



Spherical doubly warped spacetimes for radiating stars and cosmology

Carlo Alberto Mantica¹ · Luca Guido Molinari¹

Received: 23 April 2022 / Accepted: 25 August 2022
© The Author(s) 2022

Abstract

Spherically symmetric spacetimes are ambient spaces for models of stellar collapse and inhomogeneous cosmology. We obtain results for the Weyl tensor and the covariant form of the Ricci tensor on general doubly warped (DW) spacetimes. In a spherically symmetric metric, the Ricci and electric tensors become rank-2, built with the metric tensor, a velocity vector field and its acceleration. Their structure dictates the general form of the energy-momentum tensor in the Einstein equations in DW spherical metrics. The anisotropic pressure and the heat current of an imperfect fluid descend from the gradient of the acceleration and the electric part of the Weyl tensor. For radiating stellar collapse with heat flow, the junction conditions of the doubly warped metric with the Vaidya metric are reviewed, with the boundary condition for the radial pressure. The conditions for isotropy simply accommodate various models in the literature. The anisotropy of the Ricci tensor in the special case of spherical GRW space-times (geodesic velocity), gives Friedmann equations deviating from standard FRW cosmology by terms due to the electric tensor. We introduce “perfect 2-scalars” to discuss $f(R)$ gravity with anisotropic fluid source in a doubly warped spacetime, and show that the new geometric terms in the field equations do not change the tensor structure of the fluid energy-momentum tensor.

Keywords Spherical spacetime · Doubly warped spacetime · Stellar collapse · Anisotropic cosmology · $f(R)$ gravity

Mathematics Subject Classification 83C20 (Primary) · 83C55 · 83D05 (Secondary)

✉ Luca Guido Molinari
luca.molinari@mi.infn.it
Carlo Alberto Mantica
carlo.mantica@mi.infn.it

¹ Physics Department Aldo Pontremoli, Università degli Studi di Milano and I.N.F.N. sezione di Milano, Via Celoria 16, 20133 Milan, Italy

Contents

1	Introduction
2	Doubly twisted and doubly warped spacetimes
2.1	Doubly warped (DW) space-times
2.2	Special space metrics g^* in DW spacetimes
2.3	The Weyl tensor on doubly warped spacetimes
2.4	The Ricci tensor on doubly warped spacetimes
2.5	The Ricci tensor on GRW spacetimes
3	Spherical doubly warped spacetimes
3.1	The electric tensor
3.2	The Ricci tensor
4	Radiating stars
4.1	Boundary conditions
5	Pressure isotropy
6	Friedmann equations in spherical GRW spacetimes
7	Spherical DW models in $f(R)$ gravity
7.1	Spherical GRW spacetimes in $f(R)$ gravity
8	Conclusions
9	Appendixes
9.1	Proof of Proposition 2.11
9.2	Proof of equation (41)
9.3	Proof of Lemma 3.4
9.4	A new class of solutions of Eq. (74)
9.5	The doubly warped metric
9.6	Radial space metric, $d=1+N$
9.7	Space components of the electric tensor
	References

1 Introduction

Spherical symmetry is a macroscopic property of many astrophysical objects and of the universe at large. For a static spherical mass, no matter what is the internal spherical distribution, the exterior is described by the Schwarzschild metric (1916), the unique solution of the Einstein equations that is asymptotically Minkowskian. The Vaidya metric (1943) is the non static generalization that accounts for radiation emanating from a spherical mass that changes in time [36]. In dimension $n \geq 4$ it is

$$ds^2 = \left(1 - \frac{2m}{(n-3)r^{n-3}}\right)^{-1} \left[-\frac{m_t^2}{m_r^2} dt^2 + dr^2\right] + r^2 d\Omega_{n-2}^2 \quad (1)$$

$m(t, r)$ is the total mass-radiation energy at time t in a sphere of surface area $4\pi r^2$ ($n = 4$), m_t and m_r are partial derivatives [22, 43]. The interior metric solves the Einstein equations with a spherically symmetric source modelled as an imperfect fluid. In proper coordinates, the metric is specified by three functions [70]

$$ds^2 = -B^2(t, r)dt^2 + C^2(t, r)dr^2 + D^2(t, r)d\Omega_{n-2}^2$$

Examples are the Lemaitre–Bondi–Tolman metric for the gravitational collapse of a sphere of dust [11, 31], models of self-gravitating fluids [37, 38], inhomogeneous

cosmological models [10], neutron stars in extended gravity [1, 2, 17]. The functions satisfy conditions for the metric across a timelike boundary manifold.

Because of the symmetry, the velocity field u_j of the fluid is vorticity-free. It is also shear-free, i.e. $\nabla_i u_j = \varphi(u_i u_j + g_{ij}) - u_i \dot{u}_j$, if the ratio $C(t, r)/D(t, r)$ is independent of time [34, 55]. With this condition, the spacetimes are named by geometers (spherically symmetric) *doubly twisted*:

$$ds^2 = -b^2(t, r)dt^2 + a^2(t, r)[f_1(r)^2 dr^2 + f_2^2(r)d\Omega_{n-2}^2]. \quad (2)$$

The energy momentum tensor in the Einstein equations is often stated in the form

$$T_{ij} = (\mu + p_\perp)u_i u_j + p_\perp g_{ij} + u_i q_j + u_j q_i + (p_r - p_\perp)\chi_i \chi_j \quad (3)$$

where χ_j is a unit vector parallel to q_j and all functions depend on time t and r . The majority of studies are carried to the end in the simplified situation where the twist functions $b(t, r)$ and $a(t, r)$ factor. After a redefinition of time, the result is a (spherically symmetric) *doubly warped* metric:

$$ds^2 = -b^2(r)dt^2 + a^2(t)[f_1(r)^2 dr^2 + f_2^2(r)d\Omega_{n-2}^2]. \quad (4)$$

They are still ambient spaces for models of stellar collapse and certain anisotropic cosmological models that will be considered here, and evolving wormholes [52, 74].

In this study we reverse the approach and first explore the geometry of such spacetimes and only then, turn to physics. The purpose is to clarify the context and the geometric nature of physics terms, in a covariant description.

This is the structure of the paper: in Sect. 2 we investigate general features of doubly twisted and doubly warped (DW) spacetimes, with statements about the Weyl tensor, the Ricci tensor, and the space submanifold.

In Sect. 3 we specialize to spherically symmetric DW spacetimes. We show that the unit radial vector is torse-forming, and obtain the covariant expressions of the Ricci tensor and of the electric component of the Weyl tensor, with a useful identity relating the Ricci tensor of the space submanifold to the electric tensor.

In Sect. 4 the structure of the Ricci tensor, here written with functions $R_i(t, r)$,

$$R_{ij} = R_1 u_i u_j + R_2 g_{ij} + R_3 (u_i \dot{u}_j + u_j \dot{u}_i) + R_4 \left(\frac{\dot{u}_i \dot{u}_j}{\dot{u}^k \dot{u}_k} - \frac{h_{ij}}{n-1} \right)$$

entrains a similar one for the energy-momentum tensor T_{ij} . The pressure anisotropy is encoded in R_4 that results from the electric tensor and the gradient of the acceleration \dot{u}_j . The boundary conditions connecting the general spherical DW metric to the Vaidya metric are reviewed along with the prescriptions by Santos [60]. The parameters of the inner metric determine the mass function $m(t, r)$ and a differential equation for the inner scale function $a(t)$. The latter has the physical meaning that, at the boundary, the radial pressure equals the modulus of the heat current.

In Sect. 5 we study isotropic DW spherical solutions. The isotropy condition $R_4 = 0$ determines equations for the fields b , f_1 and f_2 in (2), that provide a frame for models of stellar collapse studied by various authors.

In Sect. 6 we consider the spherical spacetimes (2) with $b(r) = 1$ (equivalent to $\dot{u}_i = 0$). For such Generalized Robertson–Walker (GRW) metric we obtain the analogue of the Friedmann equations, where pressure is anisotropic because of the electric tensor.

In Sect. 7 we introduce perfect 2-scalars, in connection with the presence of two distinguished vectors, u_i and \dot{u}_i that build the relevant tensors. They extend the fruitful concept of perfect 1-scalars introduced by us in connection with a rank-1 Ricci tensor (quasi-Einstein or perfect fluid) in RW spacetimes, with a single distinguished vector [19]. We prove that, in $f(R)$ gravity with anisotropic fluid source, the geometric corrections have the same tensor form of the fluid source.

Sections 8 and 9 contain conclusions and appendixes.

We employ Latin letters for spacetime indices and Greek letters for space ones. A dot is the directional derivative along u : $\dot{X} = u^k \nabla_k X$; for $b \neq 1$ it does not coincide with the time derivative in the comoving frame. To facilitate the reading, we postponed to appendixes some long or marginal proofs.

2 Doubly twisted and doubly warped spacetimes

Doubly twisted spacetimes are a large class of Lorentzian manifolds introduced by Kentaro Yano in 1940 [72]. He showed that the metric structure¹

$$ds^2 = -b^2(t, \mathbf{x})dt^2 + a^2(t, \mathbf{x})g_{\mu\nu}^*(\mathbf{x})dx^\mu dx^\nu \tag{5}$$

is necessary and sufficient for the spatial foliation to be totally umbilical. Hereafter, the Riemannian spatial submanifold (M^*, g^*) has dimension $n - 1$.

In physics the best known examples are the Stephani universes [42, 67]. They are conformally flat solutions of the Einstein equations with perfect fluid source. In $n = 4$:

$$ds^2 = -b^2(t, \mathbf{x})dt^2 + R^2(t)(dx^2 + dy^2 + dz^2)/V^2(t, \mathbf{x})$$

with $V = 1 + \frac{1}{4} \|\mathbf{x} - \mathbf{x}_0(t)\|^2$, $b^2 = F(t)[\dot{V}/V - \dot{R}/R]$. Another example is the conformally flat solution for a perfect fluid with heat flux by Banerjee et al. [5]:

$$ds^2 = -V^2(t, \mathbf{x})dt^2 + (dx^2 + dy^2 + dz^2)/U^2(t, \mathbf{x})$$

with $UV = A(t)\|\mathbf{x}\|^2 + \mathbf{A}(t) \cdot \mathbf{x} + A_0(t)$, $U = B(t)\|\mathbf{x}\|^2 + (t) \cdot \mathbf{x} + B_0(t)$, where the functions of time are arbitrary.

As shown in [50], doubly twisted spacetimes may be covariantly characterised by the existence of a *time-like doubly torqued* vector field: $\tau_i \tau^i < 0$ and

$$\nabla_j \tau_k = \kappa g_{jk} + \alpha_j \tau_k + \tau_j \beta_k \tag{6}$$

¹ Some authors exchange the letters a^2 and b^2 . Here a^2 is minding the scale function of RW metric, a limit case in this class.

with vector fields $\alpha_j \tau^j = 0$ and $\beta_j \tau^j = 0$. Depending on α_i, β_i and κ , a classification of doubly twisted spacetimes results (Table 1 in [51]) that includes twisted, doubly warped (DW), generalized Robertson–Walker (GRW) and static spacetimes.

In the *comoving frame*, defined by space components $\tau_\mu = 0$, the spacelike vectors α_i and β_i are in simple relation to the functions a and b in the metric (5) [50]:

$$\begin{aligned} \alpha_0 &= 0 & \beta_0 &= 0 & \kappa &= \frac{\partial_t a}{b} \\ \alpha_\mu &= \partial_\mu \log a & \beta_\mu &= -\partial_\mu \log b \end{aligned} \tag{7}$$

An equivalent description is in terms of the time-like unit vector field [30]

$$u_j = \frac{\tau_j}{\sqrt{-\tau^2}}, \quad u^j u_j = -1.$$

Equation (6) becomes $\nabla_i u_j = \varphi(g_{ij} + u_i u_j) + u_i \beta_j$, with $\varphi = \kappa/\sqrt{-\tau^2}$. The contraction with u^i gives $\beta_j = -u^k \nabla_k u_j = -\dot{u}_j$. Therefore, the second characterization of doubly twisted spacetimes is:

$$\nabla_i u_j = \varphi(g_{ij} + u_i u_j) - u_i \dot{u}_j. \tag{8}$$

The following result introduces doubly warped spacetimes:

Proposition 2.1 *The vector α_i in Eq. (6) is closed if and only if $\nabla_i \varphi = -u_i \dot{\varphi} - \varphi \dot{u}_i$.*

Proof The gradient of $\varphi = \kappa/\sqrt{-\tau^2}$ and the identity $\nabla_i \tau^2 = 2\kappa \tau_i + 2\alpha_i \tau^2$ give

$$\frac{\nabla_i \varphi}{\varphi} - \varphi u_i = \frac{\nabla_i \kappa}{\kappa} - \alpha_i \tag{9}$$

Another gradient and antisymmetrization in the indices give

$$\nabla_i \alpha_j - \nabla_j \alpha_i = -\varphi(u_i \dot{u}_j - u_j \dot{u}_i) + u_j \nabla_i \varphi - u_i \nabla_j \varphi$$

If α_i is closed, contraction with u^i gives the property. The opposite is also true. □

2.1 Doubly warped (DW) space-times

A doubly warped space-time is a special case of the metric (5) where b^2 is independent of time and a^2 is independent of position:

$$ds^2 = -b^2(\mathbf{x})dt^2 + a^2(t)g_{\mu\nu}^*(\mathbf{x})dx^\mu dx^\nu \tag{10}$$

We present three equivalent covariant characterisations. We shall exploit the one based on a time-like unit vector.

- Doubly torqued vector (Mantica, Molinari [50, 51]) *A Lorentzian manifold is doubly-warped if and only if there is a time-like doubly torqued vector field (6) with α_j and β_j closed.*

With the metric (10) the relations (7) show that $\alpha_i = 0, \beta_0 = 0$ and $\beta_\mu = -\partial_\mu \log b(\mathbf{x})$. Then β_i is closed. One can relax the condition $\alpha_i = 0$ to α_i closed by noting that a rescaled vector $\tau' = \lambda \tau$ is still doubly torqued if $\tau^i \partial_i \lambda = 0$ i.e. λ does not depend on time in the frame (10). The metric changes in the factor $a'(t, \mathbf{x}) = a(t)\lambda(\mathbf{x})$, which amounts to a conformal change of the inner metric $g^*(\mathbf{x})$ of the submanifold.

- Conformal Killing vector (Ramos et al. [58], Th.1) *A space-time is doubly-warped if and only if there exists a non-null, nowhere zero and hypersurface orthogonal vector field ζ_i that is conformal Killing, $\mathfrak{L}_\zeta g_{ij} = 2\kappa g_{ij}$, with $\nabla_j \kappa = \sigma \zeta_j$ for some nonzero scalar function σ .* The equivalence is proven in the next Proposition 2.3.

- Timelike unit vector *A Lorentzian manifold is doubly-warped if and only if there is a time-like unit vector field such that:*

$$\nabla_i u_j = \varphi(u_i u_j + g_{ij}) - u_i \dot{u}_j \tag{11}$$

$$\nabla_j \varphi = -u_j \dot{\varphi} - \varphi \dot{u}_j \tag{12}$$

$$\nabla_i \dot{u}_j = \nabla_j \dot{u}_i \tag{13}$$

Everything follows from Proposition 2.1 and the identification $\beta_i = -\dot{u}_i$, with β_i closed.

- If $\varphi = 0$ then $\nabla_i u_j = -u_i \dot{u}_j$ and $\nabla_i \dot{u}_j = \nabla_j \dot{u}_i$: this is the definition of static space-time ([67], p.283).
- If $\dot{u}_j = 0$ then $\nabla_i u_j = \varphi(u_i u_j + g_{ij})$ with $\nabla_j \varphi = -u_j \dot{\varphi}$: this is a generalized Robertson–Walker spacetime [48]. Equivalently: $b = 1$ in the metric (10).

Lemma 2.2 *In a DW spacetime, φu_i is closed and orthogonal to hypersurfaces of constant time.*

Proof $\nabla_i(\varphi u_j) = \varphi[\varphi(u_i u_j + g_{ij}) - u_i \dot{u}_j] + u_j(-u_i \dot{\varphi} - \dot{u}_i \varphi) = (\varphi^2 - \dot{\varphi})u_i u_j + \varphi^2 g_{ij} - \varphi(u_i \dot{u}_j + u_j \dot{u}_i)$. Then $\nabla_i(\varphi u_j) = \nabla_j(\varphi u_i)$ and $\varphi u_j = \partial_j F$ (a gradient). In the comoving frame: $u_0 = -b, u_\mu = 0$ and (see Appendix 9.5)

$$\varphi = \frac{\nabla_j u^j}{n-1} = \frac{1}{n-1} \Gamma_{\mu 0}^\mu u^0 = \frac{a_t}{ab} \tag{14}$$

Then $F = -\log a(t)$ and u_i is orthogonal to hypersurfaces of constant time. □

Proposition 2.3 *In a DW spacetime there exists a scalar function θ such that*

$$\nabla_i(\theta u_j) = (\theta \varphi)g_{ij} + \dot{u}_i(\theta u_j) - (\theta u_i)\dot{u}_j$$

Then θu_i is a conformal Killing vector (and a torse-forming vector).

Proof The conformal Killing condition $\mathfrak{L}_{\theta u} g_{ij} = \nabla_i(\theta u_j) + \nabla_j(\theta u_i) = 2\kappa g_{ij}$ is satisfied if and only if $\kappa = \theta \varphi$ and $u_i(\nabla_j \theta + \varphi \theta u_j - \theta \dot{u}_j) + u_j(\nabla_i \theta + \varphi \theta u_i - \theta \dot{u}_i) = 0$ i.e. $\nabla_i \log \theta = -(\varphi u_i) + \dot{u}_i$. The function θ exists because both vectors φu_i and \dot{u}_i are gradients. It is also $\nabla_i(\theta \varphi) = -(\varphi^2 + \dot{\varphi})(\theta u_i)$ (the homothetic case $\varphi^2 + \dot{\varphi} = 0$ is not considered here). □

2.2 Special space metrics g^* in DW spacetimes

The presence of the distinguished spacelike vector \dot{u}_i allows for a classification of the metrics g^* of the space sub-manifold. The gradient of the vector is decomposed in its components along u_i, \dot{u}_i , and orthogonal ones, through the projector

$$N_{jk} = g_{jk} + u_j u_k - \frac{1}{\eta} \dot{u}_j \dot{u}_k, \quad \eta \equiv \dot{u}_j \dot{u}^j \tag{15}$$

Proposition 2.4 *On a doubly warped spacetime:*

$$\nabla_j \dot{u}_k = -\eta u_j u_k + \varphi(u_j \dot{u}_k + \dot{u}_j u_k) + \dot{u}_j \dot{u}_k \frac{\dot{u}^i \nabla_i \eta}{2\eta^2} + N_{jk} \frac{N_{rs} \nabla^r \dot{u}^s}{n-2} \tag{16}$$

$$+ \dot{u}_j w_k + w_j \dot{u}_k + \Pi_{jk}$$

$$w_k = \frac{1}{2\eta} N_{km} \nabla^m \eta \tag{17}$$

$$\Pi_{jk} = \left[N_{jr} N_{ks} - \frac{N_{jk} N_{rs}}{n-2} \right] \nabla^r \dot{u}^s \tag{18}$$

$$\nabla_j \eta = 2\varphi \eta u_j + \dot{u}_j \frac{\dot{u}^i \nabla_i \eta}{\eta} + 2\eta w_j \tag{19}$$

where w_k is orthogonal to u_j and \dot{u}_j , Π_{ij} is traceless symmetric and annihilates u_i and \dot{u}_i . Both w_k and Π_{ij} are spacelike tensors.

Proof The procedure is laborious but standard [26]. Write $\nabla_j \dot{u}_k = g_{jr} g_{ks} \nabla^r \dot{u}^s$ and replace the metric tensor g_{jr} with $N_{jr} - u_j u_r + \frac{1}{\eta} \dot{u}_j \dot{u}_r$ and similarly for g_{ks} . Use $u^i \dot{u}_i = 0$ and the symmetry of $\nabla_i \dot{u}_j$ to simplify terms: $u_r \nabla^r \dot{u}^s = u_r \nabla^s \dot{u}^r = -\dot{u}_r \nabla^s u^r = -\varphi \dot{u}^s + u^s \eta$. Then: $u^r \nabla_r \eta = 2u^r \dot{u}^s \nabla_r \dot{u}_s = -2\varphi \eta$. \square

Now consider the decomposition (16) in the comoving frame, where $u_\mu = 0$. There it is $\dot{u}_0 = 0$, and \dot{u}_μ are the space components. Since $\eta = g^{*\mu\nu}(\mathbf{x}) \dot{u}_\mu \dot{u}_\nu / a^2(t)$, the vector field

$$\hat{n}_\mu = \frac{\dot{u}_\mu}{a(t)\sqrt{\eta}} \tag{20}$$

is a unit vector field on M^* : $1 = g^{*\mu\nu} \hat{n}_\mu \hat{n}_\nu$. The space components of the decomposition (16) give:

Proposition 2.5

$$\nabla_\mu^* \hat{n}_\nu = \frac{\Theta}{n-2} (g_{\mu\nu}^* - \hat{n}_\mu \hat{n}_\nu) + \hat{n}_\mu w_\nu + \Pi_{\mu\nu}^*$$

$$\Theta = \frac{a}{\sqrt{\eta}} N_{rs} \nabla^r \dot{u}^s = \nabla_\mu^* \hat{n}^\mu, \quad \Pi_{\mu\nu}^* = \frac{\Pi_{\mu\nu}}{a\sqrt{\eta}} \tag{21}$$

- If $\Pi_{\mu\nu}^* = 0$ the space submanifold is doubly twisted ([30] Eq. 4, [12] table 1, [29] Eq. 53.14). There are coordinates $\mathbf{x} = (\tilde{x}, \mathbf{y})$ and functions f_1 and f_2 such that

$$g_{\mu\nu}^* dx^\mu dx^\nu = f_1^2(\tilde{x}, \mathbf{y}) dx^2 + f_2^2(\tilde{x}, \mathbf{y}) g^{**}(\mathbf{y})_{qp} dy^q dy^p$$

- If $\Pi_{\mu\nu}^* = 0$ and $w_\mu = 0$ the space submanifold is twisted [49, 54]. There are coordinates and a function f such that

$$g_{\mu\nu}^* dx^\mu dx^\nu = d\tilde{x}^2 + f^2(\tilde{x}, \mathbf{y}) g^{**}(\mathbf{y})_{qp} dy^q dy^p$$

- If $\Pi_{\mu\nu}^* = 0$, $w_\mu = 0$ and $\partial_\mu \Theta = \hat{n}_\mu \hat{n}^\nu \partial_\nu \Theta$ the space submanifold is warped [47]. There are coordinates and a function f such that

$$g_{\mu\nu}^* dx^\mu dx^\nu = d\tilde{x}^2 + f^2(\tilde{x}) g^{**}(\mathbf{y})_{qp} dy^q dy^p$$

The vector \hat{n}_μ is an eigenvector of the Ricci tensor with eigenvalue ξ^* . The integrability condition and the Weyl tensor of the space submanifold give the Ricci tensor.²

$$R_{\mu\nu}^* = \xi^* \hat{n}_\mu \hat{n}_\nu + \frac{R^* - \xi^*}{n - 2} (g_{\mu\nu}^* - \hat{n}_\mu \hat{n}_\nu) + (n - 3) \hat{n}^\tau \hat{n}^\sigma C_{\tau\mu\nu\sigma}^* \tag{22}$$

where $R^* = g^{*\mu\nu} R_{\mu\nu}^*$.

2.3 The Weyl tensor on doubly warped spacetimes

Kentaro Yano (1957, [73] Eq. 3.8 page 161) proved that if v is a proper conformal vector field, then:

- (1) the Lie derivative along v of the Weyl tensor is zero,

$$\mathfrak{L}_v C_{jkl}{}^m = 0,$$

- (2) the Lie derivative of the divergence of the Weyl tensor is (see [64]):

$$\mathfrak{L}_v (\nabla_m C_{jkl}{}^m) = (n - 3) (\nabla_m \kappa) C_{jkl}{}^m. \tag{23}$$

In Proposition 2.3 we showed that a DW space-time is always endowed with a proper conformal Killing vector θu_i . Therefore, the above two properties hold true in DW spacetimes. The second one gives an interesting result.

First note that being u_i proportional to a doubly torqued vector, it is Weyl compatible (the proof for doubly torqued vectors is given in [51]):

$$u_i C_{jklm} u^m + u_j C_{kilm} u^m + u_k C_{ijlm} u^m = 0. \tag{24}$$

² the proof is analogous to that of (27), with the difference that the metric g^* is Riemannian.

Proposition 2.6 *On a doubly warped spacetime of dimension n :*

$$\nabla_m C_{jkl}{}^m = 0 \implies C_{jklm} u^m = 0 \text{ and } C_{jklm} \dot{u}^m = 0, \quad (n \geq 4) \quad (25)$$

$$\nabla_m C_{jkl}{}^m = 0 \implies C_{jklm} = 0 \quad (n = 4) \quad (26)$$

Proof If $\nabla_m C_{jkl}{}^m = 0$, then $\mathcal{L}_{\theta u} \nabla_m C_{jkl}{}^m = 0$. According to Eq. (23) it is

$$0 = \nabla_m (\theta \varphi) C_{jkl}{}^m = -(\varphi^2 + \dot{\varphi}) C_{jkl}{}^m (\theta u_m)$$

Then, if $\varphi^2 + \dot{\varphi} \neq 0$: $C_{jkl}{}^m u_m = 0$.

With $\nabla_m C_{jkl}{}^m = 0$ the second Bianchi identity for the Weyl tensor [3] has the simpler form

$$\nabla_i C_{jklm} + \nabla_j C_{kilm} + \nabla_k C_{ijlm} = 0.$$

The contraction with u^m and the relation $(\nabla_i C_{jklm}) u^m = C_{jklm} \nabla_i u^m$ give:

$$\begin{aligned} 0 &= -C_{jklm} \nabla_i u^m - C_{kilm} \nabla_j u^m - C_{ijlm} \nabla_k u^m \\ &= u_i C_{jklm} \dot{u}^m + u_j C_{kilm} \dot{u}^m + u_k C_{ijlm} \dot{u}^m - \varphi (C_{jkli} + C_{kilj} + C_{ijlk}) \end{aligned}$$

The contraction with u^i and the first Bianchi identity conclude the first proof.

The spacetime dimension $n = 4$ is special: Lovelock proved that $C_{jklm} u^m = 0$ for a non-null vector implies $C_{jklm} = 0$ (see [44], p 128). \square

The electric tensor $E_{kl} = u^j u^m C_{jklm}$ is symmetric, traceless and $E_{ij} u^j = 0$. We'll show that it appears in the expression of the Ricci tensor. The property (24) implies $C_{jklm} u^m = u_k E_{jl} - u_j E_{kl}$. Then $C_{jklm} u^m = 0 \iff E_{jk} = 0$, and

Corollary 2.7 $\nabla_m C_{jkl}{}^m = 0 \implies E_{ij} = 0$.

We conclude the section with quotations:

- In a DW spacetime if $\nabla_i C_{jklm} = 0$ then $C_{jklm} = 0$ (Hotloś [40], Th.6).
- In a DW spacetime if $C_{jklm} = 0$ then $C_{jklm}^* = 0$ (Gębarowski [33]).
- In a GRW spacetime ($b = 1$ i.e. $\dot{u}_i = 0$): $\nabla_m C_{jkl}{}^m = 0 \iff C_{jklm} u^m = 0$ [47].
- Banerjee [4] proved that $ds^2 = -b^2(\mathbf{x}, t) dt^2 + a^2(\mathbf{x}, t) \delta_{\mu\nu} dx^\mu dx^\nu$ is conformally flat ($C_{jkl}{}^m = 0$) if and only if $b(\mathbf{x}, t) = a(\mathbf{x}, t)[A(t)\mathbf{x} \cdot \mathbf{x} + B(t) \cdot \mathbf{x} + C(t)]$ for any $A(t)$, $B(t)$ and $C(t)$. As a special case we have:

Proposition 2.8 *The doubly warped spacetime $ds^2 = -b^2(\mathbf{x}) dt^2 + a^2(t) \delta_{\mu\nu} dx^\mu dx^\nu$ is conformally flat if and only if $b(r) = Kr^2 + C$, with constants K , C .*

2.4 The Ricci tensor on doubly warped spacetimes

We obtain the general covariant expression of the Ricci tensor, and a useful identity among the Ricci tensor $R_{\mu\nu}^*$ of the space submanifold and the electric tensor.

Proposition 2.9 *The Ricci tensor on a doubly warped spacetime is*

$$\begin{aligned}
 R_{jk} = & u_j u_k \left[\frac{R - n\xi}{n - 1} + \nabla^p \dot{u}_p + \eta + \frac{\dot{u}^s \nabla_s \eta}{2\eta} \right] + g_{jk} \left[\frac{R - \xi}{n - 1} + \eta + \frac{\dot{u}^s \nabla_s \eta}{2\eta} \right] \\
 & - (n - 2)\varphi(u_j \dot{u}_k + \dot{u}_j u_k) - \frac{\dot{u}_j \dot{u}_k}{\eta} \left[(n - 1) \left(\eta + \frac{\dot{u}^i \nabla_i \eta}{2\eta} \right) - \nabla^p \dot{u}_p \right] \\
 & - (n - 2)[\dot{u}_j w_k + w_j \dot{u}_k + \Pi_{jk} + E_{jk}]
 \end{aligned} \tag{27}$$

where $\xi = (n - 1)(\varphi^2 + \dot{\varphi})$, $\eta = \dot{u}_p \dot{u}^p$ and $E_{kl} = u^j u^m C_{jklm}$ (electric tensor). The vector w_j and the tensor Π_{ij} were given in Proposition 2.4.

Proof The integrability condition $R_{jkl}{}^m u_m = [\nabla_j, \nabla_k]u_l$ with (11) becomes:

$$\begin{aligned}
 R_{jklm} u^m = & (\varphi^2 + \dot{\varphi})(u_k g_{jl} - u_j g_{kl}) - \varphi(\dot{u}_j g_{kl} - \dot{u}_k g_{jl}) \\
 & + u_j(\dot{u}_k \dot{u}_l + \nabla_k \dot{u}_l) - u_k(\dot{u}_j \dot{u}_l + \nabla_j \dot{u}_l)
 \end{aligned}$$

The contraction with g^{jl} is $R_{km} u^m = (\xi - \eta - \nabla_p \dot{u}^p)u_k + (n - 1)\varphi \dot{u}_k + u^j \nabla_k \dot{u}_j$. With the simplification $u^j \nabla_k \dot{u}_j = -\dot{u}^j \nabla_k u_j = -\varphi \dot{u}_k + \eta u_k$, the result is:

$$R_{km} u^m = (\xi - \nabla^p \dot{u}_p)u_k + (n - 2)\varphi \dot{u}_k \tag{28}$$

An expression for the Ricci tensor is now obtained with the Weyl tensor

$$C_{jklm} = R_{jklm} + \frac{g_{jm} R_{kl} - g_{km} R_{jl} + R_{jm} g_{kl} - R_{km} g_{jl}}{n - 2} - R \frac{g_{jm} g_{kl} - g_{km} g_{jl}}{(n - 1)(n - 2)}$$

The contraction with $u^j u^m$ and the previous relations give:

$$\begin{aligned}
 E_{kl} = & (\varphi^2 + \dot{\varphi})(u_k u_l + g_{kl}) - \dot{u}_k \dot{u}_l - \nabla_k \dot{u}_l - u_k(u^j \nabla_j \dot{u}_l) - \varphi u_k \dot{u}_l \\
 & - \frac{R_{kl} + (\xi - \nabla^p \dot{u}_p)(g_{kl} + 2u_k u_l)}{n - 2} + R \frac{g_{kl} + u_k u_l}{(n - 1)(n - 2)}.
 \end{aligned}$$

The Ricci tensor is obtained:

$$\begin{aligned}
 R_{kl} = & u_k u_l \left[\frac{R - n\xi}{n - 1} + 2\nabla_p \dot{u}^p \right] + g_{kl} \left[\frac{R - \xi}{n - 1} + \nabla_p \dot{u}^p \right] \\
 & - (n - 2)(\nabla_k \dot{u}_l + \dot{u}_k \dot{u}_l + u_k \ddot{u}_l + \varphi u_k \dot{u}_l + E_{kl})
 \end{aligned}$$

Note that $\ddot{u}_l = u^j \nabla_j \dot{u}_l = u^j \nabla_l \dot{u}_j = -\dot{u}^j \nabla_l u_j = -\varphi \dot{u}_l + u_l \eta$. This simplifies the last line. Next insert the decomposition (16) for $\nabla_k \dot{u}_l$:

$$\begin{aligned}
 R_{kl} = & u_k u_l \left[\frac{R - n\xi}{n - 1} + 2\nabla_\rho \dot{u}^\rho - N_{rs} \nabla^r \dot{u}^s \right] + g_{kl} \left[\frac{R - \xi}{n - 1} + \nabla_\rho \dot{u}^\rho - N_{rs} \nabla^r \dot{u}^s \right] \\
 & - (n - 2)\varphi(u_k \dot{u}_l + \dot{u}_k u_l) - (n - 2) \frac{\dot{u}_k \dot{u}_l}{\eta} \left[\eta + \frac{u^p \nabla_p \eta}{2\eta} - \frac{N_{rs} \nabla^r \dot{u}^s}{n - 2} \right] \\
 & - (n - 2)(\dot{u}_k w_l + \dot{u}_l w_k + \Pi_{kl} + E_{kl})
 \end{aligned}$$

Finally, specify the term

$$N_{rs} \nabla^r \dot{u}^s = \nabla_r \dot{u}^r + u^r u^s \nabla_r \dot{u}_s - \frac{\dot{u}_r \dot{u}_s}{\eta} \nabla^r \dot{u}^s = \nabla_\rho \dot{u}^\rho - \eta - \frac{\dot{u}^s \nabla_s \eta}{2\eta} \tag{29}$$

The expression (27) is now obtained. □

The electric tensor is spacelike. In the comoving frame (10) where $u^\mu = 0$, it is $E_{00} = E_{0\mu} = 0$. The evaluation of the space components $E_{\mu\nu}$ (in Appendix 9.7) provides the following relation with $R^*_{\mu\nu}$, the Ricci tensor on (M^*, g^*) :

Proposition 2.10 *On a doubly warped spacetime:*

$$R^*_{\mu\nu} - \frac{R^*}{n - 1} g^*_{\mu\nu} = -(n - 3) \frac{1}{b} \left[\nabla^*_\mu b_\nu - \frac{\nabla^*_\rho b^\rho}{n - 1} g^*_{\mu\nu} \right] - (n - 2) E_{\mu\nu}. \tag{30}$$

where $b_\nu = \partial_\nu b$ and $b^\mu = g^{*\mu\nu} \partial_\nu b$, $R^* = g^{*\mu\nu} R^*_{\mu\nu}$.

The identity extends a result obtained by Gȩbarowski ([32], Eq. 23) under the restriction of harmonic Weyl tensor ($\nabla_m C_{jkl}{}^m = 0$ i.e. $E_{\mu\nu} = 0$). It will be very useful in conjunction with spherical symmetry.

Proposition 2.11

$$\nabla_k \xi = -u_k \dot{\xi} - 2\dot{u}_k \xi \tag{31}$$

$$u^k \nabla_k (\nabla_\rho \dot{u}^\rho) = -2\varphi (\nabla^p \dot{u}_p) \tag{32}$$

$$\dot{R} - 2\dot{\xi} = -2\varphi (R - n\xi) \tag{33}$$

The proofs are in Appendix 9.1. Remarkably, Eq. (33) has the same simple form as in GRW and RW spacetimes [19]. Its solution Eq. (105) can be written as

$$R = \frac{R^*}{a^2} - 2\nabla_j \dot{u}^j + 2\xi + (n - 1)(n - 2)\varphi^2. \tag{34}$$

2.5 The Ricci tensor on GRW spacetimes

On GRW spacetimes ($\dot{u}_i = 0$) the Ricci tensor is [48]:

$$R_{jk} = u_j u_k \frac{R - n\xi}{n - 1} + g_{jk} \frac{R - \xi}{n - 1} - (n - 2)E_{jk} \tag{35}$$

Equation (30) shows that the Ricci tensor is perfect fluid ($E_{jk} = 0$) if and only if $R_{\mu\nu}^*$ is Einstein. The Bianchi identity, Eq.(31) (with $\dot{u}_i = 0$) and (33) give

$$\nabla_k R = -u_k \dot{R} - \frac{2(n - 1)(n - 2)}{n - 3} \nabla^j E_{jk} \tag{36}$$

By Eq.(30) the spacelike vector $\nabla^j E_{jk}$ has space components proportional to $\nabla_\mu R^*$.

Some examples are studied by Coley and McManus [45], with the energy-momentum tensor (3) with $q_i = 0$. They show that if u_i is shear-free, vorticity-free and geodesic, then the spacetime is GRW.

3 Spherical doubly warped spacetimes

From now on we restrict to DW spacetimes with spherical symmetry:

$$ds^2 = -b^2(r)dt^2 + a^2(t) \left[f_1^2(r)dr^2 + f_2^2(r)d^2\Omega_{n-2}^2 \right] \tag{37}$$

The coordinates of (M^*, g^*) are r and the $n - 2$ angles θ_a of the sphere S_{n-2} . The space metric tensor is diagonal with $g_{rr}^* = f_1^2(r)$ and $g_{aa}^* = f_2^2(r)g_a^2(\theta)$.

The main results that are obtained: the conformal flatness of M^* , the covariant forms of the electric and Ricci tensors.

Proposition 3.1 M^* is conformally flat ($C_{\mu\nu\rho\sigma}^* = 0$).

Proof The change of variable $d\rho = f_1(r)dr$, makes the metric g^* warped: $ds_\star^2 = d\rho^2 + (f_2 \circ r)(\rho)^2 d\Omega_{n-2}^2$. The fiber S_{n-2} is a constant curvature submanifold, then M^* is conformally flat by Theorem 1.i (warped manifolds) in ref. [13] by Brozos-Vazquez et al. □

In spherical coordinates, the unit radial vector on M^* is

$$n_r = f_1(r), \quad n_a = 0 \quad (a = 1, \dots, n - 2).$$

Proposition 3.2 The unit radial vector is torse-forming:

$$\nabla_\mu^* n_\nu = \frac{\Theta}{n - 2} (g_{\mu\nu}^* - n_\mu n_\nu) \tag{38}$$

$$\Theta = \frac{n - 2}{f_1} \frac{d}{dr} \log f_2 \tag{39}$$

$\partial_\mu \Theta$ is proportional to n_μ .

Proof The equations are checked in spherical components with the Christoffel symbols in Appendix 9.6:

$$\begin{aligned} \nabla_r^* n_r &= \partial_r f_1 - \Gamma_{rr}^{*r} f_1 = 0 = \partial_r f_1 - \frac{f_1}{2} g^{*rr} \partial_r g_{rr}^* = 0, \\ \nabla_r^* n_a &= -\Gamma_{ra}^{*r} f_1 = -\frac{1}{2} g^{*rr} \partial_a g_{rr}^* = 0. \\ \nabla_a^* n_{a'} &= -\Gamma_{aa'}^{*r} f_1 = \delta_{aa'} g_a^2(\theta) \frac{f_2}{f_1} \frac{df_2}{dr} \end{aligned}$$

In the last equation one reads $\Theta = (n - 2)(1/f_1)\partial_r \log f_2$. Since it is a function only of r , it is $\partial_\mu \Theta = (n_\mu/f_1)(\partial_r \Theta)$. □

In the coordinates (4) the warping function b only depends on r . If it is not a constant, then: $\dot{u}_0 = 0$ because $u^k \dot{u}_k = 0$, and

$$\dot{u}_\mu = u^0 \nabla_0 u_\mu = -u^0 \Gamma_{0,\mu}^0 u_0 = \delta_{\mu r} \frac{b'}{b}$$

i.e. \dot{u}_μ is a radial vector, with components $\dot{u}_r = b'/b, \dot{u}_a = 0, a = 1 \dots n - 2$. The parameter $\eta = \dot{u}_i \dot{u}^i = g^{rr}(\dot{u}_r)^2$ is a factored scalar function of t and r :

$$\eta = \frac{1}{a^2(t)} \frac{b'^2}{f_1(r)^2 b(r)^2}. \tag{40}$$

The vector field $\hat{n}_\mu = \dot{u}_\mu / (a\sqrt{\eta})$ in Eq.(20) is radial and coincides with n_μ . The Eqs. (38) and (39) for n_μ are compared with Eq.(2.5) for \hat{n}_μ . They show that, by the spherical symmetry,

$$w_\mu = 0, \quad \Pi_{\mu\nu} = 0, \quad \partial_\mu \Theta \propto n_\mu.$$

Therefore (M^*, g^*) is a warped manifold (the third class in Proposition 2.5).

The integrability condition in M^* and the property $C_{\mu\nu\rho\sigma}^* = 0$ yield the Ricci tensor $R_{\mu\nu}^*$ and the eigenvalue ξ^* (see Appendix 9.2):

$$R_{\mu\nu}^* = \xi^* n_\mu n_\nu + \frac{R^* - \xi^*}{n - 2} (g_{\mu\nu}^* - n_\mu n_\nu) \tag{41}$$

$$\xi^* = -\frac{n - 2}{f_1^2} [(\partial_r \log f_2)^2 + \partial_r^2 \log f_2 - (\partial_r \log f_1)(\partial_r \log f_2)] \tag{42}$$

3.1 The electric tensor

Although M^* is conformally flat, the whole spacetime in general is not. The electric tensor arises solely from the Weyl tensor of the spacetime.

Proposition 3.3

$$E_{\mu\nu} = E(r) \left[n_\mu n_\nu - \frac{g_{\mu\nu}^*}{n-1} \right] \tag{43}$$

$$E(r) = \frac{n-3}{n-2} \frac{1}{f_1^2} \left[\frac{f_1^2}{f_2^2} + \frac{d^2}{dr^2} \log f_2 - \frac{f_1' f_2'}{f_1 f_2} + \frac{b'}{b} \frac{d}{dr} \log(f_1 f_2) - \frac{b''}{b} \right] \tag{44}$$

where a prime is a derivative in r .

Proof The electric tensor is spacelike and can be obtained from the general relation (30). To evaluate $\nabla_\mu^* b_\nu$ note that $b_\nu = n_\nu b' / f_1$. Then:

$$\begin{aligned} \nabla_\mu^* b_\nu &= n_\nu \partial_\mu \frac{b'}{f_1} + \frac{b'}{f_1} \nabla_\mu^* n_\nu = n_\nu n_\mu \frac{1}{f_1} \frac{d}{dr} \frac{b'}{f_1} + \frac{b'}{f_1} \frac{\Theta}{n-2} (g_{\mu\nu}^* - n_\mu n_\nu) \\ &= n_\nu n_\mu \frac{1}{f_1^2} \left[b'' - b' \frac{d}{dr} \log(f_1 f_2) \right] + g_{\mu\nu}^* \frac{1}{f_1^2} b' \frac{d}{dr} \log f_2 \end{aligned}$$

In particular:

$$\nabla_\mu^* b^\mu = \frac{1}{f_1^2} \left[b'' - b' \frac{d}{dr} \log(f_1 f_2) + (n-1) b' \frac{d}{dr} \log f_2 \right] \tag{45}$$

Then:

$$\frac{1}{b} \left[\nabla_\mu^* b_\nu - g_{\mu\nu}^* \frac{\nabla_\sigma^* b^\sigma}{n-1} \right] = \left[n_\nu n_\mu - \frac{g_{\mu\nu}^*}{n-1} \right] \frac{1}{f_1^2} \left[\frac{b''}{b} - \frac{b'}{b} \frac{d}{dr} \log(f_1 f_2) \right] \tag{46}$$

Since $R_{\mu\nu}^*$ in (41) is quasi-Einstein and $\nabla_\mu^* b_\nu$ has the same tensor form, the space components of the electric tensor are $E_{\mu\nu}(r) = E_1 g_{\mu\nu}^* + E_2 n_\mu n_\nu$ (see Eq.(30)). Being traceless it is $0 = (n-1)E_1 + E_2$ so that $E_{\mu\nu}$ has the form (43).

The contraction of (30) with n^μ gives:

$$\xi^* - \frac{R^*}{n-1} + \frac{(n-2)(n-3)}{n-1} \frac{1}{f_1^2} \left[\frac{b''}{b} - \frac{b'}{b} \frac{d}{dr} \log(f_1 f_2) \right] = -\frac{(n-2)^2}{n-1} E(r),$$

where R^* is evaluated in Appendix 9.6 ($N = n - 2$). This and Eq.(42) for ξ^* give:

$$\xi^* - \frac{R^*}{n-1} = -\frac{(n-2)(n-3)}{(n-1)} \left[\frac{1}{f_2^2} + \frac{1}{f_1^2} \frac{d^2}{dr^2} \log f_2 - \frac{1}{f_1^2} \frac{f_1' f_2'}{f_1 f_2} \right]. \tag{47}$$

Insertion in the previous equation completes the evaluation of $E(r)$. □

Since $E_{ij}u^i = 0, E^k_k = 0$, the extension to spacetime of the expression (43) is

$$E_{ij} = \frac{E(r)}{a^2(t)} \left[N_i N_j - \frac{g_{ij} + u_i u_j}{n - 1} \right] \tag{48}$$

where in spherical coordinates $N_i = (0, an_\mu), N_i N_j g^{ij} = a^2 n_\mu n_\nu g^{\mu\nu} = 1, n_\mu n_\nu$ is the radial projector in M^* . It is

$$E_{ij} E^{ij} = \frac{n - 2}{n - 1} \frac{E(r)^2}{a^4(t)}.$$

If \dot{u}_i exists ($b(r)$ is non-constant), then $N_i N_j = \frac{\dot{u}_i \dot{u}_j}{\eta}$ and \dot{u} is an eigenvector of the electric tensor: $E_{ij} \dot{u}^i = \frac{n-2}{n-1} \frac{E(r)}{a^2(t)} \dot{u}_j$.

3.2 The Ricci tensor

The covariant expression (48) is inserted in (27) to obtain the general expression of the Ricci tensor for the metric (37).

- If $\dot{u}_i \neq 0$:

$$R_{jk} = u_j u_k (-\xi + \nabla_p \dot{u}^p) + \frac{h_{jk}}{n - 1} (R - \xi + \nabla_p \dot{u}^p) - (n - 2) \varphi (u_j \dot{u}_k + \dot{u}_j u_k) + \left[\frac{\dot{u}_j \dot{u}_k}{\eta} - \frac{h_{jk}}{n - 1} \right] \left[\nabla_p \dot{u}^p - (n - 1) \left(\eta + \frac{\dot{u}^i \nabla_i \eta}{2\eta} \right) - (n - 2) \frac{E(r)}{a^2(t)} \right] \tag{49}$$

where $h_{jk} = g_{jk} + u_j u_k$. It only involves the vectors u_i and \dot{u}_i , and scalar fields.

It is the sum of a quasi-Einstein (perfect fluid) term, a current term, and a traceless anisotropic term that comes from two sources: the acceleration and the electric tensor. This anisotropy occurs despite the metric being spherically symmetric.

- If $\dot{u}_i = 0$, Eq. (49) is the Ricci tensor on a spherical GRW spacetime:

$$R_{jk} = -\xi u_j u_k + \frac{R - \xi}{n - 1} h_{jk} - (n - 2) \left[N_j N_k - \frac{h_{jk}}{n - 1} \right] \frac{E(r)}{a^2(t)}. \tag{50}$$

The eigenvalues are: $\xi, \frac{R-\xi}{n-1} - \frac{(n-2)^2 E(r)}{n-1 a^2}$ and $\frac{R-\xi}{n-1} + \frac{n-2 E(r)}{n-1 a^2}$, with degeneracies 1, 1, $n - 2$. The anisotropy is totally due to the electric term. R_{jk} has the perfect fluid form if and only if $E(r) = 0$.

Lemma 3.4 (this formula is proven in Appendix 9.3)

$$\nabla_p \dot{u}^p - (n - 1) \left(\frac{\dot{u}^i \nabla_i \eta}{2\eta} + \eta \right) = -\frac{n - 2}{a^2(t) f_1^2 b} \left[b'' - b' \frac{d}{dr} \log(f_1 f_2) \right]. \tag{51}$$

Other useful expressions in the spherical comoving coordinates are collected below. R and R^* are (105) and (106) with $N = n - 2$ (number of angles). $\nabla_p \dot{u}^p$ is (94), with $\nabla_\mu^* b^\mu$ in (45):

$$\dot{\varphi} = u^0 \partial_t \varphi = \frac{1}{b} \partial_t \frac{a_t}{ab} = \frac{a_{tt}}{ab^2} - \frac{(a_t)^2}{a^2 b^2}, \quad \xi \equiv (n - 1)(\varphi^2 + \dot{\varphi}) = (n - 1) \frac{1}{b^2} \frac{a_{tt}}{a}, \tag{52}$$

$$\nabla_p \dot{u}^p = \frac{1}{a^2 b} \nabla_\nu^* b^\nu = \frac{1}{a^2 b} \frac{1}{f_1^2} \left[b'' - b' \frac{d}{dr} \log(f_1 f_2) + (n - 1) b' \frac{d}{dr} \log f_2 \right], \tag{53}$$

$$R = \frac{R^*}{a^2} - \frac{2}{a^2 b} \nabla_\nu^* b^\nu + \frac{n - 1}{a^2 b^2} \left[(n - 2) a_t^2 + 2 a a_{tt} \right]. \tag{54}$$

4 Radiating stars

The Ricci tensor (49) and the Einstein equations $R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}$ fix the structure of the energy-momentum tensor of matter and radiation:

$$T_{ij} = \mu u_i u_j + P h_{ij} + (u_i q_j + u_j q_i) + \left[N_i N_j - \frac{h_{ij}}{n - 1} \right] (p_r - p_\perp) \tag{55}$$

$$\mu = -\xi + \nabla_p \dot{u}^p + \frac{1}{2} R \tag{56}$$

$$(n - 1) P = -\xi + \nabla_p \dot{u}^p - \frac{1}{2} (n - 3) R \tag{57}$$

$$q_j = -(n - 2) \varphi \dot{u}_j \tag{58}$$

$$p_r - p_\perp = \nabla_p \dot{u}^p - (n - 1) \left(\eta + \frac{\dot{u}^i \nabla_i \eta}{2\eta} \right) - (n - 2) \frac{E(r)}{a^2(t)} \tag{59}$$

T_{ij} describes an anisotropic fluid with energy density μ , heat flow q_j , shear-free and vorticity-free velocity u_j , radial pressure p_r , tangential pressure p_\perp , effective pressure $P = \frac{1}{n-1} p_r + \frac{n-2}{n-1} p_\perp$.

For a radiating star the spacetime splits into an interior spherical DW manifold V^- describing the fluid history, and an exterior manifold V^+ described by the spherically symmetric Vaidya solution (1) of the Einstein equations with a pure radiation energy-momentum tensor. A change of coordinates gives the metric the convenient form (in dimension $n \geq 4$ [9]):

$$ds_+^2 = - \left(1 - \frac{2m(v)}{(n - 3) R^{n-3}} \right) dv^2 - 2dv dR + R^2 d\Omega_{n-2}^2 \tag{60}$$

where $m(v)$ is the Vaidya exterior mass.

4.1 Boundary conditions

The inner solution (37) and Vaidya solution match at the boundary Σ of the collapsing star. It is a time-like hypersurface with intrinsic metric

$$ds^2 = -d\tau^2 + \mathcal{R}^2(\tau)d\Omega_{n-2}^2$$

In the two coordinates, the boundary has equations $f^-(t, r, \theta) = r - r_\Sigma = 0$ where r_Σ is constant, and $f^+(v, R, \theta) = R - R_\Sigma(v) = 0$. The metric and its first derivatives are continuous throughout the spacetime. Then g_{ij}^\pm and the extrinsic curvature tensors in V^\pm have to match at Σ . We proceed with the prescriptions well described by Santos in ref. [60] with the generalization $n \geq 4$ in [9].

The continuity of the metric at Σ is expressed by

$$\mathcal{R}^2(\tau) = a^2(t)f_2^2(r_\Sigma) = R_\Sigma^2(v) \tag{61}$$

$$d\tau^2 = b^2(r_\Sigma)dt^2 = \left(1 - \frac{2m}{(n-3)R_\Sigma^{n-3}} + 2\frac{dR_\Sigma}{dv}\right)dv^2$$

The latter gives:

$$\frac{dt}{d\tau} = \frac{1}{b(r_\Sigma)}, \quad \left(\frac{dv}{d\tau}\right)^{-2} = 1 - \frac{2m}{(n-3)R_\Sigma^{n-3}} + 2\frac{dR_\Sigma}{dv}. \tag{62}$$

The relevant second forms for the interior solution are evaluated, while those for the Vaidya solution are given in [9]. The continuity condition $K_{\theta\theta}^- = K_{\theta\theta}^+$ is:

$$a(t)\frac{f_2}{f_1}\frac{df_2}{dr}\Big|_{r=r_\Sigma} = \frac{dv}{d\tau}R_\Sigma\left[1 - \frac{2m}{(n-3)R_\Sigma^{n-3}}\right] + R_\Sigma\frac{dR_\Sigma}{d\tau}$$

Now use $R_\Sigma = af_2: \frac{dR_\Sigma}{d\tau} = f_2\frac{da}{d\tau} = f_2(r_\Sigma)a_t/b$. The previous equation becomes:

$$\frac{1}{f_1}\frac{df_2}{dr} - \frac{a_t}{b}f_2 = \frac{dv}{d\tau}\left[1 - \frac{2m}{(n-3)R_\Sigma^{n-3}}\right] \tag{63}$$

Multiply the second equation in (62) by $\frac{dv}{d\tau}: \frac{d\tau}{dv} = \left[1 - \frac{2m}{(n-3)R_\Sigma^{n-3}}\right]\frac{dv}{d\tau} + 2\frac{dR_\Sigma}{d\tau}$ and simplify the above equation:

$$\frac{1}{f_1}\frac{df_2}{dr} + \frac{a_t}{b}f_2 = \frac{d\tau}{dv};$$

With Eq. (63) the Misner–Sharp mass function is obtained in terms of the inner solution:

$$m(t, r) = \frac{n - 3}{2} (af_2)^{n-3} \frac{f_2^2}{f_1^2} \left[\frac{f_1^2}{f_2^2} - \left(\frac{f_2'}{f_2} \right)^2 + f_1^2 \frac{a_t^2}{b^2} \right]_{\Sigma} \tag{64}$$

The continuity condition $K_{\tau\tau}^- = K_{\tau\tau}^+$ is:

$$-\frac{1}{f_1 a} \frac{b'}{b} \Big|_{\Sigma} = \frac{d^2 v}{d\tau^2} \left(\frac{dv}{d\tau} \right)^{-1} - \frac{dv}{d\tau} \frac{m}{R_{\Sigma}^{n-2}}$$

This gives, up to a common prefactor $1/(a^2 b)$, the equation

$$2aa_{tt} + (n - 3)a_t^2 - 2a_t \frac{b'}{f_1} + (n - 3) \frac{b^2}{f_1^2} \left[\frac{f_1^2}{f_2^2} - \left(\frac{f_2'}{f_2} \right)^2 \right] - 2 \frac{bb'}{f_1^2} \frac{f_2'}{f_2} = 0. \tag{65}$$

Since the functions are evaluated at r_{Σ} , it is a differential equation for $a(t)$ with constant coefficients: $2aa_{tt} + (n - 3)a_t^2 + Aa_t - B = 0$.

A simple solution is $a(t) = -Ct$. The non linear equation was studied by Paliathanasis et al. [56]. They obtained the expansion near a movable singularity (i.e. dependent on initial conditions) that, extended to $n \geq 4$ is: $a(t) = a_0(t - t_0)^{\nu} + a_1(t - t_0) + a_2(t - t_0)^{2-\nu} + \dots$ with exponent $\nu = \frac{2}{n-1}$, arbitrary a_0 , $a_1 = -\frac{1}{2} \frac{A(n-1)}{(n-2)(n-3)}$ etc. An integral is obtained with $a_t = y(x)$, $x = e^a$, with equation $2 \frac{dy}{dx} + (n - 3)y + A - B/y = 0$. Eq.(65) translates to a relation for the physical parameters at the surface of the star. A rather long evaluation shows that (65) is equivalent to

$$\left(p_r - \sqrt{q_j q^j} \right)_{\Sigma} = 0 \tag{66}$$

At the boundary, the radial pressure equals the heat current intensity.

5 Pressure isotropy

Several models for radiating stars are characterized by isotropic pressure, $p_r = p_{\perp}$, i.e. a zero anisotropic term in the Ricci tensor.

With Lemma 3.4 and Eq.(43) for $E(r)$, the condition for isotropy is the linear differential equation for the function $b(r)$:

$$0 = \frac{d^2 b}{dr^2} - \frac{db}{dr} \frac{d}{dr} \log(f_1 f_2) + (n - 3)b \left[\frac{f_1^2}{f_2^2} - \frac{f_1' f_2'}{f_1 f_2} + \frac{d^2}{dr^2} \log f_2 \right] \tag{67}$$

Solutions are obtained by setting to zero the acceleration term and the electric term of anisotropy separately, or as a whole. The following propositions give conditions for the vanishing of single terms.

Proposition 5.1 (Condition I) *If $\dot{u}_i \neq 0$:*

$$\frac{d^2b}{dr^2} - \frac{db}{dr} \frac{d}{dr} \log(f_1 f_2) = 0 \iff \frac{db(r)}{dr} = 2K f_1(r) f_2(r) \tag{68}$$

with constant K .

Proposition 5.2 (Condition II)

$$\frac{f_1^2}{f_2^2} - \frac{f_1' f_2'}{f_1 f_2} + \frac{d^2}{dr^2} \log f_2 = 0 \iff f_1^2(r) = \frac{f_2'(r)^2}{1 + C f_2(r)^2} \tag{69}$$

with a constant C ensuring $f_1^2 > 0$.

Proof Equation (69) is a Bernoulli equation of the form $f_1' + A(r) f_1 + B(r) f_1^3 = 0$. Divide by f_1^3 and set $y = 1/f_1^2$. The equation now is: $y' - 2yA - 2B = 0$. \square

Together, conditions I and II ensure isotropy. After fixing f_2 , one evaluates f_1 with Condition II and then b with Condition I.

Remark 5.3 • If $\dot{u}_i = 0$ then Condition II is necessary and sufficient for isotropy.

- If $\nabla_m C_{jkl}{}^m = 0$ or $C_{jklm} = 0$ then $E_{ij} = 0$ and Condition I assures isotropy.
- The Ricci tensor of M^* has eigenvalues ξ^* and $(R^* - \xi^*)/(n - 2)$ with degeneracy $n - 2$. When they are equal, $\xi^* = R^*/(n - 1)$, then $R_{\mu\nu}^*$ is Einstein. With Eq. (47) one obtains the equivalence:

$$R_{\mu\nu}^* = \frac{R^*}{n - 1} g_{\mu\nu}^* \iff \text{Condition II} \tag{70}$$

- By Lemma 3.4, Condition I is equivalent to $0 = \nabla_p \dot{u}^p - (n - 1) \left(\frac{\dot{u}^i \nabla_i \eta}{2\eta} + \eta \right)$.

Proposition 5.4 *If $f_2 = r f_1$ and if Condition II is true, then*

$$f_1(r) = \frac{K}{r^2 + C} \tag{71}$$

If also Condition I is true then $b(r) \propto f_1(r)$ and the metric is conformally Robertson-Walker, with flat M^ :*

$$ds^2 = \frac{K^2}{(r^2 + C)^2} \left[-dt^2 + a^2(t)(dr^2 + r^2 d^2\Omega_{n-2}) \right]$$

Proof The equation in Proposition 5.2 now is: $\frac{1}{r^2} + \partial_r^2 \log(r f_1) - (\partial_r \log f_1) \partial_r \log(r f_1) = 0$. With $Y = \partial_r \log f_1$ the equation becomes: $Y' - \frac{1}{r} Y - Y^2 = 0$. Divide by Y^2 , multiply by r and put $X = 1/Y$: $(r X)' + r = 0$. Integration gives $r X(r) + r^2/2 + C/2 = 0$. Going back to Y we obtain

$$\frac{d}{dr} \log f_1 = -\frac{2r}{r^2 + C} = -\frac{d}{dr} \log(r^2 + C)$$

Another integration yields the result. □

The following solution of Eq.(67) for isotropy was obtained by Wagh et al. [71]. It is parameterized by $f_2(r)$ with $f_2' > 0$, and is checked by substitution. The extension to $n \geq 4$ is

Proposition 5.5

$$b(r) = K f_2(r), \quad f_1(r) = \sqrt{\frac{n-2}{n-3}} \frac{df_2}{dr}, \tag{72}$$

$K > 0$ is a constant. If also $f_2 = r f_1$ then $f_2(r) = C r^\nu$, $\nu = \sqrt{\frac{n-3}{n-2}}$.

Equation (67) simplifies with $f_2(r) = r f_1(r)$, that applies to most of the physics models considered here:

$$0 = \frac{d^2 b}{dr^2} - \frac{db}{dr} \frac{d}{dr} \log(r f_1^2) + (n-3)b \left[\frac{d^2 \log f_1}{dr^2} - \frac{1}{r} \frac{d \log f_1}{dr} - \left(\frac{d \log f_1}{dr} \right)^2 \right] \tag{73}$$

The change $x = r^2$ and $F = 1/f_1(x)$ transforms it into the isotropy condition found by Glass [34] in $n = 4$, and by Banerjee [8] in higher dimensions:

$$0 = \frac{d^2 b}{dx^2} F + 2 \frac{db}{dx} \frac{dF}{dx} - (n-3)b \frac{d^2 F}{dx^2} \tag{74}$$

With an input function, the differential equation is solved for the other function. Simple pairs are respectively found with $F_x = 0$, $F_{xx} = 0$ and $b_{xx} = 0$ and are the first three lines of Table I, with other solutions in the literature.

Here are other pairs, with $f_2 = r f_1$:

$$f_1(r) = \frac{K_1}{K_2 + (b_0 + b_2 r^2)^\nu}, \quad b(r) = b_0 + b_2 r^2, \quad \nu = \frac{n-1}{n-3} \tag{75}$$

$$f_1(r) = K r^{p-1}, \quad b(r) = K_1 r^{q_1} + K_2 r^{q_2}, \quad q_{1,2} = p \pm \sqrt{p^2(n-2) - (n-3)} \tag{76}$$

$$f_1(r) = e^{-K r^2}, \quad b(r) = e^{-K(1 \pm \sqrt{n-2})r^2} \tag{77}$$

Table 1 Isotropic solutions with $f_2(r) = rf_1(r)$.

$f_1(r)$	$b(r)$	Notes	References
1	z	$E(r) = 0$	[7, 8, 53]
z^{-1}	$a + bz^{-1}$	$E(r) = 0$	[5, 28, 46, 61]
$z^{-\nu}$	z	$\nu = \frac{n-1}{n-3}$	[8]
$(a\sqrt{z} + 1)^2/(bz)$	$(a\sqrt{z} - 1)/(a\sqrt{z} + 1)$	$n = 4$	[6]
$z^{-q-\frac{1}{2}}$	z^p	$p^2 + 2qp = (n - 3)(q^2 - \frac{1}{4})$	[68]

$z \equiv 1 + Kr^2$, where K is a constant. The solutions with $E(r) = 0$ satisfy conditions I and II. Constants may change by rescaling t and r

The last pair belongs to a larger new family discussed in Appendix 9.4. Oscillating and hyperbolic solutions are found in [59]. Other solutions are studied in [69].

Example 5.6 (Banerjee and Chatterjee [8]) The model describes the gravitational collapse of a star, started at $t = -\infty$.

$$ds^2 = -(1 + Kr^2)^2 dt^2 + C^2 t^2 (dr^2 + r^2 d\Omega_{n-2}^2)$$

The subspace M^* is flat, the spacetime M is conformally flat ($E(r) = 0$). Condition I is satisfied: the model is isotropic. The particular solution $a(t) = -Ct$ is chosen, then $\xi = 0$. The scalar curvature, the energy density and the isotropic pressure are:

$$R = -\frac{4K(n-1)}{C^2 t^2 (1 + Kr^2)} + \frac{(n-1)(n-2)}{(1 + Kr^2)^2 t^2}$$

$$\mu = \frac{(n-1)(n-2)}{2t^2 (1 + Kr^2)^2}, \quad p = \frac{2K(n-2)}{C^2 t^2 (1 + Kr^2)} - \frac{1}{2} \frac{(n-2)(n-3)}{(1 + Kr^2)^2 t^2}$$

Example 5.7 (Wagh et al. [71]) The model describes the collapse of a radiating star with equation of state $p = w\mu$. The isotropy condition is solved with the metric in Proposition 5.5. We generalize their results to $n \geq 4$:

$$ds^2 = -K^2 f_2^2(r) dt^2 + a^2(t) \left[\frac{n-2}{n-3} \left(\frac{df_2}{dr} \right)^2 dr^2 + f_2(r)^2 d\Omega_{n-2}^2 \right] \tag{78}$$

The following geometric quantities are evaluated, and depend on $a(t)$ and $f_2(r)$:

$$R^* = \frac{n-3}{f_2^2}, \quad \xi = \frac{n-1}{K^2 f_2^2} \frac{a_{tt}}{a}, \quad E(r) = \left(\frac{n-3}{n-2} \right)^2 \frac{1}{f_2^2}$$

$$R = -\frac{n-3}{a^2 f_2^2} + \frac{n-1}{K^2 a^2 f_2^2} \left[(n-2)a_t^2 + 2aa_{tt} \right] \tag{79}$$

Being $E \neq 0$, this model is never conformally flat. In the Einstein equations, the fluid parameters are

$$\mu = \frac{n-3}{2a^2 f_2^2} + \frac{(n-1)(n-2)}{2K^2 f_2^2} \frac{a_t^2}{a^2}, \quad p = \frac{n-3}{2a^2 f_2^2} - \frac{n-2}{2K^2 a^2 f_2^2} \left[(n-3)a_t^2 + 2aa_{tt} \right]$$

$$q_r = -\frac{n-2}{f_2^2} \frac{df_2}{dr} \frac{a_t}{a}$$

Note that $\mu > 0$ and $\frac{d}{dr}\mu = -2\mu \frac{d}{dr} \log f_2$. The Misner–Sharp mass is

$$m(t) = \frac{1}{2}(n-3)(af_2(r_\Sigma))^{n-3} \left[\frac{1}{n-2} - \frac{a_t^2}{K^2} \right]$$

The equation of state $p = w\mu$ and the junction condition (65) are two equations for $a(t)$ and w :

$$2aa_{tt} + [(n-1)w + (n-3)]a_t^2 + \frac{n-3}{n-2}(w-1)K^2 = 0$$

$$2aa_{tt} + (n-3)a_t^2 - 2Ka_t \sqrt{\frac{n-3}{n-2}} - \frac{n-3}{n-2}K^2 = 0 \tag{80}$$

The solutions are: $w^2(n^2 - 3n + 3) - 2w - (n - 3) = 0$, and $a(t)$ with $a_t < 0$ for a collapsing star:

$$a_t = -K \sqrt{\frac{n-3}{n-2}} \sqrt{\frac{1-w}{(n-3) + (n-1)w}}$$

The linear $a(t)$ means $\xi = 0$.

Example 5.8 (Euclidean DW stars) The Euclidean metric $ds^2 = -B^2(t, r)dt^2 + R^2(t, r)dr^2 + R^2(t, r)d\Omega_{n-2}^2$ was introduced by Herrera and Santos [39]. In particular, a Euclidean DW spherical metric has the form

$$ds^2 = -b^2(r)dt^2 + a^2(t)[f'^2(r)dr^2 + f^2(r)d\Omega_2^2]$$

The surface at fixed r and time t is $4\pi f^2(r)$, then $f(r)$ is the areal radius. The feature of the Euclidean metric is that $f(r)$ coincides with the proper radius, $\int^r dr' f'(r')$.

For all these models Condition II is satisfied, i.e. $R^*_{\mu\nu}$ is Einstein. Since Eq. (106) gives $R^* = 0$, it is $R^*_{\mu\nu} = 0$.

6 Friedmann equations in spherical GRW spacetimes

Spherical GRW spacetimes are characterized by $\dot{u} = 0$ or, equivalently, by the metric

$$ds^2 = -dt^2 + a^2(t)[f_1(r)^2 dr^2 + f_2^2(r)d\Omega_{n-2}^2] \quad (81)$$

With $b = 1$, the dot coincides with the time derivative. The Einstein equations with anisotropic fluid source (55) give $q_i = 0$ (no heat current) and

$$\mu = -\xi + \frac{1}{2}R, \quad p_r + (n-2)p_\perp = -\xi - \frac{1}{2}(n-3)R, \quad p_r - p_\perp = -(n-2)\frac{E(r)}{a^2(t)}$$

It is the electric component $E(r)$ of the Weyl tensor, Eq. (44), that breaks isotropy. With (34), $\varphi = \dot{a}/a$ (the Hubble parameter), and $\xi = (n-1)\ddot{a}/a$, the equations for the energy density and the radial pressure are

$$\mu = \frac{R^*(r)}{2a^2} + \frac{1}{2}(n-1)(n-2)\frac{\dot{a}^2}{a^2} \quad (82)$$

$$(n-1)p_r + (n-3)\mu = -(n-1)(n-2)\frac{\ddot{a}}{a} - (n-2)^2\frac{E(r)}{a^2} \quad (83)$$

The equations for μ is equal to the first Friedmann equation in RW cosmology. It shows that $a^2\mu$ is the sum of a function of r and a function of time. The derivative of $2a^2\mu$ and the equation for p_r give:

$$\dot{\mu} = -\frac{\dot{a}}{a} \left[(n-1)(p_r + \mu) + (n-2)^2\frac{E}{a^2} \right] = -\frac{\dot{a}}{a}(n-1)(P + \mu) \quad (84)$$

where P is the effective pressure.

In a spherical GRW spacetime, the expressions for R^* and $E(r)$ give

$$\frac{R^*(r)}{n-2} + 2\frac{n-2}{n-3}E(r) = \frac{n-1}{f_2^2} \left[1 - \frac{f_2'^2}{f_1^2} \right]$$

It is simple to check that in a spherical GRW spacetime, the Euclidean condition $f_1 = f_2'$ (example 5.8) is necessary and sufficient for $R^* = 0$ and $E = 0$ (the spacetime is RW with flat space).

Example 6.1 In ref. [15] the authors discuss a cosmological model with a non-radiative anisotropic fluid for an effective description of barionic matter and dark energy without assuming the presence of dark matter. The pressure anisotropy is the source of the small-scale inhomogeneities of the universe. The metric in cosmic time is (81) with $f_1^2(r) = 1/f(r)$ and $f_2^2(r) = r^2$. It is found:

$$R^* = 2\frac{1-f}{r^2} - (n-2)\frac{f'}{r}, \quad E(r) = \frac{n-3}{n-2} \left[\frac{1-f}{r^2} + \frac{f'}{2r} \right]$$

In particular, the following function is used $f(r) = 1 - \frac{2G}{r}[m_B(r) + m]$ where m_B is the Misner-Sharp mass for the inhomogeneous distribution of baryonic matter.

The choice $E(r) = 0$ implies $f(r) = 1 - kr^2$, and the spacetime is RW with curvature k .

7 Spherical DW models in $f(R)$ gravity

A generalisation of Einstein’s theory are the $f(R)$ theories. In such theories, the scalar R in the gravitational action is replaced by a smooth function $f(R)$. They were introduced by Buchdahl in 1970 [14] and gained popularity with the works by Starobinsky on cosmic inflation [66]. Now they are explored as possible cosmological theories for a geometric description of effects that otherwise require the introduction of dark matter and dark energy in the energy-momentum tensor.

Variation with respect to g_{ij} , modulo surface terms, gives the field equations:

$$f'(R)R_{kl} - [f'''(R)(\nabla_k R)(\nabla_l R) + f''(R)\nabla_k \nabla_l R] + g_{kl}[f'''(R)(\nabla_k R)^2 + f''(R)\nabla^2 R - \frac{1}{2}f'(R)] = \kappa T_{kl} \tag{85}$$

In a GRW spacetime of dimension n , if the Weyl tensor is harmonic, then R_{jk} is rank-1 (perfect-fluid) and all curvature corrections in (85) are of the same form. Therefore, they can be interpreted as geometric corrections to the perfect fluid parameters of the fluid matter-radiation source [18]. A systematic study is done in [19] with the notion of *perfect scalars* to prove the status of perfect fluid of corrections in various extended gravity theories.

$f(R)$ gravity is also being considered to model stellar collapse [16, 20, 21, 35, 41]. Since the exterior solution is pure radiation ($T^k_k = 0$), the $f(R)$ equation is still solved by the Vaidya metric, where $R = 0$. The junction conditions with the inner solution have the additional constraints $R = 0$ and $N^j \partial_j R = 0$ at the boundary, that exclude some inner metrics [62].

We show that in spherical DW spacetimes the geometric terms in the field equations (85) have the same tensor form as in (55). Therefore, for an anisotropic fluid with energy-momentum tensor (55), the effect of $f(R)$ gravity is only to modify the parameters μ , p_r , p_\perp and q_j of the fluid. If the input fluid tensor is isotropic, the geometric $f(R)$ corrections may produce anisotropy.

We introduce the following:

Definition 7.1 On a spherical DW spacetime, a scalar X is a *perfect 2-scalar* if

$$\nabla_j X = -u_j \dot{X} + \dot{u}_j \tilde{X} \tag{86}$$

It follows that $\dot{X} = u^k \nabla_k X$ and $\tilde{X} = \frac{1}{\eta} \dot{u}^k \nabla_k X$.

A linear combination of the tensors $u_i u_j$, g_{ij} , $u_i \dot{u}_j$ and $\dot{u}_i \dot{u}_j$ is a rank-2 perfect tensor (also named 2-Einstein tensor [63]).

Perfect 2-scalars are φ (see Eq.(12)), η (see Eq.(19) with $w_k = 0$), ξ (see Eq.(31)). The Ricci and the electric tensors are rank-2 perfect tensors.

If X, Y are perfect 2-scalars, also $X + Y$ and XY are such (Leibnitz rule).

Remark 7.2 An equivalent characterization is $N_j^k \nabla_k X = 0$, where N_{jk} is the projection (15). Since in spherical components $N_{tt} = N_{rr} = N_{rt} = 0$, a scalar function is a perfect 2-scalar if and only if it depends only on t and r .

Lemma 7.3 *If X is a perfect 2-scalar, then \dot{X} and \tilde{X} are perfect 2-scalars, and $\dot{\tilde{X}} = \dot{X} + \tilde{X}$.*

$$\nabla_i \dot{X} = -u_i \ddot{X} + \dot{u}_i (\dot{\tilde{X}} - \dot{X}) \tag{87}$$

$$\nabla_i \tilde{X} = -u_i (\dot{X} + \dot{\tilde{X}}) + \dot{u}_i \tilde{X} \tag{88}$$

Proof Consider the Hessian $\nabla_i \nabla_j X = -\dot{X} \nabla_i u_j + \tilde{X} \nabla_i \dot{u}_j - u_j \nabla_i \dot{X} + \dot{u}_j \nabla_i \tilde{X}$. Exchange the indices and subtract, use the fact that \dot{u}_i is closed:

$$0 = -\dot{X}(-u_i \dot{u}_j + u_j \dot{u}_i) - u_j \nabla_i \dot{X} + u_i \nabla_j \dot{X} + \dot{u}_j \nabla_i \tilde{X} - \dot{u}_i \nabla_j \tilde{X}$$

Contraction with u^j gives the first statement. Contraction with \dot{u}^j gives: $0 = \dot{X} \eta u_i + u_i \dot{u}^j \nabla_j \dot{X} + \eta \nabla_i \tilde{X} - \dot{u}_i \dot{u}^j \nabla_j \tilde{X}$. Then the second statement follows. Contraction of u^i with (88) gives the last relation. □

Proposition 7.4 *The Hessian of a perfect-2 scalar is a rank-2 perfect tensor:*

$$\begin{aligned} \nabla_i \nabla_j X &= -\dot{X} \varphi g_{ij} + u_i u_j (\ddot{X} - \varphi \dot{X} - \eta \tilde{X}) - (\dot{u}_i u_j + u_i \dot{u}_j) (\tilde{X} - \varphi \tilde{X}) \\ &\quad + \dot{u}_i \dot{u}_j \left(\tilde{X} + \tilde{X} \frac{\tilde{\eta}}{2\eta^2} \right) + N_{ij} \tilde{X} \frac{-\eta + \nabla_r \dot{u}^r - \frac{1}{2\eta} \tilde{\eta}}{n-2} \end{aligned} \tag{89}$$

Proof $\nabla_i \nabla_j X = -\dot{X} \nabla_i u_j + \tilde{X} \nabla_i \dot{u}_j - u_j \nabla_i \dot{X} + \dot{u}_j \nabla_i \tilde{X} = -\dot{X} (\varphi u_i u_j + \varphi g_{ij} - u_i \dot{u}_j) + \tilde{X} \nabla_i \dot{u}_j - u_j [-u_i \ddot{X} + \dot{u}_i (\dot{\tilde{X}} - \dot{X})] + \dot{u}_j [-u_i (\dot{X} + \tilde{X}) + \dot{u}_i \tilde{X}] = -\dot{X} \varphi g_{ij} + \tilde{X} \nabla_i \dot{u}_j + u_i u_j (\tilde{X} - \varphi \dot{X}) - (\dot{u}_i u_j + u_i \dot{u}_j) \tilde{X} + \dot{u}_i \dot{u}_j \tilde{X}$.

For a spherical DW spacetime $\nabla_i \dot{u}_j$ is given by Eq.(16) with $\Pi_{ij} = 0$ and $w_j = 0$, and it is a rank-2 perfect tensor. □

The curvature scalar R of a spherical DW spacetime is a function only of r and t , then it is a perfect 2-scalar and the Hessian $\nabla_i \nabla_j R$ as well as the product $\nabla_i R \nabla_j R$ are rank-2 perfect tensors. We then conclude:

Proposition 7.5 *On a spherical DW spacetime, the $f(R)$ field equations imply an energy-momentum tensor (55) that is a rank-2 perfect tensor.*

Example 7.6 In Wagh’s metric (78) the following terms in (85) are evaluated:

$$\nabla_k R = -\dot{R} u_k - 2R \dot{u}_k \tag{90}$$

$$\begin{aligned} \nabla_l \nabla_k R &= (\ddot{R} - \varphi \dot{R}) u_k u_l - (\varphi \dot{R} + 2R \eta) g_{kl} \\ &\quad + (3\dot{R} - 2R \varphi) (u_k \dot{u}_l + \dot{u}_k u_l) + 8R \dot{u}_k \dot{u}_l \end{aligned} \tag{91}$$

Proof Being R a perfect 2-scalar: $\nabla_k R = -\dot{R}u_k + (\frac{1}{\eta}\dot{u}^r \partial_r R)\dot{u}_k$. The curvature scalar R is (79), then $\partial_r R = -2(f'_2/f_2)R$. In spherical coordinates:

$$\frac{1}{\eta}\dot{u}^r \partial_r R = \frac{b^2 a^2 f_1^2 b'}{b_r^2} \frac{b'}{b} \left(-2 \frac{f'_2}{f_2} R \right) = -2 \frac{b}{b_r} \frac{f'_2}{f_2} R = -2R$$

$\nabla_l \nabla_k R = -(\nabla_l \dot{R})u_k - \varphi \dot{R}h_{kl} + 3\dot{R}u_l \dot{u}_k + 4R\dot{u}_l \dot{u}_k - 2R\nabla_l \dot{u}_k$. Since \dot{u}_k is closed in DW spacetimes, the antisymmetric part is: $0 = -u_k \nabla_l \dot{R} + u_l \nabla_k \dot{R} + 3\dot{R}(u_l \dot{u}_k - u_k \dot{u}_l)$. Contraction with u^k gives $\nabla_l \dot{R} = -\dot{R}u_l - 3\dot{R}\dot{u}_l$. This is inserted in the Hessian:

$\nabla_l \nabla_k R = (\ddot{R} - \varphi \dot{R})u_k u_l - \varphi \dot{R}g_{kl} + 3\dot{R}(u_k \dot{u}_l + \dot{u}_k u_l) + 4R\dot{u}_k \dot{u}_l - 2R\nabla_l \dot{u}_k$ The last term is (16). In Wagh’s metric it is $\dot{u}^k \nabla_k \eta = -2\eta^2$ and $\nabla_p \dot{u}^p = (n - 2)\eta$. Eq. (29) is $N_{pq} \nabla^p \dot{u}^q = \nabla_p \dot{u}^p$ and the result follows. \square

While the Ricci tensor R_{ij} is isotropic, the $f(R) - R$ terms in (85) add a contribution $\dot{u}_i \dot{u}_j [-8Rf''(R) - 4R^2 f'''(R)]$ that implies fluid anisotropy. This term is zero only for $f(R) = R + K \log R$. However $K \neq 0$ violates the condition $f'(0)$ to be a finite constant, that guarantees a finite radiation energy density in the Vaidya solution [62]. In conclusion, full isotropy in Wagh’s metric is only possible in Einstein gravity, $f(R) = R$.

For a collapsing star, the boundary condition $R = 0$ at r_Σ also satisfies the boundary condition $\dot{u}^k \nabla_k R = 0$ because of (90).

Wagh’s metric has been investigated with $f(R) = R + \alpha R^2$ in [25] and $f(R) = \xi R^4$ in [65]. To satisfy the boundary condition (65), they set to zero the prefactor, i.e. $1/f_2(r_\Sigma) = 0$. This also makes $R = 0$ at the boundary, but implies divergence of the metric tensor, making the relation (61) among proper time τ , the Vaidya parameter v and the stellar time t unclear. On the other hand, a finite value $f_2(r_\Sigma)$ makes the conditions (65) and $R = 0$ two incompatible equations for $a(t)$. Wagh’s metric is too constrained for $f(R)$ stellar collapse.

7.1 Spherical GRW spacetimes in $f(R)$ gravity

The missing acceleration \dot{u}_i is replaced by the radial unit vector N_i , with $N_0 = 0$ and $N_\mu = a f_1 \delta_{r\mu}$ in coordinates (t, r, θ_a) . The covariant expression holds:

$$\nabla_i N_j = \frac{f'_2}{a f_1 f_2} (g_{ij} + u_i u_j - N_i N_j) + \varphi N_i u_j \tag{92}$$

where $\varphi = \dot{a}/a$. Any function of t and r has gradient $\nabla_j F = -u_j \dot{F} + \frac{N_j}{a f_1} F'$, where \dot{F} and F' are partial derivatives in t and r . Then:

$$\begin{aligned} \nabla_i \nabla_j F &= -(\nabla_i u_j) \dot{F} - u_j \nabla_i \dot{F} + \nabla_i \left(\frac{N_j}{a f_1} \right) F' + \frac{N_j}{a f_1} \nabla_i F' \\ &= u_i u_j \ddot{F} + h_{ij} \left[\frac{f'_2}{a^2 f_1^2 f_2} F' - \varphi \dot{F} \right] + \frac{N_i N_j}{a^2 f_1^2} \left[F'' - \left(\frac{f'_1}{f_1} + \frac{f'_2}{f_2} \right) F' \right] \end{aligned}$$

$$+ \frac{u_i N_j + u_j N_i}{af_1} (\varphi F' - \dot{F}') \quad (93)$$

The Hessian has the form of a perfect rank-2 tensor with basic vectors u_i and N_i . The curvature scalar (34) depends on r through R^* and on t through a :

$$R' = \frac{R^{*\prime}}{a^2}, \quad \dot{R} = -2\frac{R^*}{a^2}\dot{\varphi} + 2(n-1)(n-2)\varphi\dot{\varphi} + 2\dot{\xi}$$

The Hessian $\nabla_i \nabla_j R$ and $\nabla_i R \nabla_j R$ are rank-2 tensors built with u_i and N_i . Therefore, in spherical GRW spacetimes, $f(R)$ produces heat-like and anisotropic geometric terms in the field equations. Such terms are absent if R^* is a constant.

8 Conclusions

Spherical doubly warped spacetimes are tractable settings for models of stellar collapse, inhomogeneous cosmology, and wormholes. In general, a DW spacetime is characterised by the existence of a shear, vorticity-free velocity u_j and its orthogonal acceleration field \dot{u}_j (the latter is closed, and the expansion parameter only changes in the two directions). If $\dot{u}_j = 0$ the spacetime is Generalized Robertson-Walker. The presence of two vectors makes the geometry rich.

We obtained properties for the Weyl tensor and the covariant structure of the Ricci tensor on general DW spacetimes, that extend known results for GRW, and an identity for the Ricci and electric tensors in the space sub-manifold.

In spherical symmetry, the space sub-manifold is conformally flat, R_{ij} , $\nabla_i \dot{u}_j$ and E_{ij} are rank-2 tensors built with u_i and \dot{u}_i and the metric tensor.

The general structure of the Ricci tensor dictates that of the energy-momentum tensor of an anisotropic fluid in the Einstein equations. The deviation from the perfect fluid form is determined by the electric tensor E_{ij} and the gradient of the acceleration $\nabla_i \dot{u}_j$. In the Einstein equations they yield the general pressure anisotropy term, the general heat current with q_j proportional to the acceleration, in a spherical DW metric. The ensuing explicit expressions for the energy density, heat flux, radial and transverse pressures are displayed.

We then focused on conditions that make these models isotropic, reproduced the junction conditions by Santos for star collapse, obtained the Misner-Sharp mass, and the equality of radial pressure with the modulus of the heat current. Several features investigated in literature on stellar collapse are here framed in a simple manner.

The form of the Ricci tensor can be exploited to study expansion-free compact objects, or the anisotropy in dissipative cosmology. We obtained the Friedmann equations in spherical GRW space-times, with a departure from standard FRW cosmology.

Finally, we introduced the useful concept of perfect 2-scalars, to show that the $f(R)$ corrections to Einstein gravity amount to terms with the same tensor form as the energy-momentum of the fluid source.

Funding Open access funding provided by Università degli Studi di Milano within the CRUI-CARE Agreement.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

9. Appendixes

9.1. Proof of Proposition 2.11

$\nabla_k \xi = (n - 1)\nabla_k(\varphi^2 + u^l \nabla_l \varphi) = (n - 1)[2\varphi \nabla_k \varphi + (\nabla_k u^l) \nabla_l \varphi + u^l \nabla_l \nabla_k \varphi]$. Now use (11) and (12): $\nabla_k \xi = (n - 1)[2\varphi(-u_k \dot{\varphi} - \varphi \dot{u}_k) + (\varphi u_k u^l + \varphi \delta_k^l - u_k \dot{u}^l)(-u_l \dot{\varphi} - \varphi \dot{u}_l) + u^l \nabla_l(-u_k \dot{\varphi} - \varphi \dot{u}_k)]$. After simplifications: $\nabla_k \xi = -(n - 1)(2u_k \varphi \dot{\varphi} + 3\varphi^2 \dot{u}_k + 2\dot{u}_k \dot{\varphi} + u_k \ddot{\varphi} - \varphi \eta u_k + \varphi \ddot{u}_k)$. Contraction of (16) with u^j gives $\ddot{u}_k = \eta u_k - \varphi \dot{u}_k$. Then: $\nabla_k \xi = -(n - 1)(2\varphi \dot{\varphi} + \ddot{\varphi})u_k - 2(n - 1)(\varphi^2 + \dot{\varphi})\dot{u}_k$. This is Eq. (31).

$R_{km} \dot{u}^m = g^{jl}(\nabla_j \nabla_k \dot{u}_l - \nabla_k \nabla_j \dot{u}_l) = \nabla_j \nabla_k \dot{u}^j - \nabla_k \nabla_j \dot{u}^j$. The contraction with u^k gives: $u^k \nabla_k \nabla_j \dot{u}^j = -R_{km} u^k \dot{u}^m + u^k \nabla_j \nabla_k \dot{u}^j = -(n - 2)\varphi \eta - (\nabla_j u^k)(\nabla_k \dot{u}^j) + \nabla_j \dot{u}^j = -(n - 2)\varphi \eta - \varphi u_j \ddot{u}^j - \varphi \nabla_p \dot{u}^p + u_j \dot{u}^k \nabla_k \dot{u}^j + \nabla_j(\eta u_j - \varphi \dot{u}_j)$. Now use the fact that \dot{u} is closed: $u^k \nabla_k \nabla_j \dot{u}^j = -(n - 2)\varphi \eta - \varphi u_j \ddot{u}^j - \varphi \nabla_p \dot{u}^p + \frac{1}{2} \dot{\eta} + \dot{\eta} + (n - 1)\varphi \eta - \dot{u}^j \nabla_j \varphi - \varphi \nabla_p \dot{u}^p = \varphi \eta - \varphi u_j(\eta u^j - \varphi \dot{u}^j) - 2\varphi \nabla_p \dot{u}^p + \frac{3}{2} \dot{\eta} - \dot{u}^j(-\dot{\varphi} u_j - \varphi \dot{u}_j) = 3\varphi \eta - 2\varphi \nabla_p \dot{u}^p + \frac{3}{2} \dot{\eta}$. Finally use (19) to obtain $\dot{\eta} = -2\varphi \eta$. Then: $u^k \nabla_k \nabla_j \dot{u}^j = -2\varphi \nabla_p \dot{u}^p$. This is (32).

The divergence of (28) is $\frac{1}{2} \dot{R} + R_{km}(\nabla^k u^m) = u^k \nabla_k(\xi - \nabla_p \dot{u}^p) + (\xi - \nabla_p \dot{u}^p) \nabla^k u_k + (n - 2)\nabla_k(\varphi \dot{u}^k)$. Simplify and obtain: $\dot{R} - 2\dot{\xi} = -2\varphi(R + 2\nabla_p \dot{u}^p - n\xi) - 2u^k \nabla_k(\nabla_p \dot{u}^p)$. The last term has just been evaluated and simplifies the formula to give (33).

9.2. Proof of equation (41)

Let $h_{\mu\nu}^* = g_{\mu\nu}^* - n_\mu n_\nu$. We use $\partial_\nu \Theta = n_\nu(n^\rho \partial_\rho \Theta)$ (Proposition 3.2).

$$\begin{aligned} (n - 2)R_{\mu\nu\rho\sigma}^* n^\sigma &= (n - 2)(\nabla_\mu^* \nabla_\nu^* - \nabla_\nu^* \nabla_\mu^*) n_\rho \\ &= \nabla_\mu^* [\Theta(g_{\nu\rho}^* - n_\nu n_\rho)] - \nabla_\nu^* [\Theta(g_{\mu\rho}^* - n_\mu n_\rho)] \\ &= (h_{\nu\rho}^* \partial_\mu \Theta - h_{\mu\rho}^* \partial_\nu \Theta) - \Theta(n_\nu \nabla_\mu^* n_\rho - n_\mu \nabla_\nu^* n_\rho) \\ &= (h_{\nu\rho}^* \partial_\mu \Theta - h_{\mu\rho}^* \partial_\nu \Theta) - \Theta^2(n_\nu g_{\mu\rho}^* - n_\mu g_{\nu\rho}^*) / (n - 2) \\ &= \xi^*(g_{\mu\rho}^* n_\nu - g_{\nu\rho}^* n_\mu) \end{aligned}$$

with ξ^* given by (42). The contraction with the metric tensor $g^{*\mu\rho}$ shows that it is an eigenvalue: $R_{\nu\sigma}^* n^\sigma = \xi^* n_\nu$. Since $C_{\mu\nu\rho\sigma}^* = 0$ we have:

$$0 = R_{\mu\nu\rho\sigma}^* + \frac{g_{\mu\sigma}^* R_{\nu\rho}^* - g_{\nu\sigma}^* R_{\mu\rho}^* + R_{\mu\sigma}^* g_{\nu\rho}^* - R_{\nu\sigma}^* g_{\mu\rho}^*}{n - 3} - R^* \frac{g_{\mu\sigma}^* g_{\nu\rho}^* - g_{\nu\sigma}^* g_{\mu\rho}^*}{(n - 2)(n - 3)}$$

The contraction with $n^\mu n^\sigma$

$$0 = -\xi^*(g_{\nu\rho}^* - n^\mu n^\rho) + \frac{n - 2}{n - 3} [R_{\nu\rho}^* - n_\nu n_\rho \xi^* + \xi^* g_{\nu\rho}^* - \xi^* n_\nu n_\rho] - R^* \frac{g_{\nu\rho}^* - n_\nu n_\rho}{n - 3}$$

and the Ricci tensor $R_{\nu\rho}^*$ in (41) is obtained.

9.3. Proof of Lemma 3.4

We need: $\nabla_p \dot{u}^p = \Gamma_{0\mu}^0 \frac{b^\mu}{b} + \nabla_\mu (\frac{b^\mu}{b}) = \frac{b_\mu b^\mu}{b^2} + \frac{1}{b} \nabla_\mu b^\mu - \frac{b_\mu b^\mu}{b^2} = \frac{1}{a^2(t)b} \nabla_\mu^* b^\mu$. Then:

$$\nabla_p \dot{u}^p = \frac{1}{a^2(t)b} \nabla_\mu^* b^\mu. \tag{94}$$

In the frame (37):

$$\begin{aligned} 0 &= \nabla_p \dot{u}^p - (n - 1) \left(\frac{\dot{u}^i \nabla_i \eta}{2\eta} + \eta \right) \\ &= \nabla_p \dot{u}^p - (n - 1) \left(g^{rr} \frac{\partial_r b}{b} \partial_r \log \sqrt{\eta} + \eta \right) \\ &= \frac{1}{a^2(t)b} \nabla_\mu^* b^\mu - (n - 1) \left(\frac{1}{a^2 f_1^2} \frac{\partial_r b}{b} \partial_r \log \left(\frac{\partial_r b}{f_1 b} \right) + \frac{1}{a^2(t)} \frac{(\partial_r b)^2}{f_1^2 b^2} \right) \\ &= \frac{1}{a^2(t) f_1^2 b} \left[f_1^2 \nabla_\mu^* b^\mu - (n - 1) \left(\partial_r b \partial_r \log \left(\frac{\partial_r b}{f_1 b} \right) + \frac{(\partial_r b)^2}{b} \right) \right] \\ &= \frac{1}{a^2(t) f_1^2 b} \left[f_1^2 \nabla_\mu^* b^\mu - (n - 1) (\partial_r^2 b - \partial_r b \partial_r \log f_1) \right] \end{aligned}$$

Now we use Eq. (45): $f_1^2 \nabla_\mu^* b^\mu = \partial_r^2 b - (\partial_r b) \partial_r \log(f_1 f_2) + (n - 1) (\partial_r b) (\partial_r \log f_2)$ and obtain the result.

9.4 A new class of solutions of Eq. (74)

With $B = b_x/b$ and $G = F_x/F = -f_{1x}/f_1$, it is $b_{xx}/b = B_x + B^2$ and $F_{xx}/F = G_x + G^2$. Eq. (74) becomes: $B_x - (n - 3)G_x = -B^2 - 2BG + (n - 3)G^2$. We search solutions such that $B = (\alpha - 1)G - \beta$. Then: $(n - 2 - \alpha)G_x = [\alpha^2 - (n - 2)]G^2 -$

$2\beta\alpha G + \beta^2$. A first integral is

$$\frac{d}{dx} \log \left| \frac{G - g_+}{G - g_-} \right| = \frac{2\beta\sqrt{n-2}}{(n-2) - \alpha} \quad \text{where } g_{\pm} = \beta \frac{\alpha \pm \sqrt{n-2}}{\alpha^2 - (n-2)}$$

$$G(x) = \frac{g_+ \pm g_- K e^{\nu x}}{1 \pm K e^{\nu x}} \quad \nu = \frac{2\beta\sqrt{n-2}}{(n-2) - \alpha}, \quad K \geq 0$$

Integrate $G = -f_{1x}/f_1$ and put $x = r^2$ in the end. Once f_1 is obtained, $b = C'e^{-\beta r^2} f_1^{1-\alpha}$. The result are the pairs:

$$f_1(r) = e^{-\beta r^2 \frac{1}{\alpha - \sqrt{n-2}}} (1 \pm K e^{\nu r^2})^{-\frac{\alpha - (n-2)}{\alpha^2 - (n-2)}} \tag{95}$$

$$b(r) = e^{-\beta r^2 \frac{1 - \sqrt{n-2}}{\alpha - \sqrt{n-2}}} (1 \pm K e^{\nu r^2})^{-(1-\alpha) \frac{\alpha - (n-2)}{\alpha^2 - (n-2)}} \tag{96}$$

9.5. The doubly warped metric

$$ds^2 = -b^2(\mathbf{x})dt^2 + a^2(t)g_{\mu\nu}^*(\mathbf{x})dx^\mu dx^\nu.$$

Notation: $a_t = \partial_t a$, $b_\mu = \partial_\mu b$, $b^\mu = g^{*\mu\nu} b_\nu$.

- Metric tensor:

$$g_{00} = -b^2(\mathbf{x}), \quad g_{\mu\nu} = a^2(t)g_{\mu\nu}^*(\mathbf{x}), \quad g^{00} = -\frac{1}{b^2}, \quad g^{\mu\nu} = \frac{1}{a^2}g^{*\mu\nu} \tag{97}$$

- Christoffel symbols: $\Gamma_{ij}^k = \frac{1}{2}g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$.

$$\Gamma_{00}^0 = 0, \quad \Gamma_{\mu 0}^0 = \frac{b_\mu}{b}, \quad \Gamma_{00}^\mu = \frac{bb^\mu}{a^2}, \quad \Gamma_{\mu 0}^\lambda = \delta_\mu^\lambda \frac{a_t}{a}, \tag{98}$$

$$\Gamma_{\mu\nu}^0 = \frac{aa_t}{b^2}g_{\mu\nu}^*, \quad \Gamma_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^{*\lambda} \tag{99}$$

- Riemann tensor: $R_{ijkl}^m = -\partial_j \Gamma_{k,l}^m + \partial_k \Gamma_{j,l}^m + \Gamma_{j,l}^p \Gamma_{kp}^m - \Gamma_{k,l}^p \Gamma_{jp}^m$

$$R_{\mu 0 \rho}^0 = -\frac{1}{b} \nabla_\mu^* b_\rho + \frac{aa_{tt}}{b^2} g_{\mu\rho}^*, \quad R_{\mu\nu\rho}^0 = \frac{aa_t}{b^3} (b_\mu g_{\nu\rho}^* - b_\nu g_{\mu\rho}^*) \tag{100}$$

$$R_{\mu\nu\rho}^\sigma = R_{\mu\nu\rho}^{*\sigma} + \frac{a_t^2}{b^2} (g_{\mu\rho}^* \delta_\nu^\sigma - g_{\nu\rho}^* \delta_\mu^\sigma) \tag{101}$$

- Ricci tensor: $R_{ij} = \partial_m \Gamma_{ij}^m - \partial_i \Gamma_{mj}^m + \Gamma_{ij}^k \Gamma_{km}^m - \Gamma_{im}^k \Gamma_{kj}^m$

$$R_{00} = \frac{b}{a^2} \nabla_\mu^* b^\mu - (n-1) \frac{a_{tt}}{a} \tag{102}$$

$$R_{\mu 0} = (n-2) \frac{a_t}{a} \frac{b_\mu}{b} \tag{103}$$

$$R_{\mu\nu} = R_{\mu\nu}^* + g_{\mu\nu}^* \frac{1}{b^2} [(n-2)a_t^2 + aa_{tt}] - \frac{1}{b} \nabla_{\mu}^* b_{\nu} \tag{104}$$

- Scalar curvature: $R = g^{ij} R_{ij} = -\frac{1}{b^2} R_{00} + \frac{1}{a^2} \sum_{\mu} R_{\mu\mu} g^{*\mu\mu}$

$$R = \frac{R^*}{a^2} - \frac{2}{a^2 b} \nabla_{\mu}^* b^{\mu} + (n-1) \frac{1}{a^2 b^2} [(n-2)a_t^2 + 2aa_{tt}] \tag{105}$$

9.6. Radial space metric, d=1+N

$$dx^2 = f_1^2(r) dr^2 + f_2^2(r) \sum_{a=1}^N g_a^2(\theta) d\theta_a^2$$

$$\Gamma_{rr}^{*r} = \partial_r \log f_1, \quad \Gamma_{ra}^{*r} = 0, \quad \Gamma_{rr}^{*a} = 0, \quad \Gamma_{aa'}^{*r} = -\delta_{aa'} g_a^2(\theta) \frac{f_2^2}{f_1^2} (\partial_r \log f_2)$$

$$\Gamma_{ra'}^{*a'} = \delta_{aa'} \partial_r \log f_2, \quad \Gamma_{a'a''}^{*a} = \hat{\Gamma}_{a'a''}^a \text{ are independent of } r$$

$$R_{rr}^* = -N \left[\partial_r^2 \log f_2 - (\partial_r \log f_1)(\partial_r \log f_2) + (\partial_r \log f_2)^2 \right], \quad R_{ra}^* = 0$$

$$R_{aa'}^* = -\delta_{aa'} g_a^2 \frac{f_2^2}{f_1^2} \left[N(\partial_r \log f_2)^2 + \partial_r^2 \log f_2 - (\partial_r \log f_1)(\partial_r \log f_2) \right] + \hat{R}_{aa'}$$

($\hat{R}_{aa'}$ is the Ricci tensor for the subspace spanned by variables θ_a).

$$R^* = \frac{\hat{R}}{f_2^2} - \frac{N}{f_1^2} \left[2\partial_r^2 \log f_2 - 2(\partial_r \log f_1)(\partial_r \log f_2) + (N+1)(\partial_r \log f_2)^2 \right] \tag{106}$$

If θ_a parametrize the unit sphere S_N , then $\hat{R} = N(N-1)$ ([57] p.65).

9.7. Space components of the electric tensor

The normalization condition $u^k u_k = -1$ is $-1 = -b^2(r) u^0 u^0$. Then $u^0 = 1/b$, $u_0 = -b$.

$$\begin{aligned} E_{\mu\nu} &= u^0 C_{0\mu\nu} u_0 = -C_{0\mu\nu}^0 \\ &= R_{\mu 0\nu}^0 - \frac{R_{\mu\nu} + R_0^0 g_{\mu\nu}}{n-2} + \frac{R g_{\mu\nu}}{(n-1)(n-2)} \\ &= -\frac{1}{b} \nabla_{\mu}^* b_{\nu} + \frac{aa_{tt}}{b^2} g_{\mu\nu}^* - \frac{1}{n-2} \left[R_{\mu\nu}^* + g_{\mu\nu}^* \frac{1}{b^2} [(n-2)a_t^2 + aa_{tt}] \right. \\ &\quad \left. - \frac{1}{b} \nabla_{\mu}^* b_{\nu} - \frac{1}{b} \nabla_{\rho}^* b^{\rho} g_{\mu\nu}^* + (n-1) \frac{aa_{tt}}{b^2} g_{\mu\nu}^* \right] \\ &\quad + \left[\frac{1}{a^2} R^* - \frac{2}{a^2 b} \nabla_{\mu}^* b^{\mu} + (n-1) \frac{1}{a^2 b^2} [(n-2)a_t^2 + 2aa_{tt}] \right] \frac{a^2 g_{\mu\nu}^*}{(n-1)(n-2)} \end{aligned}$$

$$= -\frac{n-3}{n-2} \frac{1}{b} \left[\nabla_{\mu}^* b_{\nu} - \nabla_{\rho}^* b^{\rho} \frac{g_{\mu\nu}^*}{n-1} \right] - \frac{1}{n-2} \left[R_{\mu\nu}^* - R^* \frac{g_{\mu\nu}^*}{n-1} \right].$$

References

1. Astashenok, A.V., Capozziello, S., Odintsov, S.D., Oikonomou, V.K.: Extended gravity description for the GW190814 supermassive neutron star. *Phys. Lett. B* **811**, 135910 (2020). <https://doi.org/10.1016/j.physletb.2020.135910>
2. Astashenok, A.V., Capozziello, S., Odintsov, S.D., Oikonomou, V.K.: Novel stellar astrophysics from extended gravity. *Europhys. Lett.* **134**(5), 59001 (2021). <https://doi.org/10.1209/0295-5075/134/59001>
3. Adati, T., Miyazawa, T.: On a Riemannian space with recurrent conformal curvature. *Tensor (N.S.)* **18**, 348–354 (1967)
4. Banerjee, A., Som, M.M.: Conformally flat charged dust. *Int. J. Theor. Phys.* **20**(5), 349–354 (1981). <https://doi.org/10.1007/BF00669526>
5. Banerjee, A., Dutta Choudhury, S.B., Bhui, Bidyut K.: Conformally flat solution with heat flux. *Phys. Rev. D* **40**(2), 670–672 (1989). <https://doi.org/10.1103/PhysRevD.40.670>
6. Banerjee, A., Dutta Choudhury, S.B., Bhui, Bidyut K.: Spherically symmetric solutions with heat flow in general relativity. *Pramana J. Phys.* **34**(5), 397–401 (1990). <https://doi.org/10.1007/BF02880864>
7. Banerjee, A., Chatterjee, S., Dadhich, N.: Spherical collapse with heat flow and without horizon. *Mod. Phys. Lett. A* **17**(35), 2335–2339 (2002). <https://doi.org/10.1142/S0217732302008320>
8. Banerjee, A., Chatterjee, S.: Spherical collapse of a heat conducting fluid in higher dimensions without horizon. *Astrophys. Space Sci.* **299**, 219–225 (2005). <https://doi.org/10.1007/s10509-005-6849-x>
9. Bhui, B., Chatterjee, S., Banerjee, A.: Non adiabatic gravitational collapse in higher dimensional space-time and its junctions conditions. *Astrophys. Space Sci.* **226**, 7–18 (1995). <https://doi.org/10.1007/BF00626894>
10. Bolejko, K., C el erier, M.-N., Krasinski, A.: Inhomogeneous cosmological models: exact solutions and their applications. *Class. Quantum Grav.* **28**, 164002 (2011). <https://doi.org/10.1088/0264-9381/28/16/164002>
11. Bondi, H.: Spherically symmetrical models in general relativity. *MNRAS* **107**(5,6), 410–425 (1947). <https://doi.org/10.1093/mnras/107.5-6.410>
12. Borowiec, A., Wojnar, A.: Geometry of almost-product Lorentzian manifolds and relativistic observer. [arXiv:1302.1846](https://arxiv.org/abs/1302.1846)
13. Brozos-Vazquez, M., Garcia-Rio, E., Vazquez-Lorenzo, R.: Some remarks on locally conformally flat static space-times. *J. Math. Phys.* **46**, 022501 (2005). <https://doi.org/10.1063/1.1832755>
14. Buchdahl, H.A.: Non-linear Lagrangians and cosmological theory. *MNRAS* **150**(1), 1–8 (1970). <https://doi.org/10.1093/mnras/150.1.1>
15. Cadoni, M., Sanna, A.P., Tuveri, M.: Anisotropic fluid cosmology: an alternative to dark matter? *Phys. Rev. D* **102**, 023514 (10pp) (2020). <https://doi.org/10.1103/PhysRevD.102.023514>
16. Capozziello, S., De Laurentis, M., Odintsov, S.D., Stabile, A.: Hydrostatic equilibrium and stellar structure in $f(R)$ gravity. *Phys. Rev. D* **83**, 064004 (2011). <https://doi.org/10.1103/PhysRevD.83.064004>
17. Capozziello, S., De Laurentis, M., Farinelli, R., Odintsov, S.D.: The mass-radius relation for neutron stars in $f(R)$ gravity. *Phys. Rev. D* **93**, 023501 (2016). <https://doi.org/10.1103/PhysRevD.93.023501>
18. Capozziello, S., Mantica, C.A., Molinari, L.G.: Cosmological perfect fluid in $f(R)$ gravity. *Int. J. Geom. Methods Mod. Phys.* **16**(1), 1950008 (14pp) (2019). <https://doi.org/10.1142/S0219887819500087>
19. Capozziello, S., Mantica, C.A., Molinari, L.G.: Geometric perfect fluids from extended gravity. *Europhys. Lett.* **137**(1), 19001 (6pp) (2022). <https://doi.org/10.1209/0295-5075/ac525d>
20. Chakrabarty, S., Goswami, R., Maharaj, S., Banerjee, N.: Conformally flat collapsing stars in $f(R)$ gravity. *Gen. Relativ. Gravit.* **50**, 148 (23pp) (2018). <https://doi.org/10.1007/s10714-018-2472-3>
21. Chakrabarti, S., Banerjee, N.: Gravitational collapse in $f(R)$ gravity for a spherically symmetric spacetime admitting a homothetic killing vector. *Eur. Phys. J. Plus* **131**, 144 (8pp) (2016). <https://doi.org/10.1140/epjp/i2016-16144-1>
22. Chatterjee, S., Bhui, B., Banerjee, A.: Higher-dimensional Vaidya metric with an electromagnetic field. *J. Math. Phys.* **31**, 2208–2210 (1990). <https://doi.org/10.1063/1.528626>

23. Chen, B.-Y.: A simple characterization for generalized Robertson-Walker spacetimes. *Gen. Relativ. Gravit.* **46**, 1833 (2014). <https://doi.org/10.1007/s10714-014-1833-9>
24. Chen, B.-Y.: Some results on concircular vector fields and their applications to Ricci solitons. *Bull. Korean Math. Soc.* **52**(5), 1535–1547 (2015). <https://doi.org/10.4134/BKMS.2015.52.5.1535>
25. Chowdhury, S., Pal, Kunal, Pal, Kuntal, Sarkar, T.: Collapse in $f(R)$ gravity and the method of R matching. *Eur. Phys. J.* **80**, 902 (15pp) (2020). <https://doi.org/10.1140/epjc/s10052-020-08459-w>
26. Clarkson, C.: Covariant approach for perturbations of rotationally asymmetric spacetimes. *Phys. Rev. D* **76**(10), 104034 (2007). <https://doi.org/10.1103/PhysRevD.76.104034>
27. Coley, A.A., Tupper, B.O.J.: Spherically symmetric anisotropic fluid ICKV spacetimes. *Class. Quantum Grav.* **11**, 2553–2574 (1994). <https://doi.org/10.1088/0264-9381/11/10/015>
28. De Oliveira, A.H.G., Santos, N.O., Kolassis, C.A.: Collapse of a radiating star. *MNRAS* **216**, 1001–1011 (1985). <https://doi.org/10.1093/mnras/216.4.1001>
29. Eisenhart, L.P.: *Riemannian Geometry*. Princeton University Press, New York (1949)
30. Ferrando, J.J., Morales, J.A., Portilla, M.: Inhomogeneous space-times admitting isotropic radiation: vorticity-free case. *Phys. Rev. D* **46**(2), 578–584 (1992). <https://doi.org/10.1103/PhysRevD.46.578>
31. Gair, J.R.: Spherical universes with anisotropic pressure. *Class. Quantum Grav.* **18**, 4897–4919 (2001). <https://doi.org/10.1088/0264-9381/18/22/313>
32. Gȩbarowski, A.: Doubly warped products with harmonic Weyl curvature tensor. *Colloq. Math.* **67**(1), 73–89 (1994). <https://eudml.org/doc/210265>
33. Gȩbarowski, A.: On conformally flat doubly warped products. *Soochow J. Math.* **21**(1), 125–129 (1995)
34. Glass, E.N.: Shear-free gravitational collapse. *J. Math. Phys.* **20**(7), 1508–1513 (1979). <https://doi.org/10.1063/1.524210>
35. Goswami, R., Nzioki, A.M., Maharaj, S.D., Ghosh, S.G.: Collapsing spherical stars in $f(R)$ gravity. *Phys. Rev. D* **90**, 084011 (10pp) (2014). <https://doi.org/10.1103/PhysRevD.90.084011>
36. Griffith, J.B., Podolský, J.: *Exact Space-Times in Einstein's General Relativity*. Cambridge University Press, Cambridge, Cambridge Monographs on Mathematical Physics (2012)
37. Herrera, L., Santos, N.O.: Local anisotropy in self-gravitating systems. *Phys. Rep.* **286**, 53–130 (1997). [https://doi.org/10.1016/S0370-1573\(96\)00042-7](https://doi.org/10.1016/S0370-1573(96)00042-7)
38. Herrera, L., Di Prisco, A., Martín, J., Ospino, J., Santos, N.O., Troconis, O.: Spherically symmetric dissipative anisotropic fluids: a general study. *Phys. Rev. D* **69**, 084926 (12 pp) (2004). <https://doi.org/10.1103/PhysRevD.69.084026>
39. Herrera, L., Santos, N.O.: Collapsing spheres satisfying an Euclidean condition. *Gen. Relativ. Gravit.* **42**, 2382–2391 (2010). <https://doi.org/10.1007/s10714-010-0986-4>
40. Hotlos, M.: Some theorems on doubly warped manifolds. *Demonstratio Math.* **23**(1), 39–58 (1990)
41. Jaryal, S., Chatterjee, A.: Gravitationally collapsing stars in $f(R)$ gravity. *Eur. Phys. J. C* **81**, 273 (13pp) (2021). <https://doi.org/10.1140/epjc/s10052-021-09079-8>
42. Krasiński, A.: On the global geometry of the Stephani universe. *Gen. Relativ. Gravit.* **15**(7), 673–689 (1983). <https://doi.org/10.1007/BF00759044>
43. Iyer, B.R., Vishveshwara, C.V.: The Vaidya solution in higher dimensions. *Pramana J. Phys.* **32**(6), 749–752 (1989). <https://doi.org/10.1007/BF02845995>
44. Lovelock, D., Rund, H.: *Tensors, Differential Forms, and Variational Principles* 1975 (Dover reprint, 1989)
45. Mc Manus, D.J., Coley, A.A.: Shear-free, irrotational, geodesic, anisotropic fluid cosmologies. *Class. Quantum Grav.* **11**, 2045–2058 (1994). <https://doi.org/10.1088/0264-9381/11/8/011>
46. Maiti, S.R.: Fluid with heat flux in a conformally flat space-time. *Phys. Rev. D* **25**(10), 2518–2520 (1982). <https://doi.org/10.1103/PhysRevD.25.2518>
47. Mantica, C.A., Molinari, L.G.: On the Weyl and Ricci tensor of generalized Robertson–Walker spacetimes. *J. Math. Phys.* **57**, 102502 (6 pp) (2016). <https://doi.org/10.1063/1.4965714>
48. Mantica, C.A., Molinari, L.G.: Generalized Robertson–Walker spacetimes: a survey. *Int. J. Geom. Methods Mod. Phys.* **14**(3), 1730001 (27pp) (2017). <https://doi.org/10.1142/S021988781730001X>
49. Mantica, C.A., Molinari, L.G.: Twisted Lorentzian manifolds: a characterization with torse-forming time-like unit vectors. *Gen. Relativ. Gravit.* **49**, 51 (7pp) (2017). <https://doi.org/10.1007/s10714-017-2211-1>
50. Mantica, C.A., Molinari, L.G.: A simple characterization of doubly twisted spacetimes. *Int. J. Geom. Methods Mod. Phys.* **18**(5), 2150079 (7 pp) (2021). <https://doi.org/10.1142/S0219887821500791>
51. Mantica, C.A., Molinari, L.G.: Doubly torqued vectors and a classification of doubly twisted spacetimes. *Gen. Relativ. Gravit.* **53**, 48 (2021). <https://doi.org/10.1007/s10714-021-02819-x>

52. Mehdizabeh, M.R., Ziaie, A.H.: Dynamical wormholes in Lovelock gravity. *Phys. Rev. D* **104**, 104050 (2021). <https://doi.org/10.1103/PhysRevD.104.104050>
53. Modak, B.: Cosmological solution with an energy flux. *J. Astrophys.* **5**, 317–322 (1984). <https://doi.org/10.1007/BF02714547>
54. Molinari, L.G., Tacchini, A., Mantica, C.A.: On the uniqueness of a shear-vorticity-acceleration-free velocity field in space-times. *Gen. Relativ. Gravit.* **51**, 127 (13 pp) (2019). <https://doi.org/10.1007/s10714-019-2607-1>
55. Nairai, H.: On a class of non-static fluid spheres without energy flow. *Progress Theor. Phys.* **40**(5), 1013–1022 (1968). <https://doi.org/10.1143/PTP.40.1013>
56. Paliathanasis, A., Govender, M., Leon, G.: Temporal evolution of a radiating star via Lie symmetries. *Eur. Phys. J. C* **81**, 718 (8 pp) (2021). <https://doi.org/10.1140/epjc/s10052-021-09521-x>
57. Petersen, P.: *Riemannian Geometry*, Graduate Texts in Mathematics 171. Springer, Berlin (2006)
58. Ramos, M.P.M., Vaz, E.G.R., Carot, J.: Double warped space-times. *J. Math. Phys.* **44**(10), 4839–4865 (2003). <https://doi.org/10.1063/1.1605496>
59. Sanyal, A.K., Ray, D.: Cosmological solution of the Einstein equation with heat flow. *J. Math. Phys.* **25**, 1975–1976 (1984). <https://doi.org/10.1063/1.526388>
60. Santos, N.O.: Non-adiabatic radiating collapse. *MNRAS* **216**, 403–410 (1985). <https://doi.org/10.1093/mnras/216.2.403>
61. Schäfer, D., Goenner, H.: The gravitational field of a radiating and contracting spherically-symmetric body with heat flow. *Gen. Relativ. Gravit.* **32**(11), 2119–2130 (2000). <https://doi.org/10.1023/A:1001991223754>
62. Senovilla, J.M.M.: Junction conditions for $f(R)$ gravity and their consequences. *Phys. Rev. D* **88**, 064015 (11pp) (2013). <https://doi.org/10.1103/PhysRevD.88.064015>
63. Shaikh, A.A., Ali, A., Alkhalidi, A.H., Chakraborty, D., Datta, B.R.: On some curvature properties of Lemaitre–Tolman–Bondi spacetime. *Gen. Relativ. Gravit.* **54**, 6 (21pp) (2022). <https://doi.org/10.1007/s10714-021-02890-4>
64. Sharma, R.: Proper conformal symmetries of space-times with divergence-free Weyl conformal tensor. *J. Math. Phys.* **34**(8), 3582–3587 (1988). <https://doi.org/10.1063/1.530046>
65. Solanki, J.: Gravitational collapse of compact stars in $f(R) = \xi R^4$ gravity. *Int. J. Geom. Methods Mod. Phys.* **19**(2), 2250021 (2022). <https://doi.org/10.1142/S0219887822500219>
66. Starobinsky, A.A.: A new type of isotropic cosmological models without singularity. *Phys. Lett. B* **91**(1), 99–102 (1980). [https://doi.org/10.1016/0370-2693\(80\)90670-X](https://doi.org/10.1016/0370-2693(80)90670-X)
67. Sthepani, H., Kramer, D., MacCallum, M., Hoenselaers, C., Hertl, E.: *Exact Solutions of Einstein's Field Equations*, 2nd edn. Cambridge University Press, Cambridge (2003)
68. Tewari, B.C.: Collapsing shear-free radiating fluid spheres. *Gen. Relativ. Gravit.* **45**, 1547–1558 (2013). <https://doi.org/10.1007/s10714-013-1545-6>
69. Tewari, B.C., Charan, K.: Horizon free eternally collapsing anisotropic radiating star. *Astrophys. Space Sci.* **357**, 107 (7 pp) (2015). <https://doi.org/10.1007/s10509-015-2335-2>
70. Tupper, B.O.J., Keane, A.J., Carot, J.: A classification of spherically symmetric spacetimes. *Class. Quantum Grav.* **29**(14), 145016 (2012). <https://doi.org/10.1088/0264-9381/29/14/145016>
71. Wagh, S.M., Govender, M., Govinder, K.S., Maharaj, S.D., Muktibodh, P.S., Moodley, M.: Shear-free spherically symmetric spacetimes with an equation of state $p = \alpha\rho$. *Class. Quantum Grav.* **18**, 2147–2162 (2001). <https://doi.org/10.1088/0264-9381/18/11/311>
72. Yano, K.: Conformally separable quadratic differential forms. *Proc. Imp. Acad.* **16**(3), 83–86 (1940). <https://doi.org/10.3792/pia/1195579210>
73. Yano, K.: *The Theory of Lie Derivatives and Its Applications*. Interscience, New York (1957)
74. Zangeneh, M.K., Lobo, F.S.N., Riazi, N.: Higher dimensional evolving wormholes satisfying the null energy conditions. *Phys. Rev. D* **90**, 024072 (2014). <https://doi.org/10.1103/PhysRevD.90.024072>