Gaussian Quadrature for Non-Gaussian Distributions

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Abstract. Many problems of operations research or decision science involve continuous probability distributions, whose handling may be sometimes unmanageable; in order to tackle this issue, different forms of approximation methods can be used. When constructing a k-point discrete approximation of a continuous random variable, moment matching, i.e., matching as many moments as possible of the original distribution, is the most popular technique. This can be done by resorting to the so-called Gaussian quadrature procedure (originally developed by Gauss in the nineteenth century) and solving for the roots of an orthogonal polynomial or for the eigenvalues of a real symmetric tridiagonal matrix. The moment-matching discretization has been widely applied to the Gaussian distribution and more generally to symmetric distributions, for which the procedure considerably simplifies. Despite the name, Gaussian quadrature can be theoretically applied to any continuous distribution (as far as the first $2k-1$ raw moments exist), but not much interest has been shown in the literature so far. In this work, we will consider some examples of asymmetric distributions defined over the positive real line (namely, the gamma and the Weibull, for which expressions for the integer moments are available in closed form) and show how the moment-matching procedure works and its possible practical issues. Comparison with an alternative discretization technique is discussed.
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INTRODUCTION

Many problems of operations research or decision science involve quantitities that can be modelled by continuous probability distributions. However, their handling may be sometimes unmanageable; in this case, some form of approximation is used and an available solution is the approximation-by-discretization: each continuous random variable is substituted by a proper discrete random variable [1]. How to construct this discrete approximation is a question that can lead to several answers, depending on which criterion one adopts to "measure" the discrepancy between the original continuous distribution and its discrete counterpart. Moment matching, i.e., matching as many moments as possible of the original distribution, is the most popular technique. It is well-known that a k-point discrete random variable, which is characterized by $2k$ values, the k support points $(x_1, x_2, \ldots, x_k)$ and the corresponding k probabilities $(p_1, p_2, \ldots, p_k)$, can preserve up to the first $2k-1$ moments of the continuous random variable. The values of the x_i and p_i which lead to the exact matching of all the first $2k-1$ moments can be obtained by solving the corresponding non-linear system of $2k$ equations in the $2k$ unknowns:

$$\sum_{i=1}^{k} p_i \cdot x_i^r = m_r, \quad r = 0, 1, \ldots, 2k-1, \tag{1}$$

where m_r is the r-th raw moment of the continuous random variable, and for $r = 0$ we have the trivial requirement that the p_i must sum up to 1. This can be done by resorting to the so-called Gaussian quadrature procedure (originally developed by Gauss in the nineteenth century) and solving for the roots of an orthogonal polynomial [2] or computing the eigenvalues and first component of the orthornormalized eigenvectors of a symmetric tridiagonal matrix [3]. The moment-matching discretization has been widely used with the Gaussian distribution and more generally with symmetric distributions, for which the procedure considerably simplifies (see for example [4] for an application to complex stress-strength models). Despite the name, Gaussian quadrature can be theoretically applied to any continuous distribution (having the first $2k-1$ moments finite), but not much interest has been shown in the literature so far. In this work, we will consider some examples of asymmetric distributions and show possible practical issues of the moment-matching procedure. The rest of the paper is structured as follows: in the next section, we briefly recall the matching-moment procedure based on Gaussian quadrature; then, after presenting the well-known case of normal distributions, we consider, along with the Gaussian, two asymmetrical distributions (gamma and Weibull, defined over the positive real line, for which expressions for the integer moments are available in closed form) and illustrate how the procedure works through some numerical examples, by considering different values of k, also warning against some computational issues. The third section shortly reviews an alternative discretization method that overcomes the theoretical and practical pitfalls of the moment-matching procedure. Some conclusion are drawn in the last section.

MOMENT-MATCHING FOR GAUSSIAN AND NON-GAUSSIAN DISTRIBUTIONS

System (1) can be solved by resorting to a "standard" procedure, which consists of i) computing the k points x_i as the zeros of the k-degree polynomial $\pi(x) = \prod_{i=1}^{k}(x - x_i) = \sum_{j=0}^{k} C_j x^j$, by first determining the coefficients C_j through proper rearrangements of the equations in (1); and ii) finding the k probabilities p_i, by using the first k equations of (1) [5, chap.10]. System (1) can be more efficiently solved by resorting to Golub and Welsch method [3], which reduces to solving for the eigenvalues/eigenvectors of a sparse matrix (see also [6] for a detailed description of the method). The two resolution techniques are equivalent and strictly interconnected [7]. It can be shown that the Gaussian quadrature always yields k real, distinct values x_i, which all lie in the interval spanned by the continuous random variable; it also produces positive probabilities p_i. [6] provides Matlab® code for constructing a k-point discrete random distribution matching as many as possible moments of an assigned sample of numerical values. This code has been translated into R code and has been easily adapted for working with continuous random distributions belonging to known parametric families, provided that analytic expressions are available for their first $2k-1$ finite moments [8].

For the standard normal distribution $Z \sim \mathcal{N}(0, 1)$, we have that

$$m_r = \begin{cases} 0 & \text{if } r \text{ is odd} \\ (r-1)!! & \text{if } r \text{ is even} \end{cases}$$

with $r!!$ denoting the double factorial, i.e., the product of all numbers from r to 1 that have the same parity as r. We note that if the distribution of the continuous random variable X is Gaussian, then the moment-matching procedure possesses a nice and intuitive property. If the k-point discrete approximation of a standard normal random variable $Z \sim \mathcal{N}(0, 1)$ is given by the points $z_1, z_2, \ldots, z_k$ with corresponding probabilities $p_1, p_2, \ldots, p_k$, the k-point discrete approximation for any normal random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ can be easily recovered without applying the Gaussian quadrature directly: it has support points $\mu + \sigma \cdot z_i$, $i = 1, 2, \ldots, k$, with the same probabilities p_i.

The gamma distribution is characterized by the following probability density function:

$$f(x; \theta, \kappa) = \frac{1}{\Gamma(\kappa)\theta^{\kappa}} x^{\kappa-1} e^{-x/\theta}, \quad x > 0, \theta, \kappa > 0;$$

and its raw r-th integer moment is

$$m_r = \theta^r \frac{\Gamma(r+\kappa)}{\Gamma(\kappa)}, \quad r = 1, 2, \ldots \tag{2}$$

with $\Gamma(\cdot)$ being the usual gamma function: $\Gamma(\kappa) = \int_0^{\infty} x^{\kappa-1} e^{-x} \mathrm{d}x$. The Weibull distribution is characterized by the following probability density function:

$$f(x; \theta, \kappa) = \frac{\kappa}{\theta} \left(\frac{x}{\theta}\right)^{\kappa-1} e^{-(x/\theta)^{\kappa}}, \quad x > 0, \theta, \kappa > 0;$$

and its raw r-th integer moment is

$$m_r = \theta^r \Gamma(1 + r/\kappa), \quad r = 1, 2, \ldots. \tag{3}$$

Unlike the normal family, for the gamma and Weibull distributions with assigned values of their two parameters, there is no shortcut when computing the k-point discrete approximation according to the matching-moment procedure.

Let us now consider the discretization via moment-matching of a normal, gamma, and Weibull distribution, all sharing the same values of expectation and variance, $m_1 = \mu = 10$ and $\sigma^2 = m_2 - m_1^2 = 16$. The values of the gamma parameters leading to these assigned moments can be easily recovered by recalling (2) and are equal to $\theta = 8/5$ and $\kappa = 25/4$; for the Weibull distribution, by using (3), $\theta \approx 11.246$ and $\kappa \approx 2.696$. Tables 1, 2, and 3 display, for $k = 3, 5, 7$, respectively, the values of x_i and p_i, $i = 1, \dots, k$ of the discrete k-point random distribution matching the first $2k - 1$ moments of the corresponding continuous distribution.

TABLE 1. 3-point discrete approximation for Gaussian, gamma and Weibull distributions with mean 10 and standard deviation 4

Gaussian		Gamma		Weibull	
x_i	p_i	x_i	p_i	x_i	p_i
3.072	0.1667	5.839	0.4095	4.590	0.2721
10	0.6667	12.11	0.5427	10.85	0.6076
16.93	0.1667	21.65	0.0478	17.95	0.1202

TABLE 2. 5-point discrete approximation for Gaussian, gamma and Weibull distributions with mean 10 and standard deviation 4

Gaussian		Gamma		Weibull	
x_i	p_i	x_i	p_i	x_i	p_i
−1.428	0.0113	4.233	0.1455	2.644	0.0703
4.577	0.2221	8.498	0.5273	6.765	0.3626
10.00	0.5333	14.22	0.2953	11.67	0.4346
15.42	0.2221	21.96	0.0315	17.02	0.1271
21.43	0.0113	33.09	0.0004	23.00	0.0055

TABLE 3. 7-point discrete approximation for Gaussian, gamma and Weibull distributions with mean 10 and standard deviation 4

Gaussian		Gamma		Weibull	
x_i	p_i	x_i	p_i	x_i	p_i
−5.002	0.0005	3.340	0.0572	1.735	0.0235
0.5330	0.0308	6.626	0.3497	4.624	0.1628
5.382	0.2401	10.89	0.4248	8.306	0.3640
10.00	0.4571	16.32	0.1514	12.43	0.3278
14.62	0.2401	23.22	0.0165	16.84	0.1107
19.47	0.0308	32.17	0.0004	21.55	0.0110
25.00	0.0005	44.63	0.0000	26.89	0.0002

Some computational problems may occur when the variance is relatively small if compared to the expected value, as reported in [8]. A common drawback of the moment matching procedure, when applied to Gassian or non-Gaussian distributions, which is more accentuated when k becomes larger, is its tendency to return "extreme" values, located on the left or right tail, with a very small probability; this can be noticed looking at Table 3, where the point x_7 for the discretized Gamma distribution has a probability which is very near to zero (zero, at the fourth decimal digit). If we consider for the gamma distribution a new combination of parameters, say $\theta = 150$ and $\kappa = 1/15$, so that the expected value is still 10, but the variance is now 2/3; and if we apply the moment-matching discretization, the implementation in `R` returns an unfeasible solution, since the lowest value x_1 would be equal to -3.337672 (with associated probability $3.760343 \cdot 10^{-12}$), which falls outside the natural support of the continuous distribution. This should be caused by numerical issues in `R`, when the algorithm computes the eigenvalues of a tridiagonal matrix, whose determinant, although necessarily positive in this case, is very close to zero and, due also to the finite precision computation and the large values assumed by higher moments, can turn negative. In other cases, it may occur that the values x_i and p_i cannot even be computed, since an intermediate matrix, which should be positive definite, does not result so (again, due to finite precision computation), and subsequent Cholesky decomposition cannot be performed on it.

ALTERNATIVE DISCRETIZATION TECHNIQUES

The issues related to moment-matching explain why other discretization procedures are sometimes employed, which prefer to match other features than integer moments, or to restrict the focus on the first two moments only and take into account a global measure of discrepancy like the mean squared error [9]. Moments are only partial measures of the distribution form, they do not always determine the distribution univocally [10, p.106] and discretization based on moment equalization up to a finite order cannot retain the functional properties of the original distribution. The methodology proposed by [9] is theoretically appealing and strictly related to the concept of "latent variable". Although being more time-demanding, since it requires solving a non-linear constrained optimization problem numerically, it has the advantage of needing only the first two moments of the continuous random variable to be finite and is easily implemented in `R`.

CONCLUSIONS

Moment-matching by means of Gaussian quadrature is quite a common way of discretizing a continuous probability distribution into a finite number of points. Although it is more frequently and more easily applied to Gaussian or more generally symmetrical distributions, its use is not precluded to non-Gaussian and asymmetrical distributions, provided that closed-form expressions for the required moments are available. Some computational issues can however emerge, especially for high values of the number of approximating points; to avoid such hurdles, one can turn to alternative discretization procedures, for example to a recent proposal based on the matching of the first two moments only and on a minimum-mean-squared-error criterion.

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