Finite dimensional systems of free Fermions and diffusion processes on Spin groups

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Abstract

In this article we are concerned with "finite dimensional Fermions", by which we mean vectors in a finite dimensional complex space embedded in the exterior algebra over itself. These Fermions are spinless but possess the characterizing anticommutativity property. We associate invariant complex vector fields on the Lie group Spin(2n+1) to the Fermionic creation and annihilation operators. These vector fields are elements of the complexification of the regular representation of the Lie algebra $\mathfrak{so}(2n+1)$. As such, they do not satisfy the canonical anticommutation relations, however, once they have been projected onto an appropriate subspace of $L^{2}(\text{Spin}(2n+1))$, these relations are satisfied. We define a free time evolution of this system of Fermions in terms of a symmetric positive-definite quadratic form in the creation-annihilation operators. The realization of Fermionic creation and annihilation operators brought by the (invariant) vector fields allows us to interpret this time evolution in terms of a positive selfadjoint operator which is the sum of a second order operator, which generates a stochastic diffusion process, and a first order complex operator, which strongly commutes with the second order operator. A probabilistic interpretation is given in terms of a Feynman-Kac like formula with respect to the diffusion process associated with the second order operator.

1 Introduction

Probabilistic methods in Quantum Field Theory have proved to be particularly fruitful (cf. e.g. [25, 11, 15]). These methods have been almost exclusively restricted to *Bosonic* Field Theories. Some ideas of the Bosonic probabilistic methods carry over, to an extent, to the Fermionic case using the beautiful algebraic technique of Berezin integration [3]. However, the Berezin integral, being defined in terms of Grassmann variables, does not lend itself easily to an interpretation in the context of probability theory or measure theory (nevertheless, at least for the case of a discrete number of variables, a probabilistic interpretation of the Berezin formalism is possible, albeit somewhat cumbersome: cf. e.g. [5, 6]).

The aim of this work is to study a finite dimensional system of free Fermionic creation-annihilation operators in a way which parallels, in the sense explained below, the treatment of the corresponding Bosonic case of a finite dimensional quantum harmonic oscillator. It is well known that the Hamiltonian of an *n*-dimensional quantum harmonic oscillator can be interpreted, in Euclidean times, as the second order differential operator which generates an Ornstein-Uhlenbeck process on \mathbb{R}^n (cf. e.g. [26, p. 35]). Following this Bosonic parallel we study a model for a finite dimensional system of Fermions where the Fermionic Hamiltonian is replaced, in a quite natural way, by a second order differential operator. Moreover we study the possibility of interpreting the time evolution generated by such Hamiltonian in terms of stochastic processes.

The results in this work are inspired in part by the work of Schulman [24] (cf. also [23, Chapters 22-24]) who gives a description of a single $\frac{1}{2}$ -spin particle in terms of the Feynman path integral. The precedents for Schulman's idea can be found in early work on Quantum Mechanics connecting the Pauli $\frac{1}{2}$ -spin formalism with the quantum spinning top [4, 21] (cf. also the more recent work [2]). The works which, to the knowledge of the author, are the closest in spirit to the analysis given here are [8] and [7]. The situation studied here is nevertheless quite different. We are motivated by the parallel between the (Bosonic) quantum harmonic oscillator and the Ornstein-Uhlenbeck process. For this reason we associate to the Hamiltonian of a finite system of free Fermions a second order operator whereas in the works cited above the Hamiltonian was associated to a first order operator. The analysis that follows is therefore completely different. Moreover, because of our motivation, we pay particular attention to the rigorous, functional analytic details of our description.

We start our analysis by considering Fermionic creation-annihilation operators $a_i^{\dagger}, a_j, j = 1, \ldots, n, n \in \mathbb{N}$. We associate to these Fermionic operators first order differential operators. We achieve this purpose, similarly to the works cited above, by first exploiting the standard fact that the Fermionic creationannihilation operators give rise to a faithful, irreducible representation of the complex Lie algebra $\mathfrak{so}(2n+1,\mathbb{C})$ (obtained by complexifying the real Lie algebra $\mathfrak{so}(2n+1)$ of antisymmetric real $(2n+1) \times (2n+1)$ -matrices). This representation is usually called the spin representation of $\mathfrak{so}(2n+1,\mathbb{C})$. We therefore associate the Fermionic creation-annihilation operators with abstract elements in the Lie algebra $\mathfrak{so}(2n+1,\mathbb{C})$. We then use the standard fact that this Lie algebra can be realized in terms of (left-invariant) differential operators acting on smooth functions from the (real) Lie group Spin(2n+1) to \mathbb{C} . Here Spin(2n+1)is the simply-connected Lie group obtained as universal cover of the Lie group SO(2n+1) of rotations in 2n+1 dimensions. We need to pass from SO(2n+1)to Spin(2n+1) because the spin-representation of $\mathfrak{so}(2n+1,\mathbb{C})$ does not appear inside the representation of $\mathfrak{so}(2n+1,\mathbb{C})$ given by left-invariant vector fields acting on $C^{\infty}(\mathrm{SO}(2n+1))$ (indeed the spinor representation of $\mathrm{SO}(2n+1)$ is only a projective representation of SO(2n + 1) and not an actual representation). Having associated the Fermionic creation-annihilation operators a_i^{\dagger}, a_i to differential operators D_j^+, D_j^- we consider the free Fermionic Hamiltonian $H = \sum_{j=1}^{n} E_j a_j^{\dagger} a_j$ (for positive constants E_j) and we lift it to an element \tilde{H} of the universal enveloping algebra of $\mathfrak{so}(2n+1,\mathbb{C})$ which we look upon as the algebra of differential operators on $C^{\infty}(\operatorname{Spin}(2n+1))$ generated by the left-invariant vector fields together with the identity. The lift $H \rightsquigarrow \tilde{H}$ is by its very nature non-canonical. We choose to define $\tilde{H} = \sum_{j=1}^{n} D_{j}^{+} D_{j}^{-}$ by formally replacing the creation-annihilation operators in H by their associated first order differential operators. This choice differs from the one made by the references cited above. The motivation for our choice is that we want to study a situation parallel to the Bosonic case, where the free Hamiltonian of a quantum harmonic oscillator corresponds to a second order differential operator. The main results we obtain about the operator \tilde{H} are contained in theorem 4.7, which contains functional analytic properties regarding the operator \tilde{H} , and in theorem 5.4, where we give a Feynman-Kac like formula describing the evolution in Euclidean time generated by $-\tilde{H}$.

The layout of the article is as follows. In section 2 we give some basic definitions and describe the standard relation between n Fermionic creationannihilation operators and the spin representation of the Lie algebra $\mathfrak{so}(2n +$ $1, \mathbb{C}$), in a way which is well suited for our needs. In section 3 we briefly describe the standard connection between Lie algebra and left-invariant differential operators. We then specialize this general relation to our setting and describe how to recover, from this global picture, the spin representation of $\mathfrak{so}(2n+1,\mathbb{C})$. In section 4, after some remarks regarding selfadjointness and the universal enveloping algebra of a compact Lie group, we define the operator H and describe some of its most salient functional analytic properties (theorem 4.7). Finally in section 5 we introduce some standard facts regarding stochastic processes on Lie groups and we apply the general theory to our case. The main result is that the operator H splits into two parts, a strictly second order part which is hypoelliptic and generates a diffusion on Spin(2n+1) and a first order part which, as explained in section 5, does not contribute to a diffusion on Spin(2n+1). On the other hand, since it strongly commutes with the second order part, we are able to write a simple Feynman-Kac like formula for the operator H where we average over the process generated by the strictly second order part of $-\tilde{H}$. We note that our Feynman-Kac like formula resembles the Feynman-Kac formula for the Schrödinger operator of a particle in a magnetic field (cf. e.g. [26, Chapter V, Section 15.]).

2 Fermions and $\mathfrak{so}(2n+1,\mathbb{C})$

In this section, after some notational preliminaries, we describe the relation between the Fermionic creation-annihilation operators and the 1/2-spin representations of the Lie algebra $\mathfrak{so}_{\mathbb{C}}$. Since this relation is not very broadly known we provide some details.

Let us denote by $\mathbb{C}\ell(N)$, $N \in \mathbb{N}$, the complex *Clifford algebra* over \mathbb{C}^N , that is the unital associative algebra obtained as the quotient of the full tensor algebra $T(\mathbb{C}^N)$ by the following relations,

$$\{v,w\} = -2\langle v,w\rangle \mathbf{1}, \quad v,w \in \mathbb{C}^N,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard symmetric bilinear form on \mathbb{C}^n , and $\{v, w\} \stackrel{\text{def}}{=} vw + wv$ where we have denoted the product of v, w in $\mathbb{C}\ell(N)$ simply by vw.

Let V be a real or complex , finite dimensional, vector space. We denote by $\bigwedge V$ the *exterior algebra* over V, that is the algebra obtained by quotienting the full tensor algebra T(V) over V by the two sided ideal generated by elements of the form $v \otimes v, v \in V$. The finite dimensional Fermionic Fock space $\Gamma_{\land} \mathbb{C}^n \stackrel{\text{def}}{=} \bigoplus_{k=0}^n (\mathbb{C}^n)^{\land k}, n \in \mathbb{N}$, is defined as the Hilbert space realized by taking the exterior algebra $\bigwedge \mathbb{C}^n$ just as vector space and equipping it with the Hermitian scalar product $(\cdot, \cdot)_{\Gamma \mathbb{C}^n}$ which satisfies $(v_1 \land \cdots \land v_n, w_1 \land \cdots \land w_n)_{\Gamma_{\land} \mathbb{C}^n} \stackrel{\text{def}}{=} \det_{jk}(v_j, w_k)_{\mathbb{C}^n}$ for $v_j, w_k \in \mathbb{C}^n, j, k = 1, \ldots, n$, where $(\cdot, \cdot)_{\mathbb{C}^n}$ denotes the standard Hermitian scalar product on \mathbb{C}^n (antilinear in the left component).

We call the element $1 \in \mathbb{C} \hookrightarrow \bigwedge \mathbb{C}^n$ the vacuum vector. For $v \in \mathbb{C}^n$, we denote the Fermionic creation, annihilation operators on $\Gamma_{\wedge}\mathbb{C}^n$ respectively by $c^{\dagger}(v)$, c(v). Explicitly, for any $v \in \mathbb{C}^n$, c(v) and $c^{\dagger}(v)$ are defined as Hilbert adjoint of each other in $\Gamma_{\wedge}\mathbb{C}^n$ with $c^{\dagger}(v)\psi = v \land \psi$ where $\psi \in \Gamma_{\wedge}\mathbb{C}^n$. Note that by this definition $c^{\dagger}(v)$ is (complex) linear in v whereas c(v) is (complex) anti-linear. If e_j , $j \in \{1, \ldots, n\}$, denotes the standard basis of \mathbb{C}^n , then we denote $c(e_j)$, respectively $c(e_j)^{\dagger}$, by c_j , respectively c_j^{\dagger} . They satisfy the usual canonical anticommutation relations: $\{c_j, c_k^{\dagger}\} = \delta_{jk}$, $\{c_j, c_k\} = \{c_j^{\dagger}, c_k^{\dagger}\} = 0$ (where $\{A, B\} \stackrel{\text{def}}{=} AB + BA$ for any A, B in an associative algebra). Let us fix a decomposition $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ orthogonal with respect to the

Let us fix a decomposition $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ orthogonal with respect to the standard symmetric bilinear form $\langle \cdot, \cdot \rangle$ of \mathbb{C}^{2n} , and let \mathbb{P}_1 , respectively \mathbb{P}_2 , be the projection of \mathbb{C}^{2n} onto the first, respectively the second, copy of \mathbb{C}^n . Moreover let us denote by \bar{v} the vector obtained from $v \in \mathbb{C}^{2n}$ by complex conjugating each component. Then the standard Hermitian scalar product (\cdot, \cdot) on \mathbb{C}^{2n} satisfies $(v, w) = \langle \bar{v}, w \rangle, v, w \in \mathbb{C}^{2n}$. Let us define the algebra isomorphism $\gamma : \mathbb{C}\ell(2n) \to \operatorname{End}(\Lambda \mathbb{C}^n)$ by extending to the whole of $\mathbb{C}\ell(2n)$ the relations

$$\gamma(v) = c^{\dagger}(\mathbb{P}_1 v) - c(\overline{\mathbb{P}_1 v}) - i(c^{\dagger}((\mathbb{P}_2 v) + c(\overline{\mathbb{P}_2 v})), \quad v \in \mathbb{C}^{2n}.$$
 (1)

The map γ is a representation of $\mathbb{C}\ell(2n)$ on the Fock space $\Gamma_{\wedge}\mathbb{C}^n$ usually called the *Fock space representation* of $\mathbb{C}\ell(2n)$.

Let e_j , j = 1, ..., 2n, be the standard basis of \mathbb{C}^{2n} . We also denote by e_{ℓ} , $\ell = 1, ..., n$, a basis for each of the copies of \mathbb{C}^n in the decomposition $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$. To be concrete, in the following we will take \mathbb{P}_1 , to be the projection which sends e_{2j-1} to e_j and e_{2j} to zero. Then \mathbb{P}_2 sends e_{2j} into e_j and e_{2j-1} to zero.

Let $\gamma_j \stackrel{\text{def}}{=} \gamma(e_j), j = 1, \dots, 2n$. Then, with this choice of $\mathbb{P}_1, \mathbb{P}_2$ we obtain, from (1), the following relations

$$c_{j}^{\mathsf{T}} = \frac{1}{2}(\gamma_{2j-1} + i\gamma_{2j})$$

$$c_{j} = \frac{1}{2}(-\gamma_{2j-1} + i\gamma_{2j}), \quad j = 1, \dots n.$$
(2)

Note that, under these definitions, the operators $\gamma_j = \gamma(e_j)$, are anti-Hermitian as operators on the finite dimensional Hilbert space $\Gamma_{\wedge} \mathbb{C}^n$.

Let $\mathfrak{so}(N)$, $N \in \mathbb{N}$, denote the complex Lie algebra of antisymmetric $N \times N$ real matrices. and let $\mathfrak{so}(N, \mathbb{C}) = \mathfrak{so}(N) \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification.

We now define the standard 1/2-spin representation of $\mathfrak{so}(2n + 1, \mathbb{C})$ on $\bigwedge \mathbb{C}^n$. We give the definition in a form which differs slightly from the standard presentations (cf. e.g. [9, 12]) therefore we provide some details.

Consider an embedding $\iota : \mathfrak{so}(2n) \hookrightarrow \mathfrak{so}(2n+1)$ and the relative vector space decomposition $\mathfrak{so}(2n+1,\mathbb{C}) = V_{2n} \oplus \iota(\mathfrak{so}(2n,\mathbb{C}))$, where $V_{2n} = \mathfrak{so}(2n+1,\mathbb{C})/\iota(\mathfrak{so}(2n,\mathbb{C}))$ is a 2n-dimensional vector space which generates, via the Lie brackets, all of $\mathfrak{so}(2n+1,\mathbb{C})$.

Let $\mathbb{C}\ell(V_{2n})$ be the complex Clifford algebra generated by the identity **1** and by the symbols $\kappa(X), X \in V_{2n}$, which satisfy

$$\{\kappa(X),\kappa(Y)\} = \frac{1}{4}\operatorname{tr}(XY)\mathbf{1}, \quad X,Y \in V_{2n},$$

where tr denotes the trace in the defining representation of $\mathfrak{so}(2n+1,\mathbb{C})$ and XY denotes the product of X and Y as $(2n+1) \times (2n+1)$ matrices. Let us identify V_{2n} with \mathbb{C}^{2n} and denote one such isomorphism by ϕ . Then ϕ extends to an isomorphism of $\mathbb{C}\ell(V_{2n})$ with $\mathbb{C}\ell(2n)$ and the composition

$$\gamma \circ \phi : \mathbb{C}\ell(V_{2n}) \to \operatorname{End}(\bigwedge \mathbb{C}^n),$$

of the isomorphism ϕ with a Fock representation γ of $\mathbb{C}\ell(2n)$, defines a Fock representation¹ of $\mathbb{C}\ell(V_{2n})$. The following proposition shows how the Clifford algebra $\mathbb{C}\ell(V_{2n})$ with a Fock representation $\gamma \circ \phi$ gives rise to an irreducible representation $\pi^{1/2}$ of $\mathfrak{so}(2n + 1, \mathbb{C})$ which is unique up to isomorphism and coincides with the standard 1/2-spin representation of $\mathfrak{so}(2n + 1)$.

Proposition 2.1. The map κ extends to a Lie algebra homomorphism $\mathfrak{so}(2n + 1, \mathbb{C}) \to \mathbb{C}\ell(V_{2n})$, still denoted by κ , which sends the Lie brackets of $\mathfrak{so}(2n+1, \mathbb{C})$ into the commutator [A, B] = AB - BA, for $A, B \in \mathbb{C}\ell(V_{2n})$. The composition

$$\pi^{1/2} \stackrel{\text{def}}{=} \gamma \circ \phi \circ \kappa : \mathfrak{so}(2n+1,\mathbb{C}) \to End(\bigwedge \mathbb{C}^n),$$

of the homomorphism κ with the Clifford algebra isomorphism ϕ and with the Clifford algebra representation γ , defines a representation $\pi^{1/2}$ of $\mathfrak{so}(2n+1,\mathbb{C})$. This representation is isomorphic to the standard 1/2-spin representation of $\mathfrak{so}(2n+1,\mathbb{C})$, that is, the irreducible representation of $\mathfrak{so}(2n+1,\mathbb{C})$ on $\bigwedge \mathbb{C}^n$.

Proof. Without loss of generality let us fix $\{X_{jk}\}_{1 \leq j < k \leq 2n+1}$, to be the standard basis of $\mathfrak{so}(2n+1,\mathbb{C})$ given by matrices $X_{jk} = e_j \wedge e_k \stackrel{\text{def}}{=} e_j \otimes e_k - e_k \otimes e_j$, where e_ℓ , $\ell = 1, \ldots, 2n+1$, is the standard basis of \mathbb{C}^{2n+1} . Explicitly, these matrices have components $[X_{jk}]_{j'k'} = \delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}$, $1 \leq j < k \leq 2n+1$, $j', k' \in \{1, \ldots, 2n+1\}$. and satisfy the commutation relations, for $1 \leq r < i \leq 2n+1$ and $1 \leq s < j \leq 2n+1$,

$$[X_{ri}, X_{sj}] = \delta_{is} X_{rj} + \delta_{rj} X_{is} - \delta_{ij} X_{rs} - \delta_{rs} X_{ij}.$$
(3)

We consider the embedding $\iota : \mathfrak{so}(2n, \mathbb{C}) \hookrightarrow \mathfrak{so}(2n+1, \mathbb{C})$ obtained by setting $\iota(\mathfrak{so}(2n, \mathbb{C}))$ to be the subalgebra of $\mathfrak{so}(2n+1, \mathbb{C})$ spanned by $\{X_{jk}\}_{j,k\in\{1,\ldots,2n\}}$, and we let V_{2n} be the vector space spanned by $\{X_{j,2n+1}\}_{j=1,\ldots,2n}$. Then we have the decomposition $\mathfrak{so}(2n+1, \mathbb{C}) = V_{2n} \oplus \iota(\mathfrak{so}(2n, \mathbb{C}))$, indeed, $V_{2n} \cap \iota(\mathfrak{so}(2n, \mathbb{C})) = \{0\}$, and the direct sum of V_{2n} and $\mathfrak{so}(2n, \mathbb{C})$ is spanned by the whole basis above hence coincides with $\mathfrak{so}(2n+1, \mathbb{C})$.

¹One could more efficiently define a Fock representation directly of $\mathbb{C}\ell(V_{2n})$ without any reference to $\mathbb{C}\ell(2n)$, \mathbb{C}^{2n} , or the non-canonical isomorphism ϕ . We choose to introduce ϕ and replace V_{2n} by \mathbb{C}^{2n} and $\mathbb{C}\ell(V_{2n})$ by $\mathbb{C}\ell(2n)$ to make the computations in the following sections more concrete.

A straightforward computation shows that $tr(X_{a,2n+1}X_{b,2n+1}) = -2\delta_{ab}$, $a, b \in \{1, \ldots, n\}$. Hence, in this basis, we have

$$\{\kappa(X_{a,2n+1}),\kappa(X_{b,2n+1})\} = -\frac{1}{2}\delta_{ab}, \quad a,b \in \{1,\ldots,2n\}.$$

Using these anticommutation relations and the commutation relations (3) one can easily show that κ extends to a homomorphism of Lie algebras as claimed in the statement of the proposition.

Let us identify V_{2n} with \mathbb{C}^{2n} , and therefore $\mathbb{C}\ell(V_{2n})$ with $\mathbb{C}\ell(2n)$, by the map ϕ which sends, for all $j = 1, \ldots, 2n, X_{j,2n+1}$ into the standard basis element e_j of \mathbb{C}^{2n} . Then we have

$$(\phi \circ \kappa)(X_{j,2n+1}) = \frac{1}{2}e_j, \quad j = 1, \dots, 2n,$$

 $(\phi \circ \kappa)(X_{jk}) = \frac{1}{2}e_je_k, \quad 1 \le j < k \le 2n.$

We let, as in (2), γ be the Fock space representation of $\mathbb{C}\ell(2n)$ with projections $\mathbb{P}_1, \mathbb{P}_2$ such that $\mathbb{P}_1 e_{2j-1} = \mathbb{P}_2 e_{2j} = e_j$, $\mathbb{P}_1 e_{2j} = \mathbb{P}_2 e_{2j-1} = 0$. Now, since γ is a representation of $\mathbb{C}\ell(2n)$ on $\bigwedge \mathbb{C}^n$, it is clear that $\pi^{1/2} = \gamma \circ \phi \circ \kappa$ extends to a representation of $\mathfrak{so}(2n+1,\mathbb{C})$ on $\bigwedge \mathbb{C}^n$.

Note that with these conventions

$$\begin{aligned} \pi^{1/2}(X_{\ell,2n+1}) &= \frac{1}{2}\gamma_{\ell}, \\ \pi^{1/2}(X_{j\ell}) &= \frac{1}{2}\gamma_{j}\gamma_{\ell}, \quad 1 \le j < \ell \le 2n. \end{aligned}$$

Let

$$E_j = X_{2j-1,2n+1} + iX_{2j,2n+1}, \quad E_{-j} = -X_{2j-1,2n+1} + iX_{2j,2n+1}.$$
 (4)

Then, from our choice of ϕ and γ ,

$$c_j^{\dagger} = \pi^{1/2}(E_j), \quad c_j = \pi^{1/2}(E_{-j}),$$

coincide with the standard creation annihilation operators on $\Gamma_{\wedge}\mathbb{C}^n$ as in (2).

Because of this we see that $\pi^{1/2}$ is irreducible. Indeed, by repeated applications of $\pi^{1/2}(E_j) = a_j$ or $\pi^{1/2}(E_{-j}) = a_j^{\dagger}$ any invariant subspace of $\bigwedge \mathbb{C}^n$ under $\pi^{1/2}(\mathfrak{so}(2n+1,\mathbb{C}))$ has to include all of $\bigwedge \mathbb{C}^n$. Therefore this representation coincides with the standard 1/2-spin representation of $\mathfrak{so}(2n+1,\mathbb{C})$ and the proof is complete.

We will need to consider the real Lie algebra $\mathfrak{so}(2n+1)$ alongside $\mathfrak{so}(2n+1,\mathbb{C})$. Hence we note the following fact.

Corollary 2.1.1. The representation $\pi^{1/2}$ restricts to an irreducible representation, also called 1/2-spin representation, of the real Lie algebra $\mathfrak{so}(2n+1)$.

Proof. Indeed, with the same notation as in the proof of the proposition above, we have

$$\pi^{1/2}(X_{\ell,2n+1}) = \frac{1}{2}\gamma_{\ell},$$

$$\pi^{1/2}(X_{j\ell}) = \frac{1}{2}\gamma_{j}\gamma_{\ell}, \quad 1 \le j < \ell \le 2n.$$

Being $\gamma(v)$ complex linear in $v \in \mathbb{C}^{2n}$ we have that $\pi^{1/2}$ is an irreducible analytic representation of $\mathfrak{so}(2n+1,\mathbb{C})$ and it naturally restricts to a well defined representation of the real Lie algebra $\mathfrak{so}(2n+1)$. By the "Weyl unitary trick" (cf. [1, Theorem 3, SS1, Chapter 8, pp. 202-203]) the restricted representation is indeed irreducible.

Remark 2.2. We note that, under the same conventions as the proof of the proposition above, the vector $1 \in \Gamma_{\Lambda} \mathbb{C}^n$ can be seen as a lowest weight vector with relative weight $(-\frac{1}{2}, \cdots, -\frac{1}{2}) \in \mathbb{C}^n$.

To show this let us employ the same notation as in the proof above and let us fix a Cartan subalgebra \mathfrak{h} of $\mathfrak{so}(2n+1,\mathbb{C})$ generated by the elements

$$H_j = \frac{1}{2} [E_j E_{-j}] = i X_{2j-1,2j}, \quad j = 1, \dots, n.$$
(5)

These generators are normalized in such a way that, if we identify \mathfrak{h} with \mathbb{C}^n by sending H_j into e_j and we equip \mathbb{C}^n with its standard symmetric bilinear form $\langle \cdot, \cdot \rangle$, then the dual space \mathfrak{h}^* is itself isomorphic to \mathbb{C}^n and the dual of an element $H_j \cong e_j$ is the element $H_j \cong e_j$ itself. Now, $H_j 1 = -\frac{1}{2}c_j c_j^{\dagger} 1 = -\frac{1}{2}$. Hence the vector $1 \in \Gamma_{\wedge} \mathbb{C}^n$ is associated to the weight $\lambda \in \mathfrak{h}^*$ such that $\lambda(H_j) = -\frac{1}{2}$, for all $j = 1, \ldots, n$. Under the identifications given above, of $\mathfrak{h}^* \cong \mathbb{C}^n \cong \mathfrak{h}$, where \mathbb{C}^n is identified with its dual via the standard symmetric bilinear form on \mathbb{C}^n , the weight λ corresponds to the vector

$$\left(-\frac{1}{2},\ldots,-\frac{1}{2}\right)\in\mathbb{C}^n.$$

Under the natural order of \mathbb{R} this weight is the lowest weight of the representation. Hence we have shown that, under our conventions, $1 \in \Gamma_{\wedge} \mathbb{C}^n$ is the (normalized) lowest weight vector.

Similarly, under the same conventions, one can show that $e_1 \wedge \cdots \wedge e_n \in \Gamma_{\wedge} \mathbb{C}^n$ is the (normalized) highest weight vector corresponding to the highest weight $(\frac{1}{2}, \ldots, \frac{1}{2}) \in \mathbb{C}^n$.

By repeated use of the creation annihilation operators one shows that a general weight is of the form $(\underbrace{\pm \frac{1}{2}, \ldots, \pm \frac{1}{2}})$ with a given number of "plus signs"

and the complementary number of "minus signs". From a physical perspective, the plus signs in the *weight* λ denote "filled states", that is to every "plus sign" there corresponds a Fermionic particle in the respective state.

We conclude by stating the following fact which will be regularly used in the following sections.

Proposition 2.3. If we let $\mathfrak{U}(\mathfrak{so}(2n+1,\mathbb{C}))$ be the universal enveloping algebra of $\mathfrak{so}(2n+1,\mathbb{C})$ then $\pi^{(1/2)}$ extends naturally to a representation of the universal enveloping algebra and we have $\pi^{(1/2)}(\mathfrak{U}(\mathfrak{so}(2n+1,\mathbb{C}))) \cong \mathbb{C}\ell(2n)$.

Proof. This fact follows at once from the universal property of universal enveloping algebras (cf. e.g. [13, Theorem 9.7, p. 247]).

3 Fermions and $C^{\infty}(\text{Spin}(2n+1))$

The proposition 2.3 expresses the complex Clifford algebra $\mathbb{C}\ell(2n)$ as representation of the universal enveloping algebra of the complex Lie algebra $\mathfrak{so}(2n+1,\mathbb{C})$. In this section, after a short description of the standard representation of the universal enveloping algebra in terms of certain differential operators acting on a common domain of functions, we describe how to recover from this (infinite dimensional) representation the 1/2-spin representation $\pi^{(1/2)}$ of $\mathfrak{so}(2n+1,\mathbb{C})$ defined in section 2. Consider a connected, simply connected, compact Lie group **G** with Lie algebra \mathfrak{g} . Let $C^{\infty}(\mathbf{G})$ be the space of complex valued functions on **G**. We denote by $L : \mathbf{G} \to \operatorname{End}(C^{\infty}(\mathbf{G})), L : g \mapsto L_g$, the action of **G** on $C^{\infty}(\mathbf{G})$ by the *left translation* $L_g, g \in G$, where $L_gf(x) \stackrel{\text{def}}{=} f(g^{-1}x), g, x \in \mathbf{G}$. Similarly we denote by R the action of the **G** on $C^{\infty}(\mathbf{G})$ by the *right translation* $R_g, g \in \mathbf{G}$, where $R_gf(x) \stackrel{\text{def}}{=} f(xg), g, x \in \mathbf{G}$.

Let us denote by $\mathfrak{D}(\mathbf{G})$ the algebra of differential operators on $C^{\infty}(\mathbf{G})$ generated by the identity and the left-invariant vector fields on \mathbf{G} , i.e. the vector fields which commute with the left translation.

We have the following important fact (Cf. [14, Ch. II, Proposition 1.9 and its proof, p. 108]): the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is isomorphic (as a an algebra) to $\mathfrak{D}(\mathbf{G})$. Moreover, by this isomorphism, the Lie algebra \mathfrak{g} is represented on $C^{\infty}(\mathbf{G})$ by the representation $dR : \mathfrak{g} \to \mathfrak{D}(\mathbf{G})$ which associates to each element $X \in \mathfrak{g}$ the corresponding left-invariant vector field dR(X), where the linear operator $dR(X) : C^{\infty}(\mathbf{G}) \to C^{\infty}(\mathbf{G})$ is defined by

$$dR(X)f = \frac{d}{dt}R(e^{tX})f|_{t=0}, \quad f \in C^{\infty}(\mathbf{G}).$$

Remark 3.1. The fact that the universal enveloping algebra is isomorphic (as an algebra) to $\mathfrak{D}(\mathbf{G})$ means that the invariant vector fields $dR(X_1), \ldots, dR(X_n)$ associated with the generators of the Lie algebra \mathfrak{g} satisfy the Lie algebra commutation relations, that is² $[dR(X), dR(Y)]f = dR([XY])f, f \in C^{\infty}(\operatorname{Spin}(2n+1)), X, Y \in \mathfrak{g}.$

Let $\pi^{1/2} = \gamma \circ \phi \circ \kappa$ be the 1/2-spin representation of $\mathfrak{so}(2n+1,\mathbb{C})$ as given in section 2. With the same notation as in that section we have that

$$\pi^{1/2}(X_{j,2n+1}) = \gamma_j, \quad j = 1, \dots, 2n.$$

Note that $\gamma_1, \ldots, \gamma_{2n}$ satisfy the Lie algebra *commutation* relations of $\mathfrak{so}(2n+1)$ and the *anticommutation* relations of the Clifford algebra. Now, we can lift any generator $\gamma_j, j \in 1, \ldots, 2n$, of the Clifford algebra $\mathbb{C}\ell(2n)$ to an invariant (complex) vector field as a differential operator in $\mathfrak{D}(\mathbf{G})$. These vector fields will satisfy the commutation relations of the Lie algebra $\mathfrak{so}(2n+1,\mathbb{C})$ but *not* the Clifford *anticommutation* relations of the original elements $\gamma_1, \ldots, \gamma_{2n}$. To recover the Clifford anticommutation relations we will need to project onto a subspace isomorphic to the Fermionic Fock space $\Gamma_{\Lambda}\mathbb{C}^n$. We now turn to the description of this procedure.

Let $L^2(\mathbf{G})$ denote the space of functions from \mathbf{G} to \mathbb{C} which are square integrable with respect to the normalized Haar measure dg on \mathbf{G} . By slight abuse of notation we will still denote by R the extension of the representation, of \mathbf{G} on $C^{\infty}(\mathbf{G})$ by right translation, to a representation representation of \mathbf{G} on $L^2(\mathbf{G})$. Note that this extension gives a unitary representation. We now embed the Fermionic Fock space $\bigwedge \mathbb{C}^n$ into $L^2(\operatorname{Spin}(2n+1))$.

Lemma 3.2. Let $\pi^{(1/2)}(g)$ denote the 1/2-spin representation of an element $g \in \text{Spin}(2n+1)$. Then the map

$$F^{1/2}: \Gamma_{\wedge} \mathbb{C}^n \hookrightarrow L^2(\operatorname{Spin}(2n+1)), \quad F^{1/2}: \psi \mapsto \left(1, \pi^{(1/2)}(g) \psi\right)_{\wedge \mathbb{C}^n}.$$

²We denote by [XY] (no comma) the Lie brackets of the Lie algebra \mathfrak{g} and by [A, B] = AB - BA (with comma) the comutator in an associative algebra e.g. $\mathfrak{D}(\mathbf{G})$ or $\mathfrak{U}(\mathfrak{g})$.

defines an embedding of the Fermionic Fock space $\Gamma_{\wedge}\mathbb{C}^n$ into $L^2(\operatorname{Spin}(2n+1))$. Let

$$\Psi_0 \stackrel{\text{def}}{=} F^{1/2}(1) = \left(1, \pi^{(1/2)}(\cdot) 1\right)_{\bigwedge \mathbb{C}^n}, \qquad F_{\Psi_0} \stackrel{\text{def}}{=} \operatorname{Range}(F^{1/2}), \tag{6}$$

where $\operatorname{Range}(F^{1/2})$ denotes the image of $F^{1/2}$.

The restriction of the right regular representation R of $\operatorname{Spin}(2n+1)$ to F_{Ψ_0} defines a representation which coincides with the 1/2-spin representation of $\operatorname{Spin}(2n+1)$. Moreover, $F_{\Psi_0} \subset C^{\infty}(\operatorname{Spin}(2n+1))$ and the restriction of dRto F_{Ψ_0} defines a representation of the Lie algebra $\mathfrak{so}(2n+1)$ which coincides with the 1/2-spin representation of $\mathfrak{so}(2n+1)$.

Proof. Let

$$Y^{\alpha}_{(i)j}(x) \stackrel{\text{def}}{=} \sqrt{d_{\alpha}} D^{\alpha}_{ij}(x), \quad i, j = 1, \dots, d_{\alpha} \quad x \in \mathbf{G},$$

where α labels an irreducible unitary representation of $\operatorname{Spin}(2n+1)$, d_{α} denotes the dimension of such a representation, and $D_{ij}^{\alpha}(g)$ the i, j-matrix element of $g \in \operatorname{Spin}(2n+1)$ in such representation. By Peter-Weyl theorem (cf. [1, Chapter 7 §2, Theorem 1 p.172 and Theorem 2 p.174]), for $\alpha, i = 1, \ldots, d_{\alpha}$ fixed, the set of functions $(Y_{(i)j}^{\alpha})_{j=1,\ldots,d_{\alpha}}$ spans a subspace of dimension d_{α} which is invariant and irreducible for the right regular representation. Now take $\alpha = 1/2$, and let $fj, j = 1, \ldots, d_{1/2} = 2^n$ be an orthonormal basis of $\Gamma_{\wedge}\mathbb{C}^n$ with $f_1 = 1$. Then $Y_{(i)j}^{1/2} = 2^{-n/2}(f_i, \pi^{(1/2)}(g)f_j)_{\Gamma_{\wedge}\mathbb{C}^n}$. If we pick i = 1 then by the Peter-Weyl theorem, as described above, the set $(Y_{(1)j}^{1/2})_{j=1,\ldots,d_{1/2}}$ spans a subspace $\mathcal{H}_{1/2}$ which is isomorphic to $\Gamma_{\wedge}\mathbb{C}^n$. And the isomorphism is indeed the $F^{1/2}$ in the statement of the theorem. It is also clear that the right regular representation on $L^2(\operatorname{Spin}(2n+1))$ restricts on $\mathcal{H}_{1/2}$ to a representation isomorphic to the 1/2-spin representation of $\operatorname{Spin}(2n+1)$.

To prove the last part of the statement first note that any $Y_{(i)j}^{\alpha}$, as defined above, is smooth, that is $Y_{(i)j}^{\alpha} \in C^{\infty}(\text{Spin}(2n+1))$ (for a sketch of the argument cf. e.g. [10, Part I, Chapter 2, Appendix to section 2.]). Hence dL is well defined on $\mathcal{H}_{1/2}$ which is by definition the image of $F^{1/2}$. By definition of dL it is also clear that dL, restricted to $\mathcal{H}_{1/2}$, realizes a representation of the Lie algebra $\mathfrak{so}(2n+1)$ isomorphic to the 1/2-spin representation. The proof is therefore complete.

Corollary 3.2.1. The representation dR of $\mathfrak{so}(2n+1)$ extends to a representation $dR_{\mathbb{C}}$ of $\mathfrak{U}(\mathfrak{so}(2n+1,\mathbb{C}))$ on $C^{\infty}(\operatorname{Spin}(2n+1))$, where, as before, $\mathfrak{so}(2n+1,\mathbb{C})$ denotes the complexification of the Lie algebra $\mathfrak{so}(2n+1)$.

Proof. The representation dR of $\mathfrak{U}(\mathfrak{so}(2n+1))$ associates to every element $X \in \mathfrak{U}(\mathfrak{so}(2n+1))$ a differential operator acting on the complex space $C^{\infty}(\operatorname{Spin}(2n+1))$. Hence the complex-linear extension of dR is well defined on $C^{\infty}(\operatorname{Spin}(2n+1))$ and gives a representation $dR_{\mathbb{C}}$ of $\mathfrak{U}(\mathfrak{so}(2n+1,\mathbb{C}))$ isomorphic to the 1/2-spin representation.

4 Time evolution of a Fermionic state

For quantum mechanical applications it is not enough to consider an algebra of differential operators on $C^{\infty}(\text{Spin}(2n+1))$. For example, to discuss the

time evolution of the system, it is also necessary to consider the operators as unbounded operators in the Hilbert space $L^2(\mathbf{G})$. In particular the natural question is whether an operator initially defined on $C^{\infty}(\mathbf{G})$ defines a unique unbounded operator on $L^2(\mathbf{G})$. The main objective of this section is to show that we have a well defined notion of "quasi-Hamiltonian", which lifts the notion of the Hamiltonian for a system of Fermions, to an unbounded, essentially selfadjoint, positive operator on $L^2(\operatorname{Spin}(2n+1))$ with domain $C^{\infty}(\operatorname{Spin}(2n+1))$. We begin with some general considerations.

Let \mathfrak{g} be a real semisimple Lie algebra. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be one of the equivalent Cartan involutions on \mathfrak{g} . In the case where \mathfrak{g} is the Lie algebra of a compact semisimple Lie group we take θ to be the identity. Let

$$X^* = -\theta(X), \quad X \in \mathfrak{g}. \tag{7}$$

We extend this involution to an antilinear involution on $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, where $\mathfrak{g}_{\mathbb{C}}$ is the complexification of \mathfrak{g} . This operation makes $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ into a *-algebra. An element $X \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ is said to be *Hermitian* (as an element of the universal enveloping algebra) when $X = X^*$.

Consider the algebra $\mathfrak{D}(\mathbf{G})$ of left-invariant smooth differential operators in $L^2(\mathbf{G})$ with common invariant domain $C^{\infty}(\mathbf{G})$ and let $\mathfrak{D}_{\mathbb{C}}(\mathbf{G}) \stackrel{\text{def}}{=} \mathfrak{D}(\mathbf{G}) \otimes_{\mathbb{R}} \mathbb{C}$ denote its complexification. On $\mathfrak{D}_{\mathbb{C}}(\mathbf{G})$ we have an antilinear involution, which we also denote by *, which sends the unbounded operator $D \in \mathfrak{D}_{\mathbb{C}}\mathbf{G}$) to its Hilbert-adjoint D^* with respect to the scalar product in $L^2(\mathbf{G})$. This involution makes $\mathfrak{D}(\mathbf{G})$ into a *-algebra. Consider the representation dR of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$. On $C^{\infty}(\mathbf{G})$ we have indeed that $dR(X)^* = -dR(X)$, $X \in \mathfrak{g}$. But if we consider dR(X) as unbounded operator on $L^2(\mathbf{G})$ with domain $C^{\infty}(\mathbf{G})$ then the domain of $dR(X)^*$ will in general be larger than the domain of dR(X), that is, for $X \in \mathfrak{g}$, the operator dR(iX) = idR(X) is in general Hermitian but not selfadjoint³. Therefore we cannot say that $dR(iX)^* =$ dR(iX) holds when we picture dR(iX) as unbounded operators on $L^2(\mathbf{G})$ with domain $C^{\infty}(\mathbf{G})$. One could try to extend the operator dR(iX) to a selfadjoint operator by enlarging its domain. This might be possible for one operator dR(X) for a fixed $X \in \mathfrak{U}(\mathfrak{g})$. But, since different $X, Y \in \mathfrak{U}(\mathfrak{g})$ are elements of an algebra of operators, we need to have a *common* invariant domain of definition for both dR(X) and dR(Y). Hence, in general one cannot expect to find an extension of dR which sends Hermitian elements of $\mathfrak{g}_{\mathbb{C}}$ to self-adjoint operators in $L^2(\mathbf{G})$ with common domain of selfadjointness. One could argue that this requirement is too strong and not necessarily the most natural. Perhaps a more natural situation, which is obtained in the context of compact semisimple Lie groups, is the following (cf. e.g. [22, Corollary 10.2.10, p.270]). Let **G** be a *compact* Lie group with Lie algebra \mathfrak{g} . Then

$$\overline{dR(X^*)} = dR(X)^*, \quad X \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}), \tag{8}$$

³In the context of unbounded operators in a Hilbert space, an operator T with domain Dom(T) is *Hermitian* when it satisfies $\text{Dom}(T) \subset \text{Dom}(T^*)$ and $T|_{\text{Dom}(T)} = T^*_{\text{Dom}(T)}$. The operator T is selfadjoint when in addition the stronger condition $\text{Dom}(T) = \text{Dom}(T^*)$ holds. In the algebraic context of universal enveloping algebras, an element $X \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ is said to be Hermitian when $X = X^*$, where X^* is in the sense of (7). These two, in general different, concepts for an object to be Hermitian coincides when we identify the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ with the algebra $\mathfrak{D}(\mathbf{G})$ of smooth right-invariant vector fields acting on $C^{\infty}(\mathbf{G})$.

where the over-line on the left hand side denotes the operator closure. Note that this property implies in particular that any Hermitian element $D_{\mathbb{C}} \in \mathfrak{D}(\mathbf{G})$ is automatically *essentially selfadjoint*⁴.

We now turn to the notion of commuting unbounded operators. There are two natural notions of commuting unbounded operators, weakly commuting and strongly commuting. We give the precise definitions.

Given two unbounded operators A, B with common domain \mathcal{D} in a Hilbert space \mathfrak{H} , we say that A, B weakly commute on \mathcal{D} when ABv = BAv for all $v \in \mathcal{D}$. Given two selfadjoint unbounded operators A, B we say that A, Bstrongly commute when $e^{itA}e^{isB} = e^{isB}e^{itA}$ for all $s, t \in \mathbb{R}$, where e^{itC} denotes the unitary one-parameter group generated by a selfadjoint operator C (cf. [19, Theorem VIII.13] for a justification of this definition).

Regarding the relation between strong and weak commutativity of operators on a Hilbert space we have the following result due to Nelson.

Lemma 4.1 ([18, Corollary 9.2]). Let A, B be two Hermitian unbounded operators on a Hilbert space \mathcal{H} and let \mathcal{Q} be a dense linear subspace of \mathcal{H} such that \mathcal{Q} is contained in the domain of A, B, A^2, AB, BA , and B^2 , and such that A, Bweakly commute on \mathcal{Q} . If the restriction of $A^2 + B^2$ to \mathcal{Q} is essentially selfadjoint then A and B are essentially selfadjoint and their closures $\overline{A}, \overline{B}$ strongly commute.

A direct consequence of this lemma are the following facts, which will be used in this section and the following.

Proposition 4.2. Let **G** be a compact Lie group with Lie algebra \mathfrak{g} . Let $\mathfrak{g}_{\mathbb{C}}$ be the complexified Lie algebra of \mathfrak{g} , and $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ its universal enveloping algebra. Let $X, Y \in \mathfrak{U}(\mathfrak{g})$ be two commuting operators (in the algebraic sense of elements in the universal enveloping algebra). Then

- 1. the closed operators $\overline{dR(X)}, \overline{dR(Y)} \in \mathfrak{D}(\mathbf{G})$ strongly commute;
- 2. if dR(X) is positive (semi-)definite, and dR(Y) is Hermitian, then

$$\exp(-\overline{dR(X)})\exp(\mathrm{i}\overline{dR(Y)}) = \exp(\mathrm{i}\overline{dR(Y)})\exp(-\overline{dR(X)}),$$

where we recall that dR(X) and dR(Y) are the unique closed extensions of dR(x), respectively dR(Y), and dR(X) > 0.

Proof. The statement of point 1 follows from (8) and Nelson's Lemma 4.1. Indeed, if X, Y commute in the universal enveloping algebra then dR(X) and dR(Y) weakly commute on $C^{\infty}(\mathbf{G})$ because dR is a representation of $\mathfrak{U}(\mathfrak{g})$ with domain $C^{\infty}(\mathbf{G})$. Now, for \mathbf{G} a compact group, equation (8) tells us that any Hermitian element in the algebra $\mathfrak{D}(\mathbf{G})$ is essentially self adjoint on $C^{\infty}(\mathbf{G}) \subset L^2(\mathbf{G})$. Therefore in particular, for any $X, Y \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, we have that the operators dR(X), $dR(X)^2 = dR(X^2)$, dR(X)dR(Y) = dR(XY), dR(X) + dR(Y) = dR(X+Y) have the same domain $C^{\infty}(\mathbf{G})$, and are essentially selfadjoint there. Hence the hypothesis of the Lemma 4.1 are satisfied with A = dR(X) and B = dR(Y) and statement 1 follows. The statement of point 2 is a straightforward application of spectral calculus (cf [19, Section VIII.5]).

⁴That is, it admits a unique extension to a selfadjoint operator.

Remark 4.3. Because of the above proposition we only need to check whether two operators commute as elements of the universal enveloping algebra. From the Proposition 4.2 it then follows automatically that their closures are *selfadjoint* and *strongly commuting*.

With this proposition we have completed the considerations from the general theory. We can now turn to the application that we have in mind.

Definition 4.4 (Quasi-Fermionic vector fields). Let X_{ij} , $i, j = 1, \ldots, 2n + 1$, be the standard basis (cf. (3)) of the Lie algebra $\mathfrak{so}(2n + 1)$ of the Lie group $\operatorname{Spin}(2n+1)$. Let us denote by $D_{ij} \stackrel{\text{def}}{=} dR(X_{ij})$ the corresponding left-invariant vector fields on $\operatorname{Spin}(2n+1)$. We define the following operators (cf. (4))

$$D_k^+ \stackrel{\text{def}}{=} D_{2k-1,2n+1} + iD_{2k,2n+1} ,$$

$$D_k^- \stackrel{\text{def}}{=} -D_{2k-1,2n+1} + iD_{2k,2n+1} , \qquad k = 1, \dots, n ,$$

as linear operators on $C^{\infty}(\text{Spin}(2n+1), \mathbb{C}) \subset L^2(\text{Spin}(2n+1)).$

Definition 4.5 (Quasi-Hamiltonian operator). Let us fix *n* strictly positive numbers E_1, \ldots, E_n , with $0 < E_1 \leq \cdots \leq E_n$. Using for D_k^{\pm} the notation of the previous paragraph we call a *quasi-Hamiltonian* the operator

$$\tilde{H} = \sum_{k=1}^{n} E_k D_k^+ D_k^- \,,$$

acting on $C^{\infty}(\operatorname{Spin}(2n+1))$.

Remark 4.6. The operators D_k^{\pm} restricted to the finite dimensional subspace $F_{\Psi_0} \subset C^{\infty}(\text{Spin}(2n+1))$, given in (6) of 3.2, satisfy the canonical anticommutation relations. For this reason we call these operators "quasi-Fermionic". Similarly we named the operator \tilde{H} "quasi-Hamiltonian" because, restricted to the subspace $F_{\Psi_0} \cong \bigwedge \mathbb{C}^n$, it coincides with the *free Fermionic Hamiltonian* operator $H \stackrel{\text{def}}{=} \sum_k E_k c_k^{\dagger} c_k$, where the creation-annihilation operators $c_k^{\dagger}, c_k, k = 1, \ldots, n$, were defined in section 2.

Theorem 4.7. The unbounded operator \hat{H} with domain $C^{\infty}(\text{Spin}(2n+1))$ in $L^2(\text{Spin}(2n+1))$ is a positive, essentially selfadjoint operator. Moreover the quasi-Hamiltonian can be decomposed on $C^{\infty}(\text{Spin}(2n+1))$ as

$$\tilde{H} = P_0 + iB_0, \qquad P_0 \stackrel{\text{def}}{=} -\sum_{k=1}^n E_k L_k, \quad B_0 \stackrel{\text{def}}{=} \sum_{k=1}^n E_k T_k$$

with $T_k \stackrel{\text{def}}{=} D_{2k-1,2k}$, $L_k \stackrel{\text{def}}{=} D_{2k-1,2n+1}^2 + D_{2k,2n+1}^2$, $k = 1, \ldots, n$, and the following properties are satisfied:

 The operator P₀ and iB₀, with domains Dom(P₀), Dom(iB₀) both equal to C[∞](Spin(2n+1)), are essentially selfadjoint in L²(Spin(2n+1)). Moreover P₀ is positive definite. In particular −P₀ and B₀ are closable and their closures −P₀, B₀ are selfadjoint operators which generate respectively a semigroup and a unitary group (we consider −P₀ because generators of semigroups are usually taken to be negative definite). 2. The operators $\overline{P_0}$ and $\overline{iB_0}$ strongly commute. The operators $T_k, L_k, k = 1, \ldots, n$, are essentially selfadjoint. The unique selfadjoint closures $\overline{T_k}$, $k = 1, \ldots, n$ define a family of strongly commuting unbounded operators. Moreover each $\overline{T_k}, k = 1, \ldots, n$ strongly commutes with each $\overline{L_\ell}, \ell = 1, \ldots, n$. In particular $\overline{iB_0}$ strongly commutes with $\overline{P_0}$.

Proof. First note that \hat{H} is well defined on $C^{\infty}(\operatorname{Spin}(2n+1))$, since D^{\pm} are linear combinations of smooth vector fields, in particular D^{-} maps $C^{\infty}(\operatorname{Spin}(2n+1))$ into $C^{\infty}(\operatorname{Spin}(2n+1))$ (indeed $\mathfrak{D}(\operatorname{Spin}(2n+1))$ is an algebra). Using the above definition of the operators D_k^+, D_k^- in terms of the operators D_{ij} in Definition 4.4 we have, on $C^{\infty}(\operatorname{Spin}(2n+1))$,

$$\begin{split} \tilde{H} &= -\sum_{k=1}^{n} E_k (D_{2k-1,2n+1} + iD_{2k,2n+1}) (D_{2k-1,2n+1} - iD_{2k,2n+1}) \\ &= -\sum_{k=1}^{n} E_k (D_{2k-1,2n+1})^2 - \sum_{k=1}^{n} E_k (D_{2k,2n+1})^2 - i\sum_{k=1}^{n} E_k [D_{2k-1,2n+1}, D_{2k,2n+1}] \\ &= -\sum_{k=1}^{n} E_k \left((D_{2k-1,2n+1})^2 + (D_{2k,2n+1})^2 \right) - i\sum_{k=1}^{n} E_k D_{2k-1,2k}. \end{split}$$

Therefore we obtain

$$\ddot{H} = P_0 + iB_0 \tag{9}$$

where, P_0 and iB_0 are defined in the statement of the theorem. Note that by (8) all Hermitian operators we are handling are essentially selfadjoint. Moreover the operators $D_k^+ D_k^-$, k = 1, ..., n, are positive definite since D_k^+ is the formal adjoint of D_k^- . This implies that the quasi-Hamiltonian \tilde{H} is essentially selfadjoint (by (8)) and is positive definite (since is the sum of positive definite operators). This concludes the proof of the first part of the theorem.

Property 1 is proved by a similar argument. The fact that P_0 is closable and its closure defines a semigroup follows from the fact that P_0 is essentially selfadjoint (therefore closable) and positive definite (hence defines a semigroup). Similarly B_0 is closable because iB_0 is essentially selfadjoint and therefore defines a unitary one-parameter group.

We now turn to the proof of property 2. By (8) we obtain that $iT_k = iD_{2k-1,2k}$ and L_k are self-adjoint. Note that, by remark 4.3 if two elements X, Y in the universal enveloping algebra $\mathfrak{U}(\mathfrak{so}(2n+1))$ commute, then their representation dR(X), dR(X) admit closures $\overline{dR(X)}$, $\overline{dR(Y)}$ which strongly commute. Hence to prove the commutation properties of point 2, it is enough to perform the computation on the universal enveloping algebra.

Now, from the fact that the elements $X_{2k-1,2k}$, $k = 1, \ldots, n$ generate a maximal commutative subalgebra (Cartan subalgebra)⁵ of the Lie algebra $\mathfrak{so}(2n+1)$ we obtain that the operators $T_k = D_{2k-1,2k} = dR(X_{2k-1,2k})$, $k = 1, \ldots, n$ form a commuting family of operators.

As above, by remark 4.3, to show that L_{ℓ} commutes with $X_{2k-1,2k}$, for all $\ell, k \in \{1, \ldots, n\}$, it is enough to prove that the corresponding elements of the universal enveloping algebra commute. Consider

$$L_{\ell}^{\mathfrak{U}} \stackrel{\text{def}}{=} (X_{2\ell-1,2n+1})^2 + (X_{2\ell,2n+1})^2, \quad \ell = 1, \dots, n,$$

 $^{^5\}mathrm{Cf.}$ remark 2.2

be the element, associated to L_k , in the universal enveloping algebra of $\mathfrak{so}(2n + 1)$. It is enough to prove that

$$[L_{\ell}^{\mathfrak{U}}, X_{2k-1,2k}] = 0, \quad \text{for all } \ell, k = 1, \dots, n.$$
(10)

This follows from the following straightforward computations. Using the identity $[X^2, Y] = X[X, Y] + [X, Y]X$ for any $X, Y \in \mathfrak{U}(\mathfrak{so}(2n+1)_{\mathbb{C}})$ we get

$$\begin{bmatrix} L_{\ell}^{\mathfrak{U}}, X_{2k-1,2k} \end{bmatrix} = \\ = X_{2\ell-1,2n+1} \begin{bmatrix} X_{2\ell-1,2n+1}, X_{2k-1,2k} \end{bmatrix} + \begin{bmatrix} X_{2\ell-1,2n+1}, X_{2k-1,2k} \end{bmatrix} X_{2\ell-1,2n+1} + \\ + X_{2\ell,2n+1} \begin{bmatrix} X_{2\ell,2n+1}, X_{2k-1,2k} \end{bmatrix} + \begin{bmatrix} X_{2\ell,2n+1}, X_{2k-1,2k} \end{bmatrix} X_{2\ell,2n+1}.$$
(11)

Using in this expression the commutation relations (4) of the standard basis of $\mathfrak{so}(2n+1)$, we obtain for $\ell, k = 1, \ldots, n$

$$\begin{split} [L_{\ell}^{\mathfrak{U}}, X_{2k-1,2k}] &= \\ &= -X_{2\ell-1,2n-1} \,\delta_{2\ell-1,2k-1} X_{2n+1,2k} - \delta_{2\ell-1,2k-1} X_{2n+1,2k} \, X_{2\ell-1,2n+1} + \\ &\quad + X_{2\ell,2n+1} \,\delta_{2\ell,2k} X_{2n+1,2k-1} + \delta_{2\ell,2k} X_{2n+1,2k-1} \, X_{2\ell,2n+1}. \end{split}$$

Now, using in this expression the fact that $X_{ij} = -X_{ji}$ for all $1 \le i < j \le 2n+1$, and collecting the Kronecker deltas into a unique Kronecker delta which multiplies everything, we get

$$\begin{bmatrix} L_{\ell}^{\mathfrak{U}}, X_{2k-1,2k} \end{bmatrix} = \delta_{k,\ell} \left(X_{2\ell-1,2n+1} X_{2k,2n+1} + X_{2k,2n+1} X_{2\ell-1,2n+1} - X_{2\ell,2n+1} X_{2k-1,2n+1} - X_{2k-1,2n+1} X_{2\ell,2n+1} \right).$$

Finally, using the identity $\delta_{ij}f(i,j) = \delta_{ij}f(i,i)$ where f(i,j) is any function of $i, j \in \mathbb{N}$, we get

$$\begin{bmatrix} L_{\ell}^{\mathfrak{U}}, X_{2k-1,2k} \end{bmatrix} = \delta_{k,\ell} \left(X_{2k-1,2n+1} X_{2k,2n+1} + X_{2k,2n+1} X_{2k-1,2n+1} - X_{2k,2n+1} X_{2k-1,2n+1} - X_{2k-1,2n+1} X_{2k,2n+1} \right)$$

- 0

This proves (10). As a consequence it is now clear that P_0 commutes with B_0 which concludes the proof of property 2 and of the theorem.

5 Relation with stochastic processes

From theorem 4.7 we have that the quasi-Hamiltonian is

$$\tilde{H} = P_0 + \mathrm{i}B_0,$$

with $P_0 \stackrel{\text{def}}{=} \sum_{k=1}^n E_k L_k$ and $B_0 \stackrel{\text{def}}{=} \sum_{k=1}^n E_k T_k$, where all the operators are defined on $C^{\infty}(\text{Spin}(2n+1))$.

Since the operator B_0 appears in \tilde{H} multiplied by the imaginary unit i we cannot associate directly to the closure \tilde{H} a (real) stochastic process on Spin(2n + 1). For this reason we consider, together with P_0 , B_0 , and \tilde{H} above, the following operator

$$P \stackrel{\text{def}}{=} P_0 + B_0, \quad \text{Dom}(P) \stackrel{\text{def}}{=} C^{\infty}(\text{Spin}(2n+1)). \tag{12}$$

It turns out that it is possible to associate a stochastic diffusion processes on Spin(2n+1) to both closures $\overline{P_0}$ and \overline{P} in $L^2(\text{Spin}(2n+1))$. First we see that both $-\overline{P_0}$ and $-\overline{P}$ generate probability semigroups in the following sense.

Lemma 5.1. The operators -P, respectively $-P_0$ are essentially selfadjoint on $C^{\infty}(\operatorname{Spin}(2n+1))$ and their closures $-\overline{P}$, $-\overline{P_0}$ are infinitesimal generators of strongly continuous semigroups which act on $L^2(\operatorname{Spin}(2n+1))$ as convolution semigroups of probability measures with support on $\operatorname{Spin}(2n+1)$.

Proof. The statement follows from [17, Theorem 3.1].

Now we characterize the stochastic processes generated by $-\overline{P_0}$ and $-\overline{P}$ in terms of the SDEs these processes satisfy. Before doing so let us briefly introduce the notions of stochastic differential equation (SDE) on a manifold and of generator of a diffusion process (cf. e.g. [16]).

Let \mathcal{M} be a connected smooth manifold of dimension d. Moreover for convenience let us assume \mathcal{M} to be compact. This assumption simplifies somewhat the discussion and is sufficient for our purposes because we will in the sequel only deal with manifolds associated to compact Lie groups. In particular if \mathcal{M} is a compact manifold, then every C^{∞} -vector field on it is complete, that is, the flow associated to the given vector field can be extended to all times. This allows us to define a stochastic process globally on the manifold \mathcal{M} (without the need of the introduction of an explosion time).

Let us denote by $\mathcal{X}(\mathcal{M})$ the set of C^{∞} -vector fields on \mathcal{M} . Let us consider $r \in \mathbb{N}$ vector fields $A_0, A_1, \ldots, A_r \in \mathcal{X}(\mathcal{M})$ on \mathcal{M} .

Let $(\Omega, (\mathscr{F}_t)_{0 \leq t < \infty}, \mathbb{P})$ be a filtered probability space; we denote by $(W(t)) = (W^1(t), \ldots, W^r(t))$ an *r*-dimensional \mathscr{F}_t -adapted Brownian motion starting at zero, W(0) = 0. Finally, let ξ be an \mathscr{F}_0 -measurable \mathcal{M} -valued random variable.

Consider now an \mathscr{F}_t -adapted stochastic process X = X(t) on \mathcal{M} , that is an \mathscr{F}_t -adapted random variable X = (X(t)) with values in the continuous functions $C^0([0,\infty);\mathcal{M})$. Contrary to the previous sections, in this section the letter X will be reserved to denote a random variable.

Suppose that for every $f \in C^{\infty}(\mathcal{M})$ the stochastic process X = (X(t)) satisfies \mathbb{P} -almost surely the following integral equation

$$f(X(t)) - f(\xi) = \int_0^t \sum_{k=1}^r (A_k f)(X(s)) \circ dW^k(s) + \int_0^t (A_0 f)(X(s)) \,\mathrm{d}s, \quad (13)$$

for all⁶ $t \in [0, \infty)$, where $\circ dB$ denotes integration in the Stratonovich sense (see, e.g. [16]). Then we will say that the \mathcal{M} -valued stochastic process X = (X(t)) is a *solution* to (13).

Let us spend few words on the notion of strong solution regardless whether we are on a manifold \mathcal{M} or just in \mathbb{R}^d . Given a notion of solution it is natural to ask whether it satisfies some given *initial condition* which we take here to be a point $x \in \mathcal{M}$ (withoug any randomness). A solution to (13) with (non random) initial conditions $\xi = x$, is then a stochastic process X_x starting at time 0 at x.

More preciselly, we are asking for a function $F : \mathcal{M} \times C^0([0,\infty); \mathbb{R}^r) \to C^0([0,\infty); \mathcal{M})$ which maps the initial condition $x \in \mathcal{M}$ and the given realization

⁶By saying that the equality holds \mathbb{P} -almost surely, for all t, we are saying that the right hand side and the left hand side define *indistinguishable* processes.

of the Brownian motion W = (W(t)) into a realization of a process $X_x = (X_x(t))$ on the manifold \mathcal{M} . Moreover F is such that $X_x = F(x, W)$ is a solution to (13) with initial condition $\xi = x$ with probability one and with given Brownian motion W = (W(t)). Since at some point we would like to integrate X_x both with respect to $x \in \mathcal{M}$ and with respect to \mathbb{P} it is natural to ask that F be jointly measurable in x and W = (W(t)). It turns out that this is not always possible. When it is, $X_x = F(x, W)$ is called a *strong solution* to (13) with initial condition $\xi = x \in \mathcal{M}$ with probability one (cf. the discussion in [20, Section V.10] and [16, Chapter IV, section 1, esp. pp.162-163]).

In the context of smooth manifolds the situation is particularly good because we are considering SDE with smooth coefficients. Indeed one has a result (cf. [16, Chapter V, Section 1., Theorem 1.1, p.249]) which states that given an initial condition $x \in \mathcal{M}$ and an *r*-dimensional Brownian motion W = (W(t)), then a strong solution to (13) always exists and is unique⁷.

Once this important detail about how the initial condition is understood we can give meaning to the following shorthand, which we shall refer to as a *Stratonovich SDE* on the (compact) manifold \mathcal{M} :

$$\begin{cases} dX(t) = \sum_{k=1}^{r} A_k(X(t)) \circ dW^k(t) + A_0(X(t))dt, \\ X(0) = x, \quad x \in \mathcal{M}. \end{cases}$$
(14)

The meaning associated to (14) is that we consider a strong solution X of (13) (with initial conditions $\xi = x$ with probability one) and then define a solution to (14) to be the random variable $X_x = F(x, W)$, where F is the map which defines our strong solution X.

We now define the notion of stochastic diffusion process and of its generator.

First consider a more general case. For $x \in \mathcal{M}$, let X_x be a continuous stochastic process adapted to a filtration \mathscr{F}_t in the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For simplicity we consider a stochastic process defined for all $t \in [0, \infty)$ and with values in the space of continuous maps $[0, \infty) \to \mathcal{M}$ (where \mathcal{M} is always assumed to be compact) such that X(0) = x (where equality means \mathbb{P} -a.s.).

Let P_x be the probability law associated to the random variable $(X_x(t))$. This means that P_x is the image measure under the measurable mapping $X_x = (X_x(t))$ of the probability measure \mathbb{P} . Assume that $x \mapsto P_x$ is universally measurable⁸ and that P_x is uniquely determined by $x \in \mathcal{M}$.⁹ Moreover assume that there exists a linear operator \mathfrak{L} with domain $\text{Dom}(\mathfrak{L})$ in $C(\mathcal{M})$, such that for every $f \in \text{Dom}(\mathfrak{L})$,

$$X_f(t) \stackrel{\text{def}}{=} f(X(t)) - f(X(0)) - \int_0^t (\mathfrak{L}f)(X(s)) \,\mathrm{d}s$$

is a martingale with continuous sample paths and adapted to the filtration \mathscr{F}_t associated to $X_x(t)$ (cf. [16, Chapter IV, Theorem 5.2, p.207]). Then the family

⁷The idea behind this result is that the manifold \mathcal{M} is locally diffeomorphic to \mathbb{R}^d where d is the dimension of the manifold \mathcal{M} . This means that locally the SDE (14) (and hence (13)) can be written in coordinates as a standard SDE on \mathbb{R}^d . One can apply standard results about existence and uniqueness of solutions of SDEs to these local realizations. Finally one needs to patch together different local solutions into a global solution. Details can be found in the above mentioned [16].

⁸Cf., e.g. [16, p.1].

⁹These conditions are actually automatically satisfied when X_x is the strong solution to (14).

of probability measures $(P_x)_{x \in \mathcal{M}}$ is called a *diffusion generated* by the operator \mathfrak{L} .

When, for every $x \in \mathcal{M}$, X_x is the stochastic diffusion process on the manifold \mathcal{M} which is the strong solution to (14) with initial condition X(0) = x, then we have the following representation [16, Chapter V, Theorem 1.2, p.253]. The family of probability laws $(P_x)_{x \in \mathcal{M}}$, associated with the strong solutions X_x to (14) with initial conditions $x \in \mathcal{M}$, is a diffusion generated by the operator

$$\mathfrak{L} \stackrel{\text{def}}{=} \frac{1}{2} \sum_{j=1}^{r} A_k(A_k f) + A_0 f, \quad f \in C^{\infty}(\mathcal{M}),$$

(where, as before, the manifold \mathcal{M} is assumed to be compact) and the vector fields $A_0, A_1, \ldots, A_r \in \mathcal{X}(\mathcal{M})$ are interpreted as differential operators with common domain $C^{\infty}(\mathcal{M})$.

We now go back to our setting where the manifold $\mathcal{M} = \text{Spin}(2n+1)$ and collect the specialized version of the results recalled above. Doing so we give the characterization of the generators $-\overline{P_0}$ and $-\overline{P}$ (given by (12)) in terms of stochastic processes on Spin(2n+1).

Remark 5.2 (Notation). In this section we do not distinguish between an element X_{ij} in the Lie algebra and the corresponding differential operator $D_{ij} = dR(X_{ij})$ (cf. 4.4). In particular, depending on the context, we identify A_k , $k = 1, \ldots, 2n$, with either $D_{2n+1,k}$ or $X_{2n+1,k}$. Similarly, the differential operator B_0 in theorem 4.7 will be considered also as a vector field without changing notation.

Lemma 5.3 (Stochastic processes associated to P_0 and P). The following statements hold.

1. The Stratonovich SDEs on Spin(2n+1)

$$(P) \begin{cases} dY(t) = \sum_{k=1}^{2n} \sqrt{E'_k} A_k(Y(t)) \circ dW^k(t) - B_0(Y(t)) dt \\ Y(0) = x, \quad x \in \text{Spin}(2n+1) \end{cases}$$
$$(P_0) \begin{cases} dX(t) = \sum_{k=1}^{2n} \sqrt{E'_k} A_k(X(t)) \circ dW^k(t) \\ X(0) = x, \quad x \in \text{Spin}(2n+1), \end{cases}$$

with $E'_{2k+1} \stackrel{\text{def}}{=} E'_{2k} \stackrel{\text{def}}{=} E_k$, $k = 1, \ldots, n$, and $(W^k(t), k = 1, \ldots, 2n)$, a standard Brownian motion in \mathbb{R}^{2n} , are well defined and admit a unique strong solution.

2. The operators $-\overline{P}$ and $-\overline{P}_0$ (acting on $L^2(\text{Spin}(2n+1))$) are the generators of the diffusion processes given by the strong solutions of (P), (P_0) respectively.

Proof. For the first statement see [16, Chapter 5, Theorem 1.1 p.249]. The second statement follows from [16, Theorem 1.2, p.253]. \Box

The following result relates the time evolution semigroup generated by the quasi-Hamiltonian \tilde{H} in $L^2(\text{Spin}(2n+1))$ with a stochastic diffusion process on Spin(2n+1) generated by the second order part in $-\tilde{H}$.

Theorem 5.4. We have the following representations of the semigroup generated by the closure $-\tilde{H}$ of $-\tilde{H}$

$$(f, e^{-t\overline{H}}g)_{L^2(\operatorname{Spin}(2n+1))} = \mathbb{E}_X\left[\overline{f(0)} \left(e^{-it\overline{B_0}}g\right)\left(X(t)\right)\right], \qquad t \ge 0, \qquad (15)$$

where \mathbb{E}_X denotes the expectation with respect to the process generated by $-P_0$, $\overline{B_0}$, respectively $\overline{\tilde{H}}$, denotes the closure (which exists by theorem 4.7) of the operator B_0 , respectively \tilde{H} ; $\overline{f(0)}$ denotes complex conjugation, and $f,g \in C(\operatorname{Spin}(2n+1)) \subset L^2(\operatorname{Spin}(2n+1))$.

Proof. First note that $e^{-t\overline{H}}$ is a bounded operator for all $t \in \mathbb{R}^+$. Hence f, g can be taken in $L^2(\operatorname{Spin}(2n+1))$. The equality follows directly from the representation of the Hamiltonian as $\widetilde{H} = P_0 + iB_0$, the fact that $\overline{P_0}$ and B_0 strongly commute, and the Markov property of the semigroup generated by $-\overline{P_0}$ (which is a consequence of point 2 of lemma 5.3):

$$(f, e^{-t\overline{H}}g)_{L^{2}(\operatorname{Spin}(2n+1))} = (f, e^{-t(\overline{P_{0}}+i\overline{B_{0}})}g)_{L^{2}(\operatorname{Spin}(2n+1))}$$
$$= (f, e^{-t\overline{P_{0}}}e^{-it\overline{B_{0}}}g)_{L^{2}(\operatorname{Spin}(2n+1))}$$
$$= \mathbb{E}_{X}\left[\overline{f(0)} \left(e^{-it\overline{B_{0}}}g\right)\left(X(t)\right)\right].$$

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