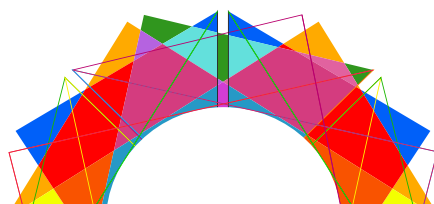


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## 2018 Conference Proceedings



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# Geometry and Origami to Share Cultural Heritage: Results of the Experimentation “The King and the Origami” at the Royal Residence of Venaria

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## Abstract

We present an innovative educational experience, introducing Mathematics and in particular geometry as tools for understanding architectural shapes. We design a set of guided tours for a cultural heritage context, the Royal Residence of Venaria, using the language of origami modeling as a medium to share spatial knowledge.

## Introduction

We present the implementation of a research activity born under a partnership agreement between Politecnico di Torino and Centro Studi delle Residenze Reali Sabaude (CSRRS). The challenge was to design new typologies of visit in a museum context, for visitors of any age and cultural background, involving new targets of users, according to the mission of the Italian *Direzione Generale per la Valorizzazione del Patrimonio Culturale*, see [9].

Our project addresses at people interested to relationships between cultural heritage and scientific disciplines. We propose indeed a new type of interaction with a museum reality, see [4]: visitors are led to read architectural shapes through origami modeling and to understand their geometry using a consistent terminology [3]. In this way we make tangible the space in which the visit takes place, by introduction of geometry as a communication tool, connecting schools and general public of any level with the different possibilities that cultural heritage offers. At the same time, museum spaces become not only educational locations but also learning topics, with the aim of sharing spatial knowledge, in the sense of Arnheim’s theory of Visual Thinking, which says that seeing is the perception of action, while perception itself is the human most active occupation, “consisting in fitting the stimulus material with templates of relatively simple shapes” [1].

## Location

Our experimentation took place at the Royal Residence of Venaria, which is a royal palace near Turin (north-western Italy): a huge architectural complex in Baroque style. It was built during the second half of the seventeenth century by several architects, among these: Amedeo di Castellamonte, Filippo Juvarra, Benedetto Alfieri. After years of degrade, it has undergone a big preservation and restoration project, since 1999. Currently, it is mainly a museum of life and rituals of the Savoy court, but it also supports the CSRRS, whose activity concerns research in history and art history of European courts (with a focus on the Savoy House), organization of cultural events, exhibitions and book publishing, collaboration with the Educational department of the Royal Residence in projects, workshops [12] and various forms of teaching.

## Goals

Unlike other museums, where the heritage observed by visitors is mainly made up of objects (paintings, furniture, etc.), the Royal Residence offers spaces that are interesting from an architectural point of view.

Our goal was to make the visitor discovering the architectural shapes and the geometry hidden in them, thus acquiring a new point of view to perceive and walk through spaces. We have therefore worked on various aspects, taking into account not only the cognitive dimension, the identity of the public and the possible motivations for the visit, but also the organizational problems involved by the inclusion of our project in the educational offer of the Royal Residence.

## Experimentation

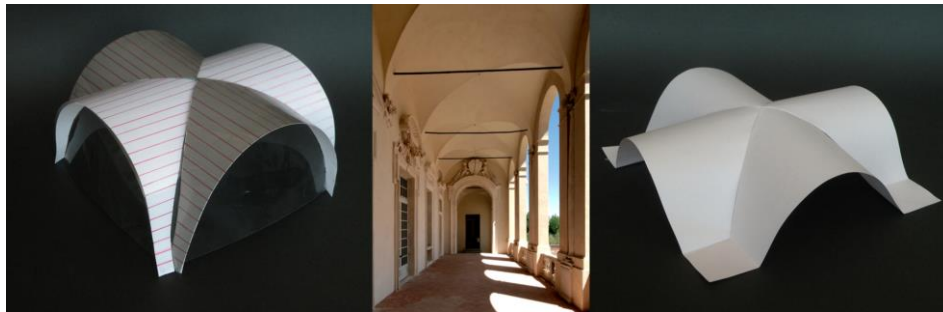
The final version of the project is the result of a two-steps experimentation conducted at the Royal Residence from December 2014 to December 2015: in the first phase, we planned and tested a set of various activities on some group of people (different by age and cultural background), in the second phase we worked with pilot classes to verify the feasibility. The next step was to verify the coherence with the standard spaces and times of visit of the Royal Residence and to control the project from a managerial point of view.

## Tools

From a cognitive point of view, our project is based on an interdisciplinary approach between representation and mathematics. We propose a set of interactive visits calibrated on the level of participants, with activities characterized by the folding of 2D and 3D models (see Table 1 below). Origami is indeed commonly recognized by teachers at all levels of school as a powerful tool to teach mathematics, specifically geometry, because it enhances spatial perception and hand-eye coordination. It helps to achieve terms, shapes and principles and puts fun into learning topics that would otherwise be too abstract – see Boakes [2] and Golan [6] – developing in students the habit to use mathematics to describe the world.

When visitors touch observed shapes with hands, they are no longer passive users, but in the spirit of learn by doing, they come into dialogue, pointing out needs and bringing suggestions or asking questions. In fact, having an origami model in one's hand, allows a tangible comparison between architectural and geometrical idea, their spatial visualization and their realization [8], example in Figure 1.

The observation of a 3D model allows to read proportional relationships, to understand the role of floor slabs and to perceive the geometries of intrados and extrados, difficult to perceive by a not expert person and also to show what the eye does not necessarily capture, due to the inaccessibility of certain areas [7]. This way of introducing architectural elements, figures and geometric properties in an intuitive, practical and unconventional manner presents undoubted advantages even for people with disability (for example, visually impaired persons or people with spatial cognition diseases).





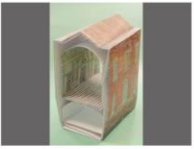



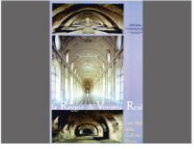

**Figure 1:** Groin vault. Acetate models; Loggia at Royal Residence of Venaria Reale; Origami model.

**Organization**

From an organizational point of view, the inclusion of our project into the Royal Residence educational activities in parallel with the normal visit tour, required attention and collaboration on various practical aspects of its feasibility. On the part of the @polito team the problems concerned: training and introduction of special tutors and their interaction with the local guides; creation of materials; timing of activities. On the part of the Royal Residence Management the problems concerned: specific training for local guides and interaction/collaboration with the new tutors; scheduling of visits; choice of spaces for workshops activities during the guided tour; evaluation of possible interferences between our activities and other visitors; timetable and length of the visit (according to planning and demand of schools); access to areas of particular interest, normally closed to visitors; dissemination of this type of visit operating budget and cost evaluation.

**Table 1: Examples of activities and theoretical focuses.**

<sup>1</sup>Pictures taken during our guided tours experimentation. <sup>2</sup>Courtesy of CSRRS.

Origami Model	2D Shapes <sup>1</sup>	Barrel Vault Translation surface <sup>1</sup>	Three-Section models Translation surface <sup>1</sup>	Cloister Vault Composition of translation surfaces <sup>1</sup>
Van Hiele Theory				
Visualization (age 3-7)	Recognize 2D geometric language/simple polygons in the GG and fold them	Recognize DD geometric language/visualize 3D geometric shapes	/	/
Analysis (age 8-11)	Visualize and describe geometric properties (e.g. symmetry)	=	Visualize and name translation surfaces	Intersection and composition of two semi-circular identical cylinders
Abstraction (age 12-13)	Describe geometric properties (e.g. proportion)	=	Visualize in scale the relative position of surfaces	Intersection and composition of two generic cylinders
Deduction (age 14-16)	Fold conics with origami	=	Design the origami model	Development of ruled surfaces
Rigor (age 17-19)	=	=	Eudclidean Geometry vs Origami: Axioms comparison	Eudclidean Geometry vs Origami: Axioms comparison
Van Hiele Theory				
Related Architecture	Galleria Grande <sup>2</sup>	Galleria dei Ritratti <sup>2</sup>	GR and GG sections <sup>2</sup>	Sala dei Valletti a piedi <sup>2</sup>

**Theoretical background and Activity Scheme**

In planning our set of activities [5], we followed the learning model designed by Pierre Van Hiele and Dina Van Hiele in 1957 and still basic, nowadays in teaching geometry [10]. Their model puts forward a hierarchy of levels that goes from small children to adults, from the simple to the complex: Visualization, Analysis, Abstraction, Deduction and Rigor. These phases outline the development of the thinking process in geometry, in addition to the geometric knowledge acquisition. They suggest an educational strategy which starts by presenting information and ends when the student summarizes what he or she has learned and links it with his daily life activities. These levels are produced by experience and instruction, rather than age; however, it is known, for example, that most elementary schoolers are at the visualization or analysis level, while some middle-school children are at the deduction level.

In Table 1 above we show examples of activities and theoretical focuses that people can tackle in the planned visits. We highlight relationships between origami models, architecture and mathematics within the framework of Van Hiele Theory. In the first column we put the minimum age in which, in general, Van Hiele levels appear. They are repeated for older ages and enriched by new tools and cognitive abilities.

### Conclusions

Since last year, the project is managed by Educational Services Department of Royal Residence of Venaria and it is possible to book this type of visits for almost all school level and general public. Moreover, specific meetings organized by the same services present our project to schools. Results confirm the general appreciation toward this kind of interactive visits, with a steady growth trend: e. g. between March and May 2017, 29 classes participated in the visits, 14 from the first level primary school, 11 from the second level primary school and 4 from the secondary level secondary school, for a total of about 600 students. Even the usual range of teachers which conduct schools to visit heritage artifacts has expanded, with the involvement of maths and physics teachers of primary and secondary schools.

After a year of activity, we observe that the set of proposed activities not only arouses interest and enthusiasm among the participants, but also changes the perception they have of the Royal Residence: no longer a building to be visited but a place where one can live a multisensorial experience. Comparison with the users and the international scientific community [11] encouraged us to look for new outputs and put in evidence additional spaces for the implementation of a training offer also in other places, such as the circuit of the Royal Residences.

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