Physica Scripta

PAPER

RECEIVED
31 October 2022

REVISED
1 February 2023

ACCEPTED FOR PUBLICATION
13 February 2023

PUBLISHED
10 March 2023

The $\mathfrak{sl}_2(\mathbb{R})$ coalgebra symmetry and the superintegrable discrete-time systems

Giorgio Gubbiotti[1] and **Danilo Latini**[2]

[1] Dipartimento di Matematica 'Federigo Enriques', Università degli Studi di milano, Via C. Saldini 50, 20133 Milano, Italy & INFN Sezione Milano, Via G. Celoria 16, 20133 Milano, Italy

[2] School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia

E-mail: giorgio.gubbiotti@unimi.it and d.latini@uq.edu.au

Keywords: coalgebra symmetry, discrete-time systems, superintegrability

Abstract

In this paper, we classify all the variational discrete-time systems in quasi-standard form in N degrees of freedom admitting coalgebra symmetry with respect to the generic realisation of the Lie–Poisson algebra $\mathfrak{sl}_2(\mathbb{R})$. This approach naturally yields several quasi-maximally and maximally superintegrable discrete-time systems, both known and new. We conjecture that this exhausts the (super)integrable cases associated with this algebraic construction.

1. Introduction

This paper is devoted to the classification and the study of a class of discrete-time systems in N degrees of freedom admitting coalgebra symmetry with respect to the Lie–Poisson algebra $\mathfrak{sl}_2(\mathbb{R})$. We make use of the notion of coalgebra symmetry for discrete-time systems we recently introduced in [29]. The main outcome of this paper is that the coalgebra symmetry approach can be fruitfully used to systematically produce superintegrable discrete-time systems in an analogous way as its continuous counterpart introduced in [11, 13]. In particular, within this paper we introduce several seemingly new discrete-time systems, including an N degrees of freedom maximally superintegrable discretisation of the celebrated Smorodinski–Winternitz system [21, 23], one of the first maximally superintegrable systems ever introduced [44].

To be more specific, we consider a class of discrete-time systems we call the *systems in quasi-standard form* which we define as the discrete Euler–Lagrange equations (dEL) of the following class of discrete Lagrangians (dLagrangian) [41]:

$$L = \sum_{k=1}^{N} \ell_k(q_k(t+h)q_k(t)) - V(\mathbf{q}(t)), \qquad \mathbf{q}(t) = (q_1(t),\ldots,q_N(t)). \tag{1.1}$$

Here $h > 0$ is a (fixed) constant and $t \in h\mathbb{Z}$, while the functions $q_k(t)$ are not supposed to be defined for all $t \in \mathbb{R}$. Following the tradition of [42, 54–56] we choose such notation to avoid confusing double indexing in the formulæ. Furthermore, we suppose that the functions $\ell_k = \ell_k(\xi)$ are smooth and locally invertible functions in a given open domain of $\mathbb{R}$. That is, the dLagrangian (1.1) is supposed to be a well-defined function on $U_1 \times U_2 \subset (\mathbb{R}^N)^{\times 2}$, where U_i, $i = 1{,}2$are open subsets of $\mathbb{R}^N$. After we fix the explicit form of the functions ℓ_k we will specify the form of the sets U_i, $i = 1{,}2$, see theorem 3.1.

Computing the discrete Euler–Lagrange equations associated to the dLagrangian (1.1) we have that a system in quasi-standard form has the following explicit expression:

$$\ell_k'(q_k(t+h)q_k(t))q_k(t+h) + \ell_k'(q_k(t)q_k(t-h))q_k(t-h) = \frac{\partial V(\mathbf{q}(t))}{\partial q_k(t)}, \tag{1.2}$$

for k=1,…,N. The number N is called the *degrees of freedom* of the system, and will be denoted throughout the paper with the shorthand notation d.o.f.. Because of the local invertibility assumptions on the functions ℓ_k we have that equation (1.2) is step-by-step solvable to determine the next iterate, as it was done in [54] in a particular case. In general, we will not show the solved equations to avoid cumbersome expressions.

© 2023 IOP Publishing Ltd

Remark 1.1. Before going further we give a few remarks on the terminology we chose to adopt and the meaning of the parameters.

- The fixed parameter $h > 0$ has the meaning of time step between two different stages of the evolution of the system. If $h \to 0^+$ we have the so-called *continuum limit*. To be more precise, the limit $h \to 0^+$ gives a (possibly trivial) continuous-time system which needs to be discussed case-by-case. Indeed, each discrete-time system need an ad-hoc scaling of the parameters. We will discuss the continuum limits of the discrete-time systems we found in this paper in section 5.
- We choose to call "system in quasi-standard form" the systems (1.2) because if $\ell_k(\xi) = \xi$ for all k=1,…,N the system is in the so-called *standard form*, as defined in [32, 57].

From the general theory of discrete variational systems, see [17, 61] we have that they are naturally symplectic. Indeed, we can introduce the following canonical momenta:

$$p_k(t) = \ell_k'(q_k(t)q_k(t-h))q_k(t-h), \quad k = 1,\ldots,N, \tag{1.3}$$

and write the the dEL equations (1.2) in canonical form as:

$$\ell_k'(q_k(t+h)q_k(t))q_k(t+h) + p_k(t) = \frac{\partial V(\mathbf{q}(t))}{\partial q_k(t)}, \tag{1.4a}$$

$$p_k(t+h) = \ell_k'(q_k(t+h)q_k(t))q_k(t), \tag{1.4b}$$

with k=1,…,N. Since in the right hand side of equation (1.4b) $\mathbf{q}(t+h)$ is present, the iteration step in equation (1.4) is intended as a two-step procedure. That is, it must be accomplished in the following way:

$$(\mathbf{q}(t), \mathbf{p}(t)) \overset{(1.4a)}{\to} (\mathbf{q}(t+h), \mathbf{p}(t)) \overset{(1.4b)}{\to} (\mathbf{q}(t+h), \mathbf{p}(t+h)). \tag{1.5}$$

We will denote the iteration from step t to $t+h$ by φ_h. As a map we will assume that φ_h: $U_1 \times U_2 \to U_1 \times U_2$, where the U_i are properly chosen open subsets of $\mathbb{R}^N$. Once the sets U_i, $i = 1, 2$, are fixed there is a one-to-one correspondence between the discrete-time system in the form (1.4) and its map form. So, in this paper we will use interchangeably the words 'discrete-time system' and 'map'.

From the general theory of discrete-time variational system the equations (1.4) preserve the canonical Poisson bracket:

$$\{q_i(t), q_j(t)\} = \{p_i(t), p_j(t)\} = 0, \quad \{q_i(t), p_j(t)\} = \delta_{i,j}, \tag{1.6}$$

see [17, 54, 63]. Here, by preservation of a Poisson bracket we mean that the following relation holds true:

$$\{q_i(t+h), q_j(t+h)\} = \{q_i(t), q_j(t)\}, \tag{1.7a}$$

$$\{p_i(t+h), p_j(t+h)\} = \{p_i(t), p_j(t)\}, \tag{1.7b}$$

$$\{q_i(t+h), p_j(t+h)\} = \{q_i(t), p_j(t)\}. \tag{1.7c}$$

This implies that the map φ_h is a *Poisson map*. In particular, since the canonical Poisson bracket (1.6) has maximal rank we have that the map φ_h is a *symplectic map*.

From the general theory of integrable symplectic maps, we recall the following definitions:

- A symplectic map φ_h: $U_1 \times U_2 \to U_1 \times U_2$ possessing N functionally independent invariants in involution with respect to a non-singular Poisson bracket for all values of $h > 0$ is said to be a *Liouville integrable map*.
- A Liouville integrable map φ_h: $U_1 \times U_2 \to U_1 \times U_2$ possessing $N + k$, with $k > 0$, functionally independent invariants in involution with respect to a non-singular Poisson bracket for all values of $h > 0$ is said to be a *superintegrable map*.
- A Liouville integrable map φ_h: $U_1 \times U_2 \to U_1 \times U_2$ possessing $2N - 2$ functionally independent invariants in involution with respect to a non-singular Poisson bracket for all values of $h > 0$ is said to be a *quasi-maximally superintegrable (QMS) map*.
- A Liouville integrable map φ_h: $U_1 \times U_2 \to U_1 \times U_2$ possessing $2N - 1$ functionally independent invariants in involution with respect to a non-singular Poisson bracket for all values of $h > 0$ is said to be a *maximally superintegrable (QMS) map*.

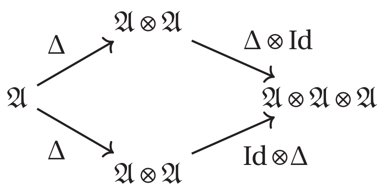
These definitions naturally generalise to the case of Poisson maps: a Poisson map φ_h: $U_1 \times U_2 \to U_1 \times U_2$ is said to be *Liouville–Poisson integrable* if, for all values of $h > 0$, it admits $M - r$ functionally independent invariants in involution, where M is the number of equations and $2r$ is the rank of the preserved Poisson structure.

Remark 1.2. Differently from the review [44] on continuous superintegrability we consider superintegrability to *require* Liouville integrability, making it a stronger property. We decide to adopt this definition because as noticed already in [29] not all discrete-time systems admitting more than N non-commuting invariants are integrable with respect to other commonly accepted notions of integrability for discrete-time systems, e.g. algebraic entropy [14].

For a complete overview on the integrability of Poisson and symplectic maps we refer to [17, 61, 63], the review part of the thesis [60], and our previous paper [29].

Before recalling the construction of coalgebra symmetry for discrete-time systems we make a final observation on continuum limits, i.e. the limits as $h \to 0^+$. Given a (super)integrable discrete-time system, if a continuum limit is known then it is expected to be (super)integrable as well. However, we observe that this does not necessarily follow from the (super)integrability of the associated discrete system. For instance, in [27] were presented several examples where in the continuum limit one invariant of a system in two d.o.f. is lost, yet the continuous-time system possesses two invariants. If both the discrete-time system for $h > 0$ and its continuum limit as $h \to 0$ are (super)integrable, we say that the discrete system is a *(super)integrable discretisation*.

We now briefly recall the definition of coalgebra symmetry for discrete-time systems we introduced in [29]. The concept of coalgebra was formulated in the theory of quantum groups [18, 19]. Precisely, a *coalgebra* is a pair of objects $(\mathfrak{A}, \Delta)$ where $\mathfrak{A}$ is a unital, associative algebra and $\Delta\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ is a *coassociative* map. That is, Δ satisfies the following condition:

$(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta \iff$ 

and it is an algebra homomorphism from $\mathfrak{A}$ to $\mathfrak{A} \otimes \mathfrak{A}$:

$$\Delta(X \cdot Y) = \Delta(X) \cdot \Delta(Y) \qquad \forall\ X, Y \in \mathfrak{A}. \tag{1.9}$$

The map Δ is called the *coproduct map*. When there is no possible confusion on the coproduct map, it is customary to denote the coalgebra simply by $\mathfrak{A}$.

In [29] we gave the following definition:

Definition 1.3. A Poisson map φ_h is said to possess the *coalgebra symmetry* with respect to the Poisson coalgebra $(\mathfrak{A}, \Delta)$ if for all $N \in \mathbb{N}$ the evolution of generators A_i, $i = 1,\ldots,K$ in a N degrees of freedom realisation of the Poisson coalgebra is:

(i) *closed* in the Poisson coalgebra, that is:

$$A_i(t + h) = a_i(A_1(t),\ldots,A_K(t)), \quad i = 1,\ldots,K, \tag{1.10}$$

with $a_i \in \mathcal{C}^\infty(\mathfrak{A})$,

(ii) it is a Poisson map with respect to the Poisson algebra $\mathfrak{A}$, i.e.:

$$\{A_i(t + h), A_j(t + h)\} = \varphi_h(\{A_i(t), A_j(t)\}), \quad i, j = 1,\ldots,K, \tag{1.11}$$

(iii) assuming that the Poisson algebra $\mathfrak{A}$ admits r *independent Casimir functions* $\{C_1(t),\ldots,C_r(t)\}$, these are *preserved as invariants* by the map φ_h, i.e.:

$$C_i(\varphi_h(t)) = C_i(t), \quad i = 1,\ldots,r. \tag{1.12}$$

This definition allows us to provide an analogue of the construction of the invariants for Hamiltonian systems given in [11, 13], a result stated in [29], theorem 3.3.

The plan of the paper is the following: in section 2 we remind the main properties of the Lie–Poisson algebra $\mathfrak{sl}_2(\mathbb{R})$ and its generic symplectic realisation. In particular, we discuss the construction of the corresponding left and right Casimir invariants. In section 3 we classify all systems in quasi-standard form admitting coalgebra symmetry with respect to the generic realisation of the Lie–Poisson algebra $\mathfrak{sl}_2(\mathbb{R})$. This result is contained in theorem 3.1. In section 4 we study the Liouville–Poisson integrability of the dynamical system of the form (1.10) associated to the generic realisation of the Lie–Poisson algebra $\mathfrak{sl}_2(\mathbb{R})$. To be more specific, we impose the existence of an additional polynomial invariant and prove in theorem 4.1 that it exists if the degree of the invariant is 1, 2, or 3. For invariants of higher degree we conjecture, based on the evidence obtained for degrees 4 and 5, that no polynomial invariants exist. In section 5 we identify the systems we found in section 4 in the explicit symplectic realisation. This gives us several integrable systems, which we put in the context of the literature. In particular, we find a discrete-time maximally superintegrable version of the Smorodinski–Winternitz system. We note that the maximal superintegrability of this discrete-time model follows from a peculiar construction of the $\mathfrak{sl}_2(\mathbb{R})$ coalgebra which is possible only in this specific case. We also find a QMS reduction of the discrete-time Wojciechowski system, introduced in [56], and a generalisation of a N degrees of freedom autonomous discrete-time Painlevé I equation we obtained in our earlier work [29]. In section 6 we provide some concluding remarks and discuss the further possible developments.

2. The $\mathfrak{sl}_2(\mathbb{R})$ Lie–Poisson coalgebra

The three-dimensional Lie–Poisson algebra $\mathfrak{sl}_2(\mathbb{R})$, spanned by the generators $\mathbf{J} := \{J_-, J_+ J_3\}$, is characterized by the following Lie–Poisson brackets:

$$\{J_-, J_+\} = 4J_3, \quad \{J_3, J_+\} = 2J_+, \quad \{J_3, J_-\} = -2J_-, \tag{2.1}$$

and it is endowed with the Casimir invariant:

$$C = C(\mathbf{J}) = J_+J_- - J_3^2. \tag{2.2}$$

This Lie–Poisson algebra can be endowed with a 'natural' coproduct map called the *primitive coproduct* [59]. Its explicit action on the basis generators and the unit element is given by ($\mu = \pm, 3$):

$$\Delta(J_\mu) = J_\mu \otimes 1 + 1 \otimes J_\mu, \quad \Delta(1) = 1 \otimes 1, \tag{2.3}$$

and extends to polynomial elements through the homomorphism property:

$$\Delta(J_\mu J_\nu) = \Delta(J_\mu)\Delta(J_\nu), \quad \mu, \nu = \pm, 3. \tag{2.4}$$

For example, the coproduct of the Casimir function (2.2) is computed as:

$$\begin{aligned}\Delta(C) &= \Delta(J_+J_- - J_3^2) = \Delta(J_+)\Delta(J_-) - \Delta(J_3)^2\\ &= (J_+ \otimes 1 + 1 \otimes J_+)(J_- \otimes 1 + 1 \otimes J_-) - (J_3 \otimes 1 + 1 \otimes J_3)^2\\ &= J_+J_- \otimes 1 + J_+ \otimes J_- + J_- \otimes J_+ + 1 \otimes J_+J_- - (J_3^2 \otimes 1 + 2J_3 \otimes J_3 + 1 \otimes J_3^2)\\ &= (J_+J_- - J_3^2) \otimes 1 + 1 \otimes (J_+J_- - J_3^2) + J_+ \otimes J_- + J_- \otimes J_+ - 2J_3 \otimes J_3,\end{aligned} \tag{2.5}$$

where we used the definition primitive coproduct (2.3) and the properties of the tensor products. From the definition of the Casimir invariant (2.2) we obtain the final result:

$$\Delta(C) = C \otimes 1 + 1 \otimes C + J_+ \otimes J_- + J_- \otimes J_+ - 2J_3 \otimes J_3. \tag{2.6}$$

Note that the Casimir of $\mathfrak{sl}_2(\mathbb{R})^{\otimes 2}$ genuinely contains new information because it is not just two tensor copies of the casimir C, but additional terms are present.

After giving this summary of the abstract properties related to the $\mathfrak{sl}_2(\mathbb{R})$ Lie–Poisson coalgebra we need a way to 'embed' these properties in the setting of (continuous or discrete-time) dynamical systems. To this end, we use the concept of *symplectic realisation of a Lie–Poisson algebra* as expressed in the following definition:

Definition 2.1. Assume we are given a Lie–Poisson algebra $\mathfrak{A}$ generated by $\{J_1,\ldots,J_m\}$, associated structure constants $c_{\mu,\nu}^{\rho}$ such as:

$$\{J_\mu, J_\nu\}_{\mathfrak{A}} = \sum_{\rho=1}^{m} c_{\mu,\nu}^{\rho} J_\rho, \quad \mu, \nu = 1,\ldots,m, \tag{2.7}$$

and Casimir invariants $\{C_1,\ldots,C_r\}$. A N *degrees of freedom symplectic realisation* of the Lie–Poisson algebra $\mathfrak{A}$ is a map

$$D\colon \mathfrak{A} \to \Omega, \tag{2.8}$$

where Ω is an open subset of a symplectic manifold (M, ω) with $\dim M = 2N$, such that it preserves the commutation relations (2.7):

$$\{D(J_\mu), D(J_\nu)\}_\omega = \sum_{\rho=1}^{m} c_{\mu,\nu}^{\rho} D(J_\rho). \quad \mu, \nu = 1,\ldots,m. \tag{2.9}$$

Moreover, the realisation $D\colon \mathfrak{A} \to \Omega$ is called *generic* if the dimension of M equals the number of generators of $\mathfrak{A}$ minus the number of its Casimir functions, i.e.:

$$2N = m - r. \tag{2.10}$$

Remark 2.2. We remark that on a symplectic manifold (M, ω), by Darboux theorem [2] and its discrete-time analog [17] we can introduce two sets of canonically conjugated variables $(\boldsymbol{\xi}, \boldsymbol{\pi}) \in U_{\xi} \times U_{\pi} \subseteq (\mathbb{R}^N)^{\times 2}$, where $\boldsymbol{\xi}$ are the canonical coordinates, and $\boldsymbol{\pi}$ their corresponding canonical momenta, defined on some open sets $U_{\xi}, U_{\pi} \subseteq \mathbb{R}^N$, and satisfying the canonical Poisson relations (analogous to (1.6)):

$$\{\xi_i, \xi_j\} = \{\pi_i, \pi_j\} = 0, \quad \{\xi_i, \pi_j\} = \delta_{i,j}. \tag{2.11}$$

Throughout the paper, we will denote the discrete-time variables in lowercase Latin letters and the continuous time variables in capital Latin letters. The Greek letters $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ will denote variables that can be either continuous or discrete-time. The difference between the discrete-time and the continuous-time cases is that in the continuous-time setting the dynamics is specified by a smooth function $H = H(\boldsymbol{\xi}, \boldsymbol{\pi})$, while in the discrete-time setting by a symplectic map φ_h: $U_{\xi} \times U_{\pi} \to U_{\xi} \times U_{\pi}$, as discussed in the Introduction.

A one degree of freedom symplectic realisation of $\mathfrak{sl}_2(\mathbb{R})$ on $(\xi_1, \pi_1) \in \mathbb{R}_+ \times \mathbb{R}$ is given by:

$$D(J_+) = \pi_1^2 + \frac{b_1}{\xi_1^2}, \quad D(J_-) = \xi_1^2, \quad D(J_3) = \xi_1\pi_1, \tag{2.12}$$

for arbitrary $b_1 \in \mathbb{R}$. Indeed, from the canonical commutation relations (2.11) it is readily verified that:

$$\{D(J_-), D(J_+)\} = 4D(J_3), \quad \{D(J_3), D(J_\pm)\} = \pm 2D(J_\pm). \tag{2.13}$$

Moreover, this realisation is *generic*, since $N = 1$, $m = 3$ and $r = 1$. So, following the literature (see for example [7, 16]), we will call this realisation *the generic one d.o.f. symplectic realisation of* $\mathfrak{sl}_2(\mathbb{R})$. That said, we will *identify* the generators with the corresponding functions on the right hand side of (2.12). The image of the Casimir functions (2.2) through the realisation (2.12) reads:

$$D(C(\mathbf{J})) = C(D(\mathbf{J})) = b_1 \in \mathbb{R}. \tag{2.14}$$

Once a realisation has been constructed, the primitive coproduct can be used to raise the number of degrees of freedom. For example, from formula (2.3) we obtain the following functions on $\mathbb{R}_+^2 \times \mathbb{R}^2$:

$$\begin{aligned} &(D \otimes D)\Delta(J_-) = \xi_1^2 + \xi_2^2, \quad (D \otimes D)\Delta(J_+) = \pi_1^2 + \pi_2^2 + \frac{b_1}{\xi_1^2} + \frac{b_2}{\xi_2^2} \\ &(D \otimes D)\Delta(J_3) = \xi_1\pi_1 + \xi_2\pi_2, \end{aligned} \tag{2.15}$$

once two copies of the one degree of freedom symplectic realisation (2.12) are considered, each of them associated to the corresponding site in the tensor product space $1 \otimes 2$ with associated free parameters $b_i \in \mathbb{R}$, $i = 1, 2$.

The functions (2.15) provide a two degrees of freedom symplectic realisation for the same Lie–Poisson algebra $\mathfrak{sl}_2(\mathbb{R})$, now w.r.t. the canonical Poisson bracket in $\mathbb{R}_+^2 \times \mathbb{R}^2$. The crucial point is that at this level the image of the Casimir element, which is now given by (2.6), turns out to be:

$$(D \otimes D)\Delta(C(\mathbf{J})) = C((D \otimes D)\Delta(\mathbf{J})) = (\xi_1\pi_2 - \xi_2\pi_1)^2 + b_1\frac{\xi_2^2}{\xi_1^2} + b_2\frac{\xi_1^2}{\xi_2^2} + b_1 + b_2, \tag{2.16}$$

that is, it is no longer a constant, but a function. Moreover, by construction, this function Poisson commutes with the generators in the two degrees of freedom symplectic realisation (2.15).

So, by applying the coproduct map iteratively, and extending its definition through the following generalization of the coassociativity property:

$$\Delta^{[N]} := (\overbrace{\mathrm{Id}\otimes \ldots \otimes \mathrm{Id}}^{N-2} \otimes \Delta^{[2]})\circ\Delta^{[N-1]} = (\Delta^{[2]} \otimes \overbrace{\mathrm{Id}\otimes \ldots \otimes\mathrm{Id}}^{N-2})\circ\Delta^{[N-1]} \tag{2.17}$$

where $\Delta^{[1]}$:=Id and $\Delta^{[2]}$:=Δ, one ends up with the N degrees of freedom symplectic realisation:

$$J_+ = \sum_{k=1}^{N}\left(\pi_k^2 + \frac{b_k}{\xi_k^2}\right), \quad J_- = \sum_{k=1}^{N}\xi_k^2, \quad J_3 = \sum_{k=1}^{N}\xi_k\pi_k, \tag{2.18}$$

where the canonical variables are defined on $\mathbb{R}_+^N \times \mathbb{R}^N$, and $b_k \in \mathbb{R}$, k=1,…,N are N associated arbitrary constants. In equation (2.18) and the following, we omit the symbol $(D \otimes \ldots \otimes D)$, because we will not consider different realisations. At this level, the crucial fact is that a total number of $(2N - 3)$ *left* and *right* Casimir invariants can be obtained from the left and right embedding of the m-th order $(2 \leqslant m \leqslant N)$ coproduct on the Casimir function (2.2):

$$\mathsf{C}^{[m]} := \Delta^{[m]}(C) \otimes \overbrace{1\otimes\ldots\otimes 1}^{N-m}, \qquad \mathsf{C}_{[m]} := \overbrace{1\otimes\ldots\otimes 1}^{N-m} \otimes \Delta^{[m]}(C). \tag{2.19}$$

The image of these elements under the N d.o.f. symplectic realisation (2.18) reads as:

$$\mathsf{C}^{[m]} = \sum_{1\leqslant i<j}^{m}\left[L_{i,j}^2 + b_i\frac{\xi_j^2}{\xi_i^2} + b_j\frac{\xi_i^2}{\xi_j^2}\right] + \sum_{j=1}^{m} b_j \qquad m = 2,\ldots, N, \tag{2.20a}$$

$$\mathsf{C}_{[m]} = \sum_{N-m+1\leqslant i<j}^{N}\left[L_{i,j}^2 + b_i\frac{\xi_j^2}{\xi_i^2} + b_j\frac{\xi_i^2}{\xi_j^2}\right] + \sum_{j=N-m+1}^{N} b_j \quad m = 2,\ldots, N, \tag{2.20b}$$

where we indicated the $N(N-1)/2$ rotation generators as:

$$L_{i,j} := \xi_i\pi_j - \xi_j\pi_i. \tag{2.21}$$

These $(2N-3)$ quadratic (in the momenta) functions Poisson commute with the generators (2.18) by construction. Moreover, they turn out to be functionally independent.

Remark 2.3. We remark that if we restrict to the case $N=2$ the left and right Casimir functions collapse to the same expression:

$$\mathsf{C}^{[2]} = \mathsf{C}_{[2]} = \Delta^{[2]}(C) \tag{2.22}$$

which is nothing but (2.6), the latter leading to the invariant (2.16) at a fixed realisation, as expected. This extends to any N, in fact for $m=N$ the two expressions collapse to:

$$\mathsf{C}^{[N]} = \mathsf{C}_{[N]} = \Delta^{[N]}(C), \tag{2.23}$$

where the action of the Nth-order coproduct is given by (2.17). This is why formulæ (2.20) give us $2N-3$ functionally independent invariants and not $2N-2$.

So, if the variables $(\boldsymbol{\xi}, \boldsymbol{\pi}) := (\mathbf{Q}, \mathbf{P}) \in U_{\mathbf{Q}} \times U_{\mathbf{P}} \subset \mathbb{R}_+^N \times \mathbb{R}^N$ are *continuous*, we can conclude that the family of Hamiltonian systems:

$$h = h(J_-, J_+, J_3) = h\left(\mathbf{Q}^2, \mathbf{P}^2 + \sum_{k=1}^{N}\frac{b_k}{Q_k^2}, \mathbf{Q}\cdot\mathbf{P}\right) \tag{2.24}$$

where h is any smooth function of the generators of the Lie–Poisson algebra $\mathfrak{sl}_2(\mathbb{R})$, is QMS. This is because the above Hamiltonian is automatically endowed with the $2N-3$ functionally independent invariants (2.20). On the other hand, in the discrete-time setting, there is no exact equivalent of the Hamiltonian function and the notion of coalgebra symmetry is replaced by definition 1.3. Thus, the problem of characterising discrete-time (symplectic) systems admitting $\mathfrak{sl}_2(\mathbb{R})$ as a hidden coalgebra symmetry is much less trivial. In the case of systems in quasi-standard form, this problem is solved in theorem 3.1. Moreover, due to the absence of the Hamiltonian, these systems are not naturally born QMS, and in fact not even Liouville integrable. The problem of integrability is tackled in section 4.

Remark 2.4. We remark that the symplectic realisation with $b_i = 0$, for $i = 1,\ldots, N$, is connected to radially symmetric systems. In this particular case, the left and right Casimir invariants are:

$$\mathsf{C}^{[m]} = \sum_{1\leqslant i<j}^{m} L_{i,j}^2, \qquad \mathsf{C}_{[m]} = \sum_{N-m+1\leqslant i<j}^{N} L_{i,j}^2 \qquad m = 2,\ldots, N, \tag{2.25}$$

which are nothing but the Casimir invariants associated with rotation subalgebras $\mathfrak{so}(m) \subseteq \mathfrak{so}(N)$. In the continuous setting, the Hamiltonian function (2.24) is given by:

$$h = h(J_-, J_+, J_3) = h(\mathbf{Q}^2, \mathbf{P}^2, \mathbf{Q}\cdot\mathbf{P}). \tag{2.26}$$

As expected, rotational symmetry is sufficient to provide quasi-maximal superintegrability. Notice that if we restrict to natural Hamiltonian systems defined in Euclidean N-space, i.e. we take:

$$h(J_-, J_+, J_3) = J_+ + V(J_-) \tag{2.27}$$

then as a consequence of Bertrand's theorem [2, 15], only two MS subcases arise. Namely, the Harmonic and Kepler-Coulomb (KC) systems, with the corresponding potentials given by:

$$V(J_-) = \alpha J_- \quad \text{and} \quad V(J_-) = -\frac{\alpha}{\sqrt{J_-}}, \tag{2.28}$$

respectively. In [29], proposition 4.2 it was proved that discrete-time radial systems in standard form admit $\mathfrak{sl}_2(\mathbb{R})$ coalgebra. In the present paper in corollary 3.3 the converse is proved.

In general, the presence of non-central terms has the effect of breaking the radial symmetry, but by preserving quasi-maximal superintegrability, which is kept thanks to the existence of the new integrals obtained through the image of the left and right Casimir invariants under the realisation (2.18). Let us conclude the section by mentioning that with the same choice of potentials (2.28), but now in terms of the new realisation involving non-central terms, would result respectively in the N d.o.f. Smorodinsky-Winternitz system [21] and the N d.o.f. generalized KC system, the latter being the N d.o.f. generalization of the (fourth-order) superintegrable Hamiltonian introduced in [62].

Although the many advances pursued in the framework of superintegrable systems with an arbitrary number of d.o.f. using the method described above, which has also been applied to the analysis of many other Lie–Poisson coalgebras [8, 16], the problem of finding MS subcases is usually left open. This is due to the fact that additional integrals, not directly obtainable following this algebraic approach, may arise. From this perspective, even for the $\mathfrak{sl}_2(\mathbb{R})$ Lie–Poisson algebra, we have recalled how it is possible to reach quasi-maximal superintegrability at most, as a total number of $2N-3$ functionally independent integrals can be constructed, besides the Hamiltonian, from the left and right Casimir invariants. Thus, for MS subcases, the search for an additional missing functionally independent constant would be required.

3. Classification results

We state and prove the following classification result:

Theorem 3.1. *A system in quasi-standard form (1.2) possesses coalgebra symmetry with respect to* $\mathfrak{sl}_2(\mathbb{R})$ *if and only if:*

$$\ell_k(\xi) = \int^{\xi}\sqrt{1-\frac{b_k}{\eta^2}}\,\mathrm{d}\eta,\quad \mathrm{b}_k \in \mathbb{R},\qquad \mathrm{V} = \mathrm{V}(\mathbf{q}^2). \tag{3.1}$$

This implies that the dLagrangian and the associated pairs of canonical variables are well-defined in the open set $U = (\mathbb{R}_+^N)^{\times 2} \subset (\mathbb{R}^N)^{\times 2}$*, or an open subset of it depending on the form of the function* $V = V(\chi)$*.*

Remark 3.2. The 'if' part of this theorem is a generalisation of [29], proposition 4.2, where it is proved that radial systems in standard form admit coalgebra symmetry. The converse will be explicitly stated in corollary 3.3.

Proof. Consider the N d.o.f. realisation of $\mathfrak{sl}_2(\mathbb{R})$ given in equation (2.18) with respect to the discrete canonical variables $(\mathbf{q}(t),\ \mathbf{p}(t))$. Then, the evolution of the $J_+(t)$ under the system (1.4) is:

$$J_+(t+h) = \sum_{k=1}^{N}\left[q_k^2(t)(\ell_k{}'(q_k(t)q_k(t+h)))^2 + \frac{b_k}{q_k^2(t+h)}\right]. \tag{3.2}$$

The right-hand side needs to be independent from $\mathbf{q}_k(t+h)$. To ensure this independence, we can differentiate equation (3.2) with respect to $q_l(t+h)$ and put the result identically equal to zero:

$$2q_l^3(t)\ell_l{}'(q_l(t)q_l(t+h))\ell_l{}''(q_l(t)q_l(t+h)) - 2\frac{b_l}{q_l^3(t+h)} = 0,\qquad l = 1,\ldots,N. \tag{3.3}$$

Interpreting the equation in terms of the variable $\xi_l = q_l(t)q_l(t+h)$, we can integrate equation (3.3) to:

$$(\ell_l{}'(\xi_l))^2 = C_l - \frac{b_l}{\xi_l^2},\qquad l = 1,\ldots,N. \tag{3.4}$$

Plugging back into equation (3.2) we obtain:

$$J_+(t+h) = \sum_{k=1}^{N} C_k q_k^2(t). \tag{3.5}$$

Imposing that the right-hand side of the equation is in $\mathcal{C}^\infty(\mathfrak{sl}_2(\mathbb{R}))$ we easily obtain that $C_k = C$ for all k. Note that, when integrating we obtain a condition of the following form:

$$\ell_l(\xi_l) = \int^{\xi_l}\sqrt{C-\frac{b_l}{\eta^2}}\,\mathrm{d}\eta + \mathrm{D}_l,\qquad \mathrm{l} = 1,\ldots,\mathrm{N}, \tag{3.6}$$

but the additive constant D_l is inessential to equations of motion (1.2) and can be safely put to zero. Then, using the scaling:

$$b_l \to Cb_l,\qquad V(\mathbf{q}) \to \sqrt{C}\,V(\mathbf{q}), \tag{3.7}$$

we see that we can choose without loss of generality $C = 1$. This proves the first part of the theorem, represented by the first identity in (3.1). Moreover, from this follows that the dLagrangian and the associated pairs of canonical variables are well-defined in the open set $U = (\mathbb{R}_+^N)^{\times 2} \subset (\mathbb{R}^N)^{\times 2}$ or an open subset of it depending on the form of the function $V = V(\mathbf{q}(t))$.

We fixed the form of the coefficients ℓ_k, so now we proceed to derive the full system in order to check the conditions of definition 1.3. By direct computations we have:

$$J_+(t+h) = J_-(t), \tag{3.8a}$$

$$J_-(t+h) = J_+(t) - 2\mathbf{p}\cdot\nabla V + (\nabla V)^2, \tag{3.8b}$$

$$J_3(t+h) = -J_3(t) + \mathbf{q}\cdot\nabla V. \tag{3.8c}$$

While it is possible to prove that the system (3.8) satisfies condition (ii), is it clear that conditions (i) and (iii) are not satisfied in general. As outlined in the last section of [29] we start by imposing that condition (iii) holds. Evaluating the translation of the Casimir function (2.2) we have:

$$C(t+h) = C(t) - [2\mathbf{q}^2(\mathbf{p}\cdot\nabla V) - 2(\mathbf{q}\cdot\mathbf{p})(\mathbf{q}\cdot\nabla V) - \mathbf{q}^2(\nabla V)^2 - (\mathbf{q}\cdot\nabla V)^2]. \tag{3.9}$$

To have the preservation of the Casimir functions, we impose that the term in square brackets is identically zero:

$$2\mathbf{q}^2(\mathbf{p}\cdot\nabla V) - 2(\mathbf{q}\cdot\mathbf{p})(\mathbf{q}\cdot\nabla V) - \mathbf{q}^2(\nabla V)^2 - (\mathbf{q}\cdot\nabla V)^2 = 0. \tag{3.10}$$

Since V does not depend on $\mathbf{p}$ we can take coefficients with respect to it. In this way we obtain that V has to satisfy the following system of equations:

$$p_i^1: \mathbf{q}^2\frac{\partial V}{\partial q_i} - q_i\,\mathbf{q}\cdot\nabla V = 0, \tag{3.11a}$$

$$p_i^0: \mathbf{q}^2(\nabla V)^2 - (\mathbf{q}\cdot\nabla V)^2 = 0. \tag{3.11b}$$

Apply Lagrange's identity [65]:

$$\sum_{k=1}^N v_k^2\sum_{k=1}^N w_k^2 - \left(\sum_{k=1}^N v_k w_k\right)^2 = \sum_{i=1}^{N-1}\sum_{j=i+1}^N (v_i w_j - v_j w_i)^2 \tag{3.12}$$

to equation (3.11b) to obtain:

$$\mathbf{q}^2(\nabla V)^2 - (\mathbf{q}\cdot\nabla V)^2 = \sum_{i=1}^{N-1}\sum_{j=i+1}^N\left(q_i\frac{\partial V}{\partial q_j} - q_j\frac{\partial V}{\partial q_i}\right)^2 = 0. \tag{3.13}$$

This readily implies:

$$q_i\frac{\partial V}{\partial q_j} - q_j\frac{\partial V}{\partial q_i} = 0,\quad i = 1,\ldots,N,\quad j = i+1,\ldots,N. \tag{3.14}$$

Not all equations in (3.14) are independent: one can only consider the subset with $i = 1$ and $j=2,\ldots,N$. The solution of this system of $N-1$ partial differential equations is given by $V = V(\mathbf{q}^2)$, as in formula (3.1). The introduction of $V = V(\mathbf{q}^2)$ makes equation (3.11a) vanish identically meaning that condition (iii) is satisfied. We observe then that further restrictions on the domain of the symplectic map can only come from the singularities of the function $V = V(\chi)$. Finally, the $\mathfrak{sl}_2(\mathbb{R})$ associated dynamical system has the following form:

$$J_+(t+h) = J_-(t), \tag{3.15a}$$

$$J_-(t+h) = J_+(t) - 4J_3(t)V'(J_-(t)) + 4J_-[V'(J_-(t))]^2, \tag{3.15b}$$

$$J_3(t+h) = -J_3(t) + 2J_-V'(J_-(t)). \tag{3.15c}$$

Hence, condition (i) is satisfied, and this ends the proof of the theorem. □

From theorem 3.1 we have that the most general form of the $\mathfrak{sl}_2(\mathbb{R})$ coalgebrically symmetric Lagrangian is:

$$L = \sum_{k=1}^N\int^{q_k(t)q_k(t+h)}\sqrt{1 - \frac{b_k}{\eta^2}}\,\mathrm{d}\eta - \mathrm{V}(\mathbf{q}^2(\mathrm{t})), \tag{3.16}$$

whose associated discrete Euler–Lagrange equations are:

$$q_k(t+h)\sqrt{1 - \frac{b_k}{q_k^2(t)q_k^2(t+h)}} + q_k(t-h)\sqrt{1 - \frac{b_k}{q_k^2(t)q_k^2(t-h)}} = 2V'(\mathbf{q}^2(t))q_k. \tag{3.17}$$

From formula (1.3) the symplectic form of the system is:

$$q_k(t+h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}}+p_k(t)=2V'(\mathbf{q}^2(t))q_k, \tag{3.18a}$$

$$p_k(t+h)=q_k(t)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}}. \tag{3.18b}$$

We conclude this section with a corollary which represents the inverse of [29], proposition 4.2:

Corollary 3.3. *A system in standard form*

$$L=\sum_{i=1}^{N}q_i(t+h)q_i(t)+V(\mathbf{q}), \tag{3.19}$$

possesses coalgebra symmetry with respect to $\mathfrak{sl}_2(\mathbb{R})$ *if and only if it is* radially symmetric, *i.e.* $V=V(\mathbf{q}^2)$.

Proof. Observe that, from equation (3.16), if $b_k=0$ for all k=1,…,N the corresponding system becomes in standard form. So, since theorem 3.1 is a necessary and sufficient condition the statement follows. □

Remark 3.4. We remark that the dynamical system associated with the $\mathfrak{sl}_2(\mathbb{R})$ (3.15) is independent from the value of the constants b_k. So, from the coalgebra point of view systems in standard form, that is $b_k=0$ for all $k=1,\ldots,N$, and in quasi-standard form can be considered in the same way: the (super)integrability of a system in standard form can be transferred to a system in quasi-standard form and vice-versa.

4. Polynomial invariants of the associated dynamical system

As discussed in [29] in the discrete-time setting the existence of a rank one coalgebra symmetry alone it is not enough to guarantee the Liouville integrability of the underlying discrete-time system. This is different from the continuum setting, when this has been proven to be true, see [7, 9].

So, in this section, we will discuss the integrability of the system (3.15) assuming the existence of an additional polynomial invariant:

$$I_d=\sum_{1\leqslant i+j+k\leqslant d}a_{i,j,k}J_+^iJ_-^jJ_3^k. \tag{4.1}$$

Then we have the following result:

Theorem 4.1. *If* $1\leqslant d\leqslant 3$ *the system (3.15) admits a polynomial invariant of the form (4.1), and in such cases we have:*

$$I_1=J_++J_--\kappa J_3, \tag{4.2a}$$

$$I_2=\lambda_3(J_++J_-)-\lambda_1J_3-\lambda_2J_3^2, \tag{4.2b}$$

$$I_3=(J_++J_-\mp 2J_3)(\tau^2-2\tau J_3+J_+J_-) \tag{4.2c}$$

with the corresponding functions $f(\xi)=V'(\xi)$*:*

$$f_1=\frac{\kappa}{2}, \tag{4.3a}$$

$$f_2=\frac{\lambda_1}{2}\frac{1}{\lambda_3-\lambda_2\xi}, \tag{4.3b}$$

$$f_3=\pm\frac{1}{2}+\frac{\tau}{2}\frac{1}{\xi}. \tag{4.3c}$$

Remark 4.2. We remark that through degeneration of parameters in (4.3) the functions f_2 and f_3 are both connected to the function f_1. Indeed, if $\lambda_2\to 0$ then f_2 degenerates to f_1, provided we make the identification $\kappa=\lambda_1/\lambda_3$. Similarly, if $\tau\to 0$ then f_3 degenerates to the particular case of f_1 with $\kappa=\pm 1$. A graphical representation of this degeneration scheme is given in figure 1. Moreover, we observe that there are countably many values of κ such that the associated evolution map, i.e. the map:

$$(J_+(t),J_-(t),J_3(t))\mapsto(J_+(t+h),J_-(t+h),J_3(t+h)) \tag{4.4}$$

from equation (3.15) with $f_1=\kappa/2$ is periodic. In appendix B we show how to find these values, but we shall not discuss these degenerate cases further.

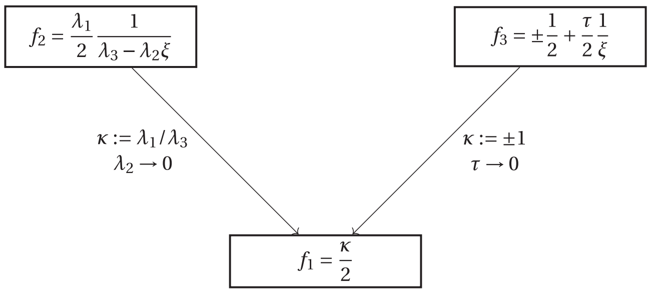


Figure 1. Degeneration scheme for the functions f_i in (4.3).

Proof. The proof of this result is through direct computation by considering the three cases $d = 1, 2, 3$. Indeed, the system already possesses one Casimir function, so if we are able to prove the existence of a functionally independent invariant, we have proven integrability.

Case $d = 1$. This case is prototypical for the other two which can be proven directly, so we will give all the details of the proof. If $d = 1$ the invariant has the following form:

$$I_1(t) = a_{1,0,0}J_+(t) + a_{0,1,0}J_-(t) + a_{0,0,1}J_3(t). \tag{4.5}$$

Then, applying the translation by the step h and using the form of the system (3.15) we obtain:

$$\begin{aligned} I_1(t+h) = {} & a_{1,0,0}J_-(t) + a_{0,1,0}(J_+(t) - 4J_3(t)f(J_-(t)) + 4J_- f^2(J_-(t))) \\ & + a_{0,0,1}(-J_3(t) + 2J_- f(J_-(t))). \end{aligned} \tag{4.6}$$

We impose that $I_1(t+h) = I_1(t)$. This yields a polynomial expression of degree one in J_+ and J_3. Taking coefficients with respect to these two variables, we obtain (after removing the non-zero common factors):

$$\begin{aligned} & a_{0,1,0} - a_{1,0,0} = 0, \quad 2f(J_-)a_{0,1,0} + a_{0,0,1} = 0, \\ & 4f^2(J_-)a_{0,1,0} + 2f(J_-)a_{0,0,1} - a_{0,1,0} + a_{1,0,0} = 0. \end{aligned} \tag{4.7}$$

The first equation in (4.7) implies $a_{1,0,0} = a_{0,1,0}$ with no other possibilities:

$$2f(J_-)a_{0,1,0} + a_{0,0,1} = 0, \quad f(J_-)(2f(J_-)a_{0,1,0} + a_{0,0,1}) = 0. \tag{4.8}$$

If $a_{0,1,0} \neq 0$ the first equation in (4.8) implies that $f =$ constant. This is not restricting because $a_{0,1,0} = 0$ yields $I_1 \equiv 0$, which must be discarded. Then, we write $f(\xi) = \kappa/2$, with $\kappa \neq 0$, and the system reduces to the single algebraic equation:

$$\kappa a_{0,1,0} + a_{0,0,1} = 0. \tag{4.9}$$

Then the invariant (4.5) has the form:

$$I_1(t) = a_{1,0,0}(J_+(t) + J_-(t) - \kappa J_3(t)). \tag{4.10}$$

Scaling away the constant $a_{1,0,0}$ we proved formula (4.2a), while formula (4.3a) was already established. The functional independence of the set $\{I_1, C\}$ is trivial, so the system is clearly integrable. This concludes the case $d = 1$.

Case $d = 2$. If $d = 2$ then the invariant has the following form:

$$\begin{aligned} I_2(t) = {} & a_{2,0,0}J_+^2 + a_{0,2,0}J_-^2(t) + a_{0,0,2}J_3^2(t) \\ & + a_{1,1,0}J_+(t)J_-(t) + a_{1,0,1}J_+(t)J_3(t) + a_{0,1,1}J_-(t)J_3(t) \\ & + a_{1,0,0}J_+(t) + a_{0,1,0}J_-(t) + a_{0,0,1}J_3(t). \end{aligned} \tag{4.11}$$

Writing $I_2(t+h) = I_2(t)$ using the system (3.15) and taking the coefficients with respect to J_+ and J_3 we have the following equations (divided by total degree D):

$$D = 2: \quad \begin{aligned} & a_{0,2,0} - a_{2,0,0} = 0, \quad f(J_-)[4a_{0,2,0}f(J_-) + a_{0,1,1}] = 0, \\ & 8a_{0,2,0}f(J_-) + a_{0,1,1} + a_{1,0,1} = 0, \end{aligned} \tag{4.12a}$$

$$D = 1: \quad \begin{aligned} & 8a_{0,2,0}J_- f^2(J_-) + 2a_{0,1,1}J_- f(J_-) + a_{0,1,0} - a_{1,0,0} = 0, \\ & 4J_- f^2(J_-)(8a_{0,2,0}f(J_-) + 3a_{0,1,1}) \\ & +4[(a_{0,0,2} + a_{1,1,0})J_- + a_{0,1,0}]f(J_-) \\ & +(a_{0,1,1} + a_{1,0,1})J_- + 2a_{0,0,1} = 0, \end{aligned} \tag{4.12b}$$

$$D=0:\quad \begin{aligned}&4J_- f^2(J_-)(4a_{0,2,0}f(J_-)^2+2a_{0,1,1}f(J_-)+a_{0,0,2})\\&+4(a_{1,1,0}J_-+a_{0,1,0})f^2(J_-)+2(a_{1,0,1}J_-+a_{0,0,1})f(J_-)\\&-(a_{0,2,0}-a_{2,0,0})J_- -(a_{0,1,0}-a_{1,0,0})=0.\end{aligned} \tag{4.12c}$$

We solve these equations starting from the highest degree: $D=2$ (4.12a). To obtain a non-trivial solution not considered for $d=1$ we have to impose that all the coefficients with respect to $f(J_-)$ in (4.12a) vanish:

$$a_{2,0,0}=a_{0,2,0}=a_{0,1,1}=a_{1,0,1}=0. \tag{4.13}$$

This reduces our system to:

$$D=1:\quad \begin{aligned}&a_{0,1,0}-a_{1,0,0}=0,\\&2[(a_{0,0,2}+a_{1,1,0})J_-+a_{0,1,0}]f(J_-)+a_{0,0,1}=0,\end{aligned} \tag{4.14a}$$

$$D=0:\quad \begin{aligned}&4[(a_{0,0,2}+a_{1,1,0})J_-+a_{0,1,0}]f^2(J_-)\\&+2a_{0,0,1}f(J_-)-a_{0,1,0}+a_{1,0,0}=0.\end{aligned} \tag{4.14b}$$

Solving the first equation for $D=1$, i.e. putting $a_{0,1,0}=a_{1,0,0}$, and discarding the trivial solutions, we have that all the equations reduce to:

$$2[(a_{0,0,2}+a_{1,1,0})J_-+a_{0,1,0}]f(J_-)+a_{0,0,1}=0. \tag{4.15}$$

That is:

$$f(J_-)=-\frac{1}{2}\frac{a_{0,0,1}}{(a_{0,0,2}+a_{1,1,0})J_-+a_{0,1,0}}. \tag{4.16}$$

Of these last four parameters we can choose three independent ones as follows:

$$a_{0,0,1}=\lambda_1,\quad a_{0,0,2}=-\lambda_2-a_{1,1,0},\quad a_{0,1,0}=\lambda_3. \tag{4.17}$$

This brings (4.16) into the form of (4.3b), and the invariant (4.11) into the following form:

$$I_2(t)=\lambda_3[J_+(t)+J_-(t)]-\lambda_1 J_3(t)-\lambda_2 J_3^2(t)+a_{1,1,0}[J_+(t)J_-(t)-J_3^2(t)]. \tag{4.18}$$

Now, note that the coefficient of the $a_{1,1,0}$ is exactly the Casimir function C of $\mathfrak{sl}_2(\mathbb{R})$, see formula (2.2). From theorem 3.1 this invariant is admitted by the system (3.15) *regardless of the function* $f=f(\xi)$. So, this invariant is trivial and we can safely discard it by putting $a_{1,1,0}=0$ in (4.18) (we are already taking into account its existence). This proves formula (4.2b). The functional independence of the set $\{I_2, C\}$ is trivial, so the system is clearly integrable. This concludes the case $d=2$.

Case $d=3$. Let us consider now the case $I_3(t)$ in formula (4.1) (we omit the explicit form because it is rather cumbersome). We employ the same strategy we employed before: from the identity $I_3(t+h)=I_3(t)$ equate all the coefficients in J_+, J_3 to zero starting from the higher degree. Taking the coefficients of degree 3 in J_+, J_3 in $I_3(t+h)=I_3(t)$ we have the following equations:

$$\begin{aligned}J_+^3&: a_{0,3,0}-a_{3,0,0}=0,\\ J_+^2J_3&: -12f(J_-)a_{0,3,0}-a_{0,2,1}-a_{2,0,1}=0,\\ J_+J_3^2&: 48f^2(J_-)a_{0,3,0}+8f(J_-)a_{0,2,1}+a_{0,1,2}-a_{1,0,2}=0,\\ J_3^3&: -32f^3(J_-)a_{0,3,0}-8f^2(J_-)a_{0,2,1}-2f(J_-)a_{0,1,2}-a_{0,0,3}=0.\end{aligned} \tag{4.19}$$

Since the only possible solution for f from this set of equations is the constant, we can safely put all the coefficients of f to zero and obtain:

$$a_{0,0,3}=a_{0,1,2}=a_{0,2,1}=a_{0,3,0}=a_{1,0,2}=a_{2,0,1}=a_{3,0,0,0}=0. \tag{4.20}$$

Inserting these values into $I_3(t+h)=I_3(t)$ the degree three terms disappear, and we can consider the degree two ones:

$$\begin{aligned}J_+^2&: (a_{1,2,0}-a_{2,1,0})J_-+a_{0,2,0}-a_{2,0,0}=0,\\ J_+J_3&: 8(J_-a_{1,2,0}-a_{0,2,0})f(J_-)+2J_-a_{1,1,1}+a_{0,1,1}+a_{1,0,1}=0,\\ J_3^2&: f(J_-)[4(J_-a_{1,2,0}+a_{0,2,0})f(J_-)+J_-a_{1,1,1}+a_{0,1,1}]=0.\end{aligned} \tag{4.21}$$

Taking the coefficients with respect to J_- the first equation in (4.21) yields:

$$a_{2,1,0}=a_{1,2,0},\quad a_{2,0,0}=a_{0,2,0}. \tag{4.22}$$

Discarding the trivial solution $f(J_-)=0$ we have that $f(J_-)$ from the second and the third equation in (4.21) must be compatible. The compatibility condition is:

$$a_{1,0,1} = a_{0,1,1}. \tag{4.23}$$

This yields a unique value for $f(J_-)$:

$$f(J_-) = -\frac{1}{4}\frac{J_- a_{1,1,1} + a_{0,1,1}}{J_- a_{1,2,0} + a_{0,2,0}}. \tag{4.24}$$

At this point we can insert this value into $I_3(t+h) = I_3(t)$ which now consists only of the linear and the degree zero terms. Now we can clear denominators and take coefficients with respect to J_-. This leaves us with a system of 26 algebraic equations we can solve with a CAS, e.g. `Maple`. We omit the explicit form of the system, but we comment that we obtain 12 different solutions with the `solve` command from `Maple 2016`. Upon inspection only two new solutions do not produce trivial solutions and satisfy the condition that $a_{1,2,0}a_{0,2,0} \neq 0$. The final form of the coefficients we obtained is shown in appendix A.

This yields the following form for the function *f*:

$$f(J_-) = \pm\frac{1}{2} - \frac{a_{0,1,1}}{4a_{1,2,0}}\frac{1}{J_-} \tag{4.25}$$

Since $a_{1,2,0} \neq 0$ we perform the scaling $a_{0,1,1} = -2a_{1,2,0}\tau$. This proves formula (4.3*c*). To conclude the proof we notice that using the results as displayed in appendix A and the scaling just presented we have:

$$\begin{aligned} I_3 = {} & a_{1,2,0}(J_+(t) + J_-(t) \mp 2J_3(t))[\tau^2 - 2\tau J_3(t) + J_+(t)J_-(t)] \\ & + a_{1,1,0}[J_+(t)J_-(t) - J_3^2(t)]. \end{aligned} \tag{4.26}$$

Now, note that the coefficient of the $a_{1,1,0}$ is exactly the Casimir function C of $\mathfrak{sl}_2(\mathbb{R})$, see formula (2.2). From theorem 3.1 this invariant is admitted by the system (3.15) *regardless of the function* $f = f(\xi)$. So, this invariant is trivial and we can safely discard it by putting $a_{1,1,0} = 0$ (we are already taking into account its existence). Then, analogously to the case $d = 1$ we can scale away the constant $a_{1,2,0}$ and we arrive at the expression (4.2*c*). The functional independence of the set $\{I_3, C\}$ is trivial, so the system is clearly integrable. This concludes the case $d = 3$ and hence concludes the proof of the theorem. □

For invariants of degree $d > 3$, by direct computation it is possible to prove the following result:

Lemma 4.3. *For all $d > 3$ the translated polynomial (4.1) on the solution of the system (3.15) is:*

$$I_d(t+h) = \sum_{1\leqslant i+j+k\leqslant d}\ \sum_{p+q+r=j}\ \sum_{s=0}^{k} A_{i,j,k;p,q,r;s} J_+^p J_-^{i+r+k-s} f^{q+2r+k-s}(J_-) J_3^{q+s}, \tag{4.27}$$

where

$$A_{i,j,k;p,q,r;s} = (-1)^{q+s} a_{i,j,k} \binom{j}{p,\ q,\ r}\binom{k}{s} 2^{2q+2r+k-s}. \tag{4.28}$$

Then, with a technique analogue to the one that we used to prove Theorem 4.1 we can see that for d=4,5 no different integrable systems are found. The calculations are of increasing complexity, so we don't show them here. However, this leads to the following conjecture:

Conjecture 1. If $d > 3$ and the function *f* is different from the ones in the formula (4.3), then only trivial invariants are possible. Here, by a trivial invariant, we mean a linear combination of the power of the Casimir function (2.2):

$$I_{\text{triv}}(t) = \sum_{k=1}^{\lfloor d/2 \rfloor} a_k C^k. \tag{4.29}$$

5. Study of the obtained systems

In this section, we present a more detailed study of the realisation in canonically conjugated coordinates ($\mathbf{q}(t)$, $\mathbf{p}(t)$) of the integrable systems we found in the previous section.

5.1. Degree one invariant

We consider the system (4.3*a*) in the realisation of $\mathfrak{sl}_2(\mathbb{R})$ (2.11) with canonical coordinates ($\mathbf{q}(t)$, $\mathbf{p}(t)$). The equation assumes the following form:

$$q_k(t+h)\sqrt{1 - \frac{b_k}{q_k^2(t)q_k^2(t+h)}} + q_k(t-h)\sqrt{1 - \frac{b_k}{q_k^2(t)q_k^2(t-h)}} = \kappa q_k. \tag{5.1}$$

From formula (1.3) the symplectic form of the system is:

$$q_k(t+h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}}+p_k(t)=\kappa q_k, \tag{5.2a}$$

$$p_k(t+h)=q_k(t)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}}. \tag{5.2b}$$

The case when $b_k = 0$ for all k=1,…,N is a linear system and it was considered in [29], Example 1, so we will consider only the case when $b_k \neq 0$.

Remark 5.1. We remark that the system (5.1) is symmetric with respect to the transformation:

$$\hat{\mathbf{q}}^{(i)}(t)=(q_1(t),\ldots,-q_i(t),\ldots,q_N(t)), \tag{5.3}$$

or its symplectic form (5.2) is symmetric under the transformation:

$$\hat{\mathbf{q}}^{(i)}(t)=(q_1(t),\ldots,-q_i(t),\ldots,q_N(t)), \tag{5.4a}$$

$$\hat{\mathbf{p}}^{(i)}(t)=(p_1(t),\ldots,-p_i(t),\ldots,p_N(t)). \tag{5.4b}$$

This implies that the system can be defined outside its natural domain as described theorem 3.1 by reflection. Reflecting with respect to each coordinate axis we obtain that the symplectic map (5.2) can be defined on the set

$$U=\left(\prod_{k=1}^{N}\{x_k\neq 0\}\right)^{\times 2}. \tag{5.5}$$

The coalgebraic invariant (4.2*a*) has the following explicit form[3]:

$$H_1=\frac{1}{2}\sum_{k=1}^{N}\left[p_k^2(t)+\frac{b_k}{q_k^2(t)}+q_k^2(t)-\kappa q_k(t)p_k(t)\right]. \tag{5.6}$$

From the coalgebra construction, we get the following two sets of invariants

$$\mathfrak{L}_1=\{H_1,\mathsf{C}^{[2]},\ldots,\mathsf{C}^{[N]}\},\qquad \mathfrak{R}_1=\{H_1,\mathsf{C}_{[2]},\ldots,\mathsf{C}_{[N]}\}, \tag{5.7}$$

By induction on the d.o.f. N, it is easy to see that both sets are functionally independent. As expected, this implies that the system (5.1) is Liouville integrable. Similarly the set

$$\mathfrak{I}_1=\{H_1,\mathsf{C}^{[2]},\ldots,\mathsf{C}^{[N]},\mathsf{C}_{[2]},\ldots,\mathsf{C}_{[N-1]}\}, \tag{5.8}$$

is made of functionally independent functions which implies the system (5.1) is QMS. As remarked in [29] this is the best we expect for general discrete-time systems with $\mathfrak{sl}_2(\mathbb{R})$ coalgebra symmetry. However, this case is special because it is possible to consider a different set of invariants constructed from the following elements:

$$\mathsf{H}_{[k]}\coloneqq\overbrace{1\otimes\ldots\otimes 1}^{k-1}\otimes\mathsf{H}\otimes\overbrace{1\otimes\ldots\otimes 1}^{N-k},\quad k=1,\ldots,N, \tag{5.9}$$

where:

$$\mathsf{H}=\frac{1}{2}(J_++J_--\kappa J_3). \tag{5.10}$$

Those N elements, at fixed realisation, result in the following functions:

$$\mathsf{H}_{[k]}=\frac{1}{2}\left(p_k^2+q_k^2+\frac{b_k}{q_k^2}-\kappa q_k p_k\right),\quad k=1,\ldots,N. \tag{5.11}$$

This gives us another set of commuting invariants:

$$\mathfrak{C}_1=\{\mathsf{H}_{[1]},\ldots,\mathsf{H}_{[N]}\}, \tag{5.12}$$

which proves Liouville's integrability again. Note that:

$$H_1=\sum_{k=1}^{N}\mathsf{H}_{[k]}. \tag{5.13}$$

Then, the following set of invariants:

$$\mathfrak{S}_1=\{\mathsf{H}_{[1]},\ldots,\mathsf{H}_{[N]},\mathsf{C}^{[2]},\ldots,\mathsf{C}^{[N]}\}, \tag{5.14}$$

is a set of $2N-1$ functionally independent invariants. For explicit commutation relations involving left and right Casimir invariants, also with these additional constants, we refer the reader to [37–39]. Functional

[3] We added a cosmetic 1/2 factor to better compare with the continuous case.

independence can be proven as follows. Define the point $\mathbf{m} = (1,\ldots,1, 0,\ldots,0)$. Then from direct computation we have:

$$\mathbf{v}_k = \nabla H_k|_{\mathbf{m}} = (\underbrace{0,\ldots,0, \overbrace{-b_m + 1}^{m\text{th element}}, 0,\ldots,0}_{N}, \underbrace{0,\ldots,0, \overbrace{-\kappa}^{m\text{th element}}, 0,\ldots,0}_{N}), \tag{5.15a}$$

$$\mathbf{w}_m = \nabla \mathsf{C}^{[m]}|_{\mathbf{m}} = (\underbrace{b_0 - mb_1,\ldots,b_0 - mb_m}_{m}, 0,\ldots,0), \tag{5.15b}$$

for $k = 1,\ldots,N$, and $m = 2,\ldots,N$ respectively and we defined $b_0 = \sum_{i=1}^{m} b_i$. Then we can define:

$$\begin{aligned} \mathbf{u}_m &= \mathbf{w}_m - \sum_{i=1}^{m} \frac{b_0 - mb_i}{-b_i + 1}\mathbf{v}_i \\ &= \kappa\left(\underbrace{0,\ldots,0}_{N}, \underbrace{\frac{b_0 - mb_1}{-b_1 + 1},\ldots,\frac{b_0 - mb_m}{-b_m + 1}}_{m}, 0,\ldots,0\right), \end{aligned} \qquad m = 2,\ldots,N. \tag{5.16}$$

So, we have that the Jacobian in $\mathbf{m}$

$$J_{\mathbf{m}} = \frac{\partial(\mathsf{H}_1,\ldots,\mathsf{H}_N, \mathsf{C}^{[2]},\ldots,\mathsf{C}^{[N]})}{\partial(\mathbf{q}, \mathbf{p})}\bigg|_{\mathbf{m}} = \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_N \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_N \end{pmatrix}, \tag{5.17}$$

is equivalent through Gauss elimination to the upper triangular matrix:

$$\tilde{J}_{\mathbf{m}} = \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_N \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_N \end{pmatrix}. \tag{5.18}$$

Since all the pivots of $\tilde{J}_{\mathbf{m}}$ are non-zero it follows that $J_{\mathbf{m}} = 2N - 1$ proving functional independence in the open set

$$U_{(+,\ldots,+)} = O_{(+,\ldots,+)} \times \mathbb{R}^N, \tag{5.19}$$

where $O_{(+,\ldots,+)}$ is the open positive orthant of $\mathbb{R}^N$ [53]. A similar argument runs in all other possible open sets

$$U_{(\pm,\ldots,\pm)} = O_{(\pm,\ldots,\pm)} \times \mathbb{R}^N, \quad O_{(\pm,\ldots,\pm)} = \{\pm q_1 > 0,\ldots,\pm q_N > 0\}, \tag{5.20}$$

evaluating the Jacobian on the points $(\pm1,\ldots,\pm 1, 0,\ldots,0) \in U_{(\pm,\ldots,\pm)}$. Noting that solutions in one orthant evolve inside the same orthant, while the coordinate lines $\{q_k = 0\}_{k=1,\ldots,N}$ are not accessible by the evolution we have that these sets exhaust all the points of the phase space, because:

$$U \subset \bigcup_{i_k \in \{\pm\}} U_{(i_1,\ldots,i_N)}, \tag{5.21}$$

where the set U was defined in formula (5.5). So, this proves the functional independence of the invariants everywhere they are defined. This allows us to conclude that the dSW system (5.1) is MS.

In figure 2 we show an orbit of (5.1) near the fixed point

$$\mathbf{q}_0 = \left(\left(\frac{b_1}{1 - (\kappa/2)^2}\right)^{1/4},\ldots,\left(\frac{b_N}{1 - (\kappa/2)^2}\right)^{1/4}\right). \tag{5.22}$$

Note that the fixed points of the map (5.1) are all of the following form:

$$\mathbf{q}_0^{(\pm,\ldots,\pm)} = \left(\pm\left(\frac{b_1}{1 - (\kappa/2)^2}\right)^{1/4},\ldots, \pm\left(\frac{b_N}{1 - (\kappa/2)^2}\right)^{1/4}\right). \tag{5.23}$$

Next, we note that the system (5.1) is a discretisation of the Smorodinsky–Winternitz (SW) oscillator (caged isotropic harmonic oscillator) [21, 23]. Indeed, under the following scaling:

$$q_k(t) = hQ_k(t), \quad b_k = h^6\beta_k, \quad \kappa = 2 - h^2\omega^2, \tag{5.24}$$

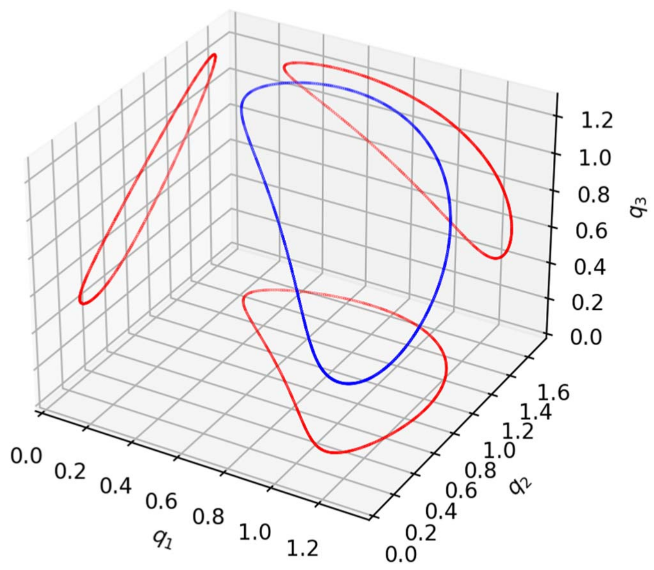


Figure 2. An orbit of the dSW system (5.1) in three d.o.f. for an initial condition near the fixed point (5.22). The value of the parameters are $b_1 = 0.01$, $b_2 = 0.03$, $b_3 = 0.02$, and $\kappa = 4\sqrt{2}/3$ with $M = 10000$ iterations. The projections of the orbit on the coordinate planes are shown.

in equation (5.1) we obtain that the leading order as $h \to 0$ is SW oscillator in Lagrangian form:

$$h^3\left(\ddot{Q}_k + \omega^2 Q_k - \frac{\beta_k}{Q_k^3}\right) + O(h^4) = 0. \tag{5.25}$$

In the same way, writing the invariant (5.2a) using the definition of the canonical momentum (3.18b) and $\dot{Q}_k = P_k$ we obtain:

$$H_1 = h^4 H_{\mathrm{SW}} + O(h^5), \quad H_{\mathrm{SW}} = \sum_{k=1}^{N}\left(P_k^2 + \omega^2 Q_k^2 + \frac{\beta_k}{Q_k^2}\right). \tag{5.26}$$

That is, at order 4 the Hamiltonian of the SW oscillator appears. The SW system is a well-known MS system whose invariants can be produced through the coalgebra symmetry method [7]. In particular, we note that the other invariants (2.20) and (5.10) are preserved in form through the scaling (5.24). Indeed, noting that in the limit $h \to 0$:

$$L_{i,j} = \ell_{i,j} h^3 + O(h^3), \quad \ell_{i,j} = P_j Q_i - P_i Q_j. \tag{5.27}$$

we have that the invariants (2.20) are self-similar at order 6 in h:

$$\mathsf{C}^{[m]}(\mathbf{q}, \mathbf{p}) = h^6 \mathsf{C}^{[m]}(\mathbf{Q}, \mathbf{P}) + O(h^7), \quad \mathsf{C}_{[m]}(\mathbf{q}, \mathbf{p}) = h^6 \mathsf{C}_{[m]}(\mathbf{Q}, \mathbf{P}) + O(h^7), \tag{5.28}$$

while:

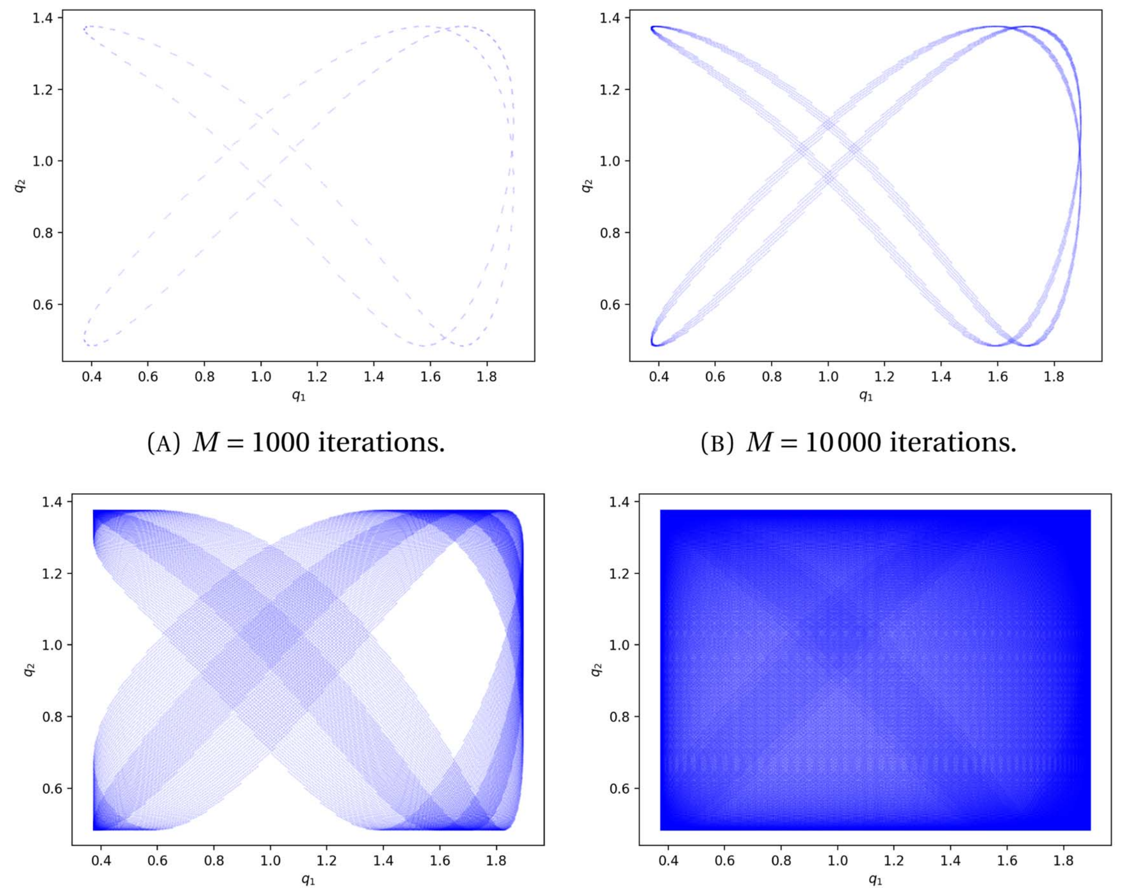
$$\mathsf{H}_k = h^4 H_{[k]} + O(h^5), \quad H_{[k]} = \frac{1}{2}\left(P_k^2 + \omega^2 Q_k^2 + \frac{\beta_k}{Q_k^2}\right). \tag{5.29}$$

A natural generalisation of the SW system arises if we consider anisotropy. That is, considering the following dynamical system:

$$H(\boldsymbol{\omega}) = \frac{1}{2}\sum_{k=1}^{N}\left(P_k^2 + \omega_k^2 Q_k^2 + \frac{\beta_k}{Q_k^2}\right). \tag{5.30}$$

This system is clearly integrable considering the separation of variables in Cartesian coordinates. Following [22], see also [28, 52], the case when $\boldsymbol{\omega} = \omega(l_1,\ldots,l_N)$ where $(l_1,\ldots,l_N) \in \mathbb{Z}^N$ and the integers l_i are coprime is MS. In the discrete-time setting we consider the Lagrangian:

$$L(\mathbf{c}) = \sum_{k=1}^{N}\left[\int^{q_k(t)q_k(t+h)} \sqrt{1 - \frac{b_k}{\xi^2}}\,\mathrm{d}\xi - \frac{c_k}{2}q_k^2(t)\right]. \tag{5.31}$$



(A) $M = 1000$ iterations. (B) $M = 10\,000$ iterations. (C) $M = 100\,000$ iterations. (D) $M = 1\,000\,000$ iterations.

Figure 3. An orbit of the two d.o.f. anisotropic dSW oscillator (5.31) scale according to (5.33) with values $h = 1/50$, $\beta_1 = 2$, $\beta_2 = 4$, $\omega_1 = 2$, and $\omega_2 = 3$ with initial datum in a neighbourhood of the fixed point (5.34).

Then, using the analogue invariants (5.10)

$$\mathsf{H}_k(c_k) = \frac{1}{2}\left(p_k^2 + q_k^2 + \frac{b_k}{q_k^2} - c_k q_k p_k\right), \quad k = 1,\ldots,N, \tag{5.32}$$

we prove that the dEL equations of the Lagrangian (5.31) are integrable. Finally, under the scaling, the associated dEL equations:

$$q_k(t) = hQ_k(t), \quad b_k = h^6\beta_k, \quad c_k = 2 - h^2\omega_k^2, \tag{5.33}$$

reduce to the Hamiltonian system associated to (5.30).

Differently from the continuum case, the experimental evidence seems to suggest that the system (5.31) is not MS. Indeed, from an experimental study of the orbits of the system (5.31) it is evident that they do not lie on a closed curve, but rather they are a space-filling curve, dense in the phase space. For instance, figure 3 shows an orbit of the system (5.31) in two d.o.f. in the scaling (5.33) with parameters such that $\omega_1/\omega_2 = 2/3 \in \mathbb{Q}$. However, despite the frequency ratio is rational after $M = 1000000$ iterations a rectangle in the phase space is almost completely filled.

The orbit is taken in a neighbourhood of the fixed point:

$$\mathbf{q}_0(\omega_1, \omega_2) = \left(\sqrt{\frac{2}{\omega_1}}\left(\frac{\beta_1}{4 - h^2\omega_1^2}\right)^{\frac{1}{4}}, \sqrt{\frac{2}{\omega_2}}\left(\frac{\beta_2}{4 - h^2\omega_2^2}\right)^{\frac{1}{4}}\right). \tag{5.34}$$

The above remarks imply that for $\boldsymbol{\omega} \in \omega\mathbb{Z}^N$, the system (5.31) is an *integrable, but not a MS discretisation* of the anisotropic SW system (5.30). At the same time, the discussion made suggests that there *might be* some MS subcases of the system (5.31) for given values of the parameters c_k. The search for this kind of MS system is outside the possibilities given by the coalgebra approach and will be the subject of future research. Here we limit ourselves to noting that in [28] the conditions for the MS of [22] for the anisotropic SW oscillator (5.30) were derived using a perturbative multiple scales approach. This hints at the possibility of a similar procedure to isolate the MS subcases using, for instance, the multiple scales method for discrete-time systems in the spirit of [30].

5.2. Degree two invariant

We consider the system (4.3*b*) in the realisation of $\mathfrak{sl}_2(\mathbb{R})$ (2.11) with canonical coordinates $(\mathbf{q}(t), \mathbf{p}(t))$. The equation assumes the following form:

$$q_k(t+h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}} + q_k(t-h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t-h)}} = \frac{\lambda_1 q_k}{\lambda_3 - \lambda_2\sum_{l=1}^N q_l^2(t)}. \tag{5.35}$$

From formula (1.3) the symplectic form of the system is:

$$q_k(t+h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}} + p_k(t) = \frac{\lambda_1 q_k}{\lambda_3 - \lambda_2\sum_{k=1}^N q_k^2(t)}. \tag{5.36a}$$

$$p_k(t+h) = q_k(t)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}}. \tag{5.36b}$$

Remark 5.2. We remark that if $\lambda_3 \neq 0$ and $b_k = 0$ for all $k = 1,\ldots,N$, this system is a generalisation of the McMillan map [43] introduced in [42]. Indeed in such a case if $\lambda_3 \neq 0$ considering the scaling $q_k \mapsto \sqrt{|\lambda_3|}\, q_k$, and $\lambda_1 = \lambda_3\sigma$ we obtain:

$$q_k(t+h) + q_k(t-h) = \frac{\sigma q_k}{1 \mp \sum_{k=1}^N q_k^2(t)}. \tag{5.37}$$

We discussed the coalgebra symmetry properties in [29], Subsection 6.2, where the method was used to prove that the system is QMS. We finally recall that the system (5.37) is an example of N d.o.f. integrable system in *standard form* [32, 57], which was later extended in a series of papers [55, 56]. For a general review on N d.o.f. integrable systems in standard form we refer also to [58], Chapter 25.

To complete the study on this system, including continuum limits and relation to known discrete-time integrable systems we distinguish the cases $\lambda_3 \neq 0$ and $\lambda_3 = 0$.

Case $\lambda_3 \neq 0$. For $b_k \neq 0$ for some $k \in \{1, 2,\ldots,N\}$ and $\lambda_3 \neq 0$ we can scale away this constant through the reparametrisation $(\lambda_1, \lambda_2) = (\lambda_3\sigma_1, \lambda_3\sigma_2)$ and obtain the system:

$$q_k(t+h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}} + q_k(t-h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t-h)}} = \frac{\sigma_1 q_k}{1 - \sigma_2\sum\limits_{l=1}^N q_l^2(t)}. \tag{5.38}$$

Remark 5.3. We remark that the discrete-time system (5.38) is symmetric with respect to the coordinate transformation (5.3), thus allowing us to extend the domain of definition for negative values of the coordinates. However, from the right hand side of equation (5.38) we might need to restrict the domain of definition of this discrete-time system. Indeed, if $\sigma_2 > 0$ we need to exclude the N-sphere $\mathbb{S}^N(\sigma_2^{-1}) = \{\mathbf{q}(t) = \sigma_2^{-1}\}$. So, in the end we have that the domain of definition of the discrete-time system (5.38) is:

$$\widehat{U} = \begin{cases} U \cap \mathbb{S}^N(\sigma_2^{-1}) & \text{if } \sigma_2 > 0, \\ U & \text{if } \sigma_2 < 0. \end{cases} \tag{5.39}$$

where the set U is given in equation (5.5).

In this case the coalgebraic invariant (4.2b) has the following explicit form:[4]

$$H_2(\lambda_3 \neq 0) = \frac{1}{2}\sum_{k=1}\left(p_k^2 + q_k^2 + \frac{b_k}{q_k^2} - \sigma_1 q_k p_k\right) - \frac{\sigma_2}{2}\left(\sum_{l=1}^{N} q_l^2\right)^2, \tag{5.40}$$

From the coalgebra construction, we get the following two sets of invariants

$$\mathfrak{L}_2 = \{H_2(\lambda_3 \neq 0),\, \mathsf{C}^{[2]},\ldots,\mathsf{C}^{[N]}\}, \tag{5.41a}$$

$$\mathfrak{R}_2 = \{H_2(\lambda_3 \neq 0),\, \mathsf{C}_{[2]},\ldots,\mathsf{C}_{[N]}\}. \tag{5.41b}$$

By induction on the d.o.f. N, it is easy to see that both sets are functionally independent. As expected, this implies that the system (5.38) is Liouville integrable. Similarly the set

$$\mathfrak{I}_2 = \{H_2(\lambda_3 \neq 0),\, \mathsf{C}^{[2]},\ldots,\mathsf{C}^{[N]},\, \mathsf{C}_{[2]},\ldots,\mathsf{C}_{[N-1]}\}, \tag{5.42}$$

is made of functionally independent functions which implies the system (5.38) is QMS.

Coming to the continuum limit, we see that applying the scaling (5.24) as $h \to 0$ we obtain from the dEL equations (5.38):

$$h^3\left[\ddot{Q}_k + \omega^2 Q_k - \frac{\beta_k}{Q_k^3} + 2\sigma_2 Q_k \sum_{l=1}^{N} Q_l^2\right] + O(h^4) = 0, \tag{5.43}$$

while for the invariant (5.40) we have:

$$H_2 = h^4 H_{\mathrm{W}}^{\mathrm{QMS}} + O(h^5), \quad H_{\mathrm{W}}^{\mathrm{QMS}} = \frac{1}{2}\sum_{k=1}^{N}\left(P_k^2 + \omega^2 Q_k^2 + \frac{\beta_k}{Q_k^2}\right) + \frac{\sigma_2}{2}\left(\sum_{k=1}^{N} Q_k^2\right)^2. \tag{5.44}$$

The classical continuum system defined by $H_{\mathrm{W}}^{\mathrm{QMS}}$ clearly possesses coalgebra symmetry with respect to $\mathfrak{sl}_2(\mathbb{R})$, and in fact it turns out to be a QMS deformation of the SW system:

$$H_{\mathrm{W}}^{\mathrm{QMS}} = H_{\mathrm{SW}} + \sigma_2 F(\mathbf{q}^2), \quad F(\xi) = \frac{\xi^2}{2}. \tag{5.45}$$

The continuum first integrals are again given by formula (5.28).

The system (5.45) is a particular case of the Wojciechowski system defined by the following Hamiltonian [66]:

$$H_{\mathrm{W}} = \frac{1}{2}\sum_{k=1}^{N}\left(P_k^2 + \omega_k^2 Q_k^2 + \frac{\beta_k}{Q_k^2}\right) + \frac{\sigma_2}{2}\left(\sum_{k=1}^{N} Q_k^2\right)^2. \tag{5.46}$$

In general, this system is proven to be integrable both through direct construction of the integrals or by the existence of a Lax pair, see [66].

This system was discretised in [54] as:

$$\begin{aligned} & q_k(t+h)\sqrt{1 - \frac{b_k}{q_k^2(t)q_k^2(t+h)}} \\ & + q_k(t-h)\sqrt{1 - \frac{b_k}{q_k^2(t)q_k^2(t-h)}} = \frac{2c_k q_k}{1 + \sum_{l=1}^{N} c_l q_l^2(t)}. \end{aligned} \tag{5.47}$$

Clearly, this system is a generalisation of the system (5.35) with $\lambda_3 \neq 0$, using the identification $c_k = \sigma_2/2$ and making the proper scaling in the coordinates q_k and the parameters b_k[5]. So, we have that the system (5.38) is a *coalgebraic QMS subcase of the discrete-time Wojciechowski system (5.47)*.

Remark 5.4. We remark that *there might be* superintegrable subcases of the Wojciechowski system (5.46) if $\omega_i/\omega_j \in \mathbb{Q}$ for some $i, j \in \{1,\ldots,N\}$. At present, we did not prove the existence of these superintegrable subcases, but we limit ourselves to notice that the proofs of integrability of both [54, 66] do not work in the QMS case. So, it is reasonable to believe that the presented first integrals are not exhaustive of all the integrable cases and there might be intermediate cases between the Liouville integrable case (N invariants) and the QMS case ($2N - 2$ invariants). The problem of the existence of superintegrable subcases of the Wojciechowski system (5.46) and its (possible) superintegrable discretisation will be the subject of future research.

Case $\lambda_3 = 0$. Let us assume now that $\lambda_3 = 0$ and that b_k are arbitrary (possibly also zero for all $k = 1, 2,\ldots,N$). In such a case, we can rescale $\lambda_1 = -\lambda_2\sigma$ and the system (5.35) reduces to:

[4] We remove the common factor λ_3 and add a cosmetic $1/2$ factor to mimic the continuum case.

[5] It is needed to go through the original system (5.35) and properly scale the parameters.

$$q_k(t+h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}}$$
$$+q_k(t-h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t-h)}} = \sigma q_k\left(\sum_{l=1}^{N} q_l^2(t)\right)^{-1}. \tag{5.48}$$

Remark 5.5. We remark that the discrete-time system (5.48) is symmetric with respect to the coordinate transformation (5.3), thus allowing us to extend the domain of definition for negative values of the coordinates. In this case the right hand side does not give any additional restriction, so the system can be defined on the whole set U given in equation (5.5).

The coalgebraic invariant (4.2*b*) has the following explicit form[6]:

$$H_2(\lambda_3=0) = \frac{\sigma}{2}\sum_{k=1}^{N} q_k p_k + \frac{1}{2}\left(\sum_{l=1}^{N} q_l^2\right)^2, \tag{5.49}$$

From the coalgebra construction, we get the same two sets of invariants as for $\lambda_3 \neq 0$ (5.41) case with $H_2(\lambda_3 \neq 0)$ replaced by $H_2(\lambda_3 = 0)$. The two sets are clearly functionally independent. In the same way replacing $H_2(\lambda_3 \neq 0)$ with $H_2(\lambda_3 = 0)$ in (5.42) we obtain a set of $2N-2$ functionally independent invariants for (5.48). Summing up, we proved that the system (5.48) is Liouville integrable and moreover is QMS.

Differently from the $\lambda_3 \neq 0$ the continuum limit of the case $\lambda_3 = 0$ is not known. Analogously to [29], Subection 5.3 it is possible to prove that no scaling of the form

$$\mathbf{q}(t) = \mathbf{q}_0 + Ah^{\gamma}\mathbf{Q}(t), \quad \sigma = \sigma(h), \quad \mathbf{q}_0 \in \mathbb{S}^N, \quad \gamma \in \mathbb{N}, \tag{5.50}$$

where $\sigma(h)$ is an analytic function of its argument, balances the terms in the systems (5.48).

5.3. Degree three invariant

We consider the system (4.3*c*) in the realisation of $\mathfrak{sl}_2(\mathbb{R})$ (2.11) with canonical coordinates $(\mathbf{q}(t), \mathbf{p}(t))$. The equation assumes the following form:

$$q_k(t+h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}} \mp q_k(t)$$
$$+q_k(t-h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t-h)}} = \tau q_k\left(\sum_{l=1}^{N} q_l^2(t)\right)^{-1}. \tag{5.51}$$

From formula (1.3) the symplectic form of the system is:

$$q_k(t+h)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}} \mp q_k(t) + p_k(t) = \tau q_k\left(\sum_{l=1}^{N} q_l^2(t)\right)^{-1}, \tag{5.52a}$$

$$p_k(t+h) = q_k(t)\sqrt{1-\frac{b_k}{q_k^2(t)q_k^2(t+h)}}. \tag{5.52b}$$

Remark 5.6. We remark that if $b_k = 0$ for all $k = 1,\ldots,N$, the system (5.51) with the plus sign

$$q_k(t+h) + q_k(t) + q_k(t-h) = \tau q_k\left(\sum_{k=1}^{N} q_k^2(t)\right)^{-1}. \tag{5.53}$$

is a particular case of an autonomous version of the discrete-time Painlevè I equation [25, 32] we introduced in [29], section 5 with the parameter $\boldsymbol{\beta} = \mathbf{0}$. The coalgebra symmetry properties of the system (5.53) were discussed in [29], where we showed that the system is QMS. We also recall that the continuum limit of the system (5.53) is unknown.

[6] We remove the common factor λ_2 and add a cosmetic 1/2 factor.

Table 1. Known integrable cases of equation (*3.17*). The novel systems are highlighted in italic.

deg$I$	$V(\xi)$	Standard form	Quasi-standard form
1	$\frac{\alpha_1\xi}{2}$	Discrete-time isotropic harmonic oscillator [29]	*Discrete-time SW system (5.1)*
2	$-\frac{\alpha_1}{2\alpha_2}\log(1-\alpha_2\xi)$	N d.o.f. McMillan system [42, 55]	QMS Wojciechowski system (5.38) [54]
2	$\frac{\alpha_1}{2}\log\xi$	*New QMS system (5.48)*	*New QMS system (5.48)*
3	$\pm\frac{\xi}{2}+\frac{\alpha_1}{2}\log\xi$	N d.o.f. aut-dPI system (*5.53*) [29]	*New QMS system (5.48)*

Considering $b_k \neq 0$ for some $k \in \{1,\ldots,N\}$ the coalgebraic invariant (4.2*c*) has the following explicit form[7]:

$$H_3 = \frac{1}{2}\sum_{k=1}^{N}\left[(p_k \mp q_k)^2 + \frac{b_k}{q_k^2}\right]\left[\left(\tau - \sum_{k=1}^{N} q_k p_k\right)^2 + \mathsf{C}^{[N]}\right]. \tag{5.54}$$

From the coalgebra construction, we get the following two sets of invariants

$$\mathfrak{L}_3 = \{H_3, \mathsf{C}^{[2]},\ldots,\mathsf{C}^{[N]}\}, \quad \mathfrak{R}_3 = \{H_3, \mathsf{C}_{[2]},\ldots,\mathsf{C}_{[N]}\}. \tag{5.55}$$

By induction on the d.o.f. N it is easy to see that both sets are functionally independent. As expected, this implies that the system (5.51) is Liouville integrable. Similarly the set

$$\mathfrak{I}_3 = \{H_3, \mathsf{C}^{[2]},\ldots,\mathsf{C}^{[N]}, \mathsf{C}_{[2]},\ldots,\mathsf{C}_{[N-1]}\}, \tag{5.56}$$

is made of functionally independent functions which implies the system (5.51) is QMS.

Based on the above discussion we have that the system (5.51) for $b_k \neq 0$ for some $k \in \{1,\ldots,N\}$ is *a novel QMS system, generalising the already known system (5.53)*.

We finally note that at present no continuous analogue of the system (5.51) is known. Indeed, analogously to the system (*5.53*) mentioned in remark 5.6 and to [29], Subection 5.3, it is possible to prove that no scaling of the form (5.50) with $\tau(h)$ analytic functions of its argument balances the terms in the systems (5.51).

6. Conclusions

In this paper, we gave the necessary and sufficient conditions for a discrete-time system in quasi-standard form (1.2) to admit coalgebra symmetry with respect to the generic symplectic realisation of the Lie–Poisson algebra $\mathfrak{sl}_2(\mathbb{R})$ (2.18) in N degrees of freedom. In particular, from theorem 3.1 we see that the most general form of this system is the one considered in [54]. In this sense, we see that the coalgebra symmetry approach naturally selects the functional form for the functions $\ell_k = \ell_k(\xi)$, $k = 1,\ldots,N$.

As already discussed in [29] the systems in quasi-standard form (1.2) admitting coalgebra symmetry with respect to the generic symplectic realisation Lie–Poisson algebra $\mathfrak{sl}_2(\mathbb{R})$ (2.18) in N degrees of freedom naturally possess $2N-3$ functionally independent invariants. However, in general these systems are not Liouville integrable. This is because we are missing an exact discrete-time equivalent of the Hamiltonian. So, to characterise the Liouville integrable cases we searched for polynomial invariants *in the variables of the algebra* $\mathfrak{sl}_2(\mathbb{R})$ of increasing degree for the associated dynamical systems (3.15). It turns out that such invariants do exist for degrees 1, 2, and 3, while for degrees 4, and 5 no new systems arise. This led us to conjecture that the only non-trivial Liouville integrable systems with a polynomial invariant are those admitting an invariant of degrees 1, 2, and 3. The proof of this conjecture is particularly challenging because, in general, it does not seem possible to resum the expression (4.27) to:

$$I_d(t+h) = \sum_{l=0}^{d} F_l(J_-)J_+^l J_3^{d-l}, \tag{6.1}$$

which will make the claim easier to prove. Unfortunately, since in general not even the degree of the function $f = f(\xi)$ is known it is not possible to apply the 'destructive' test of algebraic entropy [14, 24, 26] [8]. For the very same reason it is not possible to apply the 'constructive' singularity confinement test [24, 25]. Under the (reasonable) assumption that $f(\xi)$ is a rational function, one could try to apply a Nevanlinna theory for maps [1]. However, at present, such a theory is not enough developed to treat a system of the form (3.15). So, this topic will be the subject of further research.

[7] We added a cosmetic 1/2 factor.

[8] Quoting from [64] '*Algebraic entropy.* pro: it is canonical (invariant by birational changes of coordinates), and the vanishing of the entropy may serve as a characterisation of integrability, as the sign of catastrophic drop of the complexity, con: destructive rather than constructive, since it gives a yes/no answer to the question is this model integrable?'. Italics from the original text.

In the sequent section we carefully analysed the obtained systems in the realisation (2.18) and proved explicitly their Liouville integrability and quasi-maximal superintegrability. In addition, we found a maximally superintegrable system given by the discrete Lagrangian (5.1) which is a discretisation of the well-known SW system [21, 23]. We also discussed the possible generalisation (5.31) discretising the so-called caged anisotropic oscillator [22], which we showed to be Liouville integrable. Unfortunately, differently from its continuous counterpart, the system (5.31) does not appear to be maximally superintegrable from a numerical study of its orbits. Since this kind of analysis goes outside the applicability of the coalgebra method, this topic will be the subject of future research. Besides these cases, we found the coalgebraic subcase of the discrete-time Wojciechowski system (5.35), whose general case was constructed in [54]. Finally, we found a non-birational generalisation of a system we proposed in [29], which we deem to be new. In table 1 we give a compendium of the known Liouville integrable cases of equation (1.2).

Summing up, in this paper we proved that the coalgebra symmetry method can be applied to systematically produce N d.o.f. discrete-time superintegrable systems.

An interesting open problem is the existence of a coalgebraic discretisation of the Kepler–Coulomb system:

$$H_{\mathrm{KC}} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \alpha\left(\sum_{i=1}^{N} q_i^2\right)^{-1/2}. \tag{6.2}$$

This model is MS, through the existence of an additional integral of motion called the Laplace–Runge–Lenz vector [40, 44]. It would be interesting to show if it is possible to construct such an invariant in the discrete-time setting.

Another open problem is the generalisation to non-Euclidean manifolds. For the coalgebra approach to these systems, see for example a series of papers by Italian–Spanish school [3–7] culminating in the proof of a Bertrand-like theorem on curved space [12], linked to the so-called Perlick classification [51]. For a different, more geometric perspective on the subject, see the recent classification in [34, 35]. Finding a connection between these two approaches and building the discrete-time analogue will be subject of future research.

Finally, we note that the present construction can be applied to other Lie–Poisson algebras, especially the ones related to classified Lie algebras for which the Casimir functions are known, see [36, 45–50]. In the continuous setting this was done by Ballesteros and Blasco [8, 16]. We note that the most natural extension would be the two-photon or h_6 Lie–Poisson algebra [67], a Lie–Poisson algebra containing many other interesting Lie–Poisson algebras as subalgebras, including $\mathfrak{sl}_2(\mathbb{R})$, whose associated Hamiltonian integrable systems have been discussed in [10, 16].

Acknowledgments

This work was made in the framework of the Project "Meccanica dei Sistemi discreti" of the GNFM unit of INDAM. In particular, GG acknowledge support of the GNFM through Progetto Giovani GNFM 2023: "Strutture variazionali e applicazioni delle equazioni alle differenze ordinarie", CUP_E53C22001930001.

DL was supported by the Australian Research Council Discovery Project DP190101529 (A/Prof. Y.-Z. Zhang).

The figures in this paper are `pdf` and were produced in `python` using the libraries `numpy` [31] and `mathplotlib` [33].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Explicit form of the coefficients in the case $d = 3$

We list the explicit formulas for the coefficients obtained in the case $d = 3$ during the proof of theorem 4.1. We divide the polynomial I_3 into its homogeneous components:

$$I_3(t) = I_3^{(1)}(t) + I_3^{(2)}(t) + I_3^{(3)}(t). \tag{A1}$$

and list the coefficients by the degree of the component.

Degree three:

$$a_{0,0,3} = a_{0,1,2} = a_{0,2,1} = a_{0,3,0} = a_{1,0,2} = a_{2,0,1} = 0,\ a_{3,0,0} = 0,$$
$$a_{2,1,0} = a_{1,2,0}, \quad a_{1,1,1} = -2a_{1,2,0}, \quad a_{1,2,0} = a_{1,2,0} \tag{A2}$$

Degree two:

$$a_{2,0,0} = a_{0,2,0} = 0, \quad a_{1,0,1} = a_{0,1,1}, \quad a_{0,0,2} = -2a_{0,1,1} - a_{1,1,0}. \tag{A3}$$

Degree one:

$$a_{0,0,1} = \mp\frac{1}{2}\frac{a_{0,1,1}^2}{a_{1,2,0}}, \quad a_{0,1,0} = \frac{1}{4}\frac{a_{0,1,1}^2}{a_{1,2,0}}, \quad a_{1,0,0} = \frac{1}{4}\frac{a_{0,1,1}^2}{a_{1,2,0}}. \tag{A4}$$

Appendix B. Periodic cases of the system (3.15) for $V = \kappa\xi/2$

If $V = \kappa\xi/2$ then the system (3.15) becomes:

$$J_+(t+h) = J_-(t), \tag{B1a}$$
$$J_-(t+h) = J_+(t) - 2\kappa J_3(t) + \kappa^2 J_-(t), \tag{B1b}$$
$$J_3(t+h) = -J_3(t) + \kappa J_-(t). \tag{B1c}$$

This system is linear and it can be written in matrix form as follows:

$$\mathbf{J}(t+h) = M\mathbf{J}(t), \tag{B2}$$

where $\mathbf{J}(t) = (J_+(t), J_-(t), J_3(t))$ and

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \kappa^2 & -2\kappa \\ 0 & \kappa & -1 \end{pmatrix}. \tag{B3}$$

From standard techniques in systems of linear difference equations [20], Chap. 3 we have that the solution is given by:

$$\mathbf{J}(t_0 + lh) = M^l \mathbf{J}(t_0). \tag{B4}$$

Still following [20], Chap. 3 we can compute the lth power of the matrix M (B3) through diagonalisation or Jordan block-form reduction. The characteristic polynomial of (B3) is:

$$p_M(\mu) = (\mu - 1)[\mu^2 + (2 - \kappa^2)\mu + 1]. \tag{B5}$$

Hence, the eigenvalues are:

$$\mu_1 = 1, \quad \mu_2 = \frac{\kappa^2}{2} - 1 + \frac{\kappa}{2}\sqrt{\kappa^2 - 4}, \quad \mu_3 = \frac{\kappa^2}{2} - 1 - \frac{\kappa}{2}\sqrt{\kappa^2 - 4}. \tag{B6}$$

The eigenvalues are different for all $\kappa \neq \pm 2$. This readily implies that the matrix is diagonalisable for every $\kappa \neq \pm 2$. When $\kappa = \pm 2$ the matrix is not diagonalisable because the only eigenvalue is $\mu_1 = 1$, which has geometric multiplicity one.

Then for $\kappa \neq \pm 2$ we have:

$$M^l = G\mathrm{diag}\left(1, \left(\frac{\kappa^2}{2} - 1 + \frac{\kappa}{2}\sqrt{\kappa^2 - 4}\right)^l, \left(\frac{\kappa^2}{2} - 1 - \frac{\kappa}{2}\sqrt{\kappa^2 - 4}\right)^l\right)G^{-1}. \tag{B7}$$

The periodicity condition is that there exists a $L \in \mathbb{N}$ such that $M^L = \mathbb{I}_3$, where $\mathbb{I}_3$ is the identity matrix. From equation (B7) this is true if and only if:

$$\left(\frac{\kappa^2}{2} - 1 + \frac{\kappa}{2}\sqrt{\kappa^2 - 4}\right)^L = 1, \quad \left(\frac{\kappa^2}{2} - 1 - \frac{\kappa}{2}\sqrt{\kappa^2 - 4}\right)^L = 1. \tag{B8}$$

Since the second equation in (B8) is the same as the first up to the discrete symmetry $\kappa \mapsto -\kappa$, we can just consider the solution of the first equation in (B8). That is, the solutions of (B2) are periodic of period $L \in \mathbb{N}$ if and only if κ satisfies the following algebraic equation:

$$\left(\frac{\kappa^2}{2} - 1 + \frac{\kappa}{2}\sqrt{\kappa^2 - 4}\right)^L = 1. \tag{B9}$$

That is, the left hand side of equation (B9) must be a Lth root of unity. Recalling that Lth roots of unity can be written in complex exponential form as $z_k = \exp(2ik\pi/L)$ with $k = 0,\ldots,L-1$ we have:

$$\frac{\kappa^2}{2} - 1 + \frac{\kappa}{2}\sqrt{\kappa^2 - 4} = \exp\left(\frac{2ik\pi}{L}\right), \quad k = 0,\ldots,L-1. \tag{B10}$$

That is, the solution can be written as:

$$\kappa = \pm 2\cos\left(\frac{k\pi}{L}\right), \quad k = 0,\ldots,L-1. \tag{B11}$$

However, this is not definitive: we need to discard $k = 0$ because it yields $\kappa = \pm 2$ which is not acceptable, and since the cosine function is antiperiodic of antiperiod π we can choose the sign plus in (B11). So we denote the final expression of the solutions as $\kappa_{k,L}$ and its expression is:

$$\kappa_{k,L} = 2\cos\left(\frac{k\pi}{L}\right), \quad k = 1,\ldots,L-1. \tag{B12}$$

We underline that from formula (B12) we have $|\kappa_{k,L}| < 2$, and for all values of k and L except for

$$\frac{k}{L} = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \tag{B13}$$

the numbers $\kappa_{k,L}$ are irrational numbers. In particular, this implies that, for every L prime we will obtain new solutions. This implies that the set of values of $\kappa_{k,L}$ such that the dynamical system is periodic is a countably infinite set.

Remark B.1. We remark that for $L = 3$ we have:

$$\kappa_{0,3} = 1, \quad \kappa_{1,3} = -1. \tag{B14}$$

That is, these cases correspond to the degenerate case of the function f_3 as $\tau \to 0$, see figure 1.

ORCID iDs

Giorgio Gubbiotti https://orcid.org/0000-0003-0423-3736
Danilo Latini https://orcid.org/0000-0001-8415-1751

References

[1] Ablowitz M J, Halburd R and Herbst B 2000 On the extension of the Painlevé property to difference equations *Nonlinearity* **13** 889–905
[2] Arnol'd V I 1997 *Mathematical Methods of Classical Mechanics* vol. 60 2nd edition (Berlin: Springer-Verlag) Graduate Texts in Mathematics
[3] Ballesteros Á and Herranz F J 2007 Universal integrals for superintegrable systems on N-dimensional spaces of constant curvature *J. Phys. A: Math. Theor.* **40** F51–9
[4] Ballesteros Á, Enciso A, Herranz F J and Ragnisco O 2008 A maximally superintegrable system on an n-dimensional space of nonconstant curvature *Physica* D **237** 505–9
[5] Ballesteros Á, Enciso A, Herranz F J and Ragnisco O 2009 Superintegrability on N-dimensional curved spaces: central potentials, centrifugal terms and monopoles *Ann. Phys.* **324** 1219–33
[6] Ballesteros Á, Enciso A, Herranz F J, Ragnisco O and Riglioni D 2011 Quantum mechanics on spaces of nonconstant curvature: the oscillator problem and superintegrability *Ann. Phys.* **326** 2053–73
[7] Ballesteros Á, Herranz F J and Ragnisco O 2008 Superintegrability on *sl*(2)-coalgebra spaces *Phys. Atom. Nuclei* **71** 812–8
[8] Ballesteros A and Blasco A 2008 *n*-dimensional superintegrable systems from symplectic realizations of Lie coalgebras *J. Phys. A: Math. Theor.* **41** 304028
[9] Ballesteros A, Blasco A, Herranz F J, Musso F and Ragnisco O 2009 (Super)integrability from coalgebra symmetry: formalism and applications *J. Phys.: Conf. Ser.* **175** 012004 (26pp)
[10] Ballesteros A and Herranz F J 2001 Two-photon algebra and integrable Hamiltonian systems *J. Nonlinear Math. Phys.* **8** 18–22
[11] Ballesteros A and Ragnisco O 1998 A systematic construction of completely integrable Hamiltonians from coalgebras *J. Phys. A: Math. Gen.* **31** 3791–813
[12] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2009 Hamiltonian systems admitting a Runge-Lenz vector and an optimal extension of Bertrand's theorem to curved manifolds *Commun. Math. Phys.* **290** 1033–49
[13] Ballestreros Á, Corsetti M and Ragnisco O 1996 N-dimensional classical integrable systems from Hopf Algebras *Czechoslovak J. Phys.* **46** 1153–63
[14] Bellon M and Viallet C M 1999 Algebraic entropy *Comm. Math. Phys.* **204** 425–37
[15] Bertrand J 1873 Théorème relatif au mouvement d'un point attiré vers un centre fixe *C. R. Acad. Sci.* **77** 849–53
[16] Blasco A 2008 Integrabilidad de sistemas no lineales Hamiltonianos con *N* grados de libertad *PhD thesis* Universidad de Burgos
[17] Bruschi M, Ragnisco O, Santini P M and Tu G-Z 1991 Integrable symplectic maps *Physica* D **49** 273–94
[18] Chari V and Pressley A 1995 *A Guide to Quantum Groups* (Cambridge: Cambridge university press)
[19] Drinfel'd V G 1987 *Proc. Int. Congress Math., Berkeley 1986, AMS*
[20] Elaydi S 2005 *An introduction to Difference Equations* 3rd edition (Springer)
[21] Evans N W 1990 Super-integrability of the Winternitz system *Phys. Lett.* A **147** 483–6
[22] Evans N W and Verrier P E 2008 Superintegrability of the caged anisotropic oscillator *J. Math. Phys.* **49** 092902

[23] Friš J, Mandrosov V, Smorodinski Y A, Uhlíř M and Winternitz P 1965 On higher symmetries in quantum mechanics *Phys. Lett.* **13** 354–6

[24] Grammaticos B, Halburd R G, Ramani A and Viallet C-M 2009 How to detect the integrability of discrete systems *J. Phys A: Math. Theor.* **42** 454002 Newton Institute Preprint NI09060-DIS (41 pp)

[25] Grammaticos B, Ramani A and Papageorgiou V 1991 Do integrable mappings have the Painlevé property? *Phys. Rev. Lett.* **67** 1825

[26] Gubbiotti G 2017 Integrability of difference equations through algebraic entropy and generalized symmetries *Symmetries and Integrability of Difference Equations: Lecture Notes of the Abecederian School of SIDE 12, Montreal 2016* ed D Levi, R Verge-Rebelo and P Winternitz (Berlin: Springer International Publishing) 75–152 CRM Series in Mathematical Physics, Chap. 3

[27] Gubbiotti G 2020 Lagrangians and integrability for additive fourth-order difference equations *Eur. Phys. J. Plus* **135** 853 (30pp)

[28] Gubbiotti G and Latini D 2018 A multiple scale approach to maximal superintegrability *J. Phys. A: Math. Theor.* **51** 285201 36pp

[29] Gubbiotti G, Latini D and Tapley B K 2021 Coalgebra symmetry for discrete systems arXiv:2109.10391 Under review

[30] Hall C L and Lustri C J 2016 Multiple scales and matched asymptotic expansions for the discrete logistic equation *Nonlinear Dyn.* **85** 1345–62

[31] Harris C R *et al* 2020 Array programming with NumPy *Nature* **585** 357–62

[32] Hietarinta J, Joshi N and Nijhoff F 2016 *Discrete Systems and Integrability* (Cambridge University Press) Cambridge Texts in Applied Mathematics

[33] Hunter J D 2007 Matplotlib: a 2D graphics environment *Comput. Sci. Eng.* **9** 90–5

[34] Kress J, Schöbel K and Vollmer A 2020 Algebraic conditions for conformal superintegrability in arbitrary dimension arXiv:2006.15696

[35] Kress J, Schöbel K and Vollmer A 2019 An algebraic geometric foundation for a classification of superintegrable systems in arbitrary dimension arXiv:1911.11925

[36] Šnobl L and Winternitz P 2014 *Classification and identification of Lie algebras* (Providence (RI): American Mathematical Society) (*CRM Monograph Series*) 33

[37] Latini D 2019 Universal chain structure of quadratic algebras for superintegrable systems with coalgebra symmetry *J. Phys. A: Math. Theor.* **52** 125202

[38] Latini D, Marquette I and Zhang Y-Z 2021 Embedding of the Racah algebra $R(n)$ and superintegrability *Ann. Phys.* **426** 168397

[39] Latini D, Marquette I and Zhang Y-Z 2021 Racah algebra $R(n)$ from coalgebraic structures and chains of R(3) substructures *J. Phys. A: Math. Theor.* **54** 395202

[40] Leach P G L and Flessas G P 2003 Generalisations of the Laplace-Runge-Lenz vector *J. Nonlinear Math. Phys.* **10** 340–423

[41] Logan J D 1973 First integrals in the discrete variational calculus *Aeq. Math.* **9** 210–20

[42] McLachlan R I 1993 Integrable four-dimensional symplectic maps of standard type *Phys. Lett.* A **177** 211–4

[43] McMillan E M 1971 A problem in the stability of periodic systems *A Tribute to E.U. Condon* ed E Britton and H Odabasi (Boulder: Colorado Assoc. Univ. Press.) 219–44 Topics in Modern Physics

[44] Miller W Jr., Post S and Winternitz P 2013 Classical and quantum superintegrability with applications *J. Phys. A: Math. Theor.* **46** 423001

[45] Mubarakzyanov G M 1966 Certain theorems on solvable Lie algebras *Izv. Vys. Ucheb. Zaved. Matematika* **55** 95–8 (in Russian)

[46] Mubarakzyanov G M 1963 Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element *Izv. Vys. Ucheb. Zaved. Matematika* **35** 104–16 (in Russian)

[47] Mubarakzyanov G M 1963 The classification of the real structure of five-dimensional Lie algebras *Izv. Vys. Ucheb. Zaved. Matematika* **34** 99–106 (in Russian)

[48] Mubarakzyanov G M 1963 On solvable Lie algebras *Izv. Vys. Ucheb. Zaved. Matematika* **32** 114–23 (in Russian)

[49] Patera J, Sharp R T, Winternitz P and Zassenhaus H 1976 Invariance of real low-dimensional Lie algebras *J. Math. Phys.* **17** 986–94

[50] Patera J and Winternitz P 1977 Subalgebras of real three- and four-dimensional Lie algebras *J. Math. Phys.* **18** 1449–55

[51] Perlick V 1992 Bertrand spacetimes *Class. Quant. Grav.* **9** 1009–21

[52] Rodríguez M A, Tempesta P and Winternitz P 2009 Symmetry reduction and superintegrable Hamiltonian systems *J. Phys.: Conf. Ser.* **175** 012013

[53] Roman S 2007 *Advanced Linear Algebra* (New York: Springer-Verlag) Graduate Texts in Mathematics

[54] Suris Y B 1994 Discrete-time analogues of some nonlinear oscillators in the inverse-square potential *J. Phys. A: Math. Gen.* **27** 8161–9

[55] Suris Y B 1994 A discrete-time Garnier system *Phys. Lett.* A **189** 281–9

[56] Suris Y B 1994 A family of integrable symplectic standard-like maps related to symmetric spaces *Phys. Lett.* A **192** 9–16

[57] Suris Y B 1989 On integrable standard-like mappings *Funct. Anal. Appl* **23** 74–6

[58] Suris Y B 2003 *The Problem of Integrable Discretization: Hamiltonian Approach* (Basel: Birkhäuser)

[59] Tjin T 1992 Introduction to quantized Lie groups and algebras *Int. J. Mod. Phys.* **7** 6175–213 A

[60] Tran D T 2011 Complete integrability of maps obtained as reductions of integrable lattice equations *PhD thesis* La Trobe University

[61] Tran D T, van der Kamp P H and Quispel G R W 2016 Poisson brackets of mappings obtained as $(q, -p)$ reductions of lattice equations *Reg. Chaot. Dyn.* **21** 682–96

[62] Verrier P E and Evans N W 2008 A new superintegrable Hamiltonian *J. Math. Phys.* **49** 022902 (8pp)

[63] Veselov A P 1991 Integrable maps *Russ. Math. Surveys* **46** 1–51

[64] Viallet C-M 2021 Features of discrete integrability *Quantum Theory and Symmetries: Proceedings of the 11th International Symposium, Montreal, Canada* ed M Paranjape *et al* (New York: Springer) 21–33 CRM Series on Mathematical Physics

[65] Weisstein E W 2003 *CRC Concise Encyclopedia of Mathematics* (CRC Press)

[66] Wojciechowski S 1985 Integrability of one particle in a perturbed central quartic potential *Phys. Scr.* **31** 433–8

[67] Zhang W-M, Feng D H and Gilmore R 1990 Coherent states: theory and some applications *Rev. Mod. Phys.* **62** 867–927