On split extensions of product hoops

Manuel Mancini*†‖, Giuseppe Metere‡, Federica Piazza*§, Marco Elio Tabacchi*¶

*Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, Italy
Email: manuel.mancini@unipa.it; federica.piazza07@unipa.it; marcoelio.tabacchi@unipa.it

†Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, Belgium
Email: manuel.mancini@uclouvain.be

‡Dipartimento di Scienza per gli Alimenti la Nutrizione, l'Ambiente, Università degli Studi di Milano Statale, Italy
Email: giuseppe.metere@unimi.it

§Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra, Università degli Studi di Messina, Italy
Email: federica.piazza1@studenti.unime.it

¶Istituto Nazionale di Ricerche Demopolis, Italy

‖Corresponding author

***Abstract*—The aim of this article is to investigate internal actions/split extensions in the category of product hoops, with a focus on those that have a strong section. We provide a characterization of split extensions that strongly split in terms of strong external actions, highlighting their relevance in the categorical study of algebraic structures. Product hoops, together with their bounded counterpart, product algebras, play a significant role in the algebraic study of fuzzy logic, particularly in modeling implications and graded truth values, and the double negation retraction provides a significant example of split extensions with a strong section, thus motivating this work.**

***Index Terms*—Fuzzy logic, product logic, product algebra, product hoop, internal action, split extension.**

I. INTRODUCTION

Product algebras were introduced as the algebraic counterpart of product logic, a fundamental system within many-valued logic [18], [20]. They generalize Boolean algebras and incorporate operations that correspond to the logical connectives of *product logic*, which is a t-norm-based fuzzy logic derived from the product t-norm on the unit interval $[0, 1]$. It forms one of the three fundamental fuzzy logics (along with *Gödel logic* [20] and *Łukasiewicz logic* [26]) used in formal fuzzy reasoning. Unlike classical logic, where every statement is either true or false, product logic allows truth values to take any real number in the interval $[0, 1]$, making it a key tool in fuzzy reasoning. Product algebras capture the algebraic properties of these truth values and provide a robust framework for reasoning under uncertainty.

In a product algebra, the operations of multiplication and residuated implication play a central role, modeling conjunction and logical entailment in product logic. The interplay of these operations ensures that product algebras align with the semantics of many-valued reasoning, supporting logical inference in contexts where classical binary logic is insufficient. We refer the reader to [2], [10]–[13], [17] for more details.

Product hoops form a subclass of hoops that are closely related to product algebras, emphasizing the algebraic behavior of multiplication and residuated implication. Key properties of product hoops include the residuation property, which ensures that implication aligns with logical entailment, and their compatibility with lattice structures, facilitating logical reasoning about graded truth values.

Every product algebra can be associated with a corresponding product hoop, allowing many-valued logic to be explored from multiple algebraic perspectives. While product algebras focus on modeling logical connectives in a complete algebraic structure, product hoops provide a streamlined approach to studying implication and multiplication, making them complementary tools for analyzing logical systems.

From a categorical perspective, the variety of product algebras forms an ideally exact category (see [22], [24]). Moreover, the semi-abelian category $(\mathbf{PAlg} \downarrow L_2)$ is equivalent to the category $\mathbf{PHoops}$ of product hoops.

One of the key concepts that can be explored in the context of semi-abelian categories [23] is that of internal actions [4], which provides a unified framework to describe structures such as semidirect products, crossed extensions, and others,

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allowing the generalization of classical algebraic concepts (like group actions) in a more abstract and categorical setting. Internal actions of the objects of a category were defined by F. Borceux, G. Janelidze and G. M. Kelly with the aim of extending the correspondence between actions and split extensions from the context of groups and Lie algebras to arbitrary semi-abelian categories. However, in some cases, such as for Orzech categories of interest [27], it is more convenient to describe internal actions in terms of *external actions*, or operation, i.e., via a set of maps which satisfy a certain set of identities (see for instance [7] and [14], where internal actions are studied in the context of semi-abelian varieties of algebras).

The aim of this article is to study internal actions/split extensions in the category $\mathbf{PHoops}$ of product hoops. We focus on those split extensions which *strongly splits*, i.e., such that the corresponding split epimorphism has a *strong section* [28], [29]. We describe such an internal action/split extension in terms of a *strong external action*, which consists of a pair of maps satisfying a set of identities, which are strictly connected to the axioms of a product hoop. We provide a categorical framework that unifies and generalizes different algebraic concepts, such as semidirect products. These results may have some significant implications for the algebraic study of fuzzy logic, as they enable a deeper understanding of the structural properties of product hoops and algebras.

We conclude with some possible future directions.

II. PRELIMINARIES

Definition II-A. [19] A *BL-algebra* is an algebra $A = (A, \vee, \wedge, \cdot, \rightarrow, 0, 1)$ such that

(i) $(A, \vee, \wedge, \cdot, \rightarrow, 0, 1)$ is a bounded residuated lattice;
(ii) $x \wedge y = x \cdot (x \rightarrow y)$;
(iii) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

BL-algebras can be described in the reduced language of *hoops*. Hence we can consider the hoop reduct $(A, \cdot, \rightarrow, 1)$ of a BL-algebra A.

Recall that a *hoop* is an algebra $H = (H, \cdot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that

(i) $(H, \cdot, 1)$ is a commutative monoid;
(ii) $x \rightarrow x = 1$;
(iii) $x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x)$;
(iv) $(x \cdot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,

for every $x, y, z \in H$.

A hoop subreduct of a BL-algebras is called *basic hoop* [19], i.e., a hoop satisfying the identity

$$((x \rightarrow y) \rightarrow z) \rightarrow (((y \rightarrow x) \rightarrow z) \rightarrow z) = 1.$$

We denote by $\mathbf{Hoops}$ the category of hoops and by $\mathbf{BHoops}$ the subcategory of basic hoops.

Every hoop H is endowed with a partial order $\leqslant$, called the *natural order*, which is defined by the following equivalent conditions: for any $x, y \in H$

(i) $x \leqslant y$;
(ii) $x \rightarrow y = 1$;
(iii) there exists $z \in H$ such that $x = z \cdot y$.

Remark II-B. [24] The category $\mathbf{Hoops}$ is semi-abelian.

One of the most relevant subvarieties of the variety of BL-algebras is the category of *product algebras*. Product algebras were introduced in [20] and constitutes the equivalent algebraic semantics of *product logic*.

A product algebra is a BL-algebra satisfying

$$\neg x \vee ((x \rightarrow x \cdot y) \rightarrow y) = 1,$$

where $\neg x := x \rightarrow 0$. The class of product algebras forms an algebraic variety which we denote with $\mathbf{PAlg}$.

Example II-C. Let C be a *cancellative* hoop [19], that is, a hoop satisfying the identity

$$x \rightarrow (y \cdot x) = x.$$

The algebra $2 \oplus C$ whose underlying set is $C \cup \{0\}$, where $0 \notin C$, and whose constants are 0 and 1, and the operations $\cdot$ and $\rightarrow$ are defined by

$$x \cdot y = \begin{cases} x \cdot y, & \text{if } x, y \in C, \\ 0, & \text{otherwise}, \end{cases}$$

$$x \rightarrow y = \begin{cases} x \rightarrow y, & \text{if } x, y \in C, \\ 1, & \text{if } x = 0, \\ 0, & \text{if } x \in C,\ y = 0, \end{cases}$$

is a product algebra.

A special class of product algebras is given by the so called *product chains*, i.e., totally ordered product algebras. Every product chain has the form $2 \oplus C$, where C is a totally ordered cancellative hoop. As a consequence, every product algebra P is a subdirect product of a family $\{P_i\}_{i \in I}$ of totally ordered product algebras such that, for every $i \in I$, $P_i = 2 \oplus C_i$, where C_i is a totally ordered cancellative hoop [2]. It was proved in [1] that product algebras are term equivalent to the class of bounded *product hoops*.

Definition II-D. A *product hoop* is a basic hoop satisfying the identity

$$(y \rightarrow z) \vee ((y \rightarrow (x \cdot y)) \rightarrow x) = 1.$$

We denote by $\mathbf{PHoops}$ the variety of product hoops.

Example II-E. An example of product algebra is given by the set $A = [0, 1]$ endowed with the operations $x \cdot_{\prod} y = xy$ and

$$x \rightarrow_{\prod} y = \begin{cases} 1, & \text{if } x \leqslant y, \\ \frac{y}{x}, & \text{otherwise}. \end{cases}$$

We denote this algebra by $[0, 1]_{\prod}$. In fuzzy logic, this product algebra is called the *standard product algebra*, as it forms the standard real-valued semantics of *product logic*.

Example II-F. The two-element Boolean algebra $L_2 = \{0, 1\}$ is a product algebra. This corresponds to classical two-valued logic (true and false), making Boolean algebras special cases of product algebras.

We recall that an homomorphism of product algebras is a map $f\colon A \to B$ such that $f(a \cdot a') = f(a) \cdot f(a')$, $f(a \to a') = f(a) \to f(a')$, $f(1) = 1$ and $f(0) = 0$, for any $a, a' \in A$. Given a homomorphism of product algebras $f\colon A \to B$, its kernel $\ker f$ is a *filter* of A, i.e. a subset $F \subseteq A$ such that $(F, \cdot, 1)$ is a submonoid of $(A, \cdot, 1)$ which is upward closed with respect to the natural order $\leqslant$ of A. Conversely, every filter F of A can be seen as the kernel of the canonical projection $\pi\colon A \to A/F$.

Remark II-G. The initial object of the category $\mathbf{PAlg}$ is the two-element product algebra $L_2 = \{0, 1\}$. However, the variety $\mathbf{PAlg}$ is not semi-abelian [23], since it is not pointed, but, as shown in [24], it is protomodular (see [3], [5], [6]). As a consequence, $\mathbf{PAlg}$ is ideally exact [22] and the semi-abelian category $(\mathbf{PAlg} \downarrow L_2)$ is equivalent to the category $\mathbf{PHoops}$ of product hoops.

Our aim is now to describe *split extensions* in the category $\mathbf{PHoops}$.

Let B, X be product hoops. A split extension of B by X is a diagram

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

in $\mathbf{PHoops}$ such that $p \circ s = 1_B$ and (X, k) is a kernel of p. Since the category of product hoops is semi-abelian, there is an equivalence between split extensions and internal actions given by semidirect products. In particular, the description of semidirect products in the category of product hoops can be obtained from the results in [15].

Definition II-H. Let

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

be a split epimorphism in $\mathbf{PHoops}$. Let (X, k) be a kernel of p and let $\xi\colon B \flat X \to X$ the corresponding internal action of B on X. The semidirect product $X \rtimes_\xi B$ of X and B with respect to the action ξ is the set

$$\begin{aligned}&\{(x_1, x_2, b) \in X^2 \times B \mid ((x_1 \to s(b)) \cdot x_2) \to s(b) = x_1,\\ &((((x_1 \to s(b)) \cdot x_2) \to s(b)) \to s(b)) \to ((x_1 \to s(b)) \cdot x_2)\}\end{aligned}$$

equipped with the operations

$$\begin{aligned}(x_1, x_2, b_1) \to (y_1, y_2, b_2) = {}&((((x_1 \to s(b_1)) \cdot x_2)\\ &\to ((y_1 \to s(b_2)) \cdot y_2)) \to (s(b_1) \to s(b_2)),\\ &(((((x_1 \to s(b_1)) \cdot x_2) \to ((y_1 \to s(b_2) \cdot y_2)))\\ &\to s(b_1 \to b_2)) \to s(b_1 \to b_2)) \to\\ &(((x_1 \to s(b_1)) \cdot x_2) \to ((y_1 \to s(b_2) \cdot y_2))), b_1 \to b_2),\end{aligned}$$

$$\begin{aligned}(x_1, x_2, b_1) \cdot (y_1, y_2, b_2) = {}&((((x_1 \to s(b_1)) \cdot x_2) \cdot\\ &((y_1 \to s(b_2)) \cdot y_2)) \to (s(b_1) \cdot s(b_2)),\\ &((((((x_1 \to s(b_1)) \cdot x_2) \cdot ((y_1 \to s(b_2) \cdot y_2)))\\ &\to s(b_1 \cdot b_2)) \to s(b_1 \cdot b_2)) \to\\ &(((x_1 \to s(b_1)) \cdot x_2)) \cdot ((y_1 \to s(b_2) \cdot y_2))), b_1 \cdot b_2)\end{aligned}$$

and

$$1_{X \rtimes_\xi B} = (1, 1, 1).$$

The description of internal actions/split extensions in terms of semi-direct product is easier in the case the split epimorphism p has a *strong section* (see [28], [29]).

Definition II-I. A split extension

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

strongly splits, or has a *strong section*, if the morphism s is a *strong section* of p, i.e., if the equation

$$a \to s(b) = sp(a) \to s(b)$$

holds for every $a \in A$ and $b \in B$.

Example II-J. One may easy check that any isomorphism $p\colon A \to B$ has strong section $s = p^{-1}\colon B \to A$.

Example II-K (Double negation). Let A be a product algebra. Let

$$\mathrm{B}(A) = \{x \in A \mid \neg\neg x = x\}$$

be the set of *regular elements* of A and let

$$\mathrm{D}(A) = \{x \in A \mid \neg\neg x = 1\}$$

be the set of *dense elements* of A. One may check that $\mathrm{B}(A)$ is the greatest Boolean subalgebra of A and $\mathrm{D}(A)$ is a filter of A (see [25, Theorem 3.1]). Furthermore, the split extension

$$\begin{tikzcd} \mathrm{D}(A) \arrow[r, "i"] & A \arrow[r, shift left, "p"] & \mathrm{B}(A) \arrow[l, shift left, "j"] \end{tikzcd}$$

where $p(a) = \neg\neg a$ (see Theorem 1.2 and Lemma 1.4 of [13] where it is proved that p is a homomorphism) and i and j are the canonical inclusions, has a strong section, since the identity

$$x \to y = \neg\neg x \to y$$

holds in chains. Indeed, if $x = 0$, then

$$0 \to y = 1 = \neg\neg 0 \to y.$$

Suppose now $x \in C$, where C is a totally ordered cancellative hoop. Then

$$\begin{aligned}x \to y &= x \to ((y \to 0) \to 0)\\ &= (x \cdot (y \to 0)) \to 0\\ &= (y \to 0) \to (x \to 0)\\ &= (y \to 0) \to 0\\ &= y.\end{aligned}$$

We observe that the property of having a strong section is invariant under isomorphism of split extensions.

Thus, for any product hoop X we can define the functor $\mathrm{SplExt}_{\mathrm{ss}}(-, X)\colon \mathbf{PHoops}^{\mathrm{op}} \to \mathbf{Set}$ which assigns to any product hoop B, the set $\mathrm{SplExt}_{\mathrm{ss}}(B, X)$ of isomorphism classes of split extensions of B by X in $\mathbf{PHoops}$ that strongly splits, and to every homomorphism of product hoops $f\colon B' \to$

B the *change of base* function $f^*\colon \mathrm{SplExt}_{\mathrm{ss}}(B, X) \to \mathrm{SplExt}_{\mathrm{ss}}(B', X)$ given by pulling back along f.

Remark II-L. [15] Let

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

be a split extension in $\mathbf{PHoops}$ that strongly splits and let $\xi\colon B\flat X \to X$ be the corresponding internal action of B on X. The semidirect product $X \rtimes_\xi B$ of X and B w.r.t. the action ξ is given by the set

$$\{(x, b) \in X \times B \mid s(b) \to (s(b) \cdot x) = x\}$$

together with the operations

$$(x, b_1) \to (y, b_2) = (s(b_2 \to b_1) \to (x \to y),\ b_1 \to b_2),$$

$$(x, b_1) \cdot (y, b_1) = (s(b_1 \cdot b_2) \to (s(b_1 \cdot b_2) \cdot x \cdot y),\ b_1 \cdot b_2)$$

and

$$1_{X \rtimes_\xi B} = (1, 1, 1).$$

III. EXTERNAL ACTIONS OF PRODUCT HOOPS

The goal of this section is to describe split extensions with strong sections in the category $\mathbf{PHoops}$ in terms of *(strong) external actions*.

Definition III-A. Let B, X be product hoops. A *(strong) external action* of B on X consists of a pair of maps

$$f\colon B \times X \to X\colon (b, x) \mapsto f_b(x),$$

$$g\colon B \times X \to X\colon (b, x) \mapsto g_b(x)$$

such that

E1. $f_b(1) = g_b(1) = 1$;
E2. $f_1 = g_1 = \mathrm{id}_X$;
E3. $f_{b_1 \cdot b_2}(x \cdot g_{b_1}(x \to y)) = f_{b_1 \cdot b_2}(x \cdot (x \to y))$;
E4.

$$\begin{aligned} &g_{(b_3 \to (b_1 \cdot b_2))}(f_{b_1 \cdot b_2}(x \cdot y) \to z) = \\ &\qquad = g_{(b_2 \to b_3) \to b_1}(x \to g_{b_3 \to b_2}(y \to z)); \end{aligned}$$

E5.

$$\begin{aligned} &g_{\tilde{b}}(g_{b_3 \to (b_1 \to b_2)}(g_{b_2 \to b_1}(x \to y) \to z) \\ &\to (g_{b_3 \to (b_2 \to b_1)}(g_{b_1 \to b_2}(y \to x) \to z) \to z)) = 1, \end{aligned}$$

where

$$\tilde{b} = (((b_2 \to b_1) \to b_3) \to b_3) \to ((b_1 \to b_2) \to b_3);$$

E6.

$$\begin{aligned} &g_{((b_2 \to (b_1 b_2)) \to b_1) \to (b_2 \to b_3)}(g_{b_3 \to b_2}(y \to z) \\ &\to ((y \to f_{b_1 b_2}(xy)) \to x)) \to ((y \to f_{b_1 b_2}(xy)) \to x)) \\ &\wedge (g_{(b_2 \to b_3) \to ((b_2 \to (b_1 b_2)) \to b_1)}(((y \to f_{b_1 b_2}(xy)) \to x) \\ &\to g_{b_3 \to b_2}(y \to z)) \to g_{b_3 \to b_2}(y \to z)) = 1, \end{aligned}$$

for any $b, b_1, b_2, b_3 \in B$ and $x, y, z \in X$.

We denote by $\mathrm{EAct}_{\mathrm{ss}}(B, X)$ the set of strong external actions of B on X.

Remark III-B. It is defined a functor

$$\mathrm{EAct}_{\mathrm{ss}}(-, X)\colon \mathbf{PHoops}^{\mathrm{op}} \to \mathbf{Set}$$

which maps every product hoop B to $\mathrm{EAct}_{\mathrm{ss}}(B, X)$, and every morphism $\phi\colon B' \to B$ in $\mathbf{PHoops}$ to the function

$$\mathrm{EAct}_{\mathrm{ss}}(\phi, X)\colon \mathrm{EAct}_{\mathrm{ss}}(B, X) \to \mathrm{EAct}_{\mathrm{ss}}(B', X)$$

which maps an external action $f, g\colon B \times X \to X$ to the external action $f', g'\colon B' \times X \to X$ defined by

$$f'(b', x) = f(\phi(b'), x) \ \text{ and } \ g'(b', x) = g(\phi(b'), x).$$

We want now to show that strong external actions of product hoops are equivalent to split extensions with strong sections.

Proposition III-C. *Let B, X be product hoops. There is a bijection τ_B between* $\mathrm{SplExt}_{\mathrm{ss}}(B, X)$ *and* $\mathrm{EAct}_{\mathrm{ss}}(B, X)$.

Proof. We define τ_B as follows: given any split extensions in $\mathbf{PHoops}$ that strongly splits

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B, \arrow[l, shift left, "s"] \end{tikzcd}$$

we associate the pair of maps $f, g\colon B \times X \to X$ defined by

$$f_b(x) = s(b) \to (s(b) \cdot x) \ \text{ and } \ g_b(x) = s(b) \to x.$$

It is possibile to show that (f, g) defines a strong external action of B on X.

Now, consider the map μ_B which sends a strong external action $f, g\colon B \times X \to X$ to the split extension of B by X

$$\begin{tikzcd} X \arrow[r, "\iota_1"] & Y \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "\iota_2"] \end{tikzcd}$$

where

$$Y = \{(x, b) \in X \times B \mid f_b(x) = x\}$$

and

$$(x, b) \to (y, b') = (g_{b' \to b}(x \to y),\ b \to b'), \tag{III.1}$$

$$(x, b) \cdot (y, b') = (f_{b \cdot b'}(x \cdot y),\ b \cdot b'). \tag{III.2}$$

This split extensions strongly splits since

$$\begin{aligned} (x, b) \to \iota_2(b') &= (x, b) \to (1, b') = \\ &= (x \to g_{b \to b'}(1), b \to b') = \\ &= (x \to 1, b \to b') = (1, b \to b') \end{aligned}$$

and

$$\begin{aligned} \iota_2\pi_2(x, b) \to \iota_2(b') &= \iota_2(b) \to \iota_2(b') = \\ &= (1, b) \to (1, b') = (1, b \to b'). \end{aligned}$$

It is easy to check that μ_B is the inverse of the map τ_B. ☐

Remark III-D. Let $f, g\colon B \times X \to X$ be a strong external action of product hoops. One may check that the map g satisfies the following additional properties:

(i) g_b is monotone for any $b \in B$, i.e., if $x \leqslant y$, then $g_b(x) \leqslant g_b(y)$;
(ii) $g_{b_1 \cdot b_2} = g_{b_1} \circ g_{b_2}$, for any $b_1, b_2 \in B$;

(iii) $g_b(x \to y) = g_b(x) \to g_b(y)$, for any $b \in B$ and $x, y \in X$.
These directly follow from the fact that the set

$$Y = \{(x, b) \in X \times B \mid f_b(x) = x\}$$

endowed with the binary operations III.1 and III.2 is a hoop.

We conclude the article by proving that the bijection τ_B of Proposition III-C extends to an isomorphism of functors.

Theorem III-E. *There is a natural isomorphism*

$$\tau\colon \operatorname{SplExt}_{\mathrm{ss}}(-, X) \cong \operatorname{EAct}_{\mathrm{ss}}(-, X).$$

Proof. The bijection τ_B of Proposition III-C is natural in B, that is, for every morphism $\varphi\colon B' \to B$ in $\mathbf{PHoops}$, the diagram

$$\begin{tikzcd}
\operatorname{SplExt}_{\mathrm{ss}}(B, X) \arrow[r, "\tau_B"] \arrow[d, "\varphi^*"'] & \operatorname{EAct}_{\mathrm{ss}}(B, X) \arrow[d, "\operatorname{EAct}_{\mathrm{ss}}(\varphi, X)"] \\
\operatorname{SplExt}_{\mathrm{ss}}(B', X) \arrow[r, "\tau_{B'}"] & \operatorname{EAct}_{\mathrm{ss}}(B', X)
\end{tikzcd}$$

commutes. Indeed, both the compositions $\operatorname{EAct}_{ss}(\varphi, X) \circ \tau_B$ and $\tau_{B'} \circ \varphi^*$ sends a split extension with strong section

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] & Y \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "\iota_2"]
\end{tikzcd}$$

to the strong external action $f', g'\colon B' \times X \to X$ defined by

$$f'(b', x) = f(\varphi(b'), x) \;\text{ and }\; g'(b', x) = g(\varphi(b'), x),$$

for any $b' \in B'$ and $x \in X$. □

IV. CONCLUSIONS

In this manuscript, we analyzed the notion of internal actions and split extensions in the category of product hoops, with particular attention to those extensions that strongly split. By characterizing these split extensions in terms of strong external actions, we provided a categorical framework that extends classical algebraic concepts, such as semidirect products, into the realm of product hoops. Future research could focus on leveraging properties of product algebras or exploring computational approaches to construct new examples of split extensions with strong section. Furthermore, one could try to generalize the description of (strong) external actions in other varieties of algebras which are strictly connected with fuzzy logic, such as Gödel algebras and MV-algebras [8], [9], which represent the equivalent algebraic semantics of Gödel logic and Łukasiewicz logic respectively.

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