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Almost Global Existence for Some Hamiltonian PDEs with Small Cauchy Data on General Tori

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Abstract: In this paper we prove a result of almost global existence for some abstract nonlinear PDEs on flat tori and apply it to some concrete equations, namely a nonlinear Schrödinger equation with a convolution potential, a beam equation and a quantum hydrodinamical equation. We also apply it to the stability of plane waves in NLS. The main point is that the abstract result is based on a nonresonance condition much weaker than the usual ones, which rely on the celebrated Bourgain's Lemma which provides a partition of the "resonant sites" of the Laplace operator on irrational tori.

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1. Introduction

The problem of studying long time behaviour of solutions of Hamiltonian non linear PDEs on compact manifolds is fundamental and widely studied. In this paper we focus on the so called problem of "almost global existence", namely the problem of provingthat solutions corresponding to smooth and small initial data remain smooth and small for times of order ϵ^{-r} with arbitrary r; here ϵ is the norm of the initial datum.

We recall that there exist quite satisfactory results for semilinear equations in one space dimension [1,5,19], which have also been extended to some semilinear PDEs with unbounded perturbations [47] and to some quasilinear wave equations [24], gravity capillary water waves [12] (see also [13]), capillary water waves [42], quasi-linear Schrödinger [32] and pure gravity water waves [14] still in dimension one. We also mention that for semilinear PDEs there are some results about sub-exponentially long stability time, see for instance [16,17,22,30,34].

On the other hand for the case of higher dimensional manifolds only particular examples are known [4,11,23,26–28,33] and for PDEs in higher space dimension with unbounded perturbations only partial results have been obtained [31,35,43]. A slightly different point of view is the one developed in [46] in which the authors give some upper bounds on the possible energy transfer to high modes, for initial data Fourier supported in a box for the cubic NLS on the irrational square torus in dimension two.

To discuss the main difficulty met in order to obtain almost global existence in more than one space dimension, we recall that all the known results deal with perturbations of linear systems whose eigenvalues are of the form $\pm i\omega_j$ with ω_j real numbers playing the role of frequencies. Here j belongs to some countable set of indexes, say $\mathbb{Z}^d$, $d \geq 1$ (for instance).

The main point is that, in all known results, the frequencies are assumed to verify a certain non-resonance condition. More precisely, for some fixed $\gamma, \tau > 0$, one typically requires

$$\left| \sum_{i=1}^{\ell} \omega_{j_i} - \sum_{k=\ell+1}^{r} \omega_{j_k} \right| \geq (\max_3 \{|j_1|, ..., |j_r|\})^{-\tau} \gamma, \tag{1.1}$$

except in the case

$$r \ \text{ even}, \quad \ell = \frac{r}{2} \quad \text{and (up to permutations)} \quad j_i = j_{i+\frac{r}{2}}, \tag{1.2}$$

where $\max_3 \{|j_1|, ..., |j_r|\}$ denotes the third largest number among $|j_1|, ..., |j_r|$. Condition (1.1) is a kind of second Melnikov condition since it requires to control linear combinations involving two frequencies with index arbitrarily large. Monomials in the vector field supported on indexes satisfying (1.2) are called *resonant monomials*, which are the ones that cannot be canceled out through a Birkhoff normal form procedure. We first remark that conditions (1.1)–(1.2) are quite strong, and, in particular, (1.2) implies that the only resonant monomials are *action preserving* in the sense that $|u_j|^2$ are constants of motion. Secondly one can easily convince that there are plenty of situation in which the conditions above are violated. Just as an example, even in dimension $d = 1$

(in the case $j \in \mathbb{Z}$), and assuming ω_j even in j one can only hope to impose (1.1) except in the case

$$r \text{ even}, \quad \ell = \frac{r}{2} \quad \text{and (up to permutations)} \quad |j_i| = |j_{i+\frac{r}{2}}|, \tag{1.3}$$

which is weaker than (1.2). Indeed it is no more true that the "actions" $|u_j|^2$ are preserved. On the contrary one can only infer that the so called *super-actions* are preserved by the motion, i.e. quantities of the form (in the case $d = 1$)

$$|u_j|^2 + |u_{-j}|^2.$$

This suggests that the situation in which the linear system has *multiple* eigenvalues is more delicate. We mention, for instance, [29] where the authors deal with the multiplicity of the eigenvalues of the Laplacian on $\mathbb{T}^d$, $d > 1$, by introducing the *super-actions*

$$J_n := \sum_{k\in\mathbb{Z}^d, |k|^2=n} |u_k|^2.$$

We finally remark that, apart from the possible multiplicity of eigenvalues, to have "good" lower bounds as in (1.1) is fundamental, in classical approaches, to prove the well-posedness of the Birkhoff map. It is also well known that to prove such lower bounds one needs to have *good separation properties* of the linear eigenvalues. For instance one can think of the Laplacian on $\mathbb{T}^1 = \mathbb{S}^1$ where differences between eigenvalues *grows* at infinity since

$$||j|^2 - |k|^2| \geq |j| + |k|, \quad \forall\, j, k \in \mathbb{Z}.\ |j| \neq |k|.$$

This property holds, in some special cases also in high dimensions. For example it holds in the case of the Laplace-Beltrami operator on $\mathbb{S}^d$ and more in general holds for compact manifolds that are homogeneous with respect to a compact Lie Group of rank 1. These are special situations in high dimension in which it is still possible to prove bounds like (1.1), so essentially the problem of Birkhoff normal form can be treated as in the one dimensional case. These are the cases treated in [5,6,30]. Nevertheless, in general high dimensional settings differences of eigenvalues accumulate to zero (for example in the case of Δ on straight, irrational tori) and the Diophantine condition (1.1) is typically violated. We refer to [7] where properties of the Laplacian on general tori are discussed. In these more resonant cases it is anyway still possible to prove much *weaker Diophantine conditions* of the form

$$\left|\sum_{i=1}^{\ell} \omega_{j_i} - \sum_{k=\ell+1}^{r} \omega_{j_k}\right| \geq (\max\{|j_1|, ..., |j_r|\})^{-\tau}\gamma \tag{1.4}$$

for all possible choices of indexes $j_1, ..., j_r$, except the case (1.3). *This is a condition typically fulfilled in any space dimension*. The crucial point is that condition (1.4) allows the small divisors to accumulate to zero *very* fast (as the largest index among $|j_1|, \dots, |j_r|$ goes to infinity), and this could in principle create a *loss of derivatives* in the construction of the map used to put the system in Birkhoff normal form. We refer for instance to [11,35,43] (and reference therein) and where this problem is dealt with to prove partial long time stability results, by imposing (1.4) for small r (say $r = 3, 4$). By partial results we mean that, in the latter papers, the time scales of stability are of order at most ϵ^{-q} with a strong limitation on $q \leq 4$, and they left open the case q large.

In the present paper our aim is to develop a novel and self-contained framework in order to prove almost global existence (see Theorem 2.10 where *any* r are considered) for some Hamiltonian PDEs in which the linear frequencies are assumed to fulfil the weak condition (1.4).

The key point is that we also require the frequencies ω_j and the indexes j to fulfill a structural property ensured by a Lemma by Bourgain on the "localization of resonant sites" in $\mathbb{T}^d$. This allows to prove a theorem ensuring that the Hamiltonian of the PDE can be put in a suitable block-normal form which can be used to control the growth of Sobolev norms. For more details we refer the reader to the last paragraph of this introduction.

We emphasize that one of the points of interest of our paper is that it shows the impact of results of the kind of [15,20,25] dealing with linear time dependent systems on nonlinear systems, thus, in view of the generalizations [7–9], it opens the way to the possibility of proving almost global existence in more general systems, e.g. on some manifolds with integrable geodesic flow.

In the present paper, after proving the abstract result, we apply it to a few concrete equations for which almost global existence was out of reach with previous methods. Precisely we prove almost global existence of small amplitude solutions (1) for nonlinear Schrödinger equations with convolution potential, (2) for nonlinear beam equations and (3) for a quantum hydrodinamical model (QHD). We also prove Sobolev stability of plane waves for the Schrödinger equation (following [29]). We emphasize that these results were known only for the exceptional case of the *square torus*. We remark that our main theorem extends some partial results on the models listed above, we refer for instance to [33] for the QHD system (case (3)) and [11] for the Beam equation (case (2)). For irrational tori the only result (as far as we know) ensuring at least a quadratic lifespan of nonlinear Schrödinger equations with unbounded, quadratic nonlinearities has been proved in [35]. The present paper, at least for semilinear nonlinearity, provides a method to prove polinomially long time stability for NLS on irrational tori.

To present in a more precise way the result, we recall that an arbitrary torus can be easily identified with the standard torus endowed by a flat metric. This is the point of view we will take. For the Schrödinger equation we show that, without any restrictions on the metric of the torus, one has that if the potential belongs to a set of full measure then one has almost global existence. For the case of the beam equation, we use the metric in order to tune the frequencies and to fulfill the nonresonance condition, thus we prove that if the metric of the torus is chosen in a set of full measure then almost global existence holds. Examples of tori fulfilling our property are rectangular tori with diophantine sides, but also more general tori are allowed.

The result for the QHD model is very similar to that of the beam equation: if the metric is chosen in a set of full measure, then almost global existence holds. Also the result of Sobolev stability of plane waves in the Schrödinger equation is of the same kind: if the metric belongs to a set of full measure, one has stability of the plane waves over times longer than any inverse power of ϵ.

We also recall the result [10] in which the authors consider a nonlinear wave equation on $\mathbb{T}^d$ and prove that if the initial datum is small enough in some Sobolev norm then the solution remains small in a weaker Sobolev norm for times of order ϵ^{-r} with arbitrary r. The main difference is that this result involves a loss of smoothness of the solution which is not present in our result; however, we emphasize that at present our method does no apply to the wave equation since no generalizations of Bourgain's Lemma to systems of first order are known.

Finally we remark that our point of view is to show that solutions starting from a ball of radius ϵ do not reach the boundary of a ball of radius 2ϵ for very long time. Proving this implies both the *existence* and the *stability* of the solution over a large time scale $O(\epsilon^{-r})$. A different point of view is to give upper bounds on the possible growth of the Sobolev norm in terms of the time t. This problem, as already remarked, has been tackled widely for linear equations. However we mention [18,44,45] and the recent result [41] dealing with nonlinear equations. A dual point of view is to study possible *instability* of solutions, namely to show that even solutions evolving from *small* initial data could show a large growth of the Sobolev norm by waiting for sufficiently long time. Without trying to be exhaustive we quote [21,36–40].

Ideas of the proof of the abstract result. Our aim is to study the dynamics of a Hamiltonian system whose corresponding Hamiltonian has an elliptic fixed point at the origin. Passing to the Fourier side and in appropriate complex coordinates u_j we assume that the Hamiltonian has the form

$$H(u) = H_0 + P(u), \qquad u = (u_j)_{j\in\mathbb{Z}^d}, \quad H_0 := \sum_{j\in\mathbb{Z}^d} \omega_j |u_j|^2,$$

where ω_j are the linear frequencies of oscillations, the unknown u belongs to some scale of separable Hilbert spaces (we will work actually on scales of Sobolev spaces) and the perturbation $P = O(u^q)$ is a regular enough (say C^∞) function having a zero at the origin of order at least $q \geq 3$. We also assume that H conserves the momentum. The precise assumptions on H are given in Sect. 2.2. By classical theory one expects that the homogeneous terms of high degree (at least q in this example) give a small contribution to the dynamics of the linear Hamiltonian. In other words, for u belonging to a small ball around the origin of order ϵ one expects a bound like ϵ^{q-1} for the vector field X_P generated by the perturbation P. This would implies the stability of solutions, evolving from initial data of size ϵ, over a times scale of order $O(\epsilon^{-q})$. In classical Birkhoff normal form approach the main idea is to construct a symplectic change of coordinates Φ which transform the Hamiltonian H into

$$H \circ \Phi = H_0 + Z + O(u^{r+2}), \qquad r \gg q,$$

where Z is in *standard Birkhoff normal form*, i.e. it Poisson commutes with H_0. Under suitable non-resonance conditions on the frequencies ω_j one can also ensure that Z Poisson commutes with the *Sobolev norms*

$$\|u\|_s^2 = \sum_{j\in\mathbb{Z}^d} (1 + |j|^2)^{\frac{s}{2}} |u_j|^2, \tag{1.5}$$

which is not a priori guaranteed only by the condition $\{Z, H_0\} = 0$. However in this strong non-resonant case, one expect a time of stability of order $O(\epsilon^{-r})$, since since neither H_0 nor Z contribute to the possible *growth* of the Sobolev norm. Of course this is a very favourable situation. General settings are usually more complicated and the strategy described above fails.

Our point of view is the following. First of all, following [5], we decompose the variables in variables of large index (high modes) and variables of small index (low modes), i.e. we split

$$u = u^{\leq} + u^{\perp}, \qquad N \gg 1,$$

where N is a fixed large constant and $u^{\perp}$ is supported only on u_j with indexes $|j| > N$. The first crucial observation is that the terms in the Hamiltonian which are at least *cubic* in high variables $u^{\perp}$ (the case $q = 3$) give a very small contribution. Indeed if u is in a space of sufficiently high regularity (say H^s) one expects a tame-like bound N^{-s+s_0} for the generated vector field (see Lemma 3.8). Therefore as first step we split the Hamiltonian function as

$$H = H_0 + P_0 + P_1 + P_2 + P_{\perp},$$

where $P_{\perp}$ has a zero of order at least 3 in $u_{\perp}$, where P_j, $j = 0, 1, 2$ is homogeneuos of degree j in $u_{\perp}$. We have the following important remarks:

- First of all we remark that, thanks to the conservation of momentum the monomials (of homogeneity r) appearing in the perturbation P have the form

$$\Big(\prod_{i=1}^{\ell} u_{j_i}\Big)\Big(\prod_{i=\ell+1}^{r} \overline{u_{j_i}}\Big) \quad \text{for some} \quad 0 \le \ell \le r,$$
$$j_1 + \cdots + j_\ell - j_{\ell+1} - \cdots - j_r = 0.$$

  All the resonant monomials, i.e. the ones Poisson commuting with H_0, are those supported on indexes satisfying

$$\sum_{i=1}^{\ell} \omega_{j_i} - \sum_{k=\ell+1}^{r} \omega_{j_k} = 0.$$

  Hypothesis 2.8 guarantees that the condition above is verified if and only if up to permutation, one has (see (2.22))

$$r = 2\ell \quad \text{and} \quad \omega_{j_i} = \omega_{j_{i+\ell}}, \qquad i = 1, \ldots, \ell.$$

  This implies that resonant monomials Poisson commute both with H_0 and with the Sobolev norm $\| \cdot \|_s^2$ in (1.5).
- the term $P_{\perp}$ already gives a small contribution, at least for regular u. So we do not apply any normal form procedure to eliminate monomials belonging to $P_{\perp}$.
- By momentum conservation if a homogenous term of degree q has only one high variable u_j with $|j| > N$ then one has the bound $|j| \le qN$. This means that these monomials can be eliminate just by requiring the very weak non-resonance condition 1.4. Indeed, in this case, the right hand side of (1.4) can be bounded from below by a constant depending only on N. Then no loss of derivatives can arise from these small divisors. Only resonant monomials cannot be eliminated. See the first item for details.
- The crucial point of our strategy is to deal with the terms belonging to P_2, and here it is fundamental the second assumption on the frequencies ω_j, i.e. they fulfil the *Bourgain's clustering property*. We refer to Hypothesis 2.5 for a precise statement. Roughly speaking such property implies that the is a partition of $\mathbb{Z}^d = \cup_\alpha \Omega_\alpha$, made by clusters $\Omega_\alpha \subset \mathbb{Z}^d$ with the following properties: the clusters have a dyadic property that allows to control the H^s-norm with the L^2-norm, and indexes $j, k \in \mathbb{Z}^d$ belonging to different clusters $j \in \Omega_\alpha$, $k \in \Omega_\beta$, possesses frequencies ω_j and ω_k which are *well-separated*. See formula (2.19).

Now, consider a monomial of the form

$$u_{j_1}\overline{u_{j_2}}\Big(\prod_{i=3}^{\ell} u_{j_i}\Big)\Big(\prod_{i=\ell+1}^{q} \overline{u_{j_i}}\Big) \quad \text{with} \quad |j_1| \sim |j_2| \gg \max\{|j_3|, \dots, |j_q|\}. \tag{1.6}$$

Hypothesis 2.5 guarantees that if the two highest indexes j_1, j_2 do not belong to the same cluster then the very weak lower bounds in (1.4) can be improved. This is the content of the fundamental Lemma 3.17. Therefore one can cancel out all the monomials in P_2 with the exception of those monomials in (1.6) for which j_1, j_2 belong to the same Bourgain's cluster.

In conclusion, performing a normal form procedure takin into account the remarks above, we transform the Hamiltonian H into (see Theorem 3.3)

$$\widetilde{H} = H_0 + Z_0 + Z_2 + R_T + R_\perp,$$

where $R_\perp$ is homogeneous of degree at least 3 in $u^\perp$, R_T has large minimal degree $O(u^r)$, Z_0 is supported only on *low* modes and commutes both with H_0 and $\|\cdot\|_s^2$, while Z_2 is quadratic in the high variables, i.e. it can be seen as a quadratic form in the high variables with coefficients the low variables. In particular it is in block-diagonal normal form (according to Definition 3.2), namely the two highest indexes belong to the same Bourgain's cluster. The important consequence, proved in Lemma 4.3, is that the flow generated by Z_2 is uniformly bounded in H^s. This follows the ideas implemented in [15,20,25] to give upper bounds on the flows of linear Schrödinger equations with multiplicative potential.

2. The Abstract Theorem

2.1. Phase Space. Denote $\mathcal{Z}^d := \mathbb{Z}^d \times \{-1, 1\}$. Let g be a positive definite, symmetric, quadratic form on $\mathbb{Z}^d$ and, for $J \equiv (j, \sigma) \in \mathcal{Z}^d$, denote

$$|J|^2 \equiv |j|^2 := \sum_{i=1}^{d} |j_i|^2, \quad |J|_g^2 \equiv |j|_g^2 := g(j, j). \tag{2.1}$$

We define

$$\ell_s^2(\mathcal{Z}^d; \mathbb{C}) := \Big\{u \equiv (u_J)_{J\in\mathcal{Z}^d}, \quad u_J \in \mathbb{C}, \; : \\ \|u\|_s^2 := \sum_{J\in\mathcal{Z}^d} (1 + |J|)^{2s} \, |u_J|^2 < \infty\Big\}. \tag{2.2}$$

In the following we will simply write ℓ_s^2 for $\ell_s^2(\mathcal{Z}^d; \mathbb{C})$ and ℓ^2 for ℓ_0^2. We denote by $B_s(R)$ the open ball of radius R and center 0 in ℓ_s^2. Furthermore in the following $\mathcal{U}_s \subset \ell_s^2$ will always denote an open set containing the origin.

We endow ℓ^2 by the symplectic form $\mathrm{i}\sum_{j\in\mathbb{Z}^d} u_{(j,+)} \wedge u_{(j,-)}$, which, when restricted to ℓ_s^2 ($s > 0$), is a weakly symplectic form.

Correspondingly, given a function $H \in C^1(\mathcal{U}_s)$, for some s, its Hamilton equations are given by

$$\dot{u}_{(j,+)} = -\mathrm{i}\frac{\partial H}{\partial u_{(j,-)}}, \qquad \dot{u}_{(j,-)} = \mathrm{i}\frac{\partial H}{\partial u_{(j,+)}}, \tag{2.3}$$

or, compactly

$$\dot{u}_{(j,\sigma)} = -\sigma \mathrm{i} \frac{\partial H}{\partial u_{(j,-\sigma)}} . \tag{2.4}$$

We will also denote by

$$X_H(u) := (X_J)_{J \in \mathcal{Z}^d}, \qquad X_{(j,\sigma)} := -\sigma \mathrm{i} \frac{\partial H}{\partial u_{(j,-\sigma)}} \tag{2.5}$$

the corresponding (formal) Hamiltonian vector field.

In the following we will work on the space ℓ^2_s with s large. More precisely, all the properties we will ask will be required to hold for all s large enough.

2.2. The Class of Functions (and Perturbations). Given an index $J \equiv (j, \sigma) \in \mathcal{Z}^d$ we define the involution

$$\bar{J} := (j, -\sigma). \tag{2.6}$$

Given a multindex $\mathbf{J} \equiv (J_1, ..., J_r)$, with $J_l \in \mathcal{Z}^d$, $l = 1, ..., r$, we define $\bar{\mathbf{J}} := (\bar{J}_1, ..., \bar{J}_r)$.

On the contrary, *for a complex number the bar will simply denote the complex conjugate*.

Definition 2.1. On ℓ^2_s we define the involution I by

$$(Iu)_J := \overline{u_{\bar{J}}}. \tag{2.7}$$

The sequences such that $Iu = u$ will be called *real sequences*.

Given a multi-index $\mathbf{J} \equiv (J_1, ..., J_r)$, we also define its momentum by

$$\mathcal{M}(\mathbf{J}) := \sum_{l=1}^{r} \sigma_l j_l. \tag{2.8}$$

In particular in the following we will deal almost only with multi indexes with zero momentum, so we define

$$\mathcal{I}_r := \left\{ \mathbf{J} \in (\mathcal{Z}^d)^r \ : \ \mathcal{M}(\mathbf{J}) = 0 \right\}. \tag{2.9}$$

Given a homogeneous polynomial P of degree r, namely $P : \ell^2_s \to \mathbb{C}$ for some s, it is well known that it can be written in a unique way in the form

$$P(u) = \sum_{J_1,...,J_r \in \mathcal{Z}^d} P_{J_1,...,J_r} u_{J_1} ... u_{J_r} , \tag{2.10}$$

with $P_{J_1,...,J_r} \in \mathbb{C}$ symmetric with respect to any permutation of the indexes.

We are now ready to specify the class of functions we will consider.

Definition 2.2 (*Polynomials*). Let $r \geq 1$. We denote by $\mathcal{P}_r$ the space of formal polynomials $P(u)$ of the form (2.10) satisfying the following conditions:

P.1 *(Momentum conservation):* $P(u)$ contains only monomyals with zero momentum, namely (recall (2.9))

$$P(u) = \sum_{\mathbf{J}\in\mathcal{I}_r} P_{\mathbf{J}} u_{J_1} \ldots u_{J_r}; \tag{2.11}$$

P.2 *(Reality):* for any $\mathbf{J} \in (\mathcal{Z}^d)^r$, one has $\overline{P_{\bar{\mathbf{J}}}} = P_{\mathbf{J}}$.
P.3 *(Boundedness):* The coefficients $P_{\mathbf{J}}$ are bounded, namely

$$\sup_{\mathbf{J}\in\mathcal{I}_r} |P_{\mathbf{J}}| < \infty.$$

For $R > 0$ we endow the space $\mathcal{P}_r$ with the family of norms

$$\|P\|_R := \sup_{\mathbf{J}\in\mathcal{I}_r} |P_{\mathbf{J}}| R^r. \tag{2.12}$$

Given $r_2 \geq r_1 \geq 1$ we denote by $\mathcal{P}_{r_1,r_2} := \bigcup_{l=r_1}^{r_2} \mathcal{P}_l$ the space of polynomials $P(u)$ that may be written as

$$P = \sum_{l=r_1}^{r_2} P_l, \qquad P_l \in \mathcal{P}_l,$$

endowed with the natural norm

$$\|P\|_R := \sum_{l=r_1}^{r_2} \|P_l\|_R .$$

Of course other possible choices for the norm (2.12) are possible (see for instance the majorant norm on multilinear operators in [17]). However this choice is sufficient to prove the needed properties on the polynomials in $\mathcal{P}_r$. We refer to Sect. 3.1.

Remark 2.3. By the reality condition $(P.2)$ in Definition 2.2, one can note that if $P \in \mathcal{P}_r$ then

- $P(u) \in \mathbb{R}$ for all real sequence u (see Definition 2.1).
- Fix $J_1, J_2 \in \mathbb{Z}^d$ and define

$$A_{J_1,J_2}(u) := \sum_{\substack{J_3,\ldots,J_r\in\mathbb{Z}^d \\ (J_1,J_2,J_3,\ldots,J_r)\in\mathcal{I}_r}} P_{J_1,J_2,J_3,\ldots,J_r} u_{J_3} \ldots u_{J_r}.$$

Then, for all real sequence u, one has

$$A_{(j_1,+),(j_2,-)} = \bar{A}_{(j_2,+),(j_1,-)}; \tag{2.13}$$

this "formal selfadjointness" will play a fundamental role in the following.

Definition 2.4 (*Functions*). We say that a function $P \in C^\infty(\mathcal{U}_s; \mathbb{C})$ belongs to class $\mathcal{P}$, and we write $P \in \mathcal{P}$, if
- all the terms of its Taylor expansion at $u = 0$ are of class $\mathcal{P}_r$ for some r;
- the vector field X_P (recall (2.5)) belongs to $C^\infty(\mathcal{U}_s; \ell^2_s)$ for all $s > d/2$.

The Hamiltonian systems that we will study are of the form

$$H = H_0 + P, \tag{2.14}$$

with $P \in \mathcal{P}$ and H_0 of the form

$$H_0(u) := \sum_{j \in \mathbb{Z}^d} \omega_j u_{(j,+)} u_{(j,-)}, \tag{2.15}$$

and $\omega_j \in \mathbb{R}$ a sequence on which we are going to make some assumptions in the next subsection.

2.3. Statement of the Main Result. We need the following assumption.

Hypothesis 2.5. *The frequency vector $\omega = (\omega_j)_{j \in \mathbb{Z}^d}$ satisfies the following.*

F.1 There exist constants $C_1 > 0$ and $\beta > 1$ such that, $\forall j$ large enough one has

$$\frac{1}{C_1} |j|^\beta \le \omega_j \le C_1 |j|^\beta .$$

F.2 For any $r \ge 3$ there exist $\gamma_r > 0$ and τ_r such that the following condition holds for all N large enough

$$\begin{gathered} \forall J_1, \ldots, J_r \quad \text{with} \quad |J_l| \le N, \quad \forall l = 1, \ldots, r \\ \sum_{l=1}^{r} \sigma_{j_l} \omega_{j_l} \ne 0 \quad \Longrightarrow \quad \left| \sum_{l=1}^{r} \sigma_{j_l} \omega_{j_l} \right| \ge \frac{\gamma_r}{N^{\tau_r}} . \end{gathered} \tag{2.16}$$

F.3 There exists a partition

$$\mathbb{Z}^d = \bigcup_\alpha \Omega_\alpha , \tag{2.17}$$

with the following properties:

F.3.1 $*$ either Ω_α is finite dimensional and centered at the origin, namely there exists C_1 such that

$$j \in \Omega_\alpha \quad \Longrightarrow \quad |j| \le C_1$$

$*$ or it is dyadic, namely there exists a constant C_2 independent of α such that

$$\sup_{j \in \Omega_\alpha} |j| \le C_2 \inf_{j \in \Omega_\alpha} |j| . \tag{2.18}$$

F.3.2 There exist $\delta > 0$ and $C_3 = C_3(\delta)$ such that, if $j \in \Omega_\alpha$ and $i \in \Omega_\beta$ with $\alpha \ne \beta$, then

$$|i - j| + \left|\omega_i - \omega_j\right| \ge C_3(|i|^\delta + |j|^\delta) . \tag{2.19}$$

Remark 2.6. If in the above inequality one substitutes $|i - j|$ by a norm of $|i - j|$ which is equivalent to the norm $|.|$, then (2.19) still holds with a different constant. The same is true if one substitutes the norms at right hand side with equivalent norms. In the following we will exploit such a freedom.

Finally, we need a separation property of the resonances, namely that the resonances do not couple very low modes with very high modes. To state this precisely, we first define an equivalence relation on $\mathbb{Z}^d$

Definition 2.7. For $i, j \in \mathbb{Z}^d$, we say that $i \sim j$ if $\omega_i = \omega_j$. We denote by $[i]$ the equivalence classes with respect to such an equivalence relation.

Hypothesis 2.8. *The frequency vector $\omega = (\omega_j)_{j\in\mathbb{Z}^d}$ satisfies the following.*

(NR.1) The equivalence classes are dyadic, namely there exists $C > 0$ such that

$$C \inf_{j\in[i]} |j| \geq \sup_{j\in[i]} |j|, \quad \forall i \in \mathbb{Z}^d; \tag{2.20}$$

(NR.2) Non-resonance*: Given any sequence of multiindexes $(j_k, \sigma_k) \in \mathcal{Z}^d$, $k = 1, \ldots, l$, one has that the condition*

$$\sum_{i=1}^{l} \sigma_i \omega_{j_i} = 0 \tag{2.21}$$

implies that ℓ is even and that there exists a permutation τ of $(1, ..., l)$ such that

$$\forall\, i = 1, ..., l/2, \quad \omega_{j_{\tau(i)}} = \omega_{j_{\tau(i+l/2)}} \text{ and } \sigma_{\tau(j)} = \sigma_{\tau(j+l/2)}. \tag{2.22}$$

We say that a sequence of multiindexes satisfying (2.22) is resonant*, otherwise we say that it is* non-resonant.

Remark 2.9. We point out that the Hypothesis 2.8 is only used in Sect. 4 in order to prove energy estimates for the system in normal form, see Lemma 4.2.

Our main abstract theorem pertains the Cauchy problem

$$\begin{cases} \dot{u} = X_H(u) \\ u(0) = u_0 \end{cases}. \tag{2.23}$$

Theorem 2.10. *Consider the Cauchy problem (2.23) where H has the form (2.14) with H_0 as in (2.15) and $P \in \mathcal{P}$ vanishing at order at least 3 at $u = 0$. Assume that the frequencies ω_j fulfill Hypotheses 2.5, 2.8 and let $\beta > 1$ be the constant given by Hyp. 2.5. For any integer r there exists $s_r \in \mathbb{N}$ such that for any $s \geq s_r$ there exists $\epsilon_0 > 0$ and $c > 0$ with the following property: if the initial datum $u_0 \in \ell^2_s$ is real and small, namely if*

$$Iu_0 = u_0, \quad \epsilon := \|u_0\|_{\ell^2_s} < \epsilon_0, \tag{2.24}$$

then the Cauchy problem (2.23) has a unique solution

$$u \in C^0((-T_\epsilon, T_\epsilon), \ell^2_s) \cap C^1((-T_\epsilon, T_\epsilon), \ell^2_{s-\beta})$$

with $T_\epsilon > c\epsilon^{-r}$. Moreover there exists $C > 0$ such that

$$\sup_{|t|\leq T_\epsilon} \|u(t)\|_{\ell^2_s} \leq C\epsilon. \tag{2.25}$$

The main step for the proof of Theorem 2.10 consists in proving a suitable normal form lemma which is given in the next section.

3. Normal Form

In the following we will use the notation $a \lesssim b$ to mean there exists a constant C, independent of all the relevant parameters, such that $a \leq Cb$. If we want to emphasize the fact that the constant C depends on some parameters, say r, s, we will write $a \lesssim_{s,r} b$. We will also write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

Furthermore in order to separate low and high frequency modes in a way coherent with the resonance relations we have to measure the size of the indexes $j \in Z^d$ by the size of the corresponding frequency. Precisely, we define

$$|j|_\omega := |\omega_j|^{1/\beta}, \quad |J|_\omega \equiv |(J, \sigma)|_\omega := |j|_\omega \ . \tag{3.1}$$

Remark 3.1. In general $|.|_\omega$ is not a norm, since the triangular inequality could fail to hold, however this will not cause any problem in the forthcoming developments.

In the following we will informally say that an index j is larger then N if $|j|_\omega > N$. We need the following definition.

Definition 3.2 (*N-block normal form*). Let $\bar{r} \geq 3$ and $N \gg 1$. We say that a polynomial $Z \in \mathcal{P}_{3,\bar{r}}$ of the form

$$Z = \sum_{l=3}^{\bar{r}} \sum_{\mathbf{J} \in \mathcal{I}_l} Z_{\mathbf{J}} u_{J_1} ... u_{J_l} \ ,$$

(recall Definition 2.2)

is in *N-block normal form* if $Z_{\mathbf{J}} \neq 0$ *only* if $\mathbf{J} \equiv (J_1, ..., J_l)$ fulfills **one** of the following two conditions:

1. $|J_n|_\omega \leq N$ for any $n = 1, \dots, l$ and $\sum_{n=1}^{l} \sigma_{j_n} \omega_{j_n} = 0$;
2. there exist *exactly* 2 indexes larger than N, say J_1 and J_2 and the following two conditions hold:
   2.1 $J_1 = (j_1, \sigma_1)$, $J_2 = (j_2, \sigma_2)$ with $\sigma_1 \sigma_2 = -1$.
   2.2 there exist α such that $j_1, j_2 \in \Omega_\alpha$, namely both the large indexes belong to the same cluster[1] Ω_α.

We now state the main result of this section.

Theorem 3.3. *Fix any $N \gg 1$, $s_0 > d/2$ and consider the Hamiltonian (2.14) with ω_j fulfilling Hypothesis 2.5 and $P \in \mathcal{P}$. For any $\bar{r} \geq 3$ there are $\tau > 0$, $s_{\bar{r}} > s_0$ such that for any $s \geq s_{\bar{r}}$ there exist $R_{s,\bar{r}}, C_{s,\bar{r}} > 0$ such that for any $R < R_{s,\bar{r}}$ the following holds. If*

$$RN^\tau < R_{s,\bar{r}}, \tag{3.2}$$

then there exists an invertible canonical transformation

$$\mathcal{T}^{(\bar{r})}, \ [\mathcal{T}^{(\bar{r})}]^{-1} \ : B_s(R) \to B_s(C_{s,\bar{r}} R), \tag{3.3}$$

such that

$$H^{(\bar{r})} := H \circ \mathcal{T}^{(\bar{r})} = H_0 + Z^{(\bar{r})} + \mathcal{R}_T + \mathcal{R}_\perp \tag{3.4}$$

where

[1] Recall conditions $F.3.1$, $F.3.2$ in Hypothesis 2.5.

- *$Z^{(\bar r)} \in \mathcal{P}_{3,\bar r}$ is in N-block normal form and fulfills*

$$\left\| Z^{(\bar r)} \right\|_R \lesssim_{\bar r} R^3 \; ; \tag{3.5}$$

- *$\mathcal{R}_T$ is such that $X_{\mathcal{R}_T} \in C^\infty(B_s(R_{s,\bar r}); \ell^2_s)$ and*

$$\sup_{\|u\|_s \le R} \left\| X_{\mathcal{R}_T}(u) \right\|_s \lesssim_{\bar r, s} R^2 (RN^\tau)^{\bar r - 3}, \quad \forall R \le R_{s,\bar r} \; ; \tag{3.6}$$

- *$\mathcal{R}_\perp$ is such that $X_{\mathcal{R}_\perp} \in C^\infty(B_s(R_{s,\bar r}); \ell^2_s)$ and*

$$\sup_{\|u\|_s \le R} \left\| X_{\mathcal{R}_\perp}(u) \right\|_s \lesssim_{\bar r, s} \frac{R^2}{N^{s - s_0}}, \quad \forall R \le R_{s,\bar r} \; . \tag{3.7}$$

The rest of the section is devoted to the proof of this theorem and is split in a few subsections.

3.1. Properties of the Class of Functions $\mathcal{P}$. First we give the following lemma.

Lemma 3.4 (Estimates on the vector field). *Fix $r \ge 3$, $R > 0$. Then for any $s > s_0 > d/2$ there exists a constant $C_{r,s} > 0$ such that, $\forall P \in \mathcal{P}_r$, the following inequality holds:*

$$\|X_P(u)\|_s \le C_{r,s} \frac{\|P\|_R}{R}, \qquad \forall u \in B_s(R).$$

Proof. Let $P \in \mathcal{P}_r$. Then (recalling (2.5)) one has $X_P = ((X_P)_J)_{J \in \mathcal{Z}^d}$ with

$$\begin{aligned} (X_P)_{(j,+)} &= -\mathrm{i}\partial_{u_{(j,-)}} P \\ &= -\mathrm{i} r \sum_{\substack{J_1,\dots,J_{r-1} \in \mathcal{Z}^d,\ J=(j,+) \\ \mathcal{M}(J_1,\dots,J_{r-1})+j=0}} P_{J,J_1,\dots,J_{r-1}} u_{J_1} \dots u_{J_{r-1}} \end{aligned} \tag{3.8}$$

and similarly for $(X_P)_{(j,-)}$. Remark that the r.h.s. of (3.8) defines a unique symmetric $(r-1)$-linear form

$$(\widetilde{X_P})_{(j,+)}(u^{(1)}, \dots, u^{(r-1)}) := \mathrm{i} r \sum_{\substack{J_1,\dots,J_{r-1} \in \mathcal{Z}^d \\ \mathcal{M}(J_1,\dots,J_{r-1})+j=0}} P_{J,J_1,\dots,J_{r-1}} u^{(1)}_{J_1} \dots u^{(r-1)}_{J_{r-1}}. \tag{3.9}$$

In order to apply Lemma A.1 we decompose

$$u^{(l)} = u^{(l)}_+ + u^{(l)}_-, \text{ with } u^{(l)}_\sigma := (u^{(l)}_{(j,\sigma)})_{j \in \mathbb{Z}^d} \; . \tag{3.10}$$

Substituting in the previous expression we have

$$\begin{aligned} &(\widetilde{X_P})_+(u^{(1)}, \dots, u^{(r-1)}) = \\ &\quad = \sum_{l=0}^{r-1} \binom{r-1}{l} (\widetilde{X_P})_+(u^{(1)}_+, \dots, u^{(l)}_+, u^{(l+1)}_-, \dots, u^{(r-1)}_-) \; . \end{aligned} \tag{3.11}$$

Now each of the addenda of (3.11) fulfills the assumptions of Lemma A.1. Therefore, since $\|u\|_{s_0} \leq \|u\|_s$ one has

$$
\begin{aligned}
&\left\| (\widetilde{X_P})_+(u_+^{(1)}, \ldots, u_+^{(l)}, u_-^{(l+1)}, \ldots, u_-^{(r-1)}) \right\|_s \\
&\qquad\qquad \lesssim \sup_{(J, J_1, \ldots, J_r) \in (\mathcal{Z}^d)^r} \left| P_{J, J_1, \ldots, J_{r-1}} \right| \left\| u^{(1)} \right\|_s \ldots \left\| u^{(r-1)} \right\|_s .
\end{aligned}
$$

Taking all the $u^{(l)}$ equal to $u \in B_s(R)$ (i.e. $\|u\|_s < R$) and recalling the norm in (2.12) one gets the thesis for $(X_P)_+$. Similarly one gets the thesis for $(X_P)_-$ and this concludes the proof of the lemma. $\square$

As usual given two functions $f_1, f_2 \in C^\infty(\ell_s^2; \mathbb{C})$ we define their Poisson Brackets by

$$
\{f_1; f_2\} := \mathrm{i} \sum_{j \in \mathbb{Z}^d} \left(\frac{\partial f_1}{\partial u_{(j,-)}} \frac{\partial f_2}{\partial u_{(j,+)}} - \frac{\partial f_1}{\partial u_{(j,+)}} \frac{\partial f_2}{\partial u_{(j,-)}} \right) \equiv df_1 X_{f_2}, \tag{3.12}
$$

which could be ill defined (but will turn out to be well defined in the cases we will consider).

We recall that if both f_1 and f_2 have smooth vector field then

$$
X_{\{f_1; f_2\}} = [X_{f_1}; X_{f_2}], \tag{3.13}
$$

with $[\cdot; \ \cdot]$ denoting the commutator of vector fields.

Lemma 3.5 (Poisson brackets). *Given two polynomials $P_1 \in \mathcal{P}_{r_1}$ and $P_2 \in \mathcal{P}_{r_2}$, one has $\{P_1; P_2\} \in \mathcal{P}_{r_1+r_2-2}$ with*

$$
\|\{P_1; P_2\}\|_R \leq \frac{2r_1 r_2}{R^2} \left\| P_{r_1} \right\|_R \left\| P_{r_2} \right\|_R .
$$

Proof. It follows by formula (3.12) recalling (2.12) and exploiting the momentum conservation. $\square$

We now fix some large $N > 0$, but will track the dependence of all the constants on N. Corresponding to N we define a decomposition of u in *low* and *high* modes. Precisely, we define the projectors

$$
\Pi^{\leq} u := (u_J)_{|J|_\omega \leq N}, \quad \Pi^{\perp} u := (u_J)_{|J|_\omega > N} \tag{3.14}
$$

and denote

$$
u^{\leq} := \Pi^{\leq} u, \quad u^{\perp} := \Pi^{\perp} u, \tag{3.15}
$$

so that $u = u^{\leq} + u^{\perp}$.

As in [1,5], a particular role is played by the polynomials $P \in \mathcal{P}_r$ which are quadratic or cubic in $u^\perp$. We are now going to give a precise meaning to this formal statement. First, given $f \in C^\infty(\mathcal{U}_s; \mathbb{C})$, we denote by

$$
d^l f(u)(h_1, \ldots, h_l)
$$

the l-th differential of f evaluated at u and applied to the increments $h_1, \ldots, h_l$.

Definition 3.6. Let $P \in \mathcal{P}_r$ and recall the notation (3.15).
• We say that P has has a zero of order 0 in $u^\perp$ if $P(u^\leq)$ is not identically zero for $u \in \ell^2_s$.
• We say that P has has a zero of order at least 1 in $u^\perp$ if $P(u^\leq) = 0, \forall\, u \in \ell^2_s$.
• We say that P has has a zero of order at least $k \geq 2$ in $u^\perp$ if

$$
\begin{aligned}
& P(u^\leq) = 0, \\
& d^l P(\Pi^\leq u)(\Pi^\perp h_1, ..., \Pi^\perp h_l) = 0 \quad \forall\, u, h_1, ..., h_l \in \ell^2_s \quad \forall\, l = 1, ..., k-1
\end{aligned}
$$

We say that P is homogeneous of degree $k \geq 1$ in $u^\perp$ if it has a zero of order at least k, but not of order at least $k+1$.

We say that P is homogeneous of degree 0 if it has a zero of order 0 in $u^\perp$ and $P(u) \equiv P(u^\leq)$ for $u \in \ell^2_s$.

Remark 3.7. By the very definition of normal form, one can decompose $Z^{(r)} = Z_0 + Z_2$, with Z_0 homogeneous of degree zero in $u^\perp$ and Z_2 homogeneous of degree 2 in $u^\perp$. Furthermore Z_0 is in Birkhoff normal form in the classical sense, namely it contains only resonant monomials, i.e. monomials of the form

$$
u_{J_1} ... u_{J_r}, \quad \text{with} \quad \sum_l \sigma_l \omega_{j_l} = 0 \,.
$$

We also remark that, in view of Hypothesis 2.8-(NR.2) such monomials are *super-action preserving.*

Lemma 3.8. *For all $s > s_0 > d/2$ and all $r \geq 3$, there exists a constant $C_{r,s} > 0$ such that the following holds:*

(i) if $P \in \mathcal{P}_r$ has a zero of order at least 2 in $u^\perp$, then

$$
\sup_{\|u\|_s \leq R} \left\| \Pi^\leq X_P(u) \right\|_s \leq \frac{C_{r,s}}{N^{s-s_0}} \frac{\|P\|_R}{R} \;;
$$

(ii) if $P \in \mathcal{P}_r$ has a zero of order at least 3 in $u^\perp$, then

$$
\sup_{\|u\|_s \leq R} \|X_P(u)\|_s \leq \frac{C_{r,s}}{N^{s-s_0}} \frac{\|P\|_R}{R} \,.
$$

Proof. Consider first the case (i) and remark that, using the notation (3.10), we have $\left(\Pi^\leq X_P(u)\right)_\pm = \pm \mathrm{i} \nabla_{u^\leq_\mp} P$, so that $\Pi^\leq X_P(u)$ has a zero of order 2 in $u^\perp$. It follows that both in the case (i) and in the case (ii) we have to estimate a polynomial function $X(u)$ of the form (3.8) with a zero of second order in $u^\perp$. To exploit this fact consider first the $+$ component and consider again the multilinear form $(\widetilde{X})_+$ as in (3.9): we have

$$
X_+(u) = X_+(u^\perp + u^\leq) = \sum_{l=0}^{r-1} \binom{r-1}{l} (\widetilde{X})_+(\underbrace{u^\perp, ..., u^\perp}_{l-\text{times}}, \underbrace{u^\leq, ..., u^\leq}_{r-1-l-\text{times}}),
$$

but, since $X_+(u)$ has a zero of at least second order in $u^\perp$, one has

$$
X_+(u) = \sum_{l=2}^{r-1} \binom{r-1}{l} (\widetilde{X})_+(\underbrace{u^\perp, ..., u^\perp}_{l-\text{times}}, \underbrace{u^\leq, ..., u^\leq}_{r-1-l-\text{times}}) \,.
$$

Consider the first addendum (which is the one giving rise to worst estimates): proceeding as in the proof of Lemma 3.4 one can apply Lemma A.1 and get the estimate

$$\left\| (\widetilde{X})_+(u^\perp, u^\perp, u^\leq, \dots, u^\leq) \right\|_s \lesssim \sup_{j, j_1, \dots, j_{r-1} \in \mathbb{Z}^d} \left| P_{j, j_1, ---, j_{r-1}} \right| \times \times \left(\left\| u^\perp \right\|_s \left\| u^\perp \right\|_{s_0} \| u^\leq \|_{s_0}^{r-3} + \| u^\leq \|_s \left\| u^\perp \right\|_{s_0}^2 \| u^\leq \|_{s_0}^{r-4} \right),$$

but

$$\left\| u^\perp \right\|_{s_0}^2 = \sum_{|J|_\omega > N} \langle J \rangle^{2s_0} |u_J|^2 = \sum_{|J|_\omega > N} \frac{\langle J \rangle^{2s} |u_J|^2}{\langle J \rangle^{2(s-s_0)}} \lesssim \frac{\| u^\perp \|_s^2}{N^{2(s-s_0)}} = \frac{\| u \|_s^2}{N^{2(s-s_0)}}.$$

Since, by F.1 one has $|j|_\omega \lesssim |j|$ and therefore $\frac{1}{\langle j \rangle} \lesssim \frac{1}{N}$, it follows

$$\left\| (\widetilde{X})_+(u^\perp, u^\perp, u^\leq, \dots, u^\leq) \right\|_s \lesssim \sup_{j, j_1, \dots, j_{r-1} \in \mathbb{Z}^d} \left| P_{j, j_1, ---, j_{r-1}} \right| \frac{R^{r-1}}{N^{s-s_0}} \lesssim \frac{\| P \|_R}{R} \frac{1}{N^{s-s_0}}.$$

The other cases can be treated similarly. □

3.2. Lie Tranfsorm. Given $G \in \mathcal{P}_{r,\bar{r}}$, consider its Hamilton equations $\dot{u} = X_G(u)$, which, by Lemma 3.4, are locally well posed in a neighborhood of the origin. Denote by Φ_G^t the corresponding flow, then we have the following Lemma whose proof is equal to the finite dimensional case.

Lemma 3.9. *Consider $\bar{r} \geq r_1 \geq r \geq 3$ and $s > s_0 > d/2$. There exists $C_{r,s} > 0$ such that for any $G \in \mathcal{P}_{r,r_1}$ and any $R > 0$ satisfying*

$$\frac{\| G \|_R}{R} \leq \frac{1}{C_{r,s}}, \tag{3.16}$$

the following holds. For any $|t| \leq 1$ one has $\Phi_G^t(B_s(R)) \subset B_s(2R)$ and the estimate

$$\sup_{u \in B_s(R)} \left\| \Phi_G^t(u) - u \right\| \leq |t| \, C_{r,s} \frac{\| G \|_R}{R}, \quad \forall t \ : \ |t| \leq 1 \, .$$

Definition 3.10. The map $\Phi_G := \Phi_G^t \big|_{t=1}$ is called the *Lie transform* generated by G.

In order to describe how a function is transformed under Lie transform we define the operator

$$\begin{aligned} ad_G : C^\infty(\mathcal{U}_s, \mathbb{C}) &\to C^\infty(\mathcal{U}_s, \mathbb{C}) \\ f &\mapsto ad_G f := \{ f; G \}, \end{aligned}$$

and its k-th power $ad_G^k f := \{ ad_G^{k-1} f; G \}$ for $k \geq 1$. Also the following Lemma has a standard proof equal to that of the finite dimensional case.

Lemma 3.11. *Let $\bar{r} \geq r \geq 3$ and $s > s_0 > d/2$ and consider $G \in \mathcal{P}_{r,\bar{r}}$. There exists $C_{r,s} > 0$ such that for any $R > 0$ satisfying (3.16) the following holds. For any $f \in C^\infty(B_s(2R); \mathbb{C})$ and any $n \in \mathbb{N}$ one has*

$$f(\Phi_G^t(u)) = \sum_{k=0}^{n} \frac{t^k}{k!}(ad_G^k f)(u) + \frac{1}{n!}\int_0^t (t-\tau)^n (ad_G^{n+1} f)\left(\Phi_G^\tau(u)\right) d\tau, \tag{3.17}$$

$\forall u \in B_s(R)$ and any t with $|t| \leq 1$.

From Lemma 3.5 one has the following corollary.

Corollary 3.12. *Let $G \in \mathcal{P}_{r_1,r_2}$, $F \in \mathcal{P}_{r_3,r_4}$, with $r_1, r_2, r_3, r_4 \leq \bar{r}$ and $3 \leq r_1 \leq r_2$. Let $\bar{n} \in \mathbb{N}$ be the smallest integer such that $(\bar{n}+1)(r_1-2)+r_2 > \bar{r}$. Then there exists $C_{\bar{r}} > 0$ such that for any $k \leq \bar{n}$, one has*

$$\left\| (ad_G)^k F \right\|_R \leq \left(\frac{C_{\bar{r}} \left\| G \right\|_R}{R^2} \right)^k \left\| F \right\|_R \ .$$

A further standard Lemma we need is the following.

Lemma 3.13. *Let $G \in \mathcal{P}_{r_1,r_2}$, $3 \leq r_1, r_2 \leq \bar{r}$ and let Φ_G be the Lie transform it generates. Let R_s by the largest value of R such that (3.16) holds. Then there exists $C > 0$ such that for any $F \in C^\infty(B_s(2R_s))$ satisfying*

$$\sup_{\|u\|_s \leq 2R} \|X_F(u)\|_s =: C_R < \infty \ , \quad \forall R < R_s \ .$$

one has

$$\sup_{\|u\|_s \leq R/C} \left\| X_{F \circ \Phi_G}(u) \right\|_s =: 2C_R < \infty, \quad \forall R < R_s \ .$$

From Lemma 3.11, Corollary 3.12 and Lemma 3.13, one has the following Corollary which is the one relevant for the perturbative construction leading to the normal form lemma.

Corollary 3.14. *There exists $\mu_0 > 0$ such that for any $G \in \mathcal{P}_{r,\bar{r}}$, $3 \leq r \leq \bar{r}$, the following holds. If*

$$\mu := \frac{C_{\bar{r}} \left\| G \right\|_R}{R^2} < \mu_0,$$

with $C_{\bar{r}}$ the constant of Corollary 3.12, then, for any $F \in \mathcal{P}_{r_1,\bar{r}}$, $r_1 \leq \bar{r}$, one has

$$F \circ \Phi_G = \tilde{F} + \mathcal{R}_{F,G},$$

with $\tilde{F} \in \mathcal{P}_{r+r_1-2,\bar{r}}$ ($\tilde{F} \equiv 0$ if $r + r_1 - 2 > \bar{r}$) and $\mathcal{R}_{F,G} \in C^\infty(B_s(R/C); \mathbb{C})$ which fulfill the following estimates

$$\left\| \tilde{F} \right\|_R \lesssim_{\bar{r}} \mu \left\| F \right\|_R \ , \quad \sup_{\|u\|_s \leq R/C} \left\| X_{\mathcal{R}_{F,G}}(u) \right\|_s \lesssim \frac{\left\| F \right\|_R}{R} \mu^{\bar{n}},$$

with $\bar{n}$ as in Corollary 3.12 and C as in Lemma 3.13.

3.3. Homological Equation. In order to construct the transformation $\mathcal{T}^{(\bar{r})}$ of Theorem 3.3, we will use the Lie transform generated by auxiliary Hamiltonian functions $G_3, \dots, G_{\bar{r}}$, with $G_\ell \in \mathcal{P}_{\ell,\bar{r}}$, which in turn will be constructed by solving the homological equation

$$\{H_0; G\} + Z = F \tag{3.18}$$

with $F \in \mathcal{P}_{\ell,\bar{r}}$ a given polynomial of order 2 in $u^\perp$ and Z to be determined, but in N-block normal form. In order to solve the homological equation we need a nonresonance condition seemingly stronger than (2.16), but which actually follows from $F.1$, $F.2$, $F.3$ of Hypothesis 2.5.

First we remark that (recall Remark 2.6), by F.1, the assumptions F.2 implies

F.2' For any $r \geq 3$ there exist $\gamma_r > 0$ and τ_r such that the following condition holds for all N large enough

$$\begin{gathered} \forall J_1, \dots, J_r \quad \text{with} \quad |j_l|_\omega \leq N, \quad \forall l = 1, \dots, r \\ \sum_{l=1}^{r} \sigma_{j_l} \omega_{j_l} \neq 0 \implies \left| \sum_{l=1}^{r} \sigma_{j_l} \omega_{j_l} \right| \geq \frac{\gamma_r}{N^{\tau_r}}, \end{gathered} \tag{3.19}$$

with redefined constants.

Similarly F.3.2 implies

F.3.2' There exist $\delta > 0$ and $C_3 = C_3(\delta)$ such that, if $j \in \Omega_\alpha$ and $i \in \Omega_\beta$ with $\alpha \neq \beta$, then

$$|i - j| + \left|\omega_i - \omega_j\right| \geq C_3(|i|_\omega^\delta + |j|_\omega^\delta), \tag{3.20}$$

which is the one we will use.

To state the non-resonance condition we need the following definition.

Definition 3.15 (*Non resonant multi-indexes*). For $\ell \in \mathbb{N}$ and $N \gg 1$ we say that multi-indexes $\mathbf{J} = (J_1, \dots, J_l) \in \mathcal{I}_l$ (see (2.9)), with $J_i = (j_i, \sigma_i) \in \mathcal{Z}^d$, are *non resonant multi-indexes* if

$$\sum_{i=1}^{l} \sigma_{j_i} \omega_{j_i} \neq 0, \tag{3.21}$$

and **one** of the following conditions holds:

(I.1) there is at most one index larger than N;
(I.2) there exist *exactly* 2 indexes larger than N, say J_1 and J_2 with $\sigma_1\sigma_2 = 1$;
(I.3) there exist *exactly* 2 indexes larger than N, say J_1 and J_2 with $\sigma_1\sigma_2 = -1$ and such that there exist $\alpha \neq \beta$ such that $j_1 \in \Omega_\alpha$ and $j_2 \in \Omega_\beta$, namely if the two largest indexes are such that $\sigma_1\sigma_2 = -1$ then they belong to different clusters.[2]

We denote by $\mathcal{J}_l^N$ the subset of $\mathcal{I}_l$ of non resonant multi-indexes.

We denote by $\mathcal{S}_l^N$ the subset of $\mathcal{I}_l$ made of multi-indexes $\mathbf{J}$ such that there exist at least **three** indexes larger than N.

Remark 3.16. By Definitions 3.2 and 3.15 we notice that an Hamiltonian $Z \in \mathcal{P}_r$, $r \geq 3$, of the form (2.11) but supported only on multi-indexes $\mathbf{J} \in \mathcal{I}_r \backslash \left(\mathcal{J}_r^N \cup \mathcal{S}_r^N\right)$ is in N-block normal form.

[2] Recall conditions $F.3.1$, $F.3.2$ in Hypothesis 2.5.

Lemma 3.17. *Assume Hypothesis 2.5 and let $r \in \mathbb{N}$. Then there exist τ_r' and $\gamma_r' > 0$, such that for any $3 \le p \le r$ and any multi-index $\mathbf{J} \in \mathcal{J}_p^N$ one has the bound*

$$\left|\sum_{l=1}^{p} \sigma_l \omega_{j_l}\right| \ge \frac{\gamma_r'}{N^{\tau_r'}} . \tag{3.22}$$

Proof. Assume that we are in the case $(I.1)$ in Definition 3.15. If all the indexes j_l are smaller than N, then there is nothing to prove in view of (3.21) and $(F.2)$ in Definition 2.5.

Consider now the case where there is only one index, say J_1, larger than N and the length of the multi-index is $n+1 \le r$. The quantity to be estimated is now

$$\left|\sum_{l=2}^{n} \sigma_l \omega_{j_l} + \sigma_1 \omega_{j_1}\right| . \tag{3.23}$$

By condition $F.1$, one has

$$\left|\sum_{l=2}^{n} \sigma_l \omega_{j_l}\right| \le r N^{\beta} C_1 \quad \text{and} \quad |\omega_{j_1}| \ge C_1 |j_1|^{\beta} .$$

Therefore, if

$$|j_1| \ge 2(rC_1^2)^{1/\beta} N =: N_1$$

the estimate (3.22) is satisfied. Hence, the estimate on the quantity (3.23) is nontrivial only if all the indexes are smaller than N_1. It follows that we can use (2.16) with N replaced by N_1, getting

$$|(3.23)| \ge \frac{\gamma_r}{N_1^{\tau_r}} = \frac{\gamma_r}{2^{\tau_r}(rC_1^2)^{\tau_r/\beta} N^{\tau_r}}$$

which implies the bound (3.22) by choosing

$$\gamma_r' \le \frac{\gamma_r}{2^{\tau_r}(rC_1^2)^{\tau_r/\beta}} .$$

This concludes the case $(I.1)$.

Consider now the case $(I.2)$, i.e. when there are two indexes larger than N, say J_1 and J_2 with $\sigma_1\sigma_2 = 1$. This case is dealt with similarly to the previous case.

We discuss now to the case $\sigma_1\sigma_2 = -1$. By condition $(I.3)$ in Definition 3.15 there exist $\alpha \ne \beta$ such that $j_1 \in \Omega_\alpha$ and $j_2 \in \Omega_\beta$. It follows (recall (F.3) in Hyp. 2.5) that either

$$\left|\omega_{j_1} - \omega_{j_2}\right| \ge C(|j_1|_\omega^\delta + |j_2|_\omega^\delta) \tag{3.24}$$

or

$$|j_1 - j_2| \ge C(|j_1|_\omega^\delta + |j_2|_\omega^\delta) . \tag{3.25}$$

for some $C > 0$. Assume for concreteness that $|j_1| \ge |j_2|$ and $\sigma_1 = 1, \sigma_2 = -1$.

Consider first the case where (3.24) holds. The quantity to be estimated is

$$\left|\sum_{l=3}^{n}\sigma_l\omega_{j_l}+\omega_{j_1}-\omega_{j_2}\right| . \tag{3.26}$$

Notice that (3.24) implies $\left|\omega_{j_1}-\omega_{j_2}\right| \geq C\,|j_1|_\omega^\delta$ and that we also have

$$\left|\sum_{l=3}^{n}\sigma_l\omega_{j_l}\right| \leq (r-2)N^\beta .$$

Then it follows that (3.22) is automatic if

$$|j_1|_\omega \geq \frac{2(r-2)^{1/\delta}}{C}N^{\beta/\delta} =: N_2 .$$

Hence the bound on (3.26) is nontrivial only if all the indexes are smaller than N_2. In this case we can apply (2.16) with N_2 in place of N, getting

$$|(3.26)| \geq \frac{\gamma_r}{N_2^{\tau_r}} = \frac{\gamma_r C^{\tau_r}}{2^{\tau_r}(r-2)^{\tau_r/\delta}N^{\frac{\beta}{\delta}\tau_r}}$$

which is the wanted estimate, in particular with $\tau_r' \geq \frac{\beta}{\delta}\tau_r$.

It remains to bound (3.26) from below with indexes fulfilling (3.25).

By the zero momentum condition we have

$$\sum_{l=3}^{n}\sigma_l j_l + j_1 - j_2 = 0$$

but

$$\left|\sum_{l=3}^{n}\sigma_l j_l\right| \leq \sum_{l=3}^{n}|j_l| \leq C\sum_{l=3}^{n}|j_l|_\omega \leq CrN$$

while

$$|j_1-j_2| \geq C\,|j_1|_\omega^\delta .$$

It follows that in our set there are no indexes with $C\,|j_1|_\omega^\delta > rN$ (otherwise the zero momentum condition cannot be fulfilled), so all the indexes must be smaller than $N_3 := (rN/C)^{1/\delta}$, and again we can estimate (3.26) using (2.16) with N substituted by N_3, thus getting the thesis. □

Lemma 3.18. (Homological equation). *Consider the Homological equation (3.18) with H_0 as in (2.15) and ω_j satisfying Hypotheses 2.5 and where $F \in \mathcal{P}_{r,\bar r}$ is a polynomial having a zero of order 2 in $u^\perp$. Then equation (3.18) has solutions $Z \in \mathcal{P}_{r,\bar r}$ and $G \in \mathcal{P}_{r,\bar r}$ where Z is in N-block normal form, $N \gg 1$ and moreover*

$$\|Z\|_R \leq \|F\|_R , \tag{3.27}$$

$$\|G\|_R \leq \frac{N^{\tau_{\bar r}'}}{\gamma_{\bar r}'}\|F\|_R . \tag{3.28}$$

Proof. Notice that, denoting $u_{\mathbf{J}} := u_{J_1} ... u_{J_r}$ and recalling (3.12), one has

$$\begin{aligned}
\{H_0; u_{\mathbf{J}}\} &= \mathrm{i} \sum_j \frac{\partial H_0}{\partial u_{(j,-)}} \frac{\partial u_{\mathbf{J}}}{\partial u_{(j,+)}} - \frac{\partial H_0}{\partial u_{(j,+)}} \frac{\partial u_{\mathbf{J}}}{\partial u_{(j,-)}} \\
&= \mathrm{i} \sum_j \omega_j u_{(j,+)} \frac{\partial u_{\mathbf{J}}}{\partial u_{(j,+)}} - \omega_j u_{(j,-)} \frac{\partial u_{\mathbf{J}}}{\partial u_{(j,-)}} \\
&= \mathrm{i} \sum_j \omega_j u_{\mathbf{J}} \left(\sum_{l=1}^{r} \delta_{(j,+),J_l} \omega_j - \delta_{(j,-),J_l} \omega_j \right) \\
&= \mathrm{i} u_{\mathbf{J}} \left(\sum_{l=1}^{r} \delta_{(j,+),(j_l,\sigma_l)} \omega_{j_l} - \delta_{(j,-),(j_l,\sigma_l)} \omega_{j_l} \right) = \mathrm{i} u_{\mathbf{J}} \sum_{l=1}^{r} \sigma_l \omega_{j_l} \ .
\end{aligned}$$

It follows that, writing

$$P = \sum_{\mathbf{J} \in \mathcal{I}_r} P_{\mathbf{J}} u_{\mathbf{J}}$$

one can solve the Homological equation (3.18) by defining (recall Definition 3.15)

$$\begin{aligned}
Z(u) &:= \sum_{\mathbf{J} \in \mathcal{I}_r \setminus \mathcal{J}_r^N} P_{\mathbf{J}} u_{\mathbf{J}}, \\
G(u) &:= \sum_{\mathbf{J} \in \mathcal{J}_r^N} \frac{P_{\mathbf{J}}}{\mathrm{i} \sum_{l=1}^{r} \sigma_l \omega_{j_l}} u_{\mathbf{J}} \ .
\end{aligned}$$

By Remark 3.16 we have that Z is in N-block normal form. The estimates (3.27)–(3.28) immediately follow using Lemma 3.17. □

3.4. Proof of the Normal Form Lemma. Theorem 3.3 is an immediate consequence of the forthcoming Lemma 3.19. To introduce it, we first split

$$P = \tilde{P} + \mathcal{R}_{T,0}$$

with $\tilde{P} \in \mathcal{P}_{3,\bar{r}}$ and $\mathcal{R}_{T,0}$ having a zero of order at least $\bar{r} + 1$ at the origin. A relevant role will be played by the quantity $\|\tilde{P}\|_R$. In order to simplify the notation, we remark that, for R sufficiently small there exists $K_{s,\bar{r}}$ such that

$$\sup_{\|u\|_s \le R} \left\| X_{\mathcal{R}_{T,0}}(u) \right\|_s \le K_{s,\bar{r}} \|\tilde{P}\|_R R^{\bar{r}-3} \frac{1}{R}.$$

Lemma 3.19 (Iterative lemma). *Assume Hypothesis 2.5 and fix $\bar{r} \ge 3$. There exists $\mu_{\bar{r}} > 0$ such that for any $3 \le k \le \bar{r}$ and any $s > s_0 > d/2$ there exist $R_{s,k} > 0$, $C_{s,k}, \tau > 0$ such that for any $R < R_{s,k}$ and any $N \gg 1$ the following holds. If one has*

$$\mu := \frac{\|\tilde{P}\|_R}{R^2} N^{\tau} < \mu_{\bar{r}}, \tag{3.29}$$

then there exists an invertible canonical transformation

$$\mathcal{T}^{(k)} : B_s(R) \to B_s(C_{s,k}R), \tag{3.30}$$

with

$$[\mathcal{T}^{(k)}]^{-1} : B_s(R) \to B_s(C_{s,k}R), \tag{3.31}$$

such that

$$H^{(k)} := H \circ \mathcal{T}^{(k)} = H_0 + Z^{(k)} + P_k + \mathcal{R}_{T,k} + \mathcal{R}_{\perp,k} \tag{3.32}$$

where

- $Z^{(k)} \in \mathcal{P}_{3,k}$ *is in* N*-block normal form and fulfills*

$$\left\| Z^{(k)} \right\|_R \lesssim_{\bar{r},k} \|\tilde{P}\|_R \,; \tag{3.33}$$

- $P_k \in \mathcal{P}_{k,\bar{r}}$ *fulfills*

$$\|P_k\|_R \lesssim_{\bar{r},k} \|\tilde{P}\|_R \mu^{k-3} \,; \tag{3.34}$$

- $\mathcal{R}_{T,k}$ *is such that* $X_{\mathcal{R}_{T,k}} \in C^\infty(B_s(R_{s,k}); \ell_s^2)$ *and*

$$\sup_{\|u\|_s \le R} \left\| X_{\mathcal{R}_{T,k}}(u) \right\|_s \lesssim_{\bar{r},k,s} \|\tilde{P}\|_R \mu^{\bar{r}-3} \frac{1}{R}, \quad \forall R \le R_{s,k} \,; \tag{3.35}$$

- $\mathcal{R}_{\perp,k}$ *is such that* $X_{\mathcal{R}_{\perp,k}} \in C^\infty(B_s(R_{s,k}); \ell_s^2)$ *and*

$$\sup_{\|u\|_s \le R} \left\| X_{\mathcal{R}_{\perp,k}}(u) \right\|_s \lesssim_{\bar{r},k,s} \frac{\|\tilde{P}\|_R}{R} \frac{1}{N^{s-s_0}}, \quad \forall R \le R_{s,k} \,. \tag{3.36}$$

The proof occupies the rest of the section and is split in a few Lemmas. We reason inductively. First, we consider the Taylor expansion of P_k in $u^\perp$ and we write

$$P_k = P_{k,eff} + R_{k,\perp}, \tag{3.37}$$

with $P_{k,eff}$ containing only terms of degree 0, 1 and 2 in $u^\perp$, while $R_{k,\perp}$ has a zero of order at least 3 in $u^\perp$. Then we determine G_{k+1} and Z_{k+1} by solving the homological equation

$$\{H_0; G_{k+1}\} + P_{k,eff} = Z_{k+1}, \tag{3.38}$$

so that, by Lemma 3.18 and the inductive assumption (3.34), we get

$$\|G_{k+1}\|_R \lesssim \|P_k\|_R \, N^\tau \lesssim R^2 \|\tilde{P}\|_R \mu^{k-2} \tag{3.39}$$

$$\|Z_{k+1}\|_R \lesssim \|P_k\|_R \, N^\tau \lesssim \|\tilde{P}\|_R \mu^{k-3}. \tag{3.40}$$

Consider the Lie transform $\Phi_{G_{k+1}}$ (recall Definition 3.10) generated by G_{k+1}. By the estimate (3.39) and the condition (3.29) we have that there is $R_{s,k+1} > 0$ such that (3.16) is fulfilled for $R < R_{s,k+1}$. Hence Lemma 3.9 applies and so we deduce that the map $\Phi_{G_{k+1}}$ is well-posed.

We study now $H^{(k)} \circ \Phi_{G_{k+1}}$. To start with we prove the following Lemma.

Lemma 3.20. *Let G_{k+1} be the solution of (3.38), then one has*

$$H_0 \circ \Phi_{G_{k+1}} = H_0 + Z_{k+1} - P_{k,eff} + \tilde{H}_0 + \mathcal{R}_{H_0,G_{k+1}}, \tag{3.41}$$

with $\tilde{H}_0 \in \mathcal{P}_{k+1,\bar{r}}$, and, provided $R < R^0_{k+1}$, for some R^0_{k+1}, one has

$$\left\| \tilde{H}_0 \right\|_R \lesssim \mu^{k-2} \|\tilde{P}\|_R \ . \tag{3.42}$$

Furthermore, there exists $C_0 > 0$ such that one has

$$\sup_{\|u\|_s \le R/C_0} \left\| X_{\mathcal{R}_{H_0,G_{k+1}}}(u) \right\|_s \lesssim \mu^{\bar{r}-3} \|\tilde{P}\|_R R^{-1} \ . \tag{3.43}$$

Proof. Let $\bar{n}$ be such that $(\bar{n}+1)(k-2)+k > \bar{r}$; using the expansion (3.17) one gets

$$\begin{aligned} H_0 \circ \Phi_{G_{k+1}} &= H_0 + \{H_0; G_{k+1}\} + \sum_{l=2}^{\bar{n}} \frac{ad^l_{G_{k+1}}}{l!} H_0 + \mathcal{R}_{H_0,G_{k+1}} \\ &= H_0 + \{H_0; G_{k+1}\} + \sum_{l=2}^{\bar{n}} \frac{ad^{l-1}_{G_{k+1}}}{l!} \{H_0; G_{k+1}\} + \mathcal{R}_{H_0,G_{k+1}} \end{aligned} \tag{3.44}$$

where we can rewrite explicitly the remainder term as

$$\mathcal{R}_{H_0,G_{k+1}} = \frac{1}{\bar{n}!} \int_0^1 (1-\tau)^{\bar{n}} \left(ad^{\bar{n}}_{G_{k+1}} \{H_0, G_{k+1}\} \right) \circ \Phi^{\tau}_{G_{k+1}} d\tau \ .$$

Since G_{k+1} fulfills the Homological equation one has

$$\{H_0, G_{k+1}\} = Z_{k+1} - P_{k,eff} \in \mathcal{P}_{k,\bar{r}}$$

with

$$\|\{H_0, G_{k+1}\}\|_R \lesssim \|P_k\|_R \lesssim \mu^{k-3} \|\tilde{P}\|_R.$$

Hence, defining $\tilde{H}_0$ to be the sum in Eq. (3.44), one has

$$\begin{aligned} \left\| \tilde{H}_0 \right\|_R &\equiv \left\| \sum_{l=2}^{\bar{n}} \frac{ad^{l-1}_{G_{k+1}}}{l!} \{H_0; G_{k+1}\} \right\|_R \\ &\lesssim \sum_{l=2}^{\bar{n}} \left(\frac{C \, \|G_{k+1}\|_R}{R^2} \right)^{l-1} \frac{1}{l!} \mu^{k-3} \|\tilde{P}\|_R \lesssim \mu^{k-3} \|\tilde{P}\|_R \mu = \mu^{k-2} \|\tilde{P}\|_R, \end{aligned}$$

provided R is small enough. Analogously one gets

$$\sup_{\|u\|_s \le R/C} \left\| X_{\mathcal{R}_{H_0,G_{k+1}}}(u) \right\|_s \lesssim \mu^{k-3} \frac{\|\tilde{P}\|_R \mu^{\bar{n}}}{R},$$

and, since $k + \bar{n} \ge \bar{r}$ the thesis follows. □

In an analogous way one proves the following simpler Lemma whose proof is omitted.

Lemma 3.21. *Let $G_{k+1} \in \mathcal{P}_{k,\bar r}$ fulfills the estimate (3.39), then we have*

$$
\begin{aligned}
P_k \circ \Phi_{G_{k+1}} &= P_k + \tilde P_k + \mathcal{R}_{P_k, G_{k+1}}, \\
Z^{(k)} \circ \Phi_{G_{k+1}} &= Z^{(k)} + \tilde Z^{(k)} + \mathcal{R}_{Z^{(k)}, G_{k+1}},
\end{aligned} \tag{3.45}
$$

and the following estimates hold

$$
\begin{aligned}
\left\| \tilde P_k \right\|_R &\lesssim \|\tilde P\|_R \mu^{k-2}, \qquad \sup_{\|u\|_s \le R/C} \left\| X_{\mathcal{R}_{P_k, G_{k+1}}}(u) \right\|_s \lesssim \frac{\|\tilde P\|_R \mu^{\bar r}}{R}, \\
\left\| \tilde Z^{(k)} \right\|_R &\lesssim \|\tilde P\|_R \mu^{k-2}, \qquad \sup_{\|u\|_s \le R/C} \left\| X_{\mathcal{R}_{Z^{(k)}, G_{k+1}}}(u) \right\|_s \lesssim \frac{\|\tilde P\|_R \mu^{\bar r}}{R} .
\end{aligned}
$$

End of the proof of Lemma 3.19. We consider the Lie transform $\Phi_{G_{k+1}}$ generated by G_{k+1} determined by the equation (3.38) and we define

$$
\mathcal{T}^{(k+1)} := \mathcal{T}^{(k)} \circ \Phi_{G_{k+1}}.
$$

By estimate (3.39), condition (3.29), taking R small enough, we have that Lemma 3.9 applied to G_{k+1} and the inductive hypothesis on $\mathcal{T}^{(k)}$ imply that $\mathcal{T}^{(k+1)}$ satisfies (3.30)–(3.31) with $k \rightsquigarrow k+1$ and some constant $C_{s,k+1}$.

Recalling (3.41), (3.45) we define

$$
\begin{aligned}
& Z^{(k+1)} = Z^{(k)} + Z_{k+1}, \quad P_{k+1} = \tilde P_k + \tilde Z^{(k)} + \tilde H_0 \\
& \mathcal{R}_{T,k+1} = \mathcal{R}_{H_0, G_{k+1}} + \mathcal{R}_{P_k, G_{k+1}} + \mathcal{R}_{Z^{(k)}, G_{k+1}} + \mathcal{R}_{T,k} \circ \Phi_{G_{k+1}}, \\
& \mathcal{R}_{\perp,k+1} = R_{\perp,k} + \mathcal{R}_{\perp,k} \circ \Phi_{G_{k+1}} .
\end{aligned}
$$

Then the iterative estimates follow from the estimates of Lemmas 3.20 and 3.21. This concludes the proof. □

Proof of Theorem 3.3. Condition (3.2) implies (3.29). Then the result follows by Lemma 3.19 taking $k = \bar r$. □

An important consequence of Theorem 3.3 is the following.

Corollary 3.22. *Consider the Hamiltonian (2.14) with ω_j fulfilling Hypotheses 2.5 and $P \in \mathcal{P}$ (see Definition 2.4). For any $r \ge 3$ there exists $N_r > 0$, $\tau > 0$ and $s_r > d/2$ and a canonical transformation $\mathcal{T}_r$ such that for any $s \ge s_r$ there exists $R_s > 0$ and $C_s > 0$ such that the following holds for any $R < R_s$:*
(i) one has

$$
\mathcal{T}_r \in C^\infty(B_s(R/C_s); B_s(R)), \quad \mathcal{T}_r^{-1} \in C^\infty(B_s(R/C_s); B_s(R)), \tag{3.46}
$$
$$
H^r := H \circ \mathcal{T}_r = H_0 + Z^r + \mathcal{R}^{(r)}, \tag{3.47}
$$

where

- *$Z^r \in \mathcal{P}_{3,r}$ is in N_r-block normal form according to Definition 3.2;*
- *$\mathcal{R}^{(r)}$ is such that $X_{\mathcal{R}^{(r)}} \in C^\infty(B_s(R_s/C_s); B_s(R_s))$ and*

$$
\sup_{\|u\|_s \le R} \left\| X_{\mathcal{R}^{(r)}}(u) \right\|_s \lesssim_r R^{r+1}, \quad \forall R \le R_s/C_s . \tag{3.48}
$$

(ii) Given $u \in B_s(R)$ we write $u = (u^{\le}, u^{\perp})$ according to the splitting (3.14)–(3.15) with N replaced by N_r and we set $Z^r = Z_0 + Z_2$ (see Remark 3.7) where Z_0 is the part independent of $z^{\perp}$ and Z_2 is the part homogeneous of order 2 in $z^{\perp}$. Then we have

$$\sup_{\|u\|_s \le R} \left\| \Pi^{\le} X_{Z_2}(u^{\le}, u^{\perp}) \right\|_s \lesssim_r R^{r+1}, \quad \forall R \le R_s / C_s \,. \tag{3.49}$$

Proof. Let us fix

$$\bar{r} = 2r - 1, \tag{3.50}$$

consider $\tau = \tau_r$ given by Lemma 3.17 and fix

$$s_r = 2(s_0 + \tau(\bar{r} - 3)).$$

We now take $N_r = N$ such that

$$R N^{\tau} \simeq R^{1/2} \iff N \simeq R^{-1/2\tau} \,. \tag{3.51}$$

With this choices the assumption (3.2) holds taking $R < R_s$ with R_s small enough. Then Theorem 3.3 applies with $s \ge s_r$, $N = N_r$ and $\tau = \tau_r$ chosen above. First of all notice that

$$(R N^{\tau})^{\bar{r}-3} \simeq \frac{1}{N^{s_r - s_0}} \iff R^{\frac{\bar{r}-3}{2}} \simeq R^{\frac{s_r - s_0}{2\tau}} \,. \tag{3.52}$$

Then formulæ (3.46)–(3.47) follow by (3.3)–(3.4) setting $\mathcal{R}^{(r)} = \mathcal{R}_T + \mathcal{R}_{\perp}$. Then estimate (3.48) follows by (3.6)–(3.7) and (3.52). The estimate (3.49) follows by Lemma 3.8 and the choice of $N = N_r$ in (3.51). □

4. Dynamics and Proof of the Main Result

In this section we conclude the proof of Theorem 2.10.

Consider the Cauchy problem (2.23) (with Hamiltonian H as in (2.14)) with an initial datum u_0 satisfying (2.24) and fix any $r \ge 3$. Recalling Hypotheses 2.5, 2.8, setting

$$\epsilon \simeq R, \tag{4.1}$$

then for $s \gg 1$ large enough and ϵ small enough (depending on r), we have that the assumptions of Corollary 3.22 are fulfilled. Therefore we set

$$z_0 := \mathcal{T}_r(u_0),$$

and we consider the Cauchy problem

$$\dot{z} = X_{H^r}(z), \qquad z(0) = z_0, \tag{4.2}$$

with H^r given in (3.47). By (3.46) we have that the bound (2.25) on the solution $u(t)$ of (2.23) follows provided we show

$$\|z(t)\|_s^2 \lesssim_s \|z(0)\|_s^2 + R^{r+2}|t|, \qquad |t| < T_R \tag{4.3}$$

where $z(t)$ is the solution of the problem (4.2) and where we denoted

$$T_R := \sup \left\{ |t| \in \mathbb{R}^+ \ : \ \|z(t)\|_s < R \right\}, \tag{4.4}$$

the (possibly infinite) escape time of the solution from the ball of radius R.

The rest of the section is devoted to the proof of the claim (4.3). To do this we now analyze the dynamics of the system (4.2) obtained from the normal form procedure. To this end we write the Hamilton equations in the form of a system for the two variables $(z^{\leq}, z^{\perp})$ and also split the normal form $Z^r = Z_0 + Z_2$ as in item (ii) in Corollary 3.22. We get

$$\dot{z}^{\leq} = \Lambda z^{\leq} + X_{Z_0}(z^{\leq}) + \Pi^{\leq} X_{Z_2}(z^{\leq}, z^{\perp}) + \Pi^{\leq} X_{\mathcal{R}^{(r)}}(z^{\leq}, z^{\perp}), \tag{4.5}$$

$$\dot{z}^{\perp} = \Lambda z^{\perp} + \Pi^{\perp} X_{Z_2}(z^{\leq}, z^{\perp}) + \Pi^{\perp} X_{\mathcal{R}^{(r)}}(z^{\leq}, z^{\perp}) \, . \tag{4.6}$$

where Λ is the linear operator such that $\Lambda z = X_{H_0}(z)$. The key points to analyze the dynamics are the following:

(i) Z_0 is in standard Birkhoff normal form, namely it contains only monomyals Poisson commuting with H_0;
(ii) by item (i) of Lemma 3.8 one has that $\Pi^{\leq} X_{Z_2}(z^{\leq}, z^{\perp})$ is a remainder term (see item (ii) in Corollary 3.22);
(iii) $\Pi^{\perp} X_{Z_2}(z^{\leq}, z^{\perp})$ is linear in $z^{\perp}$. Furthermore, for any given trial solution $z^{\leq}(t)$ it is a time dependent family of linear operators, which by the property (2.13) are Hamiltonian and thus conserve the L^2 norm;
(iv) since Z_2 is in normal form it leaves invariant the dyadic decomposition Ω_α on which the ℓ^2 norm is equivalent to all the ℓ^2_s norms.

Remark 4.1. Recalling (2.15) we have that a monomial $u_{J_1} ... u_{J_l}$, $J_i = (j_i, \sigma_i) \in \mathcal{I}_l$, $i = 1, \dots, l$ Poisson commutes with the Hamiltonian H_0 if and only if condition (2.21) holds true. Therefore, by Hypothesis 2.8-(NR.2), the Hamiltonian Z_2 is supported only on monomials with indexes satisfying (2.22).

Formally we split the analysis in a few lemmas. The first is completely standard and provides a priori estimates on the low frequency part $z^{\leq}$ of the solution of (4.5).

Lemma 4.2. *There exists K_1 such that for any real initial datum $z_0 \equiv (z_0^{\leq}, z_0^{\perp})$ for (4.5), (4.6), fulfilling $\|z_0\|_s \leq R/2$ (with R small as in (4.1)) the following holds. One has that*

$$\left\| z^{\leq}(t) \|_s^2 \leq \| z^{\leq}(0) \right\|_s^2 + K_1 R^{r+2} |t| \, , \quad \forall t, \quad |t| < T_R, \tag{4.7}$$

where T_R is given in (4.4).

Proof. For $i \in \mathbb{Z}^d$, define the "superaction"

$$J_{[i]} := \sum_{j \in [i]} z_{(j,-)} z_{(j,+)} \equiv \sum_{j \in [i]} \left| z_{(j,-)} \right|^2$$

where the sum is over the indexes belonging to the equivalence class of $[i]$ according to Definition 2.7 and the second equality follows from the reality of u. Then, by the property of being in normal form and by properties (NR.1), (NR.2) in Hypothesis 2.8, we have $\left\{ J_{[i]}; Z_0 \right\} = 0$, so that $\dot{J}_{[i]} = \left\{ J_{[i]}; Z_2 \right\} + \left\{ J_{[i]}; \mathcal{R}^{(r)} \right\}$. Denote by $\mathcal{E}$ the set of all the equivalence classes of Definition 2.7, and, for $e \in \mathcal{E}$, denote

$$\langle e \rangle := \inf_{i \in e} \langle i \rangle$$

and define the norm

$$|z|_s^2 := \sum_{e \in \mathcal{E}} \langle e \rangle^{2s} J_e .$$

By using the dyadic property (2.20), one has that the norm $|\cdot|_s$ is equivalent to the standard one on $\Pi^{\leq}\ell^2_s$. Thus we have

$$\frac{d}{dt}\left|z^{\leq}\right|_s^2=\sum_{e\in\mathcal{E}}\langle e\rangle^{2s}\frac{d}{dt}J_e=\sum_{e\in\mathcal{E}}\langle e\rangle^{2s}\left(\{J_e;Z_2\}+\left\{J_e;\mathcal{R}^{(r)}\right\}\right)$$
$$=d\left(\left|z^{\leq}\right|_s^2\right)\left(X_{Z_2}(z^{\leq},z^{\perp})+X_{\mathcal{R}^{(r)}}(z^{\leq},z^{\perp})\right)\ .$$

Then by (3.48)–(3.49)

the last quantity is estimated by a constant times R^{r+2}. From this, denoting by K_0 the constant in the above inequality, one gets

$$\left|z^{\leq}(t)\right|_s^2\leq\left|z^{\leq}(0)\right|_s^2+K_0R^{r+2}|t|.$$

So we have

$$\left\|z^{\leq}(t)\right\|_s^2\leq C\left|z^{\leq}(t)\right|_s^2\leq C\left|z^{\leq}(0)\right|_s^2+CK_0R^{r+2}\,|t|\,,$$

from which, writing $K_1:=K_0C$ one gets the estimate (4.7). □

We now provide a priori estimates on the high frequencies $z^{\perp}$ which evolve according to (4.6).

Lemma 4.3. *Fix $r\gg1$. There is s_r such that for any $s>s_r$ there exists $K_3=K_3(s)$ such that for any real initial datum $z_0\equiv(z_0^{\leq},z_0^{\perp})$ for (4.5), (4.6), fulfilling $\|z_0\|_s\leq R/2$ (with R small as in (4.1)) the following holds. One has that*

$$\left\|z^{\perp}(t)\right\|_s^2\leq K_2\left\|z^{\perp}(0)\right\|_s^2+K_3R^{r+2}|t|,\quad\forall t\ ,\quad|t|<T_R,\tag{4.8}$$

where T_R is given by (4.4).

Proof. First, we denote by $\mathcal{Z}(z^{\leq}):\Pi^{\perp}\ell^2\to\Pi^{\perp}\ell^2$ the family of linear operator s.t. $X_{Z_2}(z^{\leq},z^{\perp})=\mathcal{Z}(z^{\leq})z^{\perp}$; We also write $\mathcal{Z}(t):=\mathcal{Z}(z^{\leq}(t))$, with $z^{\leq}(t)$ the projection on low modes of the considered solution. We now introduce some further notations. For any $z\in\Pi^{\perp}\ell^2$, we introduce the projector Π_α associated to the block Ω_α of the partition. More precisely, for any α, we define

$$\Pi_\alpha:\Pi^{\perp}\ell^2\to\Pi^{\perp}\ell^2,\quad\Pi_\alpha u:=\begin{cases}z_{(j,\sigma)} & \text{if } j\in\Omega_\alpha\\ 0 & \text{if } j\notin\Omega_\alpha\end{cases}\ .\tag{4.9}$$

Then any sequence $z\in\Pi^{\perp}\ell^2$ can be written as

$$z=\sum_\alpha z_\alpha,\quad z_\alpha:=\Pi_\alpha u\ .\tag{4.10}$$

By the property 2.2 of Definition 3.2, the normal form operator $\mathcal{Z}(t)$ has a block-diagonal structure, namely it can be written as

$$\mathcal{Z}(t)=\sum_\alpha\mathcal{Z}_\alpha(t),\quad\mathcal{Z}_\alpha(t):=\Pi_\alpha\mathcal{Z}(t)\Pi_\alpha\ .\tag{4.11}$$

For any block Ω_α, we define

$$n(\alpha):=\min{}_{j\in\Omega_\alpha}\langle j\rangle$$

and for any $z \in \ell^2_s$, we define the norm

$$[\![z]\!]_s := \Big(\sum_\alpha n(\alpha)^{2s} \|z_\alpha\|_0 \Big)^{\frac{1}{2}} .$$

By using the dyadic property (2.18), one has that the norm $[\![\cdot]\!]_s$ is equivalent to the ℓ^2_s-norm $\| \cdot \|_s$.

Consider now the normal form part of equation (4.6), namely

$$\dot{z}^\perp = \Lambda z^\perp + \Pi^\perp X_{Z_2}(z^\leq, z^\perp) \; ; \tag{4.12}$$

by (4.9), (4.10), (4.11), it is block diagonal, namely it is equivalent to the decoupled system

$$\partial_t z_\alpha(t) = \Lambda z_\alpha(t) + \mathcal{Z}_\alpha(t) z_\alpha(t) \; .$$

Since $\mathcal{Z}_\alpha$ is Hamiltonian, one immediately has that

$$\|z_\alpha(t)\|_0 = \|z_\alpha(t_0)\|_0, \quad \forall t, t_0 \in [-T_R, T_R], \quad \forall \alpha. \tag{4.13}$$

Therefore, for any $t \in [-T_R, T_R]$, for the solution of (4.12) one has

$$\begin{aligned} \|z(t)\|_s^2 \lesssim_s [\![z(t)]\!]_s^2 &\lesssim_s \sum_\alpha n(\alpha)^{2s} \|z_\alpha(t)\|_0^2 \\ &\overset{(4.13)}{\lesssim_s} \sum_\alpha n(\alpha)^{2s} \|z_\alpha(t_0)\|_0^2 \lesssim_s [\![z_0]\!]_s^2 \lesssim_s \|z_0\|_s^2 , \end{aligned}$$

so that, denoting by $\mathcal{U}(t, \tau)$ the flow map of (4.12), one has

$$\|\mathcal{U}(t, \tau) z_0\|_s \le K_2 \, \|z_0\|_s \, , \; \forall t \; . \tag{4.14}$$

Consider now (4.6). Using Duhamel formula one gets

$$z^\perp(t) = \mathcal{U}(t, 0) z_0 + \int_0^t \mathcal{U}(t, \tau) \Pi^\perp X_{\mathcal{R}}(z^\leq(\tau), z^\perp(\tau)) d\tau$$

which, together with (4.14) and using also (3.48), implies

$$\frac{d}{dt} \left\| z^\perp \right\|_s^2 \le K_r R^{r+2} .$$

We then deduce the estimate (4.8). □

Conclusion of the proof of Theorem 2.10. By Lemmas 4.2, 4.3 (see estimates (4.7), (4.8)) we have that the claim (4.3) holds. By a standard bootstrap argument one can show that (recall (4.4), (4.1)) $T_R \gtrsim \epsilon^{-r}$. This implies the thesis. □

5. Applications

Let $\mathbf{e}_1, ..., \mathbf{e}_d$ be a basis of $\mathbb{R}^d$ and let

$$\Gamma := \left\{ x \in \mathbb{R}^d \ : \ x = \sum_{j=1}^{d} 2\pi n_j \mathbf{e}_j, \quad n_j \in \mathbb{Z} \right\} \tag{5.1}$$

be a maximal dimensional lattice. We denote $\mathbb{T}^d_\Gamma := \mathbb{R}^d / \Gamma$.

To fit our scheme it is convenient to introduce in $\mathbb{T}^d_\Gamma$ the basis given by $\mathbf{e}_1, ..., \mathbf{e}_d$, so that the functions turn out to be defined on the standard torus $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$, but endowed by the metric $\mathrm{g}_{ij} := \mathbf{e}_j \cdot \mathbf{e}_i$. In particular the Laplacian turns out to be

$$\Delta_g := \sum_{l,n=1}^{d} g_{ln} \partial_{x_l} \partial_{x_n} , \qquad x = (x_1, \ldots, x_d) \in \mathbb{T}^d, \tag{5.2}$$

where g_{ln} is the inverse of the matrix g_{ij}. The positive definite symmetric quadratic form of equation (2.1) is then defined by

$$g(k,k) := \sum_{l,n=1}^{d} g_{ln} k_l k_n, \qquad \forall\, k \in \mathbb{Z}^d.$$

The coefficients g_{ln}, $l, n = 1, \ldots, d$, of the metric g above can be seen as parameters that will be chosen in the set we now introduce. We also assume the symmetry $g_{ij} = g_{ji}$ for any $i, j = 1, \ldots, d$, hence we identify the metric g with $(g_{ij})_{i \le j}$, namely we identify the space of symmetric metrics with $\mathbb{R}^{\frac{d(d+1)}{2}}$. We denote by $\|g\|_2^2 := \sum_{i,j} |g_{ij}|^2$

Definition 5.1. Consider the open set

$$\mathcal{G}_0 := \left\{ \left(g_{ij}\right)_{i \le j} \in \mathbb{R}^{\frac{d(d+1)}{2}} \ : \ \inf_{x \neq 0} \frac{g(x,x)}{|x|^2} > 0 \right\}.$$

Fix $\tau_* := \frac{d(d+1)}{2} + 1$ We then define the set of admissible metrics as follows.

$$\mathcal{G} := \cup_{\gamma > 0} \mathcal{G}_\gamma$$

where

$$\mathcal{G}_\gamma := \Big\{ g \in \mathcal{G}_0 : \Big| \sum_{i \le j} g_{ij} \ell_{ij} \Big| \ge \frac{\gamma}{\left(\sum_{i \le j} |\ell_{ij}| \right)^{\tau_*}} \quad \forall \ell \equiv (\ell_{ij})_{i \le j} \in \mathbb{R}^{\frac{d(d+1)}{2}} \setminus \{0\} \Big\}$$

Remark 5.2. The set $\mathcal{G}_\gamma$ above satisfies a diophantine estimate $|(\mathcal{G}_0 \cap B_R) \backslash \mathcal{G}_\gamma| \lesssim \gamma$ (B_R is the ball in $\mathbb{R}^{\frac{d(d+1)}{2}}$), implying that $\mathcal{G}$ has full measure in $\mathcal{G}_0$ (we denote by $|\cdot|$ the Lebesgue measure). We also point out that in Sect. 5.1, we only take the metric $g \in \mathcal{G}_0$ and we shall use the convolution potential in order to impose the non-resonance conditions. For the other applications, namely in Sects. 5.2, 5.3, 5.4 we shall use that the metric g is of the form $g = \beta \bar{g}$, with $\bar{g}$ in the set of the admissible metrics $\mathcal{G}$. We then use the parameter β, in order to verify the non-resonance conditions required.

5.1. Schrödinger Equations with Convolutions Potentials. We consider Schrödinger equations of the form

$$\mathrm{i}\partial_t \psi = -\Delta_g \psi + V * \psi + f(|\psi|^2)\psi, \quad x \in \mathbb{T}^d \tag{5.3}$$

where Δ_g is in (5.2) with $g \in \mathcal{G}_0$ (see Definition 5.1), V is a potential, $*$ denotes the convolution and the nonlinearity f is of class $C^\infty(\mathbb{R}, \mathbb{R})$ in a neighborhood of the origin and $f(0) = 0$. Equation (5.3) is Hamiltonian with Hamiltonian function

$$H = \int_{\mathbb{T}^d} (\nabla \psi \cdot \nabla \varphi + \varphi(V * \psi) + F(\psi\varphi))\, dx \tag{5.4}$$

where F is a primitive of f and φ is a variable conjugated to ψ. To get equation (5.3) one has to restrict to the invariant manifold $\varphi = \overline{\psi}$.

Fix $n \geq 0$ and $R > 0$, then the potential V is chosen in the space $\mathcal{V}$ given by

$$\mathcal{V} := \left\{ V(x) = \frac{1}{|\mathbb{T}^d|_g} \sum_{k \in \mathbb{Z}^d} \hat{V}_k e^{ik\cdot x} \ : \ \hat{V}_k \langle k \rangle^n \in \left[-\frac{1}{2}, \frac{1}{2} \right], \ \forall k \in \mathbb{Z}^d \right\}, \tag{5.5}$$

which we endow with the product probability measure. Here and below $|\mathbb{T}^d|_g$ is the measure of the torus induced by the metric g.

Theorem 5.3. *There exists a set $\mathcal{V}^{(res)} \subset \mathcal{V}$ with zero measure such that for any $V \in \mathcal{V} \setminus \mathcal{V}^{(res)}$ the following holds. For any $r \in \mathbb{N}$, there exists $s_r > d/2$ such that for any $s > s_r$ there is $\epsilon_s > 0$ and $C > 0$ such that if the initial datum for (5.3) belongs to H^s and fulfills $\epsilon := |\psi|_s < \epsilon_s$ then*

$$\|\psi(t)\|_s \leq C\epsilon, \quad \textit{for all} \quad |t| \leq C\epsilon^{-r}.$$

We are now going to prove this theorem. To fit our scheme simply introduce the Fourier coefficients

$$\psi(x) = \frac{1}{|\mathbb{T}^d|_g^{1/2}} \sum_{j \in \mathbb{Z}^d} u_{(j,+)} e^{ij\cdot x}, \quad \varphi(x) = \frac{1}{|\mathbb{T}^d|_g^{1/2}} \sum_{j \in \mathbb{Z}^d} u_{(j,-)} e^{-ij\cdot x}.$$

In these variables the equation (5.3) takes the form (2.4) with $H = H_0 + P$, H_0 of the form (2.15) with frequencies

$$\omega_j := |j|_g^2 + \hat{V}_j, \tag{5.6}$$

and P obtained by substituting in the F dependent term of the Hamiltonian (5.4). It is easy to see that the perturbation is of class $\mathcal{P}$ of Definition 2.4.

In order to apply our abstract Birkhoff normal form theorem, we only need to verify the Hypotheses 2.5, 2.8. The hypothesis (F.1) in Hyp. 2.5 holds trivially with $\beta = 2$ using (5.6).

The hypothesis (F.3) follows by the generalization of the Bourgain's Lemma proved in [15]. Precisely we now prove the following lemma.

Lemma 5.4. *The assumption (F.3) of Hypothesis 2.5 holds.*

Proof. Let Ω_α be the partition of $\mathbb{Z}^d$ constructed in Theorem 2.1 of [15]. It satisfies the properties

$$||j|_g^2 - |j'|_g^2| + |j - j'| \geq C_0(|j|^\delta + |j'|^\delta), \qquad j \in \Omega_\alpha, \quad j' \in \Omega_\beta, \quad \alpha \neq \beta,$$
$$\max_{j \in \Omega_\alpha} |j| \lesssim \min_{j \in \Omega_\alpha} |j|,$$

for some $C_0 > 0$ and $\delta \in (0, 1)$. Clearly, one has that if $j \in \Omega_\alpha$, $j' \in \Omega_\beta$ with $\alpha \neq \beta$, one has that

$$\begin{aligned} |\omega_j - \omega_{j'}| + |j - j'| &= ||j|_g^2 - |j'|_g^2 + \widehat{V}_j - \widehat{V}_{j'}| + |j - j'| \\ &\geq ||j|_g^2 - |j'|_g^2| + |j - j'| - 2 \sup_{j \in \mathbb{Z}^d} |\widehat{V}_j| \\ &\geq ||j|_g^2 - |j'|_g^2| + |j - j'| - 1 \\ &\geq C_0(|j|^\delta + |j'|^\delta) - 1 \geq C_0(|j|^\delta + |j'|^\delta)/2, \end{aligned}$$

provided $|j|^\delta + |j'|^\delta \geq \frac{2}{C_0}$, which is verified when $|j| + |j'| \geq C(\delta, C_0)$ for some constant $C(\delta, C_0) > 0$. □

It remains to verify conditions (F.2) in Hyp. 2.5 and (NR.1), (NR.2) in Hyp. 2.8. Given r and N we define

$$\mathbb{Z}_N^d := \left\{ j \in \mathbb{Z}^d \ : \ |j| \leq N \right\},$$
$$\mathcal{K}_N^r := \left\{ k \in \mathbb{Z}^{\mathbb{Z}_N^d} \ : \ 0 \neq |k| \leq r \right\},$$

and remark that its cardinality $\#\mathcal{K}_N^r \leq N^{dr}$. For $k \in \mathcal{K}_N^r$, consider

$$\mathcal{V}_k^N(\gamma) := \{ V \in \mathcal{V} \ \ |\omega \cdot k| < \gamma \} .$$

Lemma 5.5. *One has*

$$\left| \mathcal{V}_k^N(\gamma) \right| \leq 2\gamma N^n \tag{5.7}$$

with n the number in the definition of $\mathcal{V}$ in (5.5).

Proof. If $\mathcal{V}_k^N(\gamma)$ is empty there is nothing to prove. Assume that $\tilde{V} \in \mathcal{V}_k^N(\gamma)$. Since $k \neq 0$, there exists $\bar{j}$ such that $k_{\bar{j}} \neq 0$ and thus $\left| k_{\bar{j}} \right| \geq 1$; so we have

$$\left| \frac{\partial \omega \cdot k}{\partial \hat{V}_{\bar{j}}} \right| \geq 1.$$

It means that if $\mathcal{V}_k^N(\gamma)$ is not empty it is contained in the layer

$$\left| \widehat{\tilde{V}}_{\bar{j}}' - \hat{V}_{\bar{j}}' \right| \leq \gamma$$

whose measure is $\gamma \langle \bar{j} \rangle^n \leq 2\gamma N^n$. This implies (5.7). □

Lemma 5.6. *For any r there exists τ and a set $\mathcal{V}^{(res)} \subset \mathcal{V}$ of zero measure, s.t., if $V \in \mathcal{V} \setminus \mathcal{V}^{(res)}$ there exists $\gamma > 0$ s.t. for all $N \geq 1$ one has*

$$|\omega \cdot k| \geq \frac{\gamma}{N^\tau}, \ \forall k \in \mathcal{K}_N^r .$$

Proof. From Lemma 5.5 it follows that the measure of the set

$$\mathcal{V}^{(res)}(\gamma) := \bigcup_{N\geq 1} \bigcup_{k \in \mathcal{K}_N^r} \mathcal{V}_k^N \left(\frac{\gamma}{N^{dr+2}}\right)$$

is estimated by a constant times γ. It follows that the set

$$\mathcal{V}^{(res)} := \cap_{\gamma>0} \mathcal{V}^{(res)}(\gamma)$$

has zero measure and with this definition the lemma is proved. □

We remark that Lemma 5.6 implies that for $V \in \mathcal{V}^{(res)}$ the frequencies ω_j satisfy $\omega_j \neq \omega_i$ for any $i \neq j$. So that the equivalence class $[j]$ (see Definition 2.7) are composed by the single element $j \in \mathbb{Z}^d$.

5.2. Beam Equation. In this section we study the beam equation

$$\psi_{tt} + \Delta_g^2 \psi + m\psi = -\frac{\partial F}{\partial \psi} + \sum_{l=1}^{d} \partial_{x_l} \frac{\partial F}{\partial(\partial_l \psi)}, \tag{5.8}$$

with $F(\psi, \partial_{x_1}\psi, ..., \partial_{x_d}\psi)$ a function of class $C^\infty(\mathbb{R}^{d+1}; \mathbb{R})$ in a neighborhood of the origin and having a zero of order 2 at the origin.

Introducing the variable $\varphi = \dot{\psi} \equiv \psi_t$, it is well known that (5.8) can be seen as an Hamiltonian system in the variables (ψ, φ) with Hamiltonian function

$$H(\psi, \varphi) := \int_{\mathbb{T}^d} \left(\frac{\varphi^2}{2} + \frac{\psi(\Delta_g^2 + m)\psi}{2} + F(\psi, \partial_1\psi, ..., \partial_d\psi) \right) dx . \tag{5.9}$$

In order to fulfill the diophantine non-resonance conditions on the frequencies we need to make some restrictions on the metric g whereas, we only require that the mass $m > 0$ is strictly positive. More precisely, we consider $\bar{g}$ be a metric in the set of the admissible metrics $\mathcal{G}$ given in the definition 5.1. We consider a metric g of the form

$$g = \beta \bar{g}, \quad \beta \in \mathcal{B} := (\beta_1, \beta_2), \quad 0 < \beta_1 < \beta_2 < +\infty. \tag{5.10}$$

we shall use the parameter β in order to tune the resonances and to impose the non-resonance conditions required in order to apply Theorem 2.10. The precise statement of the main theorem of this section is the following one.

Theorem 5.7. *Let $\bar{g} \in \mathcal{G}$, There exists a set of zero measure $\mathcal{B}^{(res)} \subset \mathcal{B}$ such that if $\beta \in \mathcal{B} \setminus \mathcal{B}^{(res)}$ then for all $r \in \mathbb{N}$ there exist $s_r > d/2$ such that the following holds. For any $s > s_r$ there exist ϵ_{rs}, c, C such that if the initial datum for (5.8) fulfills*

$$\epsilon := \left\| (\psi_0, \dot{\psi}_0) \right\|_s := \|\psi_0\|_{H^{s+2}} + \left\| \dot{\psi}_0 \right\|_{H^s} < \epsilon_{sr}, \tag{5.11}$$

then the corresponding solution satisfies

$$\left\| (\psi(t), \dot{\psi}(t)) \right\|_s \leq C\epsilon, \quad for \quad |t| \leq c\epsilon^{-r}. \tag{5.12}$$

We actually state also a corollary which state that there exists a full measure set of metrics (not only constrained to a given direction $\overline{g}$) for which the statements of Theorem 5.7 hold. Let $0 < \beta_1 < \beta_2$ and define

$$\mathcal{G}_0(\beta_1, \beta_2) := \Big\{ g \in \mathcal{G}_0 : \beta_1 \le \|g\|_2 \le \beta_2 \Big\}. \tag{5.13}$$

where $\mathcal{G}_0$ is given in the definition 5.1.

Corollary 5.8. *There exists a zero measure set $\mathcal{G}^{(res)}_{\beta_1,\beta_2} \subseteq \mathcal{G}_0(\beta_1, \beta_2)$ such that for any $g \in \mathcal{G}_0(\beta_1, \beta_2) \backslash \mathcal{G}^{(res)}_{\beta_1,\beta_2}$ the conclusion of theorem 5.7 hold.*

Proof of Corollary 5.8. To shorten notations in this proof, we denote by $n := \frac{d(d+1)}{2}$. For any $\beta_1 \le \beta \le \beta_2$, we denote by σ_β the surface $n-1$ dimensional measure on the sphere $\partial B_\beta := \{\|g\|_2 = \beta\}$. We now prove the following two claims

- **Claim 1.** One has that the surface measure of all diophantine metrics $\mathcal{G}$ in $\mathcal{G}_0$ with norm equal 1 has full surface measure in $\mathcal{G}_0 \cap \partial B_1$, namely $\sigma_1(\mathcal{G} \cap \partial B_1) = \sigma_1(\mathcal{G}_0 \cap \partial B_1)$.
- **Claim 2.** Let $\overline{g} \in \mathcal{G} \cap \partial B_1$ and let $\mathcal{B}_{\overline{g}} \subset (\beta_1, \beta_2)$ the full measure set provided in Theorem 5.7. We shall prove that

$$\mathcal{G}^{(nr)}_{\beta_1,\beta_2} := \bigcup_{\overline{g} \in \partial B_1 \cap \mathcal{G}} \mathcal{B}_{\overline{g}}$$

  has full measure in $\mathcal{G}_0(\beta_1, \beta_2)$.

Proof of Claim 1. Let $E \subset \partial B_1$. Then the set

$$\beta E := \Big\{ \beta x : x \in E \Big\} \subset \partial B_\beta$$

and, by standard scaling properties,

$$\sigma_\beta(\beta E) = C_n \beta^{n-1} \sigma_1(E) \quad \text{for some constant} \quad C_n > 0. \tag{5.14}$$

By (5.13) and Remark (5.2), the set $\mathcal{G}_{\beta_1,\beta_2} := \mathcal{G}_0(\beta_1, \beta_2) \cap \mathcal{G}$ has full measure in the open set $\mathcal{G}_0(\beta_1, \beta_2)$. By Fubini one has

$$\begin{aligned} |\mathcal{G}_0(\beta_1, \beta_2)| &= \int_{\beta_1}^{\beta_2} \sigma_\beta(\mathcal{G}_0 \cap \partial B_\beta)\, d\beta \\ &\overset{(5.14)}{=} C_n \int_{\beta_1}^{\beta_2} \beta^{n-1} \sigma_1(\mathcal{G}_0 \cap \partial B_1)\, d\beta \\ &= \frac{C_n(\beta_2^n - \beta_1^n)}{n} \sigma_1\Big(\mathcal{G}_0 \cap \partial B_1\Big) \end{aligned} \tag{5.15}$$

and similarly

$$\begin{aligned} |\mathcal{G}_{\beta_1,\beta_2}| &= \int_{\beta_1}^{\beta_2} \sigma_\beta(\mathcal{G} \cap \partial B_\beta)\, d\beta \\ &\overset{(5.14)}{=} C_n \int_{\beta_1}^{\beta_2} \beta^{n-1} \sigma_1(\mathcal{G} \cap \partial B_1)\, d\beta \\ &= \frac{C_n(\beta_2^n - \beta_1^n)}{n} \sigma_1\Big(\mathcal{G} \cap \partial B_1\Big). \end{aligned} \tag{5.16}$$

Since $|\mathcal{G}_0(\beta_1, \beta_2)| = |\mathcal{G}_{\beta_1,\beta_2}|$, by comparing (5.15), (5.16), one immediately gets that $\sigma_1(\mathcal{G} \cap \partial B_1) = \sigma_1(\mathcal{G}_0 \cap \partial B_1)$.

PROOF OF CLAIM 2. By Fubini, the Lebesgue measure $|\mathcal{G}^{(nr)}_{\beta_1,\beta_2}|$ is

$$
\begin{aligned}
|\mathcal{G}^{(nr)}_{\beta_1,\beta_2}| &= \int_{\mathcal{G}\cap\partial B_1} |\mathcal{B}_{\overline{g}}|\, d\sigma_1(\overline{g}) = (\beta_2 - \beta_1) \int_{\mathcal{G}\cap\partial B_1} d\sigma_1(\overline{g}) = (\beta_1 - \beta_2)\sigma_1(\mathcal{G} \cap \partial B_1) \\
&\overset{Claim\ 1}{=} (\beta_1 - \beta_2)\sigma_1(\partial B_1 \cap \mathcal{G}_0) = |\mathcal{G}_0(\beta_1, \beta_2)|.
\end{aligned}
$$

The claimed statement has then been proved.

□

To prove Theorem 5.7 we first show how to fit our scheme and then we prove that the Hypotheses of Theorem 2.10 are verified.

To fit our scheme we first introduce new variables

$$
u_+(x) := \frac{1}{\sqrt{2}} \left(\left(\Delta_g^2 + m\right)^{1/4} \varphi + i \left(\Delta_g^2 + m\right)^{-1/4} \psi \right), \tag{5.17}
$$

$$
u_-(x) := \frac{1}{\sqrt{2}} \left(\left(\Delta_g^2 + m\right)^{1/4} \varphi - i \left(\Delta_g^2 + m\right)^{-1/4} \psi \right), \tag{5.18}
$$

and consider their Fourier series, namely, for $\sigma = \pm 1$

$$
u_\sigma(x) = \frac{1}{|\mathbb{T}^d|_g^{1/2}} \sum_{j\in\mathbb{Z}^d} u_{(j,\sigma)} e^{\sigma i j\cdot x} .
$$

In these variables the beam equation (5.8) takes exactly the form (2.4) with $H = H_0 + P$, H_0 of the form (2.15) with frequencies

$$
\omega_j := \sqrt{|j|_g^4 + m} . \tag{5.19}
$$

and P obtained by substituting (5.18)–(5.18) in the F dependent term of the Hamiltonian (5.9). Thanks to the regularity assumption on F, it is easy to see that the perturbation P is of class $\mathcal{P}$.

The verification of (F.3) in Hyp. 2.5 goes exactly as in the case of the Schrödinger equation, since the asymptotic of ω_j in (5.19) is $\omega_j = |j|_g^2 + O(1)$. The asymptotic condition (F.1) is also trivially fulfilled with $\beta = 2$. The main point is to verify the non-resonance conditions (F.2) and the conditions (NR.1), (NR.2) in Hyp. 2.8. This will occupy the rest of this subsection.

First of all we remark that the equivalence classes of Definition 2.7 are simply defined by

$$
[j] \equiv \left\{ i \in \mathbb{Z}^d \ :\ |i|_g = |j|_g \right\} .
$$

Now, recall that $g = \beta\overline{g}$ with $\overline{g} \in \mathcal{G}$ and $\beta \in \mathcal{B} = [\beta_1, \beta_2]$. One can easily verify that

$$
|j|_g = \beta |j|_{\overline{g}} \tag{5.20}
$$

implying that $|j|_g = |k|_g$ if and only if $|j|_{\overline{g}} = |k|_{\overline{g}}$. Hence the equivalence class $[j]$ is

$$
[j] \equiv \left\{ i \in \mathbb{Z}^d \ \ |i|_{\bar{g}} = |j|_{\bar{g}} \right\} .
$$

We are going to prove the following Lemma

Lemma 5.9. *Let $\overline{g} \in \mathcal{G}$. There exists a set $\mathcal{B}^{(res)} \subset \mathcal{B}$ of zero measure, s.t., if $\beta \in \mathcal{B} \backslash \mathcal{B}^{(res)}$ then (NR.1), (NR.2) and (F.2) hold for the metric $g = \beta\,\overline{g}$.*

We first need a lower bound on the distance between points with different modulus in $\mathbb{Z}^d$. The following lemma holds

Lemma 5.10. *Fix any $N > 1$ and let $\overline{g} \in \mathcal{G}$, $\beta \in \mathcal{B} = (\beta_1, \beta_2)$, $g = \beta\overline{g}$. One has that if $j, k \in \mathbb{Z}^d$ such that $|\ell|, |k| \leq N$, $|j|_g \neq |k|_g$, then there exists $\gamma > 0$ and a constant $C(\beta_1)$ such that*

$$||k|_g^2 - |\ell|_g^2| \geq \frac{C(\beta_1)\gamma}{N^{2\tau_*}}$$

where τ_* is given in the definition 5.1.

Proof. By recalling the Definition 5.1 of the admissible set $\mathcal{G}$, since $\overline{g} \in \mathcal{G}$, one has that there exists $\gamma > 0$ such that

$$\Big|\sum_{i \leq j} \overline{g}_{ij}\ell_{ij}\Big| \geq \frac{\gamma}{\left(\sum_{i \leq j} |\ell_{ij}|\right)^{\tau}} \quad \forall \ell = (\ell_{ij})_{i \leq j} \in \mathbb{R}^{\frac{d(d+1)}{2}} \setminus \{0\}.$$

By (5.20), since $g = \beta\overline{g}$, one has that

$$\begin{aligned} |\,|k|_g^2 - |\ell|_g^2\,| &= \beta\Big|\sum_{ij} \overline{g}_{ij}\left(k_i k_j - \ell_i \ell_j\right)\Big| \\ &\geq \beta_1 \frac{\gamma}{\left(\sum_{i,j} |k_i k_j - \ell_i \ell_j|\right)^{\tau}}. \end{aligned} \tag{5.21}$$

Since $|\ell|, |k| \leq N$, one has the following chain of inequalities:

$$\sum_{i,j} |k_i k_j - \ell_i \ell_j| \leq \sum_{ij} (|\ell_i||\ell_j| + |k_i||k_j|) \lesssim |\ell|^2 + |k|^2 \overset{|\ell|,|k| \leq N}{\lesssim} N^2.$$

The latter inequality, together with (5.21) imply that there exists a constant $C(\beta_1)$ such that

$$|\,|k|_g^2 - |\ell|_g^2\,| \geq \frac{C(\beta_1)\gamma}{N^{2\tau}}$$

uniformly on $\beta \in (\beta_1, \beta_2)$. The claimed statement has then been proved. □

Using the property (5.20), one can easily verify that the frequencies $\omega_j \equiv \omega_j(\beta)$ assume the form

$$\omega_j(\beta) = \beta^2 \Omega_j, \quad \Omega_j := \sqrt{|j|_{\overline{g}}^4 + \frac{m}{\beta^4}}. \tag{5.22}$$

Since $\beta_2 \geq \beta \geq \beta_1 > 0$,

$$\Big|\sum_{i=1}^{r} \sigma_i \omega_{j_i}\Big| = \beta^2 \Big|\sum_{i=1}^{r} \sigma_i \Omega_{j_i}\Big| \geq \beta_1^2 \Big|\sum_{i=1}^{r} \sigma_i \Omega_{j_i}\Big|$$

one can verify non resonance conditions on Ω_j. Since the map

$$(\beta_1, \beta_2) \to (\zeta_1, \zeta_2) := \left(\frac{m}{\beta_2^4}, \frac{m}{\beta_1^4}\right), \qquad \beta \mapsto \zeta := \frac{m}{\beta^4} \tag{5.23}$$

is an analytic diffeomorphism, we can introduce $\zeta = m/\beta^4$ as parameter in order to tune the resonances. Hence we verify non resonance conditions on the frequencies

$$\Omega_j(\zeta) = \sqrt{|j|_g^4 + \zeta}, \quad j \in \mathbb{Z}^d. \tag{5.24}$$

Lemma 5.11. *Let $\bar{g} \in \mathcal{G}$. For any $K \leq N$, consider K indexes $j_1, ..., j_K$ with $|j_1|_g < ... |j_K|_g \leq N$; and consider the determinant*

$$D := \begin{vmatrix} \Omega_{j_1} & \Omega_{j_2} & \dots & \Omega_{j_K} \\ \partial_\zeta \Omega_{j_1} & \partial_\zeta \Omega_{j_2} & \dots & \partial_\zeta \Omega_{j_K} \\ . & . & \dots & . \\ . & . & \dots & . \\ \partial_\zeta^{K-1} \Omega_{j_1} & \partial_\zeta^{K-1} \Omega_{j_2} & \dots & \partial_\zeta^{K-1} \Omega_{j_K} \end{vmatrix}$$

There exists $C > 0$ s.t.

$$D \geq \frac{C}{N^{\eta K^2}},$$

for some constant $\eta \equiv \eta_d > 0$ depending only on the dimension d.

The proof was given in [2]. For sake of completeness we insert it.

Proof. For any $i = 1, \dots, K$, for any $n = 0, \dots, K-1$, one computes

$$\partial_\zeta^n \Omega_{j_i}(\zeta) = C_n(|j_i|_g^4 + \zeta)^{\frac{1}{2}-n}$$

for some constant $C_n \neq 0$. This implies that

$$D \geq C \prod_{i=1}^{K} \sqrt{|j_i|_g^4 + \zeta}\, |\det(A)|$$

where the matrix A is defined as

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_K \\ . & . & \dots & . \\ . & . & \dots & . \\ x_1^{K-1} & x_2^{K-1} & \dots & x_K^{K-1} \end{pmatrix}$$

where

$$x_i := \frac{1}{|j_i|_g^4 + \zeta}, \quad i = 1, \dots, K.$$

This is a Van der Monde determinant. Thus we have

$$
\begin{aligned}
|\det(A)| = \prod_{1\le i<\ell\le K} |x_i - x_\ell| &= \prod_{1\le i<\ell\le K} \left| \frac{1}{|j_i|_g^{\frac{4}{g}}+\zeta} - \frac{1}{|j_\ell|_g^{\frac{4}{g}}+\zeta} \right| \\
&\geq \prod_{1\le i<\ell\le K} \frac{||j_i|^{\frac{4}{g}} - |j_\ell|^{\frac{4}{g}}|}{||j_i|^{\frac{4}{g}}+\zeta|||j_\ell|^{\frac{4}{g}}+\zeta|} \\
&\geq \prod_{1\le i<\ell\le K} \frac{(|j_i|^{\frac{2}{g}} + |j_\ell|^{\frac{2}{g}})||j_i|^{\frac{2}{g}} - |j_\ell|^{\frac{2}{g}}|}{||j_i|^{\frac{4}{g}}+\zeta|||j_\ell|^{\frac{4}{g}}+\zeta|} \\
&\overset{Lemma\ 5.10}{\geq} CN^{-K^2(\tau_*+4)}
\end{aligned}
$$

which implies the thesis. □

Exploiting this Lemma, and following step by step the proof of Lemma 12 of [2] one gets

Lemma 5.12. *Let $\overline{g} \in \mathcal{G}$. Then and for any r there exists $\tau \equiv \tau_r$ with the following property: for any positive γ small enough there exists a set $I_\gamma \subset (\zeta_1, \zeta_2)$ such that $\forall \zeta \in I_\gamma$ one has that for any $N \geq 1$ and any multi-index $J_1, ..., J_r$ with $|J_l| \leq N$ $\forall l$, one has*

$$
\sum_{l=1}^{r} \sigma_l \Omega_{j_l} \neq 0 \implies \left| \sum_{l=1}^{r} \sigma_l \Omega_{j_l} \right| \geq \frac{\gamma}{N^\tau} .
$$

Moreover,

$$
\left|(\zeta_1, \zeta_2) \setminus I_\gamma\right| \leq C\gamma^{1/r} .
$$

End of the proof of Lemma 5.9. Let $\gamma > 0$. By recalling the diffeomorphism (5.23), one has that the set

$$
\mathcal{I}_\gamma := \{\beta \in [\beta_1, \beta_2] : m/\beta^4 \in I_\gamma\}.
$$

satisfies the estimate

$$
|(\beta_1, \beta_2) \setminus \mathcal{I}_\gamma| \lesssim \gamma^{\frac{1}{r}}
$$

Now, if we take $\beta \in \mathcal{I}_\gamma$ and if $\sum_{i=1}^r \sigma_i \omega_{j_i} \neq 0$, one has that

$$
\begin{aligned}
\Big| \sum_{i=1}^{r} \sigma_i \omega_{j_i} \Big| = \beta^2 \Big| \sum_{i=1}^{r} \sigma_i \Omega_{j_i} \Big| &\overset{\beta_1 \le \beta \le \beta_2}{\geq} \beta_1^2 \Big| \sum_{i=1}^{r} \sigma_i \Omega_{j_i} \Big| \\
&\geq \frac{\beta_1^2 \gamma}{N^\tau}.
\end{aligned}
$$

By the above result, one has that, if

$$
\beta \in \bigcup_{\gamma>0} \mathcal{I}_\gamma
$$

then (NR.2) holds and furthermore $\bigcup_{\gamma>0} \mathcal{I}_\gamma$ has full measure. Hence the claimed statement follows by defining $\mathcal{B}^{(res)} := \mathcal{B} \setminus \Big(\bigcup_{\gamma>0} \mathcal{I}_\gamma \Big)$. □

5.3. The Quantum Hydrodinamical System. We consider the following quantum hydrodynamic system on an irrational torus $\mathbb{T}^d_\Gamma$

$$\begin{cases} \partial_t \rho = -\mathtt{m}\Delta_g \phi - \operatorname{div}(\rho \nabla_g \phi) \\ \partial_t \phi = -\frac{1}{2}|\nabla_g \phi|^2 - p(\mathtt{m}+\rho) + \dfrac{\kappa}{\mathtt{m}+\rho}\Delta_g \rho - \dfrac{\kappa}{2(\mathtt{m}+\rho)^2}|\nabla_g \rho|^2, \end{cases} \tag{QHD}$$

where $\mathtt{m} > 0$, $\kappa > 0$, the function p belongs to $C^\infty(\mathbb{R}_+; \mathbb{R})$ and $p(\mathtt{m}) = 0$. The function $\rho(t, x)$ is such that $\rho(t, x) + \mathtt{m} > 0$ and it has zero average in x. The variable x is on the irrational torus $\mathbb{T}^d$ (as in the previous two applications). We assume the conditions

$$p'(\mathtt{m}) > 0. \tag{5.25}$$

We shall use Theorem 2.10 in order to prove the following almost global existence result. In order to give a precise statement of the main result, we shall introduce the following notation. Given a function $u : \mathbb{T}^d \to \mathbb{C}$, we define

$$\Pi_0 u := \frac{1}{|\mathbb{T}^d_\Gamma|^{\frac{1}{2}}} \int_{\mathbb{T}^d} u(x)\, dx, \quad \Pi_0^\perp := \mathrm{Id} - \Pi_0.$$

Let $\bar{g}$ be a metric in the set of the admissible metrics $\mathcal{G}$ given in the definition 5.1. Exactly as in the case of the Beam equation, we consider a metric g of the form

$$g = \beta \bar{g}, \quad \beta \in \mathcal{B} := (\beta_1, \beta_2), \quad 0 < \beta_1 < \beta_2 < +\infty. \tag{5.26}$$

we shall use the parameter β in order to tune the resonances and to impose the non-resonance conditions required in order to apply Theorem 2.10. The precise statement of the long time existence for the QHD system is the following.

Theorem 5.13. *Let $\bar{g} \in \mathcal{G}$. There exists a set of zero measure $\mathcal{B}^{(res)} \subset \mathcal{B}$, s.t. if $\beta \in \mathcal{B}\backslash\mathcal{B}^{(res)}$ and $g = \beta\bar{g}$, then, $\forall r \geq 2$ there exist s_r and $\forall s > s_r$ $\exists \epsilon_{rs}, c, C$ with the following property. For any initial datum $(\rho_0, \phi_0) \in H^s(\mathbb{T}^d_\Gamma) \times H^s(\mathbb{T}^d_\Gamma)$ satisfying*

$$\|\rho_0\|_s + \|\Pi_0^\perp \phi_0\|_s \leq \epsilon$$

there exists a unique solution $t \mapsto (\rho(t), \phi(t))$ of the system (QHD) satisfying the bound

$$\|\rho(t)\|_s + \|\Pi_0^\perp \phi(t)\|_s \leq C\epsilon, \quad \forall\, |t| \leq c\epsilon^{-r}.$$

Arguing as in the proof of Corollary 5.8, one can show

Corollary 5.14. *Let $0 < \beta_1 < \beta_2$. There exists a zero measure set $\mathcal{G}^{(res)}_{\beta_1,\beta_2} \subseteq \mathcal{G}_0(\beta_1, \beta_2)$ (where $\mathcal{G}_0(\beta_1, \beta_2)$ is defined in (5.13)) such that for any*

$$g \in \mathcal{G}_0(\beta_1, \beta_2) \setminus \mathcal{G}^{(res)}_{\beta_1,\beta_2}$$

the statements of theorem 5.13 hold.

The key tool in order to prove the latter almost global existence result 5.13 is to use a change of coordinates (the so called Madelung transformation) which allows to reduce the system (QHD) to a semilinear Schrödinger type equation. We shall implement this in the next sections.

5.3.1. Madelung transform For $\lambda \in \mathbb{R}_+$, we define the change of variable (*Madelung transform*)

$$\psi := \mathcal{M}_{\psi}(\rho,\phi) := \sqrt{\mathtt{m}+\rho}\,e^{\mathrm{i}\lambda\phi}, \quad \bar{\psi} := \mathcal{M}_{\bar{\psi}}(\rho,\phi) := \sqrt{\mathtt{m}+\rho}\,e^{-\mathrm{i}\lambda\phi}. \tag{$\mathcal{M}$}$$

Notice that the inverse map has the form

$$\begin{aligned} \mathtt{m}+\rho &= \mathcal{M}_{\rho}^{-1}(\psi,\bar{\psi}) := |\psi|^2, \\ \phi &= \mathcal{M}_{\phi}^{-1}(\psi,\bar{\psi}) := \frac{1}{\lambda}\arctan\left(\frac{-\mathrm{i}(\psi-\bar{\psi})}{\psi+\bar{\psi}}\right). \end{aligned} \tag{5.27}$$

In the following lemma we state a well-posedness result for the Madelung transform.

Lemma 5.15. *Define $\kappa = (4\lambda^2)^{-1}$ and $\hbar := \lambda^{-1} = 2\sqrt{\kappa}$. Then the following holds.*
(i) Let $s > \frac{d}{2}$ and

$$\delta := \frac{1}{\mathtt{m}}\|\rho\|_s + \frac{1}{\sqrt{\kappa}}\|\Pi_0^{\perp}\phi\|_s, \qquad \sigma := \Pi_0\phi.$$

There is $C = C(s) > 1$ such that, if $C(s)\delta \le 1$, then the function ψ in ($\mathcal{M}$) satisfies

$$\|\psi - \sqrt{\mathtt{m}}e^{\mathrm{i}\lambda\sigma}\|_s \le 2\sqrt{\mathtt{m}}\delta.$$

(ii) Define

$$\delta' := \inf_{\sigma\in\mathbb{T}}\|\psi - \sqrt{\mathtt{m}}e^{\mathrm{i}\sigma}\|_s.$$

There is $C' = C'(s) > 1$ such that, if $C'(s)\delta'(\sqrt{\mathtt{m}})^{-1} \le 1$, then the functions ρ,

$$\frac{1}{\mathtt{m}}\|\rho\|_s + \frac{1}{\sqrt{\kappa}}\|\Pi_0^{\perp}\phi\|_s \le 8\frac{1}{\sqrt{\mathtt{m}}}\delta'.$$

Proof. see Lemma 2.1 in [33]. □

We now rewrite equation (QHD) in the variable $(\psi,\bar{\psi})$.

Lemma 5.16. *Let $(\rho,\phi) \in H_0^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$ be a solution of (QHD) defined over a time interval $[0,T]$, $T > 0$, such that*

$$\sup_{t\in[0,T)}\left(\frac{1}{\mathtt{m}}\|\rho(t,\cdot)\|_s + \frac{1}{\sqrt{\kappa}}\|\Pi_0^{\perp}\phi(t,\cdot)\|_s\right) \le \epsilon$$

for some $\epsilon > 0$ small enough. Then the function ψ defined in ($\mathcal{M}$) solves

$$\begin{cases} \partial_t\psi = -\mathrm{i}\big(-\frac{\hbar}{2}\Delta_g\psi + \frac{1}{\hbar}p(|\psi|^2)\psi\big) \\ \psi(0) = \sqrt{\mathtt{m}+\rho(0)}\,e^{\mathrm{i}\phi(0)}. \end{cases} \tag{5.28}$$

Proof. See Lemma 2.2 in [33]. □

Notice that the (5.28) is an Hamiltonian equation of the form

$$\partial_t\psi = -\mathrm{i}\partial_{\bar{\psi}}\mathcal{H}(\psi,\bar{\psi}), \qquad \mathcal{H}(\psi,\bar{\psi}) = \int_{\mathbb{T}^d}\Big(\frac{\hbar}{2}|\nabla_g\psi|^2 + \frac{1}{\hbar}P(|\psi|^2)\Big)dx, \tag{5.29}$$

where $\partial_{\bar{\psi}} = (\partial_{\Re\psi} + \mathrm{i}\partial_{\Im\psi})/2$. The Poisson bracket is defined by

$$\{\mathcal{H},\mathcal{G}\} := -\mathrm{i}\int_{\mathbb{T}^d}\partial_{\psi}\mathcal{H}\partial_{\bar{\psi}}\mathcal{G} - \partial_{\bar{\psi}}\mathcal{H}\partial_{\psi}\mathcal{G}\,dx. \tag{5.30}$$

5.3.2. Elimination of the zero mode We introduce the set of variables

$$\begin{cases} \psi_0 = \alpha e^{-\mathrm{i}\theta} & \alpha \in [0, +\infty),\ \theta \in \mathbb{T} \\ \psi_j = z_j e^{-\mathrm{i}\theta} & j \in \mathbb{Z}^d \setminus \{0\}, \end{cases} \tag{5.31}$$

which are the polar coordinates for $j = 0$ and a phase translation for $j \neq 0$. Rewriting (5.29) in Fourier coordinates one has

$$\mathrm{i}\partial_t \psi_j = \partial_{\bar{\psi}_j} \mathcal{H}(\psi, \bar{\psi}), \quad j \in \mathbb{Z}^d,$$

where $\mathcal{H}$ is defined in (5.29). We define also the zero mean variable

$$z := \sum_{j \in \mathbb{Z}^d \setminus \{0\}} z_j e^{\mathrm{i} j \cdot x}. \tag{5.32}$$

By (5.31) and (5.32) one has

$$\psi = (\alpha + z) e^{\mathrm{i}\theta}, \tag{5.33}$$

and it is easy to prove that the quantity

$$\mathtt{m} := \sum_{j \in \mathbb{Z}^d} |\psi_j|^2 = \alpha^2 + \sum_{j \in \mathbb{Z}^d \setminus \{0\}} |z_j|^2$$

is a constant of motion for (5.28). Using (5.31), one can completely recover the variable α in terms of $\{z_j\}_{j \in \mathbb{Z}^d \setminus \{0\}}$ as

$$\alpha = \sqrt{\mathtt{m} - \sum_{j \in \mathbb{Z}^d \setminus \{0\}} |z_j|^2}.$$

Note also that the (ρ, ϕ) variables in (5.27) do not depend on the angular variable θ defined above. This implies that system (QHD) is completely described by the complex variable z. On the other hand, using

$$\partial_{\bar{\psi}_j} \mathcal{H}(\psi e^{\mathrm{i}\theta}, \overline{\psi e^{\mathrm{i}\theta}}) = \partial_{\bar{\psi}_j} \mathcal{H}(\psi, \bar{\psi}) e^{\mathrm{i}\theta},$$

one obtains

$$\begin{cases} \mathrm{i}\partial_t \alpha + \partial_t \theta \alpha = \Pi_0 \left(p(|\alpha + z|^2)(\alpha + z) \right) \\ \mathrm{i}\partial_t z_j + \partial_t \theta z_j = \frac{\partial \mathcal{H}}{\partial \bar{\psi}_j}(\alpha + z, \alpha + \bar{z}). \end{cases} \tag{5.34}$$

Taking the real part of the first equation in (5.34) we obtain

$$\partial_t \theta = \frac{1}{\alpha} \Pi_0 \left(\frac{1}{\hbar} p(|\alpha + z|^2) \Re(\alpha + z) \right) = \frac{1}{2\alpha} \partial_{\bar{\alpha}} \mathcal{H}(\alpha, z, \bar{z}), \tag{5.35}$$

where

$$\tilde{\mathcal{H}}(\alpha, z, \bar{z}) := \frac{\hbar}{2} \int_{\mathbb{T}^d} (-\Delta_g) z \cdot \bar{z} \mathrm{d}x + \frac{1}{\hbar} \int_{\mathbb{T}^d} G(|\alpha + z|^2) \, \mathrm{d}x.$$

By (5.35), (5.34) and using that

$$\partial_{\bar{\psi}_j} \mathcal{H}(\alpha + z, \alpha + \bar{z}) = \partial_{\bar{z}_j} \tilde{\mathcal{H}}(\alpha, z, \bar{z}),$$

one obtains

$$\mathrm{i}\partial_t z_j = \partial_{\bar{z}_j}\tilde{\mathcal{H}}(\alpha, z, \bar{z}) - \frac{z_j}{2\alpha}\partial_\alpha\tilde{\mathcal{H}}(\alpha, z, \bar{z}) = \partial_{\bar{z}_j}\mathcal{K}_{\mathtt{m}}(z, \bar{z}), \quad j \neq 0, \tag{5.36}$$

where

$$\mathcal{K}_{\mathtt{m}}(z, \bar{z}) := \tilde{\mathcal{H}}(\alpha, z, \bar{z})_{|\alpha=\sqrt{\mathtt{m}-\sum_{j\neq 0}|z_j|^2}}. \tag{5.37}$$

We resume the above discussion in the following lemma.

Lemma 5.17. *The following holds.*

(i) Let $s > \frac{d}{2}$ and

$$\delta := \frac{1}{\mathtt{m}}\|\rho\|_s + \frac{1}{\sqrt{\kappa}}\|\Pi_0^\perp\phi\|_s, \quad \theta := \Pi_0\phi.$$

There is $C = C(s) > 1$ such that, if $C(s)\delta \leq 1$, then the function z in (5.32) satisfies

$$\|z\|_s \leq 2\sqrt{\mathtt{m}}\delta.$$

(ii) Define

$$\delta' := \|z\|_s.$$

There is $C' = C'(s) > 1$ such that, if $C'(s)\delta'(\sqrt{\mathtt{m}})^{-1} \leq 1$, then the functions ρ,

$$\frac{1}{\mathtt{m}}\|\rho\|_s + \frac{1}{\sqrt{\kappa}}\|\Pi_0^\perp\phi\|_s \leq 16\frac{1}{\sqrt{\mathtt{m}}}\delta'.$$

(iii) Let $(\rho, \phi) \in H_0^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$ be a solution of (QHD) defined over a time interval $[0, T]$, $T > 0$, such that

$$\sup_{t\in[0,T)}\left(\frac{1}{\mathtt{m}}\|\rho(t, \cdot)\|_s + \frac{1}{\sqrt{\kappa}}\|\Pi_0^\perp\phi(t, \cdot)\|_s\right) \leq \epsilon$$

for some $\epsilon > 0$ small enough. Then the function $z \in H_0^s(\mathbb{T}^d)$ defined in (5.32) solves (5.36).

Proof. See Lemma 2.4 in [33]. □

Remark 5.18. Using (5.27) and (5.33) one can study the system (QHD) near the equilibrium point $(\rho, \phi) = (0, 0)$ by studying the complex hamiltonian system

$$\mathrm{i}\partial_t z = \partial_{\bar{z}}\mathcal{K}_{\mathtt{m}}(z, \bar{z}) \tag{5.38}$$

near the equilibrium $z = 0$, where $\mathcal{K}_{\mathtt{m}}(z, \bar{z})$ is the Hamiltonian in (5.37). Note also that the natural phase-space for (5.38) is the complex Sobolev space $H_0^s(\mathbb{T}^d)$, $s \in \mathbb{R}$, of complex Sobolev functions with zero mean.

By Lemma 5.17, one has that Theorem 5.13 will be deduced by the following Proposition

Proposition 5.19. *Let $\bar{g} \in \mathcal{G}$. There exists a set of zero measure $\mathcal{B}^{(res)} \subset \mathcal{B}$, s.t. if $\beta \in \mathcal{B}\backslash\mathcal{B}^{(res)}$ and $g = \beta\bar{g}$ then, $\forall r \geq 2$ there exist s_r and $\forall s > s_r$ $\exists\epsilon_{rs}, c, C$ with the following property. For any initial datum $z_0 \in H_0^s(\mathbb{T}^d)$ satisfying*

$$\|z_0\|_s \leq \epsilon$$

there exists a unique solution $t \mapsto z(t)$ of the equation (5.36) satisfying the bound

$$\|z(t)\|_s \leq C\epsilon, \qquad \forall\, |t| \leq c\epsilon^{-r}.$$

The rest of this section is dedicated to the proof of the latter Proposition.

5.3.3. Taylor expansion of the Hamiltonian In this section we shall use the notations introduced in Sects. 2.1, 2.2. The only difference is that, since we shall restrict to the space of zero average functions, in all the definitions given in Sects. 2.1, 2.2, one has to replace $\mathbb{Z}^d$ by $\mathbb{Z}^d\backslash\{0\}$ and $\mathcal{Z}^d$ by $\mathcal{Z}^d_0 := (\mathbb{Z}^d\backslash\{0\}) \times \{+,-\}$. In order to study the stability of $z=0$ for (5.38) it is useful to expand $\mathcal{K}_{\mathtt{m}}$ at $z=0$. We have

$$\begin{aligned}\mathcal{K}_{\mathtt{m}}(z,\bar{z}) &= \frac{\hbar}{2}\int_{\mathbb{T}^d}(-\Delta_g)z\cdot\bar{z}\,\mathrm{d}x + \frac{1}{\hbar}\int_{\mathbb{T}^d} P\Big(\Big|\sqrt{\mathtt{m}-\sum_{j\neq 0}|z_j|^2}+z\Big|^2\Big)\,\mathrm{d}x \\ &= (2\pi)^d\frac{P(\mathtt{m})}{\hbar} + \mathcal{K}^{(2)}_{\mathtt{m}}(z,\bar{z}) + \sum_{r=3}^{N-1}\mathcal{K}^{(r)}_{\mathtt{m}}(z,\bar{z}) + R^{(N)}(z,\bar{z}),\end{aligned} \tag{5.39}$$

where

$$\mathcal{K}^{(2)}_{\mathtt{m}}(z,\bar{z}) = \frac{1}{2}\int_{\mathbb{T}^d}\frac{\hbar}{2}(-\Delta_g)z\cdot\bar{z}\,\mathrm{d}x + \frac{p'(\mathtt{m})\mathtt{m}}{\hbar}\int_{\mathbb{T}^d}\frac{1}{2}(z+\bar{z})^2\,\mathrm{d}x,$$

for any $r=3,\cdots,N-1$, $\mathcal{K}^{(r)}_{\mathtt{m}}(z,\bar{z})$ is an homogeneous multilinear Hamiltonian function of degree r of the form

$$\mathcal{K}^{(r)}_{\mathtt{m}}(z,\bar{z}) = \sum_{\substack{\sigma\in\{-1,1\}^r,\ j\in(\mathbb{Z}^d\backslash\{0\})^r \\ \sum_{i=1}^r\sigma_i j_i=0}} (\mathcal{K}^{(r)}_{\mathtt{m}})_{\sigma,j}\, z^{\sigma_1}_{j_1}\cdots z^{\sigma_r}_{j_r}, \qquad |(\mathcal{K}^{(r)}_{\mathtt{m}})_{\sigma,j}| \lesssim_r 1,$$

and

$$\|X_{R^{(N)}}(z)\|_s \lesssim_s \|z\|^{N-1}_{H^s}, \qquad \forall\, z\in B_1(H^s_0(\mathbb{T}^d)).$$

This implies that $\mathcal{K}^{(r)}_m$ is in the class $\mathcal{P}_r$. The vector field of the Hamiltonian in (5.39) has the form

$$\begin{aligned}\partial_t\begin{bmatrix} z\\ \bar{z}\end{bmatrix} = \begin{bmatrix}-\mathrm{i}\partial_{\bar{z}}\mathcal{K}_{\mathtt{m}}\\ \mathrm{i}\partial_z\mathcal{K}_{\mathtt{m}}\end{bmatrix} &= -\mathrm{i}\begin{pmatrix}\frac{\hbar\Delta_g}{2}+\frac{\mathtt{m}p'(\mathtt{m})}{\hbar} & \frac{\mathtt{m}p'(\mathtt{m})}{\hbar}\\ -\frac{\mathtt{m}p'(\mathtt{m})}{\hbar} & \frac{\hbar\Delta_g}{2}-\frac{\mathtt{m}p'(\mathtt{m})}{\hbar}\end{pmatrix}\begin{bmatrix} z\\ \bar{z}\end{bmatrix} \\ &\quad + \sum_{r=3}^{N-1}\begin{bmatrix}-\mathrm{i}\partial_{\bar{z}}\mathcal{K}^{(r)}_{\mathtt{m}}\\ \mathrm{i}\partial_z\mathcal{K}^{(r)}_{\mathtt{m}}\end{bmatrix} + \begin{bmatrix}-\mathrm{i}\partial_{\bar{z}}R^{(N)}\\ \mathrm{i}\partial_z R^{(N)}\end{bmatrix}.\end{aligned}$$

Let us now introduce the 2×2 matrix of operators

$$\mathcal{C} := \frac{1}{\sqrt{2\omega(D)A(D,\mathtt{m})}}\begin{pmatrix}A(D,\mathtt{m}) & -\frac{1}{2}\mathtt{m}p'(\mathtt{m})\\ -\frac{1}{2}\mathtt{m}p'(\mathtt{m}) & A(D,\mathtt{m})\end{pmatrix},$$

with

$$A(D,\mathtt{m}) := \omega(D) + \tfrac{\hbar}{2}(-\Delta_g) + \tfrac{1}{2}\mathtt{m}p'(\mathtt{m}),$$

and where $\omega(D)$ is the Fourier multiplier with symbol

$$\begin{aligned}&\sqrt{\frac{\hbar^2}{4}|j|^4_g + \mathtt{m}p'(\mathtt{m})|j|^2_g} = \frac{\hbar}{2}\omega_j, \quad j\in\mathbb{Z}^d\setminus\{0\}\\ &\omega_j := \sqrt{|j_g|^4+\delta|j_g|^2}, \qquad \delta := \frac{4\mathtt{m}p'(\mathtt{m})}{\hbar^2}.\end{aligned} \tag{5.40}$$

Notice that, by using (5.25), the matrix $\mathcal{C}$ is bounded, invertible and symplectic, with estimates

$$\|\mathcal{C}^{\pm 1}\|_{\mathcal{L}(H_0^s \times H_0^s,\ H_0^s \times H_0^s)} \leq 1 + \sqrt{k}\beta, \quad \beta := \frac{\mathtt{m}p'(\mathtt{m})}{k}.$$

Consider the change of variables

$$\begin{bmatrix} w \\ \bar{w} \end{bmatrix} := \mathcal{C}^{-1} \begin{bmatrix} z \\ \bar{z} \end{bmatrix}.$$

then the Hamiltonian (5.39) reads

$$\begin{aligned}
&\widetilde{\mathcal{K}}_{\mathtt{m}} = \widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)} + \sum_{k=3}^{N-1} \widetilde{\mathcal{K}}_m^{(r)} + \tilde{R}_N \\
&\widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}(w, \bar{w}) := \mathcal{K}_{\mathtt{m}}^{(2)}\Big(\mathcal{C}\begin{bmatrix} w \\ \bar{w} \end{bmatrix}\Big) := \frac{1}{2}\int_{\mathbb{T}^d} \omega(D)w \cdot \bar{w}\mathrm{d}x, \\
&\widetilde{\mathcal{K}}_{\mathtt{m}}^{(i)} \in \mathcal{P}_i \quad i = 3, \ldots, N-1, \\
&\|X_{\tilde{R}_N}(w)\|_s \lesssim_s \|w\|_s^{N-1}, \quad \forall \|w\|_s \ll 1.
\end{aligned}$$

From the latter properties, one deduces that the perturbation

$$P = \sum_{k=3}^{N-1} \widetilde{\mathcal{K}}_m^{(r)} + \tilde{R}_N,$$

is in the class $\mathcal{P}$ of Definition 2.4.

The verification of (F.3) goes exactly as in the case of the Schrödinger equation, since also in this case $\omega_j = |j|_g^2 + O(1)$. The asymptotic condition (F.1) is also trivially fulfilled with $\beta = 2$. The main point is to verify the nonresonance conditions (F.2) and (NR.1), (NR.2). This will be done in the next subsection.

5.3.4. Non-resonance conditions for (QHD) According to the Sect. 5.2 on the Beam equation, we fix the metric $\bar{g} \in \mathcal{G}$ and we consider $g = \beta\,\bar{g}$, $\beta_1 \leq \beta \leq \beta_2$. We shall verify the non-resonance conditions on the frequencies ω_j in (5.40). By the property (5.20),

$$\begin{aligned}
&\omega_j = \sqrt{|j|_g^4 + \delta |j|_g^2} = \sqrt{\beta^4 |j|_{\bar{g}}^4 + \beta^2 \delta |j|_{\bar{g}}^2} = \beta^2 \Omega_j, \\
&\Omega_j := |j|_{\bar{g}} \sqrt{|j|_{\bar{g}}^2 + \frac{\delta}{\beta^2}}, \quad j \in \mathbb{Z}^d \setminus \{0\}.
\end{aligned} \tag{5.41}$$

Since $\beta_2 \geq \beta \geq \beta_1 > 0$,

$$\Big|\sum_{i=1}^r \sigma_i \omega_{j_i}\Big| = \beta^2 \Big|\sum_{i=1}^r \sigma_i \Omega_{j_i}\Big| \geq \beta_1^2 \Big|\sum_{i=1}^r \sigma_i \Omega_{j_i}\Big|$$

one can verify non resonance conditions on Ω_j. Since the map

$$(\beta_1, \beta_2) \to (\zeta_1, \zeta_2) := \Big(\frac{\delta}{\beta_2^2}, \frac{\delta}{\beta_1^2}\Big), \qquad \beta \mapsto \zeta := \frac{\delta}{\beta^2} \tag{5.42}$$

is an analytic diffeomorphism, we can introduce $\zeta = \delta/\beta^2$ as parameter in order to tune the resonances. Hence we verify non resonance conditions on the frequencies

$$\Omega_j \equiv \Omega_j(\zeta) = |j|_{\bar{g}} \sqrt{|j|_{\bar{g}}^2 + \zeta}, \quad j \in \mathbb{Z}^d \setminus \{0\}. \tag{5.43}$$

Lemma 5.20. *Assume that the metric $\bar{g} \in \mathcal{G}$. For any $K \leq N$, consider K indexes $j_1, ..., j_K$ with $|j_1|_g < \ldots < |j_K|_g \leq N$; and consider the determinant*

$$D := \begin{vmatrix} \Omega_{j_1} & \Omega_{j_2} & \ldots & \Omega_{j_K} \\ \partial_\zeta \Omega_{j_1} & \partial_\zeta \Omega_{j_2} & \ldots & \partial_\zeta \Omega_{j_K} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \partial_\zeta^{K-1} \Omega_{j_1} & \partial_\zeta^{K-1} \Omega_{j_2} & \ldots & \partial_\zeta^{K-1} \Omega_{j_K} \end{vmatrix}$$

One has

$$D \geq \frac{C}{N^{\eta K^2}}$$

for some constant $\eta \equiv \eta_d > 0$ large enough, depending only on the dimension d.

Proof. The dispersion relation is slightly different w.r. to the one of the Beam equation, hence in this proof we just highlight the small differences w.r. to Lemma 5.11. For any $i = 1, \ldots, K$, for any $n = 0, \ldots, K - 1$, one computes

$$\partial_\zeta^n \Omega_{j_i}(\zeta) = C_n |j_i|_{\bar{g}} (|j_i|_{\bar{g}}^2 + \zeta)^{\frac{1}{2} - n}$$

for some constant $C_n \neq 0$. This implies that

$$D \geq C \prod_{i=1}^{K} \Big(|j_i|_{\bar{g}} \sqrt{|j_i|_{\bar{g}}^2 + \zeta} \Big) |\det(A)|$$

where the matrix A is defined as

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_K \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ x_1^{K-1} & x_2^{K-1} & \ldots & x_K^{K-1} \end{pmatrix}$$

where

$$x_i := \frac{1}{|j_i|_{\bar{g}}^2 + \zeta}, \quad i = 1, \ldots, K.$$

This is a Van der Monde determinant. Thus we have

$$\begin{aligned} |\det(A)| &= \prod_{1 \leq i < \ell \leq K} |x_i - x_\ell| = \prod_{1 \leq i < \ell \leq K} \left| \frac{1}{|j_i|_{\bar{g}}^2 + \zeta} - \frac{1}{|j_\ell|_{\bar{g}}^2 + \zeta} \right| \\ &\geq \prod_{1 \leq i < \ell \leq K} \frac{||j_i|_{\bar{g}}^2 - |j_\ell|_{\bar{g}}^2|}{||j_i|_{\bar{g}}^2 + \zeta|\,||j_\ell|_{\bar{g}}^2 + \zeta|} \overset{\text{Lemma 5.10}}{\geq} C N^{-K^2(\tau_* + 4)} \end{aligned}$$

which implies the thesis. □

Exploiting this Lemma, and following step by step the proof of Lemma 12 of [2] one gets

Lemma 5.21. *Let $\bar{g} \in \mathcal{G}$. Then for any r there exists τ_r with the following property: for any positive γ small enough there exists a set $I_\gamma \subset (\zeta_1, \zeta_2)$ such that $\forall \zeta \in I_\gamma$ one has that for any $N \geq 1$ and any set $J_1, \ldots, J_r$ with $|J_l| \leq N$ $\forall l$, one has*

$$\sum_{l=1}^{r} \sigma_l \Omega_{j_l} \neq 0 \quad \Longrightarrow \quad \left| \sum_{l=1}^{r} \sigma_l \Omega_{j_l} \right| \geq \frac{\gamma}{N^\tau} .$$

Moreover one has

$$\left| [\zeta_1, \zeta_2] \setminus I_\gamma \right| \leq C \gamma^{1/r} .$$

By recalling the diffeomorphism (5.42), one has that the set

$$\mathcal{I}_\gamma := \{\beta \in [\beta_1, \beta_2] : \delta/\beta^2 \in I_\gamma\}$$

satisfies the estimate

$$|(\beta_1, \beta_2) \setminus \mathcal{I}_\gamma| \lesssim \gamma^{\frac{1}{r}}$$

Now, if we take $\beta \in \mathcal{I}_\gamma$ and if $\sum_{i=1}^{r} \sigma_i \omega_{j_i} \neq 0$, one has that (recall (5.41))

$$\left| \sum_{i=1}^{r} \sigma_i \omega_{j_i} \right| = \beta^2 \left| \sum_{i=1}^{r} \sigma_i \Omega_{j_i} \right| \overset{\beta_1 \leq \beta \leq \beta_2}{\geq} \beta_1^2 \left| \sum_{i=1}^{r} \sigma_i \Omega_{j_i} \right|$$
$$\geq \frac{\beta_1^2 \gamma}{N^\tau} .$$

By the above result, one has that, if

$$\beta \in \bigcup_{\gamma > 0} \mathcal{I}_\gamma$$

then (NR.2) holds and furthermore $\bigcup_{\gamma>0} \mathcal{I}_\gamma$ has full measure. Hence the claimed statement follows by defining $\mathcal{B}^{(res)} := \mathcal{B} \backslash \left(\bigcup_{\gamma>0} \mathcal{I}_\gamma \right)$.

5.4. Stability of Plane Waves in NLS. Consider the NLS

$$\mathrm{i}\psi_t = -\Delta_g \psi + f(|\psi|^2)\psi, \tag{5.44}$$

with $f \in C^\infty(\mathbb{R}, \mathbb{R})$, $f(0) = 0$ and $g = \beta \bar{g}$, $\bar{g} \in \mathcal{G}$ and $\beta \in (\beta_1, \beta_2) \subset (0, +\infty)$. (recall the Definition 5.1). The equation (5.44) admits solutions of the form

$$\psi_{*,m}(x, t) = a e^{\mathrm{i}(m \cdot x - \nu t)}, \quad m \in \mathbb{Z}^d \tag{5.45}$$

with $\nu = |m|_g^2 + f(a^2)$ and $a > 0$. In order to state the next stability theorem, we need that a suitable condition between $f'(a^2)$ and the metric g is satisfied. For this reason, we slightly modify the definition of $\mathcal{G}_0$ in 5.1. We then re-define $\mathcal{G}_0$ in the following way: fix $K > 0$, we define

$$\mathcal{G}_0 := \left\{ \left(g_{ij} \right)_{i \leq j} \in \mathbb{R}^{\frac{d(d+1)}{2}} \ : \ \inf_{x \neq 0} \frac{g(x, x)}{|x|^2} > K \right\} \tag{5.46}$$

The definition of the admissible set $\mathcal{G}$ is then the same in which one replace this new set $\mathcal{G}_0$ with its hold definition. The main theorem of this section is the following.

Theorem 5.22. *Assume that* $0 < \beta_1 < \beta_2$, $\bar{g} \in \mathcal{G}$, $2f'(a^2) < \beta_1^2 K^2$, $f'(a^2) \neq 0$ *(where* $K > 0$ *is the constant appearing in (5.46)). Then there exists a set of zero measure* $\mathcal{B}^{(res)} \subset \mathcal{B} := (\beta_1, \beta_2)$, *such that for* $\beta \in \mathcal{B} \backslash \mathcal{B}^{(res)}$ *for* $g = \beta \bar{g}$, *then, for any* $r \geq 3$, *there exist* $s_r > 0$ *such that the following holds. For any* $s > s_r$ *and any* $m \in \mathbb{Z}^d$ *there exist constants* ϵ_{rsm}, c, C *such that if the initial datum* ψ_0 *for (5.44) fulfills*

$$\|\psi_0\|_{L^2} = a\sqrt{|\mathbb{T}^d|_g}, \quad \epsilon := \left\|\psi_0 - \psi_{*,m}(.,0)\right\|_{H^s} < \epsilon_{srm}, \tag{5.47}$$

then the corresponding solution fulfills

$$\left\|\psi(t) - \psi_{*,m}(.,t)\right\|_s \leq C\epsilon, \quad \forall\ |t| \leq c\epsilon^{-r}. \tag{5.48}$$

Arguing as in the proof of Corollary 5.8, one can show also in this case the following

Corollary 5.23. *Let* $0 < \beta_1 < \beta_2$. *There exists a zero measure set* $\mathcal{G}^{(res)}_{\beta_1,\beta_2} \subseteq \mathcal{G}_0(\beta_1, \beta_2)$, *where* $\mathcal{G}_0(\beta_1, \beta_2) := \left\{g \in \mathcal{G}_0 : \beta_1 \leq \|g\|_2 \leq \beta_2\right\}$, *such that for any* $g \in \mathcal{G}_0(\beta_1, \beta_2) \backslash \mathcal{G}^{(res)}_{\beta_1,\beta_2}$ *the statements of theorem 5.22 hold.*

The rest of this subsection is devoted to sketch the proof of Theorem 5.22, which follows exactly the proof of the corresponding theorem in [29] except that in the case of nonresonant tori one has to substitute the nonresonant condition by [29] with our nonresonance and structure conditions (see Hypotheses 2.5, 2.8).

We start by reducing the problem to a problem of stability of the origin of a system of the form (2.14).

First it is easy to see that introducing the new variables φ by

$$\varphi(x,t) = e^{-\mathrm{i}m\cdot x} e^{-\mathrm{i}t|m|^2} \psi(x + 2mt, t),$$

then φ still fulfills (5.44), but $\psi_{*,m}(x,t)$ is changed to $ae^{-\mathrm{i}\nu t}$ with $\nu = f(a^2)$.

The idea of [29] is to exploit that $\varphi(x) = a$ appears as an elliptic equilibrium of the reduced Hamiltonian system obtained applying Marsden Weinstein procedure to (5.44) in order to reduce the Gauge symmetry. We recall that according to Marsden Weinstein procedure (following [29]), when one has a system invariant under a one parameter symmetry group, then there exists an integral of motion (the L^2 norm in this case), and the effective dynamics occurs in the quotient of the level surface of the integral of motion with respect to the group action. This is the same procedure exploited in Sect. 5.3 for the QHD system. The effective system has a Hamiltonian which is obtained by restricting the Hamiltonian to the level surface. Such a Hamiltonian is invariant under the symmetry group associated to the integral of motion.

More precisely, consider the zero mean variable

$$z(x) := \frac{1}{|\mathbb{T}^d|_g^{1/2}} \sum_{j \in \mathbb{Z}^d \backslash \{0\}} z_j e^{\mathrm{i}j\cdot x}$$

and the substitution

$$\varphi(x) = e^{\mathrm{i}\theta}\left(\sqrt{a^2 - \left|\mathbb{T}^d\right|_g \|z\|_{L^2}^2} + z(x)\right) \tag{5.49}$$

where $\theta \in \mathbb{T}$ is a parameter along the orbit of the Gauge group, Notice that φ belongs to the level surface $\|\varphi\|_{L^2} = a\sqrt{\left|\mathbb{T}^d\right|_g}$ and $z(x)$ is the new free variable. In this case it also turns out that this is a canonical variable (as it can be verified by the theory of [3]). Thus the Hamiltonian for the reduced system turns out to be

$$H_a(z,\bar{z}) = \int_{\mathbb{T}^d} \left(\bar{\varphi}(-\Delta\varphi) + F(|\varphi|^2)\right) dx,$$

with φ given by (5.49). The explicit form of the Hamiltonian and its expansion were computed in [29] who showed that all the terms of the Taylor expansion of H_a have zero momentum and that all the nonlinear terms are bounded, so, with our language, the nonlinear part is of class $\mathcal{P}$. Considering the quadratic part, [29] showed that there exists a linear transformation preserving H^s norms and the zero momentum condition, such that the quadratic part takes the form (2.15) with

$$\omega_j = \sqrt{|j|_g^4 - f'(a^2)|j|_g^2}\,. \tag{5.50}$$

The system is now suitable for the application of Theorem 2.10. We do not give the details, since the verification of the nonresonance and structural assumptions are done exactly in the same way as in the previous cases. Indeed one can prove the nonresonance conditions on the frequencies (5.50) reasoning as done in Sect. 5.3.4.

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A. A Technical Lemma

In this section by ℓ_s^2 we mean $\ell_s^2(\mathbb{Z}^d;\mathbb{C})$.

Lemma A.1. *Let*

$$X : \underbrace{\ell_s^2 \times \ldots \times \ell_s^2}_{r-times} \to \ell_s^2,$$

be a symmetric r-linear $X(u^{(1)},\ldots,u^{(r)}) = \left(X_j(u^{(1)},\ldots,u^{(r)})\right)_{j\in\mathbb{Z}^d}$ with the property that there exist $\sigma_0,\sigma_1,\ldots,\sigma_r$, with $\sigma_l \in \{-1,1\}$ such that

$$X_j(u^{(1)},\ldots,u^{(r)}) = \sum_{\substack{j_1,\ldots,j_r\in\mathbb{Z}^d \\ \sigma_0 j + \sum_{l=1}^r \sigma_l j_l = 0}} X_{j,j_1,\ldots,j_r} u_{j_1}^{(1)} \ldots u_{j_r}^{(r)}, \tag{A.1}$$

and $X_{j,j_1,\dots,j_r}$ completely symmetric with respect to any permutation of the indexes $j, j_1, \dots, j_r$ fulfilling

$$\sup_{j,j_1,\dots,j_r\in\mathbb{Z}^d} \left|X_{j,j_1,\dots,j_r}\right| < \infty.$$

Then, for any $s > s_0 > d/2$ there exists a constant $C_{s,r} > 0$ such that one has

$$\left\|X(u^{(1)},\dots,u^{(r)})\right\|_s \leq C_{s,r} \sup_{j,j_1,\dots,j_r\in\mathbb{Z}^d} \left|X_{j,j_1,\dots,j_r}\right| \times \\ \times \sum_{l=1}^{r} \|u^{(1)}\|_{s_0}\dots\|u^{(l-1)}\|_{s_0}\|u^{(l)}\|_s\|u^{(l+1)}\|_{s_0}\dots\|u^{(r)}\|_{s_0}\ .$$

Proof. One has

$$\left\|X(u^{(1)},\dots,u^{(r)})\right\|_s = \sum_j \langle j\rangle^{2s} \left| \sum_{\substack{j_1,\dots,j_r\in\mathbb{Z}^d \\ \sigma_0 j+\sum_{l=1}^r \sigma_l j_l=0}} X_{j,j_1,\dots,j_r} u^{(1)}_{j_1}\cdots u^{(r)}_{j_r}\right|^2 \\ \leq \sup_{j,j_1,\dots,j_r} \left|X_{j,j_1,\dots,j_r}\right|^2 \sum_j \langle j\rangle^{2s} \left(\sum_{\substack{j_1,\dots,j_r\in\mathbb{Z}^d \\ \sigma_0 j+\sum_{l=1}^r \sigma_l j_l=0}} |u^{(1)}_{j_1}\cdots u^{(r)}_{j_r}| \right)^2$$

To fix ideas consider first the case $\sigma_l = 1$ $\forall l$, then the bracket is the j-th Fourier coefficient of the function $v(x) = u^{(1)}(x)\cdots u^{(r)}(x)$, with

$$u^{(l)}(x) = \sum_{j\in\mathbb{Z}^d} \frac{1}{|\mathbb{T}^d|^{1/2}} u^{(l)}_j e^{\mathrm{i} j\cdot x}$$

for which it is well known that

$$\|v\|_s \leq C_{s,r} \sum_{l=1}^{r} \|u^{(1)}\|_{s_0}\dots\|u^{(l-1)}\|_{s_0}\|u^{(l)}\|_s\|u^{(l+1)}\|_{s_0}\dots\|u^{(r)}\|_{s_0}\ .$$

then the thesis immediately follows. To deal with the case of different signs every time one has $\sigma_l = -1$ one simply substitutes $\overline{u^{(l)}}$ to $u^{(l)}$. This concludes the proof. □

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