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An Advanced Course in Probability and Stochastic Processes.

Dirk P. Kroese and Zdravko I. Botev. CRC Press, 2024. ISBN 978-1-032-32046-5, 978-1-032-32440-1, 978-1-003-31501-8. xx+357 pp.

This book provides a comprehensive, yet concise, treatment of probability theory and stochastic processes. According to the authors, it is designed for higher undergraduate to lower graduate level students with a solid mathematical foundation with the ambition to deliver a mathematically coherent course that can fit within a one-semester framework while maintaining conceptual depth.

The book moves on from measure-theoretic foundations to stochastic calculus. The text opens with a full chapter on measure theory, covering $\sigma-$algebras, measurable functions, measures, and integrals. In this respect, Kroese and Botev echo Kallenberg's observation that

> $[\cdots]$ any serious student needs a good foundation in measure theory and basic functional analysis, before moving on to the probabilistic aspects.

While this of course strengthens structural coherence, it may also raise the entry barrier. In fact, similar beginning chapters are found in some classical texts, such as Kallenberg's *Foundations of Modern Probability* or Çinlar's *Probability and Stochastics*, whose measure-theoretic approach is destined to more advanced readers.

The subsequent Chapter 2 develops probability theory: random variables, probability distributions, L_p spaces, and integral transforms are introduced here. The chapter also revisits the concept of independence by introducing filtrations and defining mutually independent $\sigma-$algebras. Important stochastic processes –Gaussian processes, Poisson random measures, compound Poisson processes, and Lévy processes–are introduced relatively early, with a first appearance as early as in this chapter.

The chapter on convergence (Ch. 3) provides a compact, yet solid discussion of almost sure convergence, convergence in probability, convergence in distribution, and convergence in L_p. Uniform integrability is pleasantly emphasized as a unifying tool and the relation between different types of convergence is thoroughly explained. The Law of Large Numbers

(LLN) and Central Limit Theorem are presented and proved succinctly at the end of the chapter, further underscoring the book's orientation toward an audience already familiar with these fundamental results. The LLN is revisited in Chapter 5 and established under weaker assumptions within the framework of martingale theory.

Chapter 4 begins with a rigorous definition of the conditional expectation of a random variable given a $\sigma-$algebra. From that definition, more elementary concepts, such as conditional probability and Bayes theorem are derived and then extended to conditioning to $\sigma-$algebras. The introduction of Markov chains and Markov jump processes is made here, and the connection to Lévy processes and Poisson random measures is highlighted. However, readers seeking extensive applied examples may find the presentation more abstract than illustrative.

Chapter 5 is dedicated to martingales, that form a central structural pillar of the book. Any detailed analysis of this topic necessarily begins with the introduction of stopping times, Doob's theorems, and martingale convergence. The martingale framework is used to revisit foundational results such as the Strong Law of Large Numbers and the Radon–Nikodym theorem. As in the first Chapters, some proofs are technically dense, and the pedagogical balance between intuition and abstraction leans toward the latter.

Chapter 6 progresses through the construction and existence of the Wiener process, and proves some of its most distinctive features such as the strong Markov property, the reflection principle, the connection with martingales and the behaviour of its samples paths. This chapter is quite dense and also includes alternative characterization results, among which the widely known relation between the transition kernel and the heat equation.

Finally, Chapter 7 starts with the definition of Itô integral and Itô formula, then proceeds to the introduction of stochastic differential equations. The construction of the Itô integral is done in stages. It begins with simple processes, then moves to predictable processes and to more general integrands, to conclude with its multivariate extension. The exposition makes clear how martingale properties and Brownian path regularity underpin the formula. The derivation is rigorous but concise, prioritizing structural understanding over heuristic motivation.

The section on stochastic differential equations develops existence and uniqueness results under standard Lipschitz and growth conditions. The authors also emphasize the Markov property of diffusion processes. While the treatment is mathematically solid, it remains focused on foundational theory rather than extensive applications to finance or physics. Readers seeking domain-specific modeling examples may therefore need supplementary texts. An interesting addition is the discussion of numerical methods, notably Euler's scheme for SDEs.

Throughout the book, several algorithms in pseudo and Matlab codes are included to enhance the understanding of the stochastic processes analyzed through simulations of their sample paths. Further, all chapters conclude with a rich set of exercises, which I personally find particularly helpful for gaining a deeper understanding of the material. In this regard, the inclusion of selected solutions to the exercises is especially welcome, and one may hope that this feature will be expanded in a future second edition.

Generally speaking, the objective of Kroese and Botev's monograph is to provide a coherent semester-length pathway through core advanced topics, with a mathematically sound approach. However, although the authors succeed in maintaining rigor and structural unity, the breadth of topics in the volume sometimes results in a compressed treatment, with some important examples and results deferred to the exercise sections, and I doubt the entire content of the book is easily covered in a one-semester course. The book assumes a strong analytical background; it is a direct immersion into modern measure-theoretic probability, given a previous exposition to probability and some stochastic processes.

From the point of view of the comparison with other graduate texts, the structure of the first chapters evidently reflects the influence of Çinlar's lecture's at Princeton, explicitly mentioned by the authors as one of the main inspirations for the content of their book. Relative to Çinlar's *Probability and stochastics*, the present book touches, in about 200 less pages, some additional topics, such as the Itô integrals and SDEs.

As other classical probability monographs, *An Advanced Course in Probability and Stochastic Processes* starts from measure-theoretic probability, but, in contrast to these, it guides the reader relatively quickly toward stochastic processes and stochastic calculus. At the same time, it remains more pedagogically oriented than advanced references. For example, colored boxes are used to highlight particularly important theorems or definition. Thus, this book does not replace the classical references for deep specialization, such as, for example, Billingsley's *Probability and Measure* or Kallenberg's *Foundations of Modern Probability*, but it could serve as an effective bridge between foundational probability courses and more research-level texts. In conclusion, the book *An Advanced Course in Probability and Stochastic Processes* is a very nice addition to the bibliography of comprehensive advanced books in the field. I believe this book would be suitable for mathematics higher-undergraduates only, or for graduate students, in some other quantitative disciplines. From a broader perspective, researchers interested in these areas–particularly those seeking a brief but rigorous overview of many of the core topics–may also find this book both useful and enjoyable.

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