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Degenerations of automorphisms on irreducible holomorphic symplectic varieties

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Degenerations of automorphisms on irreducible holomorphic symplectic varieties

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Sintesi

L'obiettivo di questa tesi è studiare la degenerazione di automorfismi speciali (detti *non-simplettici*) su certe famiglie di varietà irriducibili olomorfe simplettiche (varietà IHS).

Infatti lo spazio dei moduli delle threefold cubiche lisce è isomorfo ad uno spazio di moduli $\mathcal{N}^{\rho,\zeta}_{\langle 6\rangle}$ di varietà IHS dotate di uno speciale automorfismo di ordine 3 non-simplettico. Questo è un risultato di Boissière–Camere–Sarti, [BCS19b]. Nello stesso articolo gli autori hanno esteso il risultato al caso generico con una singolarità cosiddetta nodale. Estendere questa mappa vuol dire in primo luogo studiare il limite di una degenerazione ad un parametro che ha come punto centrale il periodo di una threefold cubica nodale. Quello che succede è che se consideriamo una famiglia ad un parametro in $\mathcal{N}^{\rho,\zeta}_{\langle 6\rangle}$ che ha come limite un periodo nodale succede che nel limite la famiglia degenera ad una famiglia di varietà IHS con un automorfismo di ordine 3 non-simplettico con un reticolo invariante più grande. In questo senso diciamo che l'automorfismo *degenera*. In secondo luogo vuol dire fornire una mappa (che in questo caso è birazionale) tra il luogo nodale ed un opportuno spazio di moduli di varietà IHS aventi un automorfismo non-simplettico di ordine 3, con appunto un invariante più grande.

La prima parte della tesi è dedicata a trovare un risultato analogo per cubiche nodali non generiche, in particolare, andando in codimensione sempre più alta, si trova una mappa birazionale tra i sottoluoghi del luogo nodale il cui elemento generico è una cubica con una sola singolarità A_i con $i = 2, 3, 4$ ed uno spazio di moduli di varietà IHS di tipo $K3^{[2]}$ con un automorfismo non-simplettico ρ_i di ordine 3. Per far ciò utilizziamo delle tecniche sviluppate da Boissière, Camere e Sarti per arrivare ad una restrizione biettiva della mappa dei periodi.

La seconda parte, in collaborazione con Boissière e Comparin, è dedicata allo studio in dettaglio della geometria del limite che può essere espresso infatti come risoluzione simplettica della varietà di Fano delle rette che giacciono sulla fourfold cubica ciclica, ovvero un ricoprimento 3 a 1 ciclico di $\mathbb{P}^4$ che ramifica su una threefold cubica che in questo caso ha singolarità isolate di tipo A_i con $i = 2, 3, 4$.

Parole chiave: geometria algebrica, varietà IHS, automorfismi, spazi di moduli, ipersuperfici cubiche, risoluzioni simplettiche, varietà grassmanniane .

Abstract

The aim of this thesis is to study the degeneration of special automorphisms (termed *non-symplectic*) on certain families of irreducible holomorphic symplectic varieties (IHS varieties).

In fact, the moduli space of smooth cubic threefolds is isomorphic to a moduli space $\mathcal{N}^{\rho,\zeta}_{\langle 6\rangle}$ of IHS varieties equipped with a special order 3 non-symplectic automorphism. This is a result by Boissière–Camere–Sarti, [BCS19b]. In the same paper, the authors extended the result to the general case with a so-called nodal singularity. Extending this map primarily means studying the limit of a one-parameter degeneration whose central point is the period of a nodal cubic threefold. What happens is that if we consider a one-parameter family in $\mathcal{N}^{\rho,\zeta}_{\langle 6\rangle}$ that has a nodal period as its limit, it turns out that at the limit the family degenerates into a family of IHS varieties with an order 3 non-symplectic automorphism with a larger invariant. In this sense, we say the automorphism *degenerates*. Secondly, it means providing a map (which in this case is birational) between the nodal locus and a suitable moduli space of IHS varieties possessing a non-symplectic order 3 automorphism, with precisely a larger invariant.

The first part of the thesis is dedicated to finding a similar result for non-generic nodal cubics. In particular, by going to increasingly higher codimension, a birational map is found between the subspaces of the nodal locus whose generic element is a cubic with a single singularity A_i for $i = 2, 3, 4$, and a moduli space of IHS varieties of type $K3^{[2]}$ with a non-symplectic ρ_i automorphism of order 3. To achieve this, we employ techniques developed by Boissière, Camere, and Sarti to arrive at a bijective restriction of the period map.

The second part, in collaboration with Boissière and Comparin, is dedicated to a detailed study of the geometry of the limit which can in fact be expressed as the symplectic resolution of the Fano variety of lines lying on the cyclic cubic fourfold, that is, a cyclic 3-to-1 cover of $\mathbb{P}^4$ branching over a cubic threefold which in this case has isolated singularities of type A_i for $i = 2, 3, 4$.

Keywords: algebraic geometry, IHS manifolds, automorphisms, moduli spaces, cubic hypersurfaces, symplectic resolutions, grassmannian varieties.

Résumé

L'objectif de cette thèse est d'étudier la dégénérescence des automorphismes spéciaux (appelés *non-symplectiques*) sur certaines familles de variétés symplectiques holomorphes irréductibles (variétés IHS).

En fait, l'espace des modules des threefolds cubiques lisses est isomorphe à un espace des modules $\mathcal{N}^{\rho,\zeta}_{\langle 6\rangle}$ de variétés IHS dotées d'un automorphisme spécial d'ordre 3 non-symplectique. Ceci est un résultat de Boissière–Camere–Sarti, [BCS19b]. Dans le même papier, les auteurs ont étendu le résultat au cas général avec une singularité dite nodale. Étendre cette application signifie d'abord étudier la limite d'une dégénérescence à un paramètre dont le point central est la période d'une threefold cubique nodale. Ce qui se passe, c'est que si nous considérons une famille à un paramètre dans $\mathcal{N}^{\rho,\zeta}_{\langle 6\rangle}$ qui a pour limite une période nodale, il s'avère qu'à la limite la famille dégénère en une famille de variétés IHS avec un automorphisme d'ordre 3 non-symplectique avec un invariant plus grand. En ce sens, nous disons que l'automorphisme *dégénère*. Deuxièmement, cela signifie fournir une application (qui dans ce cas est birationnelle) entre l'hyperplan nodal et un espace approprié de modules de variétés IHS ayant un automorphisme non-symplectique d'ordre 3, avec justement un invariant plus grand.

La première partie de la thèse est consacrée à trouver un résultat analogue pour les cubiques nodales non génériques. En particulier, en montant en codimension de plus en plus élevée, on trouve une application birationnelle entre les sous-espaces de l'hyperplan nodal dont l'élément générique est un cubique avec une seule singularité A_i avec $i = 2, 3, 4$ et un espace des modules de variétés IHS de type $K3^{[2]}$ avec un automorphisme non-symplectique ρ_i d'ordre 3. Pour ce faire, nous utilisons des techniques développées par Boissière, Camere et Sarti pour arriver à une restriction bijective de la application des périodes.

La deuxième partie, en collaboration avec Boissière et Comparin, est consacrée à l'étude détaillée de la géométrie de la limite qui peut en fait être exprimée comme la résolution symplectique de la variété de Fano des droites qui se trouvent sur la fourfold cubique cyclique, c'est-à-dire une couverture cyclique 3 à 1 de $\mathbb{P}^4$ qui se ramifie sur une threefold cubique qui dans ce cas a des singularités isolées de type A_i avec $i = 2, 3, 4$.

Mots clés: géométrie algébrique, variétés IHS, automorphismes, espaces de moduli, hypersurfaces cubiques, résolutions symplectiques, variétés grassmanniennes.

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A chi non può leggere queste parole ma mi ha accompagnato in questo viaggio
A mio padre Carlo, mia nonna Maria, mio nonno Teo

Al collega dottorando Giulio Regeni
torturato ed ucciso durante il proprio dottorato ormai quasi 8 anni fa.

Du sagst:
Es steht schlecht um unsere Sache.
Die Finsternis nimmt zu. Die Kräfte nehmen ab.
Jetzt, nachdem wir so viele Jahre gearbeitet haben,
Sind wir in schwierigerer Lage als am Anfang.

Der Feind aber steht stärker da denn jemals.
Seine Kräfte scheinen gewachsen. Er hat ein unbesiegliches Aussehen angenommen.
Wir aber haben Fehler gemacht, es ist nicht mehr zu leugnen.
Unsere Zahl schwindet hin.
Unsere Parolen sind in Unordnung. Einen Teil unserer Wörter
Hat der Feind verdreht bis zur Unkenntlichkeit.

Was ist jetzt falsch von dem, was wir gesagt haben,
Einiges oder alles?
Auf wen rechnen wir noch? Sind wir Übriggebliebene, herausgeschleudert
Aus dem lebendigen Fluß? Werden wir zurückbleiben
Keinen mehr verstehend und von keinem verstanden?

Müssen wir Glück haben?

So fragst du. Erwarte
Keine andere Antwort als die deine.

- An Den Schwankenden, *Bertolt Brecht*

Contents

Introduction

One way to characterize compact complex Kähler manifolds is with their first Chern class, or, from a differential point of view, with their Ricci curvature. Moreover, as per the famous Beauville–Bogomolov decomposition, modulo a finite étale covering, every Ricci flat compact complex Kähler manifold is a product of complex tori, Calabi–Yau manifolds and irreducible holomorphic symplectic (IHS) manifolds. For this reason the study of these manifolds has attracted many mathematicians during the last fifty years and we chose to study the last class of manifolds mentioned, i.e. IHS manifolds.

IHS manifolds are one possible higher-dimensional generalization of a well-studied class of surfaces: $K3$ surfaces. In fact IHS manifolds, like $K3$ surfaces, have many interesting algebraic and geometric properties. Notably, one of their main properties is the presence of a natural structure of even integral lattice on the second cohomology group with integer coefficients, given by the Beauville–Bogomolov–Fujiki (BBF) quadratic form. This fact has important consequences, in particular relatively to the study of automorphisms on IHS manifolds. Indeed, due to the work of Huybrechts, Markman and Verbitsky, there exist some Torelli theorems for IHS manifolds. These theorems allow to describe automorphisms of IHS manifolds from classes of isometries of their second cohomology group. The study of automorphisms on IHS manifold has been, therefore, a very rich field of study that has led to many results.

Our main interest in the presence of an automorphism on a family of IHS manifolds is that it introduces some "rigidity" at the level of moduli spaces. Let us explain this better. Consider an IHS manifold $\bar{X}$. Then, for each IHS manifold X deformation equivalent to $\bar{X}$ we can consider the pair (X, η) where η is an isometry between the second cohomology group with integer coefficients and an abstract lattice L (which is fixed once we fix the deformation type, see Remark 2.25). The moduli space of the pairs (X, η) is, by a result of Huybrechts (see [Huy04, Section 3.1]), a smooth non-Hausdorff complex manifold. Moreover, we can define a *period map* associating to each pair (X, η) the image through η of its symplectic form. If we restrict the period map to any connected component of the moduli space of the pairs (X, η) it becomes a surjection by [Huy99, Theorem 8.1]. Now, if we restrict it again to those IHS manifolds X deformation equivalent to $\bar{X}$ which additionally admit some special automorphism (a non-symplectic automorphism of prime order) and we ask some technical conditions on X then the restriction of the period map is an isomorphism into its image (called *period domain*). This was proved in [BCS19a] for the $K3^{[n]}$ type of deformation and then generalized to every known type in [BC22].

This rigidity was crucial in order to find a surprising isomorphism between the moduli space of smooth cubic threefolds and a moduli space of IHS fourfolds, namely the moduli space $\mathcal{N}^{\rho,\zeta}_{\langle 6\rangle}$ of IHS fourfolds of $K3^{[2]}$-type with a non-symplectic automorphism of order three, whose invariant lattice has rank one and is generated by a class of square 6. Indeed, Boissière–Camere–Sarti in [BCS19b] found out, using the results by [ACT11] and [LS07] about cubic threefolds, that both the above mentioned moduli spaces admit a period map which is an isomorphism on the same period domain. But this correspondence goes deeper. Indeed, Allcock–Carlson–Toledo studied in [ACT11] the GIT compactification of the moduli space of smooth cubic threefolds, providing, thus, an extension of the period map to some singular cubic threefolds, e.g. nodal cubic threefolds. Therefore, in [BCS19b] the authors posed their interest in the case where there is a one-parameter family degenerating to the period of a generic nodal cubic threefold from the point of view of the moduli space of IHS fourfolds with a special automorphism of order three. They found that, in the limit, the family "degenerates by jumping to a family with another automorphism with a bigger invariant lattice". In other words they found, roughly speaking, that in the family $\mathcal{N}^{\rho,\zeta}_{\langle 6\rangle}$ there are no IHS fourfolds having a nodal period but, in order to find one, we need to slightly modify the automorphism and consider an automorphism with a bigger invariant lattice. We call this phenomenon *degeneration of the automorphism* and it is the starting point of this thesis.

The main subject of this thesis is the study of the degeneration of the automorphism along non-generic nodal periods. Indeed, a cubic threefold is GIT stable if and only if all its singularities are of type A_i for $i = 1, \dots, 4$, so our first objective is to study the degeneration of the automorphism when the limit is a cubic threefold having one isolated singularity of type A_i for $i = 2, 3, 4$. The method proposed here uses and generalizes some techniques of [Cam16], [BCS19a], [BCS19b] and [CC20]. The idea is to define a locus $\Delta_3^{A_i}$ where a cubic threefold C_i having one isolated singularity of type A_i is generic. The period map is then a bijection between $\Delta_3^{A_i}$ and a sublocus Ω_i/Γ of the period domain modulo the action of some arithmetic group. Now, the challenge is to find an IHS manifold X of $K3^{[2]}$-type with a non-symplectic automorphism of order three, with a bigger invariant compared to the smooth case. Chosen a marking η on X we want, moreover, that (X, η) lives in a moduli space where the period map is an isomorphism onto the image. Finally, we want that this image is at least birational to Ω_i/Γ. It looks like a lot to hope for but the main result of this thesis is the following:

Theorem 0.1. *The $\Delta_3^{A_i}$ locus for $i = 1, \dots, 4$ is birational to a $(10-i)$-dimensional moduli space of fourfolds of $K3^{[2]}$-type with Picard group of the generic member isometric to R_i endowed with a non-symplectic automorphism of order three, having invariant lattice isometric to T_i. These lattices are defined in the following table.*

i	R_i	T_i
1	$U(3) \oplus \langle -2\rangle$	$U(3) \oplus \langle -2\rangle$
2	$U \oplus A_2(-2) \oplus \langle -2\rangle$	$U(3) \oplus \langle -2\rangle$
3	$U \oplus A_2(-1)^{\oplus 2} \oplus \langle -2\rangle$	$U \oplus A_2(-1)^{\oplus 2} \oplus \langle -2\rangle$
4	$U \oplus E_6(-1) \oplus \langle -2\rangle$	$U \oplus E_6(-1) \oplus \langle -2\rangle$

This relation is, as said above, "surprising" as it relates the world of cubic threefolds and the world of IHS manifolds. It becomes a little less surprising (although not at all trivial) when we look at how we define the desired IHS manifold X needed to obtain this relation. For every cubic threefold $C \subset \mathbb{P}^4$ we can associate the so-called *cyclic* cubic fourfold Y associated to C, i.e. the 3:1 cyclic cover of $\mathbb{P}^4$ branched on C. This variety is the archetypal example of *Fano variety of $K3$ type*, a class of variety which has a deep connection to the IHS world. There exist many ways to associate an IHS manifold to a cubic fourfold Y, one of them is considering the Fano variety of lines $F(Y)$ on Y. When Y is smooth, this is an IHS manifold of $K3^{[2]}$-type by the well-known result of Beauville–Donagi ([BD85]). On the other hand, the cyclic cubic fourfolds associated to nodal cubic threefolds are singular. This implies that also the Fano variety of lines on them are singular. To overcome this problem in the first part of the thesis we consider IHS manifolds which are birational to these singular varieties. In the second part we shift our focus to the singular varieties $F(Y)$ which are Fano varieties of lines on cyclic cubic fourfolds arising from cubic threefolds having isolated singularities of type A_i for $i = 2, 3, 4$. In fact the case for $i = 1$ has been studied in [BHS23], moreover their resolution has been studied, with different methods, in [Yam22]. In particular we are interested in their geometry and in the presence of an automorphism. This second part is in collaboration with S. Boissière and P. Comparin. The main result can be summarized with the following theorem.

Theorem 0.2. *Let C_i be a complex projective cubic threefold having one isolated singularity of type A_i for $i = 2, 3, 4$ and let Y_i be its associated cyclic cubic fourfold. Assume that there exist no plane $\Pi \subset Y$ such that $\Pi \cap \mathrm{Sing}(Y_i) \neq \emptyset$. Then the Fano variety of lines $F(Y_i)$ of Y_i admits a unique symplectic resolution by an IHS manifold of $K3^{[2]}$-type $\widetilde{F(Y_i)}$.*
Moreover, there exist integral lattices R_i and T_i, defined below, such that:

i) $\mathrm{Pic}\left(\widetilde{F(Y_i)}\right) \simeq R_i$*;*

ii) there exists a non-symplectic automorphism $\tau \in \mathrm{Aut}\left(\widetilde{F(Y_i)}\right)$ whose invariant sublattice is $H^2(\widetilde{F(Y_i)}, \mathbb{Z})^{\tau^*} \simeq T_i$

with T_i and R_i defined in the following table:

i	T_i	R_i
2	$\langle 6\rangle \oplus A_2(-1)$	$\langle 6\rangle \oplus D_4(-1)$
3	$\langle 6\rangle \oplus E_6(-1)$	$\langle 6\rangle \oplus E_6(-1)$
4	$\langle 6\rangle \oplus E_8(-1)$	$\langle 6\rangle \oplus E_8(-1)$

In Chapter 1 we introduce some basic notions about lattice theory and Springer theory which we will use in the following chapters. In Chapter 2 we introduce some theory of IHS manifolds. In Chapter 3 we study the relation between nodal cubic threefolds and IHS manifolds with a non-symplectic automorphism of order 3 which leads to prove Theorem 0.1. In Chapter 4 we study the geometry of the Fano variety of lines on singular cyclic cubic fourfolds which leads to prove Theorem 0.2. In Chapter 5 we write down the computations which in our opinion were important but harmful to the comprehension of the text.

Chapter 1

Basic notions and prerequisites

"Non temere, zeta reticoli on my mind"

- Meganoidi, *Zeta reticoli*

1 Basic facts about lattices

In this section we recall the most important definitions and results of lattice theory that we need in the next chapters. The general references on lattice theory used here are [Nik80] and [CS99], see also [Men19, Section 2].

Definition 1.1. *A* lattice L *is a free* $\mathbb{Z}$*-module of finite rank together with a non-degenerate symmetric bilinear form* $(-,-) : L \times L \to \mathbb{Z}$*. We denote* (x,x) *also by* x^2*.*

By *non-degenerate* we mean that for any non-zero $l \in L$ there exists $l' \in L$ such that the product $(l, l') \neq 0$. Let L be a lattice of rank n, and let $\mathcal{B} := \{e_1, \ldots, e_n\}$ be a $\mathbb{Z}$-basis of L. Then we call *Gram matrix* of L associated to $\mathcal{B}$ the $n \times n$ symmetric matrix

$$\begin{pmatrix} (e_1, e_1) & \cdots & (e_1, e_n) \\ \vdots & \ddots & \vdots \\ (e_n, e_1) & \cdots & (e_n, e_n) \end{pmatrix}.$$

Moreover, a lattice L of rank n is said:

- *even* if $(l,l) \in 2\,\mathbb{Z}$ for every $l \in L$;
- *odd* if it is not even.

The *determinant* of a lattice L is the determinant of any Gram matrix G of the lattice.

Remark 1.2. *The determinant does not depend on the choice of the Gram matrix. Indeed, if* G *and* G' *are two Gram matrices associated to two distinct* $\mathbb{Z}$*-basis of* L*, then* $G' = S^t G S$*, where* S *is an invertible matrix with integer entries, so* $\det(S) = \pm 1$ *and* $\det(G') = \det(G)$*.*

A lattice L is said *unimodular* if $\det(L) = \pm 1$. A *sublattice* of a lattice L is a free submodule $L' \subset L$ equipped with the symmetric bilinear form which is the restriction to $L' \times L'$ of the form defined on $L \times L$.
The *divisibility* of an element $l \in L$ in a lattice L is the positive generator of the ideal

$$\{(l, m) \,|\, m \in L\} \subset \mathbb{Z}\,.$$

A sublattice $L' \subset L$ is *primitive* if L/L' is a free module. Given a subset $S \subset L$ there exists an important primitive lattice associated to it which will be used in the next chapters. We call the *orthogonal complement* of S in L, the primitive sublattice of L defined as

$$S^{\perp} := \{l \in L \,|\, (l, s) = 0 \text{ for every } s \in S\}.$$

The *direct sum* of two lattices L_1 and L_2 is the lattice $L_1 \oplus L_2$ whose bilinear form is

$$(v_1 + v_2, w_1 + w_2) := (v_1, w_1)_1 + (v_2, w_2)_2$$

for every $v_1, w_1 \in L_1$ and $v_2, w_2 \in L_2$, where $(-,-)_1$ and $(-,-)_2$ are the bilinear forms of L_1 and L_2 respectively. Note that, as L is non-degenerate by definition, if M is a sublattice of L, then

$$M \oplus M^{\perp} \subset L$$

is a sublattice of maximal rank, i.e., $\mathrm{rk}(M) + \mathrm{rk}(M^{\perp}) = \mathrm{rk}(L)$.

For a lattice L of rank n we write $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ and the bilinear form is extended $\mathbb{R}$-bilinearly to $L_{\mathbb{R}}$.
Since the lattice is non-degenerate, the quadratic form associated to the bilinear form on $L_{\mathbb{R}} \cong \mathbb{R}^n$ admits an orthonormal basis by Sylvester's theorem, i.e., there is an $\mathbb{R}$-basis $\{f_1, \ldots, f_n\}$ of $L_{\mathbb{R}}$ such that

$$\left(\sum_{i=1}^{n} x_i f_i\right)^2 = \epsilon_1 x_1^2 + \cdots + \epsilon_n x_n^2 \qquad \text{with } \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}.$$

After a permutation of the basis $\{f_1, \ldots, f_n\}$ we can assume that $\epsilon_i = 1$ for $i = 1, \ldots, l_{(+)}$ and $\epsilon_i = -1$ for $i = l_{(+)} + 1, \ldots, n$ for some $l_{(+)} \in \{0, \ldots, n\}$.
Using, again, the fact that a lattice is by definition non-degenerate we define $l_{(-)} := n - l_{(+)}$, the *signature* of L will be then the pair of integers $(l_{(+)}, l_{(-)})$. A lattice is *positive definite* if $l_{(-)} = 0$, similarly it is *negative definite* if $l_{(+)} = 0$, while it is *indefinite* if $l_{(+)}, l_{(-)} \neq 0$. We now give some examples of lattices which will appear in the next chapters.

Example 1.3. *Let k be a non-zero integer, then we denote by $\langle k \rangle$ the rank one lattice $L = \mathbb{Z}\, e$ with bilinear form $(e, e) = k$.*

Example 1.4. *If L is a lattice, for every non-zero integer k we denote by $L(k)$ the lattice obtained by taking the same $\mathbb{Z}$-module and bilinear form:*

$$(v, w)_{L(k)} := k(v, w)_L$$

for every $v, w \in L$.

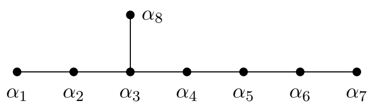
Example 1.5. *Let U denote the* hyperbolic lattice*, i.e., the unique unimodular lattice of rank 2 and signature $(1,1)$. Its Gram matrix is the following:*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Example 1.6. *We denote by E_8 be the even unimodular lattice of signature $(8,0)$ whose Gram matrix is the following:*

$$\begin{pmatrix}
2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & \\
 & -1 & 2 & -1 & & & & -1 \\
 & & -1 & 2 & -1 & & & \\
 & & & -1 & 2 & -1 & & \\
 & & & & -1 & 2 & -1 & \\
 & & & & & -1 & 2 & \\
 & & -1 & & & & & 2
\end{pmatrix}.$$

Equivalently, E_8 is represented by the following Dynkin diagram



,

where $\{\alpha_1, \dots, \alpha_8\}$ is a $\mathbb{Z}$-basis of E_8 and the bilinear form is described as follows:

- $(\alpha_i, \alpha_i) = 2$ *for every $i = 1, \dots, 8$;*
- $(\alpha_i, \alpha_j) = 0$ *if the nodes α_i and α_j in the diagram are not linked;*
- $(\alpha_i, \alpha_j) = -1$ *if the nodes α_i and α_j in the diagram are linked.*

Example 1.7. *We denote by $E_8(-1)$ be the lattice obtained by multiplying by -1 the Gram matrix of E_8, i.e., the lattice whose Gram matrix is the following:*

$$\begin{pmatrix}
-2 & 1 & & & & & & \\
1 & -2 & 1 & & & & & \\
 & 1 & -2 & 1 & & & & 1 \\
 & & 1 & -2 & 1 & & & \\
 & & & 1 & -2 & 1 & & \\
 & & & & 1 & -2 & 1 & \\
 & & & & & 1 & -2 & \\
 & & 1 & & & & & -2
\end{pmatrix}. \tag{1.7.1}$$

It is an even unimodular lattice of signature $(0,8)$.

Remark 1.8. *In the same way of Example 1.6 we can associate to any Dynkin diagram of type ADE a positive definite lattice. Therefore, in the following chapters, we will denote by A_n, D_n and E_n the lattices associated with the respective Dynkin diagrams.*

If L and L' are two lattices with bilinear forms $(-,-)$ and $(-,-)'$ respectively, we call *morphism of lattices* $\varphi : L \to L'$ a morphism of $\mathbb{Z}$-modules such that for every $l_1, l_2 \in L$ we have $(l_1, l_2) = (\varphi(l_1), \varphi(l_2))'$. Note that morphisms between two non-degenerate lattices are always injective. We say that a lattice L is *primitively embedded* in a lattice L' if there is a morphism $\varphi : L \to L'$ such that $\varphi(L)$ is a primitive sublattice of L'. An *isometry* is a bijective morphism of lattices. The group of isometries of a lattice to itself is denoted by $O(L)$.

1.1 Discriminant group and primitive embeddings

From now on, L will be a non-degenerate lattice. A fundamental tool in lattice theory is the discriminant group associated to a lattice L. In order to define it, we need to introduce the *dual* of a lattice L, which is $L^\vee := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. Consider the following morphism of lattices

$$\phi : L \hookrightarrow L^\vee, \qquad v \mapsto (v, \cdot\,).$$

Since the bilinear form is non-degenerate, ϕ is injective. We then obtain an isomorphism

$$\phi_{\mathbb{Q}} : L \otimes \mathbb{Q} \xrightarrow{\sim} L^\vee \otimes \mathbb{Q}\,.$$

The restriction of $\phi_{\mathbb{Q}}^{-1}$ to $L^\vee$ gives an embedding $L^\vee \hookrightarrow L \otimes \mathbb{Q}$, which characterizes the dual $L^\vee$ as

$$L^\vee = \{u \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid (u, v) \in \mathbb{Z} \ \text{ for every } v \in L\}.$$

We now see how to obtain a basis for the dual $L^\vee$. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of L and let M be the Gram matrix associated to $\mathcal{B}$. If $\mathcal{B}^\vee = \{v_1^\vee, \dots, v_n^\vee\}$ is the dual basis of $\mathcal{B}$, then the matrix which represents ϕ in the basis $\mathcal{B}$ and $\mathcal{B}^\vee$ is $\mathrm{mat}_{\mathcal{B},\mathcal{B}^\vee}(\phi) = M$. Since M is also the matrix of $\phi_{\mathbb{Q}}$ we have

$$\mathrm{mat}_{\mathcal{B}^\vee,\mathcal{B}}(\phi_{\mathbb{Q}}^{-1}) = M^{-1}.$$

Moreover, $\phi_{\mathbb{Q}}^{-1}$ represents the embedding $L^\vee \hookrightarrow L \otimes \mathbb{Q}$, so the columns of M^{-1} give a basis of $L^\vee$.

Lemma 1.9 (Smith normal form)**.** *Let L be a non-degenerate lattice. Then there exists a basis $\{v_1, \dots, v_n\}$ of L and non-zero integers $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ such that $\{\frac{v_1}{\lambda_1}, \dots, \frac{v_n}{\lambda_n}\}$ is a basis of $L^\vee \subset L \otimes \mathbb{Q}$.*

Since $L \subset L^\vee$ is a subgroup of maximal rank, the quotient

$$D_L := L^\vee / L$$

is a finite group: we call it the *discriminant group* of L. We denote by $\mathrm{discr}(L)$ the order of the discriminant group: this coincides with $|\det(G)|$, where G is a Gram matrix of L. Note that if D_L is trivial, then L is unimodular. We say that the lattice L is *p-elementary* if

$$D_L \cong \left(\frac{\mathbb{Z}}{p\,\mathbb{Z}}\right)^{\oplus k}$$

for a prime number p and a non-negative integer k.

In general the dual $L^\vee$ is not a lattice: the bilinear form $(-,-)_\mathbb{Q}$ obtained on $L^\vee$ by extending $\mathbb{Q}$-bilinearly the bilinear form $(-,-)$ of L can take non-integer values. Note that, for every $x_1, x_2 \in L^\vee$ and $l_1, l_2 \in L$ we have

$$\begin{aligned}(x_1+l_1, x_2+l_2)_\mathbb{Q} &= (x_1,x_2)_\mathbb{Q} + (x_1,l_2)_\mathbb{Q} + (l_1,x_2)_\mathbb{Q} + (l_1,l_2)_\mathbb{Q} \\ &\equiv (x_1,x_2)_\mathbb{Q} \pmod{\mathbb{Z}}.\end{aligned}$$

Hence D_L is equipped with a so-called *finite bilinear form*

$$b_L : D_L \times D_L \to \mathbb{Q}/\mathbb{Z}, \qquad (\bar{x}, \bar{y}) \mapsto \overline{(x,y)_\mathbb{Q}}.$$

Moreover, the $\mathbb{Q}$-extension of the quadratic form $(-)^2 : L \to \mathbb{Z}$ induces a quadratic form on D_L modulo $\mathbb{Z}$:

$$q_L : D_L \to \mathbb{Q}/\mathbb{Z}, \qquad \bar{x} \mapsto \overline{(x)^2_\mathbb{Q}}.$$

If L is an even lattice, we can say more: for every $x \in L^\vee$ and $l \in L$ we have

$$(x+l)^2_\mathbb{Q} = (x)^2_\mathbb{Q} + (l)^2_\mathbb{Q} + 2(x,l)_\mathbb{Q} \equiv (x)^2_\mathbb{Q} \pmod{2\,\mathbb{Z}}$$

Thus, if L is even, D_L is equipped with a so-called *finite quadratic form*

$$q_L : D_L \to \mathbb{Q}/2\,\mathbb{Z}, \qquad \bar{x} \mapsto \overline{(x)^2_\mathbb{Q}}.$$

Both the finite bilinear form and the finite quadratic form of D_L can be represented by a matrix: if $\{x_i\}_i$ is a system of independent generators of D_L, then:

- the matrix $M_{b_L} = (a_{i,j})$ with $a_{i,j} = b_L(x_i, x_j) \in \mathbb{Q}/\mathbb{Z}$ represents the finite bilinear form b_L;
- the matrix $M_{q_L} = (a_{i,j})$ with

  $$a_{i,j} = \begin{cases} b_L(x_i, x_j) \in \mathbb{Q}/\mathbb{Z} & \text{if } i \neq j \\ q_L(x_i) \in \mathbb{Q}/2\,\mathbb{Z} & \text{if } i = j \end{cases}$$

  represents the finite quadratic form q_L.

We conclude this part by recalling some properties of primitive embeddings.

Definition 1.10. *Two primitive embeddings $i : S \hookrightarrow L$, $j : S \hookrightarrow L'$ define isomorphic primitive sublattices if there exists an isomorphism $\varphi : L \to L'$ such that $\varphi(i(S)) = j(S)$.*

The following theorem is originally found in [Nik80, Proposition 1.15.1], but we prefer to give its reformulation in [CC20, Theorem 2.5].

Theorem 1.11 (Proposition 1.15.1 in [Nik80])**.** *Let S be an even lattice of signature $(s_{(+)}, s_{(-)})$ and discriminant form q_S. Then all the primitive embeddings of $S \hookrightarrow L$, for L the unique even lattice of invariants $(m_{(+)}, m_{(-)}, q_L)$, are determined by quintuples $\Theta_i := (H_S, H_L, \gamma, T, \gamma_T)$ such that:*

- *H_S is a subgroup of D_S , H_L is a subgroup of D_L and $\gamma : H_S \to H_L$ is an isometry $q_S|_{H_S} \simeq q_L|_{H_L}$;*

- *T is a lattice of signature $(m_{(+)} - s_{(+)}, m_{(-)} - s_{(-)})$ and discriminant form $q_T = (-q_S \oplus q_L)|_{\Gamma^\perp/\Gamma}$, with $\Gamma \subset D_S \oplus D_L$ the graph of γ and $\Gamma^\perp$ the orthogonal complement of Γ with respect to the form $(-q_S) \oplus q_L$ which has values in $\mathbb{Q}/\mathbb{Z}$;*

- *$\gamma_T \in O(q_T)$.*

In particular, T is isomorphic to the orthogonal complement of $\iota(S)$ in L. Moreover, two quintuples Θ and Θ' define isomorphic primitive sublattices if and only if $\bar{\mu}(H_S) = H'_S$ for $\mu \in O(S)$ and there exist $\phi \in O(q_L)$, $\nu : T \to T$ isomorphism such that $\gamma' \circ \bar{\mu} = \phi \circ \gamma$ and $\bar{\nu} \circ \gamma_T = \gamma'_{T'} \circ \bar{\nu}$.

1.2 Overlattices

Let L and R be two lattices such that $L \subset R$ and $\mathrm{rk}(L) = \mathrm{rk}(R)$. We say that R is an *overlattice* of L. The discriminant group D_L of a lattice L plays an important role in the study of the overlattices of L. Note that if R is an overlattice of L, then L has finite index in R. We have the following lemma.

Lemma 1.12. *Let L be a lattice and $R \supset L$ be an overlattice. Then*

$$[R : L]^2 = \frac{\mathrm{discr}(L)}{\mathrm{discr}(R)} = \frac{|D_L|}{|D_R|}.$$

Proof. Consider the following inclusions

$$L \to R \to R^\vee \to L^\vee,$$

where the composition is the canonical inclusion of L in its dual $L^\vee$. Let $\mathcal{B}_L$ and $\mathcal{B}_R$ be two basis of L and R respectively, and M_L and M_R be the Gram matrices associated. Let W be the matrix which represents the inclusion $L \hookrightarrow R$ in the basis $\mathcal{B}_L$ and $\mathcal{B}_R$. Then the transposed matrix W^t represents the inclusion $R^\vee \hookrightarrow L^\vee$, so $M_L = W^t M_R W$. Since $|\det(W)|$ is equal to the index $[R : L]$, and $|\det(M_R)|$ and $|\det(M_L)|$ are by definition the discriminants of R and L respectively, we have

$$[R : L]^2 = \frac{\mathrm{discr}(L)}{\mathrm{discr}(R)} = \frac{|D_L|}{|D_R|},$$

as we wanted. □

In particular we state an immediate corollary.

Corollary 1.13. *Let $R \supset L$ be an overlattice of a lattice L and M_R, M_L their respective Gram matrices. Then, $\frac{\det(M_L)}{\det(M_R)}$ is a perfect square.*

Let L be a lattice. We say that a subgroup $G \subset D_L$ is *isotropic* if

$$b_L(g, g') \equiv 0 \pmod{\mathbb{Z}}$$

for every $g, g' \in G$. The following result gives a relation between the overlattices of L and the isotropic subgroups of D_L.

Proposition 1.14 (Proposition 1.4.1, Item (a), in [Nik80]). *Let L be a lattice with discriminant group D_L. For every overlattice $R \supset L$, let H_R be the subgroup $H_R := R/L \subset D_L$. Then the following is a bijection.*

$$\begin{aligned} \{\textit{overlattices of } L\} &\leftrightarrow \{\textit{isotropic subgroups of } D_L\}, \\ R &\mapsto H_R. \end{aligned}$$

Proof. We refer to [Nik80, Proposition 1.4.1,(a)]. Let $\pi : L^\vee \to L^\vee/L = D_L$ be the natural projection. There is a bijection between the set of groups R such that $L \subset R \subset L^\vee$ and the set of subgroups of D_L, obtained by sending R to $H_R := \pi(R)$. Now, $\pi(R)$ is isotropic if and only if $b_L(x, y) \equiv 0$ for every $x, y \in R$, i.e., $b_{\mathbb{Q}}(x, y) \in \mathbb{Z}$ for every $x, y \in R$. This holds if and only if R is a lattice. We conclude that the bijection above gives a bijection between the overlattices of L and the isotropic subgroups of D_L. □

So, in order to determine all the overlattices of a lattice L, it is sufficient to find the subgroups of the discriminant group D_L which are isotropic. Since D_L is a finite group, this shows that a lattice has a finite number of overlattices.

2 Complex Reflection Groups and Springer Theory

In this section we will recall some notions about reflection groups and Springer theory. A deeper reference for this is [Bro10] or [BS21, Section 3].
Let V be a complex vector space of $\dim_{\mathbb{C}}(V) = n$ and $W \subset GL_n(V)$ a finite subgroup of the complex general linear group of degree n. Given an element $g \in GL_n(V)$ we define the subset $V^g := \{v \in V \mid g(v) = v\} \subset V$ of the elements fixed by g. Moreover we define also the set

$$\text{Ref}(W) := \{s \in W \mid \dim(V^s) = n - 1\}.$$

Definition 1.15. *A finite subgroup $W \subset GL_n(V)$ is called a* complex reflection group *if $W = \langle Ref(W) \rangle$.*

A classic result, stated e.g. in [Bro10, Theorem 4.1], is the following

Theorem (Serre–Chevalley, Shepherd–Todd). *Given a complex reflection group W acting on a complex vector space V of $\dim_{\mathbb{C}}(V) = n$ then there exist $f_1, \ldots, f_n$ homogeneous polynomials of degree $d_1, \ldots, d_n$ such that the invariant ring by the action of W is given by*

$$\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n].$$

The family $\{f_1, \ldots, f_n\}$ is not unique (up to permutations) but the degrees $\{d_1, \ldots, d_n\}$ are uniquely determined (up to permutations) by V and W. There exists another family which is uniquely determined (up to permutations) by V and W. Indeed, a result by Solomon [Bro10, Theorem 4.44 and Section 4.5.4] implies that the graded $\mathbb{C}[V]^W$-module of W-invariant derivations of $\mathbb{C}[V]$ admits a homogeneous $\mathbb{C}[V]^W$-basis $(g_1, \ldots, g_n)$ whose degrees $(d_1^*, \ldots, d_n^*)$ are called co-degrees. The co-degrees are invariant up to permutation.
Now, in order to state the results that we need from Springer theory, we need to define the following numbers for any $e \in \mathbb{N}$

$$\lambda(e) := |\{1 \leq i \leq n \mid e \text{ divides } d_i\}|$$

$$\lambda^*(e) := |\{1 \leq i \leq n \mid e \text{ divides } d_i^*\}|.$$

Moreover, if the primitive e-th root of the unity ζ_e is a eigenvalue for the action of $w \in W$, we set $V(w, \zeta_e)$ to be the eigenspace of V with respect to the action of w relative to ζ_e. Then, putting together various results from [LS99, Theorem C] and [Spr74, Theorem 3.4, Theorem 4.2, Theorem 6.2], we state the following

Theorem 1.16 (Springer, Lehrer-Springer). *Let W be a complex reflection group acting on V. Then for every $e \in \mathbb{N}$ it holds $\lambda(e) = \max_{w \in W} \dim(V(w, \zeta_e))$. If, moreover, e is such that $\lambda(e) = \lambda^*(e)$, then the elements w_e which attain the maximum define a unique conjugacy class in W.*

Let us explain the results in this section with a couple of examples.

2.1 Isometries of order 3 on D_4

From classical theory, that can be found e.g. in [CS99, Chapter 4, Section 7], the group of lattice isometries of D_4 is $O(D_4) \simeq G_{28} \simeq W(F_4) =: W$ where $W(F_4)$ denotes the Coxeter group F_4 and G_{28} is the 28-th group in the Shepherd–Todd classification. If we consider its action on $V = \mathbb{C}^4$ it is a complex reflection group. In order to compute the degrees and co-degrees one can use a computer algebra program like MAGMA or refer to [Bon21]. In any case we have that the degrees are

$$(d_1, d_2, d_3, d_4) = (2, 6, 8, 12)$$

and the co-degrees are

$$(d_1^*, d_2^*, d_3^*, d_4^*) = (0, 4, 6, 10).$$

So we can look at $\lambda(3) = \max_{w \in W} \dim(V(w, \zeta_3)) = 2 = \lambda^*(3)$. Suppose that this maximum is attained in $w_3 \in W = W(F_4) \subset GL_4(V)$. Therefore w_3 is an integer matrix which admits ζ_3 as eigenvalue whose eigenspace is 2-dimensional, thus the same is true also for $\bar{\zeta_3}$. Moreover, as the triple $(W, V, 3)$ satisfies the hypothesis of Theorem 1.16, w_3 is unique up to conjugation. Wrapping up everything, we proved that there exists only one isometry of D_4, up to conjugation, of order three and without fixed points (the last statement comes from the fact that the only eigenvalues are ζ_3 and $\bar{\zeta_3}$).

2.2 Isometries of order 3 on E_6

Here we want to apply again the same idea. Lattice isometries of E_6 are just $O(E_6) \simeq \mathbb{Z}/2\mathbb{Z} \times G_{35} \simeq \mathbb{Z}/2\mathbb{Z} \times W(E_6)$ (again it is a classical result which can be found on [CS99, Chapter 4, Section 8.3]) with the same convention as above. $W := W(E_6)$ acts as a complex reflection group on $V = \mathbb{C}^6$. The degrees are

$$(d_1, d_2, d_3, d_4, d_5, d_6) = (2, 5, 6, 8, 9, 12)$$

and the co-degrees are

$$(d_1^*, d_2^*, d_3^*, d_4^*, d_5^*, d_6^*) = (0, 3, 4, 6, 7, 10).$$

Now $\lambda(3) = \max_{w \in W} \dim(V(w, \zeta_3)) = 3 = \lambda^*(3)$ and we assume that w_3 attains the maximum. With the same argument as above we obtain that w_3 has ζ_3 and $\bar{\zeta}_3$ as triple eigenvalues. Therefore we proved that up to conjugation (and sign) there exists only one order three isometry without fixed points on E_6.

Chapter 2

Irreducible holomorphic symplectic manifolds

"Robando flores a la luz de la luna
Pido perdón a diestra y siniestra
Pero no me declaro culpable."

- Nicanor Parra, *Yo Pecador*

One of the classes of manifolds in algebraic geometry which has encountered a growing interest over the last forty years is surely the one of irreducible holomorphic symplectic manifolds. There exist many reasons why they have been so studied, one of them is surely the fact that they are a distinguished class of manifolds having trivial first Chern class, by Theorem 2.2. In this chapter we will give the basic definitions and first properties. There exists a vast multiplicity of good references which we used for this chapter; the main and most complete are surely [Huy99] and[Deb22], but we used also the summary made in [Ber20].

1 An introduction to irreducible holomorphic symplectic manifolds

1.1 Definition of IHS manifolds and first properties

We begin this section by writing the definition of the main object of our study. Recall that given a manifold X, we say that a holomorphic 2-form on X is symplectic if it has everywhere maximal rank.

Definition 2.1. *An irreducible holomorphic symplectic (from now on IHS) manifold X is a compact complex Kähler manifold which is simply connected and such that there exists a symplectic 2-form ω_X such that $H^0(X, \Omega_X^2) = \mathbb{C}\omega_X$.*

Consider an IHS manifold X. The fact that X admits a symplectic 2-form has important consequences. Primarily, it implies that X has even complex dimension, since the skew-symmetric form ω_x on the tangent space T_xX has maximal rank at every point $x \in X$, and the rank of a skew-symmetric form is even.

Denote the canonical bundle of X as K_X, and let the dimension of X be $2n$. As ω_X^n does not vanish on X, we conclude that the canonical bundle is trivial, with $K_X \cong \mathcal{O}_X$. Consequently, the first Chern class of X is trivial.

As the form $\omega : TX \times TX \to \mathbb{C} \otimes \mathcal{O}_X$ has maximal rank, we find an isomorphism between the tangent bundle TX and the cotangent bundle Ω_X^1.

IHS manifolds hold a distinctive position among manifolds for which the first Chern class is zero. Beauville showed that these manifolds naturally arise as fundamental components of manifolds with this property.

Theorem 2.2 (Beauville-Bogomolov). *Let M be a compact Kähler manifold such that $c_1(M)_{\mathbb{R}} = 0$. Then there exists a finite étale covering $\tilde{M}$ of M in the form*

$$\tilde{M} = T \times \prod_i C_i \times \prod_j X_j,$$

where T is a complex torus, C_i is a Calabi-Yau manifold for every i and X_j is an IHS manifold for every j.

Proof. See [Bea83b, Theorem 2]. □

Several general results can be obtained for any IHS manifold, for example at the level of cohomology. Let X be an IHS manifold. It is a consequence of Definition 2.1 that $h^{2,0}(X) = 1$. Further, for a manifold of dimension $\dim(X) = 2n$, the Hodge cohomology $H^{k,0}(X)$ is given by

$$H^{k,0}(X) = \begin{cases} \mathbb{C} \text{ if } k \text{ is even and } k \leq 2n \\ 0 \text{ if } k \text{ is odd} \end{cases}$$

as referred to in Beauville's work (see [Bea83b, Section 4, Proposition 3, Item ii)]). Moreover, again by Definition 2.1, we deduce that the H^1 is trivial by simple connectedness. In particular this implies that the Picard group $\mathrm{Pic}(X)$ and the Néron–Severi group $\mathrm{NS}(X)$ are isomorphic. Through this identification, we consider the Picard group $\mathrm{Pic}(X)$ as a subgroup of the second cohomology group $H^2(X, \mathbb{Z})$. The second cohomology group carries significant importance in the study of IHS manifolds. The corresponding Hodge numbers for X are given by

$$1 \qquad h^{1,1}(X) \qquad 1$$

Furthermore, due to the Universal Coefficient Theorem for cohomology, the second cohomology group $H^2(X, \mathbb{Z})$ is torsion-free, as the simple connectedness of X yields $H_1(X, \mathbb{Z}) = 0$. Indeed, we can say more:

Theorem 2.3. *Let X be an IHS manifold of dimension $2n$ and whose second Betti number is $b_2(X) \neq 6$. There exists a unique non-degenerate symmetric integral and non-divisible bilinear* *form* $(-,-)$ *that endows* $H^2(X,\mathbb{Z})$ *with the structure of a lattice of signature* $(3, b_2(X)-3)$ *with the following property: there exists a positive rational number* c_X *such that the equality*

$$\int_X \sigma^{2n} = c_X(\sigma,\sigma)^n$$

holds for every $\sigma \in H^2(X,\mathbb{Z})$.
Moreover, let ω *be a symplectic* 2*-form on* X*, then*

$$(\omega,\omega) = 0 \text{ and } (\omega,\bar{\omega}) > 0.$$

Proof. See [Bea83b, Section 8, Theorem 5, Item $a) - c)$]. □

Definition 2.4. *We refer to the bilinear form in Theorem 2.3 as the* Beauville–Bogomolov–Fujiki *form (BBF form). Unless specified differently, a bilinear form* $(-,-)$ *on* $H^2(X,\mathbb{Z})$ *for an IHS manifold* X *is understood to be the Beauville-Bogomolov-Fujiki form. When we discuss the lattice structure of* $H^2(X,\mathbb{Z})$*, we refer to the structure given in Theorem 2.3. We will use* c_X *to denote the Fujiki constant for* X.

Given that the orthogonal complement of $H^{2,0}(X)$ is $H^{2,0}(X) \oplus H^{1,1}(X)$, the Néron–Severi group can be described as $\mathrm{NS}(X) = \omega^\perp \cap H^2(X,\mathbb{Z})$. For a projective manifold X, the Néron-Severi group thus forms a primitive sublattice of $H^2(X,\mathbb{Z})$ with signature $(1, b_2(X)-3)$.

We refer to the Néron–Severi group equipped with the restriction of the BBF form as the *Néron–Severi lattice* or, equivalently, the *Picard lattice.* For two divisors D_1, D_2 on X, (D_1, D_2) denotes the product of the corresponding line bundles, i.e., $(D_1, D_2) = (\mathcal{O}_X(D_1), \mathcal{O}_X(D_2))$.

Definition 2.5. *We refer to a line bundle* $L \in \mathrm{Pic}(X)$ *as* primitive *if it is primitive as an element of the lattice* $\mathrm{NS}(X)$.

The Kleiman projectivity criterion for surfaces can be generalized to IHS manifolds.

Theorem 2.6. *Let* X *be an IHS manifold. Then,* X *is projective if and only if there exists a line bundle* $D \in \mathrm{Pic}(X)$ *such that* $D^2 > 0$.

Proof. See [Huy99, Theorem 3.11] □

Definition 2.7. *For a projective IHS manifold* X*, a* $\langle k\rangle$*-polarization on* X *is defined by choosing a primitive ample line bundle* $L \in \mathrm{Pic}(X)$ *such that* $L^2 = k$*. By the definition of the BBF form,* k *is always strictly positive, since an ample class is also Kähler.*

Finally, note that the morphism induced by a birational map preserves the BBF form.

Proposition 2.8. *If* $f : X_1 \dashrightarrow X_2$ *is a birational map between IHS manifolds, then* f *restricts to a biregular map* $f_{|U_1} : U_1 \to U_2$*, where* $U_i \subseteq X_i$ *is open whose complement have codimension at least two for* $i = 1, 2$*, and the map* $f_* : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$*, induced through the inclusion of* U_i *in* X_i*, is a Hodge isometry. Moreover,* f_* *does not depend on the choice of* U_i.

Proof. See [O'G97, Proposition 1.6.2]. □

This result is important as the lattice structure on the second cohomology group is a crucial tool for studying morphisms between IHS manifolds.

1.2 Examples of IHS manifolds

Every example of an IHS manifold can yield an entire set of IHS manifolds due to the following theorems. Let X denote an IHS manifold. Then [Huy99, Remark 1.12] states the following using the result known as Bogomolov–Tian–Todorov theorem (see [Bog78], [Tia87], [Tod89]).

Theorem 2.9. *There exists a universal deformation family denoted as $\mathcal{X} \to \mathrm{Def}(X)$, where $\mathrm{Def}(X)$ is smooth.*

This family $\mathcal{X} \to \mathrm{Def}(X)$ is based on X, meaning that the fiber over 0 is isomorphic to X. The Zariski tangent space of the universal deformation family at the point $0 \in \mathrm{Def}(X)$ is isomorphic to $H^1(X, TX) \cong H^1(X, \Omega_X)$. Since $\mathrm{Def}(X)$ is smooth, this implies that $\mathrm{Def}(X)$ has dimension equal to $h^{1,1}(X)$.

Proposition 2.10. *Assume that $\mathcal{X} \to B$ is a smooth and proper family over an analytic base B and that the fiber X_0 over a point $0 \in B$ is an IHS manifold. Then, for every $b \in B$ such that X_b is Kähler, X_b is also an IHS manifold.*

Proof. See [Bea83b, Proposition 9]. □

Corollary 2.11. *Every fiber of the universal deformation family $\mathcal{X} \to \mathrm{Def}(X)$ is an IHS manifold.*

Proof. This is a direct consequence of Proposition 2.10 and the fact that, by [KS60, Theorem 15], any smooth deformation of a Kähler manifold is again a Kähler manifold. □

We are now set to present some examples of IHS manifolds. If two IHS manifolds are deformation equivalent, we say that they are of the same deformation type or in the same family of deformations.

Note that IHS manifolds only appear in even dimensions. The first example we can find of an IHS manifold is a classical one: the $K3$ surface. Even though we will not use much about the geometry of $K3$ surfaces we want to give a brief description of them and some of their properties as they are an easy approach to the world of IHS manifolds and many of their features can be generalized to higher dimensions. Also, many conjectures and research directions arise from their geometry.

Definition 2.12. *A $K3$ surface is a smooth compact complex surface that is simply-connected and possesses a trivial canonical bundle.*

The definition of a 2-dimensional IHS manifold coincides with the definition of a $K3$ surface. Indeed every $K3$ surface is a Kähler manifold as proven in [Siu83].

Remark 2.13. *All $K3$ surfaces are deformation equivalent, as described in [Huy16, Chapter 7, Theorem 1.1].*

Next, as exercise, we compute $h^{1,1}(\Sigma)$ for any $K3$ surface Σ. We can derive the other Hodge numbers directly from the structure of the IHS manifold. Due to the triviality of the canonical bundle, we apply Noether's formula: $e(\Sigma) = 12\chi(\mathcal{O}_\Sigma)$, where $e(\Sigma)$ represents the topological Euler characteristic of Σ, and $\chi(\mathcal{O}_\Sigma)$ is the holomorphic Euler characteristic given by $\chi(\mathcal{O}_\Sigma) = h^{0,2} - h^{0,1} + 1 = 2$. Thus, we find $e(\Sigma) = 24$ and the second Betti number, $b_2(\Sigma) = e(\Sigma) - 2 = 22$. As, by definition of IHS manifold, $H^0(\Sigma, \Omega^2_\Sigma)$ is generated by a symplectic form, we also find $h^{1,1}(\Sigma) = 20$. The lattice $H^2(\Sigma, \mathbb{Z})$ has rank 22, and in this case, the BBF form is the cup product.

Theorem 2.14. *The second cohomology group with the BBF form can be shown to be isomorphic, as an abstract lattice, to the unimodular lattice* $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.

Proof. For a detailed proof, please refer to [Huy16, Chapter 14, example 1.4]. □

So, we are able to give the following definition.

Definition 2.15. *The abstract rank-22 unimodular lattice,* $L_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$*, is referred to as the* $K3$ lattice.

The first examples of IHS manifolds in higher dimensions were provided by Fujiki in dimension 4, and then extended to all even dimensions by Beauville [Bea83b]. The majority of the results we are going to cite can be found in *loc.cit.*. Our interest will be predominantly in one family of deformations constructed as follows. Let Σ be a $K3$ surface; for our purposes, we can restrict to the case where Σ is projective.

Definition 2.16. *We define the* Hilbert scheme of n points on Σ *as the variety that parameterizes zero-dimensional subschemes* $(Z, \mathcal{O}_Z)$ *of length* n *(i.e.,* $\dim \mathcal{O}_Z = n$*) on the surface* Σ*, denoted by* $\Sigma^{[n]}$*. Oftentimes we will denote it also by* $\mathrm{Hilb}^n(\Sigma)$*. We will refer to the Hilbert scheme* $\Sigma^{[2]}$ *of two points on* Σ *also as the* Hilbert square of Σ.

This definition sometimes can bring, from our point of view, to a lack of geometric meaning, in the sense that it is not so clear how to generalize facts about $K3$ surfaces to Hilbert schemes of points on them. In order to deal with this fact we can give another equivalent definition.

In line with [Bea83b, Section 6], we use the following notations:

- $\Sigma^{(n)}$ stands for the variety of 0-cycles of degree n, defined as the quotient of $\Sigma^n := \overbrace{\Sigma \times \cdots \times \Sigma}^{n \text{ times}}$ by the symmetric group on n elements. We will refer to it also with $\mathrm{Sym}^n(\Sigma)$.
- We label the natural mapping associating each finite scheme with the corresponding 0-cycle (termed the *Hilbert-Chow morphism*) as $\epsilon : \Sigma^{[n]} \to \Sigma^{(n)}$.
- We denote the locus of cycles in the form $p_1 + \ldots + p_n$ such that there exists $i \neq j$ with $p_i = p_j$, also called *diagonal*, as $D \subset \Sigma^{(n)}$.

Definition 2.17 (Alternative definition)**.** *Consider on* Σ^n *the action* γ *of the symmetric group* S_n *on* $\mathbb{Z}/n\mathbb{Z}$ *which permutes the factors. Then the quotient* $\Sigma^{(n)} = (\Sigma^n)/S_n$ *is singular on the diagonal. The blow-up of* $\Sigma^{(n)}$ *along the diagonal is the Hilbert scheme* $\Sigma^{[n]}$ *of* n *points on* Σ *and the blow-up morphism is identified with the* Hilbert–Chow *morphism.*

Example 2.18 (Hilbert square)**.** *There exists a nice way to characterize points on the Hilbert square $S^{[2]}$ of a surface S. Indeed the support of a closed subscheme of length two ξ can either consists of two points or one. If the support of ξ consists of two points $p \neq q$ then ξ is outside the preimage of the diagonal under the Hilbert–Chow morphism and therefore it can be identified with $p+q$. If the support of ξ consists of one point p then it can be identified with the pair (p, v) with $v \in \mathbb{P}(T_p(S))$.*

This characterization will be important to understand Chapter 4. The reason of this lies in the following proposition.

Proposition 2.19. *Let $x \in X$ be a closed point of a k-scheme X. Then the set $\mathcal{Z}_x^2$ of length two subschemes supported only at x is in bijection with $\mathbb{P}(T_x(X))$.*

Proof. We want to construct the explicit bijection. Any element $v \in T_x(X)$ defines a morphism $\phi_v \in \mathrm{Hom}_k(\mathrm{Spec}(k[\epsilon]/(\epsilon^2)), X)$. So we define the morphism:

$$\begin{aligned} \alpha : \mathrm{Hom}_k(\mathrm{Spec}(k[\epsilon]/(\epsilon^2)), X) &\to \mathcal{Z}_x^2 \\ \phi_v &\mapsto Z_{\phi_v} \end{aligned}$$

where Z_{ϕ_v} is the scheme theoretic image of ϕ_v, i.e. the smallest closed subscheme $Z_{\phi_v} \subset X$ through which ϕ_v factors. This is a subscheme of X supported at x. If v is not the zero vector then the stalk at x is given by $\mathcal{O}_{X,x}/\ker((\phi_v)_x) \simeq k[\epsilon]/(\epsilon^2)$ which is a subscheme of length two. Moreover, consider $\phi_v, \phi_w \in \mathrm{Hom}_k(\mathrm{Spec}(k[\epsilon]/(\epsilon^2)), X)$. If there exists an isomorphism of $k[\epsilon]/(\epsilon^2)$ commuting with ϕ_v and ϕ_w, then they define the same subscheme in X . But the isomorphisms of $k[\epsilon]/(\epsilon^2)$ are given by any multiplication of ϵ by a non-zero element $c \in k^*$. Therefore, the morphism α induces a morphism $\bar{\alpha} : \mathbb{P}(T_x(X)) \to \mathcal{Z}_x^2$.
Viceversa, take $Z \in \mathcal{Z}_x^2$ defined by the ideal $\mathcal{I}_Z$ and consider $\mathcal{O}_{Z,x} \simeq \mathcal{O}_{X,x}/\mathcal{I}_{Z,x}$. This is a length two module over the local ring $\mathcal{O}_{X,x}$, hence there exists a short exact sequence of $k(x)$-vector spaces:

$$0 \longrightarrow k(x) \longrightarrow \mathcal{O}_{X,x}/\mathcal{I}_{Z,x} \longrightarrow k(x) \longrightarrow 0.$$

Then $\mathcal{O}_{X,x}/\mathcal{I}_{Z,x} \simeq k(x) \oplus k(x)$. The right side of the equivalence inherits a multiplication $(a, b) \cdot (c, d) = (ab, ad + cb)$. Now, we can compose the projection $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathcal{I}_{Z,x} \simeq k(x) \oplus k(x)$ with the map $(a, b) \mapsto a + \epsilon b$ to obtain the desired tangent vector. Note that the considerations done in the first half of the proof about the zero vector and proportional vectors apply also here. □

By Fogarty's theorem [Ber12, Theorem 3.1], we know that the Hilbert scheme $\Sigma^{[n]}$ is smooth, connected, and of dimension $2n$. Additionally, the singular locus of $\Sigma^{(n)}$ is the diagonal D, and the Hilbert–Chow morphism ϵ is a birational map acting as a desingularization of $\Sigma^{(n)}$. We also know that $E = \epsilon^{-1}(D)$ is an irreducible divisor on $\Sigma^{[n]}$. This implies that a generic point on $\Sigma^{[n]}$ can be viewed as an n-tuple of non-ordered distinct points $p_1 + \ldots + p_n$ on Σ.

Theorem 2.20. *For any projective $K3$ surface Σ and for any $n \geq 2$, the Hilbert scheme of n points on Σ is a projective IHS manifold.*

Proof. See [Bea83b, Section 6, Theorem 3]. □

Remark 2.21. *It should be remarked that this result, for $n = 2$, is due to Fujiki in [Fuj83].*

Consider again a projective $K3$ surface Σ and $n \geq 2$. There is a natural primitive embedding of lattices

$$i : H^2(\Sigma, \mathbb{Z}) \to H^2(\Sigma^{[n]}, \mathbb{Z})$$

such that $H^2(\Sigma^{[n]}, \mathbb{Z}) = i(H^2(\Sigma, \mathbb{Z})) \oplus \mathbb{Z}\delta$ as lattices, with $2\delta = E$, according to Beauville ([Bea83b, Proposition 6]). Additionally, $\delta^2 = -2(n-1)$ implies $H^2(\Sigma^{[n]}, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle$.

As for the Picard group of the Hilbert scheme of n points $\Sigma^{[n]}$, the lattice inclusion i establishes an identification

$$\mathrm{Pic}(\Sigma^{[n]}) = i(\mathrm{Pic}(\Sigma)) \oplus \mathbb{Z}\delta.$$

In particular, the Picard rank of the Hilbert scheme of n points on a $K3$ surface Σ is always at least two.

Remark 2.22. *By Theorem 2.6, if Σ is projective then $\Sigma^{[n]}$ is projective as well. However, for any line bundle $L \in \mathrm{Pic}(\Sigma)$, the corresponding line bundle $i(L)$ on $\Sigma^{[n]}$ is never ample for $n \geq 2$, as it has zero product with the class of the exceptional divisor E, which means $(i(L), E) = 0$.*

Definition 2.23. *For any $n \geq 2$, the abstract lattice $L_{K3^{[n]}} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle$ is known as the $K3^{[n]}$-lattice.*

It should be noted that for any $n \geq 2$, as $h^{1,1}(\Sigma^{[n]}) = h^{1,1}(\Sigma) + 1$, the space $\mathrm{Def}(\Sigma)$ can be viewed as a proper, closed subspace of codimension one in $\mathrm{Def}(\Sigma^{[n]})$. This represents IHS manifolds of $K3^{[n]}$-type that are Hilbert schemes of a deformation of Σ. Consequently, a very general IHS manifold of $K3^{[n]}$-type is not isomorphic to the Hilbert scheme of n points on a $K3$ surface, as indicated by [Bea83b, Theorem 6].

Lemma 2.24. *The Fujiki constant for any IHS manifold X of $K3^{[n]}$-type is given by $c_X = \frac{(2n)!}{n! 2^n}$.*

Proof. See [Bea83b, Section 9] for details. □

The previously mentioned examples, including $K3$ surfaces and IHS manifolds of $K3^{[n]}$-type where $n \geq 2$, illustrate a set of IHS manifold examples for every even dimension. We list here the other known deformation families of IHS manifolds.

- *Generalized Kummer manifolds, [Bea83b, Section 7]* - These exist in every even dimension. If X represents a generalized Kummer manifold and $\dim X = 2$, then X is a Kummer surface, i.e. a $K3$ surface defined by blowing up the singularities of the quotient of a torus by $\{\pm 1\}$. For $\dim X > 2$, the second Betti number of X is $b_2(X) = 7$, hence X is not of $K3^{[n]}$-type. We define an IHS manifold as *of Kummer-type* if it is deformation equivalent to a generalized Kummer manifold. For the sake of completeness we sketch here the construction. Let A be a complex two-dimensional torus and $n \geq 1$ an integer. Then $\mathrm{Hilb}^{n+1}(A)$ is holomorphic symplectic, but it is not an IHS manifold since it is not simply connected. Then we consider the summation morphism

$$\begin{aligned} s : \mathrm{Hilb}^{n+1}(A) &\to A \\ [(Z, \mathcal{O}_Z)] &\mapsto \sum_{p \in A} l(\mathcal{O}_{Z,p})p \end{aligned}$$

and we define $K_n(A) := s^{-1}(0)$, where $0 \in A$ is the zero-point of the torus. The fiber $K_n(A)$ is then an IHS manifold of dimension $2n$, as proved by Beauville in [Bea83b].

- *O'Grady's example in dimension* 6*, [O'G03]* - This particular example only occurs in dimension 6. An X in this family possesses a second Betti number $b_2(X) = 8$, therefore, it is neither of $K3^{[3]}$-type nor of Kummer-type.

- *O'Grady's example in dimension* 10*,[O'G99]* - This specific example exists only in dimension 10. An X within this family has a second Betti number $b_2(X) = 24$, indicating that it is neither of $K3^{[5]}$-type nor of Kummer-type.

Moreover, the lattice $H^2(X, \mathbb{Z})$ is a deformation invariant and in literature there can be found an explicit description for each example (see e.g. [Deb22]) , therefore we will say that an IHS manifold X is of type L if $H^2(X, \mathbb{Z}) \simeq L$, furthermore assuming that the deformation type is fixed as one of the above.

Remark 2.25. *There exists no general proof of the fact that if $H^2(X, \mathbb{Z}) \simeq H^2(Y, \mathbb{Z}) \simeq L$, for X, Y IHS manifolds and L a lattice, then $X \sim_{def} Y$. Nevertheless, these lattices are different for every family of deformations currently known.*

2 Moduli spaces and period maps

2.1 Marked IHS manifolds and monodromy operators

An essential characteristic of $K3$ surfaces is the existence of a Torelli theorem for them, as articulated in the following theorem.

Theorem 2.26 (Global Torelli theorem for $K3$ surfaces)**.** *Two $K3$ surfaces S and Σ are isomorphic if and only if there exists a Hodge isometry $H^2(S, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z})$.*
Moreover, for any Hodge isometry $f : H^2(S, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z})$, there exists an isomorphism $\tilde{f} : \Sigma \to S$ such that $\tilde{f}^* = f$ if and only if f sends a Kähler class on S to a Kähler class on Σ. If such $\tilde{f}$ exists it is unique.

Proof. For the proof see [BR75]. □

This theorem marks a milestone in the field of IHS manifolds, providing significant impetus to explore analogous outcomes in higher dimensions. As we will show, these higher-dimensional parallels do exist, but their formal expressions are anything but straightforward.

This complexity is particularly highlighted in the flawed, oversimplified version of the Torelli theorem for higher dimensions, an error made evident through Namikawa's counterexample in [Nam02].

Consider an IHS manifold X of type L.

Definition 2.27. *A* marking *on X is an isometry $\eta : H^2(X,\mathbb{Z}) \to L$. A* marked IHS manifold *is a pair (X,η), where X is an IHS manifold together with a marking $\eta : H^2(X,\mathbb{Z}) \to L$ on X. Two marked IHS manifolds (X_1,η_1) and (X_2,η_2) are isomorphic if there exists an isomorphism $f : X_1 \to X_2$ such that $\eta_2 = \eta_1 \circ f^*$.*

Let (X,η) be a marked IHS manifold, and consider again the universal deformation family $\pi : \mathcal{X} \to \mathrm{Def}(X)$. We can extend the marking η to the family $\mathcal{X}$, in the sense that there exists a family of markings $(F_b : \mathcal{X}_b \to L)_{b\in B}$ such that $F_0 = \eta$, see [Kod86, Theorem 2.4]. Then we can define a *local period map*

$$\begin{aligned}\mathcal{P} : \mathrm{Def}(X) &\to \mathbb{P}(L)\\ b &\mapsto [F_b(H^{2,0}(\mathcal{X}_b))].\end{aligned}$$

Clearly $\mathcal{P}(0) = [\eta(H^{2,0}(X))]$.

Definition 2.28. *We call* period domain *the analytic subvariety*

$$\Omega = \{[x] \in \mathbb{P}(L\otimes\mathbb{C}) |\, x^2 = 0, (x,\bar{x}) > 0\}.$$

It is an open subvariety (in the analytic topology) on a quadric hypersurface in $\mathbb{P}(L\otimes\mathbb{C})$.

Remark 2.29. The image of $\mathcal{P}$ is contained in Ω by the properties of BBF form, see Theorem 2.3. We call *period point of* (X,η) the image $\mathcal{P}(0)$.

Theorem 2.30 (Local Torelli theorem)**.** *Let (X,η) be a marked IHS manifold; the local period map $\mathcal{P} : \mathrm{Def}(X) \to \Omega$ is a local isomorphism.*

Proof. See [Bea83b, Section 8 Theorem 5, Item b)]. □

In order to move to the global case we need the definition of parallel transport operator. These isometries play a key role in the theory of IHS manifolds.

Definition 2.31 ([Mar11, Definition 1.1])**.** *Let X_1, X_2 be IHS manifolds. An isomorphism $f : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is said to be a* parallel transport operator *if there exist*

- *a smooth and proper family $\pi : \mathcal{X} \to B$ of IHS manifolds over an analytic base B,*
- *$b_i \in B$, for $i = 1,2$, such that there exists $\psi_i : X_i \xrightarrow{\sim} \mathcal{X}_{b_i}$ isomorphisms.*
- *a continuous path $\gamma : [0,1] \to B$ with $\gamma(0) = b_1$, $\gamma(1) = b_2$*

such that f is induced, through the isomorphisms ψ_i, by the parallel transport in the local system $R^2\pi_*\mathbb{Z}$ along γ.
Given an IHS manifold X, a monodromy operator *is a parallel transport operator $f : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$.*

The BBF form is topological, hence invariant under parallel transport. Therefore, every parallel transport operator is a lattice isometry. Moreover, we can see that, given two intersecting families of deformation, they can be glued to obtain a third family of deformations, hence the composition of two parallel transport operators is a parallel transport operator; in particular monodromy operators form a subgroup inside $O(H^2(X,\mathbb{Z}))$, which we call *monodromy group* and we denote by $\mathrm{Mon}^2(X)$.

Definition 2.32. *A* Hodge monodromy operator *is a monodromy operator in $O(H^2(X,\mathbb{Z}))$ which is a Hodge isometry; we denote by $\mathrm{Mon}^2_{Hdg}(X)$ the subgroup of such operators.*

Proposition 2.33. *Let $f : X \dashrightarrow Y$ be a birational map between IHS manifolds. Then $f^* : H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is a parallel transport operator.*

Proof. See [Huy04, Lemma 2.4] □

In particular, this means that the group of birational automorphisms of an IHS manifold X acts on $H^2(X,\mathbb{Z})$ as a group of monodromy operators.

Definition 2.34. *Let (X,η) be a marked IHS manifold of type L. We denote by $\mathrm{Mon}(L)$ the subgroup of $O(L)$ given by*

$$\eta^{-1} \circ \mathrm{Mon}^2(X) \circ \eta.$$

We define the coarse moduli space of marked IHS manifolds of type L as the set of marked manifolds (X,η), modulo the equivalence relation given by isomorphism of marked manifold. We call it $\mathcal{M}_L$. By the local Torelli theorem, the universal deformations can be used as local charts for the moduli space $\mathcal{M}_L$. More specifically, by [Huy12, Proposition 4.3], for any marked pair (X,η) there exists a holomorphic embedding $\mathrm{Def}(X) \hookrightarrow \mathcal{M}_L$, identifying $\mathrm{Def}(X)$ with an open neighbourhood of the point $(X,\eta) \in \mathcal{M}_L$. The maps $\mathcal{P} : \mathrm{Def}(X) \to \Omega$.

Theorem 2.35. *The moduli space $\mathcal{M}_L$ of marked IHS manifolds which are deformation equivalent to X is a smooth non-Hausdorff complex manifold.*

Proof. See [Huy04, Section 3.1]. □

Remark 2.36. Every two marked IHS manifolds (X_1,η_1) and (X_2,η_2) are deformation equivalent if and only if they are in the same connected component of $\mathcal{M}_L$. In particular, they are in the same connected component if and only if $\eta_2^{-1} \circ \eta_1$ is a parallel transport operator, see [Mar11, Lemma 7.5]. This also implies that the number of connected components of $\mathcal{M}_L$ is equal to $[O(H^2(X,\mathbb{Z})) : \mathrm{Mon}^2(X)]$.

Then we can define a *global period map*

$$\mathcal{P} : \mathcal{M}_L \to \Omega.$$

Theorem 2.37 (Surjectivity of the period map). *Let $\mathcal{M}_L^0$ be a connected component of $\mathcal{M}_L$. Then the restriction to $\mathcal{M}_L^0$ of the period map is surjective.*

Proof. See [Huy99, Theorem 8.1]. □

The global Torelli theorem generalizes for every deformation type of IHS manifolds in the following way. The two parts of the statement were proved respectively by Huybrechts and Verbitsky.

Theorem 2.38 (Global Torelli theorem for marked IHS manifolds). *For every two inseparable points (X_1, η_1) and (X_2, η_2) in $\mathcal{M}_L$, the IHS manifolds X_1 and X_2 are bimeromorphic.*
Let $\mathcal{M}_L^0$ be a connected component of $\mathcal{M}_L$, and call $\mathcal{P}_0$ the restriction to $\mathcal{M}_L^0$ of the period map: for every $p \in \Omega_L$, the fiber $\mathcal{P}_0^{-1}(p)$ consists of pairwise inseparable points.

Proof. See [Huy99, Theorem 4.3] for the first part and [Ver13, Theorem 1.16] for the second part. □

In [Mar11] the author combines the global Torelli theorem with results on the Kähler cone of irreducible holomorphic symplectic manifolds to state a Hodge-theoretic version of the global Torelli theorem.

Theorem 2.39 (Hodge theoretic Torelli Theorem [Mar11, Theorem 1.3]). *Let X and Y be IHS manifolds of the same deformation type. Then:*

- *X and Y are bimeromorphic, if and only if there exists a parallel transport operator $f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ which is an isomorphism of integral Hodge structures.*
- *Let $f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ be a parallel transport operator, which is an isomorphism of integral Hodge structures. There exists an isomorphism $\tilde{f} : X \to Y$ inducing f if and only if f maps some Kähler class on X to a Kähler class on Y.*

This last result justifies the use of lattice theory to study automorphisms on an IHS manifold. In order to do so, it is needed a characterization of the group of monodromy operators inside the group of isometries of the second cohomology group of the manifold. We define the *positive cone* of an IHS manifold X to be the connected component $\mathcal{C}_X$ of the set $\left\{x \in H^{1,1}(X, \mathbb{R}) \mid x^2 > 0\right\}$ containing a Kähler class and its *Kähler cone* $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ as the cone consisting of Kähler classes.

2.2 Chamber decomposition

Here we give some important notions useful to the construction of a moduli space endowed with a period map which will be crucial for the following chapters. The purpose of this section is not to give a complete overview of the chamber decomposition. For a more complete review see [Mar11] and [AV15].

Let X be a projective IHS manifold whose Néron–Severi group is $\mathrm{NS}(X)$.

Remark 2.40. For $x, y \in H^{1,1}(X, \mathbb{R})$ with $x \in \mathcal{C}_X$ and $y^2 > 0$, the element y belongs to the positive cone if and only if $(x, y) > 0$.

Definition 2.41. *We call* prime exceptional divisor *an effective divisor E which is reduced, irreducible and such that $E^2 < 0$, and we denote by $\mathcal{P}_X$ the set of classes of such divisors.*

- *The* fundamental exceptional chamber of X *is the cone*

$$\mathcal{FE}_X = \{x \in \mathcal{C}_X | \ (x, E) > 0 \textit{ for every } E \in \mathcal{P}_X\}.$$

- *An* exceptional chamber *of the positive cone $\mathcal{C}_X$ is a subset in the form $g(\mathcal{FE}_X)$ for some monodromy operator $g \in \mathrm{Mon}^2_{Hdg}(X)$.*

By its definition the fundamental exceptional chamber is an exceptional chamber within the structure of $\mathcal{C}_X$. Exceptional chambers, by definition, are also cones, and given that $\mathcal{FE}_X$ is open, these chambers are open too. If two exceptional chambers have non-empty intersection then they coincide by [Mar11, Theorem 6.18, Item 2)].

Moreover, a parallel transport operator, provided it respects the Hodge decomposition, maps one exceptional chamber to another. The same is true for monodromy operators, and it is articulated in [Mar11, Lemma 5.12, Item 1)]. Therefore, the group $\mathrm{Mon}^2_{Hdg}(X)$, by its definition, acts transitively on the set of exceptional chambers.

Furthermore, the action of a Hodge monodromy operator on the set of exceptional chambers can provide insightful information regarding whether the operator originates from a birational map of X into itself.

Proposition 2.42. *Given a monodromy operator $f \in \mathrm{Mon}^2_{Hdg}(X)$, there exists a birational morphism $\tilde{f}$ such that $\tilde{f}^* = f$ if and only if $f^*(\mathcal{FE}_X) = \mathcal{FE}_X$.*

Proof. See [Mar11, Lemma 5.11, Item 6]. □

Given a Kähler class x on X, by definition of the BBF form x lies in one connected component of the positive cone $\mathcal{C}_X$. Actually we can say more: since $(x, E) > 0$ for any effective class E by the Nakai–Moishezon criterion, the Kähler cone $\mathcal{K}_X$ is contained in the fundamental exceptional chamber. Hence the fundamental exceptional chamber $\mathcal{FE}_X$ can be defined also as the exceptional chamber containing a Kähler class.

Definition 2.43 ([AV15, Definition 1.13], [Mon15, Definition 1.2])**.** *A monodromy birationally minimal (MBM) class is a rational class $\delta \in H^{1,1}(X) \cap H^2(X, \mathbb{Q})$ of negative square such that there exists a bimeromorphic map $f : X \dashrightarrow Y$ and a monodromy operator $h \in Mon^2(X)$ such that the hyperplane $\delta^\perp \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$ contains a face of $h(f^*(\mathcal{K}_Y))$. We denote with $\Delta(X)$ the set of integral MBM classes, which are also called* wall divisors.

These classes are important as they give a wall and chamber structure to the positive cone and moreover the following theorem holds.

Theorem 2.44 ([AV15, Theorem 6.2], [Mon15, Proposition 1.5]). *Given an IHS manifold X then its Kähler cone $\mathcal{K}_X$ is a connected component of $\mathcal{C}_X \setminus \mathcal{H}_\Delta$ with $\mathcal{H}_\Delta$ defined as:*

$$\mathcal{H}_\Delta := \bigcup_{\delta \in \Delta(X)} \delta^\perp \subset H^{1,1}(X) \cap H^2(X, \mathbb{R}).$$

Example 2.45 (Numerical characterization in the $K3^{[2]}$-case). *We point out that by [HT09, Theorem 22] and [Mar13, Theorem 1.2] there exists a numerical characterization of the elements in $\Delta(X)$ in the $K3^{[2]}$-type case. An effective class δ is a wall divisor on an IHS fourfold of $K3^{[2]}$-type X if and only if satisfies one of the following:*

- $(\delta, \delta) = -2$
- $(\delta, \delta) = -10$ *and it has divisibility 2, i.e.* $(\delta, H^2(X, \mathbb{Z})) \in 2\mathbb{Z}$.

Moreover, we recall that by [Mar13, Proposition 1.5] an effective class δ is monodromy reflective, i.e. the reflection by δ is an integral monodromy operator, if and only if $(\delta, \delta) = -2$.

Fixing a connected component in $\mathcal{M}_L^\circ \subset \mathcal{M}_L$ and considering a marking η, we can translate these definitions to the lattice, e.g. $\Delta(L)$ will be the set consisting of elements $\eta(\delta)$ with δ a wall divisor for $(X, \eta) \in \mathcal{M}_L^\circ$. We also define $\mathcal{C}_L := \{x \in L \otimes \mathbb{R} \mid (x, x) > 0\}$ and, given again $(X, \eta) \in \mathcal{M}_L^\circ$, the monodromy group of L is $\mathrm{Mon}^2(L) := \eta \circ \mathrm{Mon}^2(X) \circ \eta^{-1}$.

Definition 2.46. *The* Kähler-type chambers *of the positive cone $\mathcal{C}_X$ are the connected components of*

$$\mathcal{C}_X \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp.$$

By definition, Kähler-type chambers are open and they coincide if their intersection is nonempty. A parallel transport operator that respects the Hodge decomposition sends a Kähler-type chamber onto another Kähler-type chamber, see [Mar11, Lemma 5.12, Item 2)].

Now we introduce the definition of some convex cones inside $\mathrm{NS}(X) \otimes \mathbb{R} \subseteq H^{1,1}(X, \mathbb{R})$; those two vector spaces are not in general equal. A *movable line bundle* is a line bundle on X that admits a positive multiple whose base locus has codimension at least 2.

Definition 2.47. *To simplify the notation, we denote by $\mathrm{Pos}(X)$ the intersection $\mathrm{NS}(X)_\mathbb{R} \cap \mathcal{C}_X$.*

- *The movable cone $\mathrm{Mov}(X) \subseteq \overline{\mathrm{Pos}(X)}$ is the convex cone generated by classes of movable line bundles on X.*
- *The nef cone $\mathrm{Nef}(X) \subseteq \mathrm{Mov}(X)$ is the closed convex cone generated by classes of nef line bundles on X.*
- *The ample cone $\mathcal{A}(X) \subseteq \mathrm{Nef}(X)$ is the convex cone generated by classes of ample line bundles on X.*

Remark 2.48. *If X is projective and has Picard rank one, all those cones are the same, since $\mathrm{NS}(X)$ is generated by a single ample class in this situation. In general, the ample cone is open and it is the interior of the nef cone $\mathrm{Nef}(X)$, see [Laz04, Theorem 1.4.23].*

Proposition 2.49. *The interior of the movable cone $\mathrm{Mov}(X)^0$ is the intersection $\mathrm{NS}(X)_{\mathbb{R}} \cap \mathcal{FE}_X$. There is a one-to-one correspondence between the set of exceptional chambers in the positive cone $\mathcal{C}_X$ and the set of the restrictions of the exceptional chambers to $\mathrm{Pos}(X)$.*

Proof. See [Mar11, Lemma 6.22]. □

Remark 2.50. Proposition 2.49, together with Proposition 2.42, implies that a monodromy operator is induced by a birational map if and only if it fixes the movable cone.

Also the decomposition in Kähler-type chambers of $\mathcal{C}_X$ induces a decomposition on $\mathrm{Pos}(X)$, so that the chambers are the connected components of

$$\mathrm{Pos}(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^{\perp}. \tag{2.50.1}$$

The ample cone is the intersection $\mathcal{K}_X \cap \mathrm{Pos}(X)$, so it can be defined as the chamber of the movable cone containing an ample class. Moreover the movable cone $\mathrm{Mov}(X)$ can be characterized as the connected component, in the decomposition of $\mathrm{Pos}(X)$, containing an ample class.

Definition 2.51. *We call* exceptional *(resp.* Kähler-type*)* chambers *of* $\mathrm{Pos}(X)$ *the restrictions of the exceptional (resp. Kähler-type) chambers to* $\mathrm{Pos}(X)$*. The chambers of the movable cone are then the Kähler-type chambers of* $\mathrm{Pos}(X)$ *which lie inside* $\mathrm{Mov}(X)$*. The* walls *of a chamber* K *are the subspaces of* $\mathrm{Pos}(X)$ *which lie in the boundary of* K *and are a maximal open subset of a linear subspace, where maximality is taken with respect to inclusion.*

Remark 2.52. By the definition of Kähler-type chambers and by Proposition 2.42, every Kähler-type chamber inside $\mathcal{FE}_X$ corresponds to a birational model Y of X which is an IHS manifold, and it is the image through a birational map $g : X \dashrightarrow Y$ of the Kähler cone of Y. Similarly, every Kähler-type chamber inside $\mathrm{Mov}(X)$ corresponds to a birational model Y of X and is the image through a birational map $g : X \dashrightarrow Y$ of the ample cone of Y.
Finally the closure of the union of the Kähler-type chambers inside the movable cone is equal to $\overline{\mathrm{Mov}(X)}$: this is the description of the closure of the exceptional chamber of X as the closure of the *birational Kähler cone* of X given in [Mar11, Proposition 5.6].

2.3 Automorphisms of IHS manifolds

In this section we recall some results about automorphisms on IHS manifolds. Let (X, η) be a marked IHS manifold of type L. Then Proposition 2.33 implies the existence of the following maps:

$$\begin{array}{ccccc} \mathrm{Aut}(X) & \rightarrow & \mathrm{Mon}^2(X) & \rightarrow & \mathrm{Mon}^2(L) \\ \sigma & \mapsto & \sigma^* & \mapsto & \eta^{-1} \circ \sigma^* \circ \eta. \end{array}$$

where the second map is an isomorphism by definition of $\mathrm{Mon}^2(L)$. About the first one, in general we can only say that the kernel is finite and deformation invariant using [Huy99, Proposition 9.1]. On the other hand if X is of $K3^{[n]}$-type, it is injective by [Bea83a, Proposition 10]. Moreover, we can compute its image using the Hodge theoretic Torelli Theorem 2.39.

Consider S a projective $K3$ surface with an automorphism $\sigma \in \mathrm{Aut}(S)$. Then we can associate an automorphism which will be called *natural automorphism* $\sigma^{[n]} \in \mathrm{Aut}(S^{[n]})$, for any $n \geq 2$. This is obtained by sending the zero-dimensional subscheme $Z \subset S$ of length n to the zero-dimensional subscheme $\sigma(Z)$.
Equivalently, one can see this automorphism in the following way. There exists a natural action of $\sigma \times \cdots \times \sigma$ on S^n which passes to the quotient inducing an action $\sigma^{(n)}$ on $S^{(n)}$. The diagonal is mapped to itself under $\sigma^{(n)}$, therefore it exists a unique automorphism $\sigma^{[n]}$ on $S^{[n]}$ commuting with the blow-up of the diagonal (i.e. the Hilbert-Chow morphism).
The natural automorphism $\sigma^{[n]}$ is the identity on $S^{[n]}$ if and only if σ is the identity on S, so there is an injective morphism $\mathrm{Aut}(S) \hookrightarrow \mathrm{Aut}(S^{[n]})$. Moreover, $(\sigma^{[n]})^*(\delta) = \delta$. But we can say more.

Theorem 2.53. *Let S be a projective $K3$ surface, and take $n \geq 2$ an integer. An automorphism $\sigma \in \mathrm{Aut}(S^{[n]})$ is natural if and only if it leaves globally invariant the exceptional divisor E, introduced by the Hilbert–Chow morphism, on $S^{[n]}$.*

Proof. See [BS12, Theorem 1]. □

A natural way to characterize automorphisms on an IHS manifold X is to look at their action on the symplectic form ω_X. Indeed, by the Hodge theoretic Torelli Theorem 2.39 an automorphism $\sigma \in \mathrm{Aut}(X)$ induces a Hodge isometry in cohomology. Therefore it induces a morphism

$$\begin{aligned} \alpha : \mathrm{Aut}(X) &\to \mathbb{C}^* \\ \sigma &\mapsto \alpha(\sigma) \end{aligned}$$

in a way that $\sigma^*(\omega_X) = \alpha(\sigma) \cdot \omega_X$.

Remark 2.54. *If we restrict the morphism α to a finite subgroup $G < \mathrm{Aut}(X)$ then $\alpha(G)$ is a finite subgroup of $\mathbb{C}^*$, so it is cyclic and generated by a root of unity.*

Definition 2.55. *An automorphism σ is said* symplectic *if its action is trivial on ω_X, i.e. $\sigma^*_{\mathbb{C}}(\omega_X) = \omega_X$, and non-symplectic otherwise. If no non-trivial power of a non-symplectic automorphism is symplectic then the automorphism is called* purely non-symplectic.

Remark 2.56. *If a non-symplectic automorphism has prime order then it is automatically purely non-symplectic.*

Definition 2.57. *Let X be an IHS manifold. The* transcendental lattice *of X is the primitive sublattice $\mathrm{Tr}(X) = \mathrm{NS}(X)^{\perp} \subset H^2(X, \mathbb{Z})$.*

Definition 2.58. *Let $\sigma \in \operatorname{Aut}(X)$ be an automorphism of finite order on X. The* invariant lattice of *σ is the primitive sublattice $T_\sigma = H^2(X,\mathbb{Z})^{\sigma^*} = \{t \in H^2(X,\mathbb{Z}) |\, \sigma^*(t) = t\}$ and the* coinvariant lattice of *σ is the primitive sublattice $S_\sigma = T_\sigma^\perp$.*

These sublattices have a very precise relation with the Néron–Severi lattice and the transcendental lattice.

Proposition 2.59. *If $\sigma \in \operatorname{Aut}(X)$ is symplectic, then $\operatorname{Tr}(X) \subset T_\sigma$ and $S_\sigma \subset \operatorname{NS}(X)$. If $\sigma \in \operatorname{Aut}(X)$ is non-symplectic, then $T_\sigma \subset \operatorname{NS}(X)$ and $\operatorname{Tr}(X) \subset S_\sigma$.*

Proof. Let n be the order of σ and ω be the symplectic form of X.

Let σ be symplectic. Every $x \in S_\sigma$ resolves $(\operatorname{id} + \sigma^* + ... + (\sigma^*)^{n-1})(x) = 0$ by [BNWkS13, Lemma 5.1]. Then

$$0 = (x + \sigma^*(x) + ... + (\sigma^*)^{n-1}(x), \omega) = \sum_{i=0}^{n-1} ((\sigma^*)^i(x), \omega) = \sum_{i=0}^{n-1} ((\sigma^*)^i(x), (\sigma^*)^i \omega) = n(x, \omega)$$

so $x \in \operatorname{NS}(X)$ and $S_\sigma \subset \operatorname{NS}(X)$. The other inclusion is then obtained passing to the orthogonal complements.

If σ is non-symplectic, let $\zeta \neq 1$ be the root of unity such that $\sigma^*\omega = \zeta\omega$. Then, for every $x \in T_\sigma$,

$$(\omega, x) = (\sigma^*(\omega), \sigma^*(x)) = \xi(\omega, x),$$

so $x \in \operatorname{NS}(X)$. The other inclusion is then obtained from the definition of the coinvariant lattice S_σ. □

2.4 Moduli Spaces and period maps for IHS manifolds with a non-symplectic automorphism

Now, we want to specialize the constructions of coarse moduli spaces and period maps when there exist a non-symplectic automorphism on an IHS manifold X. Indeed, the presence of an automorphism allows us to provide more precise notions. The notions given in this section were first given and analyzed in [BCS19a] for the $K3^{[n]}$-type deformation family, then [BC22] generalized their results to the other deformation families.

Given an isometry ρ of a lattice L and an embedding $j : T \hookrightarrow L$ of the invariant lattice $T \simeq L^\rho$ we give the following definition.

Definition 2.60. *A (ρ, j)-polarization of an IHS manifold X of type L consists of the following data:*

i) a marking η;

ii) a primitive embedding $\iota \colon T \hookrightarrow \operatorname{Pic}(X)$ such that $\eta \circ \iota = j$;

iii) an automorphism $\sigma \in \mathrm{Aut}(X)$ such that $\sigma^*_{|H^{2,0}(X)} = \zeta \cdot \mathrm{id}$ (with ζ a primitive n-th root of unity) and η is a framing for σ, i.e. the following diagram commutes

$$\begin{array}{ccc} H^2(X,\mathbb{Z}) & \xrightarrow{\sigma^*} & H^2(X,\mathbb{Z}) \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\ L & \xrightarrow{\quad\rho\quad} & L \end{array}$$

The period domain is in this case (see [BC22, Section 3.2])

$$\Omega_T^{\rho,\zeta} := \{x \in \mathbb{P}(S_\zeta) \mid h_S(x,\bar{x}) > 0\},$$

where we denoted with S the orthogonal complement of T in L and with S_ζ the eigenspace relative to ζ inside $S_{\mathbb{C}}$.

Remark 2.61. *Note that the isotropic condition of a point in the period domain has not been dropped. Indeed take a point $x \in \mathbb{P}(S_\zeta)$, then $h_S(x,x) = h_S(\rho(x),\rho(x)) = h_S(\zeta x,\zeta x) = \zeta^2 h_S(x,x)$. Comparing the first and the last we deduce that $h_S(x,x) = 0$.*

The period map on a connected component $\mathcal{M}_T^{\rho,\zeta}$ of the moduli space of (ρ,j)-polarized IHS manifolds X of type L is surjective on

$$\Omega_T^{\rho,\zeta} \setminus \bigcup_{\delta\in\Delta(S)} (\delta^\perp \cap \Omega_T^{\rho,\zeta})$$

by [BC22, Proposition 3.12], but in order to give a bijective restriction we need to introduce another definition. Choose $K(T)$ as a connected component of

$$C_T \setminus \bigcup_{\delta\in\Delta(S)} \delta^\perp \subset T_{\mathbb{R}}.$$

Here we denoted by $C_T := \mathcal{C}_L \cap (T \otimes \mathbb{C})$.

Definition 2.62. *A (ρ,j)-polarized manifold (X,η) is $K(T)$-general if $\eta(\mathcal{K}_X^{\sigma^*}) = K(T)$ where we denote by $\mathcal{K}_X^{\sigma^*}$ the* invariant Kähler cone, *i.e. $\mathcal{K}_X^{\sigma^*} := \{x \in \mathcal{K}_X \mid \sigma^*(x) = x\}$.*

Now define the following

$$\Gamma_T^{\rho,\zeta} := \left\{\gamma_{|S} \in O(S) \mid \gamma \in O(L),\, \gamma_{|T} = \mathrm{id},\, \gamma\circ\rho = \rho\circ\gamma\right\}$$

and

$$\Delta'(L) := \left\{\nu \in \Delta(L) \mid \nu = \nu_T + \nu_S,\, \nu_T \in T_{\mathbb{Q}},\, \nu_S \in S_{\mathbb{Q}},\, \nu_T^2, \nu_S^2 < 0\right\}.$$

Moreover, we set $\mathcal{H}_T := \bigcup_{\delta\in\Delta(T)} \delta^\perp$ and $\mathcal{H}'_T := \bigcup_{\delta\in\Delta'(T)} \delta^\perp$.

Theorem 2.63. *[BCS19a, Theorem 5.6, Proposition 6.2] The restriction of the period map to the moduli space of $K(T)$-general IHS manifolds of type L*

$$\mathcal{M}^{\rho,\zeta}_{K(T)} \to \Omega^{\rho,\zeta}_T \setminus \left(\mathcal{H}_T \cup \mathcal{H}'_T\right)$$

is an isomorphism and it induces an isomorphism on the quotients:

$$\mathcal{P}^{\rho,\zeta}_{K(T)} : \mathcal{N}^{\rho,\zeta}_{K(T)} := \frac{\mathcal{M}^{\rho,\zeta}_{K(T)}}{Mon^2(T,\rho)} \to \frac{\Omega^{\rho,\zeta}_T \setminus (\mathcal{H}_T \cup \mathcal{H}'_T)}{\Gamma^{\rho,\zeta}_T}. \tag{2.63.1}$$

Here we denoted with $\mathrm{Mon}^2(T,\rho)$ the group of (ρ,T)-polarized monodromy operators:

$$\mathrm{Mon}^2(T,\rho) := \left\{g \in \mathrm{Mon}^2(L) \mid g_{|T} = \mathrm{id},\ g\circ\rho = \rho\circ g\right\}.$$

We want now to give a more general notion of polarization and introduce the (M,j)-polarization for a lattice M of signature $(1,t)$ with a primitive embedding $j: M \subset L$ defined in [Cam16, Definition 3.1] as follows.

Definition 2.64. *Given an IHS manifold X of type L we say that it carries an (M,j)-polarization if it has:*

1. *a marking $\eta\colon H^2(X,\mathbb{Z}) \to L$;*
2. *a primitive embedding $\iota\colon M \hookrightarrow \mathrm{Pic}(X)$ such that $\eta\circ\iota = j$.*

Remark 2.65. *The (ρ,j)-polarization is a special type of (M,j)-polarization with $M = T$, the invariant lattice for the isometry ρ, and on which we ask the existence of an automorphism satisfying item iii) of Definition 2.60.*

Moreover, it is immediate to see that the notion of $K(T)$-generality comes from the notion of $K(M)$-generality for an (M,j)-polarized IHS manifold given in [Cam18, Definition 3.10]. Indeed, let C_M be the connected component of the positive cone such that $\iota(C_M)$ contains the Kähler cone $\mathcal{K}_X$ of an (M,j)-polarized IHS manifold X. We define also $K(M)$ as a connected component (also called *chamber*) of

$$C_M \setminus \bigcup_{\delta\in\Delta(M)} \delta^\perp \subset M\otimes\mathbb{R}.$$

Definition 2.66. *An (M,j)-polarized IHS manifold (X,η) is $K(M)$-general if $\iota(K(M)) = \mathcal{K}_X \cap \iota(C_M)$.*

In order to see that this definition is a generalization of the one given above suppose that (X,η) is a (T,j)-polarized IHS manifold of type L where T is the invariant lattice of an automorphism $\rho \in O(L)$. Suppose moreover that the (T,j)-polarization extends in a natural way to a (ρ,j)-polarization, i.e. condition iii) of Definition 2.60 is satisfied. Then, as T is the invariant sublattice of ρ, both definitions of C_T coincide. For the same reason $\mathcal{K}_X^{\sigma^*} = \mathcal{K}_X \cap \iota(C_T)$. Looking at Definition 2.60, we see that $\mathcal{K}_X \cap \iota(C_T) = \mathcal{K}_X^{\sigma^*} = \iota(K(T))$ is equivalent to requiring that $\eta(\mathcal{K}_X \cap \iota(C_T)) = \eta(\mathcal{K}_X^{\sigma^*}) = K(T)$. In *loc. cit.* the author finds also a period map and its injective restriction with statements similar to the (ρ,j)-polarized case.

Chapter 3

Cubic threefolds and IHS manifolds

"We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time."

- T.S Eliot, *Four quartets*

In this chaper we dig into the surprising relation between cubic hypersurfaces and irreducible holomorphic symplectic manifolds.

This chapter resulted in a paper:

GIT stable cubic threefolds and certain fourfolds of $K3^{[2]}$-type, https://arxiv.org/abs/2301.11149, submitted.

1 Introduction

The relation between cubic hypersurfaces and irreducible holomorphic symplectic manifolds. has attracted the interest of many experts during the last decades. Various examples of this interest can be found in literature. The first one is the classical result, due to Beauville and Donagi [BD85], stating that the Fano variety of lines on a cubic fourfold is deformation equivalent to the Hilbert square of a $K3$ surface. There are plenty of other examples, to cite a few [Has00], [LLSvS17] and in the article of Boissière–Camere–Sarti [BCS19b] which is at the core of this chapter.

In *loc.cit.* the authors prove the existence of an isomorphism ψ between the moduli space $\mathcal{C}_3^{sm}$ of smooth cubic threefolds and the moduli space $\mathcal{N}_{\langle 6\rangle}^{\rho,\xi}$ of fourfolds of $K3^{[2]}$-type endowed with a special non-symplectic automorphism of order three. Moreover, they analyze the extension of the period map to singular cubics, given in [ACT11], in order to give a geometric interpretation of the degenerations of the automorphism ψ along either the chordal or the singular nodal hyperplanes, where the cubic threefolds either acquire a nodal singularity or they are related to the chordal cubic. In particular they find a birational morphism between the stable discriminant locus (corresponding to a generic nodal degeneration) $\Delta_3^{A_1}$ and the 9-dimensional moduli space of fourfolds of $K3^{[2]}$-type endowed with a non-symplectic automorphism of order three, having invariant lattice isometric to $U(3) \oplus \langle -2 \rangle$. In the exceptional locus of this birational morphism there are some interesting subloci, e.g. cubic threefolds having an isolated singularity of type A_i for $i = 2, 3, 4$.

The aim of this chapter is to provide a similar result also for the closed subloci $\Delta_3^{A_2}$, $\Delta_3^{A_3}$, $\Delta_3^{A_4} \subset \Delta_3^{A_1}$ where $\Delta_3^{A_i}$ is the closure of the set of cubic threefolds having an isolated singularity of type A_i for $i = 1, \dots, 4$ taken in the moduli space of cubic threefolds. These threefolds are of our interest because Allcock proved in [All03, Theorem 1.1] that a singular cubic threefold is GIT stable if and only if all its singularities are of type A_i for $i = 2, 3, 4$. Therefore, they are in the strata at the border of the GIT compactification of the moduli space of smooth cubic threefolds; different types of compactifications have been studied recently in many articles, e.g. [Yok02], [LS07], [CMGHL21], [CMGHL23] and the already cited [All03] and [ACT11].

The strategy to reach our goal will be the following. In [BCS19b, Section 4] the authors note that, in order to understand geometrically the degenerations of the automorphism along the nodal hyperplanes, one has to consider a moduli space of fourfolds of $K3^{[2]}$-type with an automorphism having an invariant lattice which is bigger than in the smooth case. So, we will start from a generic cubic threefold C in $\Delta_3^{A_i}$, then we will find a $K3$ surface $\hat{\Sigma}$ having the same period of C. The natural choice for $\hat{\Sigma}$ will be the one used in [ACT11] to define the period of a nodal cubic. Then, we want to find some conditions on the Hilbert square $\hat{\Sigma}^{[2]}$ of $\hat{\Sigma}$ such that $\hat{\Sigma}^{[2]}$ is a "good candidate" for the relation we are looking for. Indeed, our aim is to define a birational map between $\Delta_3^{A_i}$ and some moduli space of fourfolds of $K3^{[2]}$-type. Therefore a "good candidate" should be endowed with a marking, a non-symplectic automorphism and generic in a particular moduli space. Moreover, we want that the restriction of the period map to this moduli space is an isomorphism onto its image. Indeed, in general the period map for IHS manifolds is not an isomorphism because there may exist birational, non-isomorphic models in the fiber over a period. In order to ensure that this does not happen we will use the notion of $K(T)$-generality for a fourfold of $K3^{[2]}$-type, introduced by Camere in [Cam18, Definition 3.10] recalled in Definition 2.66.
The main results of this chapter can be summarized as follows.

Theorem 3.1. *The $\Delta_3^{A_i}$ locus for $i = 1, \dots, 4$ is birational to a $(10-i)$-dimensional moduli space of fourfolds of $K3^{[2]}$-type with Picard group of the generic member isometric to R_i endowed with a non-symplectic automorphism of order three, having invariant lattice isometric to T_i. These lattices are defined in the following table.*

i	T_i	R_i
1	$U(3) \oplus \langle -2 \rangle$	$U(3) \oplus \langle -2 \rangle$
2	$U \oplus A_2(-2) \oplus \langle -2 \rangle$	$U(3) \oplus \langle -2 \rangle$
3	$U \oplus A_2(-1)^{\oplus 2} \oplus \langle -2 \rangle$	$U \oplus A_2(-1)^{\oplus 2} \oplus \langle -2 \rangle$
4	$U \oplus E_6(-1) \oplus \langle -2 \rangle$	$U \oplus E_6(-1) \oplus \langle -2 \rangle$

We will find a sufficient condition on the Picard group of $\hat{\Sigma}^{[2]}$ in Section 3.3, then we will use this result in Section 3.4 and 3.5 to prove Theorem 3.1 for $i = 1, 3$ and 4. For these cases the result is proven, respectively, in Proposition 3.19, Proposition 3.21 and Proposition 3.22. Note that for $i = 1$ this coincides with the result stated in [BCS19b, Proposition 4.6]. To prove that $\hat{\Sigma}^{[2]}$ is a "good candidate" in the sense explained above we rely on the theory of moduli spaces of irreducible holomorphic symplectic manifolds with automorphism. This theory has been the object of great interest for the mathematical community; e.g. in [AS08], [Tak11] and [AST11] the authors study in depth non-symplectic automorphism of prime order on $K3$ surfaces. Other examples showing this interest are [CC20] and [BCS16]. The case $i = 2$ will be discussed in Section 3.6, where we start from a generic cubic in $\Delta_3^{A_2}$ and discover that its associated $\hat{\Sigma}^{[2]}$ does not define a "good candidate". This fact is somehow surprising at first but studying it we will find out many differences with the other cases; e.g. in order to find $\hat{\Sigma}$ in this case we need to blow-up three points which are permuted by a non-symplectic automorphism of order 3 instead of one fixed point as for $i = 3, 4$. Moreover, in the case with $i = 2$ we do not have the $K(T)$-generality property and there are multiple birational non-isomorphic models; therefore, in order to deal with the $\Delta_3^{A_2}$ locus we will introduce in Section 3.7 the notion of Kähler cone sections of K-type which generalizes the notion of $K(T)$-generality. Finally, we will show in Section 3.8 that the notion just introduced leads us to the proof of Theorem 3.1 for $i = 2$.

2 Cubic threefolds

In this section we define the objects we are mainly interested in, i.e. the nodal cubic threefold C and the $K3$ surface whose Hilbert square will be our "good candidate" as said in the introduction.
Let $C \subset \mathbb{P}^4$ be a cubic threefold described by the vanishing of a homogeneous polynomial of degree 3:

$$f(x_0 : x_1 : x_2 : x_3 : x_4) = ax_0^3 + x_0^2 f_1(x_1, x_2, x_3, x_4) + \\ + x_0 f_2(x_1, x_2, x_3, x_4) + f_3(x_1, x_2, x_3, x_4)$$

where the f_i are homogeneous polynomials of degree i in $\mathbb{C}[x_1, x_2, x_3, x_4]$ and $a \in \mathbb{C}$. We want to impose conditions on the coefficients in order to describe a *nodal* cubic, i.e. with an isolated singularity of type ADE. So, let C be singular in $p_0 \in \mathbb{P}^4$, which we may assume, after a suitable change of coordinates, to be $(1 : 0 : ... : 0)$. Hence, imposing that p_0 belongs to the cubic and the conditions on the Jacobian of f, we obtain that both the cubic part in x_0 and the linear part in $x_1, \dots, x_4$ must vanish. Moreover, in order for p_0 to be the only singularity we take $c \neq 0$ and f_2, f_3 sufficiently generic in $|\mathcal{O}_{\mathbb{P}^3}(3)|$. Modulo another change of projective coordinates we assume $c = 1$. Thus the equation becomes

$$f(x_0 : x_1 : x_2 : x_3 : x_4) = x_0 f_2(x_1, x_2, x_3, x_4) + f_3(x_1, x_2, x_3, x_4) = 0.$$

Let now $Y \subset \mathbb{P}^5$ be the triple cover of $\mathbb{P}^4$ branched over C. Therefore Y is described by the vanishing of the following polynomial

$$F(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = x_0 f_2(x_1, x_2, x_3, x_4) + f_3(x_1, x_2, x_3, x_4) + x_5^3$$

using the same notation as above. This hypersurface has again an isolated singularity of type ADE at the point $p \in \mathbb{P}^5$ of coordinates $(1 : 0 : ... : 0)$. The construction outlined here is standard and as we will use it frequently we introduce the following definition.

Definition 3.2. *We say that a cubic fourfold $Y \subset \mathbb{P}^5$ is* associated *to a cubic threefold $C \subset \mathbb{P}^4$ when $Y \to \mathbb{P}^4$ is a triple cover branched on C.*

Let us consider now the hyperplane $H \subset \mathbb{P}^5$ given by $\{x_0 = 0\}$. In this hyperplane, which we will identify with $\mathbb{P}^4$ of coordinates $(x_1 : x_2 : x_3 : x_4 : x_5)$, we consider the surface Σ given by the following intersection:

$$\begin{cases} f_2(x_1, x_2, x_3, x_4) = 0 \\ f_3(x_1, x_2, x_3, x_4) + x_5^3 = 0. \end{cases} \tag{3.2.1}$$

This is the complete intersection of a quadric Q (defined by $f_2 = 0$) and a cubic K (defined by $f_3 + x_5^3 = 0$) in $\mathbb{P}^4$ when f_2 and f_3 are sufficiently generic. This surface is deeply linked to the fourfold Y and a theorem by Wall [Wal99] links the singularities of Σ to those of the blow-up $\mathrm{Bl}_p(Y)$ of Y at p.

Theorem 3.3 ([Wal99, Theorem 2.1])**.** *Let q be a singular point of Σ. If both Q and K have a singularity in q then the whole line $\bar{pq}$ connecting p and q is singular in Y.*
*If q is not a singularity of both Q and K and is an ADE singularity of type **T** for Q or K then one of the followings holds:*

- *i) Q is smooth at q and the cubic fourfold Y has exactly two singularities, namely p and p', on the line $\bar{pq}$ and p' is of type **T**.*
- *ii) Q is singular at q and the line $\bar{pq}$ meets Y only in p and the blow-up $\mathrm{Bl}_p(Y)$ of Y in p has a singularity of type **T** at q.*

As we asked p to be the only singularity on Y, the only possibility is the one described by item ii) of the Theorem 3.3. Thus the possibilities for the singularities of Σ are exactly those that can be found in the following table, based on [DR01, Lemma 2.1].

T	$\boldsymbol{A_1}$	$\boldsymbol{A_2}$	$\boldsymbol{A_{n\geq 3}}$	$\boldsymbol{D_4}$	$\boldsymbol{D_{n\geq 5}}$	$\boldsymbol{E_6}$	$\boldsymbol{E_7}$	$\boldsymbol{E_8}$
$\hat{\boldsymbol{T}}$	$\emptyset$	$\emptyset$	$\boldsymbol{A_{n-2}}$	$3\boldsymbol{A_1}$	$\boldsymbol{A_1} + \boldsymbol{D_{n-2}}$	$\boldsymbol{A_5}$	$\boldsymbol{D_6}$	$\boldsymbol{E_7}$

Here $\hat{\boldsymbol{T}}$ is the type of the singularities that one can find on the exceptional divisor of the blow-up of a variety in a point p that has a singularity of type **T**.
Another interesting observation on Σ can be done following the argument of C. Lehn ([Leh18, Lemma 3.3, Theorem 3.6]) and Hassett ([Has00, Lemma 6.3.1]).

Theorem 3.4. *Let $Y \subset \mathbb{P}^5$ be a cubic fourfold with simple isolated singularities and suppose that it is neither reducible, nor a cone over a cubic threefold. Let $p \in Y$ be a singular point and assume that there exist no planes $\Pi \in Y$ such that $p \in \Pi$. Then the minimal resolution of $\Sigma := F(Y, p)$, the Fano variety of lines in Y passing through p, is a $K3$ surface. Moreover, $F(Y)$, the Fano variety of lines in Y, is birational to $\mathrm{Hilb}^2(\Sigma)$*

Proof. We write here the explicit morphism as it will be useful for the next sections.

For the first part, see the discussion above. Moreover, Σ is a (2,3)-complete intersection in $\mathbb{P}^4$ having only isolated ADE singularities, so it admits a minimal model which is a $K3$ surface.

For the rest of the proof, consider $W \subset Y$ the cone over Σ with vertex p. This is a Cartier divisor on Y cut out by the equation $f_2 = 0$. Hence a generic line $l \subset Y$ intersects W in exactly two points counted with multiplicity, thus defining a closed subscheme $\xi_{l \cap W}$ of length two on Σ. Therefore, we can define the birational map

$$\begin{aligned} \varphi^{-1} : F(Y) &\dashrightarrow \mathrm{Hilb}^2(\Sigma) \\ l &\mapsto \xi_{l \cap W} \end{aligned}$$

The birational inverse of φ is given by the natural map

$$\begin{aligned} \varphi : \mathrm{Hilb}^2(\Sigma) &\to F(Y) \\ \xi &\mapsto l_\xi \end{aligned}$$

where we define the residual line l_ξ as follows. The intersection between Y and $\langle \xi, p \rangle \simeq \mathbb{P}^2$ consists of a cone over ξ and a line l_ξ. □

Remark 3.5. *Note that φ has no indeterminacy points. Moreover, note that the indeterminacy locus of φ^{-1} is contained in $F(Y, p) \simeq \Sigma$. Indeed, looking at the definition of φ^{-1} in the proof above we can see that it is not defined when a line $l \subset Y$ is contained in W, the cone over Σ with vertex p. This means that either $l \subset \Sigma$ or $p \in l$, the former is impossible otherwise the plane $\Pi_{l,p} := \langle l, p \rangle$ would be contained in Y.*

Remark 3.6. *The condition of not having planes passing through the singular point of a cyclic cubic fourfold is a generic condition as computations done in Section 5.3 show. For this reason from now on we will suppose that this condition is satisfied by every cyclic cubic fourfold appearing also when not explicitly said.*

2.1 Moduli space of cubic threefold as a ball quotient

In this section we to recall Allcock–Carlson–Toledo's construction of a period map for the moduli space of GIT stable cubic threefolds as done in [ACT11]. This section is not to be intended as a complete overview of their work but as a recollection of their results useful to understand the following sections.

We denote by $\mathcal{C}_3^s := |\mathcal{O}_{\mathbb{P}^4}(3)| /\!\!/ PGL_5(\mathbb{C})$ the GIT moduli space of $PGL_5(\mathbb{C})$-stable of stable cubic threefolds and $\mathcal{C}_3^{sm}$ the sublocus of smooth cubic threefolds (this is the open set determined by the nonvanishing of the discriminant as shown in [Muk03, Chapter 5]). Now, given $C \in \mathcal{C}_3^{sm}$ the idea outlined by Allcock–Carlson–Toledo is to use the associated cyclic cubic fourfold to induce a period map on $C \in \mathcal{C}_3^{sm}$. Indeed, for any cubic fourfold Y we can define a *marking*, i.e. an isometry

$$\eta : H^4_\circ(Y, \mathbb{Z}) \to S(-1) \simeq U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$$

of the middle primitive cohomology. Moreover, the *period* of the marked pair (Y, η) is just $[\eta(H^{3,1}(Y))] \in \mathbb{P}(S(-1) \otimes \mathbb{C})$. Let σ be the covering automorphism of the associated cubic fourfold Y. Given a marking η for Y we can define the abstract isometry induced by σ as $\rho := \eta \circ \sigma \circ \eta^{-1}$ and we define a *framing* as the equivalence class of markings $\tilde{\eta}$ compatible with ρ, i.e. $\tilde{\eta} \circ \rho = \rho \circ \tilde{\eta}$, up to action of $\mu_6 := \{\pm \mathrm{id}_{S(-1)}, \pm\rho, \pm\rho^2\}$.

Now, we denote by $\mathcal{F}_3^{sm}$ the moduli space of framed smooth cubic threefolds and by $\Gamma := \{\gamma \in O(S(-1)) \mid \gamma \circ \rho = \rho \circ \gamma\}$. The latter acts on the former by composition with the framing, i.e. $(C, \eta) \mapsto (C, \gamma \circ \eta)$. As $\mu_6 \subset \Gamma$ acts trivially on $\mathcal{F}_3^{sm}$ we consider $\mathbb{P}\Gamma := \Gamma/\mu_6$ and $\mathcal{C}_3^{sm} \simeq \mathcal{F}_3^{sm}/\mathbb{P}\Gamma$. So, any framing $\eta : H^4_\circ(Y, \mathbb{Z}) \to S(-1)$ induces an isomorphism $\eta : H^4_\circ(Y, \mathbb{Z}) \to S(-1)_\xi$, where $S(-1)_\xi$ is the eigenspace of $S \otimes \mathbb{C}$ for the eigenvalue ξ of the isometry ρ. Note that if we act on a a marked cubic fourfold of period $\eta(H^{3,1}(Y))$ with an element of μ_6 we get that the period is multiplied by a non-zero scalar, therefore it remains well defined on the framed cubic threefolds. Therefore we have the following.

Theorem 3.7. *The period map sending a framed cubic threefold (C, η) to $[\eta(H^{3,1}(Y))] \in \mathbb{P}(S(-1)_\xi)$ is an isomorphism onto the image equivariant with respect to the action of $\mathbb{P}\Gamma$. Moreover, the image is the complement of an hyperplane arrangement $\mathcal{H}$ with:*

$$\mathcal{H} := \bigcup_{\delta \in S(-1),\, \delta^2 = 2} \delta^\perp.$$

Proof. See [ACT11, Theorem 1.9]. □

In *loc. cit.* the authors study also an extension of the period map for the GIT stable cubic threefolds (see [All03] for the details on GIT stability of cubic threefolds). In particular in [ACT11, Chapter 6] the authors show, by studying the limit Hodge structure of the nodal degeneration of a cubic threefold, that the period map can be extended to $\Delta_3^{A_1}$ using the period of its associated $K3$ surface, i.e. the one defined in Section 3.2.

Theorem 3.8 ([ACT11, Theorem 6.1])**.** *The period map above defined can be holomorphically extended to an isomorphism between the GIT stable locus $\mathcal{C}_3^s$ and its image, mapping $\Delta_3^{A_1}$ to a divisor.*

2.2 Motivating example

Here we introduce the example which will be the core of our analysis. Given, as in before, a ramified cyclic covering $Y \to \mathbb{P}^4$ branched along the cyclic cubic C there exists a covering automorphism σ on Y acting by multiplication by a primitive third root of the unity ζ. Any marking of the middle primitive cohomology $H^4_\circ(Y, \mathbb{Z}) \to S(-1)$ can be composed with the Abel–Jacobi map in order to induce a marking on the middle primitive cohomology of the Fano variety of lines $F(Y)$.

$$\begin{array}{ccc} H^4_\circ(Y, \mathbb{Z}) & \xrightarrow{\tilde{\eta}} & S(-1) \\ \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle -\mathrm{id}} \\ H^2_\circ(F(Y), \mathbb{Z}) & \xrightarrow{\eta} & S. \end{array}$$

From [BD85] we know that S admits a unique, up to isometries, primitive embedding in L and this embedding has $T \simeq \langle 6\rangle$ as orthogonal complement. We are interested in giving a relation between cubic threefolds and IHS manifolds of $K3^{[2]}$-type. This is classically (e.g. [ACT11]) done by looking at the cubic fourfolds which cover $\mathbb{P}^4$ and branch over a cubic threefold. Then there is a nice description of the Plücker divisor on a fourfold of $K3^{[2]}$-type associated to a generic cubic fourfold of discriminant six made by Hassett in his thesis [Has00, Section 6]: it is $\theta = 2\theta_{K3} - 3\epsilon$ where θ_{K3} is a square six class on the underlying $K3$ surface and ϵ is half of the exceptional class coming from the Hilbert–Chow morphism. In our case even though we do not ask for genericity we make the same choice; moreover, as they are all the same up to isometry, we choose $\theta_{K3} = 3u_1 + u_2$. So we have assigned an embedding $j : \langle 6\rangle \to L$ and $j(\langle 6\rangle)^\perp \simeq S$ as expected. Now using [Nik80, Cor 1.5.2] (see also Proposition 3.13 below) we can extend the isometry $H^2_\circ(F(Y),\mathbb{Z}) \oplus \langle 6\rangle \xrightarrow{\eta \oplus j} S(-1) \oplus j(\langle 6\rangle)$ to a marking $\bar\eta\colon H^2(F(Y),\mathbb{Z}) \to L$. Finally, $\sigma \in \mathrm{Aut}(Y)$ induces on $F(Y)$ an automorphism that will be also denoted with $\sigma \in \mathrm{Aut}(F(Y))$ in order to simplify the notation. Therefore there exists a natural isometry on L that is $\rho := \bar\eta \circ \sigma^* \circ \bar\eta^{-1}$. So, $F(Y)$ is an IHS manifold of $K3^{[2]}$-type and admits a (ρ, j)-polarization with the lattice $\langle 6\rangle$ playing the role of T in Definition 2.60 and a primitive third root of the unity as ζ. The period map $\mathcal{P}^{\rho,\zeta}_{\langle 6\rangle}$ relative to this space has the period domain isomorphic to a 10-dimensional complex ball

$$\Omega^{\rho,\zeta}_T := \{x \in \mathbb{P}(S_\zeta) | h_S(x,x) > 0\} \simeq \mathbb{C}B^{10}.$$

3 Degeneracy lattices

In this section we begin to study the degenerations of the automorphism ρ over the nodal hyperplane.

We want to focus on the case described in Section 2.2. Take ω a period in $\mathcal{H}_\Delta$ and $(X,\eta) \in \mathcal{P}^{-1}_{\langle 6\rangle}(\omega)$ a point in the fiber of the period map of $\langle 6\rangle$-polarized IHS manifolds of $K3^{[2]}$-type.

Definition 3.9. *The* degeneracy lattice *of (X,η) is the sublattice of S generated by those MBM classes $\delta_i \in S$ which are orthogonal to ω.*

This lattice is ρ-invariant and orthogonal to $j(\langle 6\rangle)$. So, in general, the degeneracy lattice will be $R_0 := \mathrm{Span}(\delta_1, \dots, \delta_n, \rho(\delta_1), \dots, \rho(\delta_n))$.

Remark 3.10. *For each $i \in \{1, \dots, n\}$ the sublattice $\mathrm{Span}(\delta_i, \rho(\delta_i)) =: R_{\delta_i} \subset R_0$ is just the degeneracy lattice of a polarized IHS manifold (X,η) generic in $\mathcal{H}_\Delta$ and, with a simple calculation, $R_{\delta_i} \simeq A_2(-1)$. For the sake of completeness let us show this calculation. By definition $R_{\delta_i} \simeq \langle \delta_i, \rho(\delta_i)\rangle$ with $\delta_i^2 = (\rho(\delta_i))^2 = -2$. Moreover, $(\delta_i, \rho(\delta_i)) = (\rho(\delta_i), \rho^2(\delta_i)) = (\rho(\delta_i), -\delta_i - \rho(\delta_i)) = -(\rho(\delta_i), \delta_i) + 2$ and the computation is done.*

Remark 3.11. *As explained in [BCS19b] the isometry $\rho \in O(L)$ is not represented by any automorphism of X. In fact if ρ was represented by an automorphism of X, this one would be automatically non-symplectic. Consider now $l \in \mathrm{NS}(X)$ an ample class (it always exists as X is projective) then consider $l + \rho^* l + (\rho^*)^2 l$ which is still ample and invariant. Therefore it is a multiple of the generator of the rank one invariant lattice, we call θ the primitive ample invariant class.*

Since $\delta_i \in S$, the divisor $\eta^{-1}(\delta_i)$ is orthogonal to θ, yielding a contradiction as $(\eta^{-1}(\delta_i), \theta) > 0$ by the ampleness of θ.

Recall that in Section 2.2 we provided an embedding $j : \langle 6 \rangle \to L$ such that $j(\langle 6 \rangle)^\perp \simeq S$. With the help of j we can induce an embedding of $\langle 6 \rangle \oplus R_0$ which in general will not be primitive. So, we define $T_0 := \overline{\langle 6 \rangle \oplus R_0}$ as the saturation of $\langle 6 \rangle \oplus R_0$ in L. Note that $T_0 \hookrightarrow \mathrm{Pic}(X)$ by definition of degeneracy lattice and the equality will hold for a generic element, i.e. a $\langle 6 \rangle$-polarized IHS fourfold of $K3^{[2]}$-type which has a generic period orthogonal to a fixed number of MBM classes $\delta_1, \ldots, \delta_n$.

The strategy we will use is the same used in [BCS19b] and in [DK07, §11]. We want to prove the following claim.

Claim 3.12. *The Picard group of the Hilbert square $\hat{\Sigma}^{[2]}$ of the minimal resolution of the surface defined in Section 3.2 is generated by the square six polarization and $2i$ classes of square (-2) orthogonal to it, with $i = 1, \ldots, 4$. Therefore it is generic in the above sense.*

Then we want to look for an isometry on L related to ρ which has a bigger invariant lattice. In particular, denoting with $S_0 := T_0^\perp$ in L, we look at $\mathrm{id}_{T_0} \oplus \rho_{|S_0} \in O(T_0) \oplus O(S_0)$. If it can be lifted to an isometry $\rho_0 \in O(L)$ then as $T_0 \simeq \mathrm{Pic}(\hat{\Sigma}^{[2]})$ we can find an IHS manifold of $K3^{[2]}$-type with the same period (by definition of period of a nodal cubic given in [ACT11]) generic in the space of the IHS manifolds of $K3^{[2]}$-type which are (ρ_0, j)-polarized. It is important to remark that by definition of the (ρ, j)-polarization, as the isometry $\rho \in O(L)$ comes from a non-symplectic automorphism σ on X, it can be restricted to an isometry $\rho_{S_0} \in O(S_0)$. Indeed, as $\omega \in \delta^\perp$ we deduce that $\zeta_3 \cdot (\rho(\delta), \omega) = (\rho(\delta), \rho(\omega)) = (\delta, \omega) = 0$, thus the isometry ρ can be restricted to an isometry of both the Picard lattice and its orthogonal complement. In order to lift the isometry we will apply the following result

Proposition 3.13 ([Nik80, Cor 1.5.2])**.** *Let L be a finite index overlattice of $S \oplus T \subset L$ determined by the pair (H, γ), where $H < D_S$ is a subgroup and $\gamma : H \to D_T$ is a group monomorphism. Moreover, let $\rho_S \in O(S)$ and $\rho_T \in O(T)$ be two isometries such that the induced isometry on the discriminant $\rho_S^* \in O(D_S)$ restricts to an isometry of H.*
Then the isometry $\rho_S \oplus \rho_T \in O(S) \oplus O(T)$ lifts to an isometry $\rho \in O(L)$ of L if and only if $\rho_S|_H$ is conjugate to the induced isometry ρ_T^* via γ, or equivalently $\gamma \circ \rho_S^*|_H = \rho_T^* \circ \gamma$.

Proof. The first implication is obvious as $\rho_S = \rho|_S$ and $\rho_T = \rho|_T$. The second implication is done just by noting that in the diagram

$$\begin{tikzcd} H \arrow[r, "\rho_S^*"] \arrow[d, "\gamma"] & H \arrow[d, "\gamma"] \\ \gamma(H) \arrow[r, "\rho_T^*"] & \gamma(H) \end{tikzcd}$$

the pair $\left(\rho_S^*(H) = H, \rho_T^* \circ \gamma \circ (\rho_S^*)^{-1} = \gamma\right)$ determines an overlattice isometric to L and we call this isometry ρ. $\square$

Recall now that the Picard group of an IHS fourfold of $K3^{[2]}$-type can be written as

$$\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Pic}(\hat{\Sigma}) \oplus \langle \epsilon \rangle \simeq T_{K3} \oplus \langle -2 \rangle$$

where ϵ is the (-2)-class given by half of the exceptional divisor introduced by the Hilbert–Chow morphism and $T_{K3} \subset L_{K3}$ is a sublattice of the $K3$ lattice $L_{K3} \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Thus we consider the isometry $\tilde{\rho_0} := \rho_0|_{T_{K3}}$. Then the following proposition proves that there exists an automorphism σ of the $K3$ surface $\hat{\Sigma}$ whose action on cohomology is conjugate to $\tilde{\rho_0}$.

Proposition 3.14. *Let Σ be a $K3$ surface and $\rho \in O(L_{K3})$. Suppose that $\rho(\omega) = \lambda\omega$ where $\omega \in H^2(\Sigma, \mathbb{C})$ is the period of Σ and $1 \neq \lambda \in \mathbb{C}^*$. Then if $\mathrm{Pic}(\Sigma)$ is fixed by ρ there exists $\sigma \in \mathrm{Aut}(\Sigma)$ and an element w in the Weyl subgroup of Σ such that $w\rho w^{-1} = \sigma$.*

This proposition is an immediate corollary of the following theorem by Namikawa.

Theorem 3.15 ([Nam85] Theorem 3.10)**.** *Let Σ be a $K3$ surface and G a finite subgroup of the group of isometries in $\Lambda = H^2(\Sigma, \mathbb{Z})$. Denote by ω the period of Σ, by Λ^G the sublattice of elements in Λ fixed by G and set $S_{G,\Sigma} = (\Lambda^G)^\perp \cap \{\mathbb{C}\omega\}^\perp = (\Lambda^G)^\perp$ in $H^{1,1}_{\mathbb{Z}}(\Sigma)$. Then there exists an element t in the Weyl subgroup of Σ, $W(\Sigma)$, such that $tGt^{-1} \subset \mathrm{Aut}(\Sigma)$ if and only if*

i) $\mathbb{C}\omega$ is G-invariant;

ii) $S_{G,\Sigma}$ contains no element of length -2;

iii) if $\omega \in \Lambda^G$ then $S_{G,\Sigma}$ is either 0 or nondegenerate and negative definite;

iii') if $\omega \notin \Lambda^G$ then Λ^G contains an element a with $(a, a) > 0$.

Therefore if we consider the natural automorphism on $\Sigma^{[2]}$ induced by σ then we see that the former variety is equipped with an automorphism whose action in cohomology is conjugate to ρ_0.

Theorem 3.16. *Let C be a cyclic nodal cubic and $\hat{\Sigma}$, θ as in Section 3.2. Let $(\hat{\Sigma}^{[2]}, \eta)$ the Hilbert square of $(\hat{\Sigma}, \tilde{\eta})$ and suppose that $\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq j(\theta) \oplus W \simeq \overline{\langle 6 \rangle \oplus R_0} = T_0$. If the action induced by ρ on the discriminant group D_W is trivial then the isometry $\mathrm{id}_{T_0} \oplus \rho_{|S_0} \in O(T_0) \oplus O(S_0)$ lifts to an isometry $\rho_0 \in O(L)$. Finally, if we define $K(T_0)$ as the chamber containing a Kähler class of $\Sigma^{[2]}$, then the latter is $K(T_0)$-general and defines a point in $\mathcal{M}^{\rho_0,\zeta}_{K(T_0)}$.*

Proof. In order to prove the statement it is sufficient to prove that the isometry $\mathrm{id}_{T_0} \oplus \rho_{|S_0} \in O(T_0) \oplus O(S_0)$ lifts to an isometry $\rho_0 \in O(L)$. In order to lift it the condition stated in Proposition 3.13 is that the action induced by ρ on the discriminant group D_{S_0} is trivial. This discriminant group is isomorphic to a quotient of $D_W \oplus D_{S(-1)}$, but the action of ρ is trivial on D_W by hypothesis, therefore it is enough to check the triviality on $D_{S(-1)}$. Again $D_{S(-1)}$ is isomorphic to a quotient of $D_L \oplus D_{\langle 6 \rangle}$, the action induced by ρ is trivial on the first factor because it is an order three isometry on $D_L \simeq \mathbb{Z}/2\mathbb{Z}$ and on the second by construction. The $K(T_0)$-generality of $\hat{\Sigma}^{[2]}$ follows from [BCS19a, Lemma 5.2]. ☐

Remark 3.17. *In this section we provided also an automorphism on $\hat{\Sigma}$ whose action on cohomology is conjugate to $\tilde{\rho_0}$. Therefore, we proved that, under the same hypotheses of Theorem 3.16, $(\hat{\Sigma}, \tilde{\eta}) \in \mathcal{K}^{\tilde{\rho_0},\zeta}_{T_{K3}}$, the moduli space of $(\tilde{\rho_0}, \tilde{j})$-polarized $K3$ surfaces.*

In the next sections we analyze case by case what happens. In order to perform this analysis we define $\Delta_3^{A_i}$ for $i = 1, \ldots, 4$ as the biggest sub-locus of the space of stable cubic threefolds where the generic element is a cubic with an isolated singularity of type A_i.

Remark 3.18. *One can check (an explicit computation is done in Section 5.1) that the dimension of the $\Delta_3^{A_i}$ locus is $10 - i$.*

Notation. *In the following sections every time an IHS manifold endowed with a (ρ, j)-polarization will appear we will use the following notations. The definition of the isometry playing the role of ρ will be clear in all cases. The embedding j will be constructed in the same way each time as follows. We will exhibit a square six class θ which has an embedding in L as in Section 2.1. This embedding will induce an embedding of the whole fixed lattice in L playing the role of j. Moreover, we will choose for any Hilbert square over a $K3$ surface the natural marking induced by a marking on a $K3$ surface, i.e. fixing a marking $\tilde{\eta}$ on a $K3$ surface $\hat{\Sigma}$ the natural marking η on its Hilbert square $\hat{\Sigma}^{[2]}$ induced by $\tilde{\eta}$ is the lifting of $\tilde{\eta} \oplus id_{\langle -2 \rangle}$ to the whole second cohomology group with integer coefficients. Then we will take the embedding ι in a way compatible with j, i.e. $\iota(\langle 6 \rangle) = 2\theta_{K3} - 3\epsilon$ with θ_{K3} fixed and this embedding will induce also an embedding on its orthogonal. In order to lighten the notation and the exposition we will often omit the markings where their presence is obvious.*

4 Singularity of type A_1

In this section we recall the results discussed in [BCS19b, Section 4.3] where the authors provide a birationality result for the nodal hyperplane. In order to see the analogy with the loci we are interested in we briefly review it giving a proof that fits our framework.
With the notation of Section 3.2 let p_0 be an isolated singularity of type A_1 for $C \subset \mathbb{P}^4$. This implies, for reasons of corank of the singularity (see [All03, Section 2] or directly do the computation), that f_2 is a rank 4 quadratic form and $Y \subset \mathbb{P}^5$ has an isolated singularity of type A_2. Moreover, by genericity, we can assume that the surface Σ given by the following equations:

$$\begin{cases} f_2(x_1, x_2, x_3, x_4) = 0 \\ f_3(x_1, x_2, x_3, x_4) + x_5^3 = 0. \end{cases} \tag{3.18.1}$$

is a smooth $K3$ surface. Here $x_1, \ldots, x_5$ are the homogeneous coordinates on the hyperplane $H \subset \mathbb{P}^5$ given by $\{x_0 = 0\}$. The covering automorphism σ on Y induces an automorphism τ of Σ. Explicitly, τ is given by $x_5 \mapsto \zeta_3 \cdot x_5$ and the identity on the other coordinates. The fixed locus of τ is a curve of genus 4. In [AS08] the authors show that the generic case (i.e. the one which we are considering) has $\mathrm{Pic}(\Sigma) \simeq U(3)$. Therefore for the Hilbert square of Σ it holds:

$$\mathrm{Pic}(\Sigma^{[2]}) \simeq \mathrm{Pic}(\Sigma) \oplus \langle -2 \rangle \simeq U(3) \oplus \langle -2 \rangle \simeq \langle 6 \rangle \oplus A_2(-1) := T_0.$$

Note that the last isometry is given by:

$$U(3) \oplus \langle -2 \rangle := \langle u_1, u_2 \rangle \oplus \langle \epsilon \rangle \simeq \langle 2(u_1 + u_2) - 3\epsilon \rangle \oplus \langle u_1 - \epsilon, \epsilon - u_2 \rangle \simeq j(\theta) \oplus W.$$

This proves Claim 3.12. Moreover, τ induces an order three isometry $\rho \in O(L)$ with $j(\theta)$ as fixed sublattice. So, ρ restricts to an order three isometry without fixed points of $W \simeq A_2(-1)$. There exists only one isometry of $A_2(-1)$ of order three without fixed points modulo conjugation and its action on the discriminant $D_{A_2(-1)}$ is trivial. Therefore, applying Theorem 3.16 there exists $\rho_0 \in O(L)$ lifting $\mathrm{id}_{|T_0} \oplus \rho_{|S_0}$ and $(\Sigma^{[2]}, \eta, \tau) \in \mathcal{M}^{\rho_0,\zeta}_{K(T_0)}$.

Proposition 3.19 ([BCS19b, Proposition 4.6]). *Let $R_0 = \mathrm{Span}(\delta_1, \rho(\delta_1)) \simeq A_2(-1)$ be the degeneracy lattice relative to a period ω of a generic cubic threefold having a single singularity of type A_1. Then the A_1 locus $\Delta_3^{A_1}$ is birational to the moduli space $\mathcal{N}^{\rho_0,\zeta}_{K(T_0)}$.*

Proof. The idea is to use the same structure of the proof of [BCS19b, Proposition 4.6] within our theoretical approach. First we note that the extension of the period map $\mathcal{P}^3 : \mathcal{C}_3^{\mathrm{sm}} \to \frac{\mathbb{B}^{10} \setminus (\mathcal{H}_n \cup \mathcal{H}_c)}{\mathbb{P}\Gamma}$ to the nodal locus is done by [ACT11, Section 6] defining its period as the period of its associated $K3$ surface. In our case the generic A_1 nodal cubic has the period

$$\mathcal{P}^3(C) := \mathcal{P}^{\tilde{\rho_0},\zeta}_{U(3)}((\Sigma, \eta)).$$

The latter period map is defined by taking $\mathcal{K}^{\tilde{\rho_0},\zeta}_{U(3)}$ as the moduli space of lattice polarized $K3$ surfaces with a non-symplectic automorphism of order three whose action on cohomology is conjugate to $\tilde{\rho_0}$. Following [DK07] this space comes with a period map

$$\mathcal{P}^{\tilde{\rho_0},\zeta}_{U(3)} : \mathcal{K}^{\tilde{\rho_0},\zeta}_{U(3)} \to \Omega^{\tilde{\rho_0},\zeta}_{U(3)} := \{x \in \mathbb{P}((S_0)_\zeta) \mid h_{S_0}(x, x) > 0\}$$

which induces a bijection

$$\mathcal{P}^{\tilde{\rho_0},\zeta}_{U(3)} : \mathcal{K}^{\tilde{\rho_0},\zeta}_{U(3)} \to \frac{\Omega^{\tilde{\rho_0},\zeta}_{U(3)} \setminus \mathcal{H}_{U(3)}}{\Gamma^{\tilde{\rho_0},\zeta}_{U(3)}}$$

where we denote with

$$\mathcal{H}_{U(3)} := \bigcup_{\mu \in S_0,\ \mu^2 = -2} \mu^\perp \cap \Omega^{\tilde{\rho_0},\zeta}_{U(3)}$$

and with

$$\Gamma^{\tilde{\rho_0},\zeta}_{U(3)} := \{\gamma \in O(L_{K3}) \mid \gamma \circ \tilde{\rho_0} = \tilde{\rho_0} \circ \gamma\}.$$

By definition we find the following equality:

$$\frac{\Omega^{\tilde{\rho_0},\zeta}_{U(3)}}{\Gamma^{\tilde{\rho_0},\zeta}_{U(3)}} = \frac{\Omega^{\rho_0,\zeta}_{S} \cap \delta_1^\perp}{\Gamma^{\rho_0,\zeta}_{S}}.$$

As proven above $(\Sigma^{[2]}, \eta, \tau)$ defines a point in $\mathcal{M}^{\rho_0,\zeta}_{K(T_0)}$. The period map in this space, following equation (2.63.1), descends to a bijection

$$\mathcal{P}^{\rho_0,\zeta}_{T_0} : \mathcal{N}^{\rho_0,\zeta}_{K(T_0)} = \frac{\mathcal{M}^{\rho_0,\zeta}_{K(T_0)}}{\mathrm{Mon}^2(T_0, \rho_0)} \to \frac{\Omega^{\rho_0,\zeta}_{T_0} \setminus \left(\mathcal{H}_{T_0} \cup \mathcal{H}'_{T_0}\right)}{\Gamma^{\rho_0,\zeta}_{T_0}}.$$

By their definitions we see that $\Omega^{\rho_0,\zeta}_{T_0} = \Omega^{\tilde{\rho_0},\zeta}_{U(3)}$ and $\Gamma^{\rho_0,\zeta}_{T_0} = \Gamma^{\tilde{\rho_0},\zeta}_{U(3)}$. Moreover, as $S_0 \subset L_{K3}$ is a sublattice of the unimodular $K3$ lattice the following holds:

$$\mathcal{H}_{T_0} := \bigcup_{\mu \in S_0,\ \mu^2=-2} \mu^{\perp} \cap \Omega^{\rho_0,\zeta}_{T_0} = \mathcal{H}_{U(3)}.$$

Now it is easy to see that $\frac{\Omega^{\rho_0,\zeta}_{T_0} \setminus \left(\mathcal{H}_{T_0} \cup \mathcal{H}'_{T_0}\right)}{\Gamma^{\rho_0,\zeta}_{T_0}}$ is birational to $\frac{\Omega^{\tilde{\rho_0},\zeta}_{U(3)} \setminus \mathcal{H}_{U(3)}}{\Gamma^{\tilde{\rho_0},\zeta}_{U(3)}}$. So in order to conclude the proof it is enough to show that $\Delta_3^{A_1}$ is birational to $\mathcal{K}^{\tilde{\rho_0},\zeta}_{U(3)}$ as the claim of the proposition would follow through a composition of birational morphisms. But this is true as in [ACT11] the authors show that the discriminant locus maps isomorphically to its image, the nodal hyperplane arrangement, through the period map, therefore a generic point in $\Delta_3^{A_1}$ is mapped isomorphically to the period of a generic $K3$ surface in $\mathcal{K}^{\tilde{\rho_0},\zeta}_{U(3)}$ and by [DK07] this is an isomorphism. $\square$

5 Singularity of type A_3 and A_4

In this section we prove Theorem 3.1 for $\Delta_3^{A_3}$ and $\Delta_3^{A_4}$. The cases treated in this section are very similar to the A_1 case; we will keep the same notation.
Let p_0 be a singularity of type A_k with $k \geq 2$ for $C \subset \mathbb{P}^4$. As a polynomial coincide with its Maclaurin expansion and the Hessian matrix is an analytical invariant, these are the corank 1 singularities, therefore this translates in f_2 being a rank 3 quadratic form.

Remark 3.20. *We will not use the explicit equations of the families considered. Nevertheless, in order to follow the computations it is better to pass to a handier form. As we work over $\mathbb{C}$ we may assume that, after a suitable linear change of coordinates, $f_2(x_1, x_2, x_3, x_4) = x_2x_3 + x_4^2$. Then an equation of Y becomes*

$$F(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = x_0(x_2x_3 + x_4^2) + f_3(x_1, x_2, x_3, x_4) + x_5^3.$$

Let us consider again the hyperplane $H \subset \mathbb{P}^5$ given by $\{x_0 = 0\}$. Here the surface Σ is given by the following complete intersection:

$$\begin{cases} f_2(x_1, x_2, x_3, x_4) = x_2x_3 + x_4^2 = 0 \\ f_3(x_1, x_2, x_3, x_4) + x_5^3 = 0. \end{cases} \tag{3.20.1}$$

5.1 Singularity of type A_3

If C has an singularity of type A_3 then with a standard computation (just add a cube in the new variable) one shows that Y has an singularity of type E_6, thus, this time Σ has one isolated singularity of type A_5, call it $p \in \Sigma$. Let us consider $\hat{\Sigma}$ the blow up of Σ at p. This is a $K3$ surface. The covering automorphism σ on Y descends to an automorphism τ on Σ. So, the singular point p is a fixed point for the automorphism τ. As the locus we are blowing up is fixed by the automorphism there exists a unique lift $\hat{\tau}$ such that $\hat{\tau}$ is an automorphism of $\hat{\Sigma}$

and commutes with the blow up map β. The fixed locus for $\hat{\tau}$ consists of two curves and two points. Stegmann in her PhD thesis gives a detailed description of surfaces which are complete (2,3)-intersection in $\mathbb{P}^4$ and from [Ste20, Proposition 6.3.10] we find that the Picard lattice of $\hat{\Sigma}$ is just the span of $\langle C_1, E_1, E_2, E_3, E_4, E_5\rangle \subset \mathrm{Pic}(\hat{\Sigma})$ with the following Gram matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

By genericity the Picard lattice has rank 6 (see Section 5.1 for calculations), therefore

$$\langle C_1, E_1, E_2, E_3, E_4, E_5\rangle = \mathrm{Pic}(\hat{\Sigma}).$$

As the fixed locus of $\hat{\tau}$ consists of two curves and two isolated points, we deduce, using [AS08, Table 2], that the Picard lattice admits an embedding of the invariant sublattice $U \oplus A_2(-1)^{\oplus 2}$. Moreover, this is an isomorphism of lattices. In fact, $\langle C_1 + E_3, C_1\rangle \oplus \langle E_1, E_2 - C_1\rangle \oplus \langle E_4 - C_1, E_5\rangle \simeq U \oplus A_2(-1)^{\oplus 2}$ exhibits $U \oplus A_2(-1)^{\oplus 2}$ as a primitive sublattice of the Picard lattice, the isomorphism comes from comparison of the determinants. Moreover the automorphism $\hat{\tau}$ is clearly non-symplectic (as it is the multiplication by a third root of the unity ζ_3 of the last coordinate). Now we investigate some properties of $\hat{\Sigma}^{[2]}$. Its Picard lattice is

$$\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Pic}(\hat{\Sigma}) \oplus \langle -2\rangle \simeq U \oplus A_2(-1)^{\oplus 2} \oplus \langle -2\rangle \simeq E_6(-1) \oplus \langle 6\rangle := T_0$$

where the last isomorphism is given by

$$\begin{aligned} &\mathrm{Pic}(\hat{\Sigma}) \oplus \langle -2\rangle \simeq \\ &\simeq \langle E_1, E_2, E_3, E_4, E_5, C_1 - \epsilon\rangle \oplus \langle 2(2C_1 + E_1 + 2E_2 + 3E_3 + 2E_4 + E_5) - 3\epsilon\rangle \end{aligned}$$

where ϵ is the (-2)-class which is half of the divisor introduced by the Hilbert–Chow morphism. It is useful to describe also the trascendental lattice $\mathrm{Tr}(\hat{\Sigma}^{[2]})$ of $\hat{\Sigma}^{[2]}$.

$$\mathrm{Tr}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Tr}(\hat{\Sigma}) \simeq U^{\oplus 2} \oplus E_8(-1) \oplus A_2(-1)^{\oplus 2}$$

First we note that $\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq E_6(-1) \oplus \langle 6\rangle \simeq W \oplus j(\theta)$ with $\theta = 2\theta_{K3} - 3\epsilon$ and again we assume $\theta_{K3} = 3u_1 + u_2$ with an isometry so with another choice of the marking. This proves Claim 3.12. Again we can associate to $\hat{\tau}$ an order three isometry $\rho \in O(L)$ with $j(\theta)$ as fixed sublattice with Proposition 3.14. Therefore it restricts to an order three isometry without fixed points on $W \simeq E_6(-1)$. As said in Section 2.2 there exists only one, up to conjugation and sign, order three isometry without fixed points on E_6. We can write it explicitly. Let $E_6 = \langle e_1, \dots, e_6\rangle$ with the standard Bourbaki numeration

$$\begin{tikzcd} e_1 \arrow[r, dash] & e_3 \arrow[r, dash] & e_4 \arrow[r, dash] \arrow[d, dash] & e_5 \arrow[r, dash] & e_6 \\ & & e_2 & & \end{tikzcd}$$

then ρ is given by

$$
\begin{aligned}
e_1 &\mapsto -e_1 - e_3 - e_4 - e_5 - e_6 \\
e_2 &\mapsto e_3 + e_4 + e_5 \\
e_3 &\mapsto -e_2 - e_3 - e_4 \\
e_4 &\mapsto -e_4 - e_5 \\
e_5 &\mapsto e_4 \\
e_6 &\mapsto e_1 + e_2 + e_3 + e_4 + e_5.
\end{aligned}
$$

Now, $\mathbb{Z}/3\mathbb{Z} \simeq D_W = \langle -\frac{4}{3}e_1 - e_2 - \frac{5}{3}e_3 - 2e_4 - \frac{4}{3}e_5 - \frac{2}{3}e_6 \rangle$ and therefore after substituting the expression of ρ on the generator of D_W we note that the action induced by ρ is trivial on D_W. Finally we can apply Theorem 3.16 to deduce also in this case that there exists $\rho_0 \in O(L)$ lifting $\mathrm{id}_{|T_0} \oplus \rho_{|S_0}$ and $(\hat{\Sigma}^{[2]}, \eta, \hat{\tau}) \in \mathcal{M}^{\rho_0,\zeta}_{K(T_0)}$.

Proposition 3.21. *Let $R_0 = \mathrm{Span}(\delta_1, \delta_2, \delta_3, \rho(\delta_1), \rho(\delta_2), \rho(\delta_3)) \simeq E_6(-1)$ be the degeneration lattice relative to a period ω of a generic cubic threefold having one singularity of type A_3. Then the A_3 locus $\Delta_3^{A_3}$ is birational to the moduli space $\mathcal{N}^{\rho_0,\zeta}_{K(T_0)}$.*

Proof. It is easy to see that one can adjust the same proof of Proposition 3.19 to this case and everything works. □

5.2 Singularity of type A_4

This section is analogous to the previous section so we omit the details.
If C has now a singularity of type A_4 then Y has a singularity of type E_8 and Σ an isolated singularity of type E_7 which we will call $p \in \Sigma$. Stegmann's work [Ste20, Proposition 6.3.12] and the computations in Section 5.1 provide us once again the description of its Picard lattice $\mathrm{Pic}(\hat{\Sigma}) = \langle C_1, E_1, \dots E_7 \rangle$ with the following Gram matrix:

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -2
\end{pmatrix}
$$

The fixed locus of the automorphism $\hat{\tau}$, induced by the cover automorphism, is just the union of three isolated points and three curves. Therefore we deduce from [AS08] that there exists an embedding of $U \oplus E_6(-1)$ in the Picard lattice which turns out to be an isometry. Indeed, it is immediate to see that $\mathrm{Pic}(\hat{\Sigma}) \simeq \langle C_1, C_1 + E_1 \rangle \oplus \langle E_2 - C, E_3, \dots E_7 \rangle \simeq U \oplus E_6(-1)$.

Now $\hat{\Sigma}^{[2]}$ has Picard lattice

$$\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Pic}(\hat{\Sigma}) \oplus \langle -2 \rangle \simeq U \oplus E_6(-1) \oplus \langle -2 \rangle \simeq E_8(-1) \oplus \langle 6 \rangle$$

where the last isometry is given by

$$\begin{aligned}\mathrm{Pic}(\hat{\Sigma}) \oplus \langle -2\rangle \simeq \langle E_1, \dots, E_7, C_1 - \epsilon\rangle \oplus \\ \oplus \langle 2(-2C_1 + 3E_1 + 4E_2 + 5E_3 + 6E_4 + 4E_5 + 2E_6 + 3E_7) - 3\epsilon\rangle\end{aligned}$$

and ϵ is, as usual, the (-2)-class given by half of the divisor introduced by the Hilbert–Chow morphism. Once again we note that with Proposition 3.14 we have the existence of an order 3 automorphism on $\hat{\Sigma}$ conjugate to $\tilde{\rho} = \rho_{|T_{K3}}$. This automorphism is, modulo conjugation, uniquely determined by its invariant lattice by [AS08], therefore it is $\hat{\tau}$. As in the previous section we note that $\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq j(\theta) \oplus W$, proving Claim 3.12. Moreover $W \simeq E_8(-1)$ which has trivial discriminant, therefore we can apply Theorem 3.16 and check that $(\hat{\Sigma}^{[2]}, \eta, \hat{\tau}) \in \mathcal{M}^{\rho_0,\zeta}_{K(T_0)}$. Then we can deduce the following

Proposition 3.22. *Let $R_0 = \mathrm{Span}(\delta_1, \dots, \delta_4, \rho(\delta_1), \dots, \rho(\delta_4)) \simeq E_8(-1)$ be the degeneracy lattice relative to a period ω of a generic cubic threefold having one singularity of type A_4. Then the A_4 locus $\Delta_3^{A_4}$ is birational to the moduli space $\mathcal{N}^{\rho_0,\zeta}_{K(T_0)}$.*

6 Singularity of type A_2

In this Section we begin the study of $\Delta_3^{A_2}$. As we will see in this section this case is quite different from the other cases already treated, nevertheless we will use the same notation in order to stress the similarities.

With a standard computation (using e.g the recognition principle stated in [BW79]) we can see that when C has an isolated singularity of type A_2, Y has an isolated singularity of type D_4. The family of cubic threefolds having a singularity of type A_2 corresponds to the generic family of complete $(2,3)$-intersections in $\mathbb{P}^4$ where the quadratic part f_2 has rank 3 (a proof of this fact can be found in the Section 5.1). Thus, the singular locus of the quadric hypersurface defined by $f_2 = 0$ is a line $(l_1(t) : \cdots : l_4(t) : s)$; this line intersects the cubic hypersurface in 3 points p_1, p_2, p_3 for f_3 sufficiently generic. Hence, Σ has exactly three singular points. Let σ be a covering automorphism on Y; it restricts to an order three automorphism τ on Σ, namely the one which maps $x_5 \mapsto \zeta_3 \cdot x_5$ where ζ_3 is just a primitive third root of the unity. As we are interested in the minimal resolution $\hat{\Sigma}$ of Σ (which is just the blow up of Σ at each p_i) we want to prove that the automorphism τ of Σ lifts to an automorphism $\hat{\tau}$ of $\hat{\Sigma}$ and find its fixed locus.

Proposition 3.23. *With the above notation there exists a unique lift $\hat{\tau}$ such that $\hat{\tau}$ is an automorphism of $\hat{\Sigma}$ and commutes with the map giving the minimal resolution of singularities β. Moreover $\mathrm{Fix}(\hat{\tau}) \simeq \mathrm{Fix}(\tau)$ is a curve of genus 4.*

Proof. The singular locus $\mathrm{Sing}(\Sigma)$ is a proper orbit for the automorphism τ, where proper means that $\mathrm{Fix}(\tau) \cap \mathrm{Sing}(\Sigma) = \emptyset$. Indeed the points in the singular locus are permuted by σ. Take $p_i \in \mathrm{Sing}(\Sigma)$ of coordinates $(\bar{l_1} : \cdots : \bar{l_4} : \bar{s})$ then $(\bar{l_1} : \cdots : \bar{l_4} : \zeta_3\bar{s})$ is again on the same line defined by the singular locus of $f_2 = 0$ and a different point of $\mathrm{Sing}(\Sigma)$. So $\mathrm{Sing}(\Sigma)$ is mapped to itself and therefore there exists a unique lift $\hat{\tau}$ such that $\hat{\tau}$ is an automorphism of $\hat{\Sigma}$

and commutes with β. By construction of $\hat{\tau}$, $\mathrm{Fix}(\hat{\tau}) \cap (\hat{\Sigma} \setminus \beta^{-1}(\mathrm{Sing}(\Sigma))) \simeq \mathrm{Fix}(\tau)$ as $\hat{\tau}$ acts in the same way of τ outside the exceptional divisors introduced by blowing up. Moreover, the diagram

$$\begin{tikzcd} \hat{\Sigma} \arrow[r, "\hat{\tau}"] \arrow[d, dashed, "\beta"] & \hat{\Sigma} \arrow[d, dashed, "\beta"] \\ \Sigma \arrow[r, "\tau"] & \Sigma \end{tikzcd}$$

commutes by the universal property of blow-ups, therefore $\hat{\tau}(\beta^{-1}(p_i)) = \beta^{-1}(\tau(p_i)) = \beta^{-1}(p_j)$ (where $j = i+1$ modulo 3). Therefore $\mathrm{Fix}(\hat{\tau}) \simeq \mathrm{Fix}(\tau)$ which is a curve of genus 4. In order to see that it is indeed a genus 4 curve one can look at the explicit computation done in [AS08, Proposition 4.7]. □

It follows from the results in *loc. cit.*, since the automorphism $\hat{\tau}$ on the $K3$ surface $\hat{\Sigma}$ fixes exactly one curve, that the invariant lattice $T(\hat{\tau})$ in $H^2(\hat{\Sigma}, \mathbb{Z})$ is isometric to $U(3)$ and its orthogonal complement in $H^2(\hat{\Sigma}, \mathbb{Z})$ is isometric to $U \oplus U(3) \oplus E_8(-1)^{\oplus 2}$. As we are considering a generic Y with only one singularity of type D_4 and thus a generic K3 surface Σ with exactly three A_1 singularities, the Picard lattice T of $\hat{\Sigma}$ is of rank four. We use again the description given in [Ste20, Proposition 6.3.8] and we find a lattice $\langle C_1, E_1, E_2, E_3 \rangle \subset \mathrm{Pic}(\hat{\Sigma})$ with the following Gram matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

We consider C to be generic with a singularity of type A_2 therefore by genericity (using the computations of the rank of the Picard group in Section 5.1) it holds $\langle C_1, E_1, E_2, E_3 \rangle = \mathrm{Pic}(\hat{\Sigma})$.

Remark 3.24. *Note that*

$$\langle C_1, E_1, E_2, E_3 \rangle = \langle C_1, C_1 + E_1, -2C_1 - E_1 + E_2, -E_2 + E_3 \rangle \simeq U \oplus A_2(-2).$$

From the discussion above we know that $T(\hat{\tau}) \simeq U(3)$ admits a primitive embedding $U(3) \hookrightarrow T$, so from this isomorphism we see that $U(3) \oplus A_2(-2) \subset U \oplus A_2(-2)$ can be written as a sublattice of finite index.

As in previous sections we note the following isometry:

$$\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Pic}(\hat{\Sigma}) \oplus \langle -2 \rangle \simeq U \oplus A_2(-2) \oplus \langle -2 \rangle \simeq \langle 6 \rangle \oplus D_4(-1).$$

This isometry can be described by

$$\mathrm{Pic}(\hat{\Sigma}^{[2]}) = \langle C_1, E_1, E_2, E_3, \epsilon \rangle \simeq \langle -E_1, E_2, C_1 - \epsilon, E_3, 2(2C_1 + E_1 + E_2 + E_3) - 3\epsilon \rangle.$$

Here we are implicitly giving an embedding of the square six polarization class θ into the Picard lattice as $2\theta_{K3} - 3\epsilon$ where θ_{K3} is a square six class on the $K3$ surface $\hat{\Sigma}$. As already remarked in Section 2.1, up to an isometry (so after a change of the marking), we can suppose θ_{K3} to be $3u_1 + u_2$ as in our setting. This proves Claim 3.12. It is useful to describe also the transcendental lattice $\mathrm{Tr}(\hat{\Sigma}^{[2]})$ of $\hat{\Sigma}^{[2]}$. So

$$\mathrm{Tr}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Tr}(\hat{\Sigma}) \simeq U^{\oplus 2} \oplus E_8(-1) \oplus A_2(-1) \oplus D_4(-1).$$

In this case we cannot proceed as described in Section 3.3. Indeed, let us denote with $T_0 = \eta(\mathrm{Pic}(\hat{\Sigma}^{[2]}))$ and $S_0 = T_0^{\perp} = \eta(\mathrm{Tr}(\hat{\Sigma}^{[2]}))$ and prove the following proposition.

Proposition 3.25. *There exists no order three isometry $\rho_0 \in O(L)$ on the $K3^{[2]}$ lattice L which extends the isometry $id_{T_0} \oplus \rho_{|S_0} \in O(T_0) \oplus O(S_0)$.*

Proof. In order to apply Proposition 3.13, $D_{D4(-1)} < D_{S_0}$ must be contained in the subgroup H of the proposition. Therefore, in order to lift the order three isometry without fixed points ρ, it must be the identity on H. But we know from Section 2.1 that there exists only one, up to conjugation, order three isometry without fixed points on $D_4(-1)$ and it is not trivial on $D_{D_4(-1)} \simeq (\mathbb{Z}/2\mathbb{Z})^2$. For the sake of completeness we write explicitly the order three isometry ρ of $D_4(-1)$ without fixed points.
We can express this lattice as $D_4(-1) = \langle d_1, d_2, d_3, d_4 \rangle$ with the following Gram matrix:

$$\begin{pmatrix} -2 & 0 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

Then the isometry on the generators is given by:

$$\begin{aligned} d_1 &\mapsto -d_1 + d_3 - d_4 \\ d_2 &\mapsto -d_1 + d_2 - d_3 \\ d_3 &\mapsto -d_1 - d_2 + d_3 + d_4 \\ d_4 &\mapsto -d_2 + d_3 - d_4. \end{aligned}$$

After a standard computation a basis of $D_{D_4(-1)}$ can be given by $\langle -d_1 + \frac{1}{2}d_2 + d_3 + \frac{1}{2}d_4, \frac{1}{2}d_1 - d_2 - d_3 - \frac{1}{2}d_4 \rangle = \langle a, b \rangle$ and we see that $\rho^*(a) = b$ and $\rho^*(b) = a + b$, where ρ^* is the map on $D_{D_4(-1)}$ induced by ρ. ☐

We still have a non-symplectic automorphism of order three on the $K3$ surface $\hat{\Sigma}$ and, thus, on $\hat{\Sigma}^{[2]}$. Hence, as noted in Remark 3.10, we can see the latter as a point $(\hat{\Sigma}^{[2]}, \eta)$ having a degeneracy lattice $R_{\delta_1} \simeq R_{\delta_2} \simeq A_2(-1) \subset R_0$, "forgetting" one of the two roots orthogonal to its period (cfr. with Definition 3.9). We now call $T_{\delta_1} = \langle 6 \rangle \oplus A_2(-1) \simeq \langle \theta, \delta_1 \rangle$ and S_{δ_1} its orthogonal complement in L. Then it is well defined $\rho_{\delta_1} \in O(L)$, the lifting to L of the isometry $\mathrm{id}_{|T_{\delta_1}} \oplus \rho_{|S_{\delta_1}}$ to L. In order to see this, it is enough to note that in the proof of the lifting of the isometry in Theorem 3.16 the hypothesis on the Picard lattice is not used, the only thing used is the triviality of ρ on the discriminant and in our case this holds on R_{δ_1}. So, exactly as in the case A_1 done in [BCS19b, §4.3], the isometry ρ_{δ_1} restricts to an isometry of $(\mathbb{Z}\epsilon)^{\perp}$.

Note that the fixed part of ρ_{δ_1} is by definition $\langle 6\rangle \oplus A_2(-1) \simeq U(3) \oplus \langle -2\rangle$ and if we consider a primitive embedding $T_{\delta_1} \subset \mathrm{Pic}(\hat{\Sigma}^{[2]})$ we get $T_{\delta_1} \oplus A_2(-2) \simeq \langle 6\rangle \oplus A_2(-1) \oplus A_2(-2) \simeq U(3) \oplus A_2(-2) \oplus \langle -2\rangle \subset \mathrm{Pic}(\hat{\Sigma}^{[2]})$.

Lemma 3.26. *The action of $\hat{\tau}$ on $H^2(\hat{\Sigma}, \mathbb{Z})$ is conjugate to the isometry ρ_{δ_1} restricted to $(\mathbb{Z}\epsilon)^{\perp}$.*

Proof. The proof is a straight-forward application of Theorem 3.15. Take $G = \langle \rho_{\delta_1} \rangle$, then condition $i)$ is simply verified. As $L^G \simeq U(3)$, also condition $iii')$ is easily verified. Lastly $S_{G,X} \simeq A_2(-2)$, thus also condition $ii)$ is verified, therefore there exists an automorphism $\bar{\tau}$ on $\hat{\Sigma}$ whose action is conjugate to ρ_{δ_1}.
If $\hat{\tau} = \bar{\tau}$ we are ok. Otherwise, we use the same argument of [AS08, Lemma 4.4, Proposition 4.7] which is the following. As $\bar{\tau}$ is an order 3 automorphism whose invariant lattice is isomorphic to $U(3)$ it fixes a genus 4 curve $\bar{P}$. Consider the linear system $|\bar{P}|$ associated to $\bar{P}$ and the map $\Phi : \hat{\Sigma} \to \mathbb{P}^4$. Consider an elliptic curve R on $\hat{\Sigma}$ intersecting $\bar{P}$, then $\bar{\tau}$ preserves R and has exactly 3 fixed points on it by Riemann Hurwitz formula. Using [SD74, Theorem 5.2] we deduce that Φ is an embedding. We can choose projective coordinates $(x_1 : \cdots : x_5)$ on $\mathbb{P}^4$ such that the hyperplane H whose preimage $\Phi^{-1}(H) = \bar{P}$ is given by $x_5 = 0$. Therefore the induced automorphism on the image is the automorphism of $\mathbb{P}^4$ that maps $x_5 \mapsto \zeta_3 \cdot x_5$. In other words we can make a change of coordinates on $\hat{\Sigma}$ such that $\hat{\tau} = \bar{\tau}$. $\square$

We now consider the natural automorphism $\hat{\tau}^{[2]}$ induced on $\hat{\Sigma}^{[2]}$ by $\hat{\tau}$. As its invariant lattice is T_{δ_1} we see that $(\hat{\Sigma}^{[2]}, \eta, \hat{\tau})$ defines a point in $\mathcal{N}^{\rho_{\delta_1},\zeta}_{T_{\delta_1}}$. If $\hat{\Sigma}^{[2]}$ is moreover $K(T_{\delta_1})$-general we can conclude as in Sections 3.4 and 3.5; but, it turns out not to be the case as proved by the following proposition.

Proposition 3.27. *$\hat{\Sigma}^{[2]}$ is not $K(T_{\delta_1})$-general.*

Proof. Note that by [Mar11, Theorem 6.18] the group of monodromies $\mathrm{Mon}^2(\hat{\Sigma})$ acts transitively on the set of exceptional chambers, therefore, up to taking a different birational model, in order to prove that $\hat{\Sigma}^{[2]}$ is not $K(T_{\delta_1})$-general it is sufficient to show that there exists a wall divisor $\mu \in \Delta(\hat{\Sigma})$ not fixed by the action of $\hat{\tau}$ but such that $C^{\hat{\tau}} \cap \mu^{\perp} \neq 0$, where $C^{\hat{\tau}}$ denotes a connected component of the invariant positive cone. We will prove that there exist both -2 and -10 walls which are not fixed by $\hat{\tau}$ by exhibiting them. The wall divisor E_1 is not fixed by $\hat{\tau}$ but its orthogonal hyperplane intersects non-trivially $C^{\hat{\tau}}$, e.g in $2C + E_1 + E_2 + E_3$. The same is true for the wall divisor $2E_1 + \epsilon$. $\square$

This proposition implies that $(\hat{\Sigma}^{[2]}, \eta, \hat{\tau})$ is a non-separable point in $\mathcal{M}^{\rho_{\delta_1},\zeta}_{T_{\delta_1}}$, i.e. for which the $T2$ condition of separability fails. Moreover this condition persists also if we consider $(\hat{\Sigma}^{[2]}, \eta, \hat{\tau})$ in the quotient $\mathcal{N}^{\rho_{\delta_1},\zeta}_{T_{\delta_1}}$. Indeed, in [BCS19a, Section 4] the authors show that the points in the fibre over a period $\omega \in \mathcal{N}^{\rho_{\delta_1},\zeta}_{T_{\delta_1}}$ are in a one-to-one correspondence with the number of orbits of the monodromy action on the chambers. In the proof of Proposition 3.27 we found a (-10)-class defining a wall whose orbit cuts $K(T_{\delta_1})$, therefore there exists at least two chambers which are not in the same orbit of the monodromy action. This implies the following corollary.

Corollary 3.28. *There exist at least two non biregular models in $\mathcal{N}_{T_{\delta_1}}^{\rho_{\delta_1},\zeta}$ which are over the same fiber of a period ω relative to a generic cubic threefold with a single singularity of type A_2.*

The pair $(\hat{\Sigma}^{[2]}, \eta)$ defines also a point in the moduli space of (M, j)-polarized IHS manifolds of $K3^{[2]}$-type, where $M := U \oplus A_2(-2) \oplus \langle -2 \rangle$. So we shift our interest in giving a formal characterization of marked IHS manifolds (X, ϕ) carrying the following commutative diagram in the data:

$$\begin{tikzcd} T_{\delta_1} \arrow[r, hook] \arrow[drr, "j"'] & M \arrow[r, hook, "\iota"] & \mathrm{Pic}(X) \subset H^2(X,\mathbb{Z}) \arrow[r, "\sigma^*"] \arrow[d, "\phi"] & H^2(X,\mathbb{Z}) \arrow[d, "\phi"] \\ & & L \arrow[r, "\rho_{\delta_1}"] & L. \end{tikzcd} \tag{3.28.1}$$

7 Kähler cone sections of K-type

In this section we introduce the notion of *Kähler cone sections of K-type* in order to give a more detailed description of the following situation. Let (X, ϕ) be an (M, j)-polarized IHS manifold of type L and let $T \subset M$ be a primitive sublattice. In this situation the pair (X, ϕ) is both (M, j)-polarized and $(T, j_{|T})$-polarized, so we can compare the notions of $K(M)$ and $K(T)$ generality when the two chambers are chosen in such a way that $\mathcal{K}_X \cap \iota(K(T)) \neq \emptyset \neq \mathcal{K}_X \cap \iota(K(M))$, where ι denotes the $\mathbb{C}$-linear extension of the map ι of the diagram (3.28.1). Recall that $K(T)$ and $K(M)$ are, respectively, a connected component of $C_T \setminus \bigcup_{\delta\in\Delta(T)} \delta^{\perp}$ and $C_M \setminus \bigcup_{\delta\in\Delta(M)} \delta^{\perp}$. By their definition, $\Delta(T) = \Delta(M) \cap T$ and $C_T = C_M \cap (T \otimes \mathbb{R})$, therefore our choice of the chambers $K(M)$ and $K(T)$ implies $K(M) \cap (T \otimes \mathbb{R}) \subset K(T)$ as the embedding of both via ι intersects $\mathcal{K}_X$.

Lemma 3.29. *If a pair (X, ϕ) as above is $K(T)$-general then it is also $K(M)$-general.*

Proof. Suppose that X is not $K(M)$-general: $\iota(C_M) \cap \mathcal{K}_X$ is then a proper subset of $\iota(K(M))$. By Theorem 2.44 there exists $\lambda \in \Delta(X)$ such that $\lambda^{\perp} \cap \iota(K(M)) \neq \emptyset$ and $\phi(\lambda) \notin \Delta(M)$. But remember that $\Delta(M) \supset \Delta(T)$ and by our choice of the chambers $K(M) \cap (T \otimes \mathbb{R}) \subset K(T)$; then $\phi(\lambda) \notin \Delta(T)$ and $\lambda^{\perp} \cap \iota(K(T)) \supset \lambda^{\perp} \cap \iota(K(M) \cap (T \otimes \mathbb{R})) \neq \emptyset$ implying that X is not $K(T)$-general. □

Remark 3.30. *The converse to the previous statement is not true. A counterexample is given by the IHS fourfold $\hat{\Sigma}^{[2]}$ in Section 3.6. In fact, Proposition 3.27 shows that $(\hat{\Sigma}^{[2]}, \eta)$ is not $K(T_{\delta_1})$-general while it is clearly $K(M)$-general by [BCS19a, Lemma 5.2].*

So, we give the following more general definition.

Definition 3.31. *Let K be a (connected and open) subset of a chamber $K(T)$ such that $K(T) \supset \phi(\mathcal{K}_X)$, with $\mathcal{K}_X$ denoting the Kähler cone of a (T, j)-polarized IHS manifold (X, ϕ). Then we say that X has a Kähler cone section of K-type if $K = \phi(\mathcal{K}_X) \cap C_T$.*

Remark 3.32. *Note that if we choose a subset K which is not connected or not open then no IHS manifold satisfies the above definition as its Kähler cone is connected and open.*

If the subset K chosen in Definition 3.31 is a proper subset of $K(T)$ any IHS manifold X with a Kähler section cone of K-type will not be $K(T)$-general. The following characterization gives us a link between the two definitions.

Proposition 3.33. *Let (X, ϕ) be a (T, j_T)-polarized IHS manifold of type L. If X has a Kähler cone section of K-type then there exist:*

i) a lattice M with an embedding $j_M : M \hookrightarrow L$;

ii) a primitive embedding $\tilde{\iota} : T \hookrightarrow M$;

iii) a chamber $K(M)$, i.e. a connected component of $C_M \setminus \bigcup_{\delta \in \Delta(M)} \delta^{\perp}$, with $K(M) \cap C_T = K$;

such that (X, ϕ) is a $K(M)$-general, (M, j_M)-polarized IHS manifold.
Conversely, if (X, ϕ) is a $K(M)$-general, (M, j_M)-polarized IHS manifold with $T \subset M$ then it has a Kähler cone section of K-type with $K = K(M) \cap C_T$.

Proof. By Theorem 2.44 the Kähler cone of X is a connected component of $\mathcal{C}_X \setminus \mathcal{H}_\Delta$, so denote with $\Lambda \subset \Delta(X)$ the set of those MBM classes λ for which $\lambda^{\perp}$ is an extremal ray, i.e. if $\alpha, \beta \in \mathcal{C}_X$ are such that $\alpha + \beta \in \lambda^{\perp}$ then $\alpha, \beta \in \lambda^{\perp}$. Define M as a lattice of minimal rank containing T and $\phi(\Lambda)$ and such that every embedding in the chain of inclusions $T \subset M \subset M' := \phi(\mathrm{Pic}(X))$ is primitive and use the first to define $\tilde{\iota}$; fix an embedding $j_M : M \hookrightarrow L$ such that $(j_M)_{|T} = j_T$. Then by construction there exists a chamber $K(M)$ of $C_M \setminus \mathcal{H}_{\Delta(M)}$ such that (X, ϕ) is (M, j_M)-polarized with $\iota_M := j_M \circ \phi^{-1}$ and $K(M)$-general. Moreover, denoting with ι_T the embedding $T \hookrightarrow \mathrm{Pic}(X)$ we obtain $\iota_M(K(M) \cap C_T) = \mathcal{K}_X \cap \iota_T(C_T) = \iota_M(K)$ by hypothesis and injectivity of ι_M, therefore the first part of the statement is proved. The second one is obvious using the fact that ι_T and ι_M are injective and that by construction $\iota_T = (\iota_M)_{|T}$. Indeed $\iota_T(K) = \iota_T(K(M) \cap C_T) = \iota_M(K(M)) \cap \iota_T(C_T) = \mathcal{K}_X \cap \iota_T(C_T)$. $\square$

Remark 3.34. *Note that in the proof of the above proposition we chose to exhibit a minimal $M \supset T$ for which the statement holds. In fact, the same holds for every lattice $M' \supset M \supset T$ for which (X, ϕ) admits an (M', j')-polarization by Lemma 3.29. In particular it holds for $M' \simeq \mathrm{Pic}(X)$.*

We now fix a chamber K and (X, η) a (T, j_T)-polarized IHS manifold with a Kähler cone section of K-type. According to Proposition 3.33 we can find a pair (M, j_M) such that (X, η) is (M, j_M)-polarized, $K(M)$-general IHS manifold, with $K(M) \cap C_T = K$. We define $N := j_M(M)^{\perp}$ and $S := j_T(T)^{\perp}$. On the moduli space $\mathcal{M}_{K(M), j_M}$ of (M, j_M)-polarized IHS manifolds of type L which are $K(M)$-general it is defined a period map which is an isomorphism by [Cam18, Theorem 3.13],

$$\mathcal{P}_{K(M)} : \mathcal{M}_{K(M), j_M} \to \Omega^{M, j_M} \setminus \left(\mathcal{H}_M \cup \mathcal{H}'_{K(M)} \right)$$

where a marked pair (X, ϕ) is sent to $\phi(H^{2,0}(X))$.
We define the subfamily $\mathscr{F}^T_{j_M, K}$ of the (T, j_T)-polarized IHS manifold of type L with the Kähler cone section of type K and embedding j_M. Then we state the following theorem.

Theorem 3.35. *The period map $\mathcal{P}_{K(M)}$ restricted to $\mathscr{F}^T_{j_M,K}$ defines a bijection with*

$$\Omega^T_{j_M,K} := \{\omega \in \mathbb{P}(N_{\mathbb{C}}) \mid (\omega,\bar{\omega}) > 0,\, q(\omega) = 0\} \setminus \Big((\mathcal{H}_S \cap N_{\mathbb{C}}) \cup \mathcal{H}'_{K(M)}\Big)$$

with $\mathcal{H}_{S_{\delta_1}} := \cup_{\nu\in\Delta(S)} H_\nu$ *and* $\mathcal{H}'_{K(M)} := \cup_{\nu\in\Delta'(K(M))} H_\nu$.

Proof. Consider the period map $\mathcal{P}_{K(M)}$. If we restrict this map to those IHS manifolds which are in $\mathscr{F}^T_{j_M,K}$ then the image cannot lie in $\mathcal{H}_S \cap N_{\mathbb{C}}$ because if there existed $\nu \in \mathcal{H}_S$ such that $\omega := \phi^{-1}(H^{2,0}(X)) \in H_\nu$ we would have $\phi^{-1}(\nu) \in \mathrm{NS}(X)$ by definition of $\mathrm{NS}(X)$ as the orthogonal complement of $H^{2,0}(X)$ in $H^2(X,\mathbb{Z})$. Therefore $\eta^{-1}(\nu)$ would be a wall divisor yielding a contradiction using the same argument of Remark 3.11. Being $\mathcal{P}_{K(M)}$ an injective morphism, in order to get a bijection we just need to show what is the image. Take a period $\omega \in \Omega^T_{j_M,K}$ and consider $(X,\phi) = \mathcal{P}^{-1}_{K(M)}(\omega)$ then it is immediate to see that (X,ϕ) is indeed an element of $\mathscr{F}^T_{j_M,K}$ as it is $K(M)$-general and by Proposition 3.33 it has a Kähler cone section of K-type. □

8 A moduli space for the singularity of type A_2

In this section we give the proof of Theorem 3.1 for $\Delta_3^{A_2}$ and continue the description started at the end of Section 3.6, therefore we will use the same notation resumed in the diagram 3.28.1. Moreover, we recall that $T_{\delta_1} = U(3) \oplus \langle -2\rangle$ and $M = U \oplus A_2(-2) \oplus \langle -2\rangle$.
Let us define $K(M)$ as the connected component of $C_M \setminus \cup_{\nu\in\Delta(M)} H_\nu$ which contains a Kähler class of $\hat{\Sigma}^{[2]}$, i.e. $\mathcal{K}_{\hat{\Sigma}^{[2]}} \subset \iota(K(M))\otimes\mathbb{R}$. We choose in a compatible way also $K(T_{\delta_1})$, i.e. $\mathcal{K}_{\hat{\Sigma}^{[2]}} \cap \iota(K(T_{\delta_1})) \neq \emptyset \neq \mathcal{K}_{\hat{\Sigma}^{[2]}} \cap \iota(K(M))$; we also choose $K := K(M) \cap (T_{\delta_1}\otimes\mathbb{R}) \subset K(T_{\delta_1})$. Moreover, in the case of $\hat{\Sigma}^{[2]}$, the map $i : M \to \mathrm{Pic}(\hat{\Sigma}^{[2]})$ is an isomorphism and therefore $i(K(M)) = \mathcal{K}_{\hat{\Sigma}^{[2]}}$. So, we define the family $\mathscr{F}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}} \subset \mathcal{M}^{\rho_{\delta_1},\zeta}_{T_{\delta_1}}$ of IHS manifolds in $\mathcal{M}^{\rho_{\delta_1},\zeta}_{T_{\delta_1}}$ which have a Kähler cone section of K-type and admit an embedding $T_{\delta_1} \hookrightarrow M \hookrightarrow \mathrm{Pic}(X)$ compatible with the polarization, i.e. for which the commutative diagram (3.28.1) is defined.

Remark 3.36. *We do not want to choose the embedding of M in L in the data but only the embedding j of T_{δ_1}. It is an easy exercise using [Nik80, Proposition 1.15.1] to see that there are only two possible non-isomorphic embeddings of T_{δ_1} in L and they correspond respectively to the two possible non-isomorphic embeddings of M in L (see Section 5.2). Recall that we say that two embeddings j_1 and j_2 of M in L are isomorphic if there exists an automorphism $\varphi \in O(L)$ such that $j_2 = \varphi \circ j_1$. Therefore we choose the embedding j_k such that $j_k(T_{\delta_1})^\perp = j(T_{\delta_1})^\perp =: S_{\delta_1} \simeq U(3) \oplus U \oplus E_8(-1)^{\oplus 2} \supset \mathrm{Tr}(\hat{\Sigma}^{[2]})$. Note that doing so $(\hat{\Sigma}^{[2]},\eta) \in \mathscr{F}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}$.*

Proposition 3.37. *The period map $\mathcal{P}_{K(M)}$ restricted to $\mathscr{F}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}$ defines a bijection with*

$$\Omega^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}} := \{\omega \in \mathbb{P}(N_{\mathbb{C}}(\zeta)) \mid (\omega,\bar{\omega}) > 0\} \setminus \Big((\mathcal{H}_{S_{\delta_1}} \cap N_{\mathbb{C}}) \cup \mathcal{H}'_{K(M)}\Big)$$

with $\mathcal{H}_{S_{\delta_1}} := \cup_{\nu\in\Delta(S_{\delta_1})} H_\nu$ *and* $\mathcal{H}'_{K(M)} := \cup_{\nu\in\Delta'(K(M))} H_\nu$.

Proof. This proposition states that we can apply Theorem 3.35 also with the additional structure of the $(\rho_{\delta_1}, T_{\delta_1})$-polarization. The image of $\mathcal{P}_{K(M)}$ lies in the eigenspace $\mathbb{P}(N_{\mathbb{C}}(\zeta))$ by definition of Picard group. Take now a period $\omega \in \Omega^{\rho_{\delta_1},\zeta}_{M,K(M)}$ and consider $(X, \phi) = \mathcal{P}^{-1}_{K(M)}(\omega)$, we want to show that on X there exists an automorphism σ satisfying the properties required in diagram (3.28.1). Define $\sigma^* := \phi^{-1} \circ \rho_{\delta_1} \circ \phi$. It is an isomorphism of integral Hodge structures since

$$\sigma^*(\omega_X) := \sigma^*(\mathcal{P}^{-1}_{K(M)}(\omega)) = \phi^{-1}(\rho_{\delta_1}(\omega)) = \zeta\omega_X.$$

Moreover, it is a parallel transport operator as $\rho_{\delta_1} \in \mathrm{Mon}^2(L)$. If it preserves also a Kähler class we can conclude with Markman's Torelli Theorem 2.39. But X has a Kähler cone section of K-type, therefore

$$\mathcal{K}_X \cap i(T_{\delta_1}) \supset \mathcal{K}_X \cap i(T_{\delta_1}) \cap i(C_M) = i(K(M)) \cap i(T_{\delta_1}) \neq \emptyset$$

thus, ρ_{δ_1} fixes a Kähler class. □

Remark 3.38. *Note that if (X_1, ϕ_1) and (X_2, ϕ_2) define the same point in $\mathcal{M}_{M,K(M)}$, i.e. there exists a biregular morphism $f : X_1 \to X_2$ such that $\phi_1 = \phi_2 \circ f^*$ and $i_1 = i_2 \circ f^*$, then also $\sigma_1 = f^{-1} \circ \sigma_2 \circ f$ by [BC22, Theorem 1.8].*

Let

$$\begin{aligned}\mathrm{Mon}^2(M, j, \rho_{\delta_1}) := \{&g \in \mathrm{Mon}^2(L) \mid g(M) = M,\ g(t) = t,\\ &g \circ \rho_{\delta_1} = \rho_{\delta_1} \circ g,\ \forall t \in T_{\delta_1}\}\end{aligned}$$

and denote its image in $O(N)$ via the restriction map with $\Gamma^{\rho_{\delta_1}}_{M,j}$. Note that $\mathrm{Mon}^2(M, j, \rho_{\delta_1})$ is the stabilizer of $\mathscr{F}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}$ for the action of $\mathrm{Mon}^2(M, j)$ on $\mathcal{M}_{M,K(M,T_{\delta_1})}$. Moreover, by definition, the bijection defined in Proposition 3.37 is equivariant with respect to the action of $\mathrm{Mon}^2(M, j, \rho_{\delta_1})$ and of $\Gamma^{\rho_{\delta_1}}_{M,j}$. Denoting $\mathscr{N}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}} := \mathscr{F}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}} / \mathrm{Mon}^2(M, j, \rho_{\delta_1})$ we deduce the following corollary:

Corollary 3.39. *There exists a bijection between $\mathscr{N}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}$ and $\Omega^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}/\Gamma^{\rho_{\delta_1}}_{M,j}$.*

The whole formal construction made above is "natural" in the sense that it arises from the theory of $K3$ surfaces in a compatible way. Let us clarify this sentence. Consider a diagram similar to (3.28.1) where (X, ϕ) is, this time, a marked $K3$ surface such that the following commutative diagram is defined

$$\begin{tikzcd} U(3) \arrow[r, hook] \arrow[drrr, "j"'] & U \oplus A_2(-2) \arrow[r, hook, "i"] & \mathrm{Pic}(X) \subset H^2(X,\mathbb{Z}) \arrow[r, "\sigma^*"] \arrow[d, "\phi"] & H^2(X,\mathbb{Z}) \arrow[d, "\phi"] \\ & & L_{K3} \arrow[r, "\tilde{\rho}"] & L_{K3}. \end{tikzcd}$$

Where $\tilde{\rho}$ is ρ_{δ_1} restricted to $\langle -2\rangle^\perp$. Define the subfamily of the $K3$ surfaces with an ample $U(3)$-polarization for which there exists the above diagram as $\mathscr{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$ and note that $\hat{\Sigma} \in \mathscr{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$. Then with the same techniques of Proposition 3.37 we can find a generalized version of [DK07, Theorem 11.3] which applies to our subfamily and obtain the following proposition. We denote with

$$\Gamma^{\tilde{\rho}}_{U(3),U\oplus A_2(-2)} := \{\gamma \in O(L_{K3}) \mid \gamma_{|U\oplus A_2(-2)} \in O(U \oplus A_2(-2)), \\ \gamma_{|U(3)} = \mathrm{id} \text{ and } \gamma \circ \tilde{\rho} = \tilde{\rho} \circ \gamma\}.$$

Proposition 3.40. *The period map defines a bijection between $\mathscr{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$ and*

$$\Omega^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)} := \{\omega \in \mathbb{P}(N_{\mathbb{C}}(\zeta)) \mid (\omega, \bar{\omega}) > 0\} \setminus \Big(\mathcal{H}_{S_{\delta_1}} \cap N_{\mathbb{C}}\Big)$$

with $\mathcal{H}_{S_{\delta_1}} := \cup_{\nu\in\Delta(S_{\delta_1})} H_\nu$. Moreover, this bijection descends to bijection at the level of isomorphism classes, i.e. $\Omega^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}/\Gamma^{\tilde{\rho}}_{U(3),U\oplus A_2(-2)}$ parametrizes isomorphism classes of $K3$ surfaces in $\mathscr{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$.

Proof. Consider the period map for the $K3$ surfaces which are $(\tilde{\rho}, U(3))$-ample, by [DK07, Theorem 11.2] it is a bijection with

$$\Omega^{\tilde{\rho}}_{U(3)} := \{\omega \in \mathbb{P}((S_{\delta_1} \otimes \mathbb{C})(\zeta)) \mid (\omega, \bar{\omega}) > 0\} \setminus \mathcal{H}_{S_{\delta_1}}.$$

If we moreover consider the restriction to our subfamily then it is easy to see that [DK07, Theorem 10.2] guarantees the assertion. □

We can now state the following proposition which answers our question in a formal way.

Proposition 3.41. *Let $R_0 = \mathrm{Span}(\delta_1, \delta_2, \rho(\delta_1), \rho(\delta_2)) \simeq D_4(-1)$ be the degeneracy lattice relative to a period ω of a generic cubic threefold having a single singularity of type A_2. Then the A_2 locus $\Delta_3^{A_2}$ is birational to the moduli space $\mathscr{N}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}$.*

Proof. The proof has again the same structure of the proof of Theorem 3.19, we review here the main points. Recall that the extension of the period map $\mathcal{P}^3 : \mathcal{C}_3^{\mathrm{sm}} \to \frac{\mathbb{B}^{10}\setminus(\mathcal{H}_n\cup\mathcal{H}_c)}{\mathbb{P}\Gamma}$ to the nodal locus is done by [ACT11] defining its period as the period of its associated $K3$ surface (see also Section 2.1). In our case the generic A_2 nodal cubic has the period

$$\mathcal{P}^3(C) := \mathcal{P}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}((\Sigma, \eta)).$$

The latter period map is the same of Proposition 3.40. Following Proposition 3.40 this map yields an isomorphism between $\mathscr{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$ and $\Omega^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$.

As proven in Section 3.6 $(\hat{\Sigma}^{[2]}, \eta, \tau)$ defines a point in $\mathscr{F}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}$. The period map in this space defines a bijection which, according to Corollary 3.39, descends to a bijection between $\mathscr{N}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}$ and $\Omega^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}/\Gamma^{\rho_{\delta_1}}_{M,j}$. Note that $\Omega^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}/\Gamma^{\rho_{\delta_1}}_{M,j}$ and

$$\Omega^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}/\Gamma^{\tilde{\rho}}_{U(3),U\oplus A_2(-2)}$$

are birational. So, in order to conclude the proof, it is enough to show that $\Delta_3^{A_2}$ is birational to the isomorphism classes of $K3$ surfaces in $\mathscr{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$ as the claim of the proposition would follow through a composition of birational morphisms. But this is true as in [ACT11, Section 6] the authors show that the discriminant locus maps isomorphically to its image, the nodal hyperplane arrangement, through the period map, therefore a generic point in $\Delta_3^{A_2}$ is mapped isomorphically to the period of a generic $K3$ surface in $\mathscr{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$ and by Proposition 3.40 this is an isomorphism. $\square$

Remark 3.42. *Also Kondō notes in [Kon02] this family as a codimension 1 family in the moduli space of curves of genus 4 which consists of smooth curves with a vanishing theta null.*

Chapter 4

The Fano variety of lines on a singular cyclic cubic fourfold

"Non so se quello che hai detto è giusto, ma posso provare.
Se mi aiuti"

- Federico Fellini, $8\ e\ \frac{1}{2}$

In this chapter we analyze in depth the birational relation between the Fano variety of lines on a singular cyclic cubic fourfold and the Hilbert square of a singular $K3$ surface mentioned in Section 3.2. This chapter is the result of a collaboration with S. Boissière and P. Comparin which is a work in progress.

1 Introduction

In the context of complex projective geometry, cubic hypersurfaces have been deeply studied by the mathematical community for many reasons, one of them being their rich geometry. There has been a growing interest over the last fitfty years in one class in particular: cubic fourfolds. One of the reasons why cubic fourfolds are particularly interesting resides in their Hodge structure. Indeed, they are the archetypal example of Fano varieties of $K3$ type (see [Fat22] for a survey on the subject). Because of this fact they are deeply related to the world of IHS manifolds.

As we have seen, the meaning of the degeneracy of the automorphism is that when the period point goes to the closure of the period domain, the automorphism of the family jumps to another family with a bigger invariant lattice. In view of this, in [BHS23] the authors studied in detail the geometry of the Fano variety of lines of a cuspidal cyclic cubic fourfold, i.e. a cyclic cubic fourfold having a cubic threefold with one isolated singularity of type A_1 as branch locus.

The aim of this chapter of the thesis is to supplement their results studying the Fano variety of lines $F(Y_i)$ of a cyclic cubic fourfold having as branch locus a cubic threefold with one

isolated singularity of type A_i with $i = 2, 3, 4$. In particular, using the notation of Chapter 3, we prove that $F(Y_i)$ admits a symplectic resolution by a (ρ_i, T_{δ_i})-polarized IHS manifold of $K3^{[2]}$-type.
The main theorem of this chapter is the following

Theorem 4.1. *Let C_i be a complex projective cubic threefold having one isolated singularity of type A_i for $i = 2, 3, 4$ and let Y_i be its associated cyclic cubic fourfold. Assume that there exist no plane $\Pi \subset Y$ such that $\Pi \cap \mathrm{Sing}(Y_i) \neq \emptyset$. Then the Fano variety of lines $F(Y_i)$ of Y_i admits a unique symplectic resolution by an IHS manifold of $K3^{[2]}$-type $\widehat{F(Y_i)}$.*
Moreover, there exist integral lattices R_i and T_i, defined below, such that:

i) $\mathrm{Pic}\left(\widehat{F(Y_i)}\right) \simeq R_i$;

ii) there exists a non-symplectic automorphism of order three $\tau_i \in \mathrm{Aut}\left(\widehat{F(Y_i)}\right)$ whose invariant sublattice is $H^2(\widehat{F(Y_i)}, \mathbb{Z})^{\tau_i^*} \simeq T_i$

with T_i and R_i defined in the following table:

i	T_i	R_i
2	$\langle 6 \rangle \oplus A_2(-1)$	$\langle 6 \rangle \oplus D_4(-1)$
3	$\langle 6 \rangle \oplus E_6(-1)$	$\langle 6 \rangle \oplus E_6(-1)$
4	$\langle 6 \rangle \oplus E_8(-1)$	$\langle 6 \rangle \oplus E_8(-1)$

2 Basic facts about symplectic varieties

In this section we recall some basic facts about symplectic varieties.

Let X be a normal complex projective variety and X^{reg} its regular part. The sheaf $\Omega_X^{[p]}$ of *reflexive holomorphic p-forms* on X is defined as $\iota_* \Omega^p_{X^{\mathrm{reg}}}$, where we denoted by ι the inclusion of the regular part $X^{\mathrm{reg}} \subset X$. Then a symplectic form on X is a closed reflexive 2-form ω, i.e. a global section of $\Omega_X^{[2]}$, on X which is non-degenerate at each point of X^{reg}.

Definition 4.2 ([Bea00, Definition 1.1])**.** *Assume that a normal projective variety X admits a symplectic form ω. Then X has* symplectic singularities *if for one (hence for every) resolution $f : \widehat{X} \to X$ of the singularities (i.e. a birational proper map from a smooth variety) of X, the pullback $f^*\omega_{\mathrm{reg}}$ of the holomorphic symplectic form $\omega_{\mathrm{reg}} = \omega|_{X^{\mathrm{reg}}}$ extends to a holomorphic 2-form on $\widehat{X}$. In this case X is called* symplectic variety.

From now on we denote by X a symplectic variety, ω a symplectic form on it and $\pi : \widehat{X} \to X$ a resolution of singularities. Then the regular 2-form $\pi^{[*]}\omega$ (see the discussion in [Keb13] for the definition of pullback of reflexive forms), in general, is degenerate. Therefore we give the following definition.

Definition 4.3. *A resolution of singularities $\pi : \widehat{X} \to X$ is said* symplectic *if $\pi^{[*]}\omega$ is non-degenerate.*

Recall that a birational map $\pi : Y \to X$ between normal irreducible algebraic varieties with canonical bundles K_X and K_Y is called *crepant* if the canonical map

$$\pi^* K_X \to K_Y$$

defined over the non-singular locus extends to an isomorphism over the whole manifold Y.

Proposition 4.4. *Let $\pi : \widehat{X} \to X$ be a resolution of singularities. The following are equivalent:*

- *π is crepant;*
- *π is symplectic;*
- *$K_{\widehat{X}}$ is trivial;*
- *for every symplectic form ω' on X^{reg}, its pull-back $\pi^{[*]}\omega'$ extends to a symplectic form on $\widehat{X}$.*

Proof. See [Fu06, Proposition 1.6]. □

Finally, Kaledin showed in [Kal06] that there exists a canonical stratification of a symplectic variety X.

Theorem 4.5. *Let X be a symplectic variety. Then there exists a canonical stratification of closed subschemes $X = X_0 \supset X_1 \supset \dots$ such that:*

- *X_{i+1} is the singular locus of X_i;*
- *the normalization of every irreducible component of X_i is a symplectic variety.*

Proof. See [Kal06, Theorem 2.3] □

We call *leaf* each stratum of this decomposition. In particular each irreducible component of a leaf has even dimension.

3 A reminder on cyclic cubic fourfolds

In this section we will recall the notation used in Section 3.2 which we will use throughout the rest of the chapter.

A cubic fourfold is said *cyclic* if it can be obtained as a 3:1 cyclic cover of $\mathbb{P}^4$ branched along a cubic threefold. If it has one isolated singularity of type ADE in $p := (1 : 0 : \dots : 0)$ we can give an equation for the fourfold Y with the vanishing of the following polynomial

$$F(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = x_0 f_2(x_1, x_2, x_3, x_4) + f_3(x_1, x_2, x_3, x_4) + x_5^3, \qquad (4.5.1)$$

where f_i are sufficiently generic homogeneous polynomials of degree i in $\mathbb{C}[x_1, x_2, x_3, x_4]$ such that p is the only singularity of Y. In the hyperplane $H \subset \mathbb{P}^5$ defined by $\{x_0 = 0\}$, which we identify with $\mathbb{P}^4$ of coordinates $(x_1 : x_2 : x_3 : x_4 : x_5)$, we consider the surface

Σ given as the complete intersection of the quadric Q defined by $f_2 = 0$ and the cubic K defined by $f_3 + x_5^3 = 0$. The singularities of this surface are given by the singular points of Q as shown in Theorem 3.3.

Moreover, recall that, by [DR01, Lemma 2.1], if p is a singular point of Y of type **T**, then the singular points of Σ are of type $\hat{\boldsymbol{T}}$, as described by the following table.

T	$\boldsymbol{A_1}$	$\boldsymbol{A_2}$	$\boldsymbol{A_{n\geq 3}}$	$\boldsymbol{D_4}$	$\boldsymbol{D_{n\geq 5}}$	$\boldsymbol{E_6}$	$\boldsymbol{E_7}$	$\boldsymbol{E_8}$
$\hat{\boldsymbol{T}}$	$\emptyset$	$\emptyset$	$\boldsymbol{A_{n-2}}$	$3\boldsymbol{A_1}$	$\boldsymbol{A_1} + \boldsymbol{D_{n-2}}$	$\boldsymbol{A_5}$	$\boldsymbol{D_6}$	$\boldsymbol{E_7}$

Note that there exist cases in which there is more than one singular point on Σ, indeed the number of singularities of Σ depends on the number of solutions of the equation $f_3(1,0,0,0,x) = 0$. This surface is embedded in $F(Y)$, the Fano variety of lines on Y by the results of [Leh18, Lemma 3.3, Theorem 3.6] and [Has00, Lemma 6.3.1], which we summarized in Theorem 3.4.

Throughout this chapter we will also assume that every cyclic cubic fourfold has no planes through its singular point. This hypothesis is a genericity assumption as shown in Remark 3.6.

4 A symplectic resolution for $F(Y)$

In this section we determine the existence of a symplectic resolution for $F(Y)$ when Y is a cyclic cubic fourfold branched along a cubic threefold having one isolated singularity of type A_i for $i = 2, 3, 4$.

In order to do the computations, let us consider the following equation:

$$\begin{aligned} F(x_0, \dots, x_5) = x_0 Q(x_2, x_3, x_4) + K(x_1, \dots, x_5) = x_0 q_1(x_2, x_3, x_4) + \\ + x_1^2 h_2(x_2, x_3, x_4, x_5) + x_1 q_2(x_2, x_3, x_4, x_5) + k_2(x_2, x_3, x_4, x_5). \end{aligned} \tag{4.5.2}$$

With k_2, q_i and h_2 homogeneous polynomials of degree, respectively, three, two and one. This is close to the equation studied by Boissière–Heckel–Sarti [BHS23, Section 3, Equation (3.2)] for the cyclic cubic fourfold branched over a cubic threefold with one singularity of type A_1. Here we put, following their notation, $h_1 = 0$ or, equivalently, we considered a rank 3 quadric given by $\{f_2 = 0\}$.

Remark 4.6. *Consider a cyclic cubic fourfold Y whose branch locus is a cubic threefold with one isolated singularity of type A_k and $k > 1$. The equation defining such cubic can be brought to the form of Equation* (4.5.2)*. Indeed, as noted in Section 3.2, an equation for Y can be brought to the form of Equation* (4.5.1)*. To see that the rank of f_2 for these fourfolds is three note that, in a chart containing the singular point, it is equal to the rank of the Hessian at the origin. Consequently, by drawing a comparison with the local analytic form of a singularity of type A_k (see [Arn72, Section 1]), specifically $x_1^{k+1} + x_2^2 + x_3^2 + x_4^2$, we see that this is three.*

Now, consider $F(Y)$. This is a singular variety with singular locus $F(Y, p)$ by [AK77, Corollary 1.11]. Moreover, the latter is isomorphic, by Theorem 3.4, to the singular $K3$ surface Σ. All the singularities of Σ are in the affine chart $x_1 \neq 0$, so in order to resolve its singularities we can do a local computation. The point $q_0 = (0 : 1 : 0 : 0 : 0 : 0) \in \mathrm{Sing}(\Sigma)$, so we call

l_0 the line $\overline{pq_0}$, i.e. the line corresponding to q_0 under the isomorphism $F(Y,p) \simeq \Sigma$. Now consider the Plücker embedding $\mathrm{Gr}(2,6) \hookrightarrow \mathbb{P}^{14}$, the Plücker relations yield that $\mathrm{Gr}(2,6)$ is locally given by eight complex coordinates. So, in an affine neighbourhood U of $[l_0]$ we choose Plücker coordinates $(p_{02},...,p_{05},p_{12},...,p_{15})$ characterizing the lines passing through the following points:

$$(1:0:-p_{12}:-p_{13}:-p_{14}:-p_{15}), \;\; (0:1:p_{02}:p_{03}:p_{04}:p_{05}).$$

Moreover, we put $(p_{i,2},p_{i,3},p_{i,4}) =: p_i$ for better readability.

On this chart a line in $\mathbb{P}^5$ is given by:

$$x_0 = \lambda, \; x_1 = \mu, \; x_5 = -\lambda p_{15} + \mu p_{05}, \; (x_2,x_3,x_4) = -\lambda p_1 + \mu p_0$$

with $(\lambda : \mu) \in \mathbb{P}^1$. Then to find equations for $F(Y)$ we can substitute these expressions in Equation (4.5.2) and extract the homogeneous components $\phi^{i,j}$ of degree (i,j) in (λ,μ). Then the equations become:

$$\begin{aligned}
\phi^{3,0} &= q_1(p_1) - k_2(p_1,p_{15})\\
\phi^{2,1} &= -2B_1(p_0,p_1) + q_2(p_1,p_{15}) + k_2^{2,1}((p_0,p_{05}),(p_1,p_{15}))\\
\phi^{1,2} &= q_1(p_0) - h_2(p_1,p_{15}) - 2B_2((p_0,p_{05}),(p_1,p_{15})) - k_2^{1,2}((p_0,p_{05}),(p_1,p_{15}))\\
\phi^{0,3} &= h_2(p_0,p_{05}) + q_2(p_0,p_{05}) + k_2(p_0,p_{05}).
\end{aligned}$$

Here we denoted with B_i the bilinear forms relative to q_i and with $k_2^{i,j}$ the form of weight (i,j) relative to k_2.

Proposition 4.7. *The blow-up map $\alpha : \mathrm{Bl}_\Sigma(F(Y)) \to F(Y)$ is a resolution of the indeterminacies of the rational map $\varphi^{-1} : F(Y) \to \mathrm{Hilb}^2(\Sigma)$ mentioned in Theorem 3.4 and $\mathrm{Bl}_\Sigma(F(Y)) \simeq \mathrm{Hilb}^2(\Sigma)$.*

Proof. The argument is the same of [BHS23, Theorem 3.1]. First, let us recall the definition that the map φ is a map which associates to any closed subscheme of length two ξ on Σ the residual line given by the intersection of the plane $\langle \xi, p\rangle$ and the cubic fourfold Y. We want to prove that the rational map $\alpha^{-1} \circ \varphi$ is a bijection and conclude that it is an isomorphism with Zariski's main theorem. To prove the bijection we will compute the fibres of α and φ to show that they are isomorphic.
First, note that we just need to check what happens over the singularities of Σ. Indeed, in order to compute the fibers over nonsingular points for the $K3$ surface we can reduce to the smooth case studied in [BHS23]. This is because, if we want to study the equation locally, in a neighbourhood of a line $l_{\bar{x}}$ corresponding to a point $\bar{x} := (0:\bar{x}_1:\bar{x}_2:\bar{x}_3:\bar{x}_4:\bar{x}_5) \in \mathbb{P}^5$, then we can perform a change of variable bringing $\bar{x}$ to the origin and do the same computations done in *loc.cit.*. See Section 5.4 for the explicit transformation.
Let us now compute the fibers of the morphism φ. Any plane Π_a containing the line l_0 corresponding to the origin cuts $x_0 = x_1 = 0$ in only one point of coordinates $(0:0:a_2:a_3:a_4:$

a_5) corresponding to $a := (a_2 : a_3 : a_4 : a_5) \in \mathbb{P}^3$. The intersection $\Pi_a \cap Y$ is a plane cubic in $\mathbb{P}^2$ of coordinates $(b_0 : b_1 : b_2)$ given by the equation $F(b_0 : b_1 : b_2 a) = 0$. The line l_0 has equation $b_2 = 0$ on this plane and the residual conic is given by the equation:

$$b_0 b_2 q_1(a_2, a_3, a_4) + b_1^2 h_2(a_2, a_3, a_4, a_5) + b_1 b_2 q_2(a_2, a_3, a_4, a_5) + b_2^2 k_2(a_2, a_3, a_4, a_5).$$

The fiber of φ over l_0 is given by those planes whose residual conic is the union of two lines through the singular point p. Thus it is isomorphic to the cone

$$\widetilde{C} := \{(a_2 : a_3 : a_4 : a_5) \in \mathbb{P}^3 \mid q_1(a_2, a_3, a_4) = 0\}.$$

Now, in order to compute the blow-up $\mathrm{Bl}_\Sigma(F(Y))$, we compute its local expression on the chart U. So, it is given locally as the closure of the image of the regular morphism:

$$\begin{array}{ccc} U \setminus (\Sigma \cap U) & \longrightarrow & U \times \mathbb{P}^3 \\ & & \\ ((p_0, p_{05}), (p_1, p_{15})) & \longmapsto & (((p_0, p_{05}), (p_1, p_{15})), (p_{12} : p_{13} : p_{14} : p_{15})). \end{array}$$

Denote with $a := (a_2 : a_3 : a_4 : a_5)$ the coordinates of the $\mathbb{P}^3$. Assuming $a_5 \neq 0$ put $a_5 = 1$ and the relations of the blow-up become

$$p_{1,i} = p_{15} a_j$$

for $j = 2, 3, 4$. Therefore, the equations of $\mathrm{Bl}_\Sigma(F(Y))$ on the local chart become:

$$\begin{aligned}
\widehat{\phi}^{3,0} &= q_1(a) - p_{15} k_2(a, 1) \\
\widehat{\phi}^{2,1} &= -2B_1(p_0, a) + p_{15}(q_2(a, 1) + k_2^{2,1}((p_0, p_{05}), (a, 1))) \\
\widehat{\phi}^{1,2} &= q_1(p_0) - p_{15}(h_2(a, 1) + 2B_2((p_0, p_{05}), (a, 1)) + k_2^{1,2}((p_0, p_{05}), (a, 1))) \\
\widehat{\phi}^{0,3} &= h_2(p_0, p_{05}) + q_2(p_0, p_{05}) + k_2(p_0, p_{05}).
\end{aligned}$$

The equation of the fiber under α of a line l_0 corresponding to the origin is found putting $p_0 = 0$ and $p_{15} = 0$. After homogeneization:

$$\alpha^{-1}(l_0) = \{(a_2 : a_3 : a_4 : a_5) \in \mathbb{P}^3 \mid q_1(a_2, a_3, a_4) = 0\} = \widetilde{C}.$$

Remember that there exist no planes contained in Y passing through its singular point by assumption. So, the blow-up map $\alpha : \mathrm{Bl}_\Sigma(F(Y)) \to F(Y)$ is a resolution of the indeterminacies of the rational map φ^{-1} since the coordinate a of a line l_0 selects one plane Π_a which cuts Y in three lines: l_0, l_1 and l_2. Then l_1 and l_2 (which are not necessarily distinct) define a closed subscheme of length two on Σ as seen in the proof of Theorem 3.4. Then, by Zariski's main theorem, $\mathrm{Bl}_\Sigma(F(Y)) \simeq \Sigma^{[2]}$. □

Consider now the symplectic resolution $\pi : \widehat{\Sigma} \to \Sigma$ which is just a sequence of blow-ups on the singular points. Recall that $\widehat{\Sigma}$ is a $K3$ surface and thus we consider its Hilbert square $\mathrm{Hilb}^2(\widehat{\Sigma})$ which is an IHS manifold. The map π induces a birational map $\pi^{[2]} : \mathrm{Hilb}^2(\widehat{\Sigma}) \dashrightarrow \mathrm{Hilb}^2(\Sigma)$ in the following way. To a generic closed subscheme $\xi \overset{\iota_\xi}{\hookrightarrow} \widehat{\Sigma}$ of length 2 it associates the scheme theoretic image of $\iota_\xi \circ \pi$.

Remark 4.8. *The inverse morphism* $(\pi^{[2]})^{-1}$ *restricts to an isomorphims on* $\mathrm{Hilb}^2(\Sigma) \setminus Z$ *where* $Z := \{\xi \in \mathrm{Hilb}^2(\Sigma) \mid \mathrm{supp}(\xi) \cap \mathrm{Sing}(\Sigma) \neq \emptyset\}$. *It is an easy computation to see that* Z *has dimension at most 2. Indeed, the singular locus of* Σ *consists of isolated points. Let* $r \in \mathrm{Sing}(\Sigma)$ *and* $\xi \in \mathrm{Hilb}^2(\Sigma)$ *be such that* $r \in \mathrm{supp}(\xi) \cap \Sigma$. *By the characterization of Example 2.18 we know that* ξ *can be identified either as* $r + t$ *with* $t \in \Sigma$ *and* $t \neq r$ *or* $r + v$ *with* $v \in \mathbb{P}(T_r(\Sigma))$. *In the first case we see that the set* $\{r + t \in \mathrm{Hilb}^2(\Sigma) \mid t \in \Sigma,\ t \neq r\}$ *is isomorphic to* $\Sigma \setminus \{r\}$. *In the second case,* $T_r(\Sigma) = T_r(Q) \cap T_r(K)$ *in particular* $T_r(\Sigma) \subset T_r(K)$ *and* $\dim(T_r(\Sigma)) \leq 3$ *as* K *is smooth, thus* $\dim(\mathbb{P}(T_r(\Sigma))) \leq 2$. *So* Z *is a finite union of closed subschemes of dimension at most two.*

Proposition 4.9. *If there exist a birational map* $f : X \dashrightarrow Y$ *between a normal, irreducible, projective variety* X *and an IHS manifold* Y *and a closed subscheme* $Z \subset X$ *of codimension at least 2 such that* $f_{|X \setminus Z} : X' := X \setminus Z \to Y' \subset Y$ *is an isomorphism, then* f *induces on* X *a symplectic form and all the singularities of the latter are symplectic.*

Proof. The argument is close to [Leh18, Theorem 3.6] and stated explicitly in [BHS23, Theorem 3.1], we write it here for the sake of completeness. From the fact that X' is isomorphic to an open subset of Y we deduce that the canonical bundle is trivial on X'. Then as X is a normal, irreducible variety and the codimension of Z is at least two, we obtain $K_X = 0$ as a Cartier divisor. We want to prove now that X has symplectic singularities using [Nam85, Theorem 6], i.e. we need to prove that X has rational Gorenstein singularities and the regular locus X^{reg} of X admits an everywhere non-degenerate holomorphic closed 2-form. Let W be the desingularization of an elimination of indeterminacies for f, so that the following diagram exists and commutes

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "f"] & & Y. \end{tikzcd}$$

Then $H^0(W, \mathcal{O}_W(K_W)) = H^0(W, q^*(\mathcal{O}_Y(K_Y)) = H^0(Y, \mathcal{O}_Y(K_Y)) = \mathbb{C}$ since Y is an IHS manifold, so K_W is effective. But $K_W - p^*K_X = K_W$ as we proved that K_X is trivial and thus the singularities of X are canonical. This implies that X has rational singularities by Elkik–Flenner theorem ([Rei87, Section 3, page 363]). It remains to prove that f induces an everywhere non-degenerate holomorphic closed 2-form on X^{reg}. Note that clearly $X^{\mathrm{reg}} \setminus Z \simeq X'$ as X' is isomorphic to a smooth open subset of Y. Since X' is isomorphic to an open subset of Y it admits a symplectic form inherited from that of Y'. Now, $H^0(X', \Omega^2_{X'}) \simeq H^0(X^{\mathrm{reg}}, \Omega^2_{X^{\mathrm{reg}}})$ as $\Omega^2_{X^{\mathrm{reg}}}$ is reflexive and by [Har80, Theorem 1.6] any reflexive sheaf is normal . So any symplectic 2-form on X' extends to the whole X^{reg} and it is still closed as X has rational singularities (see [KS21, Theorem 1.13]). Moreover, it is also non-degenerate otherwise it would degenerate along a divisor cutting also X'. Therefore by [Nam85, Theorem 6] we deduce that X has symplectic singularities. □

Corollary 4.10. *The varieties* $\mathrm{Hilb}^2(\Sigma)$ *and* $F(Y)$ *have symplectic singularities which admit a symplectic resolution.*

Proof. Note that both $(\pi^{[2]})^{-1}$ and $(\pi^{[2]})^{-1} \circ \varphi^{-1}$ respect the hypotheses of Proposition 4.9. The fact that they admit a symplectic resolution is proven in [Leh18, Corollary 5.6] and [Yam22, Proposition 3.5]. □

Remark 4.11. *The fact that both the varieties* $\mathrm{Hilb}^2(\Sigma)$ *and* $F(Y)$ *admit a symplectic resolution can be proven with a more general approach, see [KLSV18, Remark 5.4] for the details.*

Remark 4.12. *Note that the proof of Corollary 4.10 provides a birational map between* $\mathrm{Hilb}^2(\widehat{\Sigma_i})$ *and* $\widehat{\mathrm{Hilb}^2(\Sigma_i)}$ *and, thus, they have the same Picard group by Proposition 2.33.*

We will call $\widehat{\mathrm{Hilb}^2(\Sigma)}$ a symplectic resolution of $\mathrm{Hilb}^2(\Sigma)$ and, thus, of $F(Y)$ by Proposition 4.7. This does not imply that every symplectic resolution $\widehat{F(Y)}$ of $F(Y)$ is of this form. Indeed, a priori, it is not true that any symplectic resolution of a variety factors through its blow-up on the singular locus, but using the fact that $F(Y)$ is a four-dimensional variety we can prove the following lemma.

Lemma 4.13. *Every symplectic resolution* R *of* $F(Y)$ *factors through the blow-up* $\mathrm{Bl}_\Sigma(F(Y))$.

Proof. Consider the symplectic resolution:

$$\gamma : \widehat{\mathrm{Hilb}^2(\Sigma)} \to F(Y)$$

By [WW03, Theorem 1.1] the exceptional locus E of γ can be either a divisor or 2-dimensional. In the latter case $\dim(\gamma(E)) = 0$ by [WW03, Lemma 2.1]. Therefore, as the singular locus Σ of $F(Y)$ is a surface, the map γ contracts a divisor E into Σ. Moreover, as $\widehat{\mathrm{Hilb}^2(\Sigma)}$ is smooth the divisor E is Cartier and by the universal property of blow-up there exists a unique map $\gamma' : \widehat{\mathrm{Hilb}^2(\Sigma)} \to \mathrm{Bl}_\Sigma(F(Y))$ which factors γ through the blow-up. □

Thus, we proved that every symplectic resolution of $F(Y)$ is isomorphic to a symplectic resolution of $\mathrm{Hilb}^2(\Sigma)$. Nevertheless, in order to highlight the point of view which we are using we will also denote by $\widehat{F(Y)}$ a symplectic resolution of $F(Y)$.

5 Geometry of $F(Y)$

In this section we investigate some geometric properties of $F(Y)$ when Y is a cyclic fourfold branched along a cubic threefold C having one isolated singularity of type A_i for $i = 2, 3, 4$.

First, we want to study the nature of the singular points of $F(Y)$ on the 2-dimensional leaf, i.e.

$$\mathrm{Sing}(F(Y)) \setminus \mathrm{Sing}(\mathrm{Sing}(F(Y))) \simeq \Sigma \setminus \{\mathrm{Sing}(\Sigma)\}.$$

Proposition 4.14. *For every point of* $\Sigma \setminus \{\mathrm{Sing}(\Sigma)\} \subset F(Y)$ *there exists a neighbourhood of* $F(Y)$ *which is analytically isomorphic to* $(\mathbb{C}^2, 0) \times (\Gamma, t)$ *with* (Γ, t) *the germ of a point on a surface having an isolated singularity on t of type:*

i) D_4 if C has an isolated singularity of type A_2;

ii) E_6 if C has an isolated singularity of type A_3;

iii) E_8 if C has an isolated singularity of type A_4.

Proof. In order to do local computations we want to change the local chart given by Plücker coordinates given in Section 4.4 in a way that a point of the 2-dimensional leaf, which we can assume to be the line l_2 passing through $(1:0:0:0:0:0)$ and $(0:0:1:0:0:0)$, is the origin in the new chart. Therefore, we choose Plücker coordinates characterizing the lines passing through the points:

$$(1:-p_{11}:0:-p_{13}:-p_{14}:-p_{15}),\ (0:p_{01}:1:p_{03}:p_{04}:p_{05}).$$

Moreover, as we assumed that $l_2 \in F(Y)$, we obtain (with a slight abuse of notation) after a linear coordinate change $q_1(x_2,x_3,x_4) = x_2h_1(x_3,x_4) + q_1(x_3,x_4)$ and $K(x_1,\dots,x_5) = x_2^2h_3(x_1,x_3,x_4,x_5) + x_2q_3(x_1,x_3,x_4,x_5) + k_2(x_1,x_3,x_4,x_5)$. With computations analogous to Section 4.4 we get the following equations for $F(Y)$:

$$\begin{aligned}
\phi^{3,0} &= q_1(\bar{p_1}) - k_2(p_{11},\bar{p_1},p_{15})\\
\phi^{2,1} &= -2B_1(\bar{p_0},\bar{p_1}) + q_3(p_{11},\bar{p_1},p_{15}) + k_2^{2,1}(p_{01},\bar{p_0},p_{05},p_{11},\bar{p_1},p_{15}) - h_1(\bar{p_1})\\
\phi^{1,2} &= q_1(\bar{p_0}) + h_1(\bar{p_0}) - h_3(p_{11},\bar{p_1},p_{15}) - 2B_3(p_{01},\bar{p_0},p_{05},p_{11},\bar{p_1},p_{15}) + k_2^{1,2}(p_{01},\bar{p_0},p_{05},p_{11},\bar{p_1},p_{15})\\
\phi^{0,3} &= h_3(p_{01},\bar{p_0},p_{05}) + q_3(p_{01},\bar{p_0},p_{05}) + k_2(p_{01},\bar{p_0},p_{05}).
\end{aligned}$$

Here we put $\bar{p_i} = (p_{i,3},p_{i,4})$. In order to determine the nature of the singularity at the origin we use the same argument of [BHS23, Theorem 4.1 item (2)]. Note that in a neighbourhood of a nonsingular point of Σ the hypersurfaces $Q := \{Q=0\}$ and $K := \{K=0\}$ meet transversally, therefore h_1 and h_3 are not proportional. Hence we can suppose that after a linear change of coordinates $h_1(x_3,x_4) = x_3$ and $h_3(x_1,x_3,x_4,x_5) = x_4$. Therefore, we can use the equations $\phi^{1,2}$ and $\phi^{0,3}$ to obtain complex analytic local expressions $\widehat{p_{03}}$, $\widehat{p_{04}}$ for p_{03} and p_{04} in terms of $p_{01},p_{05},p_{11},p_{13},p_{14},p_{15}$. Thus, there exists a local biholomorphism in a neighbourhood of the origin between our variety and the variety $\chi \subset \mathbb{C}^2 \times \mathbb{C}^4$ described by the equations:

$$\begin{aligned}
\overline{\phi^{3,0}} &= q_1(\bar{p_1}) - k_2(p_{11},\bar{p_1},p_{15})\\
\overline{\phi^{2,1}} &= -2B_1(\widehat{p_{03}},\widehat{p_{04}},\bar{p_1}) + q_3(p_{11},\bar{p_1},p_{15}) + k_2^{2,1}(p_{01},\widehat{p_{03}},\widehat{p_{04}},p_{05},p_{11},\bar{p_1},p_{15}) - h_1(\bar{p_1}).
\end{aligned}$$

Here (p_{01},p_{05}) are local coordinates for Σ as the latter is given by $p_1 = 0$. By [Kal06, Theorem 2.3] (see also [LMP23, Proposition 2.2]) we know that every point in $\Sigma \setminus \mathrm{Sing}(\Sigma)$ has a neighbourhood which is locally analytically isomorphic to $(\mathbb{C}^2,0) \times (\Gamma,t)$ with (Γ,t) the germ of a smooth point or a rational double point on a surface. Therefore, as we want to understand the structure over the origin, we put $p_{01} = p_{05} = 0$. Thus we can consider the surface Γ given by the following equations in $\mathbb{C}^4$:

$$\begin{aligned}
\overline{\phi^{3,0}} &= q_1(\bar{p_1}) - k_2(p_{11},\bar{p_1},p_{15})\\
\overline{\phi^{2,1}} &= q_3(p_{11},\bar{p_1},p_{15}) - h_1(\bar{p_1}).
\end{aligned}$$

Again we can consider $h_1(\bar{p_1}) = p_{1,3}$ and use a local inversion of the second equation to obtain a local expression $\widehat{p_{13}}$ for p_{13} as a quadratic expression in p_{11}, p_{14}, p_{15}. Now to draw conclusions we need to specialize our manifold to our case. Let us begin with putting the condition of being a cyclic cubic fourfold, then an equation for Γ in $\mathbb{C}^3$ given by p_{11}, p_{14}, p_{15} becomes:

$$\overline{\phi^{3,0}} = q_1(\widehat{p_{13}}, p_{14}) - k_2(p_{11}, \widehat{p_{13}}, p_{14}) + p_{15}^3.$$

Then the following things can happen (see Section 5.5 for explicit computations):

i) p_{11}^3 appears in $k_2(p_{11}, \widehat{p_{13}}, p_{14})$: then the equation is semiquasihomogenous (SQH) of degree $(\frac{1}{3}, \frac{1}{2}, \frac{1}{3})$ thus it has a singularity of type D_4 at the origin. The condition on K is equivalent to ask an isolated singularity of type A_2 for the cubic threefold C.

ii) p_{11}^4 appears in $\overline{\phi^{3,0}}$ and p_{11}^3 does not appear in $k_2(p_{11}, \widehat{p_{13}}, p_{14})$: this monomial appears thanks to both q_1 and k_2, so if it is not eliminated then the equation is SQH of degree $(\frac{1}{4}, \frac{1}{2}, \frac{1}{3})$ yielding a singularity of type E_6. This condition is obtained when C has an isolated singularity of type A_3.

iii) p_{11}^5 appears in $k_2(p_{11}, \widehat{p_{13}}, p_{14})$, p_{11}^4 does not appear in $\overline{\phi^{3,0}}$ and p_{11}^3 does not appear in $k_2(p_{11}, \widehat{p_{13}}, p_{14})$: the equation is SQH of degree $(\frac{1}{5}, \frac{1}{2}, \frac{1}{3})$, yielding a singularity of type E_8. This condition is satisfied when C has an isolated singularity of type A_4.

The type of singularity of the germ $(\Gamma, 0)$ is, thus, determined by the above calculations. $\square$

As also noted in Chapter 3, the behaviour of $F(Y)$ when Y is branched along a cubic threefold of type A_2 is very different from the A_3 and A_4 cases, therefore, we will divide the study in two different sections.

5.1 Cubic fourfold branched along a threefold having an isolated singularity of type A_3 or A_4

We start by studying the geometry of $F(Y_i)$ in the case where Y_i is branched along a cubic threefold having one isolated singularity of type A_3 or A_4.

When a cubic threefold C has an isolated singularity of type, respectively, A_3 or A_4 then the cyclic cubic fourfold Y associated to C has an isolated singularity of type E_6 or E_8. By [DR01, Lemma 2.1] then Σ has an isolated singularity of type, respectively A_5 or E_7.

Proposition 4.15. *Let Σ be a surface with one isolated singularity q_0 of type ADE. Then*

$$\mathrm{Sing}(\mathrm{Hilb}^2(\Sigma)) \simeq \mathrm{Bl}_{q_0} \Sigma =: \widehat{\Sigma}.$$

Moreover, $\widehat{\mathrm{Hilb}^2(\Sigma)}$ is obtained by successive blow-ups along singular loci.

Proof. This is computation can be found in [Yam18, Section 2]. $\square$

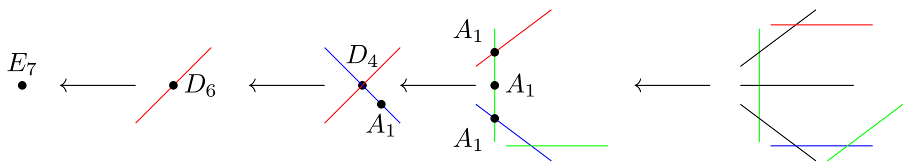
Consider now the covering automorphism σ defined on Y_i. This automorphism is just the identity on the first five coordinates and maps $x_5 \mapsto \xi_3 \cdot x_5$ with ξ_3 a third primitive root of the unity. It is a projectivity and, thus, maps lines to lines. Therefore it induces a linear automorphism on $F(Y_i)$. This is a linear automorphism mapping the singular locus to itself, therefore there exists only one automorphism on $\mathrm{Bl}_\Sigma F(Y_i)$ commuting with the blow-up morphism by the universal property of blow-ups. This automorphism corresponds via the isomorphism of Proposition 4.7 to the natural automorphism $\sigma_i^{[2]}$ induced on $\mathrm{Hilb}^2(\Sigma_i)$ by the action of σ on Σ_i (see Section 5.6 for the details). By [Yam18, Section 2] if Σ_i has an isolated singularity then $\widehat{\mathrm{Hilb}^2(\Sigma_i)}$ is obtained by a sequence of blow-ups along the successive singular loci, therefore we can iterate the above argument and induce an automorphism $\widehat{\sigma_i^{[2]}}$ on $\widehat{\mathrm{Hilb}^2(\Sigma_i)}$ (we will explain better this iterative argument in the proof of Proposition 4.16). In Chapter 3 we studied the manifolds obtained considering the Hilbert square $\mathrm{Hilb}^2(\widehat{\Sigma_i})$ and we proved that on these manifolds there exists a non-symplectic automorphism of order three whose action in cohomology is represented by $\rho_i \in O(L)$ (we use here the same notation).

Proposition 4.16. *The automorphism $\widehat{\sigma_i^{[2]}}$ induces ρ_i in cohomology, i.e. there exists a marking η_i on $\widehat{\mathrm{Hilb}^2(\Sigma_i)}$ such that $\rho_i = \eta_i^{-1} \circ (\widehat{\sigma_i^{[2]}})^* \circ \eta_i$.*

Proof. The only irreducible component E_1 of the exceptional divisor of the first blow-up α is clearly preserved by the induced automorphism. If C had one isolated singularity of, respectively, type A_3 or A_4 then Σ has, respectively:

- one isolated singularity of type A_5. Then by [Yam18, Section 2] $\widehat{\mathrm{Hilb}^2(\Sigma)}$ is obtained by $\mathrm{Hilb}^2(\Sigma)$ via 3 blow-ups, two of them introducing each one two irreducible components of the effective divisor and the third another one. At every blow-up, the induced automorphism maps the subgroup of the Picard group generated by the irreducible components of the exceptional divisor into itself because it commutes with the composition of blow-ups by the universal property of blow-ups. As the automorphism has order three it cannot swap two irreducible components, thus it preserves all the 6 irreducible components of the exceptional divisor introduced at each blow-up.

- One isolated singularity of type E_7. Then by [Yam18, Section 2] the successive blow-ups introduce on $\widehat{\mathrm{Hilb}^2(\Sigma)}$ 7 irreducible exceptional divisors in an E_7 configuration. The induced automorphism maps the subgroup of the Picard group generated by these divisors into itself as it commutes with the composition of blow-ups by the universal property of blow-ups. We recall here the picture of the blow-ups needed.



Each arrow represents a blow-up. Each line represents an irreducible component of the exceptional divisor which are coloured with a different colour at each blow-up. So, with the same argument of above we deduce that the irreducible components introduced at the first three blow-ups are preserved. Finally, the three irreducible components introduced by the last blow-up cannot be permuted as the intersection between different irreducible components needs to be preserved (as the first 4 irreducible components are preserved).

Therefore the invariant lattice has, respectively, rank at least 7 or 9. The Picard lattice is, isomorphic, respectively, to T_3 or T_4 of Theorem 3.1 which have, respectively, rank 7 or 9. By [BCS16, Corollary 5.7], the action of natural automorphisms on IHS manifolds of $K3^{[2]}$-type are uniquely determined by the action on the invariant lattice and the possibilities are listed in [BCS16, Table 1]. Thus, confronting all the possibilities, the invariant lattices must be isomorphic to the Picard lattices and the action of $\widehat{\sigma_i^{[2]}}$ induces ρ_i in cohomology. $\square$

Therefore, considered also the results of Section 4.4, we can state the following theorem.

Theorem 4.17. *Let C_i be a complex projective cubic threefold having one isolated singularity of type A_i for $i = 3, 4$ and let Y_i be its associated cyclic cubic fourfold. Assume that there exist no plane $\Pi \subset Y_i$ such that $\Pi \cap \mathrm{Sing}(Y_i) \neq \emptyset$. Then the Fano variety of lines $F(Y_i)$ of Y_i admits a unique symplectic resolution by an IHS manifold of $K3^{[2]}$-type $\widehat{F(Y_i)}$.*
Moreover, there exists an integral lattice T_i, defined below, such that:

i) $\mathrm{Pic}\left(\widehat{F(Y_i)}\right) \simeq T_i$;

ii) there exists a non-symplectic automorphism $\tau_i \in \mathrm{Aut}\left(\widehat{F(Y_i)}\right)$ whose invariant sublattice is $H^2(\widehat{F(Y_i)}, \mathbb{Z})^{\tau_i^*} \simeq T_i$

with T_i defined in the following table:

i	T_i
3	$\langle 6\rangle \oplus E_6$
4	$\langle 6\rangle \oplus E_8$

Proof. First, note that by [Yam18], the symplectic resolution $\widehat{\mathrm{Hilb}^2(\Sigma_i)}$ of $\mathrm{Hilb}^2(\Sigma_i)$ is unique. Therefore, by Lemma 4.13 the variety $F(Y_i)$ admits a unique symplectic resolution.
The Picard groups $\mathrm{Pic}\left(\widehat{F(Y_i)}\right)$ are isomorphic to those of $\mathrm{Hilb}^2(\widehat{\Sigma_i})$ as noted in Remark 4.12. The existence of the automorphism τ_i and its action in cohomology is determined by Proposition 4.16. The details on this automorphism and the Picard groups then follow from Theorem 3.1. $\square$

Moreover, we can say more about the geometry of $\widehat{F(Y)}$.

Proposition 4.18. *The variety $\mathcal{H}$, obtained by $F(Y)$ after a suitable number of successive blow-ups on the singular loci, has transversal ADE singularities, thus its blow-up on $\mathrm{Sing}(\mathcal{H})$ is a crepant resolution.*

Proof. In [Yam18] the author shows with his computations that if Σ has an isolated singularity of type T_n with T_n a singularity of type ADE then $\mathrm{Bl}_{\mathrm{Sing}(\mathrm{Hilb}^2(\Sigma))} \mathrm{Hilb}^2(\Sigma)$ has the same singularities of $\mathrm{Hilb}^2(\Gamma)$ with Γ a surface with an isolated singularity of type T'_m and $m < n$. In particular after a suitable number of blow-ups the variety $\mathcal{H}$, obtained by successive blow-ups along singular loci, will not have 0-dimensional symplectic leaves, i.e. $\mathrm{Sing}(\mathcal{H})$ will be smooth. So, by Proposition 4.14, the variety $\mathcal{H}$ has only transversal singularities. The blow-up is then a crepant resolution by [Per07, Proposition 4.2]. □

5.2 Cubic fourfold branched along a threefold having an isolated A_2

Now we want to focus to the case where C has an isolated singularity of type A_2 and thus Σ has three A_1 singularities, namely q_0, q_1 and q_2.

First, we want to describe the singular locus $\mathrm{Sing}(\mathrm{Hilb}^2(\Sigma))$.

Proposition 4.19. *Suppose that Σ has three A_1 singularities q_0, q_1 and q_2, then $\mathrm{Sing}(\mathrm{Hilb}^2(\Sigma))$ consists of three irreducible components $\widehat{\Sigma_i} \simeq \mathrm{Bl}_{q_i} \Sigma$. An irreducible component $\widehat{\Sigma_i}$ intersects the other two components in q_j, with $j \neq i$.*

Proof. Consider the Hilbert–Chow morphism $hc : \mathrm{Hilb}^2(\Sigma) \to \mathrm{Sym}^2(\Sigma)$. This morphism can be identified with the blow-up along the diagonal Δ, so

$$hc^{-1}(\mathrm{Sing}(\mathrm{Sym}^2(\Sigma)) \setminus \Delta) \subset \mathrm{Sing}(\mathrm{Hilb}^2(\Sigma)) \subset hc^{-1}(\mathrm{Sing}(\mathrm{Sym}^2(\Sigma))).$$

Now, $\mathrm{Sing}(\mathrm{Sym}^2(\Sigma)) \setminus \Delta$ consists of cycles where at least one point lies in $\mathrm{Sing}(\Sigma)$, therefore all the length two subschemes ξ such that their support consists of two different points on Σ and at least one of them is in $\mathrm{Sing}(\Sigma)$ are also singular points for $\mathrm{Hilb}^2(\Sigma)$. Therefore, the remaining points of $\mathrm{Hilb}^2(\Sigma)$ which might possibly be singular are those on the fibers over $2q_i$ for $i = 0, 1, 2$. These points are length 2 closed subschemes of Σ entirely supported on an isolated singularity of type A_1. Therefore, using the computations about $\mathrm{Hilb}^2(\Gamma)$ with Γ a surface having one isolated singularity of type A_i done in [Yam18, Section 2.1], we can see that $hc^{-1}(2q_i) \cap \mathrm{Sing}(\mathrm{Hilb}^2(\Sigma))$ is isomorphic to the exceptional divisor L_i of $\mathrm{Bl}_{q_i} \Sigma$. Thus, the embedding:

$$\begin{aligned} \Sigma &\to \mathrm{Sing}(\mathrm{Sym}^2(\Sigma)) \\ p &\mapsto p + q_i \end{aligned}$$

induces an embedding:

$$\begin{aligned} \mathrm{Bl}_{q_i} \Sigma &\to \mathrm{Sing}(\mathrm{Hilb}^2(\Sigma)) \\ p \notin L_i &\mapsto p + q_i \\ p \in L_i &\mapsto p. \end{aligned}$$

Finally, the point $q_i + q_j$ with $i \neq j$ is a point which is mapped through the two different isomorphisms to $\widehat{q_j} \in \widehat{\Sigma_i}$ and $\widehat{q_i} \in \widehat{\Sigma_j}$, where we denoted by $\widehat{q_k}$ the preimage of a point $q_k \in \Sigma$ under the blow-up $\mathrm{Bl}_{q_s} \Sigma \to \Sigma$ with $s \neq k$. □

Remark 4.20. *This proposition in particular implies that* $\mathrm{Sing}(\mathrm{Hilb}^2(\Sigma))$ *has three singular points, namely* q_0+q_1, q_0+q_2 *and* q_1+q_2. *Indeed, the three connected components* $\widehat{\Sigma_i} \simeq \mathrm{Bl}_{q_i}(\Sigma)$ *are smooth on the preimage of* q_i *under the blow-up as* q_i *is a singularity of type* A_1 *for* Σ.

Now, we want to prove that the symplectic resolution $\widehat{\mathrm{Hilb}^2(\Sigma)} \xrightarrow{\psi} \mathrm{Hilb}^2(\Sigma)$ is unique. Recall that, as noted in Section 4.4, $\widehat{\mathrm{Hilb}^2(\Sigma)}$ is obtained by $\mathrm{Hilb}^2(\widehat{\Sigma})$ via a sequence of Mukai flops $\mu : \mathrm{Hilb}^2(\widehat{\Sigma}) \dashrightarrow \widehat{\mathrm{Hilb}^2(\Sigma)}$ performed over the 2-dimensional fibers of $\mathrm{Hilb}^2(\widehat{\Sigma}) \to \mathrm{Sym}^2(\Sigma)$. Moreover, the birational map $\pi^{[2]}$ restricts to an isomorphism between $\mathrm{Hilb}^2(\widehat{\Sigma}) \setminus (\bigcup \Pi_i \cup \bigcup \Lambda_{ij})$ and $\mathrm{Hilb}^2(\Sigma) \setminus (\bigcup hc^{-1}(2q_i) \cup \bigcup hc^{-1}(q_i+q_j))$ with $i \neq j$. Here we denoted by $\Pi_i \simeq \mathbb{P}^2$ and $\Lambda_{ij} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ the subspaces of $\mathrm{Hilb}^2(\widehat{\Sigma})$ which are, respectively, $hc^{-1}(2L_i)$ and $hc^{-1}(L_i+L_j)$.

Proposition 4.21. *The symplectic resolution* $\widehat{\mathrm{Hilb}^2(\Sigma)}$ *of* $\mathrm{Hilb}^2(\Sigma)$ *is unique up to isomorphism.*

Proof. Here, we use again [WW03, Theorem 1.2] and we want to prove that the central fiber does not contain components isomorphic to $\mathbb{P}^2$. The central fiber is $\psi^{-1}(q_i+q_j)$. From the discussion above we see that $\psi^{-1}(q_i+q_j) \simeq \Lambda_{ij} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. □

In analogy with Section 5.1 we want to prove that $\widehat{\mathrm{Hilb}^2(\Sigma)}$ is obtained by $\mathrm{Hilb}^2(\Sigma)$ via successive blow-ups on the singular loci. Let $S := \mathrm{Sing}(\mathrm{Hilb}^2(\Sigma))$.

Lemma 4.22. *The blow-up map* $\beta : \mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma) \to \mathrm{Hilb}^2(\Sigma)$ *is crepant.*

Proof. As the canonical bundle of $\mathrm{Hilb}^2(\Sigma)$ is trivial the statement is equivalent to ask that $X := \mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma)$ has trivial canonical bundle. First, note that

$$\dim(\beta^{-1}(q_i+q_j)) = \mathrm{codim}(\beta^{-1}(q_i+q_j)) = 2$$

so if we prove that K_X is trivial on $X \setminus \bigcup \beta^{-1}(q_i+q_j)$ then as X is normal and irreducible we get $K_X = 0$. By Proposition 4.14 and Proposition 4.19, $\mathrm{Hilb}^2(\Sigma)$ at each point of $\widehat{\Sigma_i} \setminus q_i+q_j$ admits a local description as $\mathbb{C}^2 \times \Gamma$ with Γ a surface with a singularity of type A_1. Therefore, as shown in [Per07, Proposition 4.2], the blow-up is locally isomorphic to $\widehat{\Gamma} \times \mathbb{C}^2$ where $\widehat{\Gamma}$ denotes its blow-up which is crepant as $\widehat{\Gamma} \to \Gamma$ is so. □

We can now prove the following lemma.

Lemma 4.23. $\mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma)$ *has only symplectic singularities.*

Proof. Consider the symplectic resolution $\psi : \widehat{\mathrm{Hilb}^2(\Sigma)} \to \mathrm{Hilb}^2(\Sigma)$ and the composition of birational maps $f := \psi^{-1} \circ \beta : \mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma) \dashrightarrow \widehat{\mathrm{Hilb}^2(\Sigma)}$. As β is crepant the map f is defined and injective on a complement to a closed subset $Z \subset \mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma)$ of codimension $\mathrm{codim}(Z) \geq 2$ by [Kal01, Lemma 2.3 (i)]. Then $\mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma)$ has only symplectic singularities from Proposition 4.9. □

Remark 4.24. *By [Yam22, Proposition 3.5] we already knew also that* $\mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma)$ *has only symplectic singularities. Indeed, in the latter the author proves more: it admits a symplectic resolution.*

Indeed we can say more. As it turns out $\mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma)$ is smooth and it is the symplectic resolution of $\widehat{\mathrm{Hilb}^2(\Sigma)}$ also in this case.

Proposition 4.25. *The symplectic resolution* $\widehat{\mathrm{Hilb}^2(\Sigma)}$ *of* $\mathrm{Hilb}^2(\Sigma)$ *is isomorphic to* $\mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma)$.

Proof. Consider the symplectic resolution $\psi' : \mathcal{H}' \to \mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma)$. Then $\beta \circ \psi' : \mathcal{H}' \to \mathrm{Hilb}^2(\Sigma)$ is a symplectic resolution of $\mathrm{Hilb}^2(\Sigma)$ which by Proposition 4.21 is unique. Thus, $\mathcal{H}' \simeq \widehat{\mathrm{Hilb}^2(\Sigma)}$ is the unique symplectic resolution of $\mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma)$. The Picard group of $\widehat{\mathrm{Hilb}^2(\Sigma)}$ is isomorphic to the Picard group of $\mathrm{Hilb}^2(\widehat{\Sigma})$ as they are two symplectic resolutions of the same symplectic variety $\mathrm{Sym}^2(\Sigma)$. Moreover, by Section 3.6, it is $\mathrm{Pic}(\widehat{\mathrm{Hilb}^2(\Sigma)}) \simeq D_4(-1) \oplus \langle 6 \rangle$. Therefore, $\psi : \widehat{\mathrm{Hilb}^2(\Sigma)} \to \mathrm{Hilb}^2(\Sigma)$ has relative Picard number 3. By Proposition 4.19 we deduce that $\beta : \mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma) \to \mathrm{Hilb}^2(\Sigma)$ has at least relative Picard number 3, but the resolution ψ factors through β by Lemma 4.13, thus, the statement follows by confrontation of the relative Picard groups. Indeed, we proved that $\psi' : \widehat{\mathrm{Hilb}^2(\Sigma)} \to \mathrm{Bl}_S \mathrm{Hilb}^2(\Sigma)$ is a small symplectic contraction (see [WW03, Definition 2]) so by [WW03, Theorem 1.1] if it is not an isomorphism it can be either a sequence of Mukai flops or a contraction of some planes. Both cases are impossible as ψ factors through ψ' and in the 2-dimensional fibres of ψ there exist no planes. $\square$

Now, we are interested in the presence of an automorphism on $\widehat{\mathrm{Hilb}^2(\Sigma)}$. Indeed, with the same argument of Section 5.1, we can induce an automorphism $\widehat{\sigma^{[2]}}$ on $\widehat{\mathrm{Hilb}^2(\Sigma)}$. We are interested now in the action of $\widehat{\sigma^{[2]}}$ in cohomology. First note that by Proposition 3.23 the automorphism σ on Σ induces also a natural automorphism τ_2 on $\mathrm{Hilb}^2(\widehat{\Sigma})$, where by $\widehat{\Sigma}$ we denote the minimal resolution of Σ. As both $\sigma^{[2]}$ and τ_2 are natural automorphisms induced by σ we can see that $\pi^{[2]} \circ \tau_2 = \sigma^{[2]} \circ \pi^{[2]}$ (see Section 4.4 for the definition of $\pi^{[2]}$).

Proposition 4.26. *The automorphism* $\widehat{\sigma^{[2]}}$ *has the same action of* τ_2 *in cohomology.*

Proof. In order to prove this it is enough to show that the respective fixed loci are isomorphic, as by [BCS16, Corollary 7.5] the action of such automorphism is uniquely determined by the fixed locus. By the description of τ_2 made in Proposition 3.23 we can see that it maps L_i to L_{i+1} mod 3. Moreover, σ maps q_i to q_{i+1} mod 3. Therefore, $\mathrm{Fix}(\tau_2) \subset \mathrm{Hilb}^2(\widehat{\Sigma}) \setminus (\bigcup \Pi_i \cup \bigcup \Lambda_{ij})$ is mapped isomorphically through $\pi^{[2]}$ to $\mathrm{Fix}(\sigma^{[2]}) \subset \mathrm{Hilb}^2(\Sigma) \setminus (\bigcup hc^{-1}(2q_i) \cup \bigcup hc^{-1}(q_i + q_j))$. So we are left to prove that $\mathrm{Fix}(\sigma^{[2]}) \simeq \mathrm{Fix}(\widehat{\sigma^{[2]}})$. The automorphism $\widehat{\sigma^{[2]}}$ is defined as the only automorphism such that $\beta \circ \widehat{\sigma^{[2]}} = \sigma^{[2]} \circ \beta$. Then as σ maps q_i to q_{i+1} mod 3 it is immediate to see that $\sigma^{[2]}$ maps the irreducible component in the singular locus $\widehat{\Sigma_i}$ to $\widehat{\Sigma_{i+1}}$ mod 3. So $\mathrm{Fix}(\sigma^{[2]}) \simeq \mathrm{Fix}(\widehat{\sigma^{[2]}})$. $\square$

Then we can state the following theorem.

Theorem 4.27. *Let C_2 be a complex projective cubic threefold having one isolated singularity of type A_2 and let Y_2 be its associated cyclic cubic fourfold. Assume that there exists no plane*

$\Pi \subset Y$ *such that* $\Pi \cap \mathrm{Sing}(Y_i) \neq \emptyset$*. Then the Fano variety of lines* $F(Y_2)$ *of* Y_2 *admits a unique symplectic resolution by an IHS manifold of* $K3^{[2]}$*-type* $\widehat{F(Y_2)}$*.*
Moreover, there exists an integral lattice T_i*, defined below, such that:*

i) $\mathrm{Pic}\left(\widehat{F(Y_i)}\right) \simeq \langle 6\rangle \oplus D_4(-1)$*;*

ii) there exists a non-symplectic automorphism $\tau_2 \in \mathrm{Aut}\left(\widehat{F(Y_i)}\right)$ *whose invariant sublattice is* $H^2(\widehat{F(Y_i)}, \mathbb{Z})^{\tau_2^*} \simeq \langle 6\rangle \oplus A_2(-1)$*.*

Proof. By Proposition 4.21 and Proposition 4.26 we obtain the unicity of the symplectic resolution and the induction of the automorphism. The explicit expression of the Picard lattice and the invariant lattice follow from the description of $\mathrm{Hilb}^2(\widehat{\Sigma})$, made in Section 3.6. □

Putting together Theorem 4.27 and Theorem 4.17 we obtain Theorem 4.1.

6 Final considerations

In this section we draw some considerations on the results obtained in this chapter in the context of nodal degenerations of cubic threefolds studied in Chapter 3. This interpretation is linked to the issue brought up by [BHS23, Section 4.2] in the generic nodal case.

Consider a one parameter family $\{C_t\}_{t\neq 0}$ of smooth cubic threefolds degenerating to a nodal cubic threefold C_0. Then, consider the family of cyclic cubic fourfolds $\{Y_t\}$ where each Y_t is branched along C_t and the family of their associated Fano varieties of lines $\{F(Y_t)\}$. On each element of the family $F(Y_t)$ there exists a non-symplectic automorphism σ_t of order 3 naturally induced by the covering automorphism. Moreover, in [BCS19b, Section 3] the authors showed that for $t \neq 0$ the pair $(F(Y_t), \sigma_t)$ endowed with a properly defined marking (we recalled the details in Section 2.2) live in the moduli space $\mathcal{M}^{\rho,\zeta}_{\langle 6\rangle}$ of $(\rho, \langle 6\rangle)$-polarized IHS manifolds of $K3^{[2]}$-type. The period map $\mathcal{P}^{\rho,\zeta}_{\langle 6\rangle}$ is then surjective on the complement of the nodal hyperplane arrangement $\mathcal{H}$.

Suppose now that C_0 has one isolated singularity of type A_i for $i = 1, \dots, 4$. Then

$$\lim_{t\to 0} \mathcal{P}^{\rho,\zeta}_{\langle 6\rangle}((F(Y_t), \sigma_t) = \omega_0 \in \mathcal{H}.$$

In Theorem 3.1 we proved that the choice of the manifold $\widehat{\Sigma}^{[2]} = \mathrm{Hilb}^2(\widehat{\Sigma})$ over the period ω_0 with the automorphism $\hat{\tau}_i^{[2]}$ extends holomorphically the period map $\mathcal{P}^{\rho,\zeta}_{\langle 6\rangle}$ over the subloci $\Delta_3^{A_i}$. This choice was motivated by the analogy with the work of [BCS19b] but, as proven in Theorem 4.1, it is not the only possible choice. Indeed, the pairs $(\widehat{\Sigma}^{[2]}, \hat{\tau}^{[2]})$ and $(\widehat{F(Y_0)}, \widehat{\sigma_0})$ are equivariantly birational and, thus, if not isomorphic, they are non-separated points in $\mathcal{M}^{\rho_i,\zeta}_{T_i}$. In this case they just correspond to two different choices of Kähler chambers. We leave the question on the isomorphism between the two birational models open.

Chapter 5

Computations

“Le vent se lève! ... il faut tenter de vivre!”

- Paul Valéry, *Le Cimetière marin*

In this chapter we write down the computations which we mentioned in the previous chapters of the thesis.

1 Dimension of moduli spaces

In this section we compute the dimensions of the moduli spaces of families which are complete intersections in $\mathbb{P}^4$ of a quadric hypersurface of rank 3 or 4 and a cyclic cubic threefold. Using the generalized Morse lemma and the Recognition principle as done e.g. in [Hec20] one can arrive to a generic form for the families we are interested in and count the free parameters. But the computations are long so we will use the following result which is a direct application of the Generalised Morse Lemma (see [GLS07, I, Theorem 2.47] for a possible reference) and the Recognition Principle [BW79, Lemma 1]. As this theorem appears on a PhD dissertation which has not been published at the day we are writing this article we include here its proof.

Theorem 5.1 ([Hec20, Theorem 1.15]). *Let $Y \subset \mathbb{A}^n_{\mathbb{C}}$ be a hypersurface defined by a polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ and assume that the origin is an isolated singular point of Y of corank one. Then, there exist polynomials $C_1, \ldots, C_{k+1}$ in the coefficients of P and depending on the choice of an analytic coordinate change such that the conditions*

$$C_1 = \cdots = C_k = 0,\ C_{k+1} \neq 0$$

on the coefficients of P are equivalent to $(Y, 0)$ being of type A_k. Moreover, each C_i is homogeneous of degree $i - 2$ and fixing the analytic coordinate change they depend on, there is an explicit algorithm computing them.

Proof. Let $k \in \mathbb{N}$. Using the generalized Morse lemma we suppose that, after a suitable analytic coordinates change, P has the form:

$$P(x) = x_1^2 + \cdots + x_{n-1}^2 + P_3(x_n) + \cdots + P_{k+1}(x_n) + \sum_{i=1}^{n-1} x_i Q_i(x_1, \ldots, x_n)$$

where each P_i is a polynomial of degree i and each Q_i of degree k. In order to apply the recognition Principle we take the weight $\alpha(A_k) = \left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{k+1}\right)$ and note that the terms of degree $\alpha(A_k) < 1$ are $P_3(x_n) + \dots + P_k(x_n)$, the terms of degree $\alpha(A_k) = 1$ are $x_1^2 + \dots + x_{n-1}^2 + P_{k+1}(x_n)$ and the terms of degree $\alpha(A_k) > 1$ are $\sum_{i=1}^{n-1} x_i Q_i(x_1, \dots, x_n)$. Therefore we write $C_i x_n = P_i(x_n)$ and conclude using the recognition Principle. $\square$

This will lead us to prove the following proposition.

Proposition 5.2. *The dimension of the family $\mathcal{K}_{A_i}$ associated to the cubic threefold having one A_i singularity (and thus of the subfamily of cubic threefolds having an isolated singularity of type A_i) for $i = 1, \dots, 4$ is $10 - i$. A generic element Σ_{A_i} in $\mathcal{K}_{A_i}$ is such that rk$(\mathrm{Pic}(\Sigma_{A_i})) = 2i$*

Proof. We start from the most general case which is the A_1 case. Note that this is the only corank 0 case so the theorem does not apply in this case. The equations are given by

$$\begin{cases} f_2(x_1, x_2, x_3, x_4) = 0 \\ f_3(x_1, x_2, x_3, x_4) + a x_5^3 = 0. \end{cases} \tag{5.2.1}$$

So we have $\binom{3+2}{2} = 10$ parameters for the quadric and $\binom{3+3}{3} + 1 = 21$ for the cubic. Then we have to impose 4 conditions because if two cubic hypersurfaces differ by a multiple of the quadric they yield the same intersection. As every equation is defined up to a constant the parameters are $10+21-4-1-1 = 25$. Then we have to consider the projective transformations which preserve the family, as projectivities are up to a constant are $4 \cdot 4 + 1 - 1 = 16$. Finally, the dimension of this family is $25 - 16 = 9$. Now, we consider the family $\mathcal{K}_{A_i}$ associated to the cubic threefold having one A_i singularity for $i \geq 2$. In these cases we have the same parameters and projective transformations as before but we need to add 1 condition for being a corank 1 singularity (this is equivalent to ask that $f_2 = 0$ has rank 3 as a quadric) and $i - 2$ conditions coming from Theorem 5.1. Therefore the dimension of the moduli space of the family of $(2, 3)$-complete intersections in $\mathbb{P}^4$ associated to a cubic threefold having a singularity of type A_i is $10 - i$. Then if we take a generic element Σ_{A_i} in $\mathcal{K}_{A_i}$ then by [AST11, Section 9] we obtain $\mathrm{rk}(\mathrm{Pic}(\Sigma_{A_i})) = 22 - 2(10 - i + 1) = 2i$. $\square$

2 An easy exercise

Here we outline the execution of the exercise mentioned in Remark 3.36.
Let L be the $K3^{[2]}$ lattice. Then $D_L \simeq \mathbb{Z}/2\mathbb{Z}$ with finite quadratic form $q_L = \langle \frac{3}{2} \rangle$. Moreover let $T = U(3) \oplus \langle -2 \rangle$ and $M = U \oplus A_2(2) \oplus \langle -2 \rangle$. Clearly, $q_T \simeq q_L \oplus q_{U(3)}$ and $q_M \simeq q_L \oplus q_{A_2(2)}$, so given [Nik80, Proposition 1.15.1], recalled in Theorem 1.11, the only possibilities for the respective orthogonal complements for T in L are the following:

- the genus of the lattice with signature $(2, 18)$ and discriminant form $q_T(-1) \oplus q_L$ is non-empty. Using the notation of Conway–Sloane ([CS99]) this is $II_{(2,18)} 2_I^{+2} 3^{-2}$. There exists only one class of isomorphism represented by

$$U \oplus U(3) \oplus E_7 \oplus E_8 \oplus \langle -2 \rangle.$$

- the genus of the lattice with signature $(2,18)$ and discriminant form $q_{U(3)}(-1)$ is non-empty. Using the notation of Conway–Sloane ([CS99]) this is $II_{(2,18)}3^{-2}$. There exists only one class of isomorphism represented by
$$U \oplus U(3) \oplus E_8^{\oplus 2}.$$

Analogously for M:

- the genus of the lattice with signature $(2,16)$ and discriminant form $q_M(-1) \oplus q_L$ is non-empty. Using the notation of Conway–Sloane ([CS99]) this is $II_{(2,16)}2_I^{-4}3^{+1}$. There exists only one class of isomorphism represented by
$$U^{\oplus 2} \oplus E_8 \oplus D_4 \oplus \langle -6\rangle \oplus \langle -2\rangle.$$
- the genus of the lattice with signature $(2,16)$ and discriminant form $q_{A_2(2)}(-1)$ is non-empty. Using the notation of Conway–Sloane ([CS99]) this is $II_{(2,16)}2_{II}^{-2}3^{+1}$. There exists only one class of isomorphism represented by
$$U^{\oplus 2} \oplus E_8 \oplus D_4 \oplus A_2.$$

3 Planes through the singular point

Here we write the explicit computation of the equations needed to define a generic plane through the singular point $p = (1 : 0 : ... : 0)$ contained in the cubic fourfold Y of equation
$$F(x_0, \dots, x_5) = x_0 Q(x_1, \dots, x_5) + K(x_1, \dots, x_5) = 0.$$
Let $(a_0 : a)$ and $(b_0 : b)$ with $a, b \in \mathbb{P}^4$ be two points of Y. Then the plane Π passing through these points have equation
$$(\lambda + \mu a_0 + \nu b_0 : \mu a + \nu b) \quad (\lambda : \mu : \nu) \in \mathbb{P}^2. \tag{5.2.2}$$
Imposing the condition of being in Y we get:
$$\begin{aligned}&(\lambda + \mu a_0 + \nu b_0) Q(\mu a + \nu b) + K(\mu a + \nu b) = \\ &= (\lambda + \mu a_0 + \nu b_0)(\mu^2 Q(a) + 2\mu\nu B(a,b) + \nu^2 Q(b)) + \\ &+ \mu^3 K(a) + \mu^2 \nu K^{2,1}(a,b) + \mu\nu^2 K^{1,2}(a,b) + \nu^3 K(b) = 0\end{aligned}$$
using the same notations of Section 4.4. In order for this to be identically zero all the coefficients of the different homogeneous components have to be trivial. Therefore:
$$\begin{cases} Q(a) = 0 \\ B(a,b) = 0 \\ Q(b) = 0 \\ K(a) = 0 \\ K^{2,1}(a,b) = 0 \\ K^{1,2}(a,b) = 0 \\ K(b) = 0. \end{cases}$$

Now fix b such that $Q(b) = K(b) = 0$, the equations imply that $a = (a_1, ..., a_5)$ resolves five equations. Recall now that we want the plane Π to be non-degenerate so the points p, a and b are distinct. In particular, $a, b \in \mathbb{C}^5 \setminus \{0\}$. This shows that if Q and K are sufficiently generic the system has not solutions.

4 Translation of the equation

In this section we write the explicit translation mentioned in the proof of Proposition 4.7.

Keeping the notation of Section 4.4, note that at least one coordinate $\bar{x}_i \neq 0$, for simplicity suppose $\bar{x}_1 = 1$. Then the translation to bring $\bar{x}$ to $(0:1:0:0:0:0)$ is $t_i = x_i - \bar{x}_i x_1$ when $i \neq 0, 1$ and $t_i = x_i$ in the other cases. So we write the Equation (4.5.2) in the following way:

$$F(t_0, \dots, t_5) = t_0 Q(t_2 + \bar{x}_2, t_3 + \bar{x}_3, t_4 + \bar{x}_4) + K(t_1, t_2 + \bar{x}_2, \dots, t_5 + \bar{x}_5). \tag{5.2.3}$$

Now, remember that $(0:1:\bar{x}_2:\cdots:\bar{x}_5)$ satisfies $Q(\bar{x}_2, \bar{x}_3, \bar{x}_4) = K(1, \bar{x}_2, \dots, \bar{x}_5) = 0$ as $\bar{x} \in \Sigma$. So, we can write

$$\begin{aligned} Q(t_2 + \bar{x}_2, t_3 + \bar{x}_3, t_4 + \bar{x}_4) &= q_1(t_2 + \bar{x}_2, t_3 + \bar{x}_3, t_4 + \bar{x}_4) = q_1(t_2, t_3, t_4) + 2t_1 B_1(t_i, \bar{x}_j) \\ &= q_1(t_2, t_3, t_4) + t_1 h_1(t_2, t_3, t_4) \end{aligned}$$

and

$$\begin{aligned} K(t_1, t_2 + \bar{x}_2, \dots, t_5 + \bar{x}_5) &= t_1^2 h_2(t_2 + \bar{x}_2, \dots, t_5 + \bar{x}_5) + t_1 q_2(t_2 + \bar{x}_2, \dots, t_5 + \bar{x}_5) + \\ &+ k_2(t_2 + \bar{x}_2, \dots, t_5 + \bar{x}_5) = \\ &= t_1^2(h_2(t_2, \dots, t_5) + k^{1,2}(t_i, \bar{x}_j)) + t_1(q_2(t_2, \dots, t_5) + \\ &+ 2B_2 t_1(t_i, \bar{x}_j) + k^{2,1}(t_i, \bar{x}_j)) + \\ &+ k_2(t_2, \dots, t_5) = \\ &= t_1^2 \widetilde{h}_2(t_2, \dots, t_5) + t_1 \widetilde{q}_2(t_2, \dots, t_5) + k_2(t_2, \dots, t_5). \end{aligned}$$

If we substitute the expressions of Q and K in Equation (5.2.3) we see that it has the same form of [BHS23, Equation (3.2)].

5 Computations in Proposition 4.14

There are many ways to find explicit equations for a generic cubic threefold with one singularity of type A_i for $i = 2, 3, 4$. A very interesting approach is given by Heckel in his Ph.D. thesis [Hec20, Section 1]. In *loc. cit.*, the author writes an explicit algorithm using the Recognition Principle [BW79, Lemma 1] and the Generalized Morse Lemma [GLS07, I, Theorem 2.47]. This approach has the disadvantage of being too computational heavy in relation to the results we need in Proposition 4.14, so we propose here another one.

Keeping the notation of Proposition 4.14 we want to prove the following proposition.

Proposition 5.3. *Suppose that a cubic threefold C_i defined by the equation:*

$$\begin{aligned} F &= x_0 q_1(x_2,x_3,x_4) + K(x_1,\dots,x_4) = \\ &= x_0(x_2 h_1(x_3,x_4) + q_1(x_3,x_4)) + x_2^2 h_3(x_1,x_3,x_4) + x_2 q_3(x_1,x_3,x_4) + \\ &+ k_2(x_1,x_3,x_4) = 0. \end{aligned}$$

has one isolated singularity of type A_i for $i = 2,3,4$. Then the surface Γ locally defined by

$$\overline{\phi^{3,0}} = q_1(\widetilde{p_{13}}, p_{14}) - k_2(p_{1,1}, \widetilde{p_{13}}, p_{14}) + p_{15}^3$$

has a singularity in the origin of type $\boldsymbol{T_i}$ as defined in the following table:

i	$\boldsymbol{T_i}$
2	D_4
3	E_6
4	E_8

Proof. We consider the different cases:

- i=2. In this case the polynomial $F(1,x_1,\dots,x_4)$ is SQH of degree $(\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ if and only if the coefficient of x_1^3 is non-trivial. As noted in the proof of Proposition 4.14 this implies that Γ has a D_4 singularity in the origin.
- i=3. Clearly, the coefficient of x_1^3 is trivial otherwise we would be in the case $i = 2$. Indeed, in this case the polynomial $F(1,x_1,\dots,x_4)$ is SQH of degree $(\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2})$. So, x_1^2 must be multiplied by a non-trivial linear form in x_2,x_3,x_4. Remember that $\widetilde{p_{13}}$ is a local expression for p_{13} obtained substituting it with a local expression of $q_3(p_{1,1},p_{13},p_{14}) = p_{13}$. Therefore, in $\overline{\phi^{3,0}}$ appears, non-trivially, either the term $p_{1,1}^4$ (if in q_3 depends from x_1) or a term in $p_{1,1}^2 p_{1,4}$. In both cases $\overline{\phi^{3,0}}$ is SQH of degree $(\frac{1}{4},\frac{1}{2},\frac{1}{3})$, thus Γ has an E_6 singularity.
- i=4. Clearly we need to exclude the above cases. In this case the polynomial $F(1,x_1,\dots,x_4)$ is SQH of degree $(\frac{1}{5},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and there exist no quadratic term in x_1. In particular, $q_3(p_{1,1},p_{13},p_{14}) = q_3(p_{13},p_{14})$. So, $\overline{\phi^{3,0}} = q_1(\widetilde{p_{13}}(p_{14}),p_{14}) - k_2(p_{1,1},\widetilde{p_{13}}(p_{14}),p_{14}) + p_{15}^3$. Therefore, if $F(1,x_1,\dots,x_4)$ is SQH with weight $(\frac{1}{5})$ with respect to x_1 then the same is true for $\overline{\phi^{3,0}}$, implying that it is SQH of degree $(\frac{1}{5},\frac{1}{2},\frac{1}{3})$, thus Γ has an E_8 singularity.

□

6 The action of the automorphism on $\mathrm{Bl}_{\Sigma_i}(F(Y_i))$

In this section we give explain better the induction of the automorphism on $\mathrm{Bl}_{\Sigma_i}(F(Y_i))$.

First, by the universal property of blow-ups there exists only one automorphism τ on $\mathrm{Bl}_{\Sigma_i}(F(Y_i))$ commuting with the blow-up morphism α. Consider the following diagram:

$$\begin{tikzcd} \mathrm{Hilb}^2(\Sigma_i) \arrow[r, "\mu"] \arrow[dr, "\varphi"'] & \mathrm{Bl}_{\Sigma_i}(F(Y_i)) \arrow[d, "\alpha"] \\ & F(Y_i). \end{tikzcd}$$

Remember that μ is an isomorphism. Moreover, note that $\mu \circ \sigma^{[2]} \circ \mu^{-1} \in \mathrm{Aut}(\mathrm{Bl}_{\Sigma_i}(F(Y_i)))$, so if we show that this automorphism commutes with α we obtain that it is τ. In order to show this we prove the following lemma.

Lemma 5.4. *The automorphisms $\sigma^{[2]}$ and σ are φ-equivariant, i.e. $\varphi \circ \sigma^{[2]} = \sigma \circ \varphi$.*

Proof. This is a straight-forward computation. Keeping the notation of Chapter 4, we assume that the singular point $p \in Y$ has coordinates $(1 : 0 : \ldots : 0)$. Moreover, we assume that $p_1 := (0 : P_{11} : \ldots : P_{15})$ and $p_2 := (0 : P_{21} : \ldots : P_{25})$ are two distinct points of Σ_i. Therefore, $\varphi(\sigma^{[2]}(p_1 + p_2)) = \varphi(\sigma(p_1) + \sigma(p_2)) = l_{\sigma(p_1)\sigma(p_2)}$ with $l_{\sigma(p_1)\sigma(p_2)}$ the residual line of the intersection of the plane $\Pi := \langle p, \sigma(p_1), \sigma(p_2) \rangle$ with Y_i. As the action of σ is the identity on the first five coordinates we obtain that $\Pi = \sigma(\Pi')$ with $\Pi' = \langle p, p_1, p_2 \rangle$, thus $\varphi \circ \sigma^{[2]}(p_1 + p_2) = \sigma \circ \varphi(p_1 + p_2)$. Now it remains to check the equivariance on the pairs $p_1 + p_2$ where $p_1 \in \Sigma_i$ and $p_2 \in \mathbb{P}(T_{p_1}(\Sigma_i))$ (remember the characterization introduced in Example 2.18). Then, $\sigma^{[2]}(p_1 + p_2) = \sigma(p_1) + d_\sigma(p_2)$ with d_σ the differential of σ at the point p_1. As the map σ is linear its differential has the same action of σ component-wise. Thus the same computation of the previous part show that the automorphisms $\sigma^{[2]}$ and σ are φ-equivariant. $\square$

Now it is easy to show that $\mu \circ \sigma^{[2]} \circ \mu^{-1}$ commutes with α as:

$$\alpha \circ \mu \circ \sigma^{[2]} \circ \mu^{-1} = \varphi \circ \sigma^{[2]} \circ \mu^{-1} = \sigma \circ \varphi \circ \mu^{-1} = \sigma \circ \alpha.$$

Ringraziamenti

In this page, which is the most personal of this document, I would like to use my favourite language: Italian. So, if you see yourself mentioned, please ask somebody or some program to translate it, although probably everyone I will mention already knows the Italian needed to understand.

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