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On Riemannian 4-manifolds and their twistor spaces: A moving frame approach

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Abstract

In this paper, we study the twistor space Z of an oriented Riemannian 4-manifold M using the moving frame approach, focusing, in particular, on the Einstein, non-self-dual setting. We prove that any general first-order linear condition on the almost complex structures of Z forces the underlying manifold M to be self-dual, also recovering most of the known related rigidity results. Thus, we are naturally lead to consider first-order quadratic conditions, showing that the Atiyah–Hitchin–Singer almost Hermitian twistor space of an Einstein 4-manifold bears a resemblance, in a suitable sense, to a nearly Kähler manifold.

KEYWORDS

Einstein manifold, 4-manifold, moving frames, twistor space

1 | INTRODUCTION AND MAIN RESULTS

Let (M, g) be a Riemannian manifold of dimension $2m$, with metric g. The *twistor space* Z associated with M is defined as the set of all the pairs (p, J_p) such that $p \in M$ and J_p is a linear endomorphism of the tangent space T_pM which satisfies the following conditions:

(1) for every $X, Y \in T_pM$, $g_p(J_p(X), J_p(Y)) = g_p(X, Y)$;
(2) for every $X \in T_pM$, $J_p(J_p(X)) = -X$.

Such an endomorphism is called a *g-orthogonal complex structure* on T_pM[1].

The twistor space Z defines a fiber bundle over M via the map that assigns to every pair (p, J_p) the point $p \in M$; hence, we can define Z in an equivalent way as

$$Z = O(M)/U(m),$$

where $O(M)$ denotes the orthonormal frame bundle over M and the unitary group $U(m)$ is identified with a subgroup of $SO(2m)$ (see, for instance, [15] and [26]). Moreover, if M is oriented, Z has two connected components $Z_\pm$ defined as quotients of the bundles $O(M)_\pm$ (which are subbundles of $O(M)$) of positively and negatively oriented orthonormal frames via the action of $U(m)$.

Twistor spaces can be regarded as Riemannian manifolds: indeed, it is possible to define a natural family of Riemannian metrics g_t on them, where t is a positive parameter, by taking the pullback of a specific bilinear form defined on $O(M)$, as explained in [15, 26].

These structures, introduced by Penrose [32], have been the subject of many investigations by the mathematical community, also in virtue of the numerous geometrical and algebraic tools involved in the definition of their properties. In 1978, Atiyah et al. [4] exploited Penrose's twistor theory in the Riemannian setting, introducing the concept of twistor space associated with a Riemannian 4-manifold in their study of self-dual Yang–Mills equations. Indeed, what can be observed is that there exist strong relations between the geometry of twistor spaces and the one of the underlying Riemannian manifolds: many characterizations of certain classes of Riemannian 4-manifolds can be obtained by examining the geometrical properties of their twistor spaces.

The particular interest for the four-dimensional geometry, beside the intrinsic importance due to the obvious relation with Relativity, arises from the unique structure of the Riemann curvature operator, which cannot be realized in any other dimension. Indeed, if (M, g) is a Riemannian manifold of dimension m, the Riemann curvature tensor Riem on M admits the well-known decomposition (see, e.g., [6, 9], and Section 2)

$$\mathrm{Riem} = \mathrm{W} + \frac{1}{m-2}\,\mathrm{Ric} \owedge g - \frac{S}{2(m-1)(m-2)} g \owedge g.$$

Due to its symmetries, the Riemann curvature tensor defines a linear operator $\mathcal{R}$ from the bundle of 2-forms Λ^2 to itself. If $m = 4$ and M is oriented, Λ^2 splits, via the Hodge $\star$ operator, into the direct sum of two subbundles Λ_+ and Λ_-, which are called the bundles of *self-dual* and *anti-self-dual* forms, respectively. This implies that the Riemann curvature operator gives rise to three linear maps A, B, and C, such that A (resp, C) is a symmetric endomorphism of Λ_+ (resp., Λ_-) and B is a linear map from Λ_+ to Λ_- (see [4, 6, 18, 37] for a complete dissertation. We would like to thank Prof. Ilka Agricola for having pointed out to us this last reference); therefore, $\mathcal{R}$ is represented by a block matrix

$$\mathcal{R} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix},$$

where $A^T = A$, $C^T = C$. Moreover, $\operatorname{tr} A = \operatorname{tr} C = \frac{S}{4}$. The splitting of Λ^2 also induces a decomposition of the Weyl tensor into a sum

$$\mathrm{W} = \mathrm{W}^+ + \mathrm{W}^-,$$

where W^+ (resp., W^-) is called the *self-dual* (resp., *anti-self-dual*) *part* of W. If $\mathrm{W}^+ = 0$ (resp., $\mathrm{W}^- = 0$), we say that M is an *anti-self-dual* (resp., *self-dual*) *manifold*. A characterization of these *half conformally flat* metrics in terms of the decomposition of $\mathcal{R}$ is given by Theorem 2.1.

In the literature, many results about the relation between a Riemannian 4-manifold M and its twistor space Z were achieved starting from the hypothesis that M is half conformally flat: for instance, Atiyah et al., in [4], introduced an almost complex structure J_+ on $Z_\pm$ and showed that M is a self-dual (resp., anti-self-dual) manifold if and only if J_+ is integrable on Z_- (resp., on Z_+), that is, the associated Nijenhuis tensor N_{J_+} vanishes identically, while Eells and Salamon, in [16], defined an almost complex structure J_- on $Z_\pm$ which is never integrable. In 1985, Friedrich and Grunewald [19] characterized Einstein twistor spaces (Z, g_t) as those whose base manifold is Einstein, half conformally flat and with positive scalar curvature. Some important characterization theorems for Einstein self-dual manifolds were proved by Jensen and Rigoli [26], Friedrich and Kurke [20], and by Davidov and Muškarov (see, for instance, [11–14, 29]), starting from the classification of the almost Hermitian manifolds due to Gray and Hervella [22]. In 1985, O' Brian and Rawnsley generalized the Atiyah–Hitchin–Singer twistor theory to higher dimensions, studying the problem of integrability of certain complex structures and proving a necessary and sufficient condition of integrability which involves the vanishing of the Bochner tensor for the underlying manifold (see [31]). In the recent paper [21], the authors exploit the moving frame formalism to study, among other things, the so-called balanced and first Gauduchon metric conditions on the twistor spaces of a Riemannian 4-manifold.

In this paper, we start from the following questions:

(1) Is it possible to introduce a framework that could simplify the study of the Riemannian and Hermitian features of the twistor space associated with a Riemannian four-dimensional manifold?
(2) Given an *Einstein 4-manifold* M, is it possible to find new and interesting properties of its associated twistor space?

Our approach to the aforementioned questions is inspired by the works of Jensen and Rigoli [26] and of Fu and Zhou [21]: all our computations of the main Riemannian and Hermitian features of the twistor spaces are based on the method of moving frames *à la* Cartan, which provides an effective answer to question (1). As a consequence of our analysis, we are able to easily recover and generalize some classical results. In particular, our first main result is the following.

Theorem 1.1. *Let (M, g) be an oriented Riemannian 4-manifold and let $(Z_-, g_t, \bar{J})$ be its twistor space, with $\bar{J} = J_+$ or $\bar{J} = J_-$. Suppose that, for every X, Y smooth vector fields on Z_-,*

$$a_1(\nabla_X \bar{J})Y + a_2(\nabla_Y \bar{J})X + a_3(\nabla_{\bar{J}X}\bar{J})Y + a_4(\nabla_{\bar{J}Y}\bar{J})X + a_5(\nabla_{\bar{J}X}\bar{J})\bar{J}Y +$$
$$+ a_6(\nabla_{\bar{J}Y}\bar{J})\bar{J}X + a_7(\nabla_X \bar{J})\bar{J}Y + a_8(\nabla_Y \bar{J})\bar{J}X = 0$$

for some $a_i \in \mathbb{R}$, $i = 1, \dots, 8$, such that $a_j \neq 0$ for some j. Then, M is self-dual.

This theorem allows us to prove in an alternative way the integrability result on the Atiyah–Hitchin–Singer almost complex structure in [4] and the characterization results for Einstein, self-dual manifolds with positive scalar curvature in [29]. Concerning question (2), since one of our main goals is to study Einstein 4-manifolds whose metrics are not necessarily self-dual, the previous theorem naturally lead us to consider first-order quadratic conditions: more precisely, we are able to show a local (i.e., holding only for *orthonormal* frames/coframes), quadratic characterization of Einstein 4-manifolds:

Theorem 1.2. *An oriented Riemannian 4-manifold (M, g) is Einstein if and only if, for every orthonormal frame in $O(M)_-$ (equivalently, for every negatively oriented orthonormal coframe),*

$$\sum_{t=1}^{6} (J^t_{p,q} + J^t_{q,p})(J^t_{p,p} - J^t_{q,q}) = 0, \quad \forall p, q = 1, \dots, 6,$$

where $J^t_{p,q}$ are the components of the covariant derivative of J_+ with respect to a local orthonormal coframe on (Z_-, g_t, J_+).

Moreover, we can prove a quadratic necessary and sufficient condition for Einstein, non-self-dual manifolds:

Theorem 1.3. *Let (M, g) be an oriented Riemannian Einstein 4-manifold. Then, for every orthonormal frame in $O(M)_-$,*

$$(J^t_{q,p} + J^t_{p,q})N^t_{pq} = 0 \qquad \text{(no sum over } p, q, t),$$

where $J^t_{p,q}$ and N^t_{pq} are the components of the covariant derivative of J_+ and of the Nijenhuis tensor, respectively, with respect to a local orthonormal coframe on (Z_-, g_t, J_+). Conversely, if Equation (5.2) holds on $O(M)_-$, then, for every point $p \in M$ such that $|\mathrm{W}^-| \neq 0$ at p, the traceless Ricci tensor vanishes at p, that is $B = 0$. In particular, if $|\mathrm{W}^-| \neq 0$ on M, (M, g) is an Einstein manifold.

Remark 1.4. It is a well-known fact (see Theorem 4.5) that the twistor space (Z_-, g_t, J) of an Einstein, self-dual manifold with positive scalar curvature (M, g) is nearly-Kähler (indeed, Kähler); moreover, by Hitchin's classification result of Kähler twistor spaces (see [25]) and of Einstein, self-dual manifolds with positive scalar curvature due to Friedrich and Kurke (see [20]), this is the case if and only if (M, g) is isometric (up to quotients) to $\mathbb{S}^4$ or $\mathbb{CP}_2$. If (M, g) is Einstein, but not necessarily self-dual, the properties of its twistor space are not so very well understood: for some interesting results in this direction, see, for example, [17, 33]. Theorems 1.2 and 1.3 show that the Atiyah–Hitchin–Singer almost Hermitian twistor space of an Einstein 4-manifold bears a resemblance to a nearly Kähler manifold. Indeed, it is known that there exist relations between twistor and nearly Kähler geometries: for instance, in 1985, Belgun and Moroianu presented a homothetic classification of six-dimensional strict nearly Kähler manifolds whose canonical Hermitian connection has reduced holonomy by exploiting their twistorial structures (see [5]). Note that, in this work, we do not focus our attention on almost Hermitian manifolds (in particular, twistor space associated with a Riemannian manifold) satisfying these "weak"-nearly Kähler conditions. A natural question would be the following: is it possible to characterize the round metric on $\mathbb{S}^4$ as the unique Einstein metric, by showing that the twistor space (Z_-, g_t, J) of a 4-sphere $\mathbb{S}^4$ equipped with an Einstein metric g_{Ein} cannot satisfy the conditions in Theorems 1.2 and 1.3, unless it is Kähler (or nearly Kähler)?

This will be the subject of future investigations, together with the analysis of higher order conditions on the almost complex structures and of curvature properties of Z.

The paper is organized as follows: for the sake of completeness, and to fix the notation and conventions of the moving frame formalism, in Section 2 we recall some very well-known facts about the geometry of Riemannian 4-manifolds. Section 3 is devoted to the formal definition of the twistor space Z of a Riemannian manifold, with special attention to the case of dimension four. In Section 4, we show that, given a Riemannian 4-manifold M and its twistor space Z, any linear condition on the covariant derivative of the almost complex structures on Z implies that M is self-dual; we also show how to recover some of the classical result (due to Atiyah et al. and to Muškarov). In Section 5, we focus on quadratic conditions, in order to study the case of Einstein 4-manifolds whose metric is not necessarily self-dual.

2 | GEOMETRY OF RIEMANNIAN 4-MANIFOLDS

In this section, for the sake of completeness, we recall some useful and well-known features of Riemannian 4-manifolds (see, e.g., [6, 18, 26, 34, 37]). Throughout the paper, we adopt Einstein summation convention over repeated indices, unless it is specified otherwise.

The Hodge $\star$ operator in four dimensions. Let (M, g) be a Riemannian oriented manifold of dimension n and let Λ^k be the space (bundle) of the k-differential forms, $1 \leq k \leq n$. Given any local chart (U, φ) that contains $p \in M$, let $\{e_1, \dots, e_n\}$ be a local, positively oriented, orthonormal frame for g on U and let $\{\theta^1, \dots, \theta^n\}$ be its dual orthonormal coframe, with $\theta^i \in \Lambda^1$, $\forall i = 1, \dots, n$. Since M is oriented, we can define a *volume form* locally expressed by

$$\omega = \theta^1 \wedge \cdots \wedge \theta^n \in \Lambda^n.$$

Now, it is possible to define the *Hodge $\star$ operator*, $\forall\, 1 \leq k \leq n$, locally as

$$\begin{aligned} \star : \Lambda^k &\longrightarrow \Lambda^{n-k} \\ \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} &\longmapsto \star(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) =: \eta \end{aligned} \tag{2.1}$$

where $\eta = \theta^{j_1} \wedge \cdots \wedge \theta^{j_{n-k}} \in \Lambda^{n-k}$ is the unique $(n-k)$-form such that $(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) \wedge \eta = \omega$. By construction, $\star$ satisfies the equation

$$\star^2 = (-1)^{k(n-k)} I, \tag{2.2}$$

where I is the identity map from Λ^k to itself. Now, let $n = 4$. We have that, if $k = 2$, the $\star$ operator is an involution: indeed, by definition and Equation (2.2),

$$\star : \Lambda^2 \longrightarrow \Lambda^2 \quad \text{and} \quad \star^2 = (-1)^{2 \cdot 2} I = I.$$

If $\{\theta^1, \theta^2, \theta^3, \theta^4\}$ is an orthonormal coframe for M in a given chart, the set $\{\theta^i \wedge \theta^j\}_{1 \leq i < j \leq 4}$ is an orthonormal basis for Λ^2 with respect to the inner product of differential forms induced on Λ^2 by the metric g. Moreover, since $\star$ is an involution, its only two eigenvalues are ± 1 and it can be easily seen that

$$\begin{aligned} \star(\theta^1 \wedge \theta^2 \pm \theta^3 \wedge \theta^4) &= \theta^3 \wedge \theta^4 \pm \theta^1 \wedge \theta^2, \\ \star(\theta^1 \wedge \theta^3 \pm \theta^4 \wedge \theta^2) &= \theta^4 \wedge \theta^2 \pm \theta^1 \wedge \theta^3, \\ \star(\theta^1 \wedge \theta^4 \pm \theta^2 \wedge \theta^3) &= \theta^2 \wedge \theta^3 \pm \theta^1 \wedge \theta^4. \end{aligned} \tag{2.3}$$

This means that

$$\Lambda_{\pm} := \operatorname{span}\left\{\theta^1 \wedge \theta^2 \pm \theta^3 \wedge \theta^4, \theta^1 \wedge \theta^3 \pm \theta^4 \wedge \theta^2, \theta^1 \wedge \theta^4 \pm \theta^2 \wedge \theta^3\right\} \tag{2.4}$$

are the eigenspaces of $\star$ relative to the eigenvalue ± 1, respectively. Thus, by Equation (2.4), we have that Λ^2 decomposes in a direct sum of two three-dimensional subspaces (subbundles)

$$\Lambda^2 = \Lambda_+ \oplus \Lambda_-. \tag{2.5}$$

Note that, if $\{\theta^i\}_{i=1,\dots,4}$ is a negatively oriented orthonormal coframe, the signs $+$ and $-$ must be exchanged in the right-hand side of Equation (2.4). Moreover, it is sufficient to define

$$\begin{aligned}
\alpha^1_{\pm} &:= \frac{1}{\sqrt{2}}(\theta^1 \wedge \theta^2 \pm \theta^3 \wedge \theta^4) \\
\alpha^2_{\pm} &:= \frac{1}{\sqrt{2}}(\theta^1 \wedge \theta^3 \pm \theta^4 \wedge \theta^2) \\
\alpha^3_{\pm} &:= \frac{1}{\sqrt{2}}(\theta^1 \wedge \theta^4 \pm \theta^2 \wedge \theta^3)
\end{aligned} \tag{2.6}$$

to have orthonormal bases for Λ_+ and Λ_-, which are called, respectively, the bundle of *self-dual* and *anti-self-dual* 2-forms of M. Clearly, any 2-form η can be written in a unique way as

$$\eta = \underbrace{\frac{1}{2}(\eta + \star\eta)}_{\in\Lambda_+} + \underbrace{\frac{1}{2}(\eta - \star\eta)}_{\in\Lambda_-} =: \eta_+ + \eta_-, \tag{2.7}$$

where η_+ is the *self-dual* part of η and η_- is the *anti-self-dual* part.

There is an action of $SO(4)$ on Λ^1, defined as

$$\begin{aligned}
SO(4) \times \Lambda^1 &\longrightarrow \Lambda^1 \\
(a, \theta^i) &\longmapsto a(\theta^i) := (a^{-1})^i_j \theta^j,
\end{aligned} \tag{2.8}$$

which induces an action of $SO(4)$ on Λ^2 given by

$$a(\theta^i \wedge \theta^j) := a(\theta^i) \wedge a(\theta^j) \tag{2.9}$$

(see, e.g., [26]). Moreover, it is known that $\mathfrak{so}(4)$ and Λ^2 are isomorphic via the map

$$\begin{aligned}
f : \mathfrak{so}(4) &\longrightarrow \Lambda^2 \\
X = (X_{ij}) &\longmapsto \frac{1}{2} X_{ij} \theta^i \wedge \theta^j
\end{aligned} \tag{2.10}$$

(here, $\mathfrak{so}(n)$ denotes the Lie algebra of $SO(n)$). The isomorphism f satisfies, for every $a \in SO(4)$, $X \in \mathfrak{so}(4)$,

$$f(\mathrm{Ad}_a)(X) = a(f(X))$$

where Ad means the adjoint representation of $SO(4)$, and, since $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, f induces isomorphisms f_+, f_-

$$f_{\pm} : \mathfrak{so}(3) \longrightarrow \Lambda_{\pm}.$$

The restriction of the adjoint action of $SO(4)$ to each copy of $\mathfrak{so}(3)$ induces actions of $SO(3)$ on Λ_+ and Λ_-: namely, there exist smooth actions

$$\begin{aligned} SO(3) \times \Lambda_\pm &\longrightarrow \Lambda_\pm \\ (a, \eta_\pm) &\longmapsto a(\eta_\pm) \end{aligned}$$

such that, for every $a \in SO(3)$ and $Y \in \mathfrak{so}(3)$, $a(f_\pm(Y)) = f_\pm^{-1}(\mathrm{Ad}_a(Y))$. Moreover, there exists a surjective Lie group homomorphism

$$\mu : SO(4) \longrightarrow SO(3) \times SO(3) \tag{2.11}$$

such that, for every $a \in SO(4)$, $\mu(a) = (a_+, a_-)$, where, for every $\eta = \eta_+ + \eta_- \in \Lambda^2 = \Lambda_+ \oplus \Lambda_-$,

$$a(\eta) = a_+(\eta_+) + a_-(\eta_-).$$

Decomposition of the Riemann curvature tensor. Let (M, g) be again a Riemannian, oriented, manifold of dimension n. We denote by Riem its Riemann curvature tensor and by R_{ijkt} its components with respect to an orthonormal coframe $\{\theta^1, \dots, \theta^n\}$, with $i, j, k, t = 1, \dots, n$. We also define the *curvature forms* Ω^i_j associated with the orthonormal coframe $\{\theta^i\}$ as the 2-forms satisfying the second Cartan structure equations

$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Omega^i_j, \tag{2.12}$$

where θ^i_j are the Levi-Civita connection 1-forms which satisfy the first Cartan structure equations

$$d\theta^i = -\theta^i_j \wedge \theta^j \tag{2.13}$$

(see, e.g., [9]). Since, for every $i, j = 1, \dots, n$, $\theta^i_j + \theta^j_i = 0$, we have that $\Omega^i_j + \Omega^j_i = 0$; thus, the matrix of the curvature forms Ω takes values in $\mathfrak{so}(n)$. Moreover, the curvature forms satisfy

$$\Omega^i_j = \frac{1}{2} R_{ijkt} \theta^k \wedge \theta^t, \tag{2.14}$$

where R_{ijkt} are exactly the Riemann curvature tensor components with respect to $\{\theta^i\}$.

Let $e, \tilde{e}$ be orthonormal frames such that there exists a smooth change $A : U \cap \tilde{U} \longrightarrow O(m)$ for which $\tilde{e} = eA$ (i.e., $\tilde{e}_i = A^j_i e_j$). Recall that, if $\tilde{\Omega}$ is the matrix of the curvature forms associated with the frame $\tilde{e}$ (equivalently, to the coframe $\tilde{\theta}$ dual to $\tilde{e}$), then the following transformation law holds (see [1]):

$$\tilde{\Omega} = A^{-1} \Omega A. \tag{2.15}$$

Let us define the *Kulkarni–Nomizu product* $\owedge$: if η and κ are two (0,2)-symmetric tensors, we have that $\eta \owedge \kappa$ is the (0,4)-tensor with components

$$(\eta \owedge \kappa)_{ijkt} := \eta_{ik}\kappa_{jt} - \eta_{it}\kappa_{jk} + \eta_{jt}\kappa_{ik} - \eta_{jk}\kappa_{it}.$$

It is well known that, $\forall n \geq 3$, the Riemann curvature tensor admits the decomposition

$$\mathrm{Riem} = \mathrm{W} + \frac{1}{n-2} \mathrm{Ric} \owedge g - \frac{S}{2(n-1)(n-2)} g \owedge g, \tag{2.16}$$

where $\mathrm{Ric} = R_{ij}\theta^i \otimes \theta^j$ is the Ricci curvature tensor, S is the scalar curvature and W is the *Weyl tensor*. Equation (2.16) can be written in the local form as

$$R_{ijkt} = W_{ijkt} + \frac{1}{n-2}(R_{ik}\delta_{jt} - R_{it}\delta_{jk} + R_{jt}\delta_{ik} - R_{jk}\delta_{it}) - \frac{S}{(n-1)(n-2)}(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk}). \tag{2.17}$$

Now, let $n = 4$. It is possible to rewrite Equations (2.16) and (2.17) thanks to Equation (2.5). We know that $\{\theta^i \wedge \theta^j\}_{1\le i<j\le 4}$ is an orthonormal basis for Λ^2 and that $\{\alpha^1_\pm, \alpha^2_\pm, \alpha^3_\pm\}$, defined in Equation (2.6), is an orthonormal basis for $\Lambda_\pm$, respectively. The Riemann curvature tensor corresponds to a symmetric operator, called the *Riemann curvature operator*, defined as

$$\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2 \qquad \mathcal{R}(\gamma) = \frac{1}{4}R_{ijkt}\gamma_{kt}\theta^i \wedge \theta^j = \frac{1}{2}\gamma_{kt}\Omega^k_t, \tag{2.18}$$

where $\gamma_{kt} = \gamma(e_k, e_t)$ ($\{e_i\}$ is the orthonormal frame dual to $\{\theta^i\}$). Since Equation (2.5) holds, every 2-form γ can be written as in Equation (2.7) and, since $\mathcal{R}(\gamma) \in \Lambda^2$, it also can be expressed in a unique sum

$$\mathcal{R}(\gamma) = \mathcal{R}(\gamma)_+ + \mathcal{R}(\gamma)_-.$$

Evaluating the Riemann curvature operator on the bases (2.6) in order to find the self-dual and the anti-self-dual parts of the images, we obtain that there exist three 3×3 matrices, $A = (A_{ij})$, $B = (B_{ij})$, $C = (C_{ij})$, $i, j = 1, 2, 3$, such that, again with respect to the basis $\{\alpha^1_\pm, \alpha^2_\pm, \alpha^3_\pm\}$ of $\Lambda_\pm$, the Riemann curvature operator representative matrix takes the form

$$\mathcal{R} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \tag{2.19}$$

where $A = A^T$, $C = C^T$, and $\operatorname{tr} A = \operatorname{tr} C = S/4$ (here, tr denotes the matrix trace). More explicitly, if $\gamma = c^+_j\alpha^j_+ + c^-_k\alpha^k_-$,

$$\mathcal{R}(\gamma) = (c^+_j A_{kj} + c^-_t B_{tk})\alpha^k_- + (c^+_j B_{kj} + c^-_t C_{kt})\alpha^k_-.$$

The explicit expressions for the entries of A, B, and C in terms of the Riemann tensor can be found in [26].

Thus, we can think of A (respectively, C) as a symmetric linear map from Λ_+ (respectively, Λ_-) to itself, that is, $A \in \mathrm{End}(\Lambda_+)$, $C \in \mathrm{End}(\Lambda_-)$, and we can think of B as a linear map from Λ_+ to Λ_-, that is, $B \in \mathrm{Hom}(\Lambda_+, \Lambda_-)$.

It is also explicitly possible to write the transformation laws for A, B, and C. Recall that, if $e, \widetilde{e}$ are two orthonormal frames defined on U and $\widetilde{U}$ and $a : U \cap \widetilde{U} \longrightarrow SO(4)$ is a smooth change of frame, Equation (2.15) holds for Ω. Since, for every $a \in SO(4)$, $\mu(a) = (a_+, a_-)$ defines the restriction of the action of a on Λ_+ and Λ_-, we obtain the following transformation laws:

$$\widetilde{A} = a_+^{-1}Aa_+, \qquad \widetilde{B} = a_-^{-1}Ba_+, \qquad \widetilde{C} = a_-^{-1}Ca_-. \tag{2.20}$$

Since it will simplify all our next computations, we introduce the (purely local) quantities

$$\overset{2}{Q}_{ab} := R_{12ab} + R_{34ab}; \qquad \overset{3}{Q}_{ab} := R_{13ab} + R_{42ab}; \qquad \overset{4}{Q}_{ab} := R_{14ab} + R_{23ab}. \tag{2.21}$$

Note that, by the first Bianchi identity, we deduce

$$\overset{3}{Q}_{12} + \overset{3}{Q}_{34} = \overset{2}{Q}_{13} + \overset{2}{Q}_{42}; \quad \overset{4}{Q}_{12} + \overset{4}{Q}_{34} = \overset{2}{Q}_{14} + \overset{2}{Q}_{23}; \quad \overset{3}{Q}_{14} + \overset{3}{Q}_{23} = \overset{4}{Q}_{13} + \overset{4}{Q}_{42}. \tag{2.22}$$

Rewriting the components listed in [26], we have the following expressions for A and B:

$$
\begin{aligned}
A_{11} &= \frac{1}{2}(\overset{2}{Q}_{12} + \overset{2}{Q}_{34}); & A_{12} &= A_{21} = \frac{1}{2}(\overset{3}{Q}_{12} + \overset{3}{Q}_{34}) = \frac{1}{2}(\overset{2}{Q}_{13} + \overset{2}{Q}_{42})\\
A_{22} &= \frac{1}{2}(\overset{3}{Q}_{13} + \overset{3}{Q}_{42}); & A_{13} &= A_{31} = \frac{1}{2}(\overset{4}{Q}_{12} + \overset{4}{Q}_{34}) = \frac{1}{2}(\overset{2}{Q}_{14} + \overset{2}{Q}_{23});\\
A_{33} &= \frac{1}{2}(\overset{4}{Q}_{14} + \overset{4}{Q}_{23}); & A_{23} &= A_{32} = \frac{1}{2}(\overset{3}{Q}_{14} + \overset{3}{Q}_{23}) = \frac{1}{2}(\overset{4}{Q}_{13} + \overset{4}{Q}_{42});\\
B_{11} &= \frac{1}{2}(\overset{2}{Q}_{12} - \overset{2}{Q}_{34}); & B_{13} &= \frac{1}{2}(\overset{4}{Q}_{12} - \overset{4}{Q}_{34});\\
B_{22} &= \frac{1}{2}(\overset{3}{Q}_{13} - \overset{3}{Q}_{42}); & B_{31} &= \frac{1}{2}(\overset{2}{Q}_{14} - \overset{2}{Q}_{23});\\
B_{33} &= \frac{1}{2}(\overset{4}{Q}_{14} - \overset{4}{Q}_{23}); & B_{23} &= \frac{1}{2}(\overset{4}{Q}_{13} - \overset{4}{Q}_{42});\\
B_{12} &= \frac{1}{2}(\overset{3}{Q}_{12} - \overset{3}{Q}_{34}); & B_{32} &= \frac{1}{2}(\overset{3}{Q}_{14} - \overset{3}{Q}_{23}).\\
B_{21} &= \frac{1}{2}(\overset{2}{Q}_{13} - \overset{2}{Q}_{42}); & &
\end{aligned}
$$

As far as the Weyl tensor is concerned, by Equations (2.17) and (2.19), we have

$$
\begin{aligned}
W_{1212} &= \frac{1}{2}\left(A_{11} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{11} - \frac{S}{12}\right) & &\\
W_{1213} &= \frac{1}{2}A_{12} + \frac{1}{2}C_{12}, & W_{1214} &= \frac{1}{2}A_{13} + \frac{1}{2}C_{13},\\
W_{1223} &= \frac{1}{2}A_{13} - \frac{1}{2}C_{13}, & W_{1242} &= \frac{1}{2}A_{12} - \frac{1}{2}C_{12},\\
W_{1234} &= \frac{1}{2}\left(A_{11} - \frac{S}{12}\right) - \frac{1}{2}\left(C_{11} - \frac{S}{12}\right), & W_{1313} &= \frac{1}{2}\left(A_{22} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{22} - \frac{S}{12}\right),\\
W_{1314} &= \frac{1}{2}A_{23} + \frac{1}{2}C_{23}, & W_{1323} &= \frac{1}{2}A_{23} - \frac{1}{2}C_{23},\\
W_{1342} &= \frac{1}{2}\left(A_{22} - \frac{S}{12}\right) - \frac{1}{2}\left(C_{22} - \frac{S}{12}\right), & W_{1334} &= \frac{1}{2}A_{12} - \frac{1}{2}C_{12},\\
W_{1414} &= \frac{1}{2}\left(A_{33} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{33} - \frac{S}{12}\right), & W_{1423} &= \frac{1}{2}\left(A_{33} - \frac{S}{12}\right) - \frac{1}{2}\left(C_{33} - \frac{S}{12}\right),\\
W_{1442} &= \frac{1}{2}A_{23} - \frac{1}{2}C_{23}, & W_{1434} &= \frac{1}{2}A_{13} - \frac{1}{2}C_{13},\\
W_{2323} &= \frac{1}{2}\left(A_{33} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{33} - \frac{S}{12}\right), & W_{2342} &= \frac{1}{2}A_{23} + \frac{1}{2}C_{23},\\
W_{2334} &= \frac{1}{2}A_{13} + \frac{1}{2}C_{13}, & W_{4242} &= \frac{1}{2}\left(A_{22} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{22} - \frac{S}{12}\right),\\
W_{3442} &= \frac{1}{2}A_{12} + \frac{1}{2}C_{12}, & W_{3434} &= \frac{1}{2}\left(A_{11} - \frac{S}{12}\right) + \frac{1}{2}\left(C_{11} - \frac{S}{12}\right).
\end{aligned}
$$

It is apparent that all the components can be written as a sum of two addends

$$W_{ijkt} = W^{+}_{ijkt} + W^{-}_{ijkt}$$

This means that the Weyl tensor W splits into a sum of two (0,4)-tensors

$$W = W^{+} + W^{-}$$

called, respectively, the *self-dual* and the *anti-self-dual* components of W. A four-dimensional Riemannian manifold is called *self-dual* (respectively *anti-self-dual*) if $W^- = 0$ (resp., $W^+ = 0$). By a direct check of the entries of A, B, and C and the coefficients of W^+ and W^-, we easily obtain the following.

Theorem 2.1. *Let (M, g) be a Riemannian manifold of dimension 4. Then,*

- *M is self-dual if and only if $C - \frac{S}{12} I_3 = 0$ for every orthonormal positively oriented coframe (respectively, if and only if $A - \frac{S}{12} I_3 = 0$ for every orthonormal negatively oriented coframe);*
- *M is anti-self-dual if and only if $A - \frac{S}{12} I_3 = 0$ for every orthonormal positively oriented coframe (respectively, if and only if $C - \frac{S}{12} I_3 = 0$ for every orthonormal negatively oriented coframe);*
- *M is Einstein if and only if $B = 0$ for every orthonormal positively or negatively oriented coframe.*

3 | THE TWISTOR SPACE OF A 4-MANIFOLD

Let (M, g_M) be a connected Riemannian manifold of dimension $2m$. We define its *twistor space* Z associated with M as the set of the pairs (p, J_p), where $p \in M$ and J_p is a g-orthogonal complex structure on T_pM. It is not hard to show that the set of all g-orthogonal complex structures is diffeomorphic to $O(2m)/U(m)$, where

$$U(m) := \{A \in O(2m) : A^T J_m = J_m A\}$$

and J_m is a matrix in $O(2m) \cap \mathfrak{so}(2m)$ with entries $(J_m)_{kl} = \delta_k^{l+1} - \delta_l^{k+1}$ (see, for instance, [15]); therefore, it can be shown that, if we denote as $O(M)$ as the orthonormal frame bundle of M, Z is the associated bundle

$$Z = O(M) \times_{O(2m)} (O(2m)/U(m)) = O(M)/U(m).$$

This means that there exists a surjective smooth map $\sigma : O(M) \longrightarrow Z$ such that σ defines a principal bundle $(O(M), Z, U(m))$ with structure group $U(m)$. Moreover, the map

$$\pi_Z : Z \longrightarrow M$$

$$(p, J_p) \longmapsto p$$

determines a fiber bundle $(Z, M, O(2m)/U(m), O(2m))$ with structure group $O(2m)$ and standard fiber $O(2m)/U(m)$ (see [27]).

Now, observe that, if M is oriented, the orthonormal frame bundle is not connected: indeed, in this case $O(M)$ has two connected components

$$O(M) = O(M)_+ \sqcup O(M)_-,$$

where $O(M)_+$ (resp., $O(M)_-$) is the bundle of positively (resp., negatively) oriented frames on M. As a consequence, Z itself has two connected components

$$Z_\pm = O(M)_\pm / U(m) = SO(M)/U(m),$$

where $SO(M)$ denotes the orthonormal oriented frame bundle over M; moreover, one can obviously define the bundle projections

$$\pi_{Z_\pm} : Z_\pm \longrightarrow M \quad \text{and} \quad \sigma_\pm : O(M)_\pm \longrightarrow Z_\pm.$$

In accordance to much of the literature (see, for instance, [34]), by convention we choose Z_- to be the twistor space associated with a Riemannian manifold (M, g); therefore, from now on, $Z = Z_-$, $\pi_Z = \pi_{Z_-}$ and $\sigma = \sigma_-$.

It is known that, in general, there exists a one-parameter family of Riemannian metrics g_t on Z, with $t > 0$, constructed as pullbacks via σ of the unique (up to multiplication by a positive constant) $O(2m)$-invariant Riemannian metric on

$O(M)$ which is also horizontal with respect to σ (see [15, 20, 26]): as a consequence, the map σ becomes a Riemannian submersion and the fibers are totally geodesic submanifolds of $O(M)$.

Let $m = 2$; from now on, we adopt the index conventions $1 \le a, b, c, \ldots \le 4$ and $1 \le p, q, \ldots, \le 6$. Given a local orthonormal coframe $\{\omega^a\}_{a=1,\ldots,4}$ on an open set $U \subset M$, with Levi-Civita connection forms $\{\omega^a_b\}$, we define

$$\omega^5 := \frac{1}{2}\left(\omega^1_3 + \omega^4_2\right), \quad \omega^6 := \frac{1}{2}\left(\omega^1_4 + \omega^2_3\right);$$

a local orthonormal coframe on (Z, g_t) is obtained by considering the pullbacks of $\omega^1, \ldots, \omega^6$ via a smooth local section $u : U \longrightarrow O(M)$ of the principal $U(m)$-bundle defined by σ, where U is a suitable open subset of Z. This means that

$$g_t = \sum_{p=1}^{6} (\theta^p)^2, \tag{3.1}$$

where

$$\theta^a := u^*(\omega^a), \quad \theta^5 := 2tu^*(\omega^5), \quad \theta^6 := 2tu^*(\omega^6); \tag{3.2}$$

for the sake of simplicity, we write ω^a for $u^*(\omega^a)$ and similarly for $2t\omega^5$ and $2t\omega^6$. By Equations (3.1) and (3.2), we can write

$$g_t = g_M + 4t^2\left[\left(\omega^5\right)^2 + \left(\omega^6\right)^2\right] = g_M + (\theta^5)^2 + (\theta^6)^2 \tag{3.3}$$

(again the pullback notation is omitted). In order to compute the Levi-Civita connection forms θ^p_q and the curvature forms Θ^p_q for the orthonormal coframe defined in Equation (3.2), recall the structure equations (2.13) and (2.12). By direct computation, we obtain (see [26])

$$\begin{aligned} \theta^a_b &= \omega^a_b + \frac{1}{2}t\left(\overset{3}{Q}_{ba}\,\theta^5 + \overset{4}{Q}_{ba}\,\theta^6\right), \\ \theta^5_b &= \frac{1}{2}t\,\overset{3}{Q}_{ba}\,\theta^a, \quad \theta^6_b = \frac{1}{2}t\,\overset{4}{Q}_{ba}\,\theta^a; \\ \theta^5_6 &= \omega^1_2 + \omega^3_4. \end{aligned} \tag{3.4}$$

Now, let us denote as $\overline{\mathrm{Riem}}$, $\overline{\mathrm{Ric}}$, and $\overline{S}$ the Riemann curvature tensor, the Ricci tensor, and the scalar curvature of (Z, g_t), respectively; by Equations (2.12) and (2.14), it is easy to obtain the coefficients $\overline{R}_{pqrs}$, $\overline{R}_{pq}$ and the scalar curvature $\overline{S}$ in terms of $\overset{2}{Q}_{ab}$, $\overset{3}{Q}_{ab}$, and $\overset{4}{Q}_{ab}$ (see [8]).

It is possible to introduce two almost complex structures on (Z, g_t). Indeed, let $\{\theta^p\}$ be the orthonormal coframe defined in Equation (3.2) and $\{e_p\}$ be its dual frame; then, we define

$$J_\pm = \theta^1 \otimes e_2 - \theta^2 \otimes e_1 + \theta^3 \otimes e_4 - \theta^4 \otimes e_3 \pm \theta^5 \otimes e_6 \mp \theta^6 \otimes e_5. \tag{3.5}$$

It is clear that J_+ and J_-, which were introduced by Atiyah et al. [4] and Eells and Salamon [16], respectively, are g_t-orthogonal almost complex structures on (Z, g_t), that is, $(Z, g_t, J_\pm)$ is an almost Hermitian manifold. We can write the local expression of the almost complex structures with respect to Equation (3.2) and the dual orthonormal frame as

$$J_\pm = (J_\pm)^q_p \theta^p \otimes e_q,$$

where

$$(J_+)^2_1 = (J_+)^4_3 = (J_+)^6_5 = (J_-)^2_1 = (J_-)^4_3 = -(J_-)^6_5 = 1,$$

and all other components are zero.

We can compute the components of ∇J_+ and ∇J_- with respect to the coframe defined in Equation (3.2), recalling that

$$\nabla J_\pm = (J_\pm)^p_{q,t}\theta^t \otimes \theta^q \otimes e_p, \tag{3.6}$$

where

$$(J_\pm)^p_{q,t}\theta^t = d(J_\pm)^p_q - (J_\pm)^p_s\theta^s_q + (J_\pm)^s_q\theta^p_s;$$

the components derived from Equation (3.6), which depend on the curvature of (M, g), will be central in the proof of Theorems 4.2 and 4.7 (for the explicit expression of the components in terms of Equation (2.21), see [8]).

The almost complex structures J_+ and J_- induce the corresponding Nijenhuis tensors N_{J_+} and N_{J_-}, locally defined by

$$N_{J_\pm} = (N_\pm)^p_{tq}\theta^t \otimes \theta^q \otimes e_p, \quad (N_\pm)^p_{tq} = -(N_\pm)^p_{qt}, \tag{3.7}$$

where

$$(N_\pm)^r_{pq} = (J_\pm)^s_p(J_\pm)^r_{s,q} - (J_\pm)^s_q(J_\pm)^r_{s,p} + (J_\pm)^s_q(J_\pm)^r_{p,s} - (J_\pm)^s_p(J_\pm)^r_{q,s}; \tag{3.8}$$

we recall that, by the well-known Newlander–Nirenberg theorem [30], the vanishing of the Nijenhuis tensor is equivalent to the integrability of the almost complex structure, which, in this case, is induced by a holomorphic atlas of charts.

As said in the introduction, the problem of the integrability of J_+ and J_- was solved by Atiyah et al. [4] and Eells and Salamon [16], respectively: indeed, J_+ is integrable if and only if (M, g) is a self-dual manifold, while J_- is never integrable. However, in the same paper [16] the authors proved a correspondence between minimal two-dimensional submanifolds of (M, g) and (J_-)-holomorphic curves in Z.

For the explicit expressions of Equations (3.6) and (3.7) in terms of $\overset{2}{Q}_{ab}$, $\overset{3}{Q}_{ab}$, and $\overset{4}{Q}_{ab}$, we refer the reader to [8].

4 | LINEAR CONDITIONS ON J_+ AND J_-: PROOF OF THEOREM 1.1

In this section, we show that, given a Riemannian 4-manifold M and its twistor space Z, any linear condition on the covariant derivative of the almost complex structures J_+ and J_- on Z implies that M is self-dual. From now on, we will constantly make use of the components of $\nabla J_\pm$ and $N_{J_\pm}$ listed in [8].

Let us start with the following proposition, which should be compared with Theorem 2.1:

Proposition 4.1. *Let (M, g) be an oriented Riemannian 4-manifold. Let*

$$\mathcal{R} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$$

be the decomposition of the Riemann curvature operator on M, with $A = (A_{ij})_{i,j=1,2,3}$, $B = (B_{ij})_{i,j=1,2,3}$ and $C = (C_{ij})_{i,j=1,2,3}$. Then,

(1) M is self-dual if and only if, for every negatively oriented local orthonormal coframe, $A_{ij} = 0$ for some $i \neq j$ or $A_{kk} = A_{ll}$ for some k, l;
(2) M is Einstein if and only if, for every negatively oriented local orthonormal coframe, $B_{ij} = 0$ for some i, j.

Proof. Recall the transformation laws for A, B, and C defined in Equation (2.20) and the surjective Lie group homomorphism (2.11).

(1) If M is self-dual, then A is a scalar matrix with $A_{ij} = \frac{S}{12}\delta_{ij}$. Conversely, let us prove the claim for $A_{12} = 0$ and $A_{11} = A_{22}$, since the other cases can be shown in an analogous way. If, for every negatively oriented orthonormal coframe, $A_{12} = 0$, then the matrix A has the form

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

on $O(M)_-$. Equivalently, for every smooth change of frames $a : U \longrightarrow SO(4)$, the transformed matrix $\widetilde{A}$ is such that $\widetilde{A}_{12} = 0$. Thus, let us choose $a \in SO(4)$ such that $\mu(a) = (a_+, a_-)$, where

$$a_+ = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have that

$$\widetilde{A} = a_+^{-1} A a_+ = \begin{pmatrix} A_{33} & A_{23} & -A_{13} \\ A_{23} & A_{22} & 0 \\ -A_{13} & 0 & A_{11} \end{pmatrix};$$

thus, $\widetilde{A}_{12} = A_{23} = 0$ on $O(M)_-$, that is, A has the form

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{13} & 0 & A_{33} \end{pmatrix}.$$

Repeating the argument on A with

$$a_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

we obtain that $A_{13} = 0$; hence, A is a diagonal matrix. Choosing other suitable changes of frames, it is not hard to show that $A_{11} = A_{22} = A_{33}$, that is, A is a scalar matrix; by Theorem (2.1), M is self-dual.

Similar computations show that, if $A_{11} = A_{22}$ on $O(M)_-$, then A is a scalar matrix, that is, M is self-dual.

(2) If M is Einstein, then $B = 0$ by Theorem (2.1). Conversely, suppose, for instance, that $B_{11} = 0$ on $O(M)_-$ (the other cases can be proved analogously). Again, this means that, for every change of frames a, the transformed matrix $\widetilde{B}$ is such that $\widetilde{B}_{11} = 0$. Let us choose $a \in SO(4)$ such that $\mu(a) = (a_+, a_-)$, where

$$a_- = I_3, \qquad a_+ = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, we have that

$$\widetilde{B} = a_-^{-1} B a_+ = \begin{pmatrix} B_{12} & 0 & B_{13} \\ B_{22} & -B_{21} & B_{23} \\ B_{32} & -B_{31} & B_{33} \end{pmatrix};$$

this implies that $\widetilde{B}_{11} = B_{12} = 0$. By the same argument, if we choose

$$a_+ = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

we conclude that $B_{13} = 0$. Now, repeating the argument with suitable choices of a_+ and a_-, we obtain

$$B_{21} = B_{22} = B_{23} = B_{31} = B_{32} = B_{33} = 0,$$

that is, $B = 0$. Therefore, M is Einstein by Theorem (2.1). □

The first main result of this section is the following (see Theorem 1.1 in the introduction).

Theorem 4.2. *Let (M, g) be a Riemannian 4-manifold and let (Z, g_t, J_+) be its twistor space. If, for every $X, Y \in \mathfrak{X}(Z)$,*

$$\begin{aligned} &a_1(\nabla_X J_+)Y + a_2(\nabla_Y J_+)X + a_3(\nabla_{J_+X} J_+)Y + a_4(\nabla_{J_+Y} J_+)X + a_5(\nabla_{J_+X} J_+)J_+Y + \\ &+ a_6(\nabla_{J_+Y} J_+)J_+X + a_7(\nabla_X J_+)J_+Y + a_8(\nabla_Y J_+)J_+X = 0 \end{aligned} \tag{4.1}$$

for some $a_i \in \mathbb{R}$, $i = 1, \dots, 8$, such that $a_j \neq 0$ for some j, then, M is self-dual.

Proof. For the sake of simplicity, we will denote $(J_+)^p_{q,t} = J^p_{q,t}$. First, we rewrite the equality in Equation (4.1) with respect to a local orthonormal frame $\{e_t\}_{t=1,\dots,6}$ by putting $X = e_p, Y = e_q$. This implies that

$$a_1 J^t_{q,p} + a_2 J^t_{p,q} + a_3 J^r_p J^t_{q,r} + a_4 J^r_q J^t_{p,r} + a_5 J^r_p J^s_q J^t_{s,r} + a_6 J^r_q J^s_p J^t_{s,r} + a_7 J^s_q J^t_{s,p} + a_8 J^s_p J^t_{s,q} = 0 \tag{4.2}$$

for every $p, q, t = 1, \dots, 6$. We now proceed by steps.

(1) We start by considering Equation (4.2) with $p = 5, q = 2, t = 1$, that is,

$$(a_8 - a_4)\, \textcircled{3}Q_{12} - (a_2 + a_6)\textcircled{4}Q_{12} = 0.$$

Putting $p = 5, q = 4, t = 3$, we easily obtain

$$(a_8 - a_4)\, \textcircled{3}Q_{34} - (a_2 + a_6)\textcircled{4}Q_{34} = 0;$$

Summing these two equalities, we can write

$$(a_8 - a_4)A_{12} - (a_2 + a_6)A_{13} = 0.$$

Repeating the argument with $p = 6, q = 2, t = 1$ and $p = 6, q = 4, t = 3$ we have that

$$(a_2 + a_6)A_{12} + (a_8 - a_4)A_{13} = 0.$$

Thus, we deduce the following system of equations:

$$\begin{cases} (a_8 - a_4)A_{12} - (a_2 + a_6)A_{13} = 0, \\ (a_2 + a_6)A_{12} + (a_8 - a_4)A_{13} = 0. \end{cases}$$

If at least one of the coefficients $(a_8 - a_4)$ and $(a_2 + a_6)$ is different from 0, we must have that $A_{12} = A_{13} = 0$ on $O(M)_-$, since Equation (4.1) is a global condition. By Proposition 4.1, M is self-dual.

Note that, if we exchange the values of p and q in all the previous calculations, the following system holds:

$$\begin{cases} (a_7 - a_3)A_{12} - (a_1 + a_5)A_{13} = 0, \\ (a_1 + a_5)A_{12} + (a_7 - a_3)A_{13} = 0. \end{cases}$$

As before, if at least one of the coefficients $(a_1 + a_5)$ and $(a_7 - a_3)$ is different from zero, then M is self-dual.

(2) Now, we have to study the case

$$a_1 = -a_5, \quad a_2 = -a_6, \quad a_3 = a_7, \quad a_4 = a_8,$$

that is,

$$a_1(J^t_{q,p} - J^r_p J^s_q J^t_{s,r}) + a_2(J^t_{p,q} - J^r_q J^s_p J^t_{s,r}) + a_3(J^r_p J^t_{q,r} + J^s_q J^t_{s,p}) + a_4(J^r_q J^t_{p,r} + J^s_p J^t_{s,q}) = 0.$$

By choosing $p = 5, q = 3, t = 1$ and $p = 6, q = 3, t = 1$, we obtain

$$\begin{cases} (a_1 + a_2)(A_{22} - A_{33}) + 2(a_3 + a_4)A_{23} = 0, \\ (a_3 + a_4)(A_{22} - A_{33}) - 2(a_1 + a_2)A_{23} = 0. \end{cases}$$

Again, if not all the coefficients vanish, the system holds if and only if $A_{22} = A_{33}$ and $A_{23} = 0$. By Proposition 4.1, M is self-dual.

Finally, we have to show the claim when

$$a_1 = -a_2, \quad a_3 = -a_4.$$

Choosing $p = 3, q = 1, t = 5$ and $p = 4, q = 1, t = 5$, we get the system

$$\begin{cases} a_1(A_{22} - A_{33}) + 2a_3 A_{23} = 0, \\ -a_3(A_{22} - A_{33}) + 2a_1 A_{23} = 0. \end{cases}$$

Since, by hypothesis, at least one of the coefficients does not vanish, we conclude that $A_{22} = A_{33}$ and $A_{23} = 0$, that is, M is self-dual. □

The previous theorem allows us to easily prove a well-known result, due to Atiyah et al. [4]:

Theorem 4.3. *Let (M, g) be a Riemannian 4-manifold and let (Z, g_t, J_+) be its twistor space. Then, the almost complex structure J_+ is integrable if and only if M is self-dual.*

Proof. Recall that an almost complex structure is integrable if and only if the associated Nijenhuis tensor identically vanishes. Thus, by direct inspection of the components, if M is self-dual, the Nijenhuis tensor N_{J_+} is identically null. Conversely, note that the condition of integrability for J_+ corresponds to Equation (4.1) with

$$a_1 = a_2 = a_5 = a_6 = 0, \qquad a_4 = a_8 = -a_3 = -a_7 = 1.$$

Thus, if N_{J_+} vanishes identically, then M is self-dual. □

Remark 4.4.

(1) We point out that, as mentioned by Apostolov et al.[3], for every point $p \in M$ such that $W^- \neq 0$ at p with non-degenerate spectrum, there exist exactly two almost complex structures J_1 and J_2, determined up to sign, such that N_{J_+} vanishes at $(p, J_1), (p, J_2) \in \pi_Z^{-1}(p)$ (see also [2, 35, 36]): in other words, by a direct computation of the components of N_{J_+}, for every such $p \in M$ there exist two negatively oriented orthonormal frames e_1 and e_2 such that, with respect to those, $A_{22} = A_{33}$ and $A_{23} = 0$.
(2) A generalization of Theorem 4.3 involving the divergences of the Nijenhuis tensors was obtained in [8] (Theorem 5.5).

In the same spirit, one can provide an alternative proof for the characterization results due to Muškarov [29], which generalize a theorem due to Friedrich and Kurke [20]: for instance, we can show the following:

Theorem 4.5. (Z, g_t, J_+) *is a* q^2-Kähler manifold*, that is, for every $X \in \mathfrak{X}(Z)$,*

$$(\nabla_X J_+)X + (\nabla_{J_+X} J_+)J_+X = 0, \tag{4.3}$$

if and only if M is Einstein, self-dual, with positive scalar curvature equal to $12/t^2$. In particular, this holds if and only if (Z, g_t, J_+) is a Kähler manifold.

Remark 4.6. Note that Equation (4.3) is satisfied on any nearly Kähler manifold and on any almost Kähler manifold.

Proof. One direction is trivial: indeed, if (M, g) is Einstein, self-dual with positive scalar curvature, then (Z, g_t, J_+) is a Kähler manifold [20], which implies that it is also q^2-Kähler.

Conversely, let us suppose that (Z, g_t, J_+) is q^2-Kähler. Note that we can rewrite Equation (4.3) as

$$(\nabla_X J_+)Y + (\nabla_Y J_+)X + (\nabla_{J_+X} J_+)J_+Y + (\nabla_{J_+Y} J_+)J_+X = 0$$

for every $X, Y \in \mathfrak{X}(Z)$, which is Equation (4.1) with

$$a_3 = a_4 = a_7 = a_8 = 0, \qquad a_1 = a_2 = a_5 = a_6.$$

This immediately implies that M is self-dual. With some straightforward calculation using Equation (4.3), we obtain the system

$$\begin{cases} B_{22} + B_{33} = \dfrac{S}{6} - \dfrac{2}{t^2}, \\ B_{33} - B_{22} = -\dfrac{S}{6} + \dfrac{2}{t^2}. \end{cases}$$

Summing the two equations, we obtain $B_{33} = 0$, that is, M is Einstein by Proposition 4.1. Therefore, by the same system, we obtain that $S = 12/t^2$. □

Now, we prove the analogous of Theorem 4.2 for J_-.

Theorem 4.7. *Let (M, g) be a Riemannian 4-manifold and let (Z, g_t, J_-) be its twistor space. If ∇J_- satisfies the equation in Equation (4.1) for every $X, Y \in \mathfrak{X}(Z)$, then M is self-dual.*

Proof. The first step of the proof is identical to that in Theorem 4.2; thus, if at least one of the coefficients $a_2 + a_6$, $a_8 - a_4$, $a_1 + a_5$ or $a_7 - a_3$ is different from 0, we conclude that M is self-dual. Now, suppose that

$$a_1 = -a_5, \quad a_2 = -a_6, \quad a_3 = a_7, \quad a_4 = a_8 \ :$$

by choosing $t = 1, q = 3, p = 5$ and $t = 3, q = 5, p = 1$, we have

$$\begin{cases} a_3\left(A_{22} + A_{33} - \frac{2}{t^2}\right) + a_4(A_{22} + A_{33}) & = 0 \\ a_4\left(A_{22} + A_{33} - \frac{2}{t^2}\right) + a_3(A_{22} + A_{33}) & = 0. \end{cases}$$

If $a_3 \neq a_4$, it is immediate to prove that the previous system admits no solution: indeed, if, in addition, $a_3 \neq -a_4$, the only possible solution would be $A_{22} + A_{33} + \frac{2}{t^2} = A_{22} + A_{33} = 0$, which is impossible, while, if $a_3 = -a_4$, we would get $\frac{2}{t^2} = 0$. Hence, let $a_3 = a_4$: in this case, the system reduces to the equation

$$a_3\left(A_{22} + A_{33} - \frac{1}{t^2}\right) = 0,$$

which implies that, if $a_3 \neq 0$, (M, g) is a self-dual manifold with $S = 6/t^2$. Moreover, choosing $t = 1, q = 3, p = 6$, we conclude that $a_1 = a_2$: therefore, the relations between the a_i's are

$$a_1 = a_2 = -a_5 = -a_6, \quad a_3 = a_4 = a_7 = a_8.$$

Finally, if $a_3 = 0$, we can choose again $t = 1, q = 3, p = 6$ and $t = 3, q = 6, p = 1$ in order to find

$$\begin{cases} a_1\left(A_{22} + A_{33} - \frac{2}{t^2}\right) + a_2(A_{22} + A_{33}) & = 0 \\ a_2\left(A_{22} + A_{33} - \frac{2}{t^2}\right) + a_1(A_{22} + A_{33}) & = 0 \end{cases};$$

as before, we must have $a_1 = a_2$ and, since $a_1 \neq 0$ by hypothesis, we obtain again that (M, g) is self-dual with $S = 6/t^2$. □

5 | QUADRATIC CONDITIONS: PROOFS OF THEOREMS 1.2 and 1.3

In this section, we present some new results about the twistor space associated with an Einstein four-dimensional manifold whose metric is not necessarily self-dual; in particular, we partially generalize the characterization Theorem 4.5 and we introduce a new necessary condition for a manifold to be Einstein, which leads to a characterization of Einstein, non-self-dual manifolds.

As before, let (M, g) be a connected, oriented Riemannian manifold of dimension 4 and (Z, g_t, J_+) be its twistor space, with J_+ being the Atiyah–Hitchin–Singer almost complex structure defined in Equation (3.5). Since, in this section, we will only consider J_+, for the sake of simplicity, we will write $J^p_{q,t}$ instead of $(J_+)^p_{q,t}$ and N^t_{pq} instead of $(N_+)^t_{pq}$. We have the following (see Theorem 1.2 in the introduction):

Theorem 5.1. *(M, g) is Einstein if and only if, for every orthonormal frame in $O(M)_-$ (equivalently, for every negatively oriented orthonormal coframe),*

$$\sum_{t=1}^{6}(J^t_{p,q} + J^t_{q,p})(J^t_{p,p} - J^t_{q,q}) = 0, \quad \forall p, q = 1, \dots, 6. \tag{5.1}$$

Proof. First, suppose that M is Einstein. Then, a direct computation over the components of ∇J shows that Equation (5.1) holds; indeed, for instance, by Theorem 2.1

$$\sum_{t=1}^{6}(J^t_{1,3} + J^t_{3,1})(J^t_{1,1} - J^t_{3,3}) = B_{32}B_{12} + B_{13}B_{33} = 0.$$

Conversely, suppose that Equation (5.1) holds. By choosing $p = 1, q = 3$ and $p = 1, q = 4$, we obtain the following system:

$$\begin{cases} B_{12}B_{22} + B_{13}B_{23} = 0; \\ B_{12}B_{32} + B_{13}B_{33} = 0. \end{cases}$$

By hypothesis, the two equations must hold on all $O(M)_-$. Let us choose $e \in O(M)_-$ and suppose that $B_{ij} = 0$ for some i, j; we want to prove that $B = 0$. For instance, let $B_{11} = 0$ (the other cases can be shown analogously). Let us choose a smooth change of frames $a \in SO(4)$ such that

$$a_+ = a_- = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

by Equation (2.20), we obtain that the matrix $\widetilde{B}$ associated with the frame $\widetilde{e} = ea$ has the form

$$\widetilde{B} = \begin{pmatrix} \widetilde{B_{11}} & \widetilde{B_{12}} & \widetilde{B_{13}} \\ \widetilde{B_{21}} & \widetilde{B_{22}} & \widetilde{B_{23}} \\ \widetilde{B_{31}} & \widetilde{B_{32}} & \widetilde{B_{33}} \end{pmatrix} = \begin{pmatrix} B_{22} & -B_{21} & -B_{23} \\ -B_{12} & 0 & B_{13} \\ -B_{32} & B_{31} & B_{33}. \end{pmatrix}$$

By hypothesis, we have that

$$0 = \widetilde{B}_{12}\widetilde{B}_{22} + \widetilde{B}_{13}\widetilde{B}_{23} = -B_{23}B_{13},$$

that is, $B_{23} = 0$ or $B_{13} = 0$. In both cases, with similar computations, it can be shown that all the other B_{ij} are zero, that is, $B = 0$, which means that M is Einstein by Equation (2.20). In particular, if one of the B_{ij} in the system is 0, then M is Einstein.

Let us now suppose $B_{12}, B_{13}, B_{22}, B_{23}, B_{32}, B_{33} \neq 0$. By the previous system of equations, we obtain that

$$B_{12} = -\frac{B_{13}B_{23}}{B_{22}} = -\frac{B_{13}B_{33}}{B_{32}},$$

which implies that

$$B_{23}B_{32} = B_{22}B_{33}.$$

Choosing the matrices

$$a_+ = I_3, \qquad a_- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

in Equation (2.20), we deduce

$$B_{13}B_{22} = B_{12}B_{23} = -\frac{B_{13}B_{23}}{B_{22}}B_{23},$$

that is,

$$B_{22} = -\frac{B_{23}^2}{B_{22}}.$$

But this implies that $B_{22}^2 = -B_{23}^2$, that is, $B_{22} = B_{23} = 0$, which is a contradiction. This means that at least one of the terms in the system above has to be equal to 0; therefore, by the previous considerations, M is Einstein. □

We highlight that the fact that Theorem 5.1 is only true *locally*: this means that, if M is Einstein, the condition (5.1) holds only for orthonormal frames. Indeed, if $X, Y \in \mathfrak{X}(Z)$, the global version of Equation (5.1), namely

$$\langle (\nabla_X J_+)(Y) + (\nabla_Y J_+)(X), (\nabla_X J_+)(X) - (\nabla_Y J_+)(Y)\rangle = 0,$$

is not satisfied, in general, if the norm of X or Y is different from 1 (for instance, it is sufficient to consider $Y = 2X$). However, it is important to underline that, in order to find characterizations of the Einstein manifolds via polynomial conditions on ∇J_+, we have to investigate equations of order higher than 1, since, by Theorem 4.2, every linear condition on ∇J_+ implies that M is self-dual.

Now, we show the following (see Theorem 1.3 in the introduction):

Theorem 5.2. *Let (M, g) be an oriented Riemannian Einstein 4-manifold. Then, for every orthonormal frame in $O(M)_-$,*

$$\left(J^t_{q,p} + J^t_{p,q}\right)N^t_{pq} = 0 \qquad (\text{no sum over } p, q, t). \tag{5.2}$$

Conversely, if Equation (5.2) holds on $O(M)_-$, then, for every point $p \in M$ such that $|\mathrm{W}^-| \neq 0$ at p, $B = 0$. In particular, if $|\mathrm{W}^-| \neq 0$ on M, (M, g) is an Einstein manifold.

Proof. If (M, g) is Einstein, we obtain that

$$\begin{aligned}
(J^5_{3,1} + J^5_{1,3})N^5_{13} &= -(J^5_{4,2} + J^5_{2,4})N^5_{24} = t^2 B_{32}(A_{22} - A_{33}); \\
(J^5_{4,1} + J^5_{1,4})N^5_{14} &= -(J^5_{3,2} + J^5_{2,3})N^5_{23} = -2t^2 B_{22}A_{23}; \\
(J^6_{3,1} + J^6_{1,3})N^6_{13} &= -(J^6_{4,2} + J^6_{2,4})N^6_{24} = 2t^2 B_{33}A_{23}; \\
(J^6_{4,1} + J^6_{1,4})N^6_{14} &= -(J^6_{3,2} + J^6_{2,3})N^6_{23} = t^2 B_{23}(A_{22} - A_{33});
\end{aligned} \tag{5.3}$$

since $B = 0$ by hypothesis, Equation (5.2) holds. Conversely, let $p \in M$ such that $|\mathrm{W}^-| \neq 0$ at p: this implies that A is not a scalar matrix. Since the matrix A is symmetric, there exists $e \in O(M)_-$ such that A is diagonal, that is,

$$A = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix};$$

since A is not scalar, we can assume that $y \neq z$. Let

$$a_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \in SO(3);$$

by Equation (2.20), we have that, with respect to the transformed frame $\widetilde{e} \in O(M)_-$,

$$A = \begin{pmatrix} x & 0 & 0 \\ 0 & \frac{1}{2}(y+z) & \frac{1}{2}(z-y) \\ 0 & \frac{1}{2}(z-y) & \frac{1}{2}(y+z) \end{pmatrix}.$$

By hypothesis, Equation (5.2) holds on $O(M)_-$, which implies that $B_{22} = B_{33} = 0$ with respect to $\widetilde{e}$, by Equation (5.3) and the fact that $y \neq z$. Putting

$$a_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

by the transformation laws (2.20), we obtain

$$\widetilde{A} = \begin{pmatrix} x & 0 & 0 \\ 0 & \frac{1}{2}(y+z) & \frac{1}{2}(y-z) \\ 0 & \frac{1}{2}(y-z) & \frac{1}{2}(y+z) \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} B_{11} & B_{13} & -B_{12} \\ B_{21} & B_{23} & 0 \\ B_{31} & 0 & -B_{32} \end{pmatrix};$$

by Equation (5.2), $B_{23} = B_{32} = 0$ (note that $\widetilde{A}_{23} \neq 0$). By Equation (2.20), if $a \in SO(4)$ is a change of frames such that $\mu(a) = (a_+, a_-)$, A is invariant under the action of a_-: therefore, by similar computations on B with $a_- = I_3$, it is easy to show that $B_{12} = B_{13} = 0$. Now, let us consider two cases:

(1) **Case $\mathbf{x \neq \frac{1}{2}(y+z)}$.** Putting

$$a_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

we have

$$\widetilde{A} = \begin{pmatrix} \frac{1}{2}(y+z) & \frac{1}{2}(y-z) & 0 \\ \frac{1}{2}(y-z) & \frac{1}{2}(y+z) & 0 \\ 0 & 0 & x \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} 0 & 0 & B_{11} \\ 0 & 0 & B_{21} \\ 0 & 0 & B_{31} \end{pmatrix};$$

since $\widetilde{A}_{22} \neq \widetilde{A}_{33}$, by Equation (5.2) we obtain $B_{21} = 0$. Again, since A is invariant under the action of a_-, by analogous computations $B_{11} = 0$. Finally, putting

$$a_+ = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we conclude that $B_{31} = 0$, that is, M is Einstein.

(2) **Case $\mathbf{x = \frac{1}{2}(y+z)}$.** In this case, A and B have the form

$$A = \begin{pmatrix} x & 0 & 0 \\ 0 & x & x-z \\ 0 & x-z & x \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & 0 & 0 \\ B_{21} & 0 & 0 \\ B_{31} & 0 & 0 \end{pmatrix}$$

(note that $x \neq z$, otherwise $\mathrm{W}^- = 0$ at p). Choosing

$$a_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain

$$\tilde{A} = \begin{pmatrix} x & 0 & \frac{1}{\sqrt{2}}(x-z) \\ 0 & x & \frac{1}{\sqrt{2}}(x-z) \\ \frac{1}{\sqrt{2}}(x-z) & \frac{1}{\sqrt{2}}(x-z) & x \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} \frac{1}{\sqrt{2}}B_{11} & -\frac{1}{\sqrt{2}}B_{11} & 0 \\ \frac{1}{\sqrt{2}}B_{21} & -\frac{1}{\sqrt{2}}B_{21} & 0 \\ \frac{1}{\sqrt{2}}B_{31} & -\frac{1}{\sqrt{2}}B_{31} & 0 \end{pmatrix};$$

by $x \neq z$ and Equation (5.2), $B_{21} = 0$. As we did earlier, the invariance of A under the action of a_- implies that $B_{11} = 0$. Finally, choosing

$$a_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

we conclude that $B_{31} = 0$, that is, $B = 0$ at p. □

Remark 5.3. As Theorems 1.2 and 1.3 show, it is possible to obtain local, twistorial characterization of Einstein metrics on 4-manifolds via quadratic polynomial conditions on the Atiyah–Hitchin–Singer almost complex structure: however, it seems much more difficult to find *global* conditions for Einstein 4-manifolds in terms of their twistor spaces, without assuming self-duality as a hypothesis. To the best of our knowledge, the only result in the literature which addresses the problem of global twistorial conditions for Einstein manifolds is due to Reznikov [33], who showed that, under the further assumption that the sectional curvature is nowhere vanishing, the twistor space of an Einstein manifold is symplectic, with respect to a 2-form derived from a computation involving linear connections which preserve the complex structure. We observe that, as it is well-known, imposing self-duality on the underlying manifold implies much more rigid global conditions: for instance, it was observed by Hitchin that any Ricci-flat, self-dual manifold has a globally trivial twistor bundle [24], which, in the compact case, tells that flat manifolds and certain quotients of K3 surfaces are the only 4-manifolds with trivial twistor bundle, by a result in [23] (we point out that the authors obtained a global twistorial characterization of Ricci-flat, self-dual manifolds in [8] in terms of the Eells–Salamon almost complex structure). Also, the aforementioned results by Friedrich–Grunewald, Friedrich–Kurke, and Hitchin show that global conditions on twistor spaces can be obtained in the Einstein, self-dual case [19, 20, 25], together with the classification result due to Muškarov [29] (see also the references cited in the Introduction); another example could be the necessary and sufficient condition obtained by Davidov et al., who showed that, for every Einstein, self-dual manifold, the Nijenhuis tensor of the Eells–Salamon almost complex structure is covariantly constant with respect to the Chern connection [10]. We point out that (anti-)self-duality itself may lead to strong global conclusions in terms of twistor spaces: for instance, LeBrun [28] showed that, on the connected sum $M = m\overline{\mathbb{CP}^2}$, where $\overline{\mathbb{CP}^2}$ is the complex projective space with the opposite of the standard orientation and $m \geq 3$, there exists an anti-self-dual metric such that the twistor space of M is *Moishezon*, that is, bimeromorphic to a projective variety, while Campana managed to show a converse of this result [7].

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ENDNOTE

[1] In this paper, we call *complex structure* an endomorphism J_V of a vector space V such that $J_V^2 = -\mathrm{Id}_V$, while we call *almost complex structure* a (1,1)-tensor field J on a differentiable manifold M such that J smoothly assigns, to every point p, a complex structure J_p on T_pM.

REFERENCES

[1] L. J. Alías, P. Mastrolia, and M. Rigoli, *Maximum principles and geometric applications*, Springer Monographs in Mathematics, Springer, Cham, 2016.

[2] V. Apostolov and P. Gauduchon, *The Riemannian Goldberg–Sachs theorem*, Int. J. Math. **8** (1997), no. 4, 421–439.

[3] V. Apostolov, G. Grantcharov, and S. Ivanov, *Hermitian structures on twistor spaces*, Ann. Global Anal. Geom. **16** (1998), no. 3, 291–308.

[4] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. R. Soc. Lond. A **362** (1978), no. 1711, 425–461.

[5] F. Belgun and A. Moroianu, *Nearly Kähler 6-manifolds with reduced holonomy*, Ann. Global Anal. Geom. **19** (2001), no. 4, 307–319.

[6] A. L. Besse, *Einstein manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.

[7] F. Campana, *On twistor spaces of the class* $\mathscr{C}$, J. Differ. Geom. **33** (1991), no. 2, 541–549.

[8] G. Catino, D. Dameno, and P. Mastrolia, *Rigidity results for Riemannian twistor spaces under vanishing curvature conditions*, Ann. Global Anal. Geom. **63** (2023), no. 2, 13.

[9] G. Catino and P. Mastrolia, *A perspective on canonical Riemannian metrics*, vol. 336, Progr. Math. Birkhäuser/Springer, Cham, 2020.

[10] J. Davidov, G. Grantcharov, and O. Muškarov, *Curvature properties of the Chern connection of twistor spaces*, Rocky Mountain J. Math. **39** (2009), no. 1, 27–48.

[11] J. Davidov and O. Muškarov, *Twistor spaces with Hermitian Ricci tensor*, Proc. Amer. Math. Soc. **109** (1990), no. 4, 1115–1120.

[12] J. Davidov and O. Muškarov, *On the Riemannian curvature of a twistor space*, Acta Math. Hungar. **58** (1991), no. 3–4, 319–332.

[13] J. Davidov and O. Muškarov, *Harmonic almost-complex structures on twistor spaces*, Israel J. Math. **131** (2002), 319–332.

[14] J. Davidov, O. Muškarov, and G. Grantcharov, *Kähler curvature identities for twistor spaces*, Illinois J. Math. **39** (1995), no. 4, 627–634.

[15] P. de Bartolomeis and A. Nannicini, *Introduction to differential geometry of twistor spaces*, Geometric theory of singular phenomena in partial differential equations (Cortona, 1995), Sympos. Math., XXXVIII, Cambridge Univ. Press, Cambridge, 1998, pp. 91–160.

[16] J. Eells and S. Salamon, *Twistorial construction of harmonic maps of surfaces into four-manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), no. 4, 589–640.

[17] J. Fine and D. Panov, *Symplectic Calabi–Yau manifolds, minimal surfaces and the hyperbolic geometry of the conifold*, J. Differ. Geom. **82** (2009), no. 1, 155–205.

[18] T. Friedrich, *Self-duality of riemannian manifolds and connections*, Seminarbericht, Humboldt-Universität zu Berlin, Sektion Mathematik **25** (1980), 01.

[19] T. Friedrich and R. Grunewald, *On Einstein metrics on the twistor space of a four-dimensional Riemannian manifold*, Math. Nachr. **123** (1985), 55–60.

[20] T. Friedrich and H. Kurke, *Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature*, Math. Nachr. **106** (1982), 271–299.

[21] J. Fu and X. Zhou, *Twistor geometry of Riemannian 4-manifolds by moving frames*, Commun. Anal. Geom. **23** (2015), no. 4, 819–839.

[22] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (4) **123** (1980), 35–58.

[23] N. J. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differ. Geom. **9** (1974), 435–441.

[24] N. J. Hitchin, *Polygons and gravitons*, Math. Proc. Camb. Philos. Soc. **85** (1979), no. 3, 465–476.

[25] N. J. Hitchin, *Kählerian twistor spaces*, Proc. Lond. Math. Soc. (3) **43** (1981), no. 1, 133–150.

[26] G. R. Jensen and M. Rigoli, *Twistor and Gauss lifts of surfaces in four-manifolds*, Recent developments in geometry (Los Angeles, CA, 1987), Contemp. Math., vol. 101, Amer. Math. Soc., Providence, RI, 1989, pp. 197–232.

[27] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. I*, Wiley Classics Library, Wiley & Sons, Inc., New York, 1996, Reprint of the 1963 original, A Wiley-Interscience Publication.

[28] C. LeBrun, *Explicit self-dual metrics on* $\mathbf{C}P_2 \# \cdots \# \mathbf{C}P_2$, J. Differ. Geom. **34** (1991), no. 1, 223–253.

[29] O. Muškarov, *Almost Hermitian structures on twistor spaces and their types*, Atti Sem. Mat. Fis. Univ. Modena **37** (1989), no. 2, 285–297.

[30] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. Math. (2) **65** (1957), 391–404.

[31] N. R. O'Brian and J. H. Rawnsley, *Twistor spaces*, Ann. Global Anal. Geom. **3** (1985), no. 1, 29–58.

[32] R. Penrose, *Twistor algebra*, J. Math. Phys. **8** (1967), 345–366.

[33] A. G. Reznikov, *Symplectic twistor spaces*, Ann. Global Anal. Geom. **11** (1993), no. 2, 109–118.

[34] S. Salamon, *Topics in four-dimensional Riemannian geometry*, Geometry seminar "Luigi Bianchi" (Pisa, 1982), Lecture Notes in Mathematics, vol. 1022, Springer, Berlin, 1983, pp. 33–124.

[35] S. Salamon, *Special structures on four-manifolds*, vol. 17, Conference on Differential Geometry and Topology (Italian) Parma, 1991, pp. 109–123.

[36] S. M. Salamon, *Orthogonal complex structures*, Differential geometry and applications (Brno, 1995), Masaryk Univ., Brno, 1996, pp. 103–117.

[37] I. M. Singer and J. A. Thorpe, *The curvature of 4-dimensional Einstein spaces*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 355–365.

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