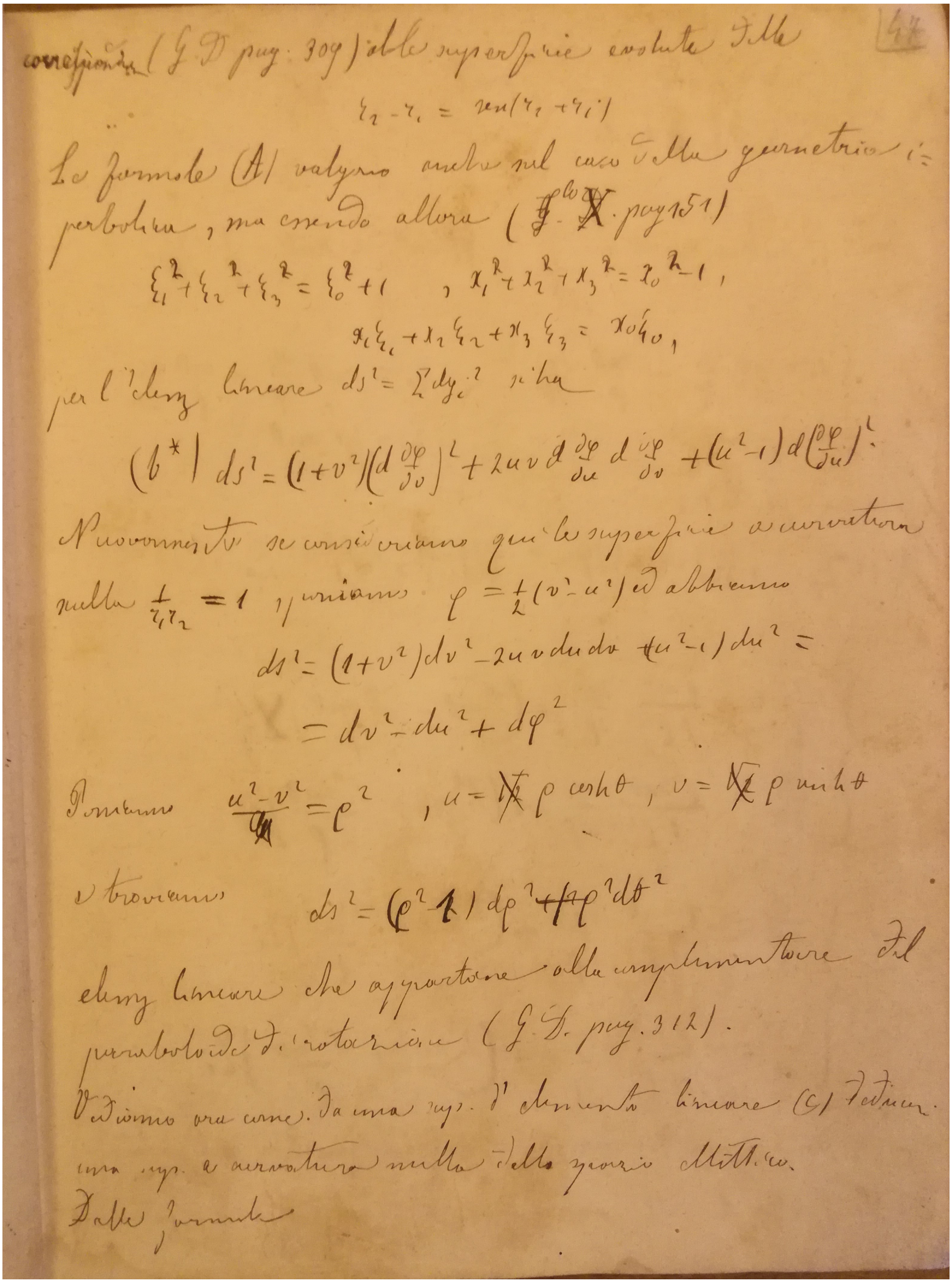
P.e. se facciamo $\varphi = \frac{u^2+v^2}{2}$ abbiamo le sup. $\frac{1}{r_1 r_2} = -1$ cioè le sup. a curvatura nulla dello spazio ellittico che danno nello spazio ordinario le superficie d'elem. lineare

$$ds^2 = (1-v^2)\,dv^2 - 2uv\,du\,dv + (1-u^2)\,du^2$$

che in coordinate polari $u=\rho\cos\theta$, $v=\rho\,\mathrm{sen}\,\theta$ diventa

$$(c)\quad ds^2 = (1-\rho^2)\,d\rho^2 + \rho^2\,d\theta^2,$$

Figure 6.1: Picture taken from Bianchi's notebook "Fascicolo 11, 1896-1897", Fondo Luigi Bianchi, Archivio della Scuola Normale Superiore di Pisa. Here and in the following pictures, we can read Bianchi's investigation on surfaces with null absolute curvature in elliptic space (see also (Bianchi 1899a))



corrispondenza (G. D. pag. 309) alle superficie evolute delle

$\xi_2 - r_1 = \text{sen}(r_1 + r_1')$

Le formole (A) valgono anche nel caso della geometria iperbolica, ma essendo allora (G. D. X pag 151)

$$\xi_1^2+\xi_2^2+\xi_3^2 = \xi_0^2+1, \quad x_1^2+x_2^2+x_3^2 = x_0^2-1,$$

$$x_1\xi_1+x_2\xi_2+x_3\xi_3 = x_0\xi_0,$$

per l'elem. lineare $ds^2 = \sum dy_i^2$ si ha

$$(b^*) \quad ds^2 = (1+v^2)\left(d\frac{\partial\varphi}{\partial v}\right)^2 + 2uv\, d\frac{\partial\varphi}{\partial u}\, d\frac{\partial\varphi}{\partial v} + (u^2-1)\, d\left(\frac{\partial\varphi}{\partial u}\right)^2.$$

Nuovamente se consideriamo qui le superficie a curvatura nulla $\frac{1}{r_1 r_2} = 1$, poniamo $\varphi = \frac{1}{2}(v^2-u^2)$ ed abbiamo

$$ds^2 = (1+v^2)dv^2 - 2uv\,du\,dv + (u^2-1)du^2 =$$
$$= dv^2 - du^2 + d\varphi^2$$

Poniamo $\frac{u^2-v^2}{} = \rho^2$, $u = \rho\cosh\theta$, $v = \rho\sinh\theta$

e troviamo $ds^2 = (\rho^2-1)\,d\rho^2 + \rho^2 d\theta^2$

elem. lineare che appartiene alla complementare del paraboloide di rotazione (G. D. pag. 312).

Vediamo ora come da una sup. d'elemento lineare (C) si deduca una sup. a curvatura nulla dello spazio ellittico.

Dalle formole

Figure 6.2: Picture taken from Bianchi's notebook "Fascicolo 11, 1896-1897", Fondo Luigi Bianchi, Archivio della Scuola Normale Superiore di Pisa.

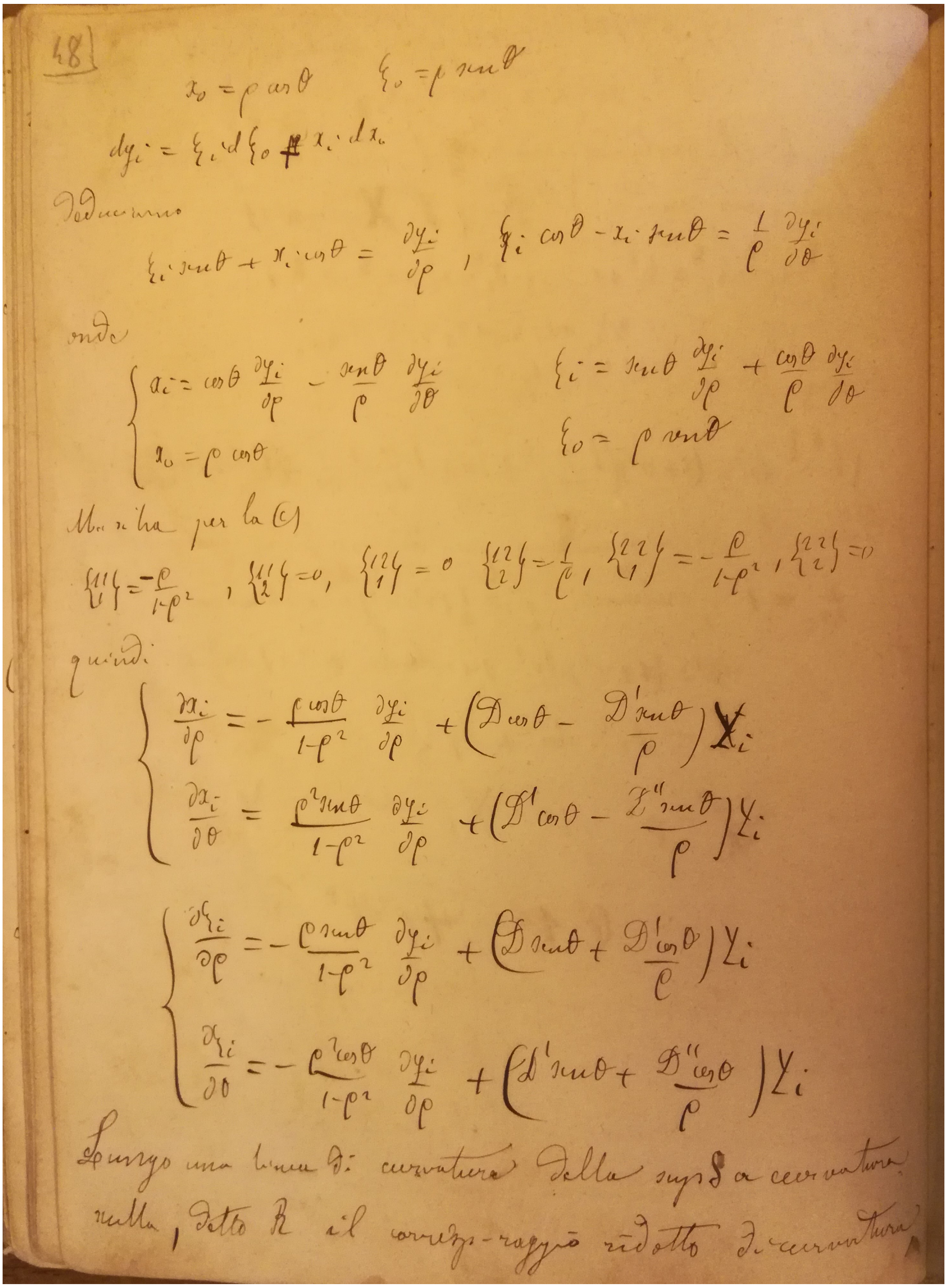
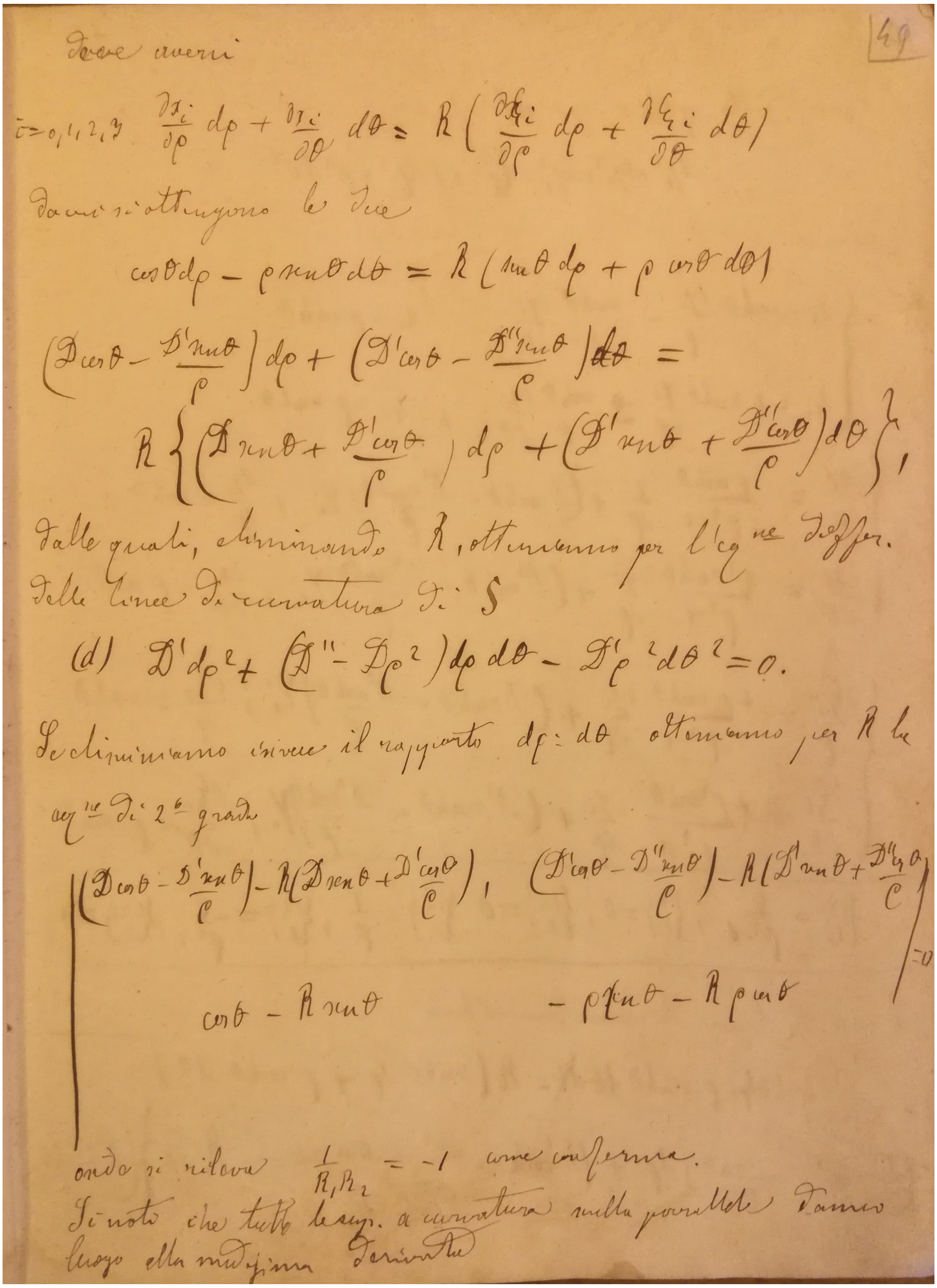


Figure 6.3: Picture taken from Bianchi's notebook "Fascicolo 11, 1896-1897", Fondo Luigi Bianchi, Archivio della Scuola Normale Superiore di Pisa.


49

dove avremo

$i=0,1,2,3 \quad \frac{\partial x_i}{\partial\rho}d\rho + \frac{\partial x_i}{\partial\theta}d\theta = R\left(\frac{\partial\xi_i}{\partial\rho}d\rho + \frac{\partial\xi_i}{\partial\theta}d\theta\right)$

da cui si ottengono le due

$$\cos\theta\, d\rho - \rho\,\mathrm{sen}\,\theta\, d\theta = R(\mathrm{sen}\,\theta\, d\rho + \rho\cos\theta\, d\theta)$$

$$\left(D\cos\theta - \frac{D'\mathrm{sen}\,\theta}{\rho}\right)d\rho + \left(D'\cos\theta - \frac{D''\mathrm{sen}\,\theta}{\rho}\right)d\theta = R\left\{\left(D\,\mathrm{sen}\,\theta + \frac{D'\cos\theta}{\rho}\right)d\rho + \left(D'\mathrm{sen}\,\theta + \frac{D''\cos\theta}{\rho}\right)d\theta\right\},$$

dalle quali, eliminando R, otteniamo per l'eq^ne differ. delle linee di curvatura di S

(d) $D'd\rho^2 + (D'' - D\rho^2)d\rho\, d\theta - D'\rho^2 d\theta^2 = 0.$

Se eliminiamo invece il rapporto $d\rho : d\theta$ otteniamo, per R le eq^ne di 2° grado

$$\begin{vmatrix} \left(D\cos\theta - \frac{D'\mathrm{sen}\,\theta}{\rho}\right) - R\left(D\,\mathrm{sen}\,\theta + \frac{D'\cos\theta}{\rho}\right), & \left(D'\cos\theta - \frac{D''\mathrm{sen}\,\theta}{\rho}\right) - R\left(D'\mathrm{sen}\,\theta + \frac{D''\cos\theta}{\rho}\right) \\ \cos\theta - R\,\mathrm{sen}\,\theta & -\rho\,\mathrm{sen}\,\theta - R\rho\cos\theta \end{vmatrix} = 0$$

onde si rileva $\frac{1}{R_1 R_2} = -1$ come conferma.

Si noti che tutte le sup. a curvatura nulla parallele danno luogo alla medesima derivata

Figure 6.4: Picture taken from Bianchi's notebook "Fascicolo 11, 1896-1897", Fondo Luigi Bianchi, Archivio della Scuola Normale Superiore di Pisa.

In this case, one has[43] $\varphi(u,v) = uv$ and (6.3) becomes

$$ds^2 = (1 \mp v^2)du^2 \mp uvdudv \pm (1-u^2)dv^2,$$

which admits a Liouville form[44]

$$ds^2 = 4^{-1}(\cos\tilde{u} + \cos\tilde{v})(\cos\tilde{u}du^2 + \cos\tilde{v}dv^2)$$
$$ds^2 = 4^{-1}(\sinh\tilde{u} + \sinh\tilde{v})(\sinh\tilde{u}du^2 + \sinh\tilde{v}dv^2) \qquad (6.7)$$

where the first corresponds to the elliptic case and the second to the hyperbolic case. This result would be insignificant if Bianchi had not reduced the determination of minimal surfaces into elliptic space to the solutions of the equation of surfaces with constant positive curvature into Euclidean space.[45] Hence, Weingarten's method gave Bianchi the opportunity to reinterpret a Euclidean problem via hyperbolic geometry: the problem of integrating

$$\frac{\partial^2\theta}{\partial\alpha^2} + \frac{\partial^2\theta}{\partial\beta^2} = -\sinh 2\theta$$

is equivalent to finding surfaces with (6.7) as line element. At that time, this connection was probably considered particularly interesting because this equation was still a real enigma. While a good theory for the pseudosphere equation was well established, an effective method for integrating the equation of surfaces with constant positive curvature was lacking. Bianchi succeeded in finding a good transformation for surfaces with positive curvature that could parallel that for the pseudosphere after reading Guichard's results, which were announced in the *Comptes rendus* of the Académie in Paris on 23th January 1899. His enthusiasm for this discovery was such that he wrote in a manuscript: *"thus, the much searched transformation of surfaces with constant positive curvature is found! on 2 March 1899"*.[46]

According to the manuscripts, Bianchi returned to the subject towards the end of the two-year period 1896-1897 when he considered surfaces with constant mean curvature, that is, surfaces with $\frac{1}{r_1} + \frac{1}{r_2} = cost$. In particular, he considered the case of surfaces in hyperbolic space with

$$\frac{1}{r_1} + \frac{1}{r_2} = 2. \qquad (6.8)$$

Once again, this class was not new for Bianchi, since in (Bianchi 1887b, p. 235) he proved they are applicable to minimal Euclidean surfaces. Condition (6.8) corresponds to equation

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[43] Indeed, in this case one has

$$\frac{\partial^2\varphi}{\partial u^2} = \frac{\partial^2\varphi}{\partial v^2} = 0 \qquad \frac{\partial^2\varphi}{\partial u\partial v} = 1.$$

[44] The change of variables is $u = \cos\frac{\tilde{u}+\tilde{v}}{2}$ and $v = \sin\frac{\tilde{u}-\tilde{v}}{2}$ in the case of elliptic space and $u = \cosh\frac{\tilde{u}+\tilde{v}}{2}$ and $v = \sinh\frac{\tilde{u}-\tilde{v}}{2}$ in the case of hyperbolic space.

[45] (Bianchi 1887b, §2)

[46] "è trovata dunque la tanto cercata trasformazione delle superficie a curvatura costante positiva! il 2 Marzo 1899" (Bianchi 1884-1927, Fascicolo XII, p.159)

(W_0) by choosing[47] $\varphi(u,v) = uv - u^2$. In this case, (6.3) reduces to

$$ds^2 = [(2u-v)^2 - 3]du^2 + 2[uv - 2(u^2-1)]dudv + (u^2-1)dv^2. \tag{6.9}$$

As emerges from the handwritten notes, at an initial stage he was not able to reduce this line element to none of the known complete class of applicable surfaces, suggesting he had found a new class of applicable surfaces (see Figure 6.5).

For this reason, he "explored" this new class by means of a specific surface in the hyperbolic space such that $\frac{1}{r_1} + \frac{1}{r_2} = 2$.[48] During these investigations, he faced another family of surfaces, for which $r_1 + r_2 = 2$. By adapting Weingarten's method to this case, one has[49] $\varphi(u,v) = uv - v^2$ and the (6.3) reduces to

$$ds^2 = (1+v^2)du^2 + 2[uv - 2(1+v^2)]dudv + [(u-2v)^2 + 3]dv^2, \tag{6.10}$$

which was initially considered to correspond to a new class of applicable surfaces by repeating the error of the preceding case. Finally, Bianchi succeeded in correcting his mistakes and found that, in the case $\frac{1}{r_1} + \frac{1}{r_2} = 2$, the line element corresponds to

$$ds^2 = d\tilde{u}^2 + (2\tilde{u} + 2\tilde{v} - e^{2\tilde{v}})d\tilde{v}^2, \tag{6.11}$$

by means of the change of variables

$$2\tilde{u} = 2u^2 - 2uv - 2\log(u-v) - 1 \qquad \tilde{v} = \log(u-v),$$

and, in the case $r_1 + r_2 = 2$, the line element corresponds to

$$ds^2 = d\tilde{u}^2 + (2\tilde{u} + 2\tilde{v} + e^{2\tilde{v}})d\tilde{v}^2, \tag{6.12}$$

by means of

$$2\tilde{u} = 2uv - 2v^2 - 2\log(u-v) - 1 \qquad \tilde{v} = \log(u-v),$$

which correspond to two classes of applicable surfaces, whose finite equations were completely known.[50]

In particular, all the surfaces of hyperbolic space with $\frac{1}{r_1} + \frac{1}{r_2} = 2$ are now all completely known in finite terms thanks to Weingarten's method. In (Bianchi 1887b), Bianchi had also shown that they are the only surfaces in hyperbolic space (along with spheres) that have

[47] Indeed, in this case one has

$$\frac{\partial^2\varphi}{\partial u^2} = -2 \qquad \frac{\partial^2\varphi}{\partial v^2} = 0 \qquad \frac{\partial^2\varphi}{\partial u \partial v} = 1.$$

[48] (Bianchi 1884-1927, Fascicolo XII, pp. 6-18)

[49] Indeed, in this case one has

$$\frac{\partial^2\varphi}{\partial u^2} = 0 \qquad \frac{\partial^2\varphi}{\partial v^2} = -2 \qquad \frac{\partial^2\varphi}{\partial u \partial v} = 1.$$

[50] They correspond to (5.45) of Section 5.5.4.

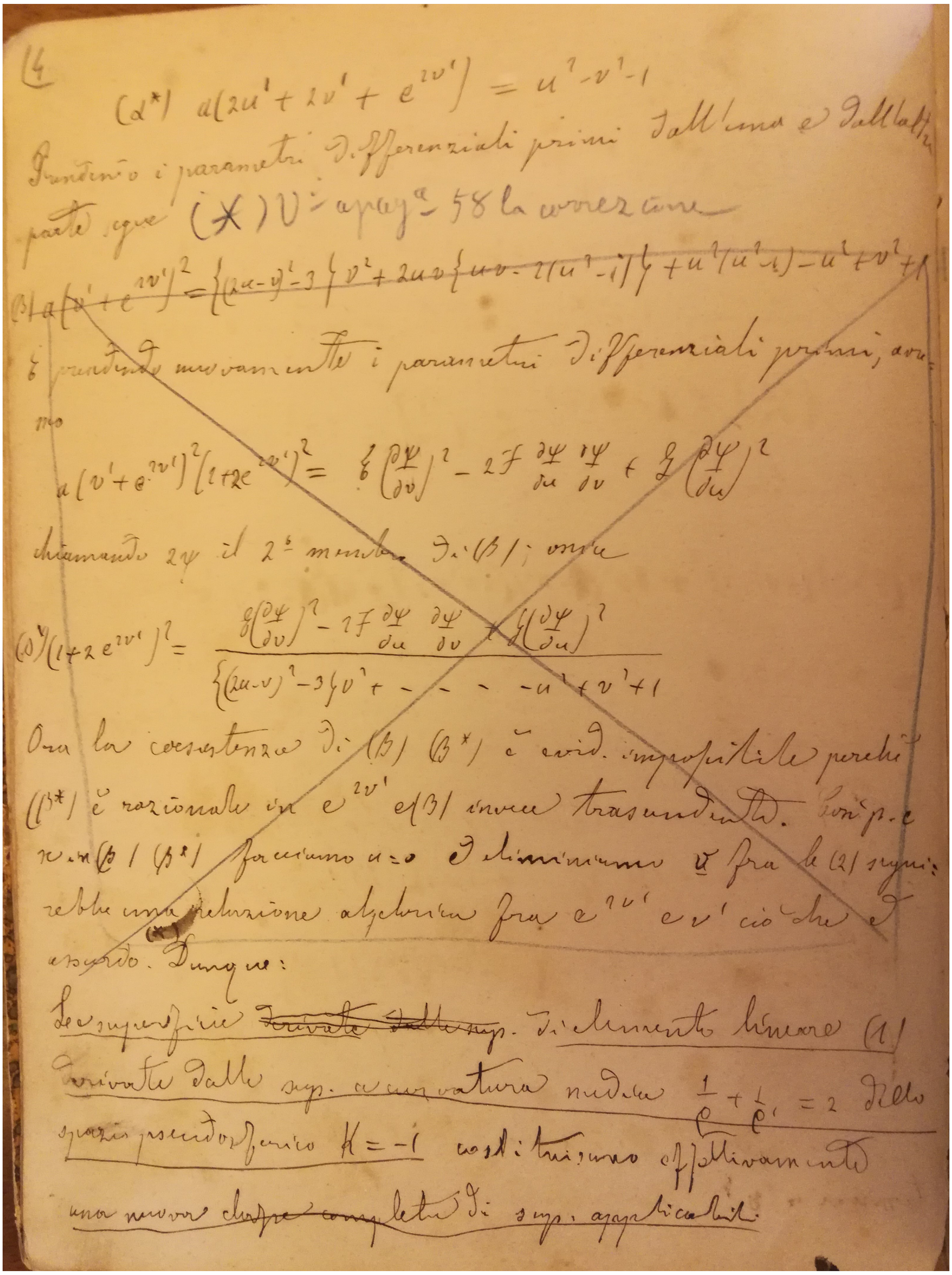

Figure 6.5: Picture taken from Bianchi's notebook "Fascicolo 12, 1898-1899", Fondo Luigi Bianchi, Archivio della Scuola Normale Superiore di Pisa. In the final lines we can read: *"Surfaces with line element* (1) *[he refers to* (6.9)*], which are derived from surfaces with mean curvature* $\frac{1}{\rho} + \frac{1}{\rho'} = 2$ *of the pseuspheric space* $K = -1$ *actually constitute a new complete class of applicable surfaces".* On this page, Bianchi indicates that a correction of this demonstration can be found on page 58 of the same Fascicolo. However, he succeeded in connecting the line elements (6.9) and (6.10) to (6.11) and (6.12) only on page 62.

isothermal curvature lines, as well as their conformal representation. In (Bianchi 1899a), these properties and Weingarten's method allowed him to determine in finite terms a new class of isothermal surfaces S, which is the conformal image in the Euclidean space of the surfaces with $\frac{1}{r_1} + \frac{1}{r_2} = 2$, that solve the following equation

$$\frac{(1+q^2)r - 2pqs + (1+p^2)t}{(1+p^2+q^2)^{3/4}} + \frac{2}{z}\left(\frac{1}{\sqrt{1+p^2+q^2}} \pm 1\right) = 0, \tag{6.13}$$

where p, q, r, s, t follow Monge's notation and S is represented as $z = z(x, y)$.

More precisely, the integration of (6.13) is reduced to the integration of Liouville's equation

$$\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} = e^{-2\theta}, \tag{6.14}$$

whose solutions are completely known.[51]

Bianchi thus succeeded in overcoming the impossibility of dealing analytically with the equation (6.13) by constructing a link between different spaces and between different problems:

> Such surfaces constitute a new class of surfaces with isothermal curvature lines. This has already been demonstrated in an earlier work of mine; but what we add here is the actual determination of all these surfaces, a result not without interest, when one considers the difficulty of the problem, which involves the general search for surfaces with isothermal lines of curvature.[52]

We have thus found a clear example of how Bianchi's and Weingarten's research intertwined: while Weingarten was inspired by the reading of (Bianchi 1887b) to develop his new method, here Weingarten's new method allowed Bianchi to complete a result obtained in (Bianchi 1887b) by finding the finite equations of surfaces that admit an isothermal system on the lines of curvature.

6.2 Ricci's interpretation of Weingarten's method

When (Weingarten 1897) appeared in the *Acta Mathematica*, Gregorio Ricci Curbastro (1853-1925)[53] was in the most active phase of his career. Ten years earlier, he had failed (no one had succeeded) to win the Royal Prize for mathematics at the *Accademia dei Lincei*, where he presented a selection of papers in which he outlined the main aspects

[51] Indeed, as we have seen in Section 5.5.4, the determination of the complete class of surfaces with (6.11) with line element can be reduced to the integration of Liouville's equation (6.14).

[52] "Tali superficie costituiscono una nuova classe di superficie a linee di curvatura isoterme. Questo trovasi già dimostrato in un mio antico lavoro; ma ciò che qui aggiungiamo di nuovo è la determinazione effettiva di tutte queste superficie, risultato non privo d'interesse, ove si consideri la difficoltà del problema che ha per oggetto la ricerca generale delle superficie a linee di curvatura isoterme." (Bianchi 1899a, p. 112)

[53] For a scientific biography of Ricci, see (Levi-Civita 1925) and (Tonolo 1954).

of a new theory that he was developing at the time, which he eventually called *absolute differential calculus*.[54]

Beginning in 1884, Ricci realised that differential geometry and several areas of mathematical physics could actually be regarded as different applications of the same theory based on quadratic forms. He therefore developed a rather complicated apparatus of definitions and algorithms that made this common basis explicit.

He based his theory on the concept of invariance. Whereas Gauss and Beltrami focused on intrinsic properties and differential invariants, on the assumption that the essential properties of surfaces are described by quantities that do not vary when the parametrization on the surface varies, Ricci, by contrast, imagined a theory in which *equations* were invariants:

> The methods, of which I have been making use for some time now [...] constitute as a whole a differential calculus, the formulas and results of which are always valid in identical form for any system of independent variables and are connected only with the nature of the line element of a manifold assumed to be fundamental. They thus seem to give us something more essential, simple and complete than the formulas of the ordinary differential calculus, [...] and which can be designated by the name of *absolute differential calculus*.[55]

In Ricci's calculus, equations always hold an identical form for any system of independent variables and depend only on the line element of a given manifold. For this purpose, equations are written in terms of particular "systems of functions", which we now call *tensors*, that vary (covariantly or contravariantly) with the chosen independent variables:

> The differential equations, which are obtained in the applications of Analysis to Geometry, Mechanics and Physics, necessarily have a character of independence of the choice of variables, which in general is not brought out by the notations commonly used because with them the form the chosen variables give to the expression of the line element of space is not taken into account. The possibility of taking this form into account depends [...] on the fact that the equations [...], precisely because of their independence of the choice of variables, always contain [certain] systems of functions [tensors].[56]

[54]The *Premio Reale per la Matematica dell'Accademia dei Lincei* was established by Umberto I in 1878. It consisted of 10,000 lire and was a *retrospective prize*.

[55]"I metodi, di cui mi valgo da qualche tempo [...] costituiscono nel loro insieme un calcolo differenziale, le cui formole e risaltati valgono sempre sotto la identica forma per qualunque sistema di variabili indipendenti e sono collegati soltanto colla natura dell'elemento lineare di una varietà assunta come fondamentale. Essi sembrano quindi darci quel qualche cosa di più essenziale, semplice e completo rispetto alle formole del calcolo differenziale ordinario,[...] e che può designarsi col nome di *Calcolo differenziale assoluto*."(Ricci Curbastro 1896, p. 1)

[56]"Le equazioni differenziali, a cui si giunge nelle applicazioni della Analisi alla Geometria, alla Meccanica e alla Fisica, hanno necessariamente un carattere di indipendenza dalla scelta delle variabili, che in generale non è messo in evidenza dalle notazioni comunemente usate perché con esse non si tiene conto della forma che le variabili prescelte danno alla espressione dell'elemento lineare dello spazio. La possibilità di tener conto di questa forma dipende [...] dal fatto che le equazioni [...], appunto per la loro indipendenza dalla scelta delle variabili, contengono sempre [determinati] sistemi di funzioni [i tensori]."(Ricci Curbastro 1888)

At the time of the 1887 Lincei Prize, Ricci's calculus already had these fundamental elements in it. In particular, he had identified Christoffel's algorithm, which he called *covariant derivation*, as the essential component of his calculus. However, the commission, which was chaired by Eugenio Beltrami, while appreciating Ricci's work, judged it insufficient to deserve the award due to the difficulty of implementing the method and the lack of results justifying its use:

> Prof. Ricci's work, rather than a sum of the ultimate results that are definitively acquired and immediately usable, represents a mighty effort that in part already appears to be leading towards an honourable goal, and in part is waiting for its final justification from further attempts, in which perhaps the primitive and highly complex analytical apparatus can be definitively replaced by simpler executive arguments.[57]

The call for *"further attempts"* prompted Ricci to fervently undertake new research, namely in the field of differential geometry and surface theory. In retrospect, his choice seems rather ingenuous. Despite the commission's request for further (i.e., new) results, he applied his calculus to a theory that was already well established. According to Ricci, differential geometry was the ideal field in which to test his calculus precisely for this reason since it gave him the opportunity to highlight the advantages of his methods over classical techniques, and to provide a concise and clear interpretation of his algorithms.[58] In his view:

> In questions concerning analysis, which are not connected with the choice of independent variables by their nature, I have for a long time made use of a tool, which I call Absolute Differential Calculus, which leads to formulae and equations that always present themselves in identical form for any system of variables. By eliminating from these questions the extraneous elements represented by the independent variables, when these are not left arbitrary at all, the methods of research assume a remarkable uniformity and spontaneity and the results a natural symmetry while, thanks also to an opportune system of notations, the same generality goes to the advantage, rather than to the disadvantage, of simplicity and evidence of the formulae and the rapidity of the deductions. And this is natural, since, if the indirect ways and expedients laboriously thought out each time bear witness to the acumen of the person who pointed them out, they also show that science has not yet found the high road that leads to the goal; which path, once discovered, is always easy and flat and

[57]"I lavori del prof. Ricci, piuttosto che una somma di ultimi risultati definitivamente acquisiti ed immediatamente utilizzabili, rappresentino un poderoso sforzo che in parte apparisce già conducente ad una meta onorevole, in parte aspetta la sua giustificazione finale da ulteriori cimenti, nei quali forse il primitivo ed assai complesso apparato analitico potrà essere definitivamente surrogato da più semplici argomenti esecutivi." (Beltrami 1889, p. 307)

[58]See (Ricci Curbastro 1893, pp. 311–312)

opens up new and wider horizons.[59]

This quote also recalls the comparison made by Finikoff between Darboux and Bianchi presented in Section 6.1.1. Just as Bianchi saw in Gaussian theory the "high road" for the study of surface theory, which, with only few elementary tools, had proved to be effective in exploring a field that was so difficult, Ricci saw in his calculus the "high road" for the study of questions related to analysis, to theory of elasticity, to theory of heat, and to differential geometry.

6.2.1 Ricci's surface theory

Ricci's main works on surface theory are (Ricci Curbastro 1893), (Ricci Curbastro 1895) and his textbook entitled *Lezioni sulla teoria delle superficie* (Ricci Curbastro 1898).

In contrast to the practice of the time, Ricci avoided the problem of the actual determination of finite equations of geometric entities. He justified his choice in this way:

> Being accustomed to the concepts of Analytical Geometry implies that any geometric entity is not considered as known until its equations are known in finite terms; but this way of representation is neither the only, nor the most appropriate, when it is a matter of studying the geometric entity in itself without relating it to others. –If the represented entity is well defined, given its analytical representation, there is an infinite number of analytical representations for the same entity that are perfectly equivalent; and even in this case the vagueness is to the detriment of elegance and conciseness. –If, however, with the aid of Differential Calculus, one eliminates the arbitrary constants or functions that represent this indeterminacy, one obtains new representations, such that there is an univocal correspondence between them and the geometric entities represented.[60]

[59]"Nelle questioni di Analisi, che per loro natura non sono collegate colla scelta delle variabili indipendenti, io mi valgo da molto tempo di uno strumento, che chiamo Calcolo Differenziale assoluto, il quale conduce a formule ed equazioni, che si presentano sempre sotto la identica forma per qualunque sistema di variabili. Eliminati da tali questioni gli elementi ad esse estranei rappresentati dalle variabili indipendenti, quando queste non siano lasciate affatto arbitrarie, i metodi di ricerca assumono una notevole uniformità e spontaneità ed i risultati una simmetria tutta loro propria, mentre, grazie anche ad un opportuno sistema di notazioni, la stessa generalità va a vantaggio, anzi che a scapito, della semplicità ed evidenza delle formole e della rapidità delle deduzioni. E ciò è naturale, dacchè, se le vie indirette e gli spedienti faticosamente pensati volta per volta fanno fede dell'acume di chi li additò, danno in pari tempo a vedere che la scienza non ha ancora trovata la via maestra, che conduce alla meta; la quale via, una volta scoperta, risulta sempre facile e piana ed apre alla vista nuovi e più larghi orizzonti." (Ricci Curbastro 1898, p. 1)

[60]"L'abitudine ai concetti della Geometria Analitica fa sì che un qualunque ente geometrico non si consideri come noto, finchè non se ne conoscano le equazioni in termini finiti; ma questo modo di rappresentazione non è nè il solo, nè il più opportuno, quando si tratti di studiare l'ente geometrico in sè stesso senza porlo in relazione con altri. –Se per esso l'ente rappresentato risulta ben definito, data la sua rappresentazione analitica, di queste invece per uno stesso ente ve ne ha un numero infinito, che si equivalgono perfettamente; ed anche in questo caso la indeterminatezza è a danno della eleganza e della concisione. –Se però col sussidio del Calcolo Differenziale si eliminano le costanti o le funzioni arbitrarie, che rappresentano

Ricci described a generic surface $S:(x^1,x^2)\mapsto y^k=(y^1(x^1,x^2),y^2(x^1,x^2),y^3(x^1,x^2))$ that is embedded in Euclidean space by the assignment of two quadratic differential forms, its first and second fundamental forms, which are, respectively,

$$\varphi=\sum_{r,s=1}^{2}a_{rs}dx^r dx^s \qquad \psi=\sum_{r,s=1}^{2}b_{rs}dx^r dx^s.$$

By means of the coefficients a_{rs} and b_{rs}, $r,s=1,2$, S is represented through the following equations, which are so-called *intrinsic equations*,

$$\begin{aligned} a_{rs}&=\sum_{h=1}^{3}\frac{\partial y^h}{\partial x^r}\frac{\partial y^h}{\partial x^s} \qquad r,s=1,2\\ D_s\left(\frac{\partial y^h}{\partial x^r}\right)&=\zeta^h b_{rs} \qquad h=1,2,3 \quad r,s=1,2 \end{aligned} \tag{6.15}$$

where D_s is the covariant derivative[61] with respect to the metric φ and $\zeta=(\zeta^1,\zeta^2,\zeta^3)$ the normal unit vector to the surface.

The intrinsic equations (6.15) can assume another form by using *congruences of lines*, a basic tool in Ricci's theory. A congruence of curves is a family of curves $f(x^1,x^2,a(x^1,x^2))$ drawn on the surface that depends on a single parameter a such that, for each point on the surface, only one curve of the family passes. Ricci did not use their description in finite terms, instead using the corresponding differential equation, of which the function f was the general integral. More precisely, Ricci specified the vector field[62] $\lambda=(\lambda^1(x^1,x^2),\lambda^1(x^1,x^2))$ of which the curve f is an integral curve.

Together with λ, Ricci introduced[63] its orthogonal unit vector field (*orthogonal canonical system*), which was defined as

$$\bar\lambda=(\bar\lambda^1,\bar\lambda^2)=\left(-\frac{1}{\sqrt{a}}\lambda_2,\frac{1}{\sqrt{a}}\lambda_1\right), \tag{6.16}$$

where a is the determinant of the matrix a_{rs} and λ_i, $i=1,2$ are the covariant components of λ, $\lambda_i=\sum_{s=1}^{2}a_{is}\lambda^s$. The definition of λ and $\bar\lambda$ allowed Ricci to introduce the covariant tensor φ_s, called *line bundle*, which is defined through the following equations[64]

$$D_s\lambda_r=\bar\lambda_r\varphi_s \qquad D_s\bar\lambda_r=-\lambda_r\varphi_s. \tag{6.17}$$

questa indeterminazione, si ottengono delle nuove rappresentazioni, tali, che vi ha corrispondenza univoca tra esse e gli enti geometrici rappresentati." (Ricci Curbastro 1898, p. 3)

[61]The covariant derivative of a tensor $T^{\beta_1\beta_2\ldots\beta_n}_{\alpha_1\alpha_2\ldots\alpha_m}$ with respect to the metric $\varphi=\sum_{r,s}a_{rs}dx^r dx^s$ and the variable x^γ is

$$\begin{aligned} T^{\beta_1\beta_2\ldots\beta_n}_{\alpha_1\alpha_2\ldots\alpha_m\gamma}:=\frac{\partial T^{\beta_1\beta_2\ldots\beta_n}_{\alpha_1\alpha_2\ldots\alpha_m}}{\partial x^\gamma}&+\Gamma^{\beta_1}_{\alpha\gamma}T^{\alpha\beta_2\ldots\beta_n}_{\alpha_1\alpha_2\ldots\alpha_m}+\Gamma^{\beta_2}_{\alpha\gamma}T^{\beta_1\alpha\ldots\beta_n}_{\alpha_1\alpha_2\ldots\alpha_m}+\cdots+\Gamma^{\beta_n}_{\alpha\gamma}T^{\beta_1\beta_2\ldots\alpha}_{\alpha_1\alpha_2\ldots\alpha_m}\\ &-\Gamma^{\alpha}_{\alpha_1\gamma}T^{\beta_1\beta_2\ldots\beta_n}_{\alpha\alpha_2\ldots\alpha_m}-\Gamma^{\alpha}_{\alpha_2\gamma}T^{\beta_1\beta_2\ldots\beta_n}_{\alpha_1\alpha\ldots\alpha_m}-\cdots-\Gamma^{\alpha}_{\alpha_m\gamma}T^{\beta_1\beta_2\ldots\beta_n}_{\alpha_1\alpha_2\ldots\alpha}. \end{aligned}$$

[62]λ is normalised, i.e., $\sum_{r=1}^{2}\lambda^r\lambda_r=1$.

[63](Ricci Curbastro 1893, p. 313)

[64]By deriving (covariantly with respect to φ) $\sum_{r=1}^{2}\lambda^r\lambda_r=1$, one obtains $\sum_{r=1}^{2}\lambda^r D_s\lambda_r=0$. Hence, there must exist a covariant tensor φ_s such that $D_s\lambda_r=\varphi_s\bar\lambda_r$, where $\bar\lambda_r$, $r=1,2$, are the covariant components of $\bar\lambda$. The same holds true for $\sum_{r=1}^{2}\bar\lambda^r\bar\lambda_r=1$.

From a geometric point of view, φ_s is the set of systems of lines depending on one parameter drawn on the surface in such a way that two lines belonging to two given systems of the bundle cut into each other at a constant angle.[65]

When any congruence of curves over the surface is given, the coefficients of the first and second fundamental forms, which constitute two symmetrical covariant systems, can be decomposed as[66]

$$a_{rs} = \lambda_r \lambda_s + \bar{\lambda}_r \bar{\lambda}_s \qquad b_{rs} = \alpha \lambda_r \lambda_s + \theta(\lambda_r \bar{\lambda}_s + \lambda_s \bar{\lambda}_r) + \beta \bar{\lambda}_r \bar{\lambda}_s. \tag{6.18}$$

To approach questions related to the immersion of surfaces in Euclidean space, Ricci translated the theory of Darboux's trihedron into his own calculus. When a system of orthogonal congruences λ and $\bar{\lambda}$ on S are given, their Euclidean tangent unit vectors $\xi = (\xi^1, \xi^2, \xi^3)$ and $\eta = (\eta^1, \eta^2, \eta^3)$, are, respectively[67]

$$\xi^h = \sum_{r=1}^{2} \lambda^r \frac{\partial y^h}{\partial x^r} \qquad \eta_h = \sum_{r=1}^{2} \bar{\lambda}^r \frac{\partial y^h}{\partial x^r}$$

where $h = 1, 2, 3$. The already defined normal vector ζ completes the trihedron.

With appropriate substitutions,[68] the intrinsic equations (6.15) of a surface S become

$$\begin{cases} \dfrac{\partial \xi^h}{\partial x^r} = +\eta^h \varphi_r + \zeta^h Q_r \\ \dfrac{\partial \eta^h}{\partial x^r} = -\xi^h \varphi_r + \zeta^h P_r \\ \dfrac{\partial \zeta^h}{\partial x^r} = -\xi^h Q_r + \eta^h P_r, \end{cases} \tag{6.19}$$

where Ricci posed $Q_r = \alpha \lambda_r + \mu \bar{\lambda}_r$ and $P_r = \mu \lambda_r + \beta \bar{\lambda}_r$ for simplicity, and φ is the bundle to which λ and $\bar{\lambda}$ belong. Finally, the immersion of the surface is given by

$$\frac{\partial y^h}{\partial x^r} = \xi^h \lambda_r + \eta^h \bar{\lambda}_r \qquad h = 1, 2, 3 \quad r = 1, 2. \tag{6.20}$$

6.2.2 Ricci's demonstration of Weingarten's method

Ricci expounded his version of Weingarten's new method first in (Ricci Curbastro 1897) and then in his *Lezioni* (Ricci Curbastro 1898, §63), at the conclusion of the final chapter dedicated to the solution of the first and second problems of applicability.[69] His interest in Weingarten's work seems to have been a natural consequence of his preoccupations of the period. In the previous years, besides including surface theory in his calculus, Ricci had tackled problems related to applicability, such as the deformation of surfaces with constant

[65] See (Ricci Curbastro 1893, §5).
[66] See (Cogliati 2022, pp. 241, 250).
[67] (Ricci Curbastro 1898, §63)
[68] (Ricci Curbastro 1898, §118)
[69] Concerning the timeline, it is possible that the drafting of (Ricci Curbastro 1897) occurred at the same time as that of (Ricci Curbastro 1898, §63), as suggested by an erroneous numeration in (Ricci Curbastro 1897, p. 87), where two formulae appear with reference to the numbering of (Ricci Curbastro 1898, §63).

curvature and the deformation of surfaces into ruled surfaces.[70] He also had somehow followed the developments that led Weingarten to win the Paris Prize, as attested by undated handwritten transcripts of (Weingarten 1891a) and (Goursat 1891).[71]

As will become clear from the following detailed exposition of the contents of (Ricci Curbastro 1897), it is therefore not surprising that, as soon as he had read (Weingarten 1897), Ricci included it in his theory of surfaces. Indeed, through the use of the moving trihedron, Weingarten's new method was well suited to Ricci's new theory and enabled him to follow the demonstration step by step as it was presented in (Weingarten 1897). Ricci also facilitated the comparison between the two demonstrations by recalling at the beginning of his memoir the changes in notations between his work and the original.[72] In addition to favouring comprehension and familiarisation with Ricci's methods, this parallel process of the two demonstrations allows us to appreciate the greater incisiveness of Ricci's proof with respect to Weingarten's.

Ricci's proof of the surface's dependence from tangent vector field $\boldsymbol{X}_1$

Before tackling the demonstration of Weingarten's theorem, Ricci explained the connection between the variation of a tangent vector of Darboux's trihedron and the deformation of surfaces by providing a demonstration that does not depend on that of equation (W).

To this end, he considered any congruence λ_r, $r = 1, 2$ and *"a spherical representation of the tangents to the lines λ_r that is analogous to that of Gauss for the normals to the surface"*.[73] Ricci denoted the line element associated to this spherical representation with χ and, by definition, one has

$$\chi = \sum_{r,s=1}^{2} e_{rs} dx^r dx^s, \tag{6.21}$$

where $e_{rs} = \frac{\partial \xi}{\partial x^r} \frac{\partial \xi}{\partial x^s}$.

Ricci essentially showed that if both a congruence of lines λ_r and its spherical representation are assigned, then the corresponding surface is also defined since the quantities Q_r, P_r, φ_r, on which its intrinsic equations (6.19) depend, are completely determined.

[70](Ricci Curbastro 1895, Chap. 2)

[71]Fondo Gregorio Ricci Curbastro, Liceo *G. Ricci Curbastro* in Lugo di Romagna (RA). According to the inventory published in (Dalmonte and Pirazzini 2002, pp. 313–334), the manuscripts mentioned correspond to the number 21.

[72]See (Ricci Curbastro 1897, pp. 81–82). With respect to 5.5.3, Darboux's moving trihedron $\boldsymbol{X}, \boldsymbol{X}', \boldsymbol{X}''$ is replaced by ξ, η, ζ; Weingarten's Euclidean coordinates $\boldsymbol{x} = (x, y, z)$ become (y^1, y^2, y^3) in Ricci. Weingarten's system (6.20) corresponds to the analogue Weingarten's system (5.30). With regard to Darboux's rotations and translations, Weingarten's notation is changed into Ricci's through the following substitutions

$$p \mapsto P_1 \qquad p_1 \mapsto P_2 \qquad q \mapsto Q_1 \qquad q_1 \mapsto Q_2$$
$$a \mapsto \lambda_1 \qquad \alpha \mapsto \lambda_2 \qquad -b \mapsto \bar{\lambda}_1 \qquad -\beta \mapsto \bar{\lambda}_2 \qquad R \mapsto \varphi_1 \qquad -R_1 \mapsto \varphi_2$$

[73]"Una rappresentazione sferica delle tangenti alle linee λ_r analoga a quella di Gauss per la normale alla superficie." (Ricci Curbastro 1897, p. 82)

Indeed, the assignment of a congruence λ defines a bundle φ_r, $r = 1, 2$ through the equations (6.17). In addition, the spherical representation (6.21) of the congruence λ_r can be written as[74]

$$\chi = \sum_{r,s=1}^{2} (\varphi_r \varphi_s + Q_r Q_s) dx^r dx^s, \tag{6.22}$$

This means that φ_r, $r = 1, 2$, are the coordinates of a system of a congruence that is drawn on the unit sphere and Q_r, $r = 1, 2$, constitute its orthogonal system, thus both Q_r and φ_r, $r = 1, 2$, can be connected to the spherical representation χ of the congruence λ_r. Finally, the congruences φ_r and Q_r define a bundle of congruences χ_r that they belong to, and Ricci showed that P_r, $r = 1, 2$, coincide with χ_r, $r = 1, 2$.[75]

Proof of Weingarten's theorem

Ricci closely followed (Weingarten 1897) to prove Weingarten's theorem. He first fixed a reduced system of coordinates on a surface S with $\varphi = \sum_{r,s=1}^{2} a_{rs} dx^r dx^s$ as first fundamental form by choosing a function z, whose derivatives are proportional to the bundle of congruences φ_r, i.e., a function z such that

$$\frac{\partial z}{\partial x^r} = \sqrt{\sigma} \varphi_r \tag{6.23}$$

holds true. In this way, one has[76] $\sigma = (\Delta_1 z)^2$ and[77]

$$\sum_{r=1}^{2} \sigma^r \bar{z}_r + 2\sqrt{\sigma^3} G = 0, \tag{6.25}$$

which express the link between Gaussian parameters of the reduced form and Gaussian curvature G as well as Weingarten's equation (5.28). In addition, (6.25) guarantees that z

[74] As a consequence of (6.19) and the orthonormality of ξ, η, ζ, one has $e_{rs} = \varphi_r \varphi_s + Q_r Q_s$. By comparing e_{rs} with a_{rs} as in (6.18), one deduces that φ_r, $r = 1, 2$, are the coordinates of a system of a congruence drawn on the unit sphere and the Q_r its orthogonal system.

[75] In the case of the sphere, the integrability conditions of the intrinsic equations (6.19) are $D_r \xi_s = D_s \xi_r$, i.e.,

$$e^{rs} Q_{rs} = -P^r \varphi_r \qquad e^{rs} \varphi_{rs} = P^r Q_r \qquad e^{rs} \bar{P}_{rs} = 1.$$

By comparing these latter with the following equations

$$e^{rs} Q_{rs} = -\chi^r \varphi_r \qquad e^{rs} \varphi_{rs} = \chi^r Q_r,$$

which are obtained by multiplying (6.17) by e^{rs}, one has that P_r coincides with χ_r.

[76] It is a consequence of the definition $(\Delta_1 z)^2 := \sum_{r=1}^{2} z^r z_r$.

[77] (6.25) is a consequence of the requirement $z_{rs} = z_{sr}$. Indeed, by deriving (6.23), one has

$$\sqrt{\sigma}(D_s \varphi_r - D_r \varphi_s) = \frac{1}{2\sqrt{\sigma}} (\sigma_s \varphi_r - \sigma_r \varphi_s). \tag{6.24}$$

As a consequence of a remarkable result in Ricci's theory, which is obtained by deriving (6.17) (see (Ricci Curbastro 1898, §41) for more details), in the left side of (6.24), one has $D_s \varphi_r - D_r \varphi_s = \sqrt{a} G$, where a is the determinant of the metric φ and G the Gaussian curvature. By multiplying and dividing the right side of (6.24) by $\sqrt{\sigma}\sqrt{a}$ and by taking into account (6.16) and (6.23), it becomes $\frac{\sqrt{a}}{2\sigma}(\sigma_r z_r + \sigma_s z_s)$. Finally, the substitution of these expressions in (6.24) gives (6.25).

and σ are independent functions and can be chosen as parameters of the surface.[78]

With respect to this parametrization, the spherical representation χ of λ becomes

$$\chi_0 = \left(\frac{1}{\sigma} + Q_1^2\right) dz^2 + 2Q_1Q_2 d\sigma dz + Q_2^2 d\sigma^2$$

by combining (6.22) and (6.23). When differential parameters are formed with respect to χ_0, Ricci obtained $\eta = \frac{\nabla(\xi,z)}{\Delta_1 z}$ (which corresponds to Weingarten's equation (5.33)) and the intrinsic equations (6.19) become

$$\begin{cases} \frac{\partial \xi}{\partial z} = \frac{\nabla(\xi,z)}{\sigma\Delta_1 z} + \zeta Q_1 & \frac{\partial \xi}{\partial \sigma} = \zeta Q_2 \\ \frac{\partial \eta}{\partial z} = -\frac{\xi}{\sigma} + \zeta P_1 & \frac{\partial \eta}{\partial \sigma} = \zeta P_2. \end{cases} \tag{6.26}$$

When ξ and η satisfy (6.26), the existence of the corresponding embedded surface depends on the integrability of

$$\frac{\partial y^h}{\partial x^r} = \xi^h \lambda_r + \eta^h \bar{\lambda}_r \qquad h = 1,2,3 \quad r = 1,2,$$

which correspond to Weingarten's equations (5.34), that is, the following equation

$$\lambda_2 Q_1 - \lambda_1 Q_2 = \bar{\lambda}_1 P_2 - \bar{\lambda}_2 P_1 \tag{6.27}$$

must hold true.[79] By an appropriate substitution of differential parameters in (6.27),[80] Ricci finally deduced Weingarten's equations

$$\lambda_1 \Delta_1 z + \bar{\lambda}_1 \sqrt{\Delta_1 z}\Delta_2 z - \left(\lambda_2 - \frac{\bar{\lambda}_2}{2\sqrt{\Delta_1 z}}\right) \Delta_1(z, \Delta_1 z) + 2\bar{\lambda}_2 \sqrt{\Delta_1 z}\Delta_{22} z. \tag{W}$$

Hence, by taking advantage of the intrinsic character of (W) Ricci concluded that the Gaussian parameter z of the reduced form of any other surface with φ as line element, satisfies the same equation (W) and $\sigma = \Delta_1 z$ holds true.

Conversely, any solution of (W) leads to surfaces with φ as line element. For the proof Ricci considered two generic functions z and σ such that $\frac{\partial z}{\partial x^r} = \sqrt{\sigma}\varphi_r$ and, consequently, $\sigma = \Delta_1^2 z$. The assignment of a congruence λ gives the following differentials

$$dy^h = \frac{\partial y^h}{\partial z} dz + \frac{\partial y^h}{\partial \sigma} d\sigma = (\xi^h \lambda_1 + \eta^h \bar{\lambda}_1) dz + (\xi^h \lambda_2 + \eta^h \bar{\lambda}_2) d\sigma$$

[78]In the considered case of non-developable surfaces, one has $G \neq 0$. Thus, as a consequence of (6.25), also the angle ω between the congruences σ_r and $\bar{z}_r$ is not zero, since $\cos\omega = \sum_{r=1}^2 \sigma^r \bar{z}_r$.

[79]Indeed, as usual, the integrability conditions are given by $\frac{\partial}{\partial x^s}\frac{\partial y^h}{\partial x^r} = \frac{\partial}{\partial x^r}\frac{\partial y^h}{\partial x^s}$. By substituting (6.19), one has

$$\zeta^h(Q_s\lambda_r + P_s\bar{\lambda}_r - Q_r\lambda_s - P_r\bar{\lambda}_s) + \eta^h(\varphi_s\lambda_r + \frac{\partial \bar{\lambda}_r}{\partial x^s} - \varphi_r\lambda_s - \frac{\partial \bar{\lambda}_s}{\partial x^r}) + \xi^h(\frac{\partial \lambda_r}{\partial x^s} - \varphi_s\bar{\lambda}_r - \frac{\partial \lambda_s}{\partial x^r} + \varphi_r\bar{\lambda}_s) = 0,$$

where the coefficients of η^h and ξ^h are automatically zero as an immediate consequence of (6.17). It is easy to note also that the coefficient of η^h and ξ^h correspond to the first two equations in (4.7) of Darboux's integrability conditions.

[80]Except for the substitutions given in the footnote 72, the differential parameters are the same as Weingarten's.

whose exactness is guaranteed by its integrability condition, which is (W) and holds true by hypothesis. Finally, the functions y_1, y_2, y_3 so defined are such that $\varphi = \sum_{i=1}^{3} y_i$ since one has $a_{ij} = \lambda_i \lambda_j + \bar{\lambda}_i \bar{\lambda}_j$.

This concludes the proof of Weingarten's theorem, which Ricci stated as follows:

Theorem 6.1. *Let S_0 be a surface parametrized by z and σ such that $z = \sqrt{\sigma}\varphi_r$ and λ be a congruence belonging to the bundle φ_r, whose tangent vector field is $\xi = (\xi_1, \xi_2, \xi_3)$. Then, every solution z of* (W) *such that $\Delta_1^2 z$ is not a function solely of z, being differential parameters formed with respect to the spherical representation of λ, corresponds to a single surface, which is applicable to S_0, whose Cartesian coordinates can be obtained from*

$$dy^h = (\xi^h \lambda_1 + \eta^h \bar{\lambda}_1)dz + (\xi^h \lambda_2 + \eta^h \bar{\lambda}_2)d\sigma,$$

by quadratures.

6.3 The case study of the 1901 Lincei Prize

At the peak of a period of intense research, in 1901 Ricci again participated in the Lincei Prize. His main competitors were Ernesto Pascal and Guido Castelnuovo and Federigo Enriques, who decided to take part to the prize together due to the synergy between their important researches in the field of algebraic surfaces. However, the competition rules were unclear on the regularity and the evaluation criteria in the case of Castelnuovo and Enriques, and the Academia voted for their exclusion from the contest in June 1903. As a result of this affair, the awarding of the prize was postponed to 5 June 1904. In the end, the commission, which was led by Luigi Bianchi and composed of Veronese, Cerruti, Dini and d'Ovidio, decided that the prize could not be awarded.

There are at least two aspects to consider in order to correctly interpret this episode in its historical context as evidence of the non-recognition of the value of absolute differential calculus by the Italian scientific community before the discovery of relativity.

Firstly, the exclusion of Castelnuovo and Enriques was a tortured decision that divided the commission. On the one hand, their works were judged to be by far the best, but, on the other hand, the commission did not want to set a precedent by allowing two competitors, who would be unmeritorious of the prize individually,[81] to be able to win the prize by competing together. According to (Bottazzini 1999), the failure to award the prize should also (but not only) be seen as a consequence of this event: it was deemed inappropriate for the prize to be awarded to others since Castelnuovo and Enriques did not win only as a result of a defect of form.

Secondly, the works of the other competitors did not stand out enough to establish a real competition with Castelnuovo and Enriques. In particular, the commission severely judged the results pursued by Ricci. From the prize report read by Bianchi, the gap in Ricci's work was as follows:

[81] This was certainly not the case for Castelnuovo and Enriques. The two won the prize individually in the 1905 and 1907 editions, respectively.

> To deserve the royal prize [one needs] at least one work of truly exceptional value [...] the algorithms developed by [Ricci] [...] certainly prove to be useful, though not indispensable, in dealing with various mathematical questions; and of this we find evidence in the works of Ricci himself and in those of a few disciples. But considering, in the works presented, the truly new results acquired by science [they] did not appear to us of such importance as to merit the highest distinction.[82]

To understand the committee's judgement, it is therefore necessary to understand what both the committee and Ricci meant by *results*. In this respect, the committee's position, which was a mirror of the Italian mathematical community at the beginning of the twentieth century, can be summarised in Bianchi's position, as he was one of the most authoritative European voices in differential geometry. A detailed study of Bianchi's and Ricci's applications of Weingarten's method, which is proposed in this chapter, can therefore be a contribution to concretely measuring the distance between the two viewpoints. As we have seen, although it is a secondary theme in both their work, considering the concreteness of their research allows us to highlight numerous peculiarities.[83]

The following consideration in (Cogliati 2022, p. 231) effectively sums up Ricci's interest:

> In contrast to many fundamental contributions in the history of mathematical thought, it [the absolute differential calculus] did not primarily consist in the discovery of new notions and new theorems, but rather in a new way of organising and understanding results that were already known and could thus be interpreted in the light of common conceptual roots. In other words, one could say that, from a historical point of view, the contribution of Ricci's calculus was predominantly metatheoretical in nature, in the sense that it became explicit as a mature (and by no means trivial) reflection on an already existing body of doctrines, such as the theory of Riemannian varieties, the theory of the equivalence of differential quadratic forms, and the theory of invariants and differential parameters.[84]

[82]"A meritare il premio reale [occorre] almeno un lavoro di un valore veramente eccezionale [...] gli algoritmi da [Ricci] sviluppati [...] si dimostrano certamente utili, sebbene non indispensabili, nel trattare varie questioni matematiche; e di ciò troviamo le prove nei lavori stessi del Ricci e in quelli di alcuni pochi seguaci. Ma considerando, nei lavori presentati, i risultati veramente nuovi acquisiti alla scienza [essi] non ci sono apparsi di tale e tanta importanza da meritare l'altissima distinzione." (Bianchi 1904a, p. 150)

[83]Another interesting comparison between Bianchi's and Ricci's work is provided in (Toscano 2000). Here, the author reconstructs the events that led to the formulation of Bianchi's identities and their recognition. While focuses on Bianchi's position on absolute differential calculus, we aim to highlight the historical context in which Ricci tried to disseminate his calculus.

[84]Diversamente da molti contributi fondamentali nella storia del pensiero matematico, esso [il calcolo differenziale assoluto] non consistette tanto nella scoperta di nuove nozioni e di nuovi teoremi, quanto piuttosto in un modo nuovo di organizzare e comprendere risultati già noti, i quali poterono così essere interpretati alla luce di radici concettuali comuni. In altri termini, si potrebbe dire che da un punto di

A mature reflection on the methods and techniques of differential geometry, for example, led Ricci to note how some geometers, such as Bianchi and Weingarten, had taken the two fundamental quadratic forms as the basis of surface theory, but had not contextualised their research *entirely* in the theory of quadratic forms, not recognising the essentially algebraic nature of the problems they tackled. As a consequence, they used indirect and artificial methods compared with his own methods.

Looking at Ricci's demonstration, it is surprising how perfectly Weingarten's method fits Ricci's intrinsic theory of surfaces. This fact has a twofold relevance: on the one hand, it attests to Weingarten's ability to carry the theory of surfaces from the 1860s to the end of the century through the second problem of applicability by grasping the invariant aspect of the problem. On the other hand, Ricci's calculus, when approaching the difficult challenge of the second problem of applicability, proved adaptable to Weingarten's demonstration, which was a sign of his well-constructed calculus. It is therefore not surprising that (Ricci Curbastro 1897) had a certain circulation: when citing Weingarten's original memoir, Gerhard Hessenberg, Pasquale Calapso and Aurel Voss, for example, also referred to Ricci's short publication.[85]

Ricci's demonstration can certainly be regarded as a copy of Weingarten's. In fact, Ricci's great effort lies in the construction of the calculus, rather than in the demonstration of Weingarten's method. The calculus did, however, allow him to obtain an elegant demonstration of the dependence existing between deformations and tangent congruence: precisely because of the way the calculus was constructed, it gave him all the necessary tools to tackle the question by following a more direct and logical route. This brevity, however, is certainly at the expense of geometric intuition, which is, for Bianchi, *"the true source of discoveries in the field of ordinary infinitesimal geometry"*.[86] It must be emphasised, however, that Bianchi himself recognised that Ricci's markedly analytical approach, which seemed to echo the lack of geometric vision that Weingarten attributed to himself, was nevertheless indispensable for constructing a calculation to explore those fields where imagination could not work. According to the report, Bianchi appreciated Ricci's results for transitive motions of three-dimensional manifolds, which was a problem he himself had solved in (Bianchi 1898) and which he reconsidered in the light of (Ricci Curbastro 1899) in (Bianchi 1918, Chap. 13).[87] Moreover, as is pointed out in Section 4.3.2, lack of geometric interpretation was not only criticised in Ricci's work, nor was it only criticised by Bianchi: Weingarten was also attacked on this point from various mathematicians.

However, the committee expected *new* and *remarkable* problems to be solved. Levi

vista storico la portata del calcolo di Ricci fu di natura prevalentemente metateorica, nel senso che esso si esplicitò come una riflessione matura (e non affatto banale) sopra un corpo di dottrine già esistenti, quali la teoria delle varietà riemanniane, la teoria dell'equivalenza delle forme quadratiche differenziali e la teoria degli invarianti e dei parametri differenziali.

[85]See (Hessenberg 1900, p. 2), (Calapso 1901, p. 3) and (Voss 1903, p. 397).

[86](Bianchi 1904a, p. 148)

[87]In this respect, see (Rivis 2018, Chap. 3).

Civita stated that relativity alone would be sufficient to recognise his master's genius,[88] but the award report seems to state that general relativity was also necessary. At that stage in the development of tensor geometry, absolute differential calculus was still the answer to an unposed problem.

In addition, Ricci encompassed more abstract issues at a time when the scientific community still demanded results that gave concreteness to geometric entities, as Bianchi's research have clearly pointed out. For instance, Ricci did not deal with problems concerning the effective integration of equations and avoided research on particular surfaces, preferring more general results. The crucial point of the equation (W) for both Weingarten and Bianchi lies in its integrability, but while Bianchi exploited its integrability to find numerous results, Ricci only mentioned it in the closing lines of his *Lezioni*: *"Moreover, in his memoirs Weingarten shows how the fundamental equation better than (A)* [Darboux's equation] *of §163 lends itself in many cases to integration by known methods"*.[89]

In agreement with Blaschke, we believe that Bianchi could also have expressed the following opinion with regard to the use of absolute differential calculus in differential geometry, which he had advanced by means of the multitude of theorems that flourished in his hands:

> At the end of the last century, Hilbert's *Foundations of Geometry* were published and under the influence of this book, the Greek fashion of axiomatics resurged. Bianchi compared the studies on fundamentals to a long train stopped at a station: everyone shouts "Depart!" but the train does not move.[90]

[88](Levi-Civita 1925, p. 1)

[89](Ricci Curbastro 1898, §165)

[90]"Alla fine del secolo scorso uscirono i *Fondamenti di geometria* di Hilbert e sotto l'influsso di questo libro risorse la moda greca dell'assiomatica. Bianchi confrontò gli studi sui fondamenti con un lungo treno fermo in una stazione: tutti gridano «partenza!» ma il treno non si muove" (Blaschke 1954, p. 47)

Appendix A

The difference between applicability and (local) isometry

As noted in Section 1.2.1, Euler defined developable surfaces as those surfaces that can be unfolded in the plane (*in planum explicare licet*). He did not explain the meaning of these words, but he later compared these surfaces to sheets of paper that were folded along straight lines. We also saw that in (Euler 1862) he paraphrased this request by requiring the existence of a transformation between surfaces that locally preserves the distance between points. In both cases he translated these demands into mathematical terms by requiring the equality of the line elements of the considered surfaces.

As we have shown in Sections 2.1 and 2.2, Gauss also interpreted the fact that a surface, which can be regarded as flexible (*flexile*), can be unfolded (*explicari posse*) without deformations onto another (this is precisely the content of the problem of applicability) in terms of the equality of the line elements of the considered surfaces.

More precisely, let a surface S be parametrized by means of

$$\begin{aligned}&\boldsymbol{x} : U \to \mathbb{R}^3\\ &(u,v) \to \boldsymbol{x}(u,v) = (x(u,v), y(u,v), z(u,v)),\end{aligned}$$

and another surface S' be parametrized by means of

$$\begin{aligned}&\boldsymbol{x}' : U' \to \mathbb{R}^3\\ &(u',v') \to \boldsymbol{x}'(u',v') = (x'(u',v'), y'(u',v'), z'(u',v')).\end{aligned}$$

Then, both Euler and Gauss required the existence of a sufficiently regular one-to-one transformation of the parameters

$$\begin{aligned}&\boldsymbol{T} : U \to U'\\ &(u,v) \to (u',v') = (T_1(u,v), T_2(u,v)),\end{aligned}$$

such that it transforms the line element $ds^2 = E(u,v)du^2 + F(u,v)dudv + G(u,v)dv^2$ of S into the line element $ds'^2 = E'(u',v')du'^2 + F'(u',v')du'dv' + G'(u',v')dv'^2]$ of S', that is, $ds^2(u,v) = ds'^2(u',v')$. This request coincides with the modern definition of *local isometric*.

However, applicability and local isometry are not equivalent concepts. While applicable surfaces are always isometric, the existence of an isometry between S and S' is not sufficient to guarantee the existence of a bending, that is of a continuous movement in which the shape of the first surface passes to the shape of the second one, without being stretched, that characterises the applicability of S and S'.[1]

However, this distinction went unnoticed for long. Euler deduced the condition for two surfaces to be applicable ($ds^2 = ds'^2$) from the isometric correspondence of infinitesimal triangles on two surfaces. Similarly, but without explicit references to infinitesimal triangles, Gauss obtained the same result in (Gauss 1825a) and, more explicitly, in (Gauss 1828). The isometric correspondence of infinitesimal triangles was probably considered sufficient to guarantee the existence of a continuous motion.

In this respect, (Minding 1838a) is of some interest. When dealing with applicability, Minding often referred to the idea of flexibility and indiscriminately used terms such as *Biegsamkeit, biegsam* and *Abwickelung, abwickeln*, as one can see in the titles of (Minding 1838a) and (Minding 1838a), which are "*Ueber die Biegung gewisser Flächen*" and "*Ueber die Biegung krummer Flächen*", respectively, and the title of (Minding 1839), which is *"Wie sich entscheiden lässt, ob zwei gegebene krumme Flächen auf einander abwickelbar sind oder nicht..."*.

Minding translated the idea of deformation into mathematical terms by requiring the equality of surfaces' line elements. Moreover, he clearly wrote that surfaces having the same line element as a given one are *"a mere deformation"* ("eine blosse Biegung" (Minding 1838a, p. 297)) of it:

> Because of the equality of corresponding line elements, every triangle between three infinitely near points of the first surface is congruent to the triangle between the three corresponding points of the second; both surfaces are thus composed of the same elementary triangles in the same order, as the concept of bending demands.[2]

A more precise description of how the isometric correspondence of infinitesimal triangles generates a bending can be found in the exposition of the 1860 Grand Prix theme that was given in (Bour 1905, p. 344). Here, the possibility of rotating the infinitesimal triangles that

[1] In modern terms, a bending is defined as follows, (see (Spivak 1999, Vol. 5, p. 170-171)). Let $f : S \to S'$ be an isometry between the two surfaces S and S'. Then, a bending of S on S' is a map $\alpha : [0,1] \times S \to S'$ such that

- for every $t \in [0,1]$, the map $\bar{\alpha}_t = \alpha(t,p) : S \to R^3$ is an embedding
- $\bar{\alpha}_0 = Id$
- $\bar{\alpha}_1 = f$.

It is clear that the existence of a bending presupposes the existence of an isometry.

[2] "Wegen der Gleichheit entsprechender Linear-Elemente, jedes Dreieck zwischen drei einander unendlich nahen Puncten der ersten Fläche congruent dem Dreiecke zwischen den drei entsprechenden Puncten der zweiten; heide Flächen sind also aus denselben Elementar-Dreiecken in derselben Ordnung zusammengesetzt, wie der Begriff der Biegung fordert."(Minding 1838a)

compose the surface along the joints that connect the edges of the triangles was explicitly used, but without requiring any special conditions to the regularity of the surfaces:

> It is known that a curved surface can always be replaced by a polyhedral surface with an infinite number of plane facets that are infinitely small, which merge with them at the limit. It is then conceivable that the facets can rotate around their edges, and that the angles formed by their planes, or the curvature of the surface, vary on condition that the neighbouring facets also change position, so that the hinges are no longer torn off and one side does not fold over the other; the new surface thus obtained will obviously be able to come back to rest on the first.[3]

Apparently, the first to draw attention to the distinction between isometry and bending was Aurel Voss, who employed the terms *Abwickelung* and *Biegung* to refer to the existence of an isometry and that of a continuous deformation, respectively.[4]

Bianchi's three editions of *Lezioni di geometria differenziale* describe the gradual awareness of this misunderstanding. While Bianchi still identified the two notions in the first lithographed edition (Bianchi 1886a), in the second edition (Bianchi 1902a), he emphasised that the existence of a continuous series of configurations that leads the first surface to coincide with the second must be proved for the unfolding to be possible. Finally, in the third edition (Bianchi 1922), Bianchi reported the first result establishing conditions under which the notion of isometry and applicability are actually equivalent.

This result is due to a student of Bianchi, Eugenio Elia Levi (1883–1917), and appeared in (Levi 1908).[5] Bianchi himself encouraged Levi to work on this subject. Levi wrote in a footnote: *"This matter was proposed to me by the eminent Prof. Bianchi; for this and for the useful advice he gave me, allow me to thank him most sincerely"*.[6]

Theorem A.1 (E. E. Levi)**.** *Let S_0 and S_1 be two isometric surfaces. If S_0 and S_1 have null* [7] *or negative curvature, then they are always applicable; in the case S_0 and S_1 have positive curvature and are analytic, they are applicable, except for a reflection.*[8]

Hence, for sufficiently regular surfaces, the notion of local isometry and that of applicability actually coincide and one can rightly disregard any difference between the require-

[3]"On sait qu'une surface courbe peut toujours être remplacée par une surface polyédrale à facettes planes infiniment petites et en nombre infini, qui se confond avec elles à la limite. On conçoit alors que les facettes puissent tournèr autour de leurs arêtes, et que les angles formés par leurs plans, où là courbure de la surface, varient à condition que les facettes voisines changent aussi de position, de telle sorte que les charnières ne soient pus arrachées et qu'un côté ne se replie pas sur l'autre; la nouvelle surface ainsi obtenue pourra évidemment revenir s'appliquer sur la première."

[4](Voss 1903, pp. 362–363).

[5]For a modern treatment of the problem, the interest reader can refer to (Spivak 1999, pp. 229–230).

[6]"Tale questione mi fu proposta dal Chiar.mo Prof. Bianchi; di ciò e degli utili consigli che mi diede mi sia concesso di ringraziarlo vivamente." (Levi 1908, p. 4)

[7]Levi did not discuss the case of surfaces with zero curvature, taking it for granted.

[8]A surface with positive curvature is analytic when the coefficients E, F, G of its line element are analytic. In this respect, see (Levi 1908, p. 4).

ment of applicability and that of (local) isometry, as was the practice in the nineteenth century.

Bianchi's influence on Levi is evident: the proof is based on several analytical and geometrical results already known in literature, many of which had been obtained by Bianchi himself and were included in (Bianchi 1902a) and (Bianchi 1905), which also bears the same title of (Levi 1908).

For the proof of Theorem A.1 in the case of surfaces with positive curvature, Levi resorted to Bianchi's result that guarantees the possibility (assuming the analyticity of surfaces) of deforming a surface so that a certain line on it assumes a predetermined shape.[9] Levi thus considered a curve Γ_0 on S_0 and a generic curve Γ_α. For Bianchi's results, the assignment of Γ_α is sufficient to determine a surface S_α that contains Γ_α and is applicable to S_0. Levi assumed that Γ_0 varies under precise regularity conditions until it coincides with Γ_1, which is a curve on S_1, by assuming all configurations Γ_α between Γ_0 and Γ_1. Then, the surface S_0 continuously deforms by assuming all configurations S_α that are generated by the curves Γ_α until reaching S_1. Moreover, since Bianchi's proof is based on the fact that the angle θ between the principal normal of the curve and the normal to the surface can assume values in two disjoint sets, $[0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi]$, Levi divided surfaces with positive curvature into two classes. He showed that a continuous deformation can only occur when the angle θ is always in $[0, \frac{\pi}{2})$ or is always in $(\frac{\pi}{2}, \pi]$ during the deformation. Finally, by noting that two symmetrical surfaces belong to distinct classes, Levi deduced that they are not applicable.

For surfaces with negative curvature, Levi proposed another method that did not require the surfaces to be analytic based on the following result given in (Bianchi 1905): given two arbitrary, non-tangent curves on a surface S' with negative curvature, there always exists a deformation that makes them the asymptotic lines of a surface S, which is a deformation of S'.

To find a continuous succession of applicable surfaces that leads S_0 to coincide with S_1, Levi therefore considered a pair of asymptotic lines (Γ_0, Γ'_0) on S_0 and their image (Γ_1, Γ'_1) on S_1 through the isometry. He applied Bianchi's theorem to a pair of curves $(\Gamma_\alpha, \Gamma'_\alpha)$ that vary continuously by assuming exactly (Γ_0, Γ'_0) and (Γ_1, Γ'_1) as initial and final configurations. These pairs of curves $(\Gamma_\alpha, \Gamma'_\alpha)$ determine the continuous succession of surfaces S_α, which realises the bending of S_0 into S_1.

Besides showing the content of Theorem A.1, Levi's demonstration highlights how a sophisticated (also analytic) background was required to more deeply investigate into the distinction between applicability and local isometry.

[9] (Bianchi 1902a, §111-112)

Appendix B

Brief excursus on the evolution of the term *surface*

In differential geometry, the term *surface* has acquired different interpretations over the centuries and this change was also influenced by the advancements in calculus.

The notion of surface as a two dimensional manifold finds its origins in the publication of Riemann's dissertation *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (On the hypotheses which lie at the bases of geometry), (Riemann 1867), which marked a turning point for differential geometry.[1] Until then—and after the very early stage of development of space geometry in which surfaces were intended as boundaries of solids—differential geometry was only concerned with regular surfaces, as well as regular curves, that were embedded in Euclidean space.

Investigation of embedded surfaces continued even after the publication of (Riemann 1867). However, whereas surfaces were previously primarily considered seen as immersed in Euclidean space, and surfaces that corresponded to the same line element were looked at in all their possible deformations, after Riemann surfaces were determined by the assignment of their line element and considered as the complete class of applicable surfaces, while specific immersions in Euclidean space were interpreted as a rigid surface. This fact clearly emerges from the title of the two main parts constituting (Ricci Curbastro 1898): the first part, which is devoted to intrinsic geometry is entitled *"On the properties of surfaces considered as flexible and inextensible veils"* (Delle proprietà delle superficie considerate come veli flessibili ed inestendibili), the second part, which deals with surfaces as described by the assignment of first and second fundamental forms, is called *"Theory of surfaces considered as having a rigid form in space"* (Teoria delle superficie considerate come dotate di forma rigida nello spazio).

In addition, until the end of the nineteenth century, curves and surfaces were supposed to be regular. During the eighteenth century, regularity was taken completely for granted. This is clear from Euler's work. Despite his strong interest in analysis, he did not consider to require an appropriate degree of regularity for functions. For example, while constructing

[1]For an account of the concept of manifold, the interested reader can refer to (Scholz 1999).

the infinitesimal triangle on the developable surface, he expanded the components of the local parametrization of the surface to the first order. However, he did not discuss the possible non-existence of the first derivatives, i.e., the possible non-existence of the tangent vector to the surface. According to (Capobianco, Enea, and Ferraro 2017), this can be explained by his geometric conception of analysis: variables were still seen as geometric quantities and *therefore* were assumed to be regular. From this point of view, Euler was unable to consider functions that were not regular:

> despite the various claims of the independence of analysis, Euler was not able to definitely break the link with geometry. It is true that he replaced the study of geometric quantities by general or abstract quantities; however, the latter—even not represented by lines in a diagram—were closely connected with the former, since they were the result of a process of abstraction upon geometric quantities; namely, they were made up of what all the geometrical quantities have in common. For this reason, abstract quantities (and functions—functions being quantities) were assumed to have the properties of a "nice" or "well-behaved" curved line (such as, the existence of the tangent, of the radius of a curve, of the area under the curve, and so on). For instance, since geometric quantities were thought to be continuous, general quantities had to be continuous, too. Thus, Euler never proved that if a variable quantity x moved from a value x_1 to a value x_2, then a function $f(x)$ assumed all the values between $f(x_1)$ and $f(x_2)$. This proposition was considered obvious, a simple consequence of the notion of quantity. In the same manner, since a tangent could usually be associated to a point of a curve, it seemed obvious that a function could be differentiated and, since an area can be associated a piece of a curved line, it seemed that obvious that any function could be associated to an integral.

During the nineteenth century, the conviction that geometric quantities were essentially regular gradually weakened. Thus, at the beginning of his *Disquisitiones*, Gauss warned that his investigations were limited to surfaces, or parts of them, that were regular. In particular, Gauss required the existence of the tangent plane and the continuity of curvature:

> A curved surface is said to possess continuous curvature at one of its points A, if the directions of all the straight lines drawn from A to points of the surface at an infinitely small distance from A are deflected infinitely little from one and the same plane passing through A. This plane is said to touch the surface at the point A. If this condition is not satisfied for any point, the continuity of the curvature is here interrupted, as happens, for example, at the vertex of a cone. The following investigations will be restricted to such surfaces, or to such parts of surfaces, as have the continuity of their curvature nowhere interrupted. We shall only observe now that the methods used to determine the position of the

Fig. 16.

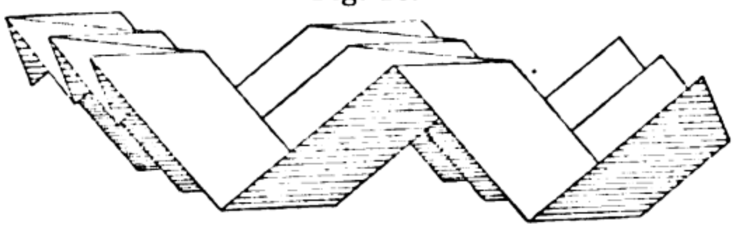

Figure B.1: Drawing of a developable surface according to Wiener's definition. It was provided by Wiener himself in (Wiener 1887, p. 30).

> tangent plane lose their meaning at singular points, in which the continuity of the curvature is interrupted, and must lead to indeterminate solutions.[2]

While Johann Bernoulli used the vertex of the cone to construct isometry with the plane and Euler certainly included pyramids and prisms among developable surfaces, Gauss stated that his theory could not be used to study such surfaces in their entirety; those regions containing singular points—the vertex of the cone and the edges of pyramids and prisms—must be excluded.

As a consequence, mathematicians began to consider surfaces as described by functions that were "sufficiently regular" or "derivable at least up to second order" or "as regular as possible". Hence, the notion of surface became similar to today's notion of *local chart*.

In the second half of the nineteenth century, mathematical analysis was enriched with new and bizarre results. Among the classic examples, we should mention the *Cantor function* (1884), which is a continuous function with zero derivative almost everywhere that grows continuously, and *Peano curve* (1890), which is a never derivable space-filling curve. Their geometric interpretation probably led to a weakening of the demand for regularity of curves and surfaces and thus broadened the class of "objects" that could be considered surfaces. It is therefore not surprising that in (Wiener 1887, Chap. 1) Wiener extended the definition of developable surfaces to surfaces that were composed of several faces, each of which was applicable in the plane, joining along curves that are singularities of the surface (see Figure B.1). Wiener also included in this definition surfaces that were composed of an infinite number of infinitely small faces obtained by a limit process. In the wake of these new viewpoints, but without explicitly referring to Wiener's work, the young Lebesgue demonstrated that developable surfaces exist that are not ruled and do

[2]"Superficies curva apud punctum A in ipsa situm curvatura continua gaudere dicitur, si directiones omnium rectarum ab A ad omnia puncta superficiei ab A infinite parum distantia ductarum infinite parum ab uno eodemque plano per A transiente deflectuntur: hoc planum superficiem curvam in puncto A *tangere* dicitur. Quodsi huic conditioni in aliquo puncto satisfieri nequit, continuitas curvaturae hic interrumpitur, uti e.g. evenit in cuspide coni. Disquisitiones praesentes ad tales superficies curvas, vel ad tales superficiei partes, restringentur, in quibus continuitas curvaturae nullibi interrumpitur. Hic tantammodo observamus, methodos, quae positioni plani tangentis determinandae inserviunt, pro punctis singularibus, in quibus continuitas curvaturae interrumpitur, vim suam perdere, et ad indeterminata perducere debere." (Gauss 1828, §3)

not have a tangent plane at every point.[3] Lebesgue considered a developable surface and a non-geodesic curve C on it, and a second developable surface passing through C. Since C divided each of the two surfaces into two parts, A_1 and A_2, B_1 and B_2, four pairs of surfaces were obtained, (A_1, A_2), (A_1, B_1), (A_1, B_2), (B_1, B_2). These pairs are developable surfaces (in the extended sense), since they consist of developable surfaces that stick together along a curve, C. By iterating the procedure, Lebesgue obtained a surface with an infinite number of singularities that can be unfolded onto the plane, but that does not contain any segments.[4]

A description of this astonishing result was provided by Picard:

> According to general practice, we suppose in the preceding analysis, as in all infinitesimal geometry of curves and surfaces, the existence of derivatives, which we need in the calculus. It may seem premature to entertain a theory of surfaces in which one does not make such hypotheses. However, a curious result has been pointed out by Mr Lebesgue, according to which one may, by the aid of continuous functions, obtain surfaces corresponding to a plane, of such sort that every rectifiable line of the plane has a corresponding rectifiable line of the same length of the surface, nevertheless the surfaces obtained are no longer ruled. If one takes a sheet of paper, and crumples it by hand, one obtains a surface applicable to the plane and made up of a finite number of pieces of developable surfaces, joined two and two by lines, along which they form a certain angle. If one imagines that the pieces become infinitely small, the crumpling being pushed everywhere to the limit, one may arrive at the conception of surfaces applicable to the plane and yet not developable and not ruled.[5]

As already pointed out by (Cajori 1929, p. 437), *"The crumpling of a sheet of paper reminds one of Euler's process of folding along straight lines. Euler's procedure was regular, systematic—too much so to yield the results of Lebesgue".*

Euler (unlike Gauss) already considered surfaces such as pyramids or prisms to be developable, but his investigations had disregarded the singularities of these surfaces. At the beginning of the twentieth century, the development of analysis, together with that of differential geometry, had made it possible not only to resume the study of such surfaces but also to conceive new surfaces.

[3](Lebesgue 1899, p. 1504)

[4]The interested reader can find a comment on Wiener's and Lebegue's work in (Friedman 2018, pp. 159–161).

[5](Picard 1922, p. 555)

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