

MAXIMUM PRINCIPLES AT INFINITY AND THE AHLFORS-KHAS'MINSKII DUALITY: AN OVERVIEW

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ABSTRACT. This note is meant to introduce the reader to a duality principle for nonlinear equations recently discovered in [74, 54, 50]. Motivations come from the desire to give a unifying potential-theoretic framework for various maximum principles at infinity appearing in the literature (Ekeland, Omori-Yau, Pigola-Rigoli-Setti), as well as to describe their interplay with properties coming from stochastic analysis on manifolds. The duality involves an appropriate version of these principles formulated for viscosity subsolutions of fully nonlinear inequalities, called the Ahlfors property, and the existence of suitable exhaustion functions called Khas'minskii potentials. Applications, also involving the geometry of submanifolds, will be discussed in the last sections. We conclude by investigating the stability of these maximum principles when we remove polar sets.

CONTENTS

1. Prelude: maximum principles at infinity	1
1.1. An example: immersions into cones	3
1.2. Parabolicity, capacity and Evans potentials	6
1.3. Link with stochastic processes	7
1.4. On weak formulations: the case of quasilinear operators	9
2. The general framework	10
3. Ahlfors, Khas'minskii properties and the AK-duality	15
4. Applications	20
4.1. Completeness, viscosity Ekeland principle and ∞ -parabolicity	20
4.2. The Hessian principle and martingale completeness	21
4.3. Laplacian principles	22
5. Partial Trace (Grassmannian) operators	22
6. AK-duality and polar sets	24
References	26

1. PRELUDE: MAXIMUM PRINCIPLES AT INFINITY

Maximum principles at infinity are a powerful tool to investigate problems in Geometry. They arose from the desire to generalize the statement that, on a compact Riemannian manifold $(X, \langle \cdot, \cdot \rangle)$

¹2010 *Mathematics Subject Classification*: Primary 31C12, 35B50; Secondary 35B53, 58J65, 58J05, 53C42.

Key words and phrases. Potential theory, Liouville theorem, Omori-Yau, maximum principles, stochastic completeness, martingale, completeness, Ekeland, Brownian motion.

The first author is supported by the grants SNS17_B_MARI and SNS_RB_MARI of the Scuola Normale Superiore.

The second author was partially supported by CNPq-Brazil.

of dimension $m \geq 2$, every function $u \in C^2(X)$ attains a maximum point x_0 and

$$(1) \quad (i) : u(x_0) = \sup_X u, \quad (ii) : |\nabla u(x_0)| = 0, \quad (iii) : \nabla^2 u(x_0) \leq 0,$$

where $\nabla^2 u$ is the Riemannian Hessian and the last relation is meant in the sense of quadratic forms. If M is noncompact and given $u \in C^2(X)$ bounded from above, although one cannot ensure the existence of a maximum point, there could still exist a sequence $\{x_k\} \subset X$ such that some of the relations in (1) hold in a limit sense as $k \rightarrow \infty$. Informally speaking, when this happens we could think that X is “not too far from being compact”. The first example of maximum principle at infinity is the famous Ekeland’s principle, [24], that can be stated as follows:

Definition 1.1. *A metric space (X, d) satisfies the Ekeland maximum principle if, for each u upper semicontinuous (USC) on X and bounded from above, there exists a sequence $\{x_k\} \subset X$ with the following properties:*

$$u(x_k) > \sup_X u - \frac{1}{k}, \quad u(y) \leq u(x_k) + \frac{1}{k}d(x_k, y) \text{ for each } y \in X.$$

The full statement of Ekeland’s principle contains, indeed, a further property that is crucial in applications, that is, the possibility to create one such $\{x_k\}$ suitably close to a given maximizing sequence $\{\bar{x}_k\}$. We will briefly touch on it later. By works of I. Ekeland, J.D. Weston and F. Sullivan, cf. [24, 75, 71], the validity of Ekeland’s principle in the form given above is in fact *equivalent* to X being a complete metric space. Therefore, in the smooth setting, the (geodesic) completeness of a manifold X enables to find $\{x_k\}$ approximating both (i) and (ii) in (1.1). However, condition (iii) requires further restrictions on the geometry of X , first investigated by H. Omori [58] and S.T. Yau [15, 77] (the second with S.Y. Cheng). They introduced the following two principles, respectively, in the Hessian case [58] and in the Laplacian case [15, 77]:

Definition 1.2. *Let $(X, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. We say that X satisfies the strong Hessian (respectively, Laplacian) maximum principle if, for each $u \in C^2(X)$ bounded from above, there exists a sequence $\{x_k\} \subset X$ with the following properties:*

$$(2) \quad \begin{array}{lll} \text{(Hessian)} & u(x_k) > \sup_X u - k^{-1}, & |\nabla u(x_k)| < k^{-1}, \quad \nabla du(x_k) \leq k^{-1}\langle \cdot, \cdot \rangle; \\ \text{(Laplacian)} & u(x_k) > \sup_X u - k^{-1}, & |\nabla u(x_k)| < k^{-1}, \quad \Delta u(x_k) \leq k^{-1}. \end{array}$$

Here, the word “strong” refers to the presence of the condition on the gradient. Historically, these principles are called the *Omori-Yau maximum principles*, and proved to be remarkably effective in a wealth of different geometric problems. Among them, we stress the striking proofs of the generalized Schwarz Lemma for maps between Kähler manifolds in [78], and of the Bernstein theorem for maximal hypersurfaces in Minkovski space in [16].

Geometric conditions to guarantee the Omori-Yau principles are often expressed in terms of growths of the curvatures of X with respect to the distance $\rho(x)$ from a fixed origin $o \in X$, but are not necessarily depending on them. The most general known condition guaranteeing the Omori-Yau principles is given by [60] (cf. also improvements in [12, 10]): the principle holds whenever X

supports a function w with the following properties¹:

$$(3) \quad \begin{aligned} &0 < w \in C^2(X \setminus K) \text{ for some compact } K, & w \rightarrow +\infty \text{ as } x \text{ diverges,} \\ &|\nabla w| \leq G(w), & \text{and } \begin{cases} \nabla^2 w \leq G(w)\langle \cdot, \cdot \rangle & \text{for Omori's principle,} \\ \Delta w \leq G(w) & \text{for Yau's principle,} \end{cases} \end{aligned}$$

for some G satisfying

$$(4) \quad 0 < G \in C^1(\mathbb{R}^+), \quad G' \geq 0, \quad \int^{+\infty} \frac{ds}{G(s)} = +\infty.$$

For instance, $G(t) = (1+t)$ gives the sharp polynomial threshold. The criterion is effective, since the function in (3) can be explicitly found in a number of geometrically relevant applications: for instance, if the radial sectional curvature² (respectively, Ricci curvature) is bounded from below as follows:

$$(5) \quad \begin{aligned} &\text{Sect}_{\text{rad}} \geq -G^2(\rho) && \text{on } X, && \text{for Omori's principle,} \\ &\text{Ric}(\nabla \rho, \nabla \rho) \geq -G^2(\rho) && \text{on } X \setminus \{\{o\} \cup \text{cut}(o)\} && \text{for Yau's principle,} \end{aligned}$$

then one can choose $w(x) = \log(1 + \rho(x))$ in (3) to deduce the validity of the strong Hessian, respectively, strong Laplacian principle (technically, ρ is not C^2 , but one can overcome the problem by using Calabi's trick, see [60]). Note that (5) includes the case when G is constant, considered in [58, 77]. However, the existence of w could be granted even without bounds like (5): for instance, in the strong Laplacian case, this happens if X is properly immersed with bounded mean curvature in \mathbb{R}^m (or in a Cartan-Hadamard ambient space with bounded sectional curvature), or if X is a Ricci soliton, see [4]. The function w in (3) is an example of what we will call a *Khas'minskii potential*. The reason for the name will be apparent in a moment.

1.1. An example: immersions into cones. There is, by now, a wealth of applications of the Omori-Yau principles in geometry, see for instance [4]. Here, we illustrate how the principles can be effectively used in geometry by means of the following example in [53], that is related to the pioneering paper by Omori [58]. Hereafter, a non-degenerate cone $\mathcal{C}_{o,v,\varepsilon}$ of center $o \in \mathbb{R}^n$, axis $v \in \mathbb{S}^{n-1}$ and width $\varepsilon \in (0, \pi/2)$ is the set of points $x \in \mathbb{R}^n$ such that

$$\left\langle \frac{x - o}{|x - o|}, v \right\rangle \geq \cos \varepsilon.$$

Theorem 1.3. [53, Cor. 1.18] *Let $\varphi : X^m \rightarrow \mathbb{R}^{2m-1}$ be an isometric immersion. Denoting with ρ the distance from a fixed origin o , assume that the sectional curvature of X satisfies*

$$(6) \quad -C(1 + \rho^2) \leq \text{Sect} \leq 0, \quad \text{on } M,$$

for some constant $C > 0$. Then, $\varphi(X)$ cannot be contained into any non-degenerate cone of \mathbb{R}^{2m-1} .

Sketch of the proof: Suppose, by contradiction, that $\varphi(X) \subset \mathcal{C}_{o,v,\varepsilon}$ for some o, v, ε . Without loss of generality, we can assume that o is the origin of \mathbb{R}^{2m-1} . The first step is to construct a

¹Here, as usual, if we write “ $w(x) \rightarrow +\infty$ as x diverges” we mean that the sublevels of w have compact closure in X , that is, that w is an exhaustion.

²The radial sectional curvature is the sectional curvature restricted to 2-planes containing $\nabla \rho$. Inequality $\text{Sect}_{\text{rad}} \geq -G^2(\rho)$ means that $\text{Sect}(\pi_x) \geq -G^2(\rho(x))$ for each $x \notin \{o\} \cup \text{cut}(o)$ and $\pi_x \leq T_x X$ 2-plane containing $\nabla \rho$.

function that encodes the geometry of the problem at hand: following [58], we fix $x_0 \in X \setminus \{o\}$ and $a \in (0, \cos \varepsilon)$, we set $T = \langle \varphi(x_0), v \rangle$ and we define

$$u(x) = \sqrt{T^2 + a^2|\varphi(x)|^2} - \langle \varphi(x), v \rangle.$$

By construction, it is easy to show that $u < T$ on X and that the non-empty upper level set $\{u > 0\}$ (that contains x_0) has bounded image $\varphi(\{u > 0\})$. In view of (6) and the discussion above, the strong Hessian principle holds and can be applied to u . However, for $x \in \{u > 0\}$ and unit vector $W \in T_x X$, a computation shows that

$$\begin{aligned} \nabla^2 u(W, W) &= \frac{a^2(1 + \langle \Pi(W, W), \varphi \rangle)}{\sqrt{T^2 + a^2|\varphi|^2}} - \langle \Pi(W, W), v \rangle - \frac{a^4 \langle W, \varphi \rangle^2}{(T^2 + a^2|\varphi|^2)^{3/2}} \\ &\geq \frac{a^2(1 - |\Pi(W, W)||\varphi|)}{\sqrt{T^2 + a^2|\varphi|^2}} - |\Pi(W, W)| - \frac{a^4|\varphi|^2}{(T^2 + a^2|\varphi|^2)^{3/2}} \\ &\geq \frac{a^2 T^2}{(T^2 + a^2|\varphi|^2)^{3/2}} - |\Pi(W, W)|g(x), \end{aligned}$$

for some continuous function g on $\{u > 0\}$. Since the codimension of φ is strictly less than m and the sectional curvature is non-positive, a useful algebraic lemma due to Otsuki [20, 60] guarantees the existence of $W \in T_x X$ such that $\Pi(W, W) = 0$. Having fixed such W , and taking into account that $\varphi(\{u > 0\})$ is bounded, there exists a uniform constant $c > 0$ such that

$$\sup_{Z \in T_x X, |Z|=1} \nabla^2 u(Z, Z) \geq \nabla^2 u(W, W) \geq \frac{a^2 T^2}{(T^2 + a^2|\varphi|^2)^{3/2}} \geq c > 0 \quad \text{on } \{u > 0\}.$$

However, evaluating the above inequality on a sequence $\{x_k\}$ realizing the strong Hessian principle we obtain a contradiction. \square

Remark 1.4. In its full strength, Ekeland's principle also guarantees that the sequence $\{x_k\}$ satisfying (1.1) can be chosen to be close to a given maximizing sequence $\{\bar{x}_k\}$ with explicit bounds, a fact that is very useful in applications to functional analysis and PDEs. On the other hand, to present no systematic investigation of an analogous property was performed for the Omori-Yau principles. A notable exception, motivated by a geometrical problem involving convex hulls of isometric immersions, appeared in [25]: there, the authors proved that if X is complete and $\text{Sect} \geq -c$ for some $c \in \mathbb{R}^+$ (resp. $\text{Ric} \geq -c$), any maximizing sequence $\{\bar{x}_k\}$ has a *good shadow*, that is, a sequence $\{x_k\}$ satisfying Omori (resp. Yau) principle and also $d(x_k, \bar{x}_k) \rightarrow 0$ as $k \rightarrow \infty$.

The properties in (2) can be rephrased as follows, say in the Hessian case: denoting with $\lambda_1(A) \leq \lambda_2(A) \leq \dots \leq \lambda_m(A)$ the eigenvalues of a symmetric matrix A , X has the strong Hessian principle if either one of the following properties holds:

- (i) it is not possible to find a function $u \in C^2(X)$ bounded from above and such that, on some non-empty upper level-set $\{u > \gamma\}$,

$$(7) \quad \max \left\{ |\nabla u| - 1, \lambda_m(\nabla^2 u) - 1 \right\} \geq 0.$$

- (ii) for every open set $U \subset M$ and every $u \in C^2(U) \cap C(\bar{U})$ bounded from above and solving the differential inequality

$$\max \left\{ |\nabla u| - 1, \lambda_m(\nabla^2 u) - 1 \right\} \geq 0,$$

it holds $\sup_{\mathbf{U}} \mathbf{u} = \sup_{\partial \mathbf{U}} \mathbf{u}$.

By rescaling, the constant 1 can be replaced by any fixed positive number. It is evident that the strong Hessian principle is equivalent to (i), while (i) \Leftrightarrow (2) can easily be proved by contradiction: if we assume that (i) fails for some \mathbf{u} , consider as \mathbf{U} the upper level set where (7) holds and contradict (ii), while if (ii) fails for some \mathbf{u} , then take any upper level set at height $\gamma \in (\sup_{\partial \mathbf{U}} \mathbf{u}, \sup_{\mathbf{U}} \mathbf{u})$ to contradict (i).

Conditions (i) and (ii) are invariant by translations $\mathbf{u} \mapsto \mathbf{u} + \text{const}$. For future use it is important to describe another characterization, not translation invariant, where the constant 1 in (ii) is replaced by a pair of functions

$$(f\xi) \quad \begin{cases} f \in C(\mathbb{R}), & f(0) = 0, & f > 0 \text{ on } \mathbb{R}^+, & f \text{ is odd and strictly increasing;} \\ \xi \in C(\mathbb{R}), & \xi(0) = 0, & \xi < 0, \text{ on } \mathbb{R}^+, & \xi \text{ is odd and strictly decreasing.} \end{cases}$$

Namely, (i) and (ii) are also equivalent to

(\mathcal{A}) for some (*equivalently*, any) pair (f, ξ) satisfying $(f\xi)$, the following holds: for every open set $\mathbf{U} \subset M$ and every $\mathbf{u} \in C^2(\mathbf{U}) \cap C(\bar{\mathbf{U}})$ bounded from above and solving the differential inequality

$$(8) \quad \max \left\{ |\nabla \mathbf{u}| - \xi(-\mathbf{u}), \lambda_m(\nabla^2 \mathbf{u}) - f(\mathbf{u}) \right\} \geq 0 \quad \text{on } \{x : \mathbf{u}(x) > 0\} \neq \emptyset,$$

it holds $\sup_{\mathbf{U}} \mathbf{u} = \sup_{\partial \mathbf{U}} \mathbf{u}$.

The choice of the upper level-set $\{\mathbf{u} > 0\}$ is related to the vanishing of f, ξ in $(f\xi)$ at zero and is, of course, just a matter of convenience. The proof of (ii) \Leftrightarrow (\mathcal{A}) proceeds by translation and rescaling arguments, and localizing on suitable upper level sets of \mathbf{u} , and is given in detail in [50, Prop. 5.1]. From the technical point of view, the dependence of (\mathcal{A}) on f, ξ just in terms of the mild properties in $(f\xi)$, and especially the possibility to check (\mathcal{A}) in terms of a *single* pair f, ξ satisfying $(f\xi)$, is useful in applications.

Characterization (\mathcal{A}) is in the form of a maximum principle on sets with boundary, for subsolutions of the fully nonlinear inequality

$$\mathcal{F}(x, \mathbf{u}(x), \nabla \mathbf{u}(x), \nabla^2 \mathbf{u}(x)) \doteq \max \left\{ |\nabla \mathbf{u}| - \xi(-\mathbf{u}), \lambda_m(\nabla^2 \mathbf{u}) - f(\mathbf{u}) \right\} \geq 0.$$

This point of view relates the principles to another property that can be seen as a replacement of the compactness of X , the *parabolicity* of X . We recall that

Definition 1.5. *A manifold X is said to be parabolic if each solution of $\Delta \mathbf{u} \geq 0$ on X that is bounded from above is constant.*

Indeed, L.V. Ahlfors (see [2, Thm. 6C]) observed that X is parabolic if and only if property (\mathcal{A}) holds with (8) replaced by $\Delta \mathbf{u} \geq 0$. The problem of deciding whether a manifold is parabolic or not is classical, and there is by now a well established theory, see [28] for a thorough account. For surfaces, the theory arose in connection to the *type problem* for Riemann surfaces, cf. [2], and arguments involving parabolicity are still crucial in establishing a number of powerful, recent results in modern minimal surface theory (see for instance [51, 55, 45] for beautiful examples).

The tight relation between parabolicity and potential theory suggests that it might be possible to treat both Ekeland and Omori-Yau principles as well in terms of a fully nonlinear potential theory. In recent years, there has been an increasing interest in the theory of fully-nonlinear PDEs on manifolds, and especially R. Harvey and B. Lawson dedicated a series of papers [29, 30, 32, 31, 34, 33] to develop a robust geometric approach for fully nonlinear PDEs well suited to do potential theory

for those equations. Their work fits perfectly to the kind of problems considered in the present paper, and will be introduced later. Our major concern in the recent [54, 50, 52] is to put the above principles, as well as other properties to be discussed in a moment, into a unified framework where new relations, in particular an underlying duality, could emerge between them.

1.2. Parabolicity, capacity and Evans potentials. There are a number of equivalent conditions characterizing the parabolicity of X , see [28, Thm. 5.1] and [65], and we now focus on two of them.

The first one describes parabolic manifolds as those for which the 2-capacity of every compact K vanishes. We recall that, for fixed $q \in (1, \infty)$, the q -capacity of a condenser (K, Ω) with $K \subset \Omega \subset X$, K compact, Ω open, is the following quantity:

$$(9) \quad \text{cap}_q(K, \Omega) = \inf \left\{ \int_{\Omega} |\nabla \phi|^q, \quad : \quad \phi \in \text{Lip}_c(\Omega), \quad \phi \geq 1 \text{ on } K \right\}.$$

If K and Ω have Lipschitz boundary, the infimum is realized by the unique solution u of the q -Laplace equation

$$(10) \quad \begin{cases} \Delta_q u \doteq \text{div}(|\nabla u|^{q-2} \nabla u) = 0 & \text{on } \Omega \setminus K, \\ u = 1 & \text{on } K, \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

called the q -capacitor of (K, Ω) . A manifold is called q -parabolic if $\text{cap}_q(K, X) = 0$ for some (equivalently, every) compact set K , see [36, 72]. By extending Ahlfors result for the Laplace-Beltrami operator [61, 64],

$$(11) \quad X \text{ is } q\text{-parabolic} \quad \iff \quad \left. \begin{array}{l} \forall U \subset X \text{ open, } \forall u \in C^2(U) \cap C(\bar{U}), \\ \text{bounded above and solving } \Delta_q u \geq 0, \\ \text{it holds } \sup_U u = \sup_{\partial U} u, \end{array} \right\}$$

and both are equivalent to the constancy of solutions of $\Delta_q u \geq 0$ that are bounded from above.

There is a further, quite useful characterization of parabolicity (in the linear setting $q = 2$), expressed in terms of suitable exhaustion functions and studied by Z. Kuramochi and M. Nakai [44, 56, 57], with previous contribution by R.Z. Khas'minskii [42]. In [44, 56, 57], the authors proved that X is parabolic if and only if, for each compact set K with smooth boundary, there exists a function w solving

$$(12) \quad \begin{aligned} w &\in C^\infty(X \setminus K), \quad w > 0 \text{ on } X \setminus K, \quad w = 0 \text{ on } \partial K, \\ w(x) &\rightarrow +\infty \text{ as } x \text{ diverges,} \quad \Delta w = 0 \text{ on } X \setminus K. \end{aligned}$$

Such a w is named an *Evans potential* on $X \setminus K$. Evans potentials proved to be useful in investigating the topology of X by means of the beautiful Li-Tam-Wang's theory of harmonic functions, cf. [47, 70] (cf. also [46, 63] for comprehensive accounts), and it is therefore of interest to see whether other Liouville type properties could be characterized in terms of Evans potentials. One quickly realizes that, for this to hold, the function \mathcal{F} replacing Δ in (12) must have a very specific form, and in fact, to present, the Laplace-Beltrami is the only operator for which solutions w in (12) *with the equality sign* have been constructed. The reason is that the proof in [44, 56, 57], see also [74], strongly uses the characterization of parabolicity in terms of the 2-capacity and the linearity of the Laplace-Beltrami operator. Even for $q \neq 2$, the equivalence of q -parabolicity with the existence of q -harmonic Evans potentials is still an open problem, although results for more general operators on rotationally symmetric manifolds (cf. the last section in [54]) indicate that it is likely to hold.

Quite differently, if we just require that w be a *supersolution*, that is, $\Delta_q w \leq 0$, things are much more flexible and are still worth interest, as we shall see later. In [74] the author proved that the k -parabolicity is equivalent to the existence of w satisfying (12) with the last condition weakened to $\Delta_q w \leq 0$. Although part of the proof uses q -capacities and is therefore very specific to the q -Laplacian, the underlying principle is general: the construction of w proceeds by “stacking” solutions of suitable obstacle problems, an idea that will be described later in a more general framework.

It is natural to ask what is the picture for $p = +\infty$, that is, setting,

$$\text{cap}_\infty(K, \Omega) = \inf \left\{ \|\nabla \phi\|_\infty, : \phi \in \text{Lip}_c(\Omega), \phi \geq 1 \text{ on } K \right\},$$

to study ∞ -parabolic manifolds, defined as those for which $\text{cap}_\infty(K, X) = 0$ for every compact K . The problem has recently been addressed in [65], where the authors proved that

$$X \text{ is } \infty\text{-parabolic} \iff X \text{ is (geodesically) complete}$$

and thus, a-posteriori, ∞ -parabolicity is equivalent to Ekeland’s principle. Below, we shall complement these characterizations as applications of our main duality principle. To do so, we exploit the existence of ∞ -capacitors for (K, Ω) , that is, *suitable* minimizers (absolutely minimizing Lipschitz extensions, cf. [17, 39]) for $\text{cap}_\infty(K, \Omega)$. Their existence was first considered by G. Aronsson [6] and proved by R. Jensen [38] when $X = \mathbb{R}^m$: the ∞ -capacitor turns out to be the (unique) solution of

$$(13) \quad \begin{cases} \Delta_\infty u \doteq \nabla^2 u(\nabla u, \nabla u) = 0 & \text{on } \Omega \setminus K, \\ u = 1 & \text{on } K, \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the equation is meant in the viscosity sense (see below). The operator Δ_∞ , called the infinity Laplacian, has recently attracted a lot of attention because of its appearance in Analysis, Game Theory and Physics, and its investigation turns out to be challenging because of its high degeneracy (see [38, 17, 18]).

1.3. Link with stochastic processes. Before introducing the duality, we mention some other important function-theoretic properties of X , coming from stochastic analysis, that fit well with our setting and give further geometric motivation. We start recalling that parabolicity can be further characterized in terms of the Brownian motion on X . Briefly, on each Riemannian manifold X one can construct the heat kernel $p(x, y, t)$ (cf. [21]), and consequently a stochastic process \mathcal{B}_t whose infinitesimal generator is Δ , called the Brownian motion, characterized by the identity

$$(14) \quad \mathbb{P}(\mathcal{B}_t \in \Omega : \mathcal{B}_0 = x) = \int_\Omega p(x, y, t) dy,$$

for every open subset $\Omega \subset X$ (see [7] for a beautiful, self-contained introduction).

Definition 1.6. *Let \mathcal{B}_t be the Brownian motion on X .*

- \mathcal{B}_t on X is called *recurrent* if, almost surely, its trajectories visit every fixed compact set $K \subset X$ infinitely many times.
- \mathcal{B}_t on X is called *non-explosive* if, almost surely, its trajectories do not escape to infinity in finite time, that is, (14) with $\Omega = X$ is identically 1 for some (equivalently, every) $(x, t) \in X \times \mathbb{R}^+$.

A manifold whose Brownian motion is non-explosive is called *stochastically complete*. The recurrence of \mathcal{B}_t is equivalent to the parabolicity of X , cf. [28], and can therefore be characterized in terms of a property of type (\mathcal{A}) above. Similarly, by [28, Thm. 6.2], X is stochastically complete if and only if there exists no open subset $U \subset X$ supporting a solution u of

$$(15) \quad \begin{cases} \Delta u \geq \lambda u & \text{on } U, \quad \text{for some } \lambda \in \mathbb{R}^+, \\ u > 0 & \text{on } U, \quad u = 0 \quad \text{on } \partial U. \end{cases}$$

for some (equivalently, any) fixed $\lambda \in \mathbb{R}^+$. The last property is (\mathcal{A}) provided that (7) is replaced by $\Delta u - \lambda u \geq 0$, that is, choosing $f(r) = \lambda r$ and removing the gradient condition. Again by translation, rescaling and localizing arguments, the function λr can be replaced by any f satisfying $(f\xi)$.

Around 15 years ago, new interest arose around the notion of stochastic completeness, after the observation in [59, 60] that the property is *equivalent* to a relaxed form of the strong Laplacian principle, called the *weak (Laplacian) maximum principle*. Namely, X has the weak Laplacian principle if, for each $u \in C^2(X)$ bounded above, there exists a sequence $\{x_k\}$ such that

$$u(x_k) > \sup_X u - k^{-1}, \quad \Delta u(x_k) \leq k^{-1},$$

that is, (2) holds with no gradient condition. The weak Hessian principle can be defined accordingly. It turns out that, in most geometric applications, the gradient condition in (2) is unnecessary, making thus interesting to study both the possible difference between weak and strong principles, and the geometric conditions guaranteeing the weak principles.

Remark 1.7 (Strong Laplacian \neq weak Laplacian). It is easy to construct *incomplete* manifolds satisfying the weak Laplacian principle but not the strong one, for instance $X = \mathbb{R}^m \setminus \{0\}$ (cf. [50, Ex. 1.21]). A nice example of a *complete*, radially symmetric surface satisfying the weak Laplacian principle but not the strong one has recently been found in [13]. Therefore, the two principles are really different. Also, the weak Laplacian principle is unrelated to the (geodesic) completeness of X , and in fact, if one removes a compact subset K that is polar for the Brownian motion on X , $X \setminus K$ is still stochastically complete (see Theorem 6.3 below).

Remark 1.8 (Geometric conditions for weak Laplacian principle). Conditions involving just the volume growth of balls in X (that, by Bishop-Gromov volume comparison, are weaker than those in (5), cf. [63]) were first considered in [40, 46], later improved in [28, Thm. 9.1]: X is stochastically complete provided that

$$(16) \quad \int^{+\infty} \frac{r}{\log \text{vol} B_r} dr = +\infty.$$

As a consequence of work of R.Z. Khas'minskii [42, 60], X is stochastically complete provided that it supports an exhaustion w outside a compact set K that satisfies

$$(17) \quad 0 < w \in C^2(X \setminus K), \quad w(x) \rightarrow +\infty \text{ as } x \text{ diverges}, \quad \Delta w \leq \lambda w \text{ on } X \setminus K,$$

for some $\lambda > 0$. The analogy with (3) and (12) is evident, and it is the reason why we call w in (3) a Khas'minskii type potential. It was first observed in [54] that, in fact, the existence of w satisfying (17) is *equivalent* to the stochastic completeness of X . It is therefore natural to ask whether this is specific to the operators Δu and $\Delta u - \lambda u$ or if it is a more general fact, and, in the latter case, how one can take advantage from such an equivalence. This is the starting point of the papers [54, 50].

A further motivation to study Khas'minskii type potentials comes from the desire to understand the link between the Hessian maximum principles and the theory of stochastic processes. It has

been suggested in [62, 64] that a good candidate to be a probabilistic counterpart of a Hessian principle is the *martingale completeness* of X . In fact, one can study the non-explosion property for a natural class of stochastic processes that includes the Brownian motion: the class of martingales (cf. [23, 69]).

- X is called *martingale complete* if and only if each martingale on X has infinite lifetime almost surely.

Differently from the case of stochastic completeness, there is not much literature on the interplay between martingale completeness and geometry, with the notable exception of [23]. The picture is still fragmentary and seems to be quite different from the Laplacian case: for instance, a martingale complete manifold must be (geodesically) complete ([23, Prop. 5.36]). In [23, Prop. 5.37], by using probabilistic tools M. Emery proved that X is martingale complete provided that there exists $w \in C^2(X)$ satisfying

$$(18) \quad \begin{aligned} 0 < w \in C^2(X), \quad w(x) \rightarrow +\infty \text{ as } x \text{ diverges,} \\ |\nabla w| \leq C, \quad \nabla dw \leq C \langle \cdot, \cdot \rangle \text{ on } X, \end{aligned}$$

for some $C > 0$. Evidently, this is again a Khas'minskii type property. Although the gradient condition in (18) might suggest that the martingale completeness of X be related to the strong Hessian principle, in [62, 64] the authors give some results to support a tight link to the *weak* Hessian principle. Which Hessian principle relates to martingale completeness, and why? Is (18) equivalent to the martingale completeness of X ?

The picture described above for Omori-Yau and Ekeland principles, and for parabolicity, stochastic and martingale completeness, suggest that there might be a general “duality principle” relating an appropriate maximum principle in the form of (A) on open sets, to the existence of suitable Khas'minskii type potentials. This is in fact the case, and the rest of this note aims to settle the problem in the appropriate framework, to explain our main result (the AK-duality) and describe its geometric consequences. An important starting point is to reduce the regularity of solutions of the relevant differential inequalities.

1.4. On weak formulations: the case of quasilinear operators. Formulations of maximum principles at infinity for functions with less than C^2 regularity have already been studied in depth in recent years, see [60, 61, 64, 4, 11], by using distributional solutions. Due to the appearance of quasilinear operators in Geometric Analysis, a natural class of inequalities to investigate is the following quasilinear one:

$$(19) \quad \Delta_{\mathbf{a}} \mathbf{u} \doteq \operatorname{div} \left(\mathbf{a}(|\nabla \mathbf{u}|) \nabla \mathbf{u} \right) \geq \mathbf{b}(x) f(\mathbf{u}) \mathbf{l}(|\nabla \mathbf{u}|),$$

for $\mathbf{a} \in C(\mathbb{R}^+)$, $0 < \mathbf{b} \in C(X)$, $f \in C(\mathbb{R})$, $\mathbf{l} \in C(\mathbb{R}_0^+)$, considered in [11] in full generality. For instance, the study of graphs with prescribed mean curvature and of mean curvature solitons in warped product ambient space leads to inequalities like

$$\operatorname{div} \left(\frac{\nabla \mathbf{u}}{\sqrt{1 + |\nabla \mathbf{u}|^2}} \right) \geq \frac{\mathbf{b}(x) f(\mathbf{u})}{\sqrt{1 + |\nabla \mathbf{u}|^2}},$$

see Section 1 in [11]. The weak and strong maximum principles are stated in terms of functions solving (19) on some upper level set, much in the spirit of (i) at page 4, and are summarized in the next

Definition 1.9. *We say that*

- $(\mathbf{bl})^{-1}\Delta_{\mathbf{a}}$ satisfies the weak maximum principle at infinity if for each non-constant $\mathbf{u} \in \text{Lip}_{\text{loc}}(X)$ bounded above, and for each $\eta < \sup_X \mathbf{u}$,

$$\inf_{\{\mathbf{u} > \eta\}} \left\{ \left(\mathbf{b}(x)\mathbf{l}(|\nabla \mathbf{u}|) \right)^{-1} \Delta_{\mathbf{a}} \mathbf{u} \right\} \leq 0,$$

and the inequality has to be intended in the following sense: if \mathbf{u} solves

$$(20) \quad \Delta_{\mathbf{a}} \mathbf{u} \geq K \mathbf{b}(x)\mathbf{l}(|\nabla \mathbf{u}|) \quad \text{weakly on } \{\mathbf{u} > \eta\},$$

for some $K \in \mathbb{R}$, then necessarily $K \leq 0$.

- $(\mathbf{bl})^{-1}\Delta_{\mathbf{a}}$ satisfies the strong maximum principle at infinity if for each non-constant $\mathbf{u} \in C^1(X)$ bounded above, and for each $\eta < \sup_X \mathbf{u}$, $\varepsilon > 0$,

$$(21) \quad \Omega_{\eta, \varepsilon} = \{x \in X : \mathbf{u}(x) > \eta, |\nabla \mathbf{u}(x)| < \varepsilon\} \quad \text{is non-empty,}$$

and

$$\inf_{\Omega_{\eta, \varepsilon}} \left\{ \left(\mathbf{b}(x)\mathbf{l}(|\nabla \mathbf{u}|) \right)^{-1} \Delta_{\mathbf{a}} \mathbf{u} \right\} \leq 0,$$

where, again, the inequality has to be intended in the way explained above.

To present, there exist sharp sufficient conditions both to guarantee the weak and the strong principles for $(\mathbf{bl})^{-1}\Delta_{\mathbf{a}}$. These are explicit, and expressed in terms of the growth of the Ricci curvature (of the type in (5)) or of the volume of geodesic balls in X (resembling (16)). The conditions enable to deduce, among others, sharp Liouville theorems for entire graphs with controlled mean curvature. The interested reader is referred to Theorems 1.2 and 1.7 in [11] for the most up-to-date results, and to [60, 4, 11] for applications. While working with distributional solutions is quite effective for the weak principle, it seems not an optimal choice in the presence of a gradient condition because $\Omega_{\eta, \varepsilon}$ in (21) needs to be open and thus forces to restrict to C^1 functions \mathbf{u} . For our purposes, we found more appropriate to work with upper semicontinuous (USC) viscosity solutions.

2. THE GENERAL FRAMEWORK

We summarize the picture both for the weak and the strong principles. By property (\mathcal{A}) above, they can be rephrased in terms of solutions of a fully nonlinear PDE of the type

$$(22) \quad \mathcal{F}(x, \mathbf{u}(x), \nabla \mathbf{u}(x), \nabla^2 \mathbf{u}(x)) \geq 0$$

on an open subset $U \subset X$, where \mathcal{F} is continuous in its arguments, elliptic and proper, in the following sense:

$$\begin{aligned} (\text{degenerate ellipticity}) \quad & \mathcal{F}(x, r, \mathbf{p}, \mathbf{A}) \geq \mathcal{F}(x, r, \mathbf{p}, \mathbf{B}) \quad \text{if } \mathbf{A} \geq \mathbf{B} \text{ as a quadratic form,} \\ (\text{properness}) \quad & \mathcal{F}(x, r, \mathbf{p}, \mathbf{A}) \leq \mathcal{F}(x, s, \mathbf{p}, \mathbf{A}) \quad \text{if } r \geq s. \end{aligned}$$

For instance,

$$\mathcal{F} = \text{Tr}(\mathbf{A}) - f(r) \quad \text{for the weak Laplacian case, or}$$

$$\mathcal{F} = \max \{ \lambda_{\mathbf{m}}(\mathbf{A}) - f(r), |\mathbf{p}| - \xi(-r) \} \quad \text{for the strong Hessian case,}$$

for some (any) f, ξ satisfying $(f\xi)$. Note that the 4-ple $(x, r, \mathbf{p}, \mathbf{A})$ lies in the set

$$J^2(X) = \left\{ (x, r, \mathbf{p}, \mathbf{A}) : x \in X, r \in \mathbb{R}, \mathbf{p} \in T_x X, \mathbf{A} \in \text{Sym}^2(T_x X) \right\},$$

called the 2-jet bundle of X , and in what follows, with $J_x^2 \mathbf{u}$ we denote the 2-jet of \mathbf{u} at x , i.e. the 4-ple $(x, \mathbf{u}(x), \nabla \mathbf{u}(x), \nabla^2 \mathbf{u}(x))$.

Remark 2.1 (Regularity of solutions). Although in the above discussion we dealt with C^2 solutions, it will be crucial for us to relax the regularity requirements and consider viscosity solutions: an upper semicontinuous (USC) function $u : X \rightarrow [-\infty, +\infty)$ solves (22) in the viscosity sense provided that, for every x and for every test function ϕ of class C^2 in a neighbourhood of x and touching u from above at x , that is, satisfying

$$\begin{cases} \phi \geq u & \text{around } x, \\ \phi(x) = u(x), \end{cases} \quad \text{it holds} \quad \mathcal{F}(x, \phi(x), \nabla\phi(x), \nabla^2\phi(x)) \geq 0,$$

To state the property that encompasses the maximum principles discussed above, it is more convenient for us to exploit the geometric approach to fully-nonlinear PDEs pioneered by N.V. Krylov [43] and systematically developed by R. Harvey and B. Lawson Jr. in recent years ([29, 30, 32]). To the differential inequality (22), we associate the closed subset

$$(23) \quad F = \left\{ (x, r, p, A) : \mathcal{F}(x, r, p, A) \geq 0 \right\} \subset J^2(X).$$

The ellipticity and properness of \mathcal{F} imply a positivity and negativity property for F (properties (P) and (N) in [30]). To avoid some pathological behaviour in the existence-uniqueness theory for the Dirichlet problem for $\mathcal{F} = 0$, one also needs a mild topological requirement on F (assumption (T) in [30]). A subset $F \subset J^2(X)$ satisfying (P), (N), (T) is called a *subequation*: it might be given in terms of a function $\mathcal{F} : J^2(X) \rightarrow \mathbb{R}$, as in (23), but not necessarily.

A function $u \in C^2(X)$ is said to be F -subharmonic if $J_x^2 u \in F$ for each $x \in X$. If $u \in \text{USC}(X)$, as in Remark 2.1 we say that u is F -subharmonic if, for every test function $\phi \in C^2$ at any point x , $J_x^2 \phi \in F$. Given an open subset $\Omega \subset M$, we define

$$F(\Omega) = \left\{ u \in \text{USC}(\Omega) : u \text{ is } F\text{-subharmonic on } \Omega \right\},$$

while, for closed K , we set $F(K)$ to denote the functions $u \in \text{USC}(K)$ that are F -subharmonics on $\text{Int } K$.

Examples that are relevant for us include the following subsets for $f \in C(\mathbb{R})$ non-decreasing and denoting with $\lambda_1(A) \leq \dots \leq \lambda_m(A)$ the eigenvalues of A .

(E1) (**Eikonal type**). The eikonal $E = \{|p| \leq 1\}$, and its modified version $E_\xi = \{|p| \leq \xi(r)\}$ for ξ satisfying $(f\xi)$. In view of the properties of ξ , note that E_ξ -subharmonics must be non-positive.

(E2) (**k-subharmonics**). $F = \{\lambda_1(A) + \dots + \lambda_k(A) \geq f(r)\}$, $k \leq m$. When $f = 0$, F -subharmonic functions are called k -plurisubharmonic; these subequations, which naturally appear in the theory of submanifolds, have been investigated for instance in [76, 68, 31, 32]. The class encompasses the subequation $\{\text{Tr}(A) \geq f(r)\}$, related both to the stochastic completeness of X (if f satisfies $(f\xi)$) and to the parabolicity of X (if $f = 0$).

(E3) (**Prescribing eigenvalues**). $F = \{\lambda_k(A) \geq f(r)\}$, for $k \in \{1, \dots, \dim X\}$. For $k = m$ (and, as we shall see by duality, $k = 1$) this is related to the Hessian principles. In particular, if $f \equiv 0$, the subequations

$$F_j = \{\lambda_j(A) \geq 0\}$$

describe the m -branches associated to the Monge-Ampère equation $\det(\nabla^2 u) = 0$.

($\mathcal{E}4$) (**Branches of k-Hessian subequation**). For $\lambda \doteq (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ and $k \in \{1, \dots, m\}$, consider the elementary symmetric function

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq m} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

Since σ_k is invariant by permutation of coordinates of λ , we can define $\sigma_k(\mathbf{A})$ as σ_k being applied to the ordered eigenvalues $\{\lambda_j(\mathbf{A})\}$. According to Gårding's theory in [26], $\sigma_k(\lambda)$ is a hyperbolic polynomial with respect to the vector $\mathbf{v} = (1, \dots, 1) \in \mathbb{R}^m$. Denote with

$$\mu_1^{(k)}(\lambda) \leq \dots \leq \mu_k^{(k)}(\lambda)$$

the ordered eigenvalues³ of σ_k . Clearly, $\mu_j^{(k)}$ is permutation invariant, thus the expression $\mu_j^{(k)}(\mathbf{A})$ is meaningful. As a matter of fact,

$$F_j = \left\{ \mu_j^{(k)}(\mathbf{A}) \geq f(\mathbf{r}) \right\}, \quad 1 \leq j \leq k$$

is a subequation. In particular, if $f \equiv 0$, F_1, \dots, F_k are called the branches of the k-Hessian equation $\sigma_k(\nabla^2 \mathbf{u}) = 0$. The smallest branch F_1 can be equivalently described as

$$F_1 = \left\{ \sigma_1(\mathbf{A}) \geq 0, \dots, \sigma_k(\mathbf{A}) \geq 0 \right\}.$$

Many more examples of this kind arise from hyperbolic polynomials $q(\lambda)$, cf. [35].

($\mathcal{E}5$) (**Subequations on complex, quaternionic and Cayley manifolds**). If X is an almost complex, Hermitian manifold, the complexified Hessian matrix \mathbf{A} splits into pieces of type $(2, 0)$, $(1, 1)$ and $(0, 2)$, and it makes sense to consider the last three examples in terms of the eigenvalues of the hermitian symmetric matrix $\mathbf{A}^{(1,1)}$. In particular this includes plurisubharmonic functions, that is, solutions of

$$\left\{ \lambda_1(\mathbf{A}^{(1,1)}) \geq 0 \right\}.$$

Analogous examples can be given on quaternionic and octonionic manifolds.

($\mathcal{E}6$) (**Pucci operators**). For $0 < \lambda \leq \Lambda$, the Pucci operators (cf. [14]) are classically defined as

$$\begin{aligned} \mathcal{P}_{\lambda, \Lambda}^+(\nabla^2 \mathbf{u}) &= \sup \left\{ \text{Tr}(X \cdot \nabla^2 \mathbf{u}) \quad : \quad X \in \text{Sym}^2(\text{TX}) \text{ with } \lambda \mathbf{I} \leq X \leq \Lambda \mathbf{I} \right\}, \\ \mathcal{P}_{\lambda, \Lambda}^-(\nabla^2 \mathbf{u}) &= \inf \left\{ \text{Tr}(X \cdot \nabla^2 \mathbf{u}) \quad : \quad X \in \text{Sym}^2(\text{TX}) \text{ with } \lambda \mathbf{I} \leq X \leq \Lambda \mathbf{I} \right\}. \end{aligned}$$

The subequations describing solutions of $\mathcal{P}_{\lambda, \Lambda}^\pm(\nabla^2 \mathbf{u}) \geq f(\mathbf{u})$ can be defined as follows: denoting with $\mathbf{A}^+ \geq 0$ and $\mathbf{A}^- \leq 0$ the positive and negative part of a symmetric matrix $\mathbf{A} = \mathbf{A}^+ + \mathbf{A}^-$, we can set

$$\begin{aligned} F_{\lambda, \Lambda}^+ &= \left\{ \lambda \text{Tr}(\mathbf{A}^-) + \Lambda \text{Tr}(\mathbf{A}^+) \geq f(\mathbf{r}) \right\}, \\ F_{\lambda, \Lambda}^- &= \left\{ \Lambda \text{Tr}(\mathbf{A}^-) + \lambda \text{Tr}(\mathbf{A}^+) \geq f(\mathbf{r}) \right\}. \end{aligned}$$

³That is, the opposite of the roots of $\mathcal{P}(t) \doteq \sigma_k(\lambda + t\mathbf{v}) = 0$.

($\mathcal{E}7$) (**Quasilinear**). We can also consider viscosity solutions of

$$(24) \quad \Delta_{\mathbf{a}} \mathbf{u} \doteq \operatorname{div}(\mathbf{a}(|\nabla \mathbf{u}|) \nabla \mathbf{u}) \geq f(\mathbf{u}),$$

for $\mathbf{a} \in C^1(\mathbb{R}^+)$ satisfying

$$(25) \quad \theta_1(t) \doteq \mathbf{a}(t) + t\mathbf{a}'(t) \geq 0, \quad \theta_2(t) \doteq \mathbf{a}(t) > 0.$$

Examples include

- the mean curvature operator, describing the mean curvature of the graph hypersurface $\{(x, v(x)) : x \in M\}$ into the Riemannian product $M \times \mathbb{R}$. In this case, $\mathbf{a}(t) = (1+t^2)^{-1/2}$;
- the q -Laplacian Δ_q , $q > 1$, where $\mathbf{a}(t) = t^{q-2}$;
- the operator of exponentially harmonic functions, where $\mathbf{a}(t) = \exp(t^2)$, considered for instance in [22];

Indeed, expanding the divergence we can set

$$F = \overline{\{\mathbf{p} \neq 0, \operatorname{Tr}(\mathbb{T}(\mathbf{p})\mathcal{A}) > f(r)\}},$$

where

$$\mathbb{T}(\mathbf{p}) \doteq \mathbf{a}(|\mathbf{p}|) \langle \cdot, \cdot \rangle + \frac{\mathbf{a}'(|\mathbf{p}|)}{|\mathbf{p}|} \mathbf{p} \otimes \mathbf{p} = \theta_1(|\mathbf{p}|) \Pi_{\mathbf{p}} + \theta_2(|\mathbf{p}|) \Pi_{\mathbf{p}^\perp},$$

and $\Pi_{\mathbf{p}}, \Pi_{\mathbf{p}^\perp}$ are, respectively, the $(2,0)$ -versions of the orthogonal projections onto the spaces $\langle \mathbf{p} \rangle$ and \mathbf{p}^\perp . Similarly, we can consider the non-variational, *normalized* quasilinear operator given by

$$F = \left\{ \overline{\mathbf{p} \neq 0, \frac{\operatorname{Tr}(\mathbb{T}(\mathbf{p})\mathcal{A})}{\max\{\theta_1(|\mathbf{p}|), \theta_2(|\mathbf{p}|)\}} > f(r)} \right\}.$$

In the case of the mean curvature operator, the last subequation represents viscosity solutions of

$$\operatorname{div} \left(\frac{\nabla \mathbf{u}}{\sqrt{1 + |\nabla \mathbf{u}|^2}} \right) \geq \frac{f(\mathbf{u})}{\sqrt{1 + |\nabla \mathbf{u}|^2}},$$

that are related to prescribed mean curvature graphs and mean curvature solitons in warped product spaces, see [11, Chapter 1].

($\mathcal{E}8$) (**∞ -Laplacian**). The normalized ∞ -Laplacian $F = \overline{\{\mathbf{p} \neq 0, |\mathbf{p}|^{-2} \mathcal{A}(\mathbf{p}, \mathbf{p}) > f(r)\}}$.

In ($\mathcal{E}7$) and ($\mathcal{E}8$), the necessity to take as F the closure of its interior is made necessary to match property (T), due to the possible singularity of the operator at $\mathbf{p} = 0$. The above examples can be defined on each Riemannian (complex, quaternionic, octonionic) manifold, since there is no explicit dependence of F from the point x , and are therefore called *universal* subequations. To include large classes of subequations with coefficients depending on the point x , that can be seen as “deformations” of universal ones, Harvey and Lawson in [30] introduced the concept of *local jet-equivalence* between subequations. Without going into the details here, we limit to say that, for instance, any semilinear inequality of the type

$$\mathbf{a}^{ij}(x) u_{ij} + \mathbf{b}^i(x) u_i \geq c(x) f(\mathbf{u})$$

for smooth $\mathbf{a}^{ij}, \mathbf{b}^i, c$ with $c > 0$ on X and $[\mathbf{a}^{ij}]$ positive definite at every point, is locally jet-equivalent to the universal example describing solutions of

$$\Delta \mathbf{u} \geq f(\mathbf{u}).$$

The key fact is that local jet-equivalence allows to transfer properties from the universal example to the subequations locally jet equivalent to it.

Supersolutions for \mathcal{F} , that is, solutions of

$$(26) \quad \mathcal{F}(x, u(x), \nabla u(x), \nabla^2 u(x)) \leq 0,$$

are taken into account starting from the observation that $w = -u$ solves

$$(27) \quad \widetilde{\mathcal{F}}(x, w(x), \nabla w(x), \nabla^2 w(x)) \geq 0,$$

with

$$\widetilde{\mathcal{F}}(x, r, p, A) = -\mathcal{F}(x, -r, -p, -A).$$

This suggests to define the *dual subequation*

$$\widetilde{F} = - \sim \text{Int}(F).$$

In particular,

$$\text{if } F = \{\mathcal{F}(x, r, p, A) \geq 0\}, \quad \text{then } \widetilde{F} = \{\widetilde{\mathcal{F}}(x, r, p, A) \geq 0\}.$$

Therefore, u is \widetilde{F} -subharmonic if $-u$ is a supersolution in the standard, viscosity sense, and u is F -harmonic on Ω if $u \in F(\Omega)$ and $-u \in \widetilde{F}(\Omega)$. The above operator is in fact a duality, in particular

$$(28) \quad \widetilde{\widetilde{F \cap G}} = \widetilde{F} \cup \widetilde{G}, \quad \widetilde{\widetilde{F}} = F$$

for each subequations F, G , and \widetilde{F} is a subequation if F is so. Concerning examples (E1) to (E8),

- In (E1), the dual of the eikonal equation is $\{|p| \geq 1\}$, that of E_ξ is $\widetilde{E}_\xi = \{|p| \geq \xi(-r)\}$.
- In (E2),

$$\widetilde{F} = \left\{ \sum_{j=m-k+1}^m \lambda_j(A) \geq f(r) \right\};$$

in particular, $\{\text{Tr}(A) \geq f(r)\}$ is self-dual: $\widetilde{F} = F$.

- In (E3), $\widetilde{F} = \{\lambda_{m-k+1}(A) \geq f(r)\}$.
- In (E4), if $F = \{\mu_j^{(k)}(A) \geq f(r)\}$ then $\widetilde{F} = \{\mu_{k-j+1}^{(k)}(A) \geq f(r)\}$.
- In (E6), the dual of $F_{\lambda, \Lambda}^\pm$ is $F_{\lambda, \Lambda}^\mp$.
- Examples (E7) and (E8) are self-dual: $\widetilde{F} = F$.

Given a subequation F and for $g \in C(X)$ we shall introduce the *obstacle subequation*

$$F^g = F \cap \{r \leq g(x)\},$$

that describes F -subharmonic functions lying below the obstacle g . Note that, since the dual of $\{r \leq g(x)\}$ is $\{r \leq -g(x)\}$, \widetilde{F}^g describes functions u that are \widetilde{F} -subharmonic on the upper set $\{u(x) > g(x)\}$. As we shall see in a moment, functions in \widetilde{F}^0 will be used to describe the property that unifies the maximum principles at infinity described above, and it is therefore expectable, by duality, that obstacles subequations play an important role in our main result.

Definition 2.2. *Let $F \subset J^2(X)$ be a subequation. Given $\Omega \Subset X$ open, $g \in C(\overline{\Omega})$ and $\phi \in C(\partial\Omega)$ with $\phi \leq g$ on $\partial\Omega$, a function $u \in C(\overline{\Omega})$ is said to solve the obstacle problem with obstacle g and boundary value ϕ if*

$$\begin{cases} u & \text{is } F^g\text{-harmonic on } \Omega \\ u = \phi & \text{on } \partial\Omega. \end{cases}$$

3. AHLFORS, KHAS'MINSKII PROPERTIES AND THE AK-DUALITY

The definition of the next property is inspired by the original work of Ahlfors [2], as well as by the recent improvements in [4, 5, 37].

Definition 3.1. *A subequation $H \subset J^2(X)$ is said to satisfy the Ahlfors property if, having set $H_0 = H \cup \{r \leq 0\}$, for each $U \subset X$ open with non-empty boundary and for each $u \in H_0(\bar{U})$ bounded from above and positive somewhere, it holds*

$$\sup_{\partial U} u^+ \equiv \sup_{\bar{U}} u.$$

Roughly speaking, when u is H -subharmonic on the set $\{u > 0\}$, the Ahlfors property means that its supremum is attained on the boundary of U .

Example 3.2. We consider the following subequations:

- 1) If $F = \{\text{Tr}(A) \geq 0\}$, in view of Ahlfors' characterization [2], the Ahlfors property for \tilde{F} ($= F$) is a version, for viscosity solutions, of the property characterizing the parabolicity of X in [2]. Similarly, if $F = \{\text{Tr}(A) \geq f(r)\}$ for f satisfying (f ξ), the Ahlfors property for \tilde{F} can be seen as a viscosity version of the weak Laplacian principle, that is, of the stochastic completeness of X . As a matter of fact (cf. [50]), in both of the cases the property still *characterizes* the parabolicity, respectively the stochastic completeness, of X .
- 2) The Ahlfors property for the dual eikonal $\tilde{E} = \{|p| \geq 1\}$ can be viewed as a viscosity version of Ekeland's principle. Its equivalence to the original Ekeland's principle, hence to geodesic completeness, is one of the applications of our main result below.
- 3) Consider the subequations $F = \{\lambda_1(A) \geq f(r)\}$ and $E_\xi = \{|p| \leq \xi(r)\}$, for (f, ξ) satisfying (f ξ). Then, the Ahlfors property for the dual

$$\begin{aligned} \widetilde{F \cap E_\xi} &= \tilde{F} \cup \widetilde{E_\xi} = \{\lambda_m(A) \geq f(r)\} \cup \{|p| \geq \xi(-r)\} \\ &= \left\{ \max\{|p| - \xi(-r), \lambda_m(A) - f(r)\} \geq 0 \right\} \end{aligned}$$

can be seen as a viscosity analogue of the strong Hessian principle. Analogously, the Ahlfors property for $\tilde{F} \cup \widetilde{E_\xi}$ with $F = \{\text{Tr}(A) \geq f(r)\}$ is a natural, viscosity version of Yau's strong Laplacian principle. Differently from the examples in 1), it is *not known* whether these Ahlfors properties are, in fact, equivalent to the classical strong Hessian and Laplacian principles for C^2 solutions.

A comment is in order: although the above viscosity versions might be strictly stronger than the corresponding classical ones, all of the known geometric conditions that guarantee the weak and strong Hessian or Laplacian principles in the C^2 case *also ensure* their viscosity counterparts. Therefore, passing to the viscosity realm does not prevent from geometric applications, and indeed is able to uncover new relations. To see them, we shall introduce the Khas'minskii properties, that generalize (3) and (17). Hereafter, a pair (K, h) consists of

- a smooth, relatively compact open set $K \subset X$;
- a function $h \in C(X \setminus K)$ satisfying $h < 0$ on $X \setminus K$ and $h(x) \rightarrow -\infty$ as x diverges.

Definition 3.3. *A subequation $F \subset J^2(X)$ satisfies the Khas'minskii property if, for each pair (K, h) , there exists a function w satisfying:*

$$(29) \quad w \in F(X \setminus K), \quad h \leq w \leq 0 \quad \text{on } X \setminus K, \quad \text{and} \quad w(x) \rightarrow -\infty \quad \text{as } x \text{ diverges.}$$

Such a function w is called a Khas'minskii potential for (K, h) .

Loosely speaking, F has the Khas'minskii property if it is possible to construct F -subharmonic exhaustions that decay to $-\infty$ as slow as we wish. In practice, checking the Khas'minskii property might be a hard task, and often, from the geometric problem under investigation, one is just able to extract *some* of the Khas'minskii potentials. This motivates the following definition (cf. the recent [52]).

Definition 3.4. A subequation $F \subset J^2(X)$ satisfies the weak Khas'minskii property if there exist a relatively compact, smooth open set K and a constant $C \in \mathbb{R} \cup \{+\infty\}$ such that, for each $x_0 \notin \bar{K}$ and each $\varepsilon > 0$, there exists w satisfying

$$(30) \quad w \in F(X \setminus K), \quad w \leq 0 \quad \text{on } X \setminus K, \quad w(x_0) \geq -\varepsilon, \quad \limsup_{x \rightarrow \infty} w(x) \leq -C.$$

We call such a w a weak Khas'minskii potential for the triple $(\varepsilon, K, \{x_0\})$.

Remark 3.5. When $C = +\infty$ and F is scale invariant (that is, it is fiber-wise a cone), the condition over ε in (30) can be avoided by a simple rescaling.

Example 3.6. The existence of w in (3), for G satisfying (4), implies that a weak Khas'minskii property holds for the subequation

$$\{\operatorname{Tr}(A) \geq G(-r)\} \cap \{|p| \leq G(-r)\}.$$

Indeed, the weak Khas'minskii potentials can be constructed by suitably rescaling and modifying the function $-w$. Up to playing with w and G , the above is equivalent to the weak Khas'minskii property for

$$\{\operatorname{Tr}(A) \geq f(r)\} \cap \{|p| \leq \xi(r)\},$$

for some (any) (f, ξ) satisfying $(f\xi)$. Similarly, the existence of w in (17) implies the weak Khas'minskii property for $F = \{\operatorname{Tr}(A) \geq f(r)\}$.

We are ready to state our main result, the Ahlfors-Khas'minskii duality (shortly, AK-duality), Theorems 4.3 and 4.10 in [50]. It applies to subequations F on X that are locally jet-equivalent to a universal one and satisfy a few further assumptions. Some of them are merely technical and will not be described here. Their validity characterizes the set of *admissible* subequations, that is still quite general. For instance, each of the examples in $(\mathcal{E}2), \dots, (\mathcal{E}6)$, and the subequations locally jet-equivalent to them, are admissible provided that f satisfies $(f\xi)$.

Theorem 3.7. Let $F \subset J^2(X)$ be an admissible subequation, locally jet-equivalent to a universal one. Assume that

($\mathcal{H}1$) negative constants are strictly F -subharmonic;

($\mathcal{H}2$) F satisfies the comparison theorem: whenever $\Omega \Subset X$ is open, $u \in F(\Omega)$, $v \in \tilde{F}(\Omega)$,

$$u + v \leq 0 \quad \text{on } \partial\Omega \implies u + v \leq 0 \quad \text{on } \Omega.$$

Then, AK-duality holds for F and for $F \cap E_\xi$ for some (any) ξ satisfying $(f\xi)$, i.e.,

$$\begin{array}{ccc} F \text{ satisfies } (K) & \iff & F \text{ satisfies } (K_w) \\ (\text{Khas'minskii prop.}) & & (\text{weak Khas'minskii prop.}) \iff \tilde{F} \text{ satisfies } (A) \\ & & (\text{Ahlfors prop.}) \end{array},$$

and

$$F \cap E_\xi \text{ satisfies } (K) \iff F \cap E_\xi \text{ satisfies } (K_w) \iff \tilde{F} \cup \tilde{E}_\xi \text{ satisfies } (A).$$

Seeking to clarify the role of each assumption in the AK-duality, we briefly examine the importance of each one.

Remark 3.8 (On assumption $(\mathcal{H}1)$). This property holds for each of $(\mathcal{E}2), \dots, (\mathcal{E}6)$ provided that f satisfies $(f\xi)$, and it is important to ensure the validity of the finite maximum principle: functions $u \in \tilde{F}^0(Y)$ cannot achieve a local positive maximum. The latter is crucial for our proof to work.

Remark 3.9 (On assumption $(\mathcal{H}2)$). This is delicate to check, and curiously enough it plays a role just in the proof of $(K_w) \Rightarrow (A)$. Comparison holds for uniformly continuous subequations which are strictly increasing in the r variable, a case that covers examples $(\mathcal{E}2), \dots, (\mathcal{E}5)$ as well as $(\mathcal{E}6)$, see [50, Thm. 2.25] for details⁴. The uniform continuity resembles the classical condition 3.14 in [19]. Regarding examples $(\mathcal{E}7)$ and $(\mathcal{E}8)$, the worse dependence on the gradient term makes comparison much subtler. One can check the comparison theorem for the universal subequation in $(\mathcal{E}8)$, even with $f \equiv 0$, by means of other interesting methods, cf. [50, Thm. 2.27]. As for $(\mathcal{E}7)$, on Euclidean space the validity of comparison for strictly increasing f is a direct application of the classical theorem on sums (i.e. Ishii's Lemma, [19] and [30, Thm. C.1]). In a Riemannian setting, Ishii's Lemma uses the infimal convolutions with the squared distance function $r^2(x, y)$ on $X \times X$, and in neighbourhoods where the sectional curvature is negative the Hessian of r^2 is *positive* to second order on pairs of parallel vectors. The error term produced by such positivity can be easily controlled for normalized quasilinear operators, but in the unnormalized case one has to require the boundedness of the eigenvalues θ_1, θ_2 in order to avoid further a-priori bounds on the subsolutions and supersolutions (like Lipschitz continuity of either one of them). Nevertheless, we note that the boundedness of θ_1, θ_2 notably includes the mean curvature operator.

Summarizing, we have

Corollary 3.10. *The AK-duality holds both for F and for $F \cap E_\xi$, with ξ satisfying $(f\xi)$, in each of the following cases:*

- F is locally jet-equivalent to any of $(\mathcal{E}2), \dots, (\mathcal{E}6)$;
- F is the normalized quasilinear example in $(\mathcal{E}7)$, or F is the unnormalized example with $\theta_1, \theta_2 \in L^\infty(\mathbb{R}^+)$;
- F is the universal example in $(\mathcal{E}8)$,

and, in each case, f satisfies $(f\xi)$. Furthermore, the AK-duality holds for the eikonal subequations in $(\mathcal{E}1)$.

Sketch of the proof of the AK-duality: Since $(K) \Rightarrow (K_w)$ is obvious, we shall prove $(K_w) \Rightarrow (A)$ and $(A) \Rightarrow (K)$. The proof of the first implication is inspired by a classical approach that dates back to Phragmén-Lindelöf type theorems in classical complex analysis, and we therefore concentrate on $(A) \Rightarrow (K)$.

Fix a pair (K, h) , and a smooth exhaustion $\{D_j\}$ of X with $K \subset D_1$. Our desired Khas'minskii

⁴In [50], the uniform continuity of the Pucci operators in $(\mathcal{E}6)$ is not explicitly stated but can be easily checked. For instance, in the case of $\mathcal{P}_{\lambda, \Lambda}^+$, referring to Definition 2.23 in [50] and using the min-max definition,

$$\mathcal{P}_{\lambda, \Lambda}^+(B) \geq \mathcal{P}_{\lambda, \Lambda}^+(A) - \mathcal{P}_{\lambda, \Lambda}^+(A - B) \geq \mathcal{P}_{\lambda, \Lambda}^+(A) - \Lambda \operatorname{Tr}((A - B)_+).$$

If $\|(A - B)_+\| < \delta$, then $\mathcal{P}_{\lambda, \Lambda}^+(B) \geq \mathcal{P}_{\lambda, \Lambda}^+(A) - m\Lambda\delta$, that proves the uniform continuity of $\mathcal{F}_{\lambda, \Lambda}^+$. The case of $\mathcal{F}_{\lambda, \Lambda}^-$ is analogous.

potential w will be constructed as a locally uniform limit of a decreasing sequence of USC functions $\{w_i\}$, such that $w_0 = 0$ and for each $i \geq 1$ we have:

$$(31) \quad \begin{aligned} & \text{(a)} \quad w_i \in F(X \setminus K), \quad w_i = (w_i)_* = 0 \quad \text{on } \partial K; \\ & \text{(b)} \quad w_i \geq -i \quad \text{on } X \setminus K, \quad w_i = -i \quad \text{outside a compact set } C_i \text{ containing } D_i; \\ & \text{(c)} \quad (1 - 2^{-i-2})h < w_{i+1} \leq w_i \leq 0 \quad \text{on } X \setminus K, \quad \|w_{i+1} - w_i\|_{L^\infty(D_i \setminus K)} \leq \frac{\varepsilon}{2^i}. \end{aligned}$$

With the above properties, the sequence $\{w_i\}$ is locally uniformly convergent on $X \setminus K$ to some function $w \in F(X \setminus K)$ with $h \leq w \leq 0$ on $X \setminus K$ and satisfying $w(x) \rightarrow -\infty$ as x diverges, that is, to the desired Khas'minskii potential.

Fix $w = w_i$. We build w_{i+1} inductively via a sequence of obstacle problems, an idea inspired by [74, 54]: we fix obstacles $g_j = w + \lambda_j$, for some sequence $\{\lambda_j\} \subset C(X \setminus K)$ such that

$$(32) \quad \begin{cases} 0 \geq \lambda_j \geq -1, \quad \lambda_j = 0 \quad \text{on } K, \quad \lambda_j = -1 \quad \text{on } X \setminus D_{j-1}, \\ \{\lambda_j\} \text{ is an increasing sequence, and } \lambda_j \uparrow 0 \text{ locally uniformly,} \end{cases}$$

and search for solutions of the obstacle problem

$$(33) \quad \begin{cases} u_j & \text{is } F^{g_j}\text{-harmonic on } D_j \setminus K, \\ u_j = 0 & \text{on } \partial K, \quad u_j = -i - 1 \quad \text{on } \partial D_j. \end{cases}$$

However, in some relevant cases we cannot fully solve (33). The first problem we shall consider is the absence of barriers, needed to prove the existence of u_j via Perron's method. No problem arise on ∂D_j , since the constant function $-i - 1$ is F^{g_j} -subharmonic by $(\mathcal{H}1)$. However, since we are working in the complement of a compact set K (think of K being a small geodesic ball, for instance), ∂K might be concave in the outward direction, that in general prevents from having barriers there. To overcome this problem, we modify X inside of K by gluing a compact manifold Y that is Euclidean in a sufficiently small ball \mathbb{B} . The gluing only involves small annuli inside of \mathbb{B} and K , with the new metric coinciding with those of X and Y outside of the gluing region. In particular, the new manifold is Euclidean in a neighbourhood of $\partial \mathbb{B}$. In this way, replacing K by $K' = Y \setminus \mathbb{B}$, $X \setminus K$ embeds isometrically into $X \setminus K'$ and the latter has a *convex* boundary isometric to $\partial \mathbb{B}$. Because of a technical assumption included in those defining the admissibility of F , this is enough to produce barriers on $\partial K'$. Once we perform this change, we suitably modify the subequation F preserving it outside the gluing region, and making it, on K' , the universal Riemannian subequation to which F is locally jet equivalent. Although these modifications change in several ways the manifold and the subequation, they are stable to preserve the Ahlfors property for \tilde{F} as well as the assumption $(\mathcal{H}1)$. The price to pay is that we may lose the comparison property $(\mathcal{H}2)$, since comparison is very sensitive to the geometry of X , at least for some relevant operators like those in $(\mathcal{E}2)$, $(\mathcal{E}3)$, $(\mathcal{E}4)$. This is the main reason why, generally, we cannot fully solve (33). However, with barriers finally available, Perron's method yields an "almost solution" u_j of the obstacle problem on $X \setminus K'$ (see [30], [50, Thm. 3.3]), that is, u_j solves

$$(34) \quad \begin{cases} u_j \in F^{g_j}(\overline{D_j \setminus K'}), \quad (-u_j)^* \in \tilde{F}^{g_j}(\overline{D_j \setminus K'}), \\ u_j = (u_j)_* = 0 \quad \text{on } \partial K', \\ u_j = (u_j)_* = -i - 1 \quad \text{on } \partial D_j. \end{cases}$$

We extend u_j outside D_j by setting $u_j \doteq -i - 1$, and define $v_j \doteq (-u_j)^* - i$. By the definition of Perron's solution, the sequence $\{v_j\}$ is decreasing on $X \setminus K'$. Thus, passing to the limit using that

$g_j \rightarrow w \geq -i$ as $j \rightarrow \infty$, $w = -i$ outside of C_i , we get

$$v_j \downarrow v \in \tilde{F}^0(X \setminus K'), \quad \text{with} \quad \begin{cases} -i \leq v \leq 1 & \text{on } X \setminus K', \\ v = -i < 0 & \text{on } \partial K', \\ v \geq 0 & \text{on } X \setminus C_i. \end{cases}$$

Here is the crucial point where the Ahlfors property enters: in fact, using Ahlfors on $X \setminus K'$ we infer that $v \equiv 0$ outside of C_i , and by the USC-version of Dini's theorem,

$$v_j \downarrow 0 \quad \text{locally uniformly on } X \setminus C_i.$$

Then, the definition of v_j yields

$$(35) \quad u_j \uparrow -i \quad \text{locally uniformly on } X \setminus C_i.$$

It remains to investigate the convergence of u_j on the bounded set $\overline{C}_i \setminus K'$. Although comparison might fail on this set, what guarantees the convergence $u_j \uparrow w$ is that each u_j , being a Perron's solution, is maximal in the set of F^{g_j} -subharmonic functions whose boundary values do not exceed 0 (on $\partial K'$) and $-i - 1$ (on ∂D_j). Concluding, $u_j \uparrow w$ locally uniformly on $X \setminus K'$, hence on $X \setminus K$. For j large enough, if we set $w_{i+1} = u_j$ it is therefore possible to meet all of (a), (b), (c) in (31), as desired.

To treat the case when F is coupled to the eikonal equation E_ξ , the issue is again the absence of barriers on $\partial K'$ to solve the obstacle problem for $F^{g_j} \cap E_\xi$. Indeed, even if, after the gluing, $\partial K'$ is convex in the direction pointing towards $X \setminus K'$, barriers must be E_ξ -subharmonic and the gradient control may prevent to build barriers up to height $-i$ at step i . To overcome this problem, the idea is to modify the subequation E_ξ in the gluing region in a different way at each step i , weakening the bound $\xi(r)$ by means of a cut-off function ϕ_i supported in a neighbourhood of $\partial K'$. The size of ϕ_i depends on the L^∞ norm of the gradient of the barriers on $\partial K'$ joining zero to $-i$, and therefore it diverges as $i \rightarrow \infty$. In this way, we clearly lose the gradient control in the limit in a neighbourhood of K' , but since $K' \Subset K$, for suitable ϕ_i no property of w_i on $X \setminus K$ get lost. \square

It is worth to remark that an important case was left uncover by Theorem 3.7. For instance, when F is independent on r (examples $(\mathcal{E}2), \dots, (\mathcal{E}6)$ with $f \equiv 0$), assumption $(\mathcal{H}1)$ does not hold. However $(\mathcal{H}1)$ is just used to ensure the strong maximum principle for functions in \tilde{F}^0 on any manifold. Therefore, we can state the following alternative version of our main theorem.

Theorem 3.11. *Let $F \subset J^2(X)$ be a universal subequation satisfying $(\mathcal{H}2)$ and*

$(\mathcal{H}1')$ \tilde{F} has the strong maximum principle on each manifold Y where it is defined: \tilde{F}^0 -subharmonic functions on Y are constant if they attain a local maximum.

Then, AK-duality holds for F .⁵

The strong maximum principle for viscosity subsolutions is a classical subject that has been investigated by many authors, in particular we quote [33, 8, 41] (cf. also [66, 67, 11] for the quasilinear case). Particularizing Theorems 3.7 and 3.11 to the mean curvature operator and its normalized version, for which the strong maximum principle is proved in [41], we have the following:

⁵Theorem 3.11 can be stated for F locally jet-equivalent to a universal example, provided that the strong maximum principle in $(\mathcal{H}1')$ holds for each manifold Y and each $\tilde{F} \subset J^2(Y)$ constructed by gluing as in the theorem.

Theorem 3.12. *The AK-duality holds for the subequation in (E7) describing solutions of*

$$(36) \quad \operatorname{div} \left(\frac{\nabla \mathbf{u}}{\sqrt{1 + |\nabla \mathbf{u}|^2}} \right) \geq f(\mathbf{u}) \quad \text{and} \quad \operatorname{div} \left(\frac{\nabla \mathbf{u}}{\sqrt{1 + |\nabla \mathbf{u}|^2}} \right) \geq \frac{f(\mathbf{u})}{\sqrt{1 + |\nabla \mathbf{u}|^2}},$$

for every non-decreasing, odd function $f \in C(\mathbb{R})$.

Other quasilinear operators.

As said, the lack of a strong enough comparison theorem forces us to require, in the unnormalized version of (E7) of Corollary 3.10, the boundedness of the eigenvalues θ_1, θ_2 on \mathbb{R}_0^+ . We believe that the AK-duality holds for each subequation locally jet-equivalent to (E7), both normalized and unnormalized, independently of θ_1, θ_2 . More information can be found in Section 2.5 and Appendix A of [50], where the authors investigate classes of quasilinear operators where comparison holds. For inequalities of the type

$$\operatorname{div} \mathcal{A}(x, \nabla \mathbf{u}) \geq \mathcal{B}(x, \mathbf{u}),$$

with \mathcal{A} a Carathéodory map that locally behaves like a q -Laplacian, and \mathcal{B} non-decreasing in \mathbf{u} with $\mathbf{u} \mathcal{B}(x, \mathbf{u}) \geq 0$, the AK-duality in a slightly less general version was first established in [54]. The use of weak instead of viscosity solutions allows to work with very general \mathcal{A}, \mathcal{B} , since a comparison theorem is easy to show and the obstacle problem is solvable by classical results. Nevertheless, the method does not allow to include a gradient dependence and thus investigate the “strong” versions of the corresponding Ahlfors property.

Liouville property.

As the cases of parabolicity and stochastic completeness show, the Ahlfors property is also related to the next Liouville one:

Definition 3.13. *A subequation $F \subset J^2(X)$ has the Liouville property if any $\mathbf{u} \in F(X)$ bounded from above and non-negative is constant.*

Indeed, in [54] the main result itself is expressed as a duality between Khas’minskii and Liouville properties. It is not difficult to show that the Ahlfors property implies the Liouville one, and that the two are equivalent provided that

$$(37) \quad \mathbf{u} \equiv 0 \quad \text{is } F\text{-harmonic,}$$

cf. [50, Prop. 4.2] and previous work in [2, 28, 4, 5]. While (37) holds in many instances, there are notable exceptions, for example the eikonal subequation. For such subequations, it is the Ahlfors property the one that actually realizes duality.

4. APPLICATIONS

4.1. Completeness, viscosity Ekeland principle and ∞ -parabolicity. Let $\mathbf{u} \in C^1(X)$ be a function bounded from above and assume that there exists a classical C^1 -Khas’minskii potential w , that is, satisfying only the exhaustion and the gradient properties in (18). Up to a rescaling, w is a Khas’minskii potential for the eikonal subequation $E = \{|p| \leq 1\}$. Following the original argument that goes back to Ahlfors [1], we consider a sequence of functions $\mathbf{u} + \frac{1}{k}w$ each of which attains a maximum at some point $x_k \in X$. Up to choosing a subsequence, it is easy to see that

$$u(x_k) > \sup_X u - \frac{1}{k}, \quad \text{and} \quad u(y) \leq u(x_k) + \frac{1}{k}d(x_k, y) \quad \text{for } y \text{ nearby } x_k.$$

Thus, recalling the AK-duality, one can see the Ahlfors property for the dual eikonal subequation $\tilde{E} = \{|p| \geq 1\}$ as a sort of *viscosity version of Ekeland principle*. Clearly, the above argument does not give a formal proof of the equivalence between the Ahlfors property for \tilde{E} and the Ekeland principle stated in Definition 1.1. In fact, it follows from the next application of Theorem 3.7:

Theorem 4.1. *Let X be a Riemannian manifold. Then, the following statements are equivalent:*

- (1) X is complete.
- (2) the dual eikonal $\tilde{E} = \{|p| \geq 1\}$ has the Ahlfors property (viscosity Ekeland principle).
- (3) the infinity Laplacian $F_\infty \doteq \{A(p, p) > 0\}$ has the Ahlfors property.
- (4) F_∞ has the next strengthened Liouville property:
 Any F_∞ -subharmonic function $u \geq 0$ such that $|u(x)| = o(\rho(x))$ as x diverges ($\rho(x)$ the distance from a fixed origin) is constant.

Sketch of the proof: The key implications are (2) \Rightarrow (1) and (3) \Rightarrow (1). Both proceed by contradiction, so assume the existence of a unit speed geodesic γ defined on a maximal finite interval $[0, T)$, and pick a small compact set K not intersecting $\gamma([0, T))$ (this is possible since γ is diverging).

(2) \Rightarrow (1). Apply the AK-duality to produce a Khas'minskii potential $w \in E(X \setminus K)$. By restriction, the function $u \doteq w \circ \gamma$ is E -subharmonic on $[0, T)$, that is, any C^2 test ϕ touching u from above shall satisfy $|\phi'| \leq 1$ at touching points. However, since $u \leq 0$ and $T < +\infty$, we can choose a line with derivative strictly less than -1 lying above the graph of u : translating the line downwards up to the first touching point we get a contradiction. Thus, $T = +\infty$ and X is complete.

(3) \Rightarrow (1): Pick an exhaustion of X by smooth, relatively compact domains Ω_j with $K \Subset \Omega_1$. As we said before, we exploit the existence of a (unique) ∞ -capacitor u_j for (K, Ω_j) (see [39, 17]), that satisfies

$$(38) \quad \begin{cases} u_j \text{ is } F_\infty\text{-harmonic on } \Omega_j \setminus K, \\ u_j = 1 \text{ on } \partial K, \quad u_j = 0 \text{ on } \partial \Omega_j. \end{cases}$$

By comparison (Theorem 2.27 in [50]), and since $\{u_j\}$ is equi-Lipschitz because of the minimization properties of u_j , the sequence $v_j = 1 - u_j$ subconverges locally uniformly to a F_∞ -harmonic, Lipschitz function $v_\infty \geq 0$. Applying the Ahlfors property on $X \setminus K$ we get that $v_\infty = 0$ on $X \setminus K$. Now, setting $w_j = v_j \circ \gamma$, we have $w_j(0) = 0$, $w_j = 1$ after some $T_j < T$, and by integration, $1/T \leq \|w_j'\|_\infty \leq C$ on $[0, T)$, for each j . This is impossible, since $w_j \rightarrow 0$ locally uniformly. \square

4.2. The Hessian principle and martingale completeness. According to 3) in Example 3.2, we formally define the viscosity, weak and strong Hessian principles in terms of Ahlfors properties. Let us consider the subequations $F = \{\lambda_1(A) \geq -1\}$ and $E = \{|p| \leq 1\}$, whose duals are $\tilde{F} = \{\lambda_m(A) \geq 1\}$ and $\tilde{E} = \{|p| \geq 1\}$. Then, X satisfies:

- the viscosity, weak Hessian principle if the Ahlfors property holds for \tilde{F} ;
- the viscosity, strong Hessian principle if the Ahlfors property holds for $\tilde{F} \cup \tilde{E}$.

The r independence on F and E in the above definition are just for convenience. The properties could be stated as in Example 3.2 by making use of a pair of functions (f, ξ) satisfying $(f\xi)$. As discussed in the introduction, there are evidences that an Hessian principle, either weak or strong, be related with the martingale completeness. Perhaps surprisingly, exploiting the low regularity and the AK-duality, we found that the two Hessian principles are equivalent, and that the martingale completeness is necessary for the validity of them. Apart from a regularity issue, this answers a question (Question 70) raised in [62] (see also [64]).

Theorem 4.2. *Let X be a Riemannian manifold. Then, the following properties are equivalent:*

- (1) X satisfies the viscosity, weak Hessian principle;
- (2) X satisfies the viscosity, strong Hessian principle;
- (3) $F \cap E$ has the Khas'minskii property with C^∞ potentials.

In particular, all the above assertions imply that X is martingale (and so, geodesically) complete.

Idea of the proof: As a consequence of AK-duality and the flexibility in the choice of (f, ξ) , each of (1) and (2) is equivalent to the corresponding Khas'minskii property for the dual subequation, that is, for F and $F \cap E$, respectively. The key facts, very specific to such an F , are the following:

- by exploiting Greene-Wu's techniques in [27], we can approximate a Khas'minskii potential for F with a smooth Khas'minskii potential, call it w . Up to playing with (f, ξ) and extending w on the entire X , we can assume that w satisfies

$$(39) \quad w < 0 \quad \text{on } X, \quad w(x) \rightarrow -\infty \quad \text{if } x \text{ diverges,} \quad \nabla^2 w \geq -w \langle \cdot, \cdot \rangle \quad \text{on } X.$$

- Integrating along the flow lines of ∇w and applying ODE comparison, it is possible to prove that $|\nabla w| \leq w$ on X . Starting from w , it is therefore easy to construct a weak Khas'minskii potential for $F \cap E$ that is smooth. AK-duality again and Greene-Wu approximation yields the full Khas'minskii property.

As said above, by work of M. Emery [23] property (3) is known to imply the martingale completeness of X . \square

Remark 4.3. In order to check the viscosity Hessian principle, we can only consider semiconcave⁶ functions, which are locally Lipschitz and 2-times differentiable a.e.. Thus, regarding to regularity, the viscosity Hessian principle is very close to the classical C^2 Hessian principle.

4.3. Laplacian principles. Differently from the Hessian principle, in view of elliptic estimates for semilinear equations, the viscosity weak Laplacian principle is equivalent to its corresponding classical C^2 principle, that is, to the stochastic completeness of X . In this case, the AK-duality improves on the original results in [42, 54]. Regarding the viscosity, strong Laplacian principle, that is, the Ahlfors property for the subequation $\{\text{Tr}(A) \geq 1\} \cup \{|p| \geq 1\} = \tilde{F} \cup \tilde{E}$, its equivalence with the classical, C^2 one (that is, Yau's principle) seems quite delicate and is currently unknown. However, the AK-duality in Corollary 3.10 guarantees the following:

Theorem 4.4. *Let X be a Riemannian manifold. Then, the following statements are equivalent:*

- (1) X satisfies the viscosity, strong Laplacian principle;
- (2) $F \cap E$ has the (weak) Khas'minskii property.

In particular, any manifold satisfying the viscosity, strong Laplacian principle must be (geodesically) complete.

5. PARTIAL TRACE (GRASSMANNIAN) OPERATORS

In the context of submanifolds it is interesting to consider extrinsic conditions instead of constrain directly the geometry of the submanifold. For instance, many applications of the Omori-Yau maximum principles (cf. [4, 3, 10]) have been investigated in that spirit. Specifically, when $\sigma: X^m \rightarrow Y^n$ is an isometric immersion, and $F \subset J^2(Y)$ is a subequation, the pull-back σ^*F induces a subset $H \doteq \overline{\sigma^*F}$, maybe only satisfying the conditions (P) and (N). In some relevant examples, like those

⁶By definition, a function u is semiconcave if and only if, locally, there exists $v \in C^2$, such that $u + v$ is concave when restricted to geodesics.

in $(\mathcal{E}2)$, $(\mathcal{E}3)$ and their complex analogues in $(\mathcal{E}5)$, the induced H is nontrivial and the following question is therefore natural:

can we transplant the Ahlfors property from \tilde{F} on Y to \tilde{H} on X ?

Trying to address the problem by contradiction, that is, assuming that the Ahlfors property does not hold for \tilde{H} , one would need to extend a nontrivial \tilde{H} -subharmonic function on X to the entire Y . This seems quite difficult to achieve, especially if X is merely immersed. On the contrary, the use of AK-duality makes the problem feasible. In particular, we can obtain the following result:

Theorem 5.1. *Let $\sigma : X^m \rightarrow Y^n$ be a proper isometric immersion. Assume that either*

- i) F_f is the universal subequation in $(\mathcal{E}2)$, $(\mathcal{E}5)$ with $k \leq m$, and σ has bounded second fundamental form II ;
- ii) F_f is the universal subequation in $(\mathcal{E}3)$ with $k \leq m$, and

$$\sup \left\{ |\mathrm{Tr}_{\mathcal{V}} II(x)| : x \in X, \mathcal{V} \leq T_x X \text{ } k\text{-dimensional} \right\} < +\infty.$$

Then,

$$\tilde{F}_f \cup \tilde{E}_\xi \text{ has the Ahlfors property on } Y \implies \tilde{F}_f \cup \tilde{E}_\xi \text{ has the Ahlfors property on } X,$$

for some (any) pair (f, ξ) satisfying $(f\xi)$.

Idea of the proof: It is conceptually quite simple: in our assumptions, AK-duality holds for each of $(\mathcal{E}2)$, $(\mathcal{E}3)$ and $(\mathcal{E}5)$, and therefore, $F_f \cap E_\xi$ has the Khas'minskii property for some (any) such (f, ξ) . Given an arbitrary potential \tilde{w} for $F_f \cap E_\xi$ on Y , by the flexibility in the choice of (f, ξ) and the properness of σ , the composition $w \doteq \tilde{w} \circ \sigma$ should correspond to a weak Khas'minskii potential for $F_g \cap E_\xi$ on X , where g just depends on (f, ξ) . To check this claim, one uses the standard chain rule formula

$$(40) \quad \nabla^2 w(X, Y) = \bar{\nabla}^2 w(\sigma_* X, \sigma_* Y) + \langle \bar{\nabla} w, II(X, Y) \rangle,$$

where $\nabla, \bar{\nabla}$ are the connections on X and Y , respectively. The adaptation to viscosity solutions, however, makes the proof of the claim subtler from the technical point of view. The arbitrariness of \tilde{w} and of these choices guarantees the validity of the weak Khas'minskii property on X . Then, AK-duality again implies the desired conclusion. \square

Remark 5.2. The presence of $\bar{\nabla} w$ in (40) forces to include the eikonal in the Ahlfors properties, otherwise more restrictive assumptions have to be imposed on X . In fact, without a gradient bound, it is possible to control the last term in (40) if and only if the second fundamental form II is trace-free on suitable subspaces \mathcal{V} . For instance, if $\sigma : X^m \rightarrow Y^n$ is a proper minimal immersion and F_f is the subequation described in $(\mathcal{E}2)$ with $k = m$, that is, $\tilde{F}_f = \{\lambda_{n-m+1} + \dots + \lambda_n(A) \geq f(\tau)\}$, the validity of the Ahlfors property for \tilde{F}_f on Y implies that X is stochastically complete (i.e. X has the viscosity, weak Laplacian principle).

Remark 5.3. A result similar to Theorem 5.1 can be stated for Riemannian submersions, cf. [50, Thm. 7.8].

The class of partial trace operators, example $(\mathcal{E}2)$, helps to understand the geometry of submanifolds. Thus, having in mind Theorem 5.1, it is important to investigate sufficient geometric conditions that imply the validity of the Ahlfors property for this kind of operators. Inspired by the seminal papers of Omori [58] and Yau [77] (that correspond to cases $k = 1$ and $k = m$ in $(\mathcal{E}2)$, respectively), we will focus on conditions involving the k -th Ricci curvature

Definition 5.4. Let Y^n be an n -dimensional manifold, and let $k \in \{1, \dots, n-1\}$. The k -th Ricci curvature is the function

$$\begin{aligned} \text{Ric}^{(k)} : TY &\longrightarrow \mathbb{R} \\ v &\longmapsto \inf_{\substack{\mathcal{W}_k \leq v^\perp \\ \dim \mathcal{W}_k = k}} \left(\frac{1}{k} \sum_{j=1}^k \text{Sect}(v \wedge e_j) \right), \end{aligned}$$

where $\{e_j\}$ is an orthonormal basis of \mathcal{W}_k .

We recall that bounding from below the k -th Ricci curvature is an intermediate condition between the corresponding bounding for the sectional and Ricci curvature. In the next result,

$$F_f = \{\lambda_1(A) + \dots + \lambda_{k+1}(A) \geq f(r)\}$$

and the functions f and ξ satisfying (f, ξ) . Having fixed an origin \mathfrak{o} , we denote with $\rho(x)$ the distance from \mathfrak{o} and with $\text{cut}(\mathfrak{o})$ the cut-locus of \mathfrak{o} , cf. [20].

Theorem 5.5. Let Y^n be complete, and assume that

$$(41) \quad \text{Ric}_x^{(k)}(\nabla \rho) \geq -G^2(\rho(x)) \quad \forall x \notin \text{cut}(\mathfrak{o}),$$

for some $k \in \{1, \dots, n-1\}$ and some G satisfying

$$0 < G \in C^1(\mathbb{R}_0^+), \quad G' \geq 0, \quad G^{-1} \notin L^1(+\infty).$$

Then, Y has the Ahlfors property for $\tilde{F}_f \cap \tilde{E}_\xi$. Moreover, if $\sigma : X^m \rightarrow Y^n$ is a proper isometric immersion, $k+1 \leq m \leq n-1$ and the eigenvalues $\mu_1 \leq \dots \leq \mu_m$ of the second fundamental form Π satisfy

$$(42) \quad \max \left\{ |\mu_1 + \dots + \mu_{k+1}|, |\mu_{m-k} + \dots + \mu_m| \right\} \leq CG(\rho \circ \sigma) \quad \text{on } X,$$

for some constant $C > 0$, then the Ahlfors property for $\tilde{F}_f \cap \tilde{E}_\xi$ holds on X . In particular, if $m = k+1$ and the mean curvature satisfies

$$|H| \leq CG(\rho \circ \sigma),$$

then X has the viscosity, strong Laplacian principle.

6. AK-DUALITY AND POLAR SETS

With the aid of AK-duality, we can characterize polar (hence, removable) sets for subequations in terms of preservation of the Ahlfors property. The study of removable sets for linear and nonlinear equations is a classical subject with a long history, and the interested reader can consult the recent [34] and the references therein for further insight. There, the problem is set in the language of subequations, and to introduce our application we first need to recall some terminology. We say that a subequation $F \subset J^2(X)$ is a

- truncated cone subequation if each fiber F_x over a point $x \in X$ is a truncated cone, that is, it satisfies the following property:

$$\text{if } J \in F_x, \quad \text{then } tJ \in F_x \quad \forall t \in [0, 1].$$

- convex cone subequation if each fiber F_x over a point $x \in X$ is a convex cone.

A subequation M is called a *monotonicity cone* for F if M is a convex cone subequation and $F + M \subset F$, that is, $J_1 + J_2 \in F$ whenever $J_1 \in F$ and $J_2 \in M$. In this case, we say that F is M -monotone. By duality, also \widetilde{F} is M -monotone, that is,

$$F + M \subset F \quad \Longrightarrow \quad \widetilde{F} + M \subset \widetilde{F}.$$

In particular, since M is a convex cone subequation, $M + M \subset M$ and thus \widetilde{M} is M -monotone and $M \subset \widetilde{M}$. In general, \widetilde{M} is a cone subequation much larger than M and it is non-convex. Moreover, it is maximal among M -monotone cone subequations: indeed, if F is a cone subequation that is M -monotone, then $0 \in F$ and thus $M = 0 + M \subset F$. Duality gives $\widetilde{F} \subset \widetilde{M}$.

Example 6.1.

- 1) On an almost complex, Hermitian manifold X , consider the subequations

$$F_j = \{\lambda_j(A^{(1,1)}) \geq 0\}, \quad 1 \leq j \leq m,$$

that are the branches of the complex Monge-Ampère equation $\det(\nabla^2 u)^{(1,1)} = 0$. Then, F_1 (the only branch that is convex) is a monotonicity cone for each F_j .

- 2) Let F_1, \dots, F_k be the branches of the k -Hessian equation $\sigma_k(\nabla^2 u) = 0$. Then, the smallest branch F_1 is a monotonicity cone for F_j for each j .
- 3) Let F be a universal subequation, and let M_k be the subequation in (E2) describing k -subharmonic functions:

$$M_k = \{\lambda_1(A) + \dots + \lambda_k(A) \geq 0\}.$$

It is proved in [34] that M_k is a monotonicity cone for F if and only if the *Riesz characteristic* of F , p_F , satisfies $p_F \geq k$. The Riesz characteristic of a universal subequation F is an explicitly computable quantity defined as follows:

$$p_F = \sup \left\{ t > 0 : I - t\Pi_v \in F \quad \forall x \in X, v \in T_x X \text{ with } |v| = 1 \right\},$$

where Π_v is the orthogonal projection onto the span of v . The interested reader can consult Section 11 of [34] for further information.

Definition 6.2. Let F be a subequation. We say that a function $\psi \in \text{USC}(X)$ is polar for a set Σ if $\Sigma \equiv \{x : \psi(x) = -\infty\}$. A closed subset $\Sigma \subset X$ is called F -polar if there exists an open neighbourhood $\Omega \supset \Sigma$ and $\psi \in F(\Omega)$ polar for Σ . The set Σ is called C^2 F -polar if, moreover, $\psi \in C^2(\Omega \setminus \Sigma)$.

Sets that are C^2 M -polar are removable for subequations having M as a monotonicity cone, cf. [34, Thm. 6.1]. The result is particularly effective when $F = \widetilde{M}$. Concerning Example 2) above, M_k -polar sets are very well understood in the Euclidean space (and, with some technical modifications, also on manifolds). In particular, if Σ has locally finite $(p-2)$ -dimensional Hausdorff measure for some $p < p_F$, then Σ is M_k -polar (cf. [34], Theorems 11.4, 11.5 and 11.13). Moreover, if $\Sigma \subset \mathbb{R}^m$, having locally finite $(p_F - 2)$ -dimensional Hausdorff measure is enough to guarantee the M_k -polarity, cf. [34, Thm. A].

In our setting, we consider subequations for which the AK-duality holds and thus we restrict to assume at least (H1). Consequently, since F is a closed subset, the constant function 0 is F -subharmonic. Any monotonicity cone M for F satisfies

$$M = \{0\} + M \subset F,$$

thus any M -polar subset is automatically F -polar.

Theorem 6.3. *Let F be an admissible subequation that is locally jet-equivalent to a universal subequation and satisfies $(\mathcal{H}1), (\mathcal{H}2)$. Suppose that \tilde{F} has the Ahlfors property on X , and let $\Sigma \subset X$ be a compact subset. Then, the following holds:*

(i) *if F is a truncated cone subequation, then*

$$\begin{array}{ccc} \tilde{F} \text{ has the Ahlfors} & \iff & \Sigma \text{ is } F\text{-polar;} \\ \text{property on } X \setminus \Sigma & & \end{array}$$

(ii) *if M is a monotonicity cone for F and Σ is M -polar, then \tilde{F} has the Ahlfors property on $X \setminus \Sigma$.*

Proof. We first prove (i).

(\Rightarrow) By AK-duality, any fixed pair (K, h) with $K \Subset X \setminus \Sigma$ admits a Khas'minskii potential ψ . Since $\psi(x) \rightarrow -\infty$ as $x \rightarrow \Sigma$, extending ψ on $X \setminus K$ by setting $\psi = -\infty$ on Σ gives a USC and F -subharmonic function on $X \setminus K$. Hence, Σ is F -polar.

(\Leftarrow) By F -polarity, fix $\Omega \supset \Sigma$ open and $\psi \in F(\Omega)$ satisfying $\Sigma = \{\psi = -\infty\}$. The upper semicontinuity of ψ implies that $\psi(x) \rightarrow -\infty$ as $x \rightarrow \Sigma$. By AK-duality, we can consider a Khas'minskii potential z for a pair (Ω', h) with $\Omega \Subset \Omega' \Subset X$. Then, for $\delta \in (0, 1]$, the family of functions $\{\delta w\}$ with

$$(43) \quad w(x) = \begin{cases} \psi(x) & \text{if } x \in \Omega, \\ z(x) & \text{if } x \in X \setminus \Omega', \end{cases}$$

realizes the weak Khas'minskii property on $X \setminus \Sigma$. By AK-duality, \tilde{F} has the Ahlfors property on $X \setminus \Sigma$, as claimed.

To show (ii), fix $\Omega \supset \Sigma$ open and $\psi \in M(\Omega)$ that satisfies $\Sigma = \{\psi = -\infty\}$. By $(\mathcal{H}1)$ and since F is a closed subset, the constant $0 \in F(X)$. Therefore, being M a monotonicity cone, $\delta\psi = 0 + \delta\psi \in F(\Omega)$ for each $\delta \in (0, 1]$. Defining w as in (43), the family $\{\delta w\}$ give again the desired weak Khas'minskii potentials. \square

Acknowledgement. *This work was completed when the second author was visiting the Abdus Salam International Center for Theoretical Physics (ICTP), Italy. He is grateful for the warm hospitality and for financial support. The authors would also like to thank the organizing and local committees of the INdAM workshop ‘‘Contemporary Research in elliptic PDEs and related topics’’ (Bari, May 30/31, 2017) for the friendly and pleasant environment.*

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