

A Hawkes model with CARMA(p,q) intensity

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Abstract

In this paper we introduce a new model, named CARMA(p,q)-Hawkes, as the Hawkes model with exponential kernel implies a strictly decreasing behaviour of the autocorrelation function while empirical evidences reject its monotonicity. The proposed model is a Hawkes process where the intensity follows a Continuous Time Autoregressive Moving Average (CARMA) process and specifically is able to reproduce more realistic dependence structures. We also study the conditions of stationarity and positivity for the intensity and the strong mixing property for the increments. Furthermore we compute the likelihood, present a simulation method and discuss an estimation approach based on the autocorrelation function.

Keywords: Point processes, Autocorrelation, CARMA, Hawkes

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1. Introduction

Point processes are useful mathematical models that describe the dynamics of observed event times and have been studied and applied in several fields of study from queueing theory to forestry statistics. Among the family of point processes the Hawkes (1971a,b) process is widely the most established and widespread model in different areas, especially in quantitative finance, actuarial science and seismology (see Ogata 1988 and references therein for further details). Indeed the Hawkes process is particularly interesting since it is a self-exciting process, which means that each arrival excites the intensity such that the probability of the next arrival is increased for some period after the jump, and consequently it is well-suited to investigate, for instance, natural clustering effects and bank default in time. To show the versatility of the Hawkes process we mention a

few other possible non-financial and non-insurance applications: a) social science area e.g., Mohler et al. (2011) for the modeling of urban crime and Boumezoued (2016) for the population dynamics; b) social media sector as done in Rizoiu et al. (2017); and c) the modeling of disease spreading such as COVID-19 transmission as discussed in Chiang et al. (2022).

Recently the Hawkes process has gained a relevant role in financial modeling, in particular in the field of market microstructure. As a matter of fact it is used to model market activity, especially order arrivals in the limit order book (e.g., Bacry et al., 2013; Muni Toke and Yoshida, 2017; Clinet and Yoshida, 2017). For a complete overview of applications of the Hawkes process in finance, we suggest the works of Bacry et al. (2015) and Hawkes (2018). The Hawkes process has aroused its appeal among researchers and practitioners as well as in the insurance area. Indeed, as mentioned in Lesage et al. (2022), insurance companies are interested in point processes for the quantification of regulatory capital and in managing risks (e.g., computing ruin probabilities and measuring the effect of cyber-attacks as discussed respectively in Cheng and Seol 2020 and Bessy-Roland et al. 2021). Swishchuk et al. (2021) show that the use of a Hawkes process with exponential kernel for modeling insurance claim occurrences provides an improvement over the fit of a classical Poisson model. However, they are not able to fit different empirical autocorrelation functions as exhibited in Swishchuk et al. (2021, Figures 3 and 5, p. 112).

As stated in Errais et al. (2010), the Hawkes process with exponential kernel is Markovian and shows a good level of tractability that makes it useful for real applications in the presence of large data sets (e.g., high-frequency market data). The specification of the kernel restricts the shape of the time dependence structure of the number of jumps observed in intervals with same length. Indeed, as observed in Da Fonseca and Zaatour (2014), the autocorrelation in a Hawkes model is a decaying function of lags which is not flexible enough to represent the dependence structure observed in many data sets (e.g., wind speed data in which the exponential autocorrelation overshoots the empirical one for small lags and vice versa for large lags as documented in Benth and Rohde 2019; and, as shown in Hitaj et al. 2019, mortality rates where the empirical autocorrelation function of the shock term appears to be non-monotonic).

To overcome the aforementioned drawback, in this paper we introduce a new model named CARMA(p,q)-Hawkes process. The proposed model is a Hawkes process where the intensity follows a Continuous Time Autoregressive Moving Average (CARMA) process and it is able to provide several shapes of the autocorrelation function as it removes the monotonicity constraint detected in the standard Hawkes process. The greater flexibility relies on the CARMA(p,q) component of our model, especially in the choice of the autoregressive and moving average parameters. The CARMA process has been introduced in Doob (1944) and it is the continuous time version of the ARMA model. The advantage of the CARMA process, other than to design different shapes of autocorrelation functions, is to handle better irregular time series with respect to the ARMA process, especially for high-frequency market data, as discussed in Marquardt and Stelzer (2007) and Tómasson (2015). As a matter of fact, the CARMA model has found many applications in the

literature. Here, we list a few of these applications: a) Andresen et al. (2014) use a CARMA(p,q) model for short and forward interest rates, while b) Hitaj et al. (2019) employ such a model in order to capture the dynamics of the shock term in mortality modeling; c) Benth et al. (2014) consider a non-Gaussian CARMA process for the dynamics of spot and derivative prices in electricity markets; and d) Mercuri et al. (2021) provide formulas for the futures term structure and options written on futures in the framework of a CARMA(p,q) model driven by a time-changed Brownian motion. As remarked in Iacus and Mercuri (2015), CARMA models have manifold interests: they can be used to describe directly the dynamics of time series and to construct the variance process in continuous time models (see Brockwell et al. 2006 and Iacus et al. 2017, 2018 for further details). Our paper presents a different type of application as we use CARMA(p,q) models for the intensity of a point process.

In this paper, after reviewing the basic notions of the Hawkes and the CARMA processes, we introduce the CARMA(p,q)-Hawkes process and study the conditions of stationarity and positivity for the intensity, the autocorrelation function of the process and prove the strong mixing property of increments that leads us to the asymptotic distribution of the empirical autocorrelation function.

The remainder of the paper is organized as follows. Section 2 reviews the Hawkes process with exponential kernel while Section 3 presents the CARMA(p,q) model in the Lévy setting. Section 4 introduces the CARMA(p,q)-Hawkes process. Section 5 focuses on the autocorrelation function of the jumps in the proposed model and its asymptotic distribution, while Section 6 presents a simulation and an estimation exercise. Section 7 concludes the paper.

2. The Hawkes Process

Point processes are useful to describe the dynamics of observed event times, i.e., a collection of realizations $\{t_i\}_{i=1}^{\infty}$, $t_i \geq 0$ for $i = 1, 2, \dots$ with $t_0 := 0$ of the non-decreasing non-negative process $\{T_i\}_{i \geq 1}$ called the time arrival process. The counting process N_t , representing the number of events up to time t , is obtained from the time arrival process as follows:

$$N_t := \sum_{i \geq 1} \mathbb{1}_{\{T_i \leq t\}} \quad (1)$$

for $t \geq 0$ with associated filtration $(\mathcal{F}_t)_{t \geq 0}$ that contains the information of the counting process N_t up to time t . An important quantity when dealing with a point process N_t is the conditional intensity λ_t defined as:

$$\lambda_t = \lim_{\Delta \rightarrow 0^+} \frac{\Pr[N_{t+\Delta} - N_t = 1 | \mathcal{F}_t]}{\Delta}.$$

It then follows that the counting process satisfies the following properties

$$\Pr [N_{t+\Delta} - N_t = \eta | \mathcal{F}_t] = \begin{cases} 1 - \lambda_t \Delta + o(\Delta) & \text{if } \eta = 0 \\ \lambda_t \Delta + o(\Delta) & \text{if } \eta = 1 \\ o(\Delta) & \text{if } \eta > 1 \end{cases}$$

The conditional intensity λ_t of a general self-exciting process has the following form:

$$\lambda_t = \mu + \int_0^t h(t-s) dN_s \quad (2)$$

with baseline intensity parameter $\mu > 0$ and (excitation) kernel function $h(t) : [0, +\infty) \rightarrow [0, +\infty)$ that represents the contribution to the intensity at time t that is made by an event occurred at a previous time $T_i < t$. Following the general results about the Hawkes process in Brémaud and Massoulié (1996), the stationary condition reads:

$$\int_0^{+\infty} h(t) dt < 1. \quad (3)$$

The most used kernel is the exponential function and specifically $h(t) = \alpha e^{-\beta t}$ with $\alpha, \beta \geq 0$. The stationary condition in (3) implies $\beta > \alpha$ while to prove the Markovianity of the couple (λ_t, N_t) it is enough to rewrite the intensity for any $s < t$ as

$$\lambda_t = \mu + e^{-\beta(t-s)} \int_0^s \alpha e^{-\beta(s-l)} dN_l + \int_s^t \alpha e^{-\beta(t-l)} dN_l.$$

Observing that $\int_0^s \alpha e^{-\beta(s-l)} dN_l = \lambda_s - \mu$, thus

$$\lambda_t = \mu + e^{-\beta(t-s)} (\lambda_s - \mu) + \int_s^t \alpha e^{-\beta(t-l)} dN_l. \quad (4)$$

From (4) we have that the distribution of the intensity λ_t given the information at time s depends only upon λ_s and on the increments of the counting process over the interval $[s, t)$, which depend on the conditional intensity itself implying that the couple (λ_t, N_t) is itself Markovian. The intensity λ_t is the solution of the following differential equation:

$$d\lambda_t = \beta(\mu - \lambda_t) dt + \alpha dN_t, \quad \text{with } \lambda_0 = \mu.$$

Exploiting the Markovianity of the process $X_t := (\lambda_t, N_t)$, it is possible to get the infinitesimal generator (see Errais et al. 2010 and Da Fonseca and Zaatour 2014 for further details) associated to a function $f : \mathbb{R}_+ \times \mathbb{N} \rightarrow \mathbb{R}$ with continuous partial derivatives with respect to the first argument

$\frac{\partial f}{\partial \lambda}(x)$. Starting from the definition of the infinitesimal operator for a Markov process X_t , that is:

$$\mathcal{A}f := \lim_{\Delta \rightarrow 0^+} \frac{\mathbb{E}[f(X_{t+\Delta}) | \mathcal{F}_t] - f(X_t)}{\Delta},$$

Errais et al. (2010) compute the infinitesimal generator for the Hawkes process with exponential kernel that reads

$$\mathcal{A}f = \beta(\mu - \lambda_t) \frac{\partial f}{\partial \lambda}(\lambda_t, N_t) + \lambda_t [f(\lambda_t + \alpha, N_t + 1) - f(\lambda_t, N_t)]. \quad (5)$$

For every function f in the domain of the infinitesimal generator it is possible to build a martingale process M_t with respect to the natural filtration in the following way:

$$M_t = f(\lambda_t, N_t) - f(\lambda_0, N_0) - \int_0^t \mathcal{A}f(\lambda_s, N_s) ds$$

that leads to the well-known Dynkin's formula

$$\mathbb{E}[f(\lambda_t, N_t) | \mathcal{F}_s] = f(\lambda_s, N_s) + \mathbb{E} \left[\int_s^t \mathcal{A}f(\lambda_u, N_u) du \middle| \mathcal{F}_s \right], \quad \forall t > s.$$

The above formula for $f \equiv N_t$ has been used in Da Fonseca and Zaatour (2014) to compute the moments and the autocovariance function of jump increments observed in intervals of length τ with lag δ .

Proposition 1. *Consider four time instants $t_1 = t$, $t_2 = t + \tau$, $t_3 = t + \tau + \delta$ and $t_4 = t + 2\tau + \delta$, the following equalities for the Hawkes model are obtained (see Da Fonseca and Zaatour 2014 for further details).*

1. *The long-run expected value of the number of jumps during an interval of length τ is*

$$\mathbb{E}(\Delta_\tau N_\infty) := \lim_{t \rightarrow +\infty} \mathbb{E}[N_{t+\tau} - N_t] = \frac{\mu}{1 - \frac{\alpha}{\beta}} \tau. \quad (6)$$

2. *The long-run variance of the increments reads*

$$\begin{aligned} \text{Var}(\tau) &:= \lim_{t \rightarrow +\infty} \mathbb{E}[(N_{t+\tau} - N_t)^2] - \mathbb{E}[N_{t+\tau} - N_t]^2 \\ &= \frac{\mu}{1 - \frac{\alpha}{\beta}} \left(\tau \left(\frac{1}{1 - \frac{\alpha}{\beta}} \right)^2 + \left(1 - \left(\frac{1}{1 - \frac{\alpha}{\beta}} \right)^2 \right) \frac{1 - e^{\tau(\beta - \alpha)}}{\beta - \alpha} \right). \end{aligned} \quad (7)$$

3. *The long-run covariance of the number of arrivals for two non-overlapping intervals of length*

τ with lag $\delta > 0$ is

$$\begin{aligned} Cov(\tau, \delta) &:= \lim_{t \rightarrow +\infty} \mathbb{E}[(N_{t+\tau} - N_t)(N_{t+2\tau+\delta} - N_{t+\tau+\delta})] - \mathbb{E}[N_{t+\tau} - N_t] \mathbb{E}[N_{t+2\tau+\delta} - N_{t+\tau+\delta}] \\ &= \frac{\mu\beta\alpha(2\beta - \alpha)(e^{(\alpha-\beta)\tau} - 1)^2}{2(\alpha - \beta)^4} e^{(\alpha-\beta)\delta}. \end{aligned} \quad (8)$$

4. The long-run autocorrelation function of the number of jumps over intervals of length τ separated by a time lag of δ reads

$$Acf(\tau, \delta) = \frac{e^{-2\beta\tau}(e^{\alpha\tau} - e^{\beta\tau})^2\alpha(\alpha - 2\beta)}{2(\alpha(\alpha - 2\beta)(e^{(\alpha-\beta)\tau} - 1) + \beta^2\tau(\alpha - \beta))} e^{(\alpha-\beta)\delta}, \quad (9)$$

and is always positive for $\alpha < \beta$ (stationarity condition) and exponentially decaying with the lag δ .

From (8) and (9) it is clear that the Hawkes model with exponential kernel can reproduce only strictly decreasing autocorrelation functions for varying lag values δ . An interesting extension is given in Boswijk et al. (2018) where self-excitation is identified through the modeling of common jumps between the log price process and its own jump intensity.

3. Lévy CARMA(p,q) models

The formal definition of a Lévy CARMA(p,q) model Y_t with $p > q \geq 0$ is based on the continuous version of the state-space representation of an autoregressive moving average ARMA(p,q) model. In particular we have that

$$Y_t = \mathbf{b}^\top X_t \quad (10)$$

where X_t satisfies the following stochastic differential equation

$$dX_t = \mathbf{A}X_t dt + \mathbf{e}dZ_t, \quad (11)$$

and $\{Z_t\}_{t \geq 0}$ is a Lévy process. The $p \times p$ matrix \mathbf{A} has the following form

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \dots & -a_1 \end{bmatrix}_{p \times p}, \quad (12)$$

and the $p \times 1$ vectors \mathbf{e} and \mathbf{b} are defined as follows

$$\mathbf{e} = [0, 0, \dots, 1]^\top \quad (13)$$

$$\mathbf{b} = [b_0, b_1, \dots, b_{p-1}]^\top \quad (14)$$

with $b_{q+1} = \dots = b_{p-1} = 0$. Given a starting value for X_s , the solution of (11) is

$$X_t = e^{\mathbf{A}(t-s)} X_s + \int_s^t e^{\mathbf{A}(t-u)} dZ_u, \quad \forall t > s$$

where $e^{\mathbf{A}} = \sum_{h=0}^{+\infty} \frac{1}{h!} \mathbf{A}^h$.

As reported in Brockwell et al. (2011, Section 2, p. 251), under the assumption that ensures the stationarity of X_t (i.e., all eigenvalues $\tilde{\lambda}_1, \dots, \tilde{\lambda}_p$ of matrix \mathbf{A} are distinct with negative real part¹), the CARMA(p,q) model can be written as a summation of a finite number of continuous autoregressive models of order 1, which are also known as CAR(1) models. Specifically,

$$Y_t = \mathbf{b}^\top e^{\mathbf{A}(t-s)} X_s + \int_0^{+\infty} \sum_{i=1}^p \left[\alpha(\tilde{\lambda}_i) e^{\tilde{\lambda}_i(t-u)} \right] \mathbb{1}_{s \leq u \leq t} dZ_u \quad (15)$$

where $\alpha(z) = \frac{b(z)}{a^{(1)}(z)}$ and the polynomials $a(z)$ and $b(z)$ are defined as

$$a(z) := z^p + a_1 z^{p-1} + \dots + a_p \quad \text{and} \quad b(z) := b_0 + b_1 z + \dots + b_{p-1} z^{p-1}.$$

Note that $a^{(1)}(z)$ is the first derivative of the polynomial $a(z)$.

Remark 1. The eigenvalues of matrix \mathbf{A} denoted by $\tilde{\lambda}_1, \dots, \tilde{\lambda}_p$ are the same as the zeros of the autoregressive polynomial $a(z)$. As observed in Tsai and Chan (2005), the assumption that the zeros of $a(z)$ have negative real parts is a necessary condition for the stationarity of the CARMA(p,q) process Y_t .

Definition 1. A stationary CARMA(p,q) process Y_t where Z is a second-order subordinator can be equivalently defined as:

$$Y_t = \int_{-\infty}^{+\infty} h(t-u) dZ_u$$

where the function $h(t) = \mathbf{b}^\top e^{\mathbf{A}} \mathbb{1}_{[0,+\infty)}(t) \mathbf{e}$ is the kernel of the CARMA(p,q) process. As Y_t is independent of $Z_s - Z_t$, $\forall s \geq t$, the process Y_t is said to be a casual function of the subordinator Z_t , also known as casual CARMA(p,q) model.

In Brockwell and Marquardt (2005) it is shown that the function $h(u)$ can be written as

$$h(u) = \sum_{i=1}^p \frac{b(\tilde{\lambda}_i)}{a^{(1)}(\tilde{\lambda}_i)} e^{\tilde{\lambda}_i u} \mathbb{1}_{\{0 < u < +\infty\}}. \quad (16)$$

¹The eigenvalues are sorted based on their real part in an increasing order.

The positivity conditions for the kernel and for the process itself, which is for instance required for modeling the volatility using CARMA(p,q) models, have been deeply investigated in Tsai and Chan (2005) and in Benth and Rohde (2019) for the case of positive subordinators.

4. CARMA(p,q)-Hawkes model

In this section, we introduce a point process where the intensity follows a CARMA(p,q) process that is a generalization of the Hawkes process with an exponential kernel.

4.1. CARMA(p,q)-Hawkes: stationarity and positivity conditions for the intensity

Definition 2. A vector process $[X_{1,t}, \dots, X_{1,p}, N_t]^\top$ of dimension $p+1$ is a CARMA(p,q)-Hawkes process if the conditional intensity λ_t of the counting process N_t is a CARMA(p,q) process driven by N_t and has the following form:

$$\lambda_t = \mu + \mathbf{b}^\top X_t, \quad (17)$$

in which the baseline parameter μ is strictly positive and the vector \mathbf{b} is defined as in (14). The vector $X_t = [X_{1,t}, \dots, X_{1,p}]^\top$ satisfies the linear stochastic differential equation

$$dX_t = \mathbf{A}X_t - dt + \mathbf{e}dN_t \text{ with } X_0 = \mathbf{0} \quad (18)$$

where the companion matrix \mathbf{A} and the vector \mathbf{e} have respectively the form in (12) and (13).

The dynamics of the state space process X_t in (18) is described through a linear stochastic differential equation and it is a Markov process. Consequently, a CARMA(p,q)-Hawkes process is in turn a Markov process.

Remark 2. The stochastic differential equation in (18) has an analytical solution given the initial condition, that is

$$X_t = \int_0^t e^{\mathbf{A}(t-s)} \mathbf{e} dN_s. \quad (19)$$

The non-decreasing and non-negative trajectories of the counting process N_t imply the positiveness of λ_t for non-negative kernel functions.

To investigate the stationary regime of a CARMA(p,q)-Hawkes model, it is necessary to determine the conditions required for a non-negative kernel, i.e., $\mathbf{b}^\top e^{\mathbf{A}t} \mathbf{e} \geq 0, \forall t \geq 0$. In case of a CARMA(p,q) driven by a non-negative Lévy process the conditions of a non-negative kernel are presented in Tsai and Chan (2005, Theorem 1, p. 592). In a similar fashion such conditions can be applied directly to our case due to the non-negative trajectories of the counting process N_t . Indeed, as done in Brockwell et al. (2006, Theorem 5.2) for COGARCH(p,q) models, we rephrase their results for a generic CARMA(p,q)-Hawkes process when $b_0 > 0$ in the next proposition.

Proposition 2.

- (a) For a CARMA(p, q)-Hawkes process such that the real part of all eigenvalues of \mathbf{A} is negative, the kernel function $h(t) := \mathbf{b}^\top e^{\mathbf{A}t} \mathbf{e} \mathbb{1}_{\{t \geq 0\}}$ is non-negative if and only if the ratio function $\frac{b(z)}{a(z)}$ is completely monotone² on $(0, +\infty)$.
- (b) A sufficient condition for the kernel function of a CAR(p)-Hawkes process to be non-negative is that all eigenvalues of \mathbf{A} are real and negative.
- (c) A sufficient condition for the kernel function of a CAR(p)-Hawkes process to be non-negative is that if $(\tilde{\lambda}_{i_1}, \tilde{\lambda}_{i_1+1}), \dots, (\tilde{\lambda}_{i_r}, \tilde{\lambda}_{i_r+1})$ is a partition of the set of all pairs of complex conjugate eigenvalues of \mathbf{A} (counted with multiplicity), then there exists an injective mapping $u : \{1, \dots, r\} \rightarrow \{1, \dots, p\}$ such that $\tilde{\lambda}_{u(j)}$ real eigenvalue of \mathbf{A} satisfies $\tilde{\lambda}_{u(j)} \geq \operatorname{Re}(\tilde{\lambda}_{i_j})$.
- (d) For a non-negative kernel $h(t)$ in a CAR(p)-Hawkes process, it is necessary to find a real eigenvalue $\tilde{\lambda}_i$ such that $\tilde{\lambda}_i \geq \operatorname{Re}(\tilde{\lambda}_j)$ where $j = 1, \dots, p$ with $j \neq i$.
- (e) Suppose all eigenvalues of \mathbf{A} are negative real numbers sorted as follows $\tilde{\lambda}_p \leq \dots \leq \tilde{\lambda}_1$ and that all the roots of $b(z) = 0$ are negative real numbers such that $\gamma_q \leq \dots \leq \gamma_1 < 0$. If $\sum_{i=1}^k \gamma_i \leq \sum_{i=1}^k \tilde{\lambda}_i$ for $1 \leq k \leq q$, then the kernel of a CARMA(p, q)-Hawkes process is non-negative.
- (f) A necessary and sufficient condition for a non-negative $h(t)$ in a CARMA(2,1)-Hawkes process is that $\tilde{\lambda}_2 \leq \tilde{\lambda}_1 < 0$ and $b_0 + \tilde{\lambda}_1 b_1 \geq 0$ with $b_1 \geq 0$.

We remark that the non-negativity requirement for the kernel implies a strictly positive intensity process λ_t as the baseline parameter μ is strictly positive.

Without loss of generality, we assume that matrix \mathbf{A} is diagonalizable which corresponds to the assumption that the eigenvalues of \mathbf{A} are distinct. The eigenvectors of \mathbf{A} are

$$\left[1, \tilde{\lambda}_j, \dots, \tilde{\lambda}_j^{p-1}\right]^\top, \quad j = 1, \dots, p$$

used to define a $p \times p$ matrix \mathbf{S} as

$$\mathbf{S} := \begin{bmatrix} 1 & \dots & 1 \\ \tilde{\lambda}_1 & \dots & \tilde{\lambda}_p \\ \tilde{\lambda}_1^2 & \dots & \tilde{\lambda}_p^2 \\ \vdots & & \vdots \\ \tilde{\lambda}_1^{p-1} & \dots & \tilde{\lambda}_p^{p-1} \end{bmatrix}.$$

²A function $f(x)$ defined on $(0, +\infty)$ is said to be completely monotone if and only if it has derivatives of all orders and $(-1)^n \frac{\partial^n f(t)}{\partial x^n} \geq 0$ for $n = 0, 1, 3, \dots$

It follows that \mathbf{S} satisfies $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \text{diag}(\tilde{\lambda}_1, \dots, \tilde{\lambda}_p)$, a result used to prove the next proposition on the stationarity conditions for a CARMA(p,q)-Hawkes process.

Proposition 3. *Let us consider a non-negative kernel function and suppose $\mu > 0$. Then a CARMA(p,q)-Hawkes $(X_{1,t}, \dots, X_{p,t}, N_t)$ is a stationary process if all eigenvalues of \mathbf{A} are distinct with non-negative real part and $-\mathbf{b}^\top \mathbf{A}^{-1} \mathbf{e} < 1$.*

Proof. For a non-negative kernel function, the stationary condition in (3) for a CARMA(p,q)-Hawkes process becomes

$$\int_0^{+\infty} \mathbf{b}^\top e^{\mathbf{A}t} \mathbf{e} dt = \lim_{T \rightarrow +\infty} \int_0^T \mathbf{b}^\top e^{\mathbf{A}t} \mathbf{e} dt = \lim_{T \rightarrow +\infty} \mathbf{b}^\top \mathbf{A}^{-1} (e^{\mathbf{A}T} - \mathbf{I}) \mathbf{e}, \quad (20)$$

where \mathbf{I} is the identity matrix with dimension p . As \mathbf{A} is diagonalizable,

$$e^{\mathbf{A}T} = \mathbf{S} e^{\mathbf{\Lambda}T} \mathbf{S}^{-1}$$

where $\mathbf{\Lambda} := \text{diag}(\tilde{\lambda}_1, \dots, \tilde{\lambda}_p)$. Thus the limit in (20) is

$$\lim_{T \rightarrow +\infty} \mathbf{b}^\top \mathbf{A}^{-1} (e^{\mathbf{A}T} - \mathbf{I}) \mathbf{e} = \mathbf{b}^\top \mathbf{A}^{-1} \left[\mathbf{S} \left(\lim_{T \rightarrow +\infty} e^{\mathbf{\Lambda}T} \right) \mathbf{S}^{-1} - \mathbf{I} \right] \mathbf{e}.$$

Recalling that all eigenvalues of \mathbf{A} have negative real part, we notice that $e^{\mathbf{\Lambda}T}$ tends to a $p \times p$ zero matrix. The integral in (20) becomes

$$\int_0^{+\infty} \mathbf{b}^\top e^{\mathbf{A}t} \mathbf{e} dt = -\mathbf{b}^\top \mathbf{A}^{-1} \mathbf{e}. \quad (21)$$

The stationarity condition in (3) implies $-\mathbf{b}^\top \mathbf{A}^{-1} \mathbf{e} < 1$. □

Assumption 1. *We shall assume for the remainder of the paper that: i) the kernel is a non-negative function and $\mu > 0$; and ii) all eigenvalues of \mathbf{A} are distinct with negative real part and $\mathbf{b}^\top \mathbf{A}^{-1} \mathbf{e} > -1$.*

For practical applications, instead of checking ex-post signs of eigenvalues of matrix \mathbf{A} , it is possible to enforce ex-ante the negativity of the real part for eigenvalues using some transformations on the parameters space as done, for example, in Tómasson (2015). As a CARMA(p,q)-Hawkes process is Markovian, we are able to calculate the infinitesimal operator as described in the following proposition.

Proposition 4. *Let $f : \mathbb{R}^p \times \mathbb{N} \rightarrow \mathbb{R}$ be a function with continuous partial derivatives with respect to the first p arguments. Under the same conditions in Assumption 1, the infinitesimal generator*

of function f for a CARMA(p,q)-Hawkes process is:

$$\begin{aligned} \mathcal{A}f_t &= \lambda_t [f(X_{1,t}, \dots, X_{p,t} + 1, N_t + 1) - f(X_{1,t}, \dots, X_{p,t}, N_t)] \\ &+ \sum_{i=1}^{p-1} \frac{\partial f}{\partial X_{i,t}} X_{i+1,t} + \frac{\partial f}{\partial X_{p,t}} \mathbf{A}_{[p,]} X_t \end{aligned} \quad (22)$$

where $\mathbf{A}_{[p,]}$ is the p -th row of the companion matrix \mathbf{A} and the intensity process λ_t is defined as in (17). Alternatively, the infinitesimal generator can be written as

$$\mathcal{A}f_t = \lambda_t [f(X_{1,t}, \dots, X_{p,t} + 1, N_t + 1) - f(X_{1,t}, \dots, X_{p,t}, N_t)] + \nabla_p f^\top \mathbf{A} X_t \quad (23)$$

where $\nabla_p f := \left[\frac{\partial f}{\partial X_{1,t}}, \dots, \frac{\partial f}{\partial X_{p,t}} \right]^\top$.

Proof. Let us consider two cases. If $N_{T+h} - N_T = 0$, the vector $X_T = [X_{1,t}, \dots, X_{p,t}]^\top$ becomes $X_{T+h} = X_{T+h}^{\text{NJ}}$ where X_{T+h}^{NJ} means no jump (NJ) occurred in the interval $(T, T+h]$ and can be written in the following way

$$X_{T+h}^{\text{NJ}} = e^{\mathbf{A}(T+h-t_0)} X_{t_0} + \int_{t_0}^T e^{\mathbf{A}(T+h-t)} \mathbf{e} dN_t$$

as the quantity $\int_T^{T+h} e^{\mathbf{A}(T+h-t)} \mathbf{e} dN_t$ is zero due to the absence of jumps in the interval $(T, T+h]$. From

$$X_{T+h}^{\text{NJ}} = e^{\mathbf{A}h} \left[e^{\mathbf{A}(T-t_0)} X_{t_0} + \int_{t_0}^T e^{\mathbf{A}(T-t)} \mathbf{e} dN_t \right] = e^{\mathbf{A}h} X_T$$

we have that

$$\lim_{h \rightarrow 0} X_{T+h}^{\text{NJ}} = X_T. \quad (24)$$

If $N_{T+h} - N_T = 1$ then $X_{T+h} := X_{T+h}^{\text{1J}}$ is computed as

$$X_{T+h}^{\text{1J}} = e^{\mathbf{A}(T+h-t_0)} X_{t_0} + \int_{t_0}^T e^{\mathbf{A}(T+h-t)} \mathbf{e} dN_t + \int_T^{T+h} e^{\mathbf{A}(T+h-t)} \mathbf{e} dN_t.$$

Defining the jump time T_h in the time interval $(T, T+h]$ we get

$$\int_T^{T+h} e^{\mathbf{A}(T+h-t)} \mathbf{e} dN_t = e^{\mathbf{A}(T+h-T_h)} \mathbf{e}.$$

As $\lim_{h \rightarrow 0} T_h = T$, we observe that

$$\lim_{h \rightarrow 0} X_{T+h}^{\text{1J}} = \left[e^{\mathbf{A}(T-t_0)} X_{t_0} + \int_{t_0}^T e^{\mathbf{A}(T-t)} \mathbf{e} dN_t \right] + \mathbf{e} = X_T + \mathbf{e}. \quad (25)$$

Note that $X_t + \mathbf{e} = [X_{t,1}, \dots, X_{t,p} + 1]^\top$ and consider the following quantity:

$$\begin{aligned} \mathbb{E} [f(X_{1,t+h}, \dots, X_{p,t+h}, N_{t+h}) | \mathcal{F}_t] &= f(X_{1,t+h}^{\text{NJ}}, \dots, X_{p,t+h}^{\text{NJ}}, N_t) (1 - \lambda_t h) \\ &+ f(X_{1,t+h}^{\text{IJ}}, \dots, X_{p,t+h}^{\text{IJ}}, N_t + 1) \lambda_t h + o(h). \end{aligned}$$

The infinitesimal generator is:

$$\begin{aligned} \mathcal{A}f_t &:= \lim_{h \rightarrow 0} \frac{\mathbb{E} [f(X_{1,t+h}, \dots, X_{p,t+h}, N_{t+h}) | \mathcal{F}_t] - f(X_{1,t}, \dots, X_{p,t}, N_t)}{h} \\ &= \lim_{h \rightarrow 0} \lambda_t [f(X_{1,t+h}^{\text{IJ}}, \dots, X_{p,t+h}^{\text{IJ}}, N_t + 1) - f(X_{1,t+h}^{\text{NJ}}, \dots, X_{p,t+h}^{\text{NJ}}, N_t)] \\ &+ \lim_{h \rightarrow 0} \frac{f(X_{1,t+h}^{\text{NJ}}, \dots, X_{p,t+h}^{\text{NJ}}, N_t) - f(N_t, X_{1,t}, \dots, X_{p,t})}{h}. \end{aligned}$$

From (24) and (25) we obtain

$$\begin{aligned} \mathcal{A}f_t &:= \lambda_t [f(X_{1,t}, \dots, X_{p,t} + 1, N_t + 1) - f(X_{1,t}, \dots, X_{p,t}, N_t)] \\ &+ \lim_{h \rightarrow 0} \frac{f(X_{1,t+h}^{\text{NJ}}, \dots, X_{p,t+h}^{\text{NJ}}, N_t) - f(X_{1,t}, \dots, X_{p,t}, N_t)}{h}. \end{aligned} \quad (26)$$

To compute the limit (26) we use De l'Hôpital's rule

$$\begin{aligned} \lim_{h \rightarrow 0} \sum_{i=1}^p \frac{\partial f}{\partial X_{i,t+h}^{\text{NJ}}} \frac{\partial X_{i,t+h}^{\text{NJ}}}{\partial h} &= \lim_{h \rightarrow 0} \left[\frac{\partial f}{\partial X_{1,t+h}^{\text{NJ}}}, \dots, \frac{\partial f}{\partial X_{p,t+h}^{\text{NJ}}} \right] \mathbf{A} e^{\mathbf{A}h} X_t \\ &= \sum_{i=1}^{p-1} \frac{\partial f}{\partial X_{i,t}} X_{i+1,t} + \frac{\partial f}{\partial X_{p,t}} \mathbf{A}_{[p, \cdot]} X_t, \end{aligned} \quad (27)$$

and substituting (27) in (26), we finally obtain the result in (23). \square

The conditional expected value for $f(X_{1,T}, \dots, X_{p,T}, N_T)$ can be computed applying the Dynkin's formula:

$$\mathbb{E} [f(X_{1,T}, \dots, X_{p,T}, N_T) | \mathcal{F}_{t_0}] = f(X_{1,t_0}, \dots, X_{p,t_0}, N_{t_0}) + \mathbb{E} \left[\int_{t_0}^T \mathcal{A}f_t dt | \mathcal{F}_{t_0} \right] \quad (28)$$

that has a representation of the following form

$$d\mathbb{E} [f(X_{1,t}, \dots, X_{p,t}, N_t) | \mathcal{F}_{t_0}] = \mathbb{E} [\mathcal{A}f_t | \mathcal{F}_{t_0}] dt, \quad (29)$$

with initial condition $f(X_{1,t_0}, \dots, X_{p,t_0}, N_{t_0})$. We use the infinitesimal generator (23) and the result in (28) to obtain the following proposition for the computation of the first moment of the counting process N_t . In the remainder of the paper, we use $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]$.

Proposition 5. Let $\tilde{\mathbf{A}}$ be a $p \times p$ companion matrix where the last row has the following structure

$$\tilde{\mathbf{A}}_{[p,\cdot]} = [b_0 - a_p, b_1 - a_{p-1}, \dots, b_{p-1} - a_1]. \quad (30)$$

Under Assumption 1 and supposing that all eigenvalues of $\tilde{\mathbf{A}}$ are distinct with negative real part, for any $T > t_0 \geq 0$, the conditional first moment of the counting process is

$$\mathbb{E}_{t_0} [N_T] = N_{t_0} + \mu \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) (T - t_0) + \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}(T-t_0)} - \mathbf{I} \right] \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right], \quad (31)$$

while the conditional expected value of the state process X_T is

$$\mathbb{E}_{t_0} [X_T] = e^{\tilde{\mathbf{A}}(T-t_0)} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] - \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu. \quad (32)$$

The quantities in (31) and (32) satisfy respectively the following ordinary differential equations:

$$d\mathbb{E}_{t_0} [N_t] = \left[\mu \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) + \mathbf{b}^\top e^{\tilde{\mathbf{A}}(t-t_0)} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \right] dt \quad (33)$$

and

$$d\mathbb{E}_{t_0} [X_t] = \left(\tilde{\mathbf{A}} \mathbb{E}_{t_0} [X_t] + \mu \mathbf{e} \right) dt \quad (34)$$

with initial conditions³ X_{t_0} and N_{t_0} . The long-run value for $\mathbb{E}_{t_0} [X_T]$ is obtained as follows

$$\mathbb{E} [X_\infty] := \lim_{T \rightarrow +\infty} \mathbb{E}_{t_0} [X_T] = -\tilde{\mathbf{A}}^{-1} \mathbf{e} \mu. \quad (35)$$

Moreover, the expected number of events that occurs in an interval with length τ , i.e., $(T, T + \tau]$, given the information at time $t_0 < T$ is

$$\mathbb{E}_{t_0} [(N_{T+\tau} - N_T)] = \mu \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \tau + \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} e^{\tilde{\mathbf{A}}(T-t_0)} \left(e^{\tilde{\mathbf{A}}\tau} - \mathbf{I} \right) \left(X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right) \quad (36)$$

and the stationary behaviour of (36) is

$$\mathbb{E} [\Delta_\tau N_\infty] := \lim_{T \rightarrow +\infty} \mathbb{E}_{t_0} [N_{T+\tau} - N_T] = \mu \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \tau, \quad \forall \tau > 0. \quad (37)$$

Proof. To determine the expected number of jumps in (31) we obtain first the infinitesimal generator of the function $f(X_{1,t}, \dots, X_{p,t}, N_t) = N_t$, that is $\mathcal{A}f_t = \lambda_t$ where the conditional intensity λ_t is defined in (18). Applying the Dynkin's formula in (29) we obtain the following ODE

$$d\mathbb{E}_{t_0} [N_t] = \left[\mu + \mathbf{b}^\top \mathbb{E}_{t_0} (X_t) \right] dt. \quad (38)$$

³For $t_0 = 0$, then $\mathbb{E}_{t_0} [X_T] = \left(e^{\tilde{\mathbf{A}}(T-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu$ and $\mathbb{E}_{t_0} [N_T] = \mu \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) (T - t_0) + \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}(T-t_0)} - \mathbf{I} \right] \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu.$

Then, we compute $\mathbb{E}_{t_0}[X_t]$ that requires a system of infinitesimal generators. In particular, for $i = 1, \dots, p-1$, we have

$$\mathcal{A}X_{t,i} = X_{t,i+1}$$

and

$$\mathcal{A}X_{t,p} = \left(\mu + \mathbf{b}^\top X_t\right) + \mathbf{A}_{[p,\cdot]}X_t = \mu + \sum_{i=1}^p (b_{i-1} - a_{p+1-i}) X_{t,i}.$$

Applying (29), we get

$$d\mathbb{E}_{t_0}[X_t] = \left(\tilde{\mathbf{A}}\mathbb{E}_{t_0}[X_t] + \mu\mathbf{e}\right) dt \quad (39)$$

where $\tilde{\mathbf{A}}$ is defined in (30). With the initial condition X_{t_0} , the solution of the system in (39) is (32). Substituting (32) in (38) we obtain the following ODE for the expected number of jumps

$$d\mathbb{E}_{t_0}[N_t] = \left[\mu \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1}\mathbf{e}\right) + \mathbf{b}^\top e^{\tilde{\mathbf{A}}(t-t_0)} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1}\mathbf{e}\mu\right]\right] dt$$

whose solution is in (31) with initial condition N_{t_0} . Using the result in (31) we observe by straightforward calculations that the expected number of jumps in an interval of length τ reads as in (36). Due to the negativity assumption for the real part of the eigenvalues of matrix $\tilde{\mathbf{A}}$, we obtain the asymptotic behaviour in (35) and (37) as $\lim_{T \rightarrow +\infty} e^{\tilde{\mathbf{A}}T} = \mathbf{0}$ where $\mathbf{0}$ is a $p \times p$ zero matrix (see (C.5)). \square

Remark 3. *The result in (37) becomes (6) if we consider a CAR(1)-Hawkes with $b_0 = \alpha$ and $a_1 = \beta$.*

Using the same arguments in Brockwell et al. (2006, proof of Proposition 4.1, p. 815), all eigenvalues of matrix $\tilde{\mathbf{A}}$ have negative real parts if for some positive integer $r \geq 1$ the following inequality holds

$$\left\| \mathbf{S}^{-1}\mathbf{e}\mathbf{b}^\top \mathbf{S} \right\|_r < \operatorname{Re}(\tilde{\lambda}_1) \quad (40)$$

where, in this context, $\|\cdot\|_r$ denotes the natural matrix norm induced by the vector \mathbb{L}^r -norm. This result comes directly from an application of the Bauer-Fike Theorem (see Bauer and Fike 1960 for further details) since $\tilde{\mathbf{A}}$ is obtained by perturbing matrix \mathbf{A} as $\tilde{\mathbf{A}} = \mathbf{A} + \mathbf{e}\mathbf{b}^\top$.

A sufficient condition for (40) is

$$\frac{\sigma_{\max}(\mathbf{S})}{\sigma_{\min}(\mathbf{S})} \|\mathbf{b}\|_2 < \operatorname{Re}(\tilde{\lambda}_1) \quad (41)$$

where $\|\mathbf{b}\|_2 := \sqrt{\sum_{i=1}^p b_{i-1}^2}$ is the Euclidean norm of \mathbf{b} , $\sigma_{\max}(\mathbf{S})$ and $\sigma_{\min}(\mathbf{S})$ are maximal and minimal singular values of \mathbf{S} . In particular, we observe that

$$\left\| \mathbf{S}^{-1}\mathbf{e}\mathbf{b}^\top \mathbf{S} \right\|_2 \leq k_2(\mathbf{S}) \left\| \mathbf{e}\mathbf{b}^\top \right\|_2 \quad (42)$$

and that $k_2(\mathbf{S}) := \|\mathbf{S}\|_2 \|\mathbf{S}^{-1}\|_2$, the condition number in 2-norm, can be written as

$$k_2(\mathbf{S}) = \frac{\sigma_{\max}(\mathbf{S})}{\sigma_{\min}(\mathbf{S})}. \quad (43)$$

Moreover, denoting with $\|\mathbf{e}\mathbf{b}^\top\|_{\mathbb{F}}$ the Frobenius norm of $\mathbf{e}\mathbf{b}^\top$, we obtain $\|\mathbf{e}\mathbf{b}^\top\|_2 \leq \|\mathbf{e}\mathbf{b}^\top\|_{\mathbb{F}}$. Applying the definition of the Frobenius norm we have

$$\|\mathbf{e}\mathbf{b}^\top\|_2 \leq \|\mathbf{b}\|_2, \quad (44)$$

and combining (42), (43) and (44) we get

$$\|\mathbf{S}^{-1}\mathbf{e}\mathbf{b}^\top\mathbf{S}\|_2 \leq \frac{\sigma_{\max}(\mathbf{S})}{\sigma_{\min}(\mathbf{S})} \|\mathbf{b}\|_2. \quad (45)$$

Thus, the inequality in (41) implies (40).

4.2. Simulation and Likelihood Estimation of the CARMA(p,q)-Hawkes

We propose a simulation method for the CARMA(p,q)-Hawkes model following the same idea presented in Ozaki (1979, Section 4, p. 148).

Suppose that T_1, \dots, T_k , which correspond to time arrivals, are already observed. Then it is possible to simulate the next time arrival T_{k+1} by generating a random number from a standard uniform distribution, i.e., $U \sim \text{Unif}(0, 1)$, and by solving this equation with respect to u :

$$\ln(U) = - \int_{T_k}^u \lambda_t dt. \quad (46)$$

The conditional intensity λ_t can be replaced by (17) and X_t can be substituted by (19) obtaining so

$$\ln(U) = - \int_{T_k}^u \left[\mu + \mathbf{b}^\top \int_0^t e^{\mathbf{A}(t-s)} \mathbf{e} dN_s \right] dt. \quad (47)$$

Developing and rearranging the right-hand side of (47) we have

$$\begin{aligned} \ln(U) &= -\mu(u - T_k) - \mathbf{b}^\top \int_{T_k}^u \sum_{i=1}^k e^{\mathbf{A}(t-T_i)} \mathbf{e} dt \\ &= -\mu(u - T_k) - \mathbf{b}^\top \sum_{i=1}^k e^{\mathbf{A}(T_k-T_i)} \int_{T_k}^u e^{\mathbf{A}(t-T_k)} dt \mathbf{e} \\ &= -\mu(u - T_k) - \mathbf{b}^\top \left[\sum_{i=1}^k e^{\mathbf{A}(T_k-T_i)} \right] \mathbf{A}^{-1} \left[e^{\mathbf{A}(u-T_k)} - \mathbf{I} \right] \mathbf{e} \end{aligned}$$

where the integral in the second equality is computed using the results in (A.1). Defining $S(k) :=$

$\sum_{i=1}^k e^{\mathbf{A}(T_k - T_i)}$, we finally get

$$\ln(U) = -\mu(u - T_k) - \mathbf{b}^\top S(k) \mathbf{A}^{-1} \left[e^{\mathbf{A}(u - T_k)} - \mathbf{I} \right] \mathbf{e}. \quad (48)$$

The quantity $S(k)$ can be obtained recursively as follows

$$\begin{aligned} S(1) &= \mathbf{I} \\ S(i) &= e^{\mathbf{A}(T_i - T_{i-1})} [S(i-1)] + \mathbf{I}, \quad i \geq 2. \end{aligned}$$

Note that a similar recursive expression has been obtained in Ozaki (1979) for a Hawkes process with exponential kernel.

As follows we present the likelihood of a CARMA(p,q)-Hawkes model. Consider that $\theta = (b_0, \dots, b_q, a_1, \dots, a_p)$, then the likelihood of a CARMA(p,q)-Hawkes model is given by

$$\mathcal{L}(\theta, \mu) = -\int_0^{T_k} \lambda_t dt + \int_0^{T_k} \ln(\lambda_t) dN_t. \quad (49)$$

Exploiting the fact that $\int_0^{T_k} \ln(\lambda_t) dN_t = \sum_{i=1}^k \ln(\lambda_{T_i})$, then (49) can be written as

$$\mathcal{L}(\theta, \mu) = -\int_0^{T_k} \left[\mu + \mathbf{b}^\top X_t \right] dt + \sum_{i=1}^k \ln(\lambda_{T_i}) \quad (50)$$

and recalling once again that X_t can be expressed by (19) and rearranging the expression we have

$$\mathcal{L}(\theta, \mu) = -\mu T_k - \mathbf{b}^\top \int_0^{T_k} \int_0^t e^{\mathbf{A}(t-s)} \mathbf{e} dN_s dt + \sum_{i=1}^k \ln(\lambda_{T_i}). \quad (51)$$

Working on the inner integral, the likelihood becomes

$$\mathcal{L}(\theta, \mu) = -\mu(T_k) - \mathbf{b}^\top \int_0^{T_k} \left[\int_s^{T_k} e^{\mathbf{A}(t-s)} dt \right] dN_s \mathbf{e} + \sum_{i=1}^k \ln(\lambda_{T_i}), \quad (52)$$

while using the results in (A.1) we get

$$\mathcal{L}(\theta, \mu) = -\mu T_k - \mathbf{b}^\top \int_0^{T_k} \mathbf{A}^{-1} \left[e^{\mathbf{A}(T_k - s)} - \mathbf{I} \right] dN_s \mathbf{e} + \sum_{i=1}^k \ln(\lambda_{T_i}). \quad (53)$$

Developing the integral in (53) and recalling that $S(k) := \sum_{i=1}^k e^{\mathbf{A}(T_k - T_i)}$, we finally obtain that

the likelihood of a CARMA(p,q)-Hawkes model writes

$$\mathcal{L}(\theta, \mu) = -\mu T_k - \mathbf{b}^\top \mathbf{A}^{-1} S(k) \mathbf{e} + k \mathbf{b}^\top \mathbf{A}^{-1} \mathbf{e} + \sum_{i=1}^k \ln(\lambda_{T_i}). \quad (54)$$

5. Autocovariance and Autocorrelation of a CARMA(p,q)-Hawkes process

In this section we compute the stationary autocorrelation and autocovariance functions for the number of jumps in non-overlapping time intervals of length τ . To this aim we introduce some quantities that are useful to compute the asymptotic covariance of a CARMA(p,q)-Hawkes process. The first quantity we introduce is the $\frac{p(p+1)}{2} \times \frac{p(p+1)}{2}$ matrix $\tilde{\tilde{\mathbf{A}}}$ defined as follows

$$\tilde{\tilde{\mathbf{A}}} := \begin{bmatrix} D_{[p,p]}^1 & U_{[p,p-1]}^{1,2} & 0_{[p,p-2]} & \dots & \dots & \dots \\ L_{[p-1,p]}^{2,1} & D_{[p-1,p-1]}^2 & U_{[p-1,p-2]}^{2,3} & 0_{[p-1,p-3]} & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \dots \\ L_{[p-j+1,p]}^{j,1} & \dots & L_{[p-j+1,p-j+2]}^{j,j-1} & D_{[p-j+1,p-j+1]}^j & U_{[p-j+1,p-j]}^{j,j+1} & 0_{[p-j+1,p-j-1]} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \dots \\ L_{[1,p]}^{p,1} & \dots & \dots & \dots & \dots & D_{[1,1]}^p \end{bmatrix} \quad (55)$$

where the square matrices $D_{[p-j+1,p-j+1]}^j$, $j = 1, \dots, p-1$, have the following structure

$$D_{[p-j+1,p-j+1]}^j = \begin{bmatrix} 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 1 \\ b_{j-1} - a_{p-j+1} & b_j - a_{p-j} & \dots & \dots & b_{p-1} - a_1, \end{bmatrix}$$

with $D_{[1,1]}^p = 2(b_{p-1} - a_1)$. Matrices $L_{[p-j+1,p-i+1]}^{j,i}$ for $j = 2, \dots, p$ and $i = 1, \dots, j-1$ are characterized by the entries with the form

$$L^{j,i}(h, l) = \begin{cases} b_{j-2+i} - a_{p-j+1+(i-1)} & \text{if } h = p-j+1, l = j-i+1 \text{ and } j \neq p \\ 2(b_{j-2+i} - a_{p-j+1+(i-1)}) & \text{if } h = p-j+1, l = j-i+1 \text{ and } j = p \\ 0 & \text{otherwise} \end{cases}$$

while matrices $U_{[p-i+1,p-i]}^{i,i+1}$ for $i = 1, \dots, p-1$ have form

$$U_{[p-i+1,p-i]}^{i,i+1} = \begin{bmatrix} \mathbf{0}_{[1,p-i]} \\ \mathbf{I}_{[p-i,p-i]} \end{bmatrix}.$$

Here an example of the matrix $\tilde{\mathbf{A}}$ for a CARMA(3,2)-Hawkes model

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ b_0 - a_3 & b_1 - a_2 & b_2 - a_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & b_0 - a_3 & 0 & b_1 - a_2 & b_2 - a_1 & 1 \\ 0 & 0 & 2(b_0 - a_3) & 0 & 2(b_1 - a_2) & 2(b_2 - a_1) \end{bmatrix}.$$

The second quantity introduced is the $p \times \frac{p(p+1)}{2}$ matrix \mathbf{B} defined as:

$$\mathbf{B} := \begin{bmatrix} b_0 & b_1 & \dots & b_{p-1} & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & b_0 & \dots & 0 & b_1 & \dots & b_{p-1} & 0 & \dots & 0 \\ \vdots & \ddots & \dots & 0 \\ 0 & \dots & 0 & b_0 & 0 & \dots & b_1 & 0 & \dots & b_{p-1} \end{bmatrix} \quad (56)$$

where the generic i -th row is the result of a row concatenation of p vectors with dimensions $p, p-1, \dots, p-i, \dots, 1$, respectively. The first $i-1$ vectors have zero entries except the element in position i that coincides with b_{i-1} , the vector with dimension $p-i$ contains the elements b_i, \dots, b_{p-i} and the remaining vectors have zero entries.

For example, in the case of a CARMA(3,2)-Hawkes model, the structure of matrix \mathbf{B} reads

$$\mathbf{B} = \begin{bmatrix} b_0 & b_1 & b_2 & 0 & 0 & 0 \\ 0 & b_0 & 0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & 0 & b_1 & b_2 \end{bmatrix}.$$

The third quantity is the $\frac{p(p+1)}{2} \times p$ matrix $\tilde{\mathbf{C}}$ in which the entry in the i -th row and in j -th column has the following structure

$$c_{i,j} := \begin{cases} 0 & \text{if } i \neq j \left(p - \frac{j-1}{2} \right) \text{ and } i \neq \frac{p(p+1)}{2} \\ \mu & \text{if } i = j \left(p - \frac{j-1}{2} \right) \text{ and } i \neq \frac{p(p+1)}{2} \\ b_{j-1} & \text{if } i = \frac{p(p+1)}{2} \text{ and } j \neq p \\ 2\mu + b_{p-1} & \text{if } i = \frac{p(p+1)}{2} \text{ and } j = p \end{cases}. \quad (57)$$

Let H be a $p \times 1$ vector. Then we define the operator $vl(\cdot)$ as a function that transforms the $p \times p$ matrix HH^\top into a $\frac{p(p+1)}{2}$ vector containing the lower triangular part of the product HH^\top .

Specifically:

$$vlt\left(HH^\top\right) := \left[\underbrace{H_1H_1, \dots, H_pH_1}_{p \text{ entries}}, \underbrace{H_2H_2, \dots, H_pH_2}_{p-1 \text{ entries}}, \dots, \underbrace{H_iH_i, \dots, H_pH_i}_{p-i+1 \text{ entries}}, \dots, H_pH_p \right]^\top. \quad (58)$$

5.1. Conditions for existence of stationary autocovariance function

We rewrite the quantity $\mathbb{E}_{t_0} [X_T X_T^\top] \mathbf{b}$ using the $vlt(\cdot)$ operator defined in (58).

Lemma 1. *The following identity holds true*

$$\mathbb{E}_{t_0} [X_T X_T^\top] \mathbf{b} = \mathbf{B} vlt\left(\mathbb{E}_{t_0} (X_T X_T^\top)\right) \quad (59)$$

where the matrix \mathbf{B} is defined in (56) and the operator $vlt(\cdot)$ is defined as in (58). Moreover:

$$\begin{aligned} vlt\left(\mathbb{E}_{t_0} (X_T X_T^\top)\right) &= e^{\tilde{\mathbf{A}}(T-t_0)} vlt\left(X_{t_0} X_{t_0}^\top\right) + \left[e^{\tilde{\mathbf{A}}(T-t_0)} - \mathbf{I}\right] \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e}\right) \\ &+ e^{\tilde{\mathbf{A}}T} \left[\int_{t_0}^T e^{-\tilde{\mathbf{A}}t} \tilde{\mathbf{C}} e^{\tilde{\mathbf{A}}t} dt\right] e^{-\tilde{\mathbf{A}}t_0} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu\right]. \end{aligned} \quad (60)$$

Proof. Using the definition of matrix \mathbf{B} in (56), the identity in (59) is straightforward. To show the result in (60), we need first to compute the infinitesimal generator for each component of $vlt(X_t X_t^\top)$. From the definition in (58) we identify p blocks where the dimension of each block decreases by one unit. More precisely, the j -th block has $p - j + 1$ elements. Considering the first block (i.e., $j = 1$) we have p infinitesimal generators obtained applying the result in (22) of Proposition 4. For the first element in the first block, we have $\mathcal{A}X_{t,1}^2 = 2X_{t,2}X_{t,1}$. While for the i -th element in the first block with $i = 2, \dots, p - 1$ we get $\mathcal{A}X_{t,i}X_{t,1} = X_{t,i}X_{t,2} + X_{t,i+1}X_{t,1}$ and finally

$$\begin{aligned} \mathcal{A}X_{t,p}X_{t,1} &= \lambda_t [(X_{t,p} + 1)X_{t,1} - X_{t,p}X_{t,1}] + X_{t,p}X_{t,2} + A_{[p,\cdot]}X_tX_{t,1} \\ &= \mu X_{t,1} + X_{t,p}X_{t,2} + \left(\mathbf{b}^\top + A_{[p,\cdot]}\right) X_tX_{t,1}. \end{aligned}$$

For a generic j -th block, we get $p - j + 1$ infinitesimal generators. In particular for $i = j$ we have $\mathcal{A}X_{t,j}^2 = 2X_{t,j}X_{t,j+1}$. For $i = j + 1, \dots, p - 1$ we have $\mathcal{A}X_{t,i}X_{t,j} = X_{t,i}X_{t,j+1} + X_{t,j}X_{t,i+1}$ and

$$\begin{aligned} \mathcal{A}X_{t,p}X_{t,j} &= \lambda_t [(X_{t,p} + 1)X_{t,j} - X_{t,p}X_{t,j}] + X_{t,p}X_{t,j+1} + A_{[p,\cdot]}X_tX_{t,j} \\ &= \mu X_{t,j} + X_{t,p}X_{t,j+1} + \left(\mathbf{b}^\top + A_{[p,\cdot]}\right) X_tX_{t,j}. \end{aligned}$$

The last block contains only one infinitesimal generator of the form

$$\begin{aligned}\mathcal{A}X_{t,p}^2 &= \lambda_t \left[(X_{t,p} + 1)^2 - X_{t,p}^2 \right] + 2A_{[p,\cdot]} X_t X_{t,p} \\ &= \mu + \mathbf{b}^\top X_t + 2\mu X_{t,p} + 2 \left(\mathbf{b}^\top + A_{[p,\cdot]} \right) X_t X_{t,p}.\end{aligned}$$

Using the Dynkin's formula in (29) we obtain the following system of linear ODE's:

$$d\text{vlt} \left(\mathbb{E}_{t_0} \left(X_t X_t^\top \right) \right) = \left[\mu \tilde{\mathbf{e}} + \tilde{\mathbf{C}} \mathbb{E}_{t_0} (X_t) + \tilde{\mathbf{A}} \text{vlt} \left(\mathbb{E}_{t_0} \left(X_t X_t^\top \right) \right) \right] dt \quad (61)$$

where the $\frac{p(p+1)}{2}$ vector $\tilde{\mathbf{e}}$ is composed of zero entries except the last position where the element is one; $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{C}}$ are defined in (55) and (57) respectively.

The first step is to solve the ODE defined in (61) whose solution has the following form

$$\begin{aligned}\text{vlt} \left(\mathbb{E}_{t_0} \left(X_T X_T^\top \right) \right) &= e^{\tilde{\mathbf{A}}(T-t_0)} \text{vlt} \left(X_{t_0} X_{t_0}^\top \right) + e^{\tilde{\mathbf{A}}T} \int_{t_0}^T e^{-\tilde{\mathbf{A}}t} \left[\mu \tilde{\mathbf{e}} + \tilde{\mathbf{C}} \mathbb{E}_{t_0} (X_t) \right] dt \\ &= e^{\tilde{\mathbf{A}}(T-t_0)} \text{vlt} \left(X_{t_0} X_{t_0}^\top \right) + \left[e^{\tilde{\mathbf{A}}(T-t_0)} - \mathbf{I} \right] \tilde{\mathbf{A}}^{-1} \mu \tilde{\mathbf{e}} \\ &+ e^{\tilde{\mathbf{A}}T} \int_{t_0}^T e^{-\tilde{\mathbf{A}}t} \tilde{\mathbf{C}} \mathbb{E}_{t_0} (X_t) dt.\end{aligned} \quad (62)$$

We also observe that

$$\begin{aligned}e^{\tilde{\mathbf{A}}T} \int_{t_0}^T e^{-\tilde{\mathbf{A}}t} \tilde{\mathbf{C}} \mathbb{E}_{t_0} (X_t) dt &= e^{\tilde{\mathbf{A}}T} \int_{t_0}^T e^{-\tilde{\mathbf{A}}t} \tilde{\mathbf{C}} \left[e^{\tilde{\mathbf{A}}(t-t_0)} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] - \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] dt \\ &= e^{\tilde{\mathbf{A}}T} \int_{t_0}^T e^{-\tilde{\mathbf{A}}t} \tilde{\mathbf{C}} e^{\tilde{\mathbf{A}}t} dt e^{-\tilde{\mathbf{A}}t_0} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \\ &- \left[e^{\tilde{\mathbf{A}}(T-t_0)} - \mathbf{I} \right] \tilde{\mathbf{A}}^{-1} \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu.\end{aligned} \quad (63)$$

Substituting (63) into (62) we obtain the result in (60). \square

Appendix D contains the proofs of the following two propositions on the variance and covariance of the number of jumps that occur in two non-overlapping time intervals of the same length for a CARMA(p,q)-Hawkes model.

Proposition 6. *Under Assumption 1 and supposing that all eigenvalues of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}$ have negative real parts, the long-run covariance $\text{Cov}(\tau, \delta)$ defined as in (8) for a CARMA(p,q)-Hawkes process has the following form:*

$$\text{Cov}(\tau, \delta) = \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}\tau} - \mathbf{I} \right] e^{\tilde{\mathbf{A}}\delta} g_\infty(\tau) \quad (64)$$

where $g_\infty(\tau)$ is defined as

$$g_\infty(\tau) := (\mathbf{I} - e^{\tilde{\mathbf{A}}\tau}) \tilde{\mathbf{A}}^{-1} \mu \left[\mathbf{e} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} - \mathbf{e} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \left(\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) + \mathbf{B} \tilde{\mathbf{A}}^{-1} \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \right]. \quad (65)$$

Proposition 7. *Under the same assumptions as in Proposition 6, the long-run variance $\text{Var}(\tau)$ of the number of jumps in a interval with length τ for a CARMA(p, q)-Hawkes process, defined as in (8), has the following form:*

$$\begin{aligned} \text{Var}(\tau) &= \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \left(1 - 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \mu \tau + 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \tilde{\mathbf{A}}^{-1} \mathbf{e} \tau \mu^2 \left(\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \\ &+ 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{B} \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \tau - 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}\tau} - \mathbf{I} \right] h_\infty(0) \end{aligned} \quad (66)$$

where $h_\infty(0)$ is defined as

$$h_\infty(0) := -\tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) + \tilde{\mathbf{A}}^{-1} \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu^2 \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} + \tilde{\mathbf{A}}^{-1} \mathbf{B} \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right). \quad (67)$$

Remark 4. *Combining the results in Propositions 6 and 7, we determine the asymptotic autocorrelation function of number of jumps in non-overlapping time intervals of length τ , i.e., $\rho_\tau(d)$, for a CARMA(p, q)-Hawkes in a closed-form formula:*

$$\rho_\tau(d) = \frac{\text{Cov}(\tau, d-1)}{\text{Var}(\tau)}, \quad d = 1, 2, \dots \quad (68)$$

where d denotes the lag order.

5.2. Strong mixing property for the increments of a CARMA(p, q)-Hawkes and asymptotic distribution of the autocorrelation function

The asymptotic distribution of the autocorrelation function of a CARMA(p, q)-Hawkes process can be easily obtained if we show that the increments of the process are strongly mixing.

Definition 3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and \mathcal{A}, \mathcal{B} two sub σ -algebras of \mathcal{F} . The strong-mixing coefficient is defined as:*

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup \{ |\mathbb{P}(A \cap B) - \mathbb{P}(A) \mathbb{P}(B)| \mid A \in \mathcal{A}, B \in \mathcal{B} \}. \quad (69)$$

Following Poinas et al. (2019), the quantity in (69) can be reformulated for a point process N_t in the following way:

$$\alpha_N(r) := \sup_{t \in \mathbb{R}} \alpha(\xi_{-\infty}^t, \xi_{t+r}^\infty) \quad (70)$$

where ξ_a^b denotes the σ -algebra generated by the cylinder sets on the interval $(a, b]^4$. Considering the sequence $(\Delta_1 N_k)_{k \in \mathbb{Z}}$ where $\Delta_1 N_k := N_{k+1} - N_k$ is the number of jumps in the interval of length

⁴Let N be a counting process defined as a map from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a measurable space $(\mathbb{M}, \mathcal{M})$

1 and extremes $k, k + 1$, the strong-mixing coefficient has the form

$$\alpha_{\Delta_1 N}(r) := \sup_{n \in \mathbb{Z}} \alpha(\mathcal{F}_{-\infty}^n, \mathcal{F}_{n+r}^\infty) \quad (71)$$

where \mathcal{F}_a^b is the σ -algebra generated by the sequence $(\Delta_1 N_k)_{a \leq k \leq b}$. If $\alpha_N(r) \rightarrow 0$ (respectively $\alpha_{\Delta_1 N_k}(r) \rightarrow 0$) as $r \rightarrow +\infty$, the point process N_t (respectively $\Delta_1 N_k$) is said to be strongly-mixing. Using Theorem 1 in Cheyssson and Lang (2020), we obtain the following proposition.

Proposition 8. *A CARMA(p, q)-Hawkes process satisfying Assumption 1 is strongly mixing with exponential rate.*

Proof. We first prove the existence of a positive constant $a_0 > 0$ such that the kernel function satisfies the condition

$$\int_{\mathbb{R}} e^{a_0|t|} h(t) dt < +\infty. \quad (72)$$

We notice that Assumption 1 implies that

$$\int_{\mathbb{R}} e^{a_0|t|} h(t) dt = \mathbf{b}^\top \int_0^{+\infty} e^{a_0 t} e^{\mathbf{A}t} dt \mathbf{e} = \mathbf{b}^\top \mathbf{S} \int_0^{+\infty} e^{a_0 t} e^{\mathbf{A}t} dt \mathbf{S}^{-1} \mathbf{e}.$$

Choosing $a_0 \in (0, |\operatorname{Re}(\lambda_1)|)$ the condition in (72) is ensured and thus we can apply the result in Theorem 1 proved by Cheyssson and Lang (2020), and the strong-mixing coefficient results to be $\alpha_N(r) = O(e^{-ar})$ where $a \in (0, a_0)$. \square

As shown in Cheyssson and Lang (2020), we have that $\alpha_{\Delta_1 N}(r) \leq \alpha_N(r)$ and the result in Proposition 8 implies that the sequence $(\Delta_1 N_k)_{k \in \mathbb{Z}}$ is strongly mixing. This result is useful to determine the asymptotic distribution of the sample autocovariance and autocorrelation functions associated to the sequence $(\Delta_1 N_k)_{k \in \mathbb{Z}}$. Following the result in Ibragimov and Linnik (1971), we obtain the following result for the asymptotic distribution of the sample mean, the sample variance and the sample autocovariance function.

Proposition 9. *Let $(N_t)_{t \geq 0}$ be a stationary CARMA(p, q)-Hawkes process that satisfies the assumptions in Proposition 8. We assume the existence of a positive constant ϕ such that $\mathbb{E} \left[(\Delta_1 N_1)^{4+\phi} \right] <$*

of locally finite counting measures on Ω . Then the σ -algebra ξ_a^b is defined as:

$$\xi_a^b := \sigma(\{N \in \mathbb{M} : N(A) = n\}; A \in \mathcal{B}((a, b]), n \in \mathbb{N}).$$

$+\infty$. Denoting with

$$V_k := \begin{bmatrix} \Delta_1 N_k \\ (\Delta_1 N_k - \mathbb{E}(\Delta_1 N_\infty))^2 \\ (\Delta_1 N_k - \mathbb{E}(\Delta_1 N_\infty))(\Delta_1 N_{k+1} - \mathbb{E}(\Delta_1 N_\infty)) \\ \vdots \\ (\Delta_1 N_k - \mathbb{E}(\Delta_1 N_\infty))(\Delta_1 N_{k+d} - \mathbb{E}(\Delta_1 N_\infty)) \end{bmatrix}, \text{ with } k = 1, \dots, n \text{ and } d < n$$

as $n \rightarrow +\infty$, we have:

$$\sqrt{n} \left(\frac{1}{n} \sum_{k=1}^n V_k - \begin{pmatrix} \mathbb{E}(\Delta_1 N_\infty) \\ \text{Var}(\Delta_1 N_\infty) \\ \text{Acv}(1) \\ \vdots \\ \text{Acv}(d) \end{pmatrix} \right) \rightarrow \mathcal{N}_{d+2}(\mathbf{0}, \Sigma) \quad (73)$$

where $\text{Acv}(d) := \text{Cov}(1, d-1)$ and

$$\Sigma := \mathbb{E}(V_1 V_1^\top) + 2 \sum_{k=2}^{+\infty} \mathbb{E}(V_1 V_k^\top). \quad (74)$$

Proof. The proof is quite standard and is an application of Theorem 18.5.3 in Ibragimov and Linnik (1971) and Cramér-Wold device. Denoting with

$$\vartheta := [\mathbb{E}(\Delta_1 N_\infty), \text{Var}(\Delta_1 N_\infty), \text{Acv}(1), \dots, \text{Acv}(d)]^\top$$

we apply Theorem 18.5.3 in Ibragimov and Linnik (1971) to the linear combination $(c^\top V_k)_{k=1,2,\dots,n}$ where c is a generic $d+2$ real vector such that $c^\top \Sigma c > 0$. Since the strong mixing property is preserved under linear transformations as well as the rate we have

$$\sqrt{n} \left(\frac{1}{n} \sum_{k=1}^n c^\top V_k - c^\top \vartheta \right) \rightarrow \mathcal{N} \left(0, c^\top \mathbb{E}(V_1 V_1^\top) c + 2 \sum_{k=1}^{+\infty} c^\top \mathbb{E}(V_1 V_k^\top) c \right),$$

that is

$$\sqrt{n} \left(\frac{1}{n} \sum_{k=1}^n c^\top V_k - c^\top \vartheta \right) \rightarrow \mathcal{N}(0, c^\top \Sigma c).$$

Applying Cramér-Wold device we obtain the asymptotic behavior in (73). \square

Applying the Delta method, it is possible to use the result in Proposition 9 to obtain the asymptotic distribution of the autocorrelation function.

6. Application to simulated series

This section examines the time series of the counting process N_t generated by the simulation of two different stochastic processes: the standard Hawkes process and the CARMA(3,1)-Hawkes process. We show the behaviour of the autocorrelation function and its 95% confidence interval obtained applying the results of Subsection 5.2. Furthermore, we investigate the estimation of parameters by means of the Maximum Likelihood Estimation (MLE) method described in Subsection 4.2 and the Moment Matching Estimation (MME) method which we describe below.

Consider a sequence of empirical observations for the increments of a counting process $(\Delta_\tau N_k)_{k=1, \dots, n}$, then the MME method is composed of two steps. The first step is to compute the least squares estimator:

$$\hat{\theta}_{n,\tau} := \underset{\hat{\theta}_{n,\tau} \in \Theta \subseteq \mathbb{R}^{p+q+1}}{\operatorname{argmin}} M(\hat{\rho}_{n,\tau}, \theta)$$

where Θ is a subset of \mathbb{R}^{p+q+1} such that the stationary condition is guaranteed, the kernel function is non-negative defined and higher order moments of a CARMA(p,q)-Hawkes process exist. For a fixed $m \geq p + q + 1$, $M : \mathbb{R}_+^m \times \Theta \rightarrow \mathbb{R}$ is defined as:

$$M(\hat{\rho}_{n,\tau}, \theta) := \sum_{d=1}^m (\hat{\rho}_{n,\tau}(d) - \rho_\tau(d))^2$$

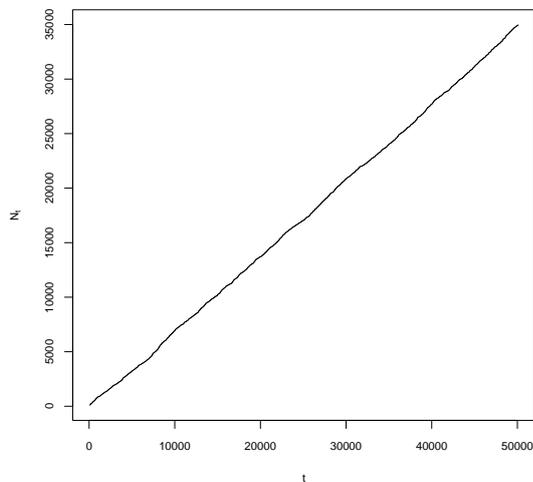
in which d denotes the lag order, $\hat{\rho}_{n,\tau}(d)$ represents a vector containing the empirical autocorrelations and $\rho_\tau(d)$ is a vector of theoretical autocorrelations described in Section 5. The vector θ includes only the autoregressive (a_1, \dots, a_p) and moving average (b_0, \dots, b_q) parameters. Once obtained the autoregressive and moving average parameters, the parameter μ can be estimated, which corresponds to the second step, from Equation (37) using the empirical first moment of $\Delta_\tau N_t$ with $\tau = 1$.

Note that all chosen parameters for the Hawkes process and the CARMA(3,1)-Hawkes process ensure two conditions: the stationarity of the process (see Section 2) and the existence of the asymptotic autocorrelation function (see Section 5).

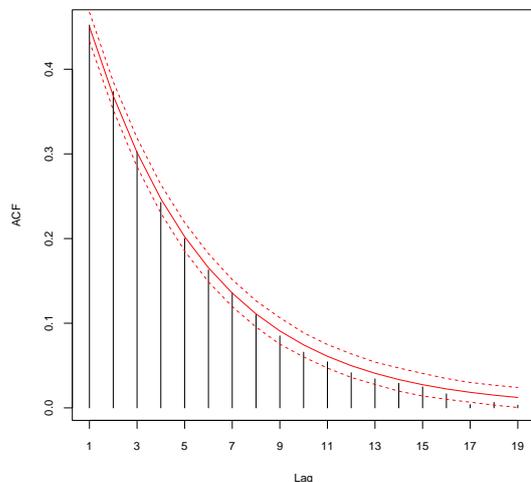
6.1. The Hawkes process

We simulate the Hawkes process with the following parameters: $\mu = 0.2$, $a_1 = 0.7$ and $b_0 = 0.5$. For the sake of clarity, we recall that the Hawkes process can be seen also as a CAR(1)-Hawkes process as mentioned in the Remark 3.

Figure 1(a) shows a simulated trajectory of the counting process N_t , while Figure 1(b) exhibits the autocorrelation function at different lags. From Figure 1(b) we observe that all empirical autocorrelations belong to the 95% confidence interval and, as expected, the ACF depicts an exponential monotonic decay (decreasing) behaviour which is typical of the Hawkes processes.



(a) Trajectory of the counting process N_t



(b) Autocorrelation function

Figure 1: Trajectory of the counting process N_t and autocorrelation function (ACF) using a Hawkes process. Input parameters: $\mu = 0.2$, $a_1 = 0.7$ and $b_0 = 0.5$.

Table 1 exhibits the MLE estimates for the parameters and the number of occurred events for different levels of final time T , whereas Table 2 shows the estimated parameters using the MME method.

$\hat{\mu}$	\hat{a}_1	\hat{b}_0	N_t	T
0.2101	0.7270	0.4883	3199	5000
0.2014	0.7389	0.5211	10240	15000
0.1987	0.7054	0.4997	17042	25000
0.2011	0.7028	0.5004	34914	50000

Table 1: Parameter estimates from MLE and number of occurred events N_T for a Hawkes process. True parameters are $\mu = 0.2$, $a_1 = 0.7$ and $b_0 = 0.5$.

$\hat{\mu}$	\hat{a}_1	\hat{b}_0	T
0.2440	0.7540	0.4877	5000
0.2121	0.7127	0.4954	15000
0.2044	0.7016	0.4951	25000
0.1992	0.7042	0.4990	50000

Table 2: Parameter estimates from MME for a Hawkes process. True parameters are $\mu = 0.2$, $a_1 = 0.7$ and $b_0 = 0.5$.

6.2. CARMA(3,1)-Hawkes process

We simulate the counting process N_t using a CARMA(3,1)-Hawkes model with the following parameters: $\mu = 0.30$, $a_1 = 1.3$, $a_2 = 0.34 + \pi^2/4 \approx 2.807$, $a_3 = 0.025 + 0.025\pi^2 \approx 0.2717$, $b_0 = 0.2$ and $b_1 = 0.3$. This set of parameters ensures the stationary conditions since $-\mathbf{b}^\top \mathbf{A}^{-1} \mathbf{e} \approx 0.7359973 < 1$ and the largest eigenvalue of \mathbf{A} is $\tilde{\lambda}_1 \approx -0.1012$. We can easily verify the condition for the existence of $\mathbb{E}(\Delta_\tau N_\infty)$ in (37). Indeed, the real part of the largest eigenvalue of $\tilde{\mathbf{A}}$ is negative (-0.02903). Moreover, it is possible to verify that all eigenvalues of $\tilde{\mathbf{A}}$ have negative real part, thus the long-run autocorrelation function exists (the real part of the largest eigenvalue of $\tilde{\mathbf{A}}$ is -0.0290). In order to analyze the nonnegativity of the kernel $h(t)$, Figure 2 presents the behaviour of $h(t)$ with $t \in [0, 30]$. From (16) the tail behaviour of $h(t)$ is proportional to $e^{-0.1012t}$ (i.e., $h(t) \sim 1.9800e^{-0.1012t}$ as $t \rightarrow +\infty$).

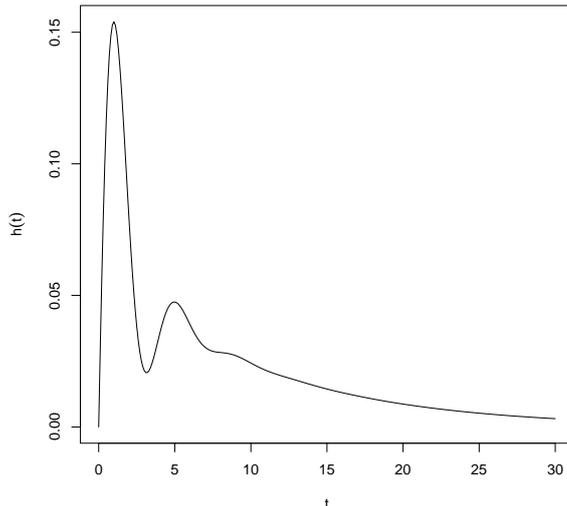
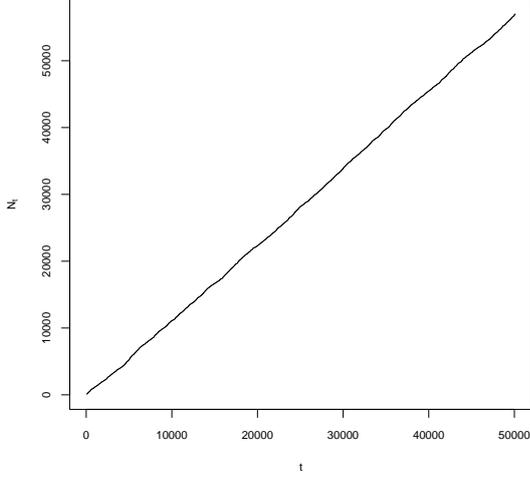


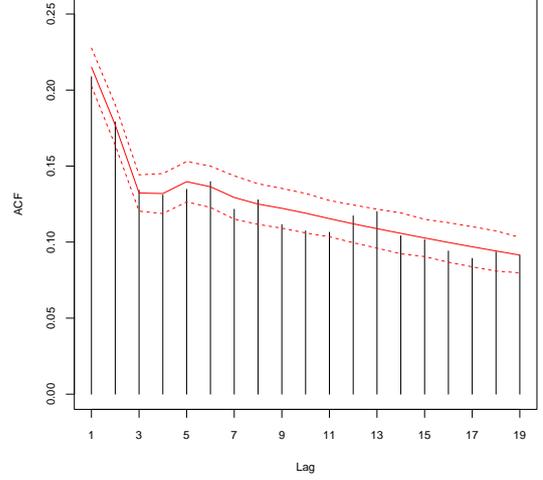
Figure 2: Kernel function of a CARMA(3,1)-Hawkes. Input parameters: $\mu = 0.30$, $a_1 = 1.3$, $a_2 = 0.34 + \pi^2/4 \approx 2.807$, $a_3 = 0.025 + 0.025\pi^2 \approx 0.2717$, $b_0 = 0.2$ and $b_1 = 0.3$.

Figure 3(a) exhibits a simulated trajectory of the counting process N_t , while Figure 3(b) displays the ACF at different lags. Figure 3(b) is a clear example of the fact that a CARMA(3,1)-Hawkes process can accommodate different shapes of autocorrelation structures rather than an exponential monotonic decay behaviour as it is the case of a standard Hawkes process.

Table 3 shows the estimated parameters obtained using the MLE method and the number of occurred events for different levels of final time T , whereas Table 4 exhibits the estimated parameters using the MME method for different levels of final time T .



(a) Trajectory of the counting process N_t



(b) Autocorrelation function

Figure 3: Trajectory of the counting process N_t and autocorrelation function (ACF) using a CARMA(3,1)-Hawkes. Input parameters: $\mu = 0.30$, $a_1 = 1.3$, $a_2 = 0.34 + \pi^2/4 \approx 2.807$, $a_3 = 0.025 + 0.025\pi^2 \approx 0.2717$, $b_0 = 0.2$ and $b_1 = 0.3$.

$\hat{\mu}$	\hat{a}_1	\hat{a}_2	\hat{a}_3	\hat{b}_0	\hat{b}_1	N_T	T
0.3166	1.5372	2.8565	0.2869	0.2011	0.3345	5229	5000
0.3063	1.3521	2.6686	0.2758	0.1998	0.3023	16666	15000
0.3068	1.0811	2.4380	0.2480	0.1808	0.2508	28154	25000
0.2949	1.4177	2.6901	0.2550	0.1889	0.3138	56815	50000

Table 3: Parameter estimates from MLE and number of occurred events N_T for a CARMA(3,1)-Hawkes process. True parameters are: $\mu = 0.30$, $a_1 = 1.3$, $a_2 = 0.34 + \pi^2/4 \approx 2.807$, $a_3 = 0.025 + 0.025\pi^2 \approx 0.2717$, $b_0 = 0.2$ and $b_1 = 0.3$.

$\hat{\mu}$	\hat{a}_1	\hat{a}_2	\hat{a}_3	\hat{b}_0	\hat{b}_1	T
0.2946	2.4941	2.9148	0.2228	0.1600	0.4954	4999
0.2788	2.8938	3.2043	0.2440	0.1827	0.6283	14999
0.2934	1.5754	2.8956	0.2634	0.1947	0.3592	24999
0.3104	1.2584	2.7107	0.2749	0.1998	0.2797	49999

Table 4: Parameter estimates from MME for a CARMA(3,1)-Hawkes process. True parameters are: $\mu = 0.30$, $a_1 = 1.3$, $a_2 = 0.34 + \pi^2/4 \approx 2.807$, $a_3 = 0.025 + 0.025\pi^2 \approx 0.2717$, $b_0 = 0.2$ and $b_1 = 0.3$.

7. Conclusion

In this paper we introduce a new Hawkes process where the intensity is a CARMA(p,q) model. We analyze the statistical properties of this process and obtain a closed-form expression for the autocorrelation function of the number of jumps observed in non-overlapping time intervals of same

length. The model is a generalization of the standard Hawkes with exponential kernel but it is able to reproduce more complex dependence structures similar to those documented in several data sets.

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Appendix A. Integration of matrix exponentials

Let \mathbf{A} be a square matrix and $\mathbf{A}^{(i)} := \underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{i \text{ times}}$. As the exponential of the matrix \mathbf{A} can be computed as

$$\exp(\mathbf{A}t) = \mathbf{I} + \sum_{i=1}^{+\infty} \frac{\mathbf{A}^{(i)}t^i}{i!},$$

it is straightforward to show that

$$\int_{t_0}^T e^{\mathbf{A}(T-t)} dt = \mathbf{A}^{-1} \left(e^{\mathbf{A}(T-t_0)} - \mathbf{I} \right) = \left(e^{\mathbf{A}(T-t_0)} - \mathbf{I} \right) \mathbf{A}^{-1}. \quad (\text{A.1})$$

Appendix B. Solution of a general Linear Ordinary differential Equation

To solve $dY_t = (b_t + AY_t) dt$, we consider the transformation $X_t = e^{-At}Y_t$ and observe that

$$dX_t = -Ae^{-At}Y_t dt + e^{-At}dY_t = e^{-At}b_t dt,$$

from where we have $X_T = X_{t_0} + \int_{t_0}^T e^{-At}b_t dt$ that in terms of Y_t reads

$$Y_T = e^{A(T-t_0)}Y_{t_0} + \int_{t_0}^T e^{A(T-t)}b_t dt. \quad (\text{B.1})$$

Appendix C. Computation of integrals with matrix exponentials

Some useful results for computing integrals that involve matrix exponentials are provided in Van Loan (1978) and Carbonell et al. (2008). In particular, we recall the result that deals with the computation of the following two integrals:

$$\int_0^t e^{\mathbf{H}_{11}(t-u)} \mathbf{H}_{12} e^{\mathbf{H}_{22}u} du \quad (\text{C.1})$$

$$\int_0^t \int_0^u e^{\mathbf{H}_{11}(t-u)} \mathbf{H}_{12} e^{\mathbf{H}_{22}(u-r)} \mathbf{H}_{23} e^{\mathbf{H}_{33}r} dr du \quad (\text{C.2})$$

where \mathbf{H}_{11} , \mathbf{H}_{12} , \mathbf{H}_{22} , \mathbf{H}_{23} and \mathbf{H}_{33} have dimension $d_1 \times d_1$, $d_1 \times d_2$, $d_2 \times d_2$, $d_2 \times d_3$ and $d_3 \times d_3$, respectively. We need to define a block triangular matrix \mathbf{H} as follows

$$\mathbf{H} := \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{22} & \mathbf{H}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{33} \end{pmatrix}. \quad (\text{C.3})$$

The integrals (C.1) and (C.2) coincide with the elements $\mathbf{B}_{12}(t)$ and $\mathbf{B}_{13}(t)$ in the matrix exponential:

$$e^{\mathbf{H}t} = \begin{pmatrix} \mathbf{B}_{11}(t) & \mathbf{B}_{12}(t) & \mathbf{B}_{13}(t) \\ \mathbf{0} & \mathbf{B}_{22}(t) & \mathbf{B}_{23}(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{33}(t) \end{pmatrix} \quad (\text{C.4})$$

while $\mathbf{B}_{11}(t) := e^{\mathbf{H}_{11}t}$, $\mathbf{B}_{22}(t) := e^{\mathbf{H}_{22}t}$ and $\mathbf{B}_{33}(t) := e^{\mathbf{H}_{33}t}$.

Remark 5. *The eigenvalues of \mathbf{H} coincide with the eigenvalues of \mathbf{H}_{11} , \mathbf{H}_{22} and \mathbf{H}_{33} . If the real part of all eigenvalues of \mathbf{H}_{11} , \mathbf{H}_{22} and \mathbf{H}_{33} is negative, the following result holds*

$$\lim_{t \rightarrow +\infty} e^{\mathbf{H}t} = \mathbf{0}$$

that implies

$$\lim_{t \rightarrow +\infty} \mathbf{B}_{12}(t) = \mathbf{0} \quad (\text{C.5})$$

and

$$\lim_{t \rightarrow +\infty} \mathbf{B}_{13}(t) = \mathbf{0}. \quad (\text{C.6})$$

Appendix D. Proofs of propositions on long-run covariance and variance of the number of jumps

We provide below the proof of Proposition 6 on the long-run covariance of the number of jumps in a CARMA(p,q)-Hawkes model.

Proof. We first determine the covariance of number of jumps in two non-overlapping time intervals given the information at time t_0 . This quantity is formally defined as

$$\begin{aligned} \text{Cov}_{t_0}(\tau, \delta) &:= \mathbb{E}_{t_0} [(N_{t+\tau} - N_t)(N_{t+2\tau+\delta} - N_{t+\tau+\delta})] \\ &\quad - \mathbb{E}_{t_0} [(N_{t+\tau} - N_t)] \mathbb{E}_{t_0} [(N_{t+2\tau+\delta} - N_{t+\tau+\delta})]. \end{aligned} \quad (\text{D.1})$$

Using the iteration property of the conditional expected value, (D.1) becomes

$$\begin{aligned} \text{Cov}_{t_0}(\tau, \delta) &= \mathbb{E}_{t_0} [(N_{t+\tau} - N_t) \mathbb{E}_{t+\tau} [(N_{t+2\tau+\delta} - N_{t+\tau+\delta})]] \\ &\quad - \mathbb{E}_{t_0} [(N_{t+\tau} - N_t)] \mathbb{E}_{t_0} [(N_{t+2\tau+\delta} - N_{t+\tau+\delta})]. \end{aligned}$$

Applying the result (36) in Proposition 5, we get

$$\text{Cov}_{t_0}(\tau, \delta) = \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}(\tau+\delta)} - e^{\tilde{\mathbf{A}}\delta} \right] g_{t_0}(t, \tau) \quad (\text{D.2})$$

where

$$\begin{aligned}
g_{t_0}(t, \tau) &= \mathbb{E}_{t_0} [(N_{t+\tau} - N_t) X_{t+\tau}] - e^{\tilde{\mathbf{A}}(t+\tau-t_0)} \mathbb{E}_{t_0} [N_{t+\tau} - N_t] X_{t_0} \\
&+ \left(\mathbf{I} - e^{\tilde{\mathbf{A}}(t+\tau-t_0)} \right) \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_{t+\tau} - N_t] \\
&= \mathbb{E}_{t_0} [N_{t+\tau} X_{t+\tau}] + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_t] - e^{\tilde{\mathbf{A}}\tau} \left[\mathbb{E}_{t_0} (N_t X_t) + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_t] \right] \\
&- e^{\tilde{\mathbf{A}}(t+\tau-t_0)} \mathbb{E}_{t_0} [N_{t+\tau} - N_t] X_{t_0} + \left(\mathbf{I} - e^{\tilde{\mathbf{A}}(t+\tau-t_0)} \right) \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_{t+\tau} - N_t]. \quad (\text{D.3})
\end{aligned}$$

In the *rhs* of (D.3), the last two terms are stationary due to the result in (37) and to the negativity of the real part for the eigenvalues of $\tilde{\mathbf{A}}$; the third term converges to zero as $t \rightarrow +\infty$ while the fourth term has the following limit behaviour

$$\left(\mathbf{I} - e^{\tilde{\mathbf{A}}(t+\tau-t_0)} \right) \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_{t+\tau} - N_t] \rightarrow \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu^2 \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \tau \text{ a.s. } t \rightarrow +\infty. \quad (\text{D.4})$$

For the first two terms in the *rhs* (D.3) consider the quantity:

$$h_{t_0}(t, \tau) := \mathbb{E}_{t_0} [N_{t+\tau} X_{t+\tau}] + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_t], \quad \forall \tau \geq 0, t > t_0 \quad (\text{D.5})$$

as $t \rightarrow +\infty$. In (D.5) the vector $\mathbb{E}_{t_0} [N_t X_t]$ requires the calculation of p infinitesimal generators. We then observe that for $i = 1, \dots, p-1$, the infinitesimal generator of the function $N_t X_{t,i}$ is:

$$\begin{aligned}
\mathcal{A} N_t X_{t,i} &= \left(\mu + \mathbf{b}^\top X_t \right) [(N_t + 1) X_{t,i} - N_t X_{t,i}] + N_t X_{t,i+1} \\
&= \left(\mu X_{t,i} + X_{t,i} X_t^\top \mathbf{b} \right) + N_t X_{t,i+1}
\end{aligned}$$

while for $i = p$

$$\begin{aligned}
\mathcal{A} N_t X_{t,p} &= \left(\mu + \mathbf{b}^\top X_t \right) [(N_t + 1) (X_{t,p} + 1) - N_t X_{t,p}] + N_t A_{[p,\cdot]} X_t \\
&= \left(\mu + \mathbf{b}^\top X_t + \mu N_t \right) + \left(\mu X_{t,p} + X_{t,p} X_t^\top \mathbf{b} \right) + \left(\mathbf{b}^\top + A_{[p,\cdot]} \right) N_t X_t,
\end{aligned}$$

that implies

$$d\mathbb{E}_{t_0} [X_t N_t] = \left[\left(\mu + \mathbf{b}^\top \mathbb{E}_{t_0} [X_t] + \mu \mathbb{E}_{t_0} [N_t] \right) \mathbf{e} + \mu \mathbb{E}_{t_0} [X_t] + \mathbb{E}_{t_0} [X_t X_t^\top] \mathbf{b} + \tilde{\mathbf{A}} \mathbb{E}_{t_0} [X_t N_t] \right] dt \quad (\text{D.6})$$

from where we get

$$\begin{aligned}
\mathbb{E}_{t_0} [X_T N_T] &= e^{\tilde{\mathbf{A}}(T-t_0)} X_{t_0} N_{t_0} + \int_{t_0}^T e^{\tilde{\mathbf{A}}(T-t)} \left(\mu + \mathbf{b}^\top \mathbb{E}_{t_0} [X_t] + \mu \mathbb{E}_{t_0} [N_t] \right) \mathbf{e} dt \\
&+ \int_{t_0}^T e^{\tilde{\mathbf{A}}(T-t)} \left[\mu \mathbb{E}_{t_0} [X_t] + \mathbb{E}_{t_0} [X_t X_t^\top] \mathbf{b} \right] dt. \quad (\text{D.7})
\end{aligned}$$

The quantity $\mathbb{E}_{t_0} [X_T N_T]$ is not stationary but it is useful as it appears in the *rhs* of the function $h_{t_0}(t, \tau)$ introduced in (D.5) that can be rewritten as

$$\begin{aligned}
h_{t_0}(t, \tau) &= e^{\tilde{\mathbf{A}}(t+\tau-t_0)} X_{t_0} N_{t_0} + \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \boldsymbol{\mu} \mathbf{e} \, du + \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{b}^\top \mathbb{E}_{t_0} [X_u] \mathbf{e} \, du \\
&+ \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} (\boldsymbol{\mu} \mathbb{E}_{t_0} [N_u]) \mathbf{e} \, du + \tilde{\mathbf{A}}^{-1} \mathbf{e} \boldsymbol{\mu} \mathbb{E}_{t_0} [N_t] \\
&+ \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \left[\boldsymbol{\mu} \mathbb{E}_{t_0} [X_u] + \mathbb{E}_{t_0} [X_u X_u^\top] \mathbf{b} \right] \, du.
\end{aligned} \tag{D.8}$$

We analyze the long-run behaviour of each term in the *rhs* of (D.8). We first observe that

$$\int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \, du \boldsymbol{\mu} \mathbf{e} = \left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} \boldsymbol{\mu} \mathbf{e}$$

with

$$\lim_{t \rightarrow +\infty} \left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} \boldsymbol{\mu} \mathbf{e} = -\tilde{\mathbf{A}}^{-1} \boldsymbol{\mu} \mathbf{e}. \tag{D.9}$$

The formula for the conditional expected value of the state process in (32) allows us to rewrite the third term in the *rhs* of (D.8) as follows

$$\begin{aligned}
\int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{e} \mathbf{b}^\top \mathbb{E}_{t_0} [X_u] \, du &= e^{\tilde{\mathbf{A}}(t+\tau)} \int_{t_0}^{t+\tau} e^{-\tilde{\mathbf{A}}u} \mathbf{e} \mathbf{b}^\top e^{\tilde{\mathbf{A}}u} \, du e^{-\tilde{\mathbf{A}}t_0} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \boldsymbol{\mu} \right] \\
&- \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \, du \mathbf{e} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \boldsymbol{\mu} \\
&= e^{\tilde{\mathbf{A}}(t+\tau)} \int_{t_0}^{t+\tau} e^{-\tilde{\mathbf{A}}u} \mathbf{e} \mathbf{b}^\top e^{\tilde{\mathbf{A}}u} \, du e^{-\tilde{\mathbf{A}}t_0} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \boldsymbol{\mu} \right] \\
&- \left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} \mathbf{e} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \boldsymbol{\mu}.
\end{aligned} \tag{D.10}$$

To compute the integral $e^{\tilde{\mathbf{A}}(t+\tau)} \int_{t_0}^{t+\tau} e^{-\tilde{\mathbf{A}}u} \mathbf{e} \mathbf{b}^\top e^{\tilde{\mathbf{A}}u} \, du e^{-\tilde{\mathbf{A}}t_0}$ we use the result in (C.4) and exploiting its limit behaviour (C.5), the long-run behaviour of (D.10) becomes

$$\lim_{t \rightarrow +\infty} \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{e} \mathbf{b}^\top \mathbb{E}_{t_0} [X_u] \, du = \tilde{\mathbf{A}}^{-1} \mathbf{e} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \boldsymbol{\mu}. \tag{D.11}$$

The fourth term in the *rhs* of (D.8) can be written as

$$\begin{aligned}
& \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} du \mathbf{e} \mu N_{t_0} + \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} (u - t_0) du \mathbf{e} \mu^2 \left(\mathbf{1} - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \\
& + \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{e} \mu \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}(u-t_0)} - \mathbf{I} \right] du \left(X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right) + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_t] \\
& = \left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu N_{t_0} + \left[\int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} (u - t_0) du \right] \mathbf{e} \mu^2 \left(\mathbf{1} - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \\
& + e^{\tilde{\mathbf{A}}(t+\tau)} \int_{t_0}^{t+\tau} e^{-\tilde{\mathbf{A}}u} \mathbf{e} \mathbf{b}^\top e^{\tilde{\mathbf{A}}u} du e^{-\tilde{\mathbf{A}}t_0} \tilde{\mathbf{A}}^{-1} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \mu \\
& - \tilde{\mathbf{A}}^{-1} \left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \mathbf{e} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \mu + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_t]. \tag{D.12}
\end{aligned}$$

Integrating by parts we get

$$\int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} (u - t_0) du = \tilde{\mathbf{A}}^{-1} \left[\left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} - \mathbf{I} (t + \tau - t_0) \right].$$

Thus (D.12) becomes

$$\begin{aligned}
& \left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu N_{t_0} + \tilde{\mathbf{A}}^{-1} \left[\left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} - \mathbf{I} (t + \tau - t_0) \right] \mathbf{e} \mu^2 \left(\mathbf{1} - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \\
& + e^{\tilde{\mathbf{A}}(t+\tau)} \int_{t_0}^{t+\tau} e^{-\tilde{\mathbf{A}}u} \mathbf{e} \mathbf{b}^\top e^{\tilde{\mathbf{A}}u} du e^{-\tilde{\mathbf{A}}t_0} \tilde{\mathbf{A}}^{-1} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \mu \\
& - \tilde{\mathbf{A}}^{-1} \left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \mathbf{e} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \mu + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_t].
\end{aligned}$$

Using the formula for the conditional expected value of the counting process in (31) we get

$$\begin{aligned}
& \left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu N_{t_0} + \tilde{\mathbf{A}}^{-1} \left[\left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} - \mathbf{I} (t + \tau - t_0) \right] \mathbf{e} \mu^2 \left(\mathbf{1} - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \\
& + e^{\tilde{\mathbf{A}}(t+\tau)} \int_{t_0}^{t+\tau} e^{-\tilde{\mathbf{A}}u} \mathbf{e} \mathbf{b}^\top e^{\tilde{\mathbf{A}}u} du e^{-\tilde{\mathbf{A}}t_0} \tilde{\mathbf{A}}^{-1} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \mu \\
& - \tilde{\mathbf{A}}^{-1} \left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \mathbf{e} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \mu \\
& + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \left[N_{t_0} + \mu \left(\mathbf{1} - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) (t - t_0) + \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left(e^{\tilde{\mathbf{A}}(t_1-t_0)} - \mathbf{I} \right) \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \right] \\
& = e^{\tilde{\mathbf{A}}(t+\tau-t_0)} \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu N_{t_0} + \tilde{\mathbf{A}}^{-1} \left[\left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} - \mathbf{I} \tau \right] \mathbf{e} \mu^2 \left(\mathbf{1} - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \\
& + e^{\tilde{\mathbf{A}}(t+\tau)} \int_{t_0}^{t+\tau} e^{-\tilde{\mathbf{A}}u} \mathbf{e} \mathbf{b}^\top e^{\tilde{\mathbf{A}}u} du e^{-\tilde{\mathbf{A}}t_0} \tilde{\mathbf{A}}^{-1} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \mu \\
& - \tilde{\mathbf{A}}^{-1} e^{\tilde{\mathbf{A}}(t+\tau-t_0)} \mathbf{e} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \mu + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} e^{\tilde{\mathbf{A}}(t-t_0)} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right]
\end{aligned}$$

and its long-run behaviour is established considering $t \rightarrow +\infty$, that is

$$-\tilde{\mathbf{A}}^{-1} \left[\tilde{\mathbf{A}}^{-1} + \mathbf{I}\tau \right] \mathbf{e}\mu^2 \left(\mathbf{1} - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right). \quad (\text{D.13})$$

The fifth term in the right-hand side of (D.8) can be rewritten as

$$\begin{aligned} \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbb{E}_{t_0} [X_u] du \mu &= \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} e^{\tilde{\mathbf{A}}(u-t_0)} du \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e}\mu \right] \mu \\ &- \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} du \tilde{\mathbf{A}}^{-1} \mathbf{e}\mu^2 \\ &= e^{\tilde{\mathbf{A}}(t+\tau-t_0)} (t + \tau - t_0) \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e}\mu \right] \mu \\ &- \tilde{\mathbf{A}}^{-1} \left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} \mathbf{e}\mu^2, \end{aligned}$$

that has the following long-run behaviour

$$\lim_{t \rightarrow +\infty} \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbb{E}_{t_0} [X_u] du \mu = \tilde{\mathbf{A}}^{-1} \tilde{\mathbf{A}}^{-1} \mathbf{e}\mu^2. \quad (\text{D.14})$$

Lemma 1 suggests that the last term in the *rhs* of (D.8) can be written as

$$\begin{aligned} \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbb{E}_{t_0} \left[X_u X_u^\top \right] \mathbf{b} du &= \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{B} e^{\tilde{\mathbf{A}}(u-t_0)} du v l t \left(X_{t_0} X_{t_0}^\top \right) \\ &+ \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{B} e^{\tilde{\mathbf{A}}(u-t_0)} du \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \\ &- \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} du \mathbf{B} \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \\ &+ \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{B} e^{\tilde{\mathbf{A}}u} \left[\int_{t_0}^u e^{-\tilde{\mathbf{A}}h} \tilde{\mathbf{C}} e^{\tilde{\mathbf{A}}h} dh \right] e^{-\tilde{\mathbf{A}}t_0} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e}\mu \right] du. \end{aligned}$$

The result in (A.1) implies that

$$\begin{aligned} \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbb{E}_{t_0} \left[X_u X_u^\top \right] \mathbf{b} du &= \left[\int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{B} e^{\tilde{\mathbf{A}}(u-t_0)} du \right] v l t \left(X_{t_0} X_{t_0}^\top \right) \\ &+ \left[\int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{B} e^{\tilde{\mathbf{A}}(u-t_0)} du \right] \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \\ &- \left(e^{\tilde{\mathbf{A}}(t+\tau-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} \mathbf{B} \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \\ &+ \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{B} e^{\tilde{\mathbf{A}}u} \left[\int_{t_0}^u e^{-\tilde{\mathbf{A}}h} \tilde{\mathbf{C}} e^{\tilde{\mathbf{A}}h} dh \right] e^{-\tilde{\mathbf{A}}t_0} du \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e}\mu \right]. \end{aligned}$$

To determine the asymptotic behaviour of this term, we analyze the long-run behaviour of the

integral $\int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{B} e^{\tilde{\mathbf{A}}(u-t_0)} du$. Exploiting the result in Appendix C, we have

$$\lim_{t \rightarrow +\infty} \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{B} e^{\tilde{\mathbf{A}}(u-t_0)} du = \mathbf{0}$$

as all eigenvalues of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}$ have negative real part. Using the Fubini-Tonelli's Theorem the last integral becomes

$$\int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{B} e^{\tilde{\mathbf{A}}u} \left[\int_{t_0}^u e^{-\tilde{\mathbf{A}}h} \tilde{\mathbf{C}} e^{\tilde{\mathbf{A}}h} dh \right] e^{-\tilde{\mathbf{A}}t_0} du = \int_{t_0}^{t+\tau} \int_{t_0}^u e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{B} e^{\tilde{\mathbf{A}}(u-h)} \tilde{\mathbf{C}} e^{\tilde{\mathbf{A}}(h-t_0)} dh du.$$

Its long-run behaviour is obtained using the result in (C.6), that is

$$\lim_{t \rightarrow +\infty} \int_{t_0}^{t+\tau} \int_{t_0}^u e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbf{B} e^{\tilde{\mathbf{A}}(u-h)} \tilde{\mathbf{C}} e^{\tilde{\mathbf{A}}(h-t_0)} dh du = \mathbf{0}. \quad (\text{D.15})$$

Finally, we have

$$\lim_{t \rightarrow +\infty} \int_{t_0}^{t+\tau} e^{\tilde{\mathbf{A}}(t+\tau-u)} \mathbb{E}_{t_0} \left[X_u X_u^\top \right] \mathbf{b} du = \tilde{\mathbf{A}}^{-1} \mathbf{B} \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right). \quad (\text{D.16})$$

From (D.9), (D.11), (D.13), (D.14) and (D.16) we obtain the limit behaviour for the quantity in (D.8)

$$\begin{aligned} h_\infty(\tau) &:= \lim_{t \rightarrow +\infty} h_{t_0}(t, \tau) \\ &= -\tilde{\mathbf{A}}^{-1} \mu \mathbf{e} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu - \tilde{\mathbf{A}}^{-1} \left[\tilde{\mathbf{A}}^{-1} + \mathbf{I} \tau \right] \mathbf{e} \mu^2 \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \\ &+ \tilde{\mathbf{A}}^{-1} \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu^2 \tilde{\mathbf{A}}^{-1} \mathbf{B} \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right). \end{aligned} \quad (\text{D.17})$$

Using (D.17) we can determine the asymptotic behaviour of (D.3) and we get

$$\begin{aligned} g_\infty(\tau) &:= \lim_{t \rightarrow +\infty} g_{t_0}(t, \tau) \\ &= \lim_{t \rightarrow +\infty} h_{t_0}(t, \tau) - e^{\tilde{\mathbf{A}}\tau} \left[\lim_{t \rightarrow +\infty} h_{t_0}(t, 0) \right] + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu^2 \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \tau \\ &= h_\infty(\tau) - e^{\tilde{\mathbf{A}}\tau} h_\infty(0) + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu^2 \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \tau. \end{aligned} \quad (\text{D.18})$$

By straightforward calculations (D.18) becomes (65) and the covariance reads as in (64). \square

Here we provide the proof of Proposition 7.

Proof. For the asymptotic variance we need to compute the conditional variance of the number of

jumps in an interval with length τ . First we observe that

$$\sigma_{t_0}^2(t, \tau) := \text{Var}_{t_0}(N_{t+\tau} - N_t) = \mathbb{E}_{t_0} \left[(N_{t+\tau} - N_t)^2 \right] - \mathbb{E}_{t_0}^2 [N_{t+\tau} - N_t].$$

We then compute the second moment of the increments

$$\begin{aligned} \mathbb{E}_{t_0} \left[(N_{t+\tau} - N_t)^2 \right] &= \mathbb{E}_{t_0} [N_{t+\tau}^2] + \mathbb{E}_{t_0} [N_t^2] - 2\mathbb{E}_{t_0} [N_t \mathbb{E}_t [N_{t+\tau}]] \\ &= \mathbb{E}_{t_0} [N_{t+\tau}^2] - \mathbb{E}_{t_0} [N_t^2] - 2\mathbb{E}_{t_0} [N_t] \mu \left(\mathbf{1} - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \tau \\ &\quad - 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}\tau} - \mathbf{I} \right] \left[\mathbb{E}_{t_0} [N_t X_t] + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_t] \right]. \end{aligned}$$

For $\mathbb{E}_{t_0} [N_t^2]$ it is useful to compute the infinitesimal operator for the function $f(X_{1,t}, \dots, X_{p,t}, N_t) = N_t^2$, that reads

$$\mathcal{A}f_t = \mu(2N_t + 1) + 2\mathbf{b}^\top N_t X_t + \mathbf{b}^\top X_t.$$

Applying the Dynkin's formula, we have

$$\mathbb{E}_{t_0} [N_t^2] = N_{t_0}^2 + 2\mu \int_{t_0}^t \mathbb{E}_{t_0} [N_u] du + \mu(t - t_0) + 2\mathbf{b}^\top \int_{t_0}^t \mathbb{E}_{t_0} [N_u X_u] du + \mathbf{b}^\top \int_{t_0}^t \mathbb{E}_{t_0} [X_u] dt.$$

Therefore

$$\begin{aligned} \mathbb{E}_{t_0} \left[(N_{t+\tau} - N_t)^2 \right] &= 2\mu \int_t^{t+\tau} \mathbb{E}_{t_0} [N_u] du + \mu\tau + 2\mathbf{b}^\top \int_t^{t+\tau} \mathbb{E}_{t_0} [N_u X_u] du + \mathbf{b}^\top \int_t^{t+\tau} \mathbb{E}_{t_0} [X_u] du \\ &\quad - 2\mathbb{E}_{t_0} [N_t] \mu \left(\mathbf{1} - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \tau - 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}\tau} - \mathbf{I} \right] \left[\mathbb{E}_{t_0} [N_t X_t] + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_t] \right] \\ &= 2\mu \int_t^{t+\tau} \mathbb{E}_{t_0} [N_u - N_t] du + \mu\tau + \mathbf{b}^\top \int_t^{t+\tau} \mathbb{E}_{t_0} [X_u] du \\ &\quad + 2\mathbf{b}^\top \int_t^{t+\tau} \left[\mathbb{E}_{t_0} [N_u X_u] + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_t] \right] du \\ &\quad - 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}\tau} - \mathbf{I} \right] \left[\mathbb{E}_{t_0} [N_t X_t] + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_t] \right]. \end{aligned} \tag{D.19}$$

We study the asymptotic behaviour of the terms in (D.19). We denote with $a_{t_0}(t, \tau) := \int_t^{t+\tau} \mathbb{E}_{t_0} [N_u - N_t] du$

$$\begin{aligned} a_{t_0}(t, \tau) &= \int_t^{t+\tau} \mu \left(\mathbf{1} - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) (u - t) du + \int_t^{t+\tau} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}(u-t)} - e^{\tilde{\mathbf{A}}(t-t)} \right] du \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \right] \\ &= \mu \left(\mathbf{1} - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \frac{\tau^2}{2} + \int_t^{t+\tau} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}(u-t)} - \mathbf{I} \right] du e^{\tilde{\mathbf{A}}(t-t)} \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \right]. \end{aligned}$$

We observe that the following integral is finite

$$\int_t^{t+\tau} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}(u-t)} - \mathbf{I} \right] du < +\infty$$

from where we deduce that

$$a_\infty(\tau) := \lim_{t \rightarrow +\infty} a_{t_0}(t, \tau) = \mu \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \frac{\tau^2}{2}. \quad (\text{D.20})$$

We then concentrate on the quantity $b_{t_0}(t, \tau) := \mu\tau + \mathbf{b}^\top \int_t^{t+\tau} \mathbb{E}_{t_0} [X_u] du$ that through straightforward computations can be written as

$$\begin{aligned} b_{t_0}(t, \tau) &= \mu\tau + \mathbf{b}^\top \int_t^{t+\tau} \left[e^{\tilde{\mathbf{A}}(u-t_0)} \left(X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right) - \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] du \\ &= \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \mu\tau + \mathbf{b}^\top e^{\tilde{\mathbf{A}}(t-t_0)} \int_t^{t+\tau} e^{\tilde{\mathbf{A}}(u-t)} \left(X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right) du. \end{aligned}$$

Since we have a continuous integrand in a compact support

$$\int_t^{t+\tau} e^{\tilde{\mathbf{A}}(u-t)} du < +\infty,$$

we have

$$b_\infty(\tau) := \lim_{t \rightarrow +\infty} b_{t_0}(t, \tau) = \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \mu\tau. \quad (\text{D.21})$$

Denoting with $c_{t_0}(t, \tau) := \int_t^{t+\tau} \left[\mathbb{E}_{t_0} [N_u X_u] + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0} [N_t] \right] du$, we obtain

$$c_{t_0}(t, \tau) = I_{0,t_0}(t, \tau) + I_{1,t_0}(t, \tau) + I_{2,t_0}(t, \tau) + I_{3,t_0}(t, \tau) + I_{4,t_0}(t, \tau) + I_{5,t_0}(t, \tau)$$

where $I_{0,t_0}(t, \tau) := \int_t^{t+\tau} e^{\tilde{\mathbf{A}}(u-t_0)} X_{t_0} N_{t_0} du$ is rewritten as

$$I_{0,t_0}(t, \tau) = e^{\mathbf{A}(t-t_0)\tau} \int_t^{t+\tau} e^{\tilde{\mathbf{A}}(u-t)} X_{t_0} N_{t_0} du$$

and using the same arguments as above, we get

$$I_{0,\infty}(t, \tau) := \lim_{t \rightarrow +\infty} I_{0,t_0}(t, \tau) = \mathbf{0}.$$

The quantity $I_{1,t_0}(t, \tau) := \int_t^{t+\tau} \left(e^{\tilde{\mathbf{A}}(u-t_0)} - \mathbf{I} \right) \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu du$ can be rewritten as

$$I_{1,t_0}(t, \tau) = \int_t^{t+\tau} e^{\tilde{\mathbf{A}}(u-t_0)} \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu du - \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \tau$$

while taking the limit as $t \rightarrow +\infty$, we have

$$I_{1,\infty}(t, \tau) := \lim_{t \rightarrow +\infty} I_{1,t_0}(t, \tau) = -\tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \tau. \quad (\text{D.22})$$

The quantity

$$\begin{aligned} I_{2,t_0}(t, \tau) &:= \int_t^{t+\tau} \int_{t_0}^u e^{\tilde{\mathbf{A}}(u-s)} \mathbf{e} \mathbf{b}^\top e^{\tilde{\mathbf{A}}(s-t_0)} \mathrm{d}s \mathrm{d}u \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \\ &\quad - \int_t^{t+\tau} \left(e^{\tilde{\mathbf{A}}(u-t_0)} - \mathbf{I} \right) \mathrm{d}u \tilde{\mathbf{A}}^{-1} \mathbf{e} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \end{aligned} \quad (\text{D.23})$$

depends on the integral $\int_t^{t+\tau} \int_{t_0}^u e^{\tilde{\mathbf{A}}(u-s)} \mathbf{e} \mathbf{b}^\top e^{\tilde{\mathbf{A}}(s-t_0)} \mathrm{d}s \mathrm{d}u$ where from the substitutions $s - t_0 = h$ and $r = u - t$ we get

$$\int_0^\tau \int_0^{t+r-t_0} e^{\tilde{\mathbf{A}}(t-t_0+r-h)} \mathbf{e} \mathbf{b}^\top e^{\tilde{\mathbf{A}}h} \mathrm{d}h \mathrm{d}r. \quad (\text{D.24})$$

Defining

$$\ddot{\mathbf{A}} := \begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{e} \mathbf{b}^\top \\ \mathbf{0}_{p,p} & \tilde{\mathbf{A}} \end{bmatrix}$$

and applying the result in Appendix C, the inner integral in (D.24) becomes

$$[\mathbf{I}_{p,p}; \mathbf{0}_{p,p}] e^{\ddot{\mathbf{A}}(t-t_0+r)} \begin{bmatrix} \mathbf{0}_{p,p} \\ \mathbf{I}_{p,p} \end{bmatrix}. \quad (\text{D.25})$$

Thus the integral in (D.24) can be computed as follows

$$[\mathbf{I}_{p,p}; \mathbf{0}_{p,p}] e^{\ddot{\mathbf{A}}(t-t_0)} \int_0^\tau e^{\ddot{\mathbf{A}}r} \mathrm{d}r \begin{bmatrix} \mathbf{0}_{p,p} \\ \mathbf{I}_{p,p} \end{bmatrix}. \quad (\text{D.26})$$

We notice that as $\int_0^\tau e^{\ddot{\mathbf{A}}r} \mathrm{d}r < +\infty$ and all eigenvalues of $\ddot{\mathbf{A}}$ have negative real part, then

$$I_{2,\infty}(\tau) := \lim_{t \rightarrow +\infty} I_{2,t_0}(t, \tau) = \tilde{\mathbf{A}}^{-1} \mathbf{e} \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \tau.$$

Similarly, we get the limit for the term $I_{3,t_0}(t, \tau) := \int_t^{t+\tau} \left[\int_{t_0}^u e^{\tilde{\mathbf{A}}(u-s)} \mu \mathbb{E}_{t_0}(N_s) \mathbf{e} \mathrm{d}s + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \mathbb{E}_{t_0}(N_u) \right] \mathrm{d}u$ as $t \rightarrow +\infty$:

$$I_{3,\infty}(t, \tau) := \lim_{t \rightarrow +\infty} I_{3,t_0}(t, \tau) = -\tilde{\mathbf{A}}^{-1} \left[\mathbf{I} \frac{\tau^2}{2} + \tilde{\mathbf{A}}^{-1} \tau \right] \mathbf{e} \mu^2 \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right).$$

We define the following quantity

$$I_{4,t_0}(t, \tau) := \left[\int_t^{t+\tau} e^{\tilde{\mathbf{A}}(u-t_0)} (u-t_0) du \right] \left[X_{t_0} + \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \right] \mu + \tilde{\mathbf{A}}^{-1} \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu^2 \tau \\ - \tilde{\mathbf{A}}^{-1} \int_t^{t+\tau} e^{\tilde{\mathbf{A}}(u-t_0)} du \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu^2$$

and observe that the first integral can be rewritten as

$$\int_t^{t+\tau} e^{\tilde{\mathbf{A}}(u-t_0)} (u-t_0) du = e^{\tilde{\mathbf{A}}(t-t_0)} \int_t^{t+\tau} e^{\tilde{\mathbf{A}}(u-t)} (u-t) du + e^{\tilde{\mathbf{A}}(t-t_0)} (t-t_0) \int_t^{t+\tau} e^{\tilde{\mathbf{A}}(u-t)} du$$

where both terms in the *rhs* tend to zero as $t \rightarrow +\infty$ thus

$$I_{4,\infty}(\tau) = \lim_{t \rightarrow +\infty} I_{4,t_0}(t, \tau) = \tilde{\mathbf{A}}^{-1} \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu^2 \tau.$$

Similar arguments are used to determine the limit as $t \rightarrow +\infty$ for the quantity $I_{5,t_0}(\tau) := \int_t^{t+\tau} \int_{t_0}^u e^{\tilde{\mathbf{A}}(u-s)} \mathbb{E}_{t_0} [X_s, X_s^\top] \mathbf{b} ds du$ as follows

$$I_{5,\infty}(\tau) = \lim_{t \rightarrow +\infty} I_{5,t_0}(t, \tau) = \tilde{\mathbf{A}}^{-1} \mathbf{B} \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \tau.$$

Combining all results, we get the stationary behaviour for the quantity $c_\infty(\tau) := \lim_{t \rightarrow +\infty} c_{t_0}(t, \tau)$ that reads

$$c_\infty(\tau) = -\tilde{\mathbf{A}}^{-1} \mathbf{e} \mu \tau \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) - \tilde{\mathbf{A}}^{-1} \left[\mathbf{I} \frac{\tau^2}{2} + \tilde{\mathbf{A}}^{-1} \tau \right] \mathbf{e} \mu^2 \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) + \tilde{\mathbf{A}}^{-1} \tilde{\mathbf{A}}^{-1} \mathbf{e} \mu^2 \tau \\ + \tilde{\mathbf{A}}^{-1} \mathbf{B} \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \tau. \quad (\text{D.27})$$

Furthermore

$$\lim_{t \rightarrow +\infty} \mathbb{E}_{t_0} \left[(N_{t+\tau} - N_t)^2 \right] = 2\mu a_\infty(\tau) + b_\infty(\tau) + 2\mathbf{b}^\top c_\infty(\tau) - 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}\tau} - \mathbf{I} \right] h_\infty(0) \\ = \mu^2 \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right)^2 \tau^2 + \left(1 - \mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \left(1 - 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \mu \tau \\ + 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \tilde{\mathbf{A}}^{-1} \mathbf{e} \tau \mu^2 \left(\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) + 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \mathbf{B} \tilde{\mathbf{A}}^{-1} \mu \left(\tilde{\mathbf{e}} - \tilde{\mathbf{C}} \tilde{\mathbf{A}}^{-1} \mathbf{e} \right) \tau \\ - 2\mathbf{b}^\top \tilde{\mathbf{A}}^{-1} \left[e^{\tilde{\mathbf{A}}\tau} - \mathbf{I} \right] h_\infty(0).$$

By straightforward calculations, we obtain the result in (66) for the asymptotic variance. \square