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Black holes in Sol minore

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ABSTRACT: We consider black holes in five-dimensional $N = 2$ U(1)-gauged supergravity coupled to vector multiplets, with horizons that are homogeneous but not isotropic. We write down the equations of motion for electric and magnetic ansätze, and solve them explicitly for the case of pure gauged supergravity with magnetic U(1) field strength and Sol horizon. The thermodynamics of the resulting solution, which exhibits anisotropic scaling, is discussed. If the horizon is compactified, the geometry approaches asymptotically a torus bundle over AdS₃. Furthermore, we prove a no-go theorem that states the nonexistence of supersymmetric, static, Sol-invariant, electrically or magnetically charged solutions with spatial cross-sections modelled on solvegeometry. Finally, we study the attractor mechanism for extremal static non-BPS black holes with nil- or solvegeometry horizons. It turns out that there are no such attractors for purely electric field strengths, while in the magnetic case there are attractor geometries, where the values of the scalar fields on the horizon are computed by extremization of an effective potential V_{eff} , which contains the charges as well as the scalar potential of the gauged supergravity theory. The entropy density of the extremal black hole is then given by the value of V_{eff} in the extremum.

KEYWORDS: AdS-CFT Correspondence, Black Holes, Classical Theories of Gravity, Supergravity Models

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1 Introduction and summary of results

In the seventies of the last century Hawking proved his famous theorem [1, 2] on the topology of black holes, which asserts that event horizon cross sections of 4-dimensional asymptotically flat stationary black holes obeying the dominant energy condition are topologically S^2 . This result extends to outer apparent horizons in black hole spacetimes that are not necessarily stationary [3]. Such restrictive uniqueness theorems do not hold in higher dimensions, the most famous counterexample being the black ring of Emparan and Reall [4], with horizon topology $S^2 \times S^1$. Nevertheless, Galloway and Schoen [5] were able to show that, in arbitrary dimension, cross sections of the event horizon (in the stationary case) and outer apparent horizons (in the general case) are of positive Yamabe type, i.e., admit metrics of positive scalar curvature.

Instead of increasing the number of dimensions, one can relax some of the assumptions that go into Hawking's theorem in order to have black holes with nonspherical topology. One such possibility is to add a negative cosmological constant Λ . Interpreting the term $-\Lambda g_{\mu\nu}$ as $8\pi G$ times the energy-momentum tensor $T_{\mu\nu}$, one has obviously that $-T_{\mu\nu}\xi^\nu$ is past-pointing for every future-pointing causal vector ξ^ν , and thus a violation of the dominant energy condition. Moreover, since for $\Lambda < 0$ the solutions generically asymptote to anti-de Sitter (AdS) spacetime, also asymptotic flatness does not hold anymore. In this

case, the horizon of a black hole can indeed be a compact Riemann surface Σ_g of any genus g [6–9]. It should be noted that, unless $g = 0$, these spacetimes are asymptotically only locally AdS; their global structure is different. This is in contrast to the black rings in five dimensions, which are asymptotically Minkowski, in spite of their nontrivial horizon topology. Notice in addition that the solutions of [6–9] do not exhaust the spectrum of black holes in AdS₄, since one can also have horizons that are noncompact manifolds with yet finite area (and thus finite entropy), topologically spheres with two punctures [10, 11].¹

In this paper, we will allow for both of the possibilities described above, i.e., we shall consider the case $D = 5$ and include a negative cosmological constant. More generally, our model contains scalar fields with a potential that admits AdS₅ vacua. A class of uncharged black holes in Einstein-Lambda gravity was obtained by Birmingham in [13] for arbitrary dimension D . These solutions have the property that the horizon is a $(D - 2)$ -dimensional Einstein manifold of positive, zero, or negative curvature. In our case, $D = 5$, and three-dimensional Einstein spaces have necessarily constant curvature, i.e., are homogeneous and isotropic. Similar to what is done in Bianchi cosmology, one can try to relax these conditions by dropping the isotropy assumption. The horizon is then a homogeneous manifold, and belongs thus to the nine ‘Bianchi cosmologies’, which are in correspondence with the eight Thurston model geometries, cf. appendix A for details. For two of these cases, namely Nil and Sol, the corresponding black holes in five-dimensional gravity with negative cosmological constant were constructed in [14] for the first time. Asymptotically, these solutions are neither flat nor AdS, but exhibit anisotropic scaling.

Here we go one step further with respect to [14] and add also charge. Some attempts in this direction include [15], where an intrinsically dyonic black hole with Sol horizon in Einstein-Maxwell-AdS gravity was found² and [16], which considers different models that are not directly related to gauged supergravity theories. There are various reasons for the addition of charge. First of all, charged black holes generically have an extremal limit, and a subclass of these zero-temperature solutions might preserve some fraction of supersymmetry, which is instrumental in holographic computations of the number of microstates. Moreover, in the extremal limit we expect to find an attractor mechanism [17–21], according to which the horizon values of the scalar fields in the theory are determined by the electromagnetic charges alone, and do not depend on the asymptotic values of the moduli. In our case, the corresponding attractor geometry would be AdS₂ × M , where M denotes a three-dimensional homogeneous manifold.³ These issues will be addressed in the following.

We start in section 2 by setting up the gauged supergravity model that will be considered throughout the paper. In 3 we write down the equations of motion for electric and magnetic ansätze. These are then solved explicitly in section 4 for the case of pure gauged

¹These solutions can be generalized to $D > 4$ [12].

²Since the authors of [15] do not include a Chern-Simons term, their solution does not solve the equations of motion of pure gauged supergravity.

³Note in this context that [22] considers near-horizon geometries with various homogeneous horizons in theories containing massive vector fields, while [23] constructs solutions interpolating between some Bianchi cosmologies and Lifshitz geometries or AdS₂ × S³. The authors of [23] do not show that these solutions can be obtained from some particular theories, but prove that the corresponding energy-momentum tensor satisfies the weak energy condition.

supergravity with magnetic U(1) field strength and Sol horizon. Moreover, the thermodynamics of the resulting solution, which exhibits anisotropic scaling, is discussed. If the horizon is compactified, the geometry approaches asymptotically a torus bundle over AdS₃. Section 5 is dedicated to the proof of a no-go theorem that states the nonexistence of supersymmetric, static, Sol-invariant, electrically or magnetically charged solutions with spatial cross-sections modelled on solvegeometry. Finally, in 6 we study the attractor mechanism for extremal static non-BPS black holes with nil- or solvegeometry horizons. It turns out that there are no such attractors for purely electric field strengths, while in the magnetic case there are attractor geometries, where the values of the scalar fields on the horizon are computed by extremization of an effective potential V_{eff} , which contains the charges as well as the scalar potential of the gauged supergravity theory. The entropy density of the extremal black hole is then given by the value of V_{eff} in the extremum.

2 $N = 2, D = 5$ U(1)-gauged supergravity

We consider $N = 2, D = 5$ U(1)-gauged supergravity coupled to n abelian vector multiplets, whose bosonic field content includes the fünfbein e_μ^a , the vectors A_μ^I with $I = 0, \dots, n$ and the real scalars ϕ^i , where $i = 1, \dots, n$. The gauging of the U(1) subgroup of the SU(2) R-symmetry is achieved through the vector field $A_\mu = V_I A_\mu^I$ with coupling constant g , where the V_I are constant parameters. In order to preserve supersymmetry the introduction of a scalar potential is required. The bosonic part of the Lagrangian is given by [24]

$$e^{-1} \mathcal{L} = \frac{R}{2} - \frac{1}{2} \mathcal{G}_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - \frac{1}{4} G_{IJ} F_{\mu\nu}^I F^{J\mu\nu} + \frac{e^{-1}}{48} C_{IJK} \varepsilon^{\mu\nu\rho\sigma\tau} F_{\mu\nu}^I F_{\rho\sigma}^J A_\tau^K - g^2 U, \quad (2.1)$$

where $F_{\mu\nu}^I$ are the abelian field strength tensors. The scalar potential U reads

$$U = V_I V_J \left(\frac{9}{2} \mathcal{G}^{ij} \partial_i h^I \partial_j h^J - 6 h^I h^J \right), \quad (2.2)$$

where \mathcal{G}^{ij} is the inverse of the target space metric \mathcal{G}_{ij} , ∂_i denotes the partial derivative with respect to ϕ^i and the functions $h^I = h^I(\phi^i)$ satisfy the condition

$$\mathcal{V} := \frac{1}{6} C_{IJK} h^I h^J h^K = 1, \quad (2.3)$$

with C_{IJK} a fully symmetric, constant and real tensor. The kinetic matrices \mathcal{G}_{ij} and G_{IJ} are given by

$$G_{IJ} = -\frac{1}{2} \frac{\partial}{\partial h^I} \frac{\partial}{\partial h^J} \log \mathcal{V} \Big|_{\mathcal{V}=1}, \quad \mathcal{G}_{ij} = \partial_i h^I \partial_j h^J G_{IJ} \Big|_{\mathcal{V}=1}. \quad (2.4)$$

The Einstein-, Maxwell-Chern-Simons- and scalar field equations following from (2.1) are respectively

$$R_{\mu\nu} = \mathcal{G}_{ij} \partial_\mu \phi^i \partial_\nu \phi^j + G_{IJ} \left(F_\mu^I{}^\rho F_{\nu\rho}^J - \frac{1}{6} g_{\mu\nu} F_{\rho\sigma}^I F^{J\rho\sigma} \right) + \frac{2}{3} g^2 U g_{\mu\nu}, \quad (2.5)$$

$$\nabla_\lambda (G_{IJ} F^{J\lambda\tau}) + \frac{e^{-1}}{16} C_{IJK} \varepsilon^{\mu\nu\rho\sigma\tau} F_{\mu\nu}^J F_{\rho\sigma}^K = 0, \quad (2.6)$$

$$\nabla_\mu (\mathcal{G}_{ij} \partial^\mu \phi^j) - \frac{1}{2} \partial_i \mathcal{G}_{kj} \partial_\mu \phi^k \partial^\mu \phi^j - \frac{1}{4} \partial_i G_{IJ} F_{\mu\nu}^I F^{J\mu\nu} - g^2 \partial_i U = 0. \quad (2.7)$$

3 Equations of motion for electric and magnetic ansätze

In order to solve the equations of motion (2.5)–(2.7) we use an ansatz inspired by [14], with homogeneous sections $\Sigma_{t,r}$ of constant t and r . Without loss of generality, we take the line element to be

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + \sum_{A=1}^3 e^{2T_A(r)}(\theta^A)^2, \quad (3.1)$$

where the induced metric on $\Sigma_{t,r}$ is written in terms of G -invariant 1-forms θ^A , which satisfy

$$d\theta^A = \frac{1}{2}C^A_{BC}\theta^B \wedge \theta^C, \quad (3.2)$$

with C^A_{BC} the structure constants of the Lie algebra of the isometry group G . A list of all the possible isometry groups with related structure constants and invariant 1-forms can be found in [25], while in appendix A we present those for solve- and nilgeometries along with a brief discussion of homogeneous manifolds. Henceforth we shall restrict our discussion to class A Bianchi models, which contain the most exotic cases, such as solve- and nilgeometry (cf. table 1 in appendix A).

The scalar fields are assumed to depend on the radial coordinate only,

$$\phi^i = \phi^i(r). \quad (3.3)$$

3.1 Electric ansatz

For a purely electric ansatz the vector fields are given by

$$A^I = A^I_t(r)dt, \quad (3.4)$$

and the Maxwell equations (2.6) imply

$$F^I_{rt} = \partial_r A^I_t = e^{-\sum_A T_A} G^{IJ} q_J, \quad (3.5)$$

where G^{IJ} denotes the inverse of G_{IJ} , and the constants q_I represent essentially the electric charge densities.

Using (3.5) and the Bianchi class A condition (A.4), the Einstein equations (2.5) boil down to

$$\begin{aligned} \frac{V''}{2} + \frac{V'}{2} \sum_A T'_A &= \frac{2}{3} e^{-2\sum_A T_A} G^{IJ} q_I q_J - \frac{2}{3} g^2 U, \\ \sum_A T''_A + \sum_A (T'_A)^2 &= -g_{ij} \phi^{i'} \phi^{j'}, \quad \sum_B C^B_{AB} T'_B = 0, \\ -V' T'_A - V T''_A - V T'_A \sum_B T'_B + \mathcal{J}_A &= \frac{1}{3} e^{-2\sum_B T_B} G^{IJ} q_I q_J + \frac{2}{3} g^2 U, \end{aligned} \quad (3.6)$$

where we defined

$$\mathcal{J}_A := \sum_{B,C} \left[-\frac{1}{2} D^B_{AC} (D^C_{AB} + D^B_{AC}) + \frac{1}{4} (D^A_{BC})^2 \right], \quad (3.7)$$

with

$$D^A_{BC} := e^{T_A - T_B - T_C} C^A_{BC}. \quad (3.8)$$

The third equation in (3.6) is a constraint, which is trivially satisfied for all the class A Bianchi cosmologies except for solvegeometry; in this case it reduces to

$$T'_1 = T'_2. \quad (3.9)$$

Finally, using (3.5), the equations (2.7) for the scalars become

$$\begin{aligned} V\mathcal{G}_{ij}\phi^{j'} \sum_A T'_A + V \frac{d\mathcal{G}_{ij}}{dr} \phi^{j'} + V'\mathcal{G}_{ij}\phi^{j'} + V\mathcal{G}_{ij}\phi^{j''} - \frac{1}{2}V\partial_i\mathcal{G}_{kj}\phi^{k'}\phi^{j'} \\ - \frac{1}{2}e^{-2\sum_A T_A}\partial_i G^{IJ}q_{IqJ} - g^2\partial_i U = 0. \end{aligned} \quad (3.10)$$

3.2 Magnetic ansatz

In the magnetically charged case we take for the field strength

$$F^I = p^I\theta^1 \wedge \theta^2, \quad (3.11)$$

where the p^I are magnetic charge densities. Note that F^I is closed due to the Bianchi class A condition (A.4), so locally there exists a gauge potential A^I such that $F^I = dA^I$. In the following we shall consider the case of solvegeometry, for which

$$F^I = p^I dx \wedge dy, \quad A^I = p^I x dy. \quad (3.12)$$

Using (A.7) the line element (3.1) becomes

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + e^{2(T_1(r)+z)}dx^2 + e^{2(T_2(r)-z)}dy^2 + e^{2T_3(r)}dz^2. \quad (3.13)$$

The Maxwell equations (2.6) are automatically satisfied by (3.12) and (3.13), while the nontrivial Einstein equations (2.5) read

$$\begin{aligned} \frac{V''}{2} + \frac{V'}{2}(2T'_1 + T'_3) &= \frac{1}{3}e^{-4T_1}G_{IJ}p^I p^J - \frac{2}{3}g^2U, \\ 2T''_1 + T''_3 + 2(T'_1)^2 + (T'_3)^2 &= -\mathcal{G}_{ij}\phi^{i'}\phi^{j'}, \\ -V'T'_1 - V(T''_1 + T'_1(2T'_1 + T'_3)) &= \frac{2}{3}e^{-4T_1}G_{IJ}p^I p^J + \frac{2}{3}g^2U, \\ -V'T'_3 - V(T''_3 + T'_3(2T'_1 + T'_3)) - 2e^{-2T_3} &= -\frac{1}{3}e^{-4T_1}G_{IJ}p^I p^J + \frac{2}{3}g^2U, \end{aligned} \quad (3.14)$$

where we have used the condition $T'_1 = T'_2$ and the freedom to rescale y in order to set $T_1 = T_2$. The scalar field equations (2.7) become

$$\begin{aligned} V\mathcal{G}_{ij}\phi^{j'} \sum_A T'_A + V \frac{d\mathcal{G}_{ij}}{dr} \phi^{j'} + V'\mathcal{G}_{ij}\phi^{j'} + V\mathcal{G}_{ij}\phi^{j''} - \frac{1}{2}V\partial_i\mathcal{G}_{kj}\phi^{k'}\phi^{j'} \\ - \frac{1}{2}\partial_i G_{IJe^{-4T_1}}p^I p^J - g^2\partial_i U = 0. \end{aligned} \quad (3.15)$$

4 Magnetic black hole in pure gauged supergravity

In order to study the above equations in a simplified setting, we restrict our attention to pure gauged supergravity, i.e., the theory (2.1) without vector multiplets ($n = 0$). For a purely electric or magnetic configuration, the Chern-Simons term can be consistently truncated, and (2.1) boils down to

$$e^{-1} \mathcal{L} = \frac{R}{2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \Lambda, \quad (4.1)$$

where $\Lambda = -6g^2 < 0$, $F_{\mu\nu} = F_{\mu\nu}^0$ and we fixed C_{000} in (2.3) and V_0 in (2.2) such that $G_{00} = 1$ and $U = -6$.

The field strength (3.12) becomes simply $F_{xy} = p$, while the Einstein equations (3.14) reduce to

$$\begin{aligned} \frac{V''}{2} + \frac{V'}{2} (2T'_1 + T'_3) &= \frac{1}{3} e^{-4T_1} p^2 - \frac{2}{3} \Lambda, \\ 2T''_1 + T''_3 + 2(T'_1)^2 + (T'_3)^2 &= 0, \\ -V'T'_1 - V(T''_1 + T'_1(2T'_1 + T'_3)) &= \frac{2}{3} e^{-4T_1} p^2 + \frac{2}{3} \Lambda, \\ -V'T'_3 - V(T''_3 + T'_3(2T'_1 + T'_3)) - 2e^{-2T_3} &= -\frac{1}{3} e^{-4T_1} p^2 + \frac{2}{3} \Lambda. \end{aligned} \quad (4.2)$$

One easily checks that in the uncharged case $p = 0$ the above equations are satisfied by the solvegeometry solution constructed in [14].

(4.2) can be easily solved by taking T_1 to be constant.⁴ With this assumption, a particular black hole solution is given by

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + \sqrt{\frac{p^2}{-\Lambda}} (e^{2z} dx^2 + e^{-2z} dy^2) + \frac{r^2}{A} dz^2, \quad (4.3)$$

$$F = p dx \wedge dy, \quad (4.4)$$

with

$$V(r) = -\frac{\Lambda}{2} r^2 - 2A \ln\left(\frac{r}{B}\right), \quad (4.5)$$

where A and B are two positive integration constants. It is worth noting that this solution is singular in the limit $p \rightarrow 0$, and it is thus disconnected from the one in [14]. The metric (4.3) and field strength (4.4) are invariant under the scale transformations

$$t \rightarrow t/\nu, \quad r \rightarrow \nu r, \quad z \rightarrow z + \ln \alpha, \quad x \rightarrow \lambda x, \quad y \rightarrow \pm \lambda \alpha^2 y, \quad (4.6)$$

accompanied by

$$p \rightarrow \pm \frac{p}{\lambda^2 \alpha^2}, \quad A \rightarrow \nu^2 A, \quad B \rightarrow \nu B. \quad (4.7)$$

This can be used to set e.g. $p = B = 1/g$ without loss of generality. B and the magnetic charge density p are thus not true parameters of the solution, which is specified completely by choosing A . Notice that the scaling symmetries with $\nu = 1$, $\lambda = 1/\alpha$ belong to the

⁴Note that this is not the case for the solution of [14].

Lie group Sol. If the horizon is compactified (cf. [14] for details on the compactification procedure), the transformations in (4.6) involving α and λ are broken down to a discrete subgroup ($\alpha = \lambda^{-1} = e^{na}$, where a is the constant appearing in (II.22) of [14] and $n \in \mathbb{Z}$), which does no more allow to scale p to any value. In this case, p can become actually a genuine parameter of the black hole.

(4.3) exhibits anisotropic scaling. If the horizon is compactified, the geometry approaches asymptotically for $r \rightarrow \infty$ a torus bundle over AdS_3 . In $r = 0$ there is a curvature singularity, since the Kretschmann scalar behaves as $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \sim (\ln r)/r^4$ for $r \rightarrow 0$. Horizons are determined by the roots of the function $V(r)$, which diverges both for $r \rightarrow 0$ and $r \rightarrow +\infty$ and has a unique minimum in

$$r = r_{\min} = \sqrt{\frac{2A}{-\Lambda}}. \tag{4.8}$$

If $V(r_{\min}) > 0$ the solution represents a naked singularity. For $V(r_{\min}) = 0$, i.e., $A = 3e$, we have an extremal black hole, while for $V(r_{\min}) < 0$ ($A > 3e$) there is an inner and an outer horizon and the solution is nonextremal.

Requiring the absence of conical singularities in the Euclidean section gives the Hawking temperature

$$T = \frac{-\Lambda r_h^2 - 2A}{4\pi r_h}, \tag{4.9}$$

where r_h denotes the radial coordinate of the horizon. The entropy density can be computed by means of the Bekenstein-Hawking formula and is given by

$$s = \frac{S}{V_{\text{solve}}} = \frac{(\ln(gr_h))^{1/2}}{12g^3}, \tag{4.10}$$

where we set Newton's constant $G = 1$, and V_{solve} is the volume of the compactified manifold modelled on solvegeometry.

The standard Komar integral for the mass goes like Λr^2 for large r and thus diverges for $r \rightarrow +\infty$ due to the presence of the vacuum energy, as was to be expected. Moreover, there is no obvious background to subtract, and the conditions for the applicability of the Ashtekar-Magnon-Das formalism [26, 27] are not satisfied. In spite of these difficulties, we can associate a mass to the black hole (4.3) by simply integrating the first law. Since p is not a dynamical parameter of the solution, we do not expect a term containing the variation of the magnetic charge in the first law. The mass density m satisfies thus

$$dm = Tds, \tag{4.11}$$

which gives (up to an integration constant, that can be fixed by requiring e.g. the extremal solution to have zero energy)

$$m = \frac{r_h}{16\pi g(\ln(gr_h))^{1/2}} = \frac{1}{8\pi g^2} \sqrt{\frac{A}{6}}. \tag{4.12}$$

Notice that in five dimensions, magnetic charge is actually carried by strings rather than by point particles, since the string world volume naturally couples to an electric two-form

potential, which is dual to the magnetic ansatz (3.12). The latter is characteristic of a charged string along the z -direction. In this sense, the solution constructed in this section is perhaps more correctly referred to as a black string rather than a black hole. In spite of this, we shall use the terminology ‘black hole’, since the near-horizon geometry of the extremal limit of (4.3) contains an AdS_2 factor (it is $\text{AdS}_2 \times \text{Sol}$) instead of AdS_3 ,⁵ which would be typical for extremal black strings. Note also that, if we think of (4.3) as a black string, then this is not straightforwardly related to a black hole in four dimensions, since the explicit z -dependence of the metric prevents a canonical Kaluza-Klein reduction along z .

To close this section, we remark that a generalization of the solution (4.3), (4.4) as well as the one of [14] to the stu model of $N = 2$ gauged supergravity, together with a numerical analysis of the equations of motion (4.2), is currently under investigation.

5 Existence of static, Sol-invariant BPS solutions

A simpler method to construct solutions to a given supergravity theory is based on solving the Killing spinor equations. These are of first order, and are generically much easier to solve than the full second order equations of motion. At least in the case where the Killing vector constructed as a bilinear from the Killing spinor is timelike, the latter are implied by the Killing spinor equations [28].

The supersymmetry variations for the gravitino ψ_μ and the gauginos λ_i in a bosonic background are given by (see e.g. [29])

$$\delta\psi_\mu = \left[\mathcal{D}_\mu + \frac{i}{8} h_I (\Gamma_\mu^{\nu\rho} - 4\delta_\mu^\nu \Gamma^\rho) F_{\nu\rho}^I + \frac{g}{2} \Gamma_\mu h^I V_I - \frac{3i}{2} g V_I A_\mu^I \right] \epsilon, \quad (5.1)$$

$$\delta\lambda_i = \left[\frac{3}{8} \Gamma^{\mu\nu} F_{\mu\nu}^I \partial_i h_I - \frac{i}{2} \mathcal{G}_{ij} \Gamma^\mu \partial_\mu \phi^j + \frac{3i}{2} g V_I \partial_i h^I \right] \epsilon, \quad (5.2)$$

where ϵ is the supersymmetry parameter, $h_I = \frac{1}{6} C_{IJK} h^J h^K$ and \mathcal{D}_μ denotes the Lorentz-covariant derivative.⁶ The vanishing of the gravitino supersymmetry transformations (5.1) leads to the Killing spinor equations, whose integrability conditions imply a set of constraints for the metric and the matter fields. Given $\delta\psi_\mu \equiv \hat{\mathcal{D}}_\mu \epsilon = 0$, the first integrability conditions read

$$\hat{\mathcal{R}}_{\mu\nu} \epsilon \equiv [\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu] \epsilon = 0, \quad (5.3)$$

which is a set of algebraic equations that admit a nontrivial solution ϵ iff $\det(\hat{\mathcal{R}}_{\mu\nu}) = 0$.

In what follows we shall specify to solve geometry with electric or magnetic ansatz. For the metric (3.13) the tetrad can be chosen as

$$e_t^0 = \sqrt{V}, \quad e_x^1 = e^{T_1+z}, \quad e_y^2 = e^{T_2-z}, \quad e_z^3 = e^{T_3}, \quad e_r^4 = \frac{1}{\sqrt{V}}. \quad (5.4)$$

⁵As we said, AdS_3 occurs for $r \rightarrow \infty$.

⁶Our conventions are $\mathcal{D}_\mu = \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab}$, $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$, $\Gamma^{a_1 a_2 \dots a_n} = \Gamma^{[a_1} \Gamma^{a_2} \dots \Gamma^{a_n]}$, where we antisymmetrize with unit weight.

5.1 Electric ansatz

In the case of solvegeometry and electric ansatz, the vanishing of the gravitino variations (5.1) leads to

$$\begin{aligned}
 & \left[\partial_t + \frac{V'}{4} \Gamma_{04} + \frac{i}{2} h_I F_{rt}^I \sqrt{V} \Gamma_4 + \frac{g}{2} V_I h^I \sqrt{V} \Gamma_0 - i \frac{3g}{2} V_I A_t^I \right] \epsilon = 0, \\
 & \left[\partial_r + \frac{i}{2} h_I F_{rt}^I \frac{1}{\sqrt{V}} \Gamma_0 + \frac{g}{2} V_I h^I \frac{1}{\sqrt{V}} \Gamma_4 \right] \epsilon = 0, \\
 & \left[\partial_x + e^{T_1+z} \left(\frac{1}{2} \sqrt{V} T_1' \Gamma_{14} + \frac{1}{2} e^{-T_3} \Gamma_{13} - \frac{i}{4} h_I F_{rt}^I \Gamma_{014} + \frac{g}{2} V_I h^I \Gamma_1 \right) \right] \epsilon = 0, \\
 & \left[\partial_y + e^{T_2-z} \left(\frac{1}{2} \sqrt{V} T_2' \Gamma_{24} - \frac{1}{2} e^{-T_3} \Gamma_{23} - \frac{i}{4} h_I F_{rt}^I \Gamma_{024} + \frac{g}{2} V_I h^I \Gamma_2 \right) \right] \epsilon = 0, \\
 & \left[\partial_z + e^{T_3} \left(\frac{1}{2} \sqrt{V} T_3' \Gamma_{34} - \frac{i}{4} h_I F_{rt}^I \Gamma_{034} + \frac{g}{2} V_I h^I \Gamma_3 \right) \right] \epsilon = 0.
 \end{aligned} \tag{5.5}$$

The integrability conditions (5.3) with (μ, ν) equal to (t, x) , (t, y) and (t, z) are, respectively,

$$\begin{aligned}
 & \left[\frac{1}{2} V' T_1' - g^2 (V_I h^I)^2 + i \sqrt{V} T_1' h_I F_{rt}^I \Gamma_0 + ig V_I h^I h_J F_{rt}^J \Gamma_{04} \right] \epsilon = 0, \\
 & \left[\frac{1}{2} V' T_2' - g^2 (V_I h^I)^2 + i \sqrt{V} T_2' h_I F_{rt}^I \Gamma_0 + ig V_I h^I h_J F_{rt}^J \Gamma_{04} \right] \epsilon = 0, \\
 & \left[\frac{1}{2} V' T_3' - g^2 (V_I h^I)^2 + i \sqrt{V} T_3' h_I F_{rt}^I \Gamma_0 + ig V_I h^I h_J F_{rt}^J \Gamma_{04} \right] \epsilon = 0,
 \end{aligned} \tag{5.6}$$

while for (x, y) , (x, z) and (y, z) we have

$$\begin{aligned}
 & \left[VT_1' T_2' - g^2 (V_I h^I)^2 + \frac{1}{4} (h_I F_{rt}^I)^2 - e^{-2T_3} - \frac{i}{2} \sqrt{V} (T_1' + T_2') h_I F_{rt}^I \Gamma_0 \right. \\
 & \qquad \qquad \qquad \left. - ig V_I h^I h_J F_{rt}^J \Gamma_{04} \right] \epsilon = 0, \\
 & \left[VT_1' T_3' - g^2 (V_I h^I)^2 + \frac{1}{4} (h_I F_{rt}^I)^2 + e^{-2T_3} - \frac{i}{2} \sqrt{V} (T_1' + T_3') h_I F_{rt}^I \Gamma_0 \right. \\
 & \qquad \qquad \qquad \left. - ig V_I h^I h_J F_{rt}^J \Gamma_{04} + \sqrt{V} (T_1' - T_3') e^{-T_3} \Gamma_{34} \right] \epsilon = 0, \\
 & \left[VT_2' T_3' - g^2 (V_I h^I)^2 + \frac{1}{4} (h_I F_{rt}^I)^2 + e^{-2T_3} - \frac{i}{2} \sqrt{V} (T_2' + T_3') h_I F_{rt}^I \Gamma_0 \right. \\
 & \qquad \qquad \qquad \left. - ig V_I h^I h_J F_{rt}^J \Gamma_{04} - \sqrt{V} (T_2' - T_3') e^{-T_3} \Gamma_{34} \right] \epsilon = 0.
 \end{aligned} \tag{5.7}$$

The difference of eqs. (5.6) taken in (all the three possible) pairs leads to

$$\begin{aligned}
 & (T_1' - T_2') \left[\frac{1}{2} V' + i \sqrt{V} h_I F_{rt}^I \Gamma_0 \right] \epsilon = 0, \\
 & (T_1' - T_3') \left[\frac{1}{2} V' + i \sqrt{V} h_I F_{rt}^I \Gamma_0 \right] \epsilon = 0, \\
 & (T_2' - T_3') \left[\frac{1}{2} V' + i \sqrt{V} h_I F_{rt}^I \Gamma_0 \right] \epsilon = 0,
 \end{aligned} \tag{5.8}$$

whereas $(x, y) - (x, z)$ and $(x, y) - (y, z)$ read

$$\begin{aligned} \left[VT'_1(T'_2 - T'_3) - 2e^{-2T_3} - \frac{i}{2}\sqrt{V}(T'_2 - T'_3)h_I F_{rt}^I \Gamma_0 - \sqrt{V}(T'_1 - T'_3)e^{-T_3} \Gamma_{34} \right] \epsilon &= 0, \\ \left[VT'_2(T'_1 - T'_3) - 2e^{-2T_3} - \frac{i}{2}\sqrt{V}(T'_1 - T'_3)h_I F_{rt}^I \Gamma_0 + \sqrt{V}(T'_2 - T'_3)e^{-T_3} \Gamma_{34} \right] \epsilon &= 0. \end{aligned} \tag{5.9}$$

We can distinguish between two different cases in which (5.8) hold.

- Case A

$$T'_1 = T'_2 = T'_3. \tag{5.10}$$

In this case (5.9) leads directly to the trivial solution $\epsilon = 0$.

- Case B

$$\left[\frac{1}{2}V' + i\sqrt{V}h_I F_{rt}^I \Gamma_0 \right] \epsilon = 0. \tag{5.11}$$

Writing this condition schematically as $\mathcal{M}\epsilon = 0$, a necessary condition to have non-trivial solutions is $\det \mathcal{M} = 0$, and thus

$$\frac{1}{2}V' = \pm\sqrt{V}h_I F_{rt}^I, \tag{5.12}$$

which, once plugged back into (5.11) gives the projection

$$\Gamma_0 \epsilon = \pm i \epsilon. \tag{5.13}$$

Using (5.13) in (5.9), we get

$$\begin{aligned} \left[VT'_1(T'_2 - T'_3) - 2e^{-2T_3} \pm \frac{1}{2}\sqrt{V}(T'_2 - T'_3)h_I F_{rt}^I - \sqrt{V}(T'_1 - T'_3)e^{-T_3} \Gamma_{34} \right] \epsilon &= 0, \\ \left[VT'_2(T'_1 - T'_3) - 2e^{-2T_3} \pm \frac{1}{2}\sqrt{V}(T'_1 - T'_3)h_I F_{rt}^I + \sqrt{V}(T'_2 - T'_3)e^{-T_3} \Gamma_{34} \right] \epsilon &= 0. \end{aligned}$$

To have nontrivial solutions, the determinants of the two coefficient matrices in these linear systems must vanish, leading to $T'_1 = T'_3$ and $T'_2 = T'_3$, which brings us back to case A.

We can thus state the following

Proposition 1. *There are no static, Sol-invariant solutions to the Killing spinor equations with solvegeometry spatial cross-sections at fixed r and purely electric field strengths.*

5.2 Magnetic ansatz

In this case, the Killing spinor equations become

$$\begin{aligned}
 & \left[\partial_t + \frac{V'}{4} \Gamma_{04} + \frac{i}{4} h_{IP}^I \sqrt{V} e^{-T_1 - T_2} \Gamma_{012} + \frac{g}{2} V_I h^I \sqrt{V} \Gamma_0 \right] \epsilon = 0, \\
 & \left[\partial_r + \frac{i}{4} h_{IP}^I \frac{1}{\sqrt{V}} e^{-T_1 - T_2} \Gamma_{124} + \frac{g}{2} V_I h^I \frac{1}{\sqrt{V}} \Gamma_4 \right] \epsilon = 0, \\
 & \left[\partial_x + e^{T_1 + z} \left(\frac{1}{2} \sqrt{V} T_1' \Gamma_{14} + \frac{1}{2} e^{-T_3} \Gamma_{13} - \frac{i}{2} h_{IP}^I e^{-T_1 - T_2} \Gamma_2 + \frac{g}{2} V_I h^I \Gamma_1 \right) \right] \epsilon = 0, \\
 & \left[\partial_y + e^{T_2 - z} \left(\frac{1}{2} \sqrt{V} T_2' \Gamma_{24} - \frac{1}{2} e^{-T_3} \Gamma_{23} + \frac{i}{2} h_{IP}^I e^{-T_1 - T_2} \Gamma_1 + \frac{g}{2} V_I h^I \Gamma_2 \right) - i \frac{3g}{2} V_{IP}^I x \right] \epsilon = 0, \\
 & \left[\partial_z + e^{T_3} \left(\frac{1}{2} \sqrt{V} T_3' \Gamma_{34} + \frac{i}{4} h_{IP}^I e^{-T_1 - T_2} \Gamma_{123} + \frac{g}{2} V_I h^I \Gamma_3 \right) \right] \epsilon = 0. \tag{5.14}
 \end{aligned}$$

We have thus the following first integrability conditions:

- (t,x)

$$\begin{aligned}
 & \left[\frac{1}{2} V' T_1' - g^2 (V_I h^I)^2 + \frac{i}{2} h_{IP}^I \sqrt{V} T_1' e^{-T_1 - T_2} \Gamma_{124} + \frac{i}{2} h_{IP}^I e^{-T_1 - T_2 - T_3} \Gamma_{123} \right. \\
 & \qquad \qquad \qquad \left. + i g h_{IP}^I V_J h^J e^{-T_1 - T_2} \Gamma_{12} \right] \epsilon = 0, \tag{5.15}
 \end{aligned}$$

- (t,y)

$$\begin{aligned}
 & \left[\frac{1}{2} V' T_2' - g^2 (V_I h^I)^2 + \frac{i}{2} h_{IP}^I \sqrt{V} T_2' e^{-T_1 - T_2} \Gamma_{124} - \frac{i}{2} h_{IP}^I e^{-T_1 - T_2 - T_3} \Gamma_{123} \right. \\
 & \qquad \qquad \qquad \left. + i g h_{IP}^I V_J h^J e^{-T_1 - T_2} \Gamma_{12} \right] \epsilon = 0, \tag{5.16}
 \end{aligned}$$

- (t,z)

$$\left[\frac{1}{2} V' T_3' - g^2 (V_I h^I)^2 - i g h_{IP}^I V_J h^J e^{-T_1 - T_2} \Gamma_{12} - \frac{1}{4} (h_{IP}^I)^2 e^{-2(T_1 + T_2)} \right] \epsilon = 0, \tag{5.17}$$

- (x,y)

$$\begin{aligned}
 & \left[V T_1' T_2' - g^2 (V_I h^I)^2 + (h_{IP}^I)^2 e^{-2(T_1 + T_2)} - e^{-2T_3} \right. \\
 & \qquad \qquad \qquad \left. - i h_{IP}^I \sqrt{V} (T_1' + T_2') e^{-T_1 - T_2} \Gamma_{124} - 3 i g V_{IP}^I e^{-T_1 - T_2} \Gamma_{12} \right] \epsilon = 0, \tag{5.18}
 \end{aligned}$$

- (x,z)

$$\begin{aligned}
 & \left[V T_1' T_3' - g^2 (V_I h^I)^2 + e^{-2T_3} + \sqrt{V} (T_1' - T_3') e^{-T_3} \Gamma_{34} - i h_{IP}^I e^{-T_1 - T_2 - T_3} \Gamma_{123} \right. \\
 & \qquad \qquad \qquad \left. + \frac{i}{2} h_{IP}^I \sqrt{V} T_1' e^{-T_1 - T_2} \Gamma_{124} + i g h_{IP}^I V_J h^J e^{-T_1 - T_2} \Gamma_{12} \right] \epsilon = 0, \tag{5.19}
 \end{aligned}$$

- (y,z)

$$\left[VT_2'T_3' - g^2(V_I h^I)^2 + e^{-2T_3} - \sqrt{V}(T_2' - T_3')e^{-T_3}\Gamma_{34} + i h_{IP}^I e^{-T_1 - T_2 - T_3}\Gamma_{123} \right. \\ \left. + \frac{i}{2} h_{IP}^I \sqrt{V} T_2' e^{-T_1 - T_2}\Gamma_{124} + i g h_{IP}^I V_J h^J e^{-T_1 - T_2}\Gamma_{12} \right] \epsilon = 0, \quad (5.20)$$

- (r,t)

$$\left[\frac{1}{2} V'' - g^2(V_I h^I)^2 - \frac{1}{4} (h_{IP}^I)^2 e^{-2(T_1 + T_2)} - \frac{i}{2} h_{IP}^I \sqrt{V} (T_1' + T_2') e^{-T_1 - T_2}\Gamma_{124} \right. \\ \left. - \frac{i}{2} \partial_r (h_{IP}^I) \sqrt{V} e^{-T_1 - T_2}\Gamma_{124} + g \partial_r (V_I h^I) \sqrt{V} \Gamma_4 \right. \\ \left. - i g h_{IP}^I V_J h^J e^{-T_1 - T_2}\Gamma_{12} \right] \epsilon = 0, \quad (5.21)$$

- (r,x)

$$\left[VT_1'' + VT_1'^2 - i \partial_r (h_{IP}^I) \sqrt{V} e^{-T_1 - T_2}\Gamma_{124} + g \partial_r (V_I h^I) \sqrt{V} \Gamma_4 \right. \\ \left. - \frac{i}{2} h_{IP}^I \sqrt{V} (T_1' - 2T_2') e^{-T_1 - T_2}\Gamma_{124} \right] \epsilon = 0, \quad (5.22)$$

- (r,y)

$$\left[VT_2'' + VT_2'^2 - i \partial_r (h_{IP}^I) \sqrt{V} e^{-T_1 - T_2}\Gamma_{124} + g \partial_r (V_I h^I) \sqrt{V} \Gamma_4 \right. \\ \left. - \frac{i}{2} h_{IP}^I \sqrt{V} (T_2' - 2T_1') e^{-T_1 - T_2}\Gamma_{124} \right] \epsilon = 0, \quad (5.23)$$

- (r,z)

$$\left[VT_3'' + VT_3'^2 + \frac{i}{2} \partial_r (h_{IP}^I) \sqrt{V} e^{-T_1 - T_2}\Gamma_{124} + g \partial_r (V_I h^I) \sqrt{V} \Gamma_4 \right. \\ \left. - \frac{i}{2} h_{IP}^I \sqrt{V} (T_1' + T_2') e^{-T_1 - T_2}\Gamma_{124} \right] \epsilon = 0. \quad (5.24)$$

From the vanishing of the gaugino variation (5.2) one gets

$$\left[\frac{1}{3} \mathcal{G}_{ij} \sqrt{V} \partial_r \phi^j \Gamma_4 - g \partial_i (V_I h^I) + \frac{i}{2} \partial_i (h_{IP}^I) e^{-T_1 - T_2}\Gamma_{12} \right] \epsilon = 0. \quad (5.25)$$

The combination (5.19) + (5.20) - (5.15) - (5.16) gives

$$\left[(T_1' + T_2') \left(VT_3' - \frac{1}{2} V' \right) + 2e^{-2T_3} + \sqrt{V} (T_1' - T_2') e^{-T_3}\Gamma_{34} \right] \epsilon = 0. \quad (5.26)$$

The determinant of the coefficient matrix of this linear system vanishes if

$$(T_1' + T_2') \left(VT_3' - \frac{1}{2} V' \right) + 2e^{-2T_3} = 0 \quad \wedge \quad \sqrt{V} (T_1' - T_2') e^{-T_3} = 0, \quad (5.27)$$

which implies

$$T_1' \left(VT_3' - \frac{1}{2} V' \right) + e^{-2T_3} = 0, \quad T_1' = T_2'. \quad (5.28)$$

From the combination (5.22) – (5.23) + (5.19) – (5.20) + 2 · ((5.15) – (5.16)) we obtain

$$\left[V(T_1'' - T_2'') + (T_1' - T_2')(V(T_1' + T_2' + T_3') + V') + \sqrt{V}(T_1' + T_2' - 2T_3')e^{-T_3}\Gamma_{34} \right] \epsilon = 0. \quad (5.29)$$

Using $T_1' = T_2'$, it turns out that the vanishing of the determinant associated to (5.29) requires $T_3' = T_1'$. (5.15) – (5.16) yields

$$ih_{IP}^I e^{-T_1 - T_2 - T_3} \Gamma_{123} \epsilon = 0, \quad (5.30)$$

and thus

$$h_{IP}^I = 0. \quad (5.31)$$

Taking into account the above results and defining $T' \equiv T_1' = T_2' = T_3'$, the first integrability conditions become

- (t,x), (t,y), (t,z)

$$\left[\frac{1}{2} V' T' - g^2 (V_I h^I)^2 \right] \epsilon = 0, \quad (5.32)$$

- (x,y)

$$\left[VT'^2 - g^2 (V_I h^I)^2 - e^{-2T_3} - 3igV_{IP}^I e^{-T_1 - T_2} \Gamma_{12} \right] \epsilon = 0, \quad (5.33)$$

- (x,z), (y,z)

$$\left[VT'^2 - g^2 (V_I h^I)^2 + e^{-2T_3} \right] \epsilon = 0, \quad (5.34)$$

- (r,t)

$$\left[\frac{1}{2} V'' - g^2 (V_I h^I)^2 + g \partial_r (V_I h^I) \sqrt{V} \Gamma_4 \right] \epsilon = 0, \quad (5.35)$$

- (r,x), (r,y), (r,z)

$$\left[VT'' + VT'^2 + g \partial_r (V_I h^I) \sqrt{V} \Gamma_4 \right] \epsilon = 0. \quad (5.36)$$

(5.33) – (5.34) leads to

$$\left[2e^{-2T_3} + 3igV_{IP}^I e^{-T_1 - T_2} \Gamma_{12} \right] \epsilon = 0, \quad (5.37)$$

which implies the Dirac-type quantization condition

$$V_{IP}^I = \sigma_1 \frac{2}{3g} e^{T_1 + T_2 - 2T_3}, \quad (5.38)$$

where $\sigma_1 = \pm 1$. Plugging this back into (5.37) gives

$$\Gamma_{12}\epsilon = i\sigma_1\epsilon. \quad (5.39)$$

With (5.39), the gaugino equation (5.25) becomes

$$\left[\frac{1}{3}\mathcal{G}_{ij}\sqrt{V}\partial_r\phi^j\Gamma_4 - g\partial_i(V_I h^I) - \sigma_1\frac{1}{2}\partial_i(h_I p^I)e^{-T_1-T_2} \right]\epsilon = 0. \quad (5.40)$$

If the scalar fields were constant, $\partial_r\phi^j = 0 \forall j$, this would imply

$$g\partial_i(V_I h^I) + \sigma_1\frac{1}{2}\partial_i(h_I p^I)e^{-T_1-T_2} = 0, \quad (5.41)$$

and thus T_1 and T_2 must be constant as well, which leads to a contradiction with the first equation of (5.27). Note that this conclusion is valid provided $\partial_i(V_I h^I)$ and $\partial_i(h_I p^I)$ do not both vanish. In the latter case, however, using one of the very special geometry relations, we have

$$0 = \mathcal{G}^{ij}\partial_i(h_I p^I)\partial_j(h_J p^J) = \frac{4}{9}G_{IJP^I}p^J - \frac{2}{3}h_I p^I h_J p^J = \frac{4}{9}G_{IJP^I}p^J, \quad (5.42)$$

where the last step follows from (5.31). Since G_{IJ} is positive definite, (5.42) leads to a contradiction. If $\partial_r\phi^i \neq 0$ for at least one i , one can multiply (5.40) with $\partial_r\phi^i$ and sum over i to get⁷

$$\left[\frac{1}{3}\mathcal{G}_{ij}\sqrt{V}\partial_r\phi^i\partial_r\phi^j\Gamma_4 - g\partial_r(V_I h^I) \right]\epsilon = 0. \quad (5.43)$$

We see immediately that one needs $\partial_r(V_I h^I) \neq 0$, since otherwise $\mathcal{G}_{ij}\partial_r\phi^i\partial_r\phi^j = 0$, which is impossible because \mathcal{G}_{ij} is a definite matrix.

To proceed, we require the determinants associated to the linear systems (5.35) and (5.36) to vanish, which implies the projection condition $\Gamma_4\epsilon = -\sigma_2\epsilon$ ($\sigma_2 = \pm 1$) as well as

$$\begin{aligned} \sigma_2 g\partial_r(V_I h^I)\sqrt{V} &= \frac{1}{2}V'' - g^2(V_I h^I)^2, \\ \sigma_2 g\partial_r(V_I h^I)\sqrt{V} &= VT'' + VT'^2. \end{aligned} \quad (5.44)$$

Deriving the prefactor of ϵ in (5.32) w.r.t. r , one obtains, using also (5.44) and (5.32),

$$\begin{aligned} 2g^2(V_J h^J)\partial_r(V_I h^I) &= \frac{1}{2}(V''T' + V'T'') \\ &= \left(\sigma_2 g\partial_r(V_I h^I)\sqrt{V} + g^2(V_I h^I)^2 \right)T' + \frac{1}{2}V' \left(\sigma_2 g\partial_r(V_I h^I)\frac{1}{\sqrt{V}} - T'^2 \right) \\ &= \sigma_2 g\partial_r(V_I h^I) \left(\sqrt{V}T' + \frac{V'}{2\sqrt{V}} \right). \end{aligned} \quad (5.45)$$

Thus, since $\partial_r(V_I h^I) \neq 0$,

$$g(V_I h^I) = \sigma_2 \frac{1}{2} \left(\sqrt{V}T' + \frac{V'}{2\sqrt{V}} \right). \quad (5.46)$$

⁷Notice that $\partial_r(h_I p^I) = 0$.

Derive this w.r.t. r and then subtract the sum of the two eqs. in (5.44), divided by two, to get

$$0 = \sigma_2 \frac{1}{2} \left(\frac{V'T'}{\sqrt{V}} - \frac{V'^2}{4V^{3/2}} - \sqrt{V}T'^2 \right) = -\sigma_2 \frac{\sqrt{V}}{2} \left(\frac{V'}{2V} - T' \right)^2 = -\sigma_2 \frac{1}{2V^{3/2}T'^2} e^{-4T_3}, \tag{5.47}$$

where the last step follows from the first eq. of (5.27). Evidently, (5.47) leads to a contradiction, which implies

Proposition 2. *There are no static, Sol-invariant solutions to the Killing spinor equations with solvegeometry spatial cross-sections at fixed r and purely magnetic field strengths.*

In particular, there is no BPS limit of the black hole constructed in section 4. Note in this context that rotating supersymmetric Nil and $\widetilde{\text{SL}}(2, \mathbb{R})$ near-horizon geometries were found in [30]. Moreover, it is worth mentioning that the near-horizon limit of all supersymmetric extremal black holes in gauged (and ungauged) five-dimensional supergravity coupled to abelian vector multiplets must admit an $\text{SL}(2, \mathbb{R})$ symmetry group [31]. This follows from an index theory argument and extends earlier results of [32] for minimal gauged supergravity.

6 Attractor mechanism

According to the attractor mechanism [17–21], the entropy of an extremal black hole and the scalar fields on the event horizon are insensitive to the asymptotic values of the moduli and depend only on the electric and magnetic charges.⁸ This phenomenon was first discovered in four-dimensional ungauged supergravity for BPS black holes [17] and proved in [21]. It was subsequently extended to higher dimensions, non-supersymmetric or rotating solutions, and gauged supergravities, cf. [33–41] for an (incomplete) list of references. In particular, the generalization of the proof of [21] to five dimensions was given in [42], and to general spatial and worldvolume dimensions in [43]. Notice that [42, 43] are valid for asymptotically flat black holes only. There is no generalization that covers at the same time dimensions higher than four and non-flat asymptotics.

A recurrent feature in all these cases is that the scalar configuration on the horizon can be determined by extremizing an effective potential and that the entropy is given by the value of this potential at its extremum.

In this section, we study the attractor mechanism for extremal static black holes with nil- or solvegeometry horizons in the theory (2.1). It will turn out that there are no such attractors for purely electric field strengths, while in the magnetic case there are attractor geometries, for which we explicitly determine the effective potential V_{eff} , which contains the charges as well as the scalar potential of the gauged supergravity theory.

⁸This is valid in the absence of flat directions in the effective potential for the scalars. In the generic situation not all the moduli are stabilized on the horizon. Nevertheless, the black hole entropy is still independent of the values of the scalars that are not stabilized.

6.1 Magnetic ansatz

As a first step to extend the black hole solution (4.3) to the matter-coupled case, we consider the near-horizon limit of the ansatz (3.1). Following closely the argument presented in [41], we are interested in magnetically charged, static and extremal black holes with Sol horizon, but without referring to any particular model of very special geometry. Extremality implies that the near-horizon geometry is the product manifold $\text{AdS}_2 \times \text{Sol}$. Assuming the horizon to be located at $r = 0$, we have thus for $r \rightarrow 0$

$$V(r) \sim \left(\frac{r}{r_{\text{AdS}}}\right)^2, \quad T_1(r) \sim \frac{1}{4} \ln A, \quad T_3(r) \sim \frac{1}{2} \ln B, \quad \phi^i(r) \sim \phi_0^i, \quad (6.1)$$

with r_{AdS} the curvature radius of the AdS_2 part, A and B positive constants and ϕ_0^i the horizon values of the scalar fields. The Einstein equations (3.14) become then algebraic and admit the solution

$$A = -\frac{\Sigma_0}{g^2 U_0}, \quad B = -\frac{2}{g^2 U_0}, \quad r_{\text{AdS}}^2 = -\frac{1}{g^2 U_0}, \quad (6.2)$$

where $U_0 \equiv U(\phi_0^i) < 0$ and $\Sigma_0 \equiv G_{IJ}(\phi_0^i) p^I p^J$. Using (6.1) and (6.2), the equations (3.15) for the scalars boil down to

$$\partial_i V_{\text{eff}}|_{\phi_0^i} = 0, \quad (6.3)$$

where

$$V_{\text{eff}}(\phi^i) = \frac{\sqrt{G_{IJ}(\phi^i) p^I p^J}}{2\sqrt{2}g^2|U(\phi^i)|} \quad (6.4)$$

is an effective potential whose normalization has been chosen for later convenience. Thus, the attractor solution reads

$$ds^2 = -g^2|U_0|r^2 dt^2 + \frac{dr^2}{g^2|U_0|r^2} + \sqrt{\frac{\Sigma_0}{g^2|U_0|}} (e^{2z} dx^2 + e^{-2z} dy^2) + \frac{2}{g^2|U_0|} dz^2, \quad (6.5)$$

$$F^I = p^I dx \wedge dy, \quad \phi^i(r) = \phi_0^i. \quad (6.6)$$

The horizon values ϕ_0^i of the scalars are computed by extremization of the effective potential (6.4) and (unless V_{eff} has flat directions) are completely fixed by the magnetic charges and the constants V_I , in accordance with the attractor mechanism. Finally, the entropy density is given by

$$s = V_{\text{eff}}(\phi_0^i). \quad (6.7)$$

Notice that, even if V_{eff} has flat directions, and thus (some of) the moduli at the horizon are not stabilized, (6.7) implies that the Bekenstein-Hawking entropy is given by the value of V_{eff} at its minimum, which depends only on the magnetic charges p^I and the parameters V_I . As a consequence of the results of section 5.2, the attractor geometry (6.5), (6.6) breaks all the supersymmetries.

As an example, we consider the stu model, which involves two vector multiplets, and has $C_{012} = 1$ and its permutations as only nonvanishing components of C_{IJK} . We define $t = \phi^1$, $u = \phi^2$, and choose the parametrization $h^1 = t$, $h^2 = u$ and $h^0 = s = (tu)^{-1}$,

where the last relation follows from (2.3). Using the expressions of section 2 and taking $V_I = 1/3 \forall I$, we get

$$G_{IJ} = \frac{1}{2} \text{diag}(s^{-2}, t^{-2}, u^{-2}), \quad \mathcal{G}_{ij} = \begin{bmatrix} \frac{1}{t^2} & \frac{1}{2tu} \\ \frac{1}{2tu} & \frac{1}{u^2} \end{bmatrix}, \quad (6.8)$$

$$U(t, u) = -2 \left(tu + \frac{1}{t} + \frac{1}{u} \right). \quad (6.9)$$

The effective potential (6.4) becomes

$$V_{\text{eff}}(t, u) = \frac{\sqrt{(p^0)^2 t^2 u^2 + (p^1)^2 t^{-2} + (p^2)^2 u^{-2}}}{8g^2 (tu + t^{-1} + u^{-1})}, \quad (6.10)$$

and the eqs. (6.3) boil down to

$$\begin{aligned} (p^0)^2 t^3 u^3 (t + 2u) - (p^1)^2 (u + 2tu^3) - (p^2)^2 (t^3 u - t) \Big|_{t_0, u_0} &= 0, \\ (p^0)^2 t^3 u^3 (2t + u) - (p^1)^2 (tu^3 - u) - (p^2)^2 (t + 2t^3 u) \Big|_{t_0, u_0} &= 0. \end{aligned} \quad (6.11)$$

6.2 Electric ansatz

We now consider the case of purely electric field strengths. For a horizon modelled on solvegeometry, by means of the constraint (3.9) and the structure constants (A.6), the fourth eq. of (3.6) reduces to ($A = 1, 3$)

$$\begin{aligned} -V'T'_1 - VT''_1 - VT'_1(2T'_1 + T'_3) &= \frac{1}{3} e^{-2(2T_1+T_3)} G^{IJ} q_I q_J + \frac{2}{3} g^2 U, \\ -V'T'_3 - VT''_3 - VT'_3(2T'_1 + T'_3) - 2e^{-2T_3} &= \frac{1}{3} e^{-2(2T_1+T_3)} G^{IJ} q_I q_J + \frac{2}{3} g^2 U, \end{aligned} \quad (6.12)$$

which immediately implies that a configuration with T_1 and T_3 constant is not acceptable.⁹ One can try to relax the ansatz on T_1 and T_3 by assuming a generic power dependence like

$$e^{2T_1} \sim k_1 r^{\alpha_1}, \quad e^{2T_3} \sim k_3 r^{\alpha_3}, \quad (6.13)$$

with k_A and α_A constants, but consistency of eqs. (6.12) requires $\alpha_A = 0$ and we fall into the previous contradictory case.

For a horizon modelled on nilgeometry, cf. (A.9)–(A.11), the fourth eq. of (3.6) gives

$$\begin{aligned} -V'T'_1 - VT''_1 - VT'_1 \sum_{B=1}^3 T'_B + \frac{1}{2} e^{2(T_1-T_2-T_3)} &= \frac{q^2}{3} e^{-2(T_1+T_2+T_3)} + \frac{2}{3} g^2 U, \\ -V'T'_2 - VT''_2 - VT'_2 \sum_{B=1}^3 T'_B - \frac{1}{2} e^{2(T_1-T_2-T_3)} &= \frac{q^2}{3} e^{-2(T_1+T_2+T_3)} + \frac{2}{3} g^2 U, \\ -V'T'_3 - VT''_3 - VT'_3 \sum_{B=1}^3 T'_B - \frac{1}{2} e^{2(T_1-T_2-T_3)} &= \frac{q^2}{3} e^{-2(T_1+T_2+T_3)} + \frac{2}{3} g^2 U, \end{aligned} \quad (6.14)$$

⁹For $T'_1 = T'_3 = 0$, the difference of the two eqs. leads to $e^{-2T_3} = 0$.

where $q^2 = G^{IJ}q_Iq_J$. Again, an ansatz with T_1 , T_2 and T_3 constant does not work, since in that case the difference of the first and the second eq. of (6.14) yields

$$e^{2(T_1-T_2-T_3)} = 0. \tag{6.15}$$

If we assume $e^{2T_A} \sim k_A r^{\alpha_A}$ and plug this ansatz into (6.14), we end up with $\alpha_A = 0$, which we have just seen to lead to a contradiction. One obtains thus the following

Proposition 3. *There are no static attractors with Sol or Nil horizons and purely electric field strengths.*

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A Homogeneous manifolds

Let M be a (pseudo)-Riemannian manifold with isometry group G . M is said to be homogeneous if G acts transitively on M , i.e. if $\forall p, q \in M$ there exists an isometry $\phi \in G$ such that $\phi(p) = q$. The action of G on M is called simply transitive if the element ϕ is unique or, equivalently, if $\dim M = \dim G$. In this case, M itself is said to be simply transitive.

Let us restrict our discussion to a simply transitive manifold. Since $\dim M = \dim G$, the Killing vectors ξ_A ($A = 1, \dots, \dim M$) form a basis of the tangent space. However, it is more convenient [25] to choose a G -invariant basis X_A , i.e., a basis such that

$$\mathcal{L}_{\xi_B} X_A = [\xi_B, X_A] = 0 \quad \forall A, B, \tag{A.1}$$

with $\mathcal{L}_{\xi_B} X_A$ the Lie derivative of the vector field X_A along ξ_B . The dual basis θ^A of a G -invariant basis X_A is also G -invariant, $\mathcal{L}_{\xi_B} \theta^A = 0$, and satisfies

$$d\theta^A = \frac{1}{2} C^A_{BC} \theta^B \wedge \theta^C, \tag{A.2}$$

with C^A_{BC} the structure constants of the Lie algebra of G . Furthermore, a simply transitive homogeneous manifold can be equipped with a metric

$$ds^2 = g_{AB} \theta^A \theta^B, \tag{A.3}$$

where the components g_{AB} are constant on M .

Bianchi showed that in total there are nine three-dimensional Lie algebras, the so-called nine Bianchi cosmologies, labelled from type I to type IX. The name ‘cosmologies’ comes from the fact that these manifolds are used as spatial sections in many spatially homogeneous but anisotropic cosmological models. The Bianchi cosmologies are divided into two classes, A and B, according to the way the structure constants C^A_{BC} can be expanded (see table 6.2 of [25] for details). In particular, class A spacetimes satisfy

$$\sum_A C^A_{AB} = 0. \tag{A.4}$$

Bianchi	Thurston
I, VII ₀	E ³
II	Nil
VI ₋₁	Sol
VIII	$\widetilde{\text{SL}}(2, \mathbb{R})$
IX	S ³

Bianchi	Thurston
III	H ² × ℝ
V, VII _{h≠0}	H ³

Table 1. Class A (left) and B (right) spacetimes and corresponding Thurston geometries.

An important result in geometric topology is the Thurston conjecture [44], which states that every three-dimensional closed and orientable manifold has a geometric structure modelled on one of the eight model geometries

$$S^3, \quad E^3, \quad H^3, \quad S^2 \times \mathbb{R}, \quad H^2 \times \mathbb{R}, \quad \text{Nil}, \quad \text{Sol}, \quad \widetilde{\text{SL}}(2, \mathbb{R}), \quad (\text{A.5})$$

where $\widetilde{\text{SL}}(2, \mathbb{R})$ is the universal covering of $\text{SL}(2, \mathbb{R})$. In [45] it was shown that there exists a correspondence, not necessarily one to one, between the nine Bianchi cosmologies and the eight Thurston model geometries, which is summarized in table 1.¹⁰ In the following, we list explicitly the metrics for solvegeometry/VI₋₁ and nilgeometry/II in terms of G -invariant one-forms θ^A , as well as the nonvanishing structure constants of the related Lie algebras.

- Solvegeometry:

$$C^1_{13} = -C^1_{31} = 1, \quad C^2_{23} = -C^2_{32} = -1, \quad (\text{A.6})$$

$$\theta^1 = e^z dx, \quad \theta^2 = e^{-z} dy, \quad \theta^3 = -dz, \quad (\text{A.7})$$

$$ds^2 = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2. \quad (\text{A.8})$$

- Nilgeometry:

$$C^1_{23} = -C^1_{32} = 1, \quad (\text{A.9})$$

$$\theta^1 = dz - x dy, \quad \theta^2 = dy, \quad \theta^3 = dx, \quad (\text{A.10})$$

$$ds^2 = (dz - x dy)^2 + dy^2 + dx^2. \quad (\text{A.11})$$

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¹⁰The Bianchi types IV and VI_{h≠-1} are not contained in this correspondence. Moreover, the Thurston geometry $S^2 \times \mathbb{R}$ is missing since it corresponds to the Kantowski-Sachs model, in which G does not act simply transitively or does not possess a subgroup with simply transitive action.

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