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A Model Structure on the Category of A_∞-Categories with Strict Morphisms

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Abstract
We prove that the category of (strictly unital) A_∞-categories, linear over a commutative ring R, with strict A_∞-morphisms has a cofibrantly generated model structure. In this model structure every object is fibrant and the cofibrant objects have cofibrant morphisms. As a consequence we prove that the semi-free A_∞-categories (resp. resolutions) are cofibrant objects (resp. resolution) in this model structure.

Keywords A-infty-categories · DG-categories · Model Categories

Mathematics Subject Classification 14F08 · 18G70 · 18N40

1 Introduction and Statement of Results

We fix a commutative ring R, an A_∞-category is a R-linear DG-category associative up to homotopy. In a few words, an A_∞-category $\mathscr{A}$ is a graded category equipped with multilinear maps

$$m^n_{\mathscr{A}} : \mathscr{A}(x_{n-1}, x_n) \otimes \cdots \otimes \mathscr{A}(x_0, x_1) \to \mathscr{A}(x_0, x_n)[2-n],$$

for every integer $n \geq 1$ and sequence of objects $x_0, , ..., x_n \in \mathscr{A}$, satisfying axioms (see [13, Definition 1.1.1 (1.1)] or [16, §2.1]). Considering $m^1_{\mathscr{A}}$ as the differential, we can associate to $\mathscr{A}$ the (graded) category $H(\mathscr{A})$ whose hom spaces are given by:

$$H(\mathscr{A})(x, y) := \bigoplus_{n\in\mathbb{Z}} H^n(\mathscr{A}(x, y)). \tag{1}$$

Taking two A_∞-categories $\mathscr{A}$ and $\mathscr{B}$, we call *A_∞-functor* a family of multilinear maps $\{\mathscr{F}^n\}_{n\geq 0}$ of the form

$$\mathscr{F}^n : \mathscr{A}(x_{n-1}, x_n) \otimes \cdots \otimes \mathscr{A}(x_0, x_n) \to \mathscr{B}(\mathscr{F}^0(x_n), \mathscr{F}^0(x_0))[1-n] \tag{2}$$

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for every integer $n \geq 1$ and sequence of objects $x_0, ..., x_n \in \mathscr{A}$, satisfying axioms (see [13, Definition 1.2.1 (1.2)]). There are several notions of unit in the framework of A_∞-categories, in this paper we consider the strictly unital A_∞-categories with strictly unital A_∞-functors (see [13, Definitions 1.1.4]). From now on, we denote by A_∞Cat the category of A_∞-categories with the A_∞-functors. Note that A_∞Cat is not complete (since it does not admit equalizers [2, Lemma 1.28]).

We say that an A_∞-functor $\mathscr{F}$ is a *quasi-equivalence* if it induces an equivalence of (graded) categories:

$$[\mathscr{F}] : H(\mathscr{A}) \to H(\mathscr{B}) \tag{3}$$

and an equivalence $H^0(\mathscr{F}) : H^0(\mathscr{A}) \to H^0(\mathscr{B})$. We denote by $\mathrm{Ho}(A_\infty\mathrm{Cat})$ the homotopy category, i.e. the (Gabriel–Zisman) localization of A_∞Cat with respect to the class of quasi-equivalences.

A_∞-categories were introduced in the early 1990s in the context of Homological Mirror Symmetry since the Fukaya category of a symplectic manifold, comes naturally equipped with a structure of this kind. Note that the pioneers of this field, such as Kontsevich, Fukaya, Oh, Ono, Ohta, Seidel, Soibelman, etc, assumed by definition a *flatness* hypothesis on the hom-spaces of an A_∞-category. For example, in [6, Definition 1.1] and [7, §3.2.1] an A_∞-category $\mathscr{A}$ is such that the hom-spaces $\mathscr{A}(x, y)$ are graded free R-modules, or the base commutative ring R is assumed to be a field, see [11] or [17].

On the other hand, the definition of A_∞-category makes sense without any restriction on the hom-spaces. For this reason, more recently, many people such as Ganatra, Pardon, Shende, Oh, Tanaka started to used the term *cofibrant* A_∞*-category* to indicate an A_∞-category whose hom-spaces are h-projective DG-modules (see [8, Definition 2.6], [21], [22, Definition 1.2]). Note that, despite the name, this has nothing to do with a model structure on A_∞Cat, it is well known that A_∞Cat has no model structure [2, §1.5]. The term *cofibrant* is inherited by the DG-categories. Indeed a cofibrant DG-category (in the model structure of Example 2.2), has cofibrant hom-spaces. In particular it is a h-projective DG-category.

The first goal of this paper is to give a precise definition of *cofibrant* A_∞*-category*, namely we provided a model structure on the category of A_∞-categories (taking a subset of A_∞-functors). In this model structure, if an A_∞-category is cofibrant then it has cofibrant hom-spaces, in particular it is h-projective (see Theorem A).

Despite the lack of a model structure, we can describe the hom-spaces of the homotopy category of A_∞Cat as follows:

$$\mathrm{Ho}(A_\infty\mathrm{Cat})(\mathscr{A}, \mathscr{B}) \simeq \mathrm{Ho}(A_\infty\mathrm{Cat})(\mathscr{A}^{\mathrm{hps}}, \mathscr{B})/\approx . \tag{4}$$

Here $\mathscr{A}^{\mathrm{hps}}$ denotes a h-projective with splits unit A_∞-category which is quasi-equivalent to $\mathscr{A}$ and $\approx$ denotes the weakly equivalence relation [14, Theorem A].

In order to prove that every A_∞-category $\mathscr{A}$ has a resolution of the form $\mathscr{A}^{\mathrm{hps}}$, the notion of semi-free A_∞-category was introduced in [14], where it was proven that the semi-free resolution of $\mathscr{A}$ is an h-projective A_∞-category with splits units. Note that, in the case of DG-categories, the semi-free resolutions correspond to the cofibrant resolutions in Tabuada model structure (see Example 2.2). For this reason in *loc. cit.* it was left as an open question if the semi-free A_∞-categories are a kind of *cofibrant resolutions* in an appropriate category.

We recall that an A_∞-functor $\mathsf{F} = \{\mathsf{F}^n\}_{n\geq 0}$ is called *strict* if $\mathsf{F}^{n\geq 2} = 0$, in particular F^1 preserves the underlying graded quivers, *id est*:

$$\mathsf{F}^1(m^n_{\mathscr{A}}(a_1, ..., a_n)) = m^n_{\mathscr{B}}(\mathsf{F}^1(a_1), ..., \mathsf{F}^1(a_n))$$

for every $n \geq 1$ and $a_1, ..., a_n \in \mathscr{A}$. We denote by $\mathrm{A}_\infty\mathrm{Cat}_{\mathrm{strict}}$ the category of A_∞-categories with the strict A_∞-functors. In this note we prove the following:

Theorem A *There is a cofibrantly generated model structure on $\mathrm{A}_\infty Cat_{strict}$ whose weak equivalences are the quasi-equivalences. In such a model structure the fibrations are the isofibrations (see Definition 3.2) strict A_∞-functors which are surjective on the morphisms. Every category is a fibrant object and every cofibrant object $\mathscr{A}$ is such that $\mathscr{A}(a_1, a_2)$ is a cofibrant object in $Ch(R)$.*

Despite the category $\mathrm{A}_\infty\mathrm{Cat}$ is not complete, $\mathrm{A}_\infty\mathrm{Cat}_{\mathrm{strict}}$ is complete and cocomplete (see [14, Theorem 4.5]). As a consequence of Theorem A, we have that the semi-free A_∞-categories are cofibrant (cf. [14, Lemma D]). Moreover, consider the category I so defined:

Definition 1.1 I is the DG-category with two objects and two closed morphisms of degree zero

$$1 \underset{j_{10}}{\overset{j_{01}}{\rightleftarrows}} 2 \tag{5}$$

such that $j_{01} \cdot j_{10} = \mathrm{Id}_2$ and $j_{10} \cdot j_{01} = \mathrm{Id}_1$.

It is not difficult to prove that, given an A_∞-category $\mathscr{A}$, the *path object* of $\mathscr{A}$

$$P(\mathscr{A}) := \mathrm{A}_\infty\mathrm{Cat}(I, \mathscr{A})$$

described in [14, 2.5] is exactly the same path object of $\mathscr{A}$ in $\mathrm{A}_\infty\mathrm{Cat}_{\mathrm{strict}}$, with the model structure of Theorem A. Cf. also [20, §3] and [20, Remark 2.7].

We conclude by saying that, given an A_∞-category $\mathscr{A}$, we can take the cofibrant resolution $\mathscr{A}^{\mathrm{cof}}$ (with respect to the model structure of Theorem A) which is cofibrant in the sense of Ganatra, Pardon, Shende et al.

1.1 State of Art

By the work of Lefevré-Hasegawa [12] the category Alg_∞ of (non unital) A_∞-algebras linear over a field, has a model structure whose weak equivalences are the quasi-isomorphisms. In this model structure a morphism f is a fibration (resp. cofibration) if f^1 is surjective (resp. injective).

Note that Alg_∞ has no equalizers (see [2]) and coproducts. To see that Alg_∞ has no coproduct we use the fact that (the bar-cobar functor) $\mathrm{U} : \mathrm{Alg}_\infty \to \mathrm{DG\text{-}Alg}$ is right adjoint to the inclusion (see [3]). Since the tensor product is the coproduct of two DG-algebras, we must have $\mathrm{U}(A) \otimes \mathrm{U}(B) \simeq \mathrm{U}(A \otimes B)$, for any A_∞-algebras. Using this fact, it is easy to see that Alg_∞ has no coproduct (it has to do with the fact that there is no a good notion of tensor product of A_∞-algebras [15]). This is no longer true if we consider the category of A_∞-categories, namely, it has coproducts (which is the disjoint union) but it does not have equalizers [2, Lemma 1.28].

On the other hand, if we consider the category of A_∞-algebras with strict A_∞-morphisms we have a model structure by [9, 2.2.1. Theorem]. In this model structure the fibrations are the morphisms f such that f^1 is surjective.

Moreover, in [4] we proved that the category of A_∞-categories, linear over a field, is a fibrant category. The fibrations are the A_∞-functors $\mathscr{F}$ which are isofibrations and such that $\mathscr{F}^1$ is degree-wise surjective.

We conclude by saying that we do not known if Alg_∞ and $\mathrm{A}_\infty\mathrm{Cat}$ have coequalizers. This is an interesting question since, if it was true (at least in some cases i.e. along the cofibrations), one could try to prove that the category $\mathrm{A}_\infty\mathrm{Cat}$ (linear over a commutative ring) has a structure of cofibrant category.

2 Model Structures and Recognition Theorem

We give two examples of model structures cofibrantly generated. It will be crucial the following result which goes under the name of *Recognition Theorem* and corresponds to [10, Theorem 2.1.19]:

Theorem 2.1 (Recognition Theorem) *Let $\mathscr{C}$ be a complete and cocomplete category with $\mathscr{W}$ a subcategory of $\mathscr{C}$, and I and J sets of maps of $\mathscr{C}$. There is a cofibrantly generated model structure on $\mathscr{C}$ with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $\mathscr{W}$ as the subcategory of weak equivalences, if and only if the following conditions are satisfied:*

1. *The subcategory $\mathscr{W}$ has the two out of three property and is closed under retracts.*
2. *The domains of I are small relative to I-cell.*
3. *The domains of J are small relative to J-cell.*
4. *J-cell $\subset \mathscr{W} \cap I$-cof.*
5. *I-inj $\subset \mathscr{W} \cap J$-inj.*
6. *Either $\mathscr{W} \cap I$-cof $\subset J$-cof or $\mathscr{W} \cap J$-inj $\subset I$-inj.*

We recall that $f \in I$-inj if f has the right lifting property with respect to any morphism in I. We say that $f \in I$-cof if f has the left lifting property for any I-injective morphism.

Example 2.1 We denote by $\mathrm{Ch}(R)$ the category of unbounded chain complexes over a commutative ring R. The category $\mathrm{Ch}(R)$ has a model structure whose weak-equivalences are the quasi-isomorphisms (see [10, Definition 2.3.3 and Theorem 2.3.11]). In order to define such a model structure we define the chain complexes $\mathbb{S}^{n-1}$ and $\mathbb{D}^n$. Fixed an integer n,

$$\mathbb{S}^{n-1} := \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow \cdots$$

where R is in degree $-n+1$, and

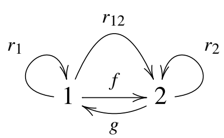
$$\mathbb{D}^n = \cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\mathrm{Id}_R} R \longrightarrow 0 \longrightarrow \cdots$$

where the only nonzero components are in degree $-n$ and $-n+1$.

The set of generating cofibrations I is given by the maps $i_n : \mathbb{S}^{n-1} \to \mathbb{D}^n$, for each $n \in \mathbb{Z}$. The set of generation trivial cofibrations J consists of $j_n : 0 \to \mathbb{D}^n$, for each $n \in \mathbb{Z}$. In this model structure the fibrations are the maps f such that f_n is surjective for all $n \in \mathbb{Z}$ and the cofibrant objects are the h-projective degreewise projective modules (see [1, Theorem 9.6.1 (ii′) iff (v′)]).

Before continuing, we define by $\mathcal{K}$ the Kontsevich category:

Definition 2.1 $\mathcal{K}$ is the DG-category with two objects generated by the following morphisms:



with the relations:

1. $d(r_{12}) = r_2 \cdot f - f \cdot r_1$.
2. $d(r_1) = g \cdot f - \mathrm{Id}_1$.
3. $d(r_2) = f \cdot g - \mathrm{Id}_2$.

Note that $\mathcal{K}$ is a semi-free resolution of the DG-category I, where I is defined in (1.1); see [5, 3.7.6. Remark]. To prove that $\mathcal{K}$ is semi-free, we consider the filtration:

$$0 \hookrightarrow I_0 \hookrightarrow I_1 \hookrightarrow I_2 \hookrightarrow I_3 := \mathcal{K}.$$

Here

1. I_0 is the discrete category with two objects: 1 and 2.
2. I_1 is the DG-category freely generated by two closed generators f and g
3. I_2 is the DG-category freely generated on I_1 by the generators r_1 and r_2, of degree 1, such that $d(r_1) = g * f - \mathrm{Id}_1$ and $d(r_2) = f * g - \mathrm{Id}_2$.
4. I_3 is the DG-category freely generated on I_2 adding the generator r_{12} and taking the quotient by the DG-ideal $(d(r_{12}) - r_2 * f + f * r_1)$.

We have the DG-functors $\Psi_0 : I_0 \to I$, $\Psi_1 : I_1 \to I$ and $\Psi_2 : I_2 \twoheadrightarrow I$, $\Psi : K \twoheadrightarrow I$. In particular, Ψ_2 and Ψ are surjective on the morphisms and Ψ is a quasi-equivalence. In particular $\Psi(j_{01}) = f$, $\Psi(j_{10}) = g$ and $\Psi(r_1) = \Psi(r_2) = \Psi(r_{12}) = 0$.

Remark 2.1 Note that we added r_{12} in I_3, since

$$r_2 * f - f * r_1 \tag{6}$$

is a closed morphism which vanish in I via Ψ. It means that, if we want to make Ψ a quasi-equivalence then (6) must be a coboundary in cohomology.

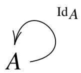
On the other hand, also

$$r_1 * g - g * r_2 \tag{7}$$

is a closed morphism whose image via Ψ_2 is 0. Nevertheless we do not need to add a new generator in I_3, since (7) is the differential of the following morphism:

$$(g * r_{12} * g + r_1 * g * r_2 - g * r_2 * r_2 + r_1 * r_1 * g).$$

Example 2.2 In [18], Tabuada proved that the category of DG-categories has a cofibrantly generated model structure whose weak-equivalences are the quasi-equivalences. The set of generating cofibrations I consists of the DG-functors Q and $\mathsf{S}(n)$ described as follows. First, we denote by $\mathcal{A}$ the DG-category:



Q is the (only) DG-functor

$$\mathsf{Q} : \emptyset \to \mathcal{A} \tag{8}$$

from the empty set (which is the initial category of DG-cats and $\mathrm{A}_\infty\mathrm{Cat}$). Fixed an integer n, $\mathcal{C}(n)$ and $\mathcal{P}(n)$ are the two DG-categories having two objects C_1, C_2 and P_1, P_2 and whose hom-spaces are given by

$$\mathcal{C}(n)(C_1, C_2) := \mathbb{S}^{n-1} \tag{9}$$

and

$$\mathcal{P}(n)(P_1, P_2) := \mathbb{D}^n, \tag{10}$$

$\mathsf{S}(n)$ is the DG-functor

$$\begin{aligned} \mathsf{S}(n) : \mathcal{C}(n) &\to \mathcal{P}(n) \\ \mathbb{S}^{n-1} &\mapsto \mathbb{D}^n. \end{aligned}$$

Now we define the trivial cofibrations. First, we denote by $\mathcal{B}$ the DG-category which has two objects B_1 and B_2 and no non trivial morphisms.

Fixed an integer n, the DG-functor $\mathsf{R}(n) : \mathcal{B} \to \mathcal{P}(n)$ is defined as follows:

$$\begin{array}{ccc} B_1 & & B_2 \\ \big\downarrow & & \big\downarrow \\ P_1 & \xrightarrow[\mathbb{D}^n]{} & P_2 \end{array} \tag{11}$$

The functor $\mathsf{F} : \mathcal{A} \to \mathcal{K}$ is the DG-functor sending the object A of $\mathcal{A}$ to the object 1 of $\mathcal{K}$.

The trivial cofibrations are generated by $J := \{\mathsf{F}, \mathsf{R}(n)\}$. In this model structure the fibrations are the isofibrations (see Definition 3.2) which are degreewise surjective [19, Proposition 1.13]. Moreover, every object is fibrant and the cofibrant objects are such that the hom-spaces are cofibrant in $\mathrm{Ch}(R)$, see [23, Proposition 2.3 (3)].

3 Proof of Theorem A

In this section we prove the main result using the Recognition Theorem 2.1.

First, we want to find an appropriate "$\mathcal{K}$" (defined in Definition 2.1) in the case of A_∞-categories. We recall that, given an A_∞-category $\mathscr{A}$, we can take a semi-free resolution of $\mathscr{A}$. In [14, §5], we give a procedure to find such a resolution. We define

Definition 3.1 The category $\mathcal{K}^{\mathrm{A}_\infty}$ is the A_∞ semi-free resolution of the category I (see Definition 1.1) defined as follows:

$$0 \hookrightarrow I_0 \hookrightarrow I_1 \hookrightarrow I_2 =: \mathcal{K}^{\mathrm{A}_\infty}.$$

0. I_0 is the discrete (strictly unital) category with two objects: 1 and 2.
1. I_1 is the A_∞-category freely generated by the closed morphisms j_{12} and j_{21}.
2. I_2 is the A_∞-category freely generated on I_1 adding the generators r_1 and r_2, such that
   (i) $m^1(r_1) = (\bigvee; f, g) - 1$.
   (ii) $m^1(r_2) = (\bigvee; g, f) - 1$.

Note that:

$$\begin{aligned} m^1_{\mathcal{K}^{\mathrm{A}_\infty}}\big((\bigvee; r_1, f) - (\bigvee; f, r_2)\big) &= \big((\bigvee; m^1(r_1), f) - (\bigvee; f, m^1(r_2))\big) \\ &= (\bigvee\!\!\!\bigvee; f, g, f) - (\bigvee\!\!\!\bigvee; f, g, f) \\ &\neq 0. \end{aligned}$$

As we already noted, this is not the same in the case of DG-categories, since

$$\begin{aligned} d_{\mathcal{K}}(r_1 * f - f * r_2) &= \big(d(r_1) * f) - (f * d(r_2))\big) \\ &= (f * g * f) - (f * g * f) \\ &= 0. \end{aligned}$$

This is why we need to add the free generator r_{12} in $\mathcal{K}$ which is the coboundary of the cocycle $r_1 * f - f * r_2$. Nevertheless $\mathcal{K}^{A\infty}$ and $\mathcal{K}$ are weakly equivalent since both are h-projective with splits unit and quasi-equivalent to I (see [14, Theorem 5.2] and [3, Definition 3.4]). It is important to say that, since $\mathcal{K}^{A\infty}$ is h-projective, $I(n, m) = (R, 0)$ is a free R-module (for any $n, m = \{1, 2\}$) and they are quasi-equivalent, then $(\mathcal{K}^{A\infty}, m^1_{\mathcal{K}^{A\infty}})$ is homotopy equivalent to the complex $(R, 0)$. In particular, since $\mathcal{K}^{A\infty}$ has split unit, it implies that the short exact sequence of graded complexes

$$0 \longrightarrow R \cdot 1_1 \hookrightarrow K^{A\infty}(1,1) \twoheadrightarrow K^{A\infty}(1,1)/R \cdot 1_1 \longrightarrow 0$$

splits.

Theorem 3.1 *Let* F' *to be the strict* A_∞*-functor from* $\mathcal{A}$ *to* $\mathcal{K}^{A\infty}$ *such that* $\mathsf{F}'(A) := 1$. *The category* $A_\infty Cat_{strict}$ *has a model structure cofibrantly generated by* $J' := \{\mathsf{F}', \mathsf{R}(n)\}$ *(defined in (11)) and* I *(defined in Example 2.2).*

First we give two definitions and a Lemma.

Definition 3.2 An A_∞-functor (resp. a DG-functor) $F : \mathscr{A} \to \mathscr{B}$ is an *isofibration* if, given $a \in \mathscr{A}$ and an isomorphism $g : \mathscr{F}(a) \to b$ in $H^0(\mathscr{B})$, there exists an isomorphism $f : a \to a'$ in $H^0(\mathscr{A})$, such that $\mathscr{F}^1(f) = g$.

Definition 3.3 An A_∞-functor (resp. a DG-functor) $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ is *surjective* if, $\mathscr{F}^0$ is surjective (as a map of sets) and $\mathscr{F}^1$ is a surjective quasi-isomorphism of complexes.

We denote by Surj the set of strict surjective A_∞-functors. The following Lemma which will be useful also to characterize the fibrations of the model structure provided by Theorem 3.1.

Lemma 3.2 *A strict* A_∞*-functor* $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ *has right lifting property with respect to*

(F1) $\mathsf{R}(n)$ *if and only if* $\mathscr{F}^1$ *is surjective on the morphisms.*
(F1′) $\mathsf{S}(n)$ *if and only if* $\mathscr{F}^1$ *is a quasi-isomorphism surjective on the morphisms.*
(F2) F *if and only if* $\mathscr{F}^1$ *is an isofibration.*
(F3) Q *if and only if* $\mathscr{F}^0$ *is surjective (as a map of sets).*

Proof Suppose that $\mathscr{F}$ is surjective on the morphisms, consider the diagram below:

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathscr{C} \\ {\scriptstyle \mathsf{R}(n)}\downarrow & \overset{\tilde{\mathsf{T}}}{\nearrow} & \downarrow{\scriptstyle \mathscr{F}} \\ \mathcal{P}(n) & \underset{\mathsf{T}}{\longrightarrow} & \mathscr{D} \end{array}$$

It is easy to find a $\tilde{\mathsf{T}}$ since $\mathsf{P}(n)$ is surjective on the morphisms and $\mathsf{P}(n)$ is uniquely determined by the image of $\mathsf{T}(R[1] \oplus R)$.

Suppose now that $\mathscr{F}$ has the right lifting property with respect to $\mathsf{R}(n)$. For every morphism $g \in \mathscr{D}$ we take the functor

$$\mathsf{T} : R[1] \oplus R \to \mathscr{D}$$
$$(a, b) \mapsto a \cdot g + b \cdot dg.$$

If there exists $\tilde{\mathsf{T}} : \mathcal{P}(n) \to \mathscr{C}$ such that

$$\mathscr{F}\big(\tilde{\mathsf{T}}((a, b))\big) = \mathsf{T}\big((a, b)\big) = a \cdot g + b \cdot dg.$$

Taking $\tilde{\mathsf{T}}(1, 0) \in \mathscr{C}$, we have $\mathscr{F}(\tilde{\mathsf{T}}(1, 0)) = \mathsf{T}((1, 0)) = g$, so $\mathscr{F}$ is surjective on morphisms.

Now we prove F1′). We note that a diagram of the form:

$$\begin{array}{ccc} \mathcal{C}(n) & \longrightarrow & \mathscr{C} \\ {\scriptstyle \mathsf{S}(n)}\downarrow & \overset{\tilde{\mathsf{T}}}{\nearrow} & \downarrow{\scriptstyle \mathscr{F}} \\ \mathcal{P}(n) & \longrightarrow & \mathscr{D} \end{array} \tag{12}$$

corresponds to the datum

$$\{(f, \tilde{g}) \in \mathscr{C} \times \mathscr{D} \text{ such that: } \deg(f)=-n+1, m^1_{\mathscr{C}}(f) = 0 \text{ and } \mathscr{F}^1(f) = m^1_{\mathscr{D}}(\tilde{g})\}. \tag{13}$$

Moreover, the existence of a lift $\tilde{\mathsf{T}}$, corresponds to the datum

$$\{\tilde{f} \in \mathscr{C} \text{ such that } m^1_{\mathscr{C}}(\tilde{f}) = f \text{ and } \mathscr{F}^1(\tilde{f}) = \tilde{g}\}. \tag{14}$$

Suppose that every diagram of the form (12) has a lift, we show that $\mathscr{F}^1$ is a surjective quasi-isomorphism. First, we prove that $[\mathscr{F}^1]$ is surjective. Let $\tilde{g} \in \mathscr{D}$ be a closed morphism, the pair $(0, \tilde{g})$ gives rise to a diagram of the form (12). Since it admits a lift then, by the characterization (14), there is a closed morphism $\tilde{f}$ such that $\mathscr{F}^1(\tilde{f}) = \tilde{g}$.

Now we prove that $\mathscr{F}^1$ is surjective. Given a morphism $g \in \mathscr{D}$, we can take the datum $(0, m^1_{\mathscr{D}}(g))$, we just shown that there exists a closed morphism $\tilde{f} \in \mathscr{C}$ such that $\mathscr{F}^1(\tilde{f}) = m^1_{\mathscr{D}}(g)$. Using the characterization (13), the pair $(\tilde{f}, g)$ gives rise to a new diagram of the form (12). Since it has a lift, then it exists $f \in \mathscr{C}$ such that $m^1_{\mathscr{C}}(f) = \tilde{f}$ and $\mathscr{F}^1(f) = g$, and we are done.

We prove that $[\mathscr{F}^1]$ is injective. Suppose that f is a closed morphism in $\mathscr{C}$ such that $\mathscr{F}^1(f)$ vanishes in cohomology. It means that there exists g such that $\mathscr{F}^1(f) = m^1_{\mathscr{D}}(g)$. Then (f, g) forms a diagram of the form (12). Since it is liftable, there exists $\tilde{f}$ such that $m^1_{\mathscr{C}}(\tilde{f}) = f$ so f is zero in cohomology.

To conclude, we need to prove that, if $\mathscr{F}^1$ is a surjective quasi-isomorphism, then every diagram of the form

$$\begin{array}{ccc} \mathcal{C}(n) & \longrightarrow & \mathscr{C} \\ {\scriptstyle \mathsf{S}(n)}\downarrow & & \downarrow{\scriptstyle \mathscr{F}} \\ \mathcal{P}(n) & \longrightarrow & \mathscr{D} \end{array} \tag{15}$$

has a lift. Using the characterization (13), we have $(f, \tilde{g})$ such that $m^1_{\mathscr{C}}(f) = 0$ and $\mathscr{F}^1(f) = m^1_{\mathscr{D}}(\tilde{g})$. We consider the short exact sequence of chain complexes:

$$0 \longrightarrow \operatorname{Ker}(\mathscr{F}^1) \hookrightarrow \mathscr{C}(x, y) \xrightarrow[\mathscr{F}^1]{\simeq} \mathscr{D}(\mathscr{F}^0(x), \mathscr{F}^0(y)) \longrightarrow 0. \tag{16}$$

Since $\mathscr{F}^1$ is a quasi-isomorphism then the cohomology of $\operatorname{Ker}(\mathscr{F}^1)$ is trivial. Moreover $\mathscr{F}^1$ is surjective, so we can take a morphism $h \in \mathscr{C}(x, y)$ such that $\mathscr{F}^1(h) = \tilde{g}$. We note that the morphism $m^1_{\mathscr{C}}(h) - f \in \mathscr{C}$ is closed and $\mathscr{F}^1(m^1_{\mathscr{C}}(h) - f) = m^1_{\mathscr{D}}(\tilde{g}) - \mathscr{F}^1(f) = 0$, so $m^1_{\mathscr{C}}(h) - f \in \mathrm{Ker}(\mathscr{F}^1)$. To conclude, since the cohomology of $\mathrm{Ker}(\mathscr{F}^1)$ is trivial, it exists $j \in \mathrm{Ker}(\mathscr{F}^1)$ such that $m^1_{\mathscr{C}}(j) = m^1_{\mathscr{C}}(h) - f$. Thanks to the characterization (14), the morphism $\tilde{f} = h - j$ corresponds to a lift of (15) and we are done.

To prove F2), note that every strict A_∞-functor $\mathcal{K}^{\mathrm{A}\infty} \to \mathscr{D}$ is (uniquely) determined by the image of the generators. Suppose that F has right lifting property with respect to $\mathscr{F}$, so for every commutative diagram of the form:

$$\begin{array}{ccc} \mathcal{R} & \longrightarrow & \mathscr{C} \\ {\scriptstyle \mathsf{F}}\downarrow & \overset{\tilde{\mathsf{T}}}{\nearrow} & \downarrow{\scriptstyle \mathscr{F}} \\ \mathcal{K}^{\mathrm{A}\infty} & \underset{\mathsf{T}}{\longrightarrow} & \mathscr{D} \end{array}$$

there exists a lift $\tilde{\mathsf{T}}$.

Suppose there exists $g : \mathscr{F}^0(c) \to d \in H^0(\mathscr{D})$. It means that there exist g^{-1}, h, t such that:

$$m^2_{\mathscr{D}}(g, g^{-1}) = \mathrm{Id} + m^1_{\mathscr{D}}(h)$$

and

$$m^2_{\mathscr{D}}(g^{-1}, g) = \mathrm{Id} + m^1_{\mathscr{D}}(t).$$

Then we can take the functor $\mathsf{T} : K^{\mathrm{A}\infty} \to \mathscr{D}$ so defined:

$$\mathsf{T}(j_{12}) = g\ , \mathsf{T}(j_{21}) = g^{-1}\ , \mathsf{T}(r_1) = t \text{ and } \mathsf{T}(r_2) = h.$$

Since F has a right lifting property, there exists $\tilde{\mathsf{T}} : \mathcal{K}^{\mathrm{A}\infty} \to \mathscr{C}$. It implies that there exists an isomorphism $\tilde{\mathsf{T}}(j_{12}) =: f$ and an object $\tilde{\mathsf{T}}(2) =: c'$ in $H^0(\mathscr{C})$ such that $\mathscr{F}^1(f) = \mathsf{T}(j_{12}) = g$, and we are done. The proof of viceversa is similar.

Now we prove F3). We consider the commutative diagram:

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathscr{C} \\ {\scriptstyle \mathsf{Q}}\downarrow & \overset{\tilde{\mathsf{T}}}{\nearrow} & \downarrow{\scriptstyle \mathscr{F}} \\ \mathcal{A} & \underset{\mathsf{T}}{\longrightarrow} & \mathscr{D} \end{array}$$

since every functor from $\mathcal{A}$ is uniquely determined by the image of the object A. It is easy to see that if there exists a lifting $\tilde{\mathsf{T}}$ for every T then $\mathscr{F}$ must be surjective on the objects. On the other hand, if $\mathscr{F}^0$ is surjective then we can find a $\tilde{\mathsf{T}}$ for every T. □

Proof of Theorem 3.1 We want to use Theorem 2.1, so we must verify the six conditions of the theorem.

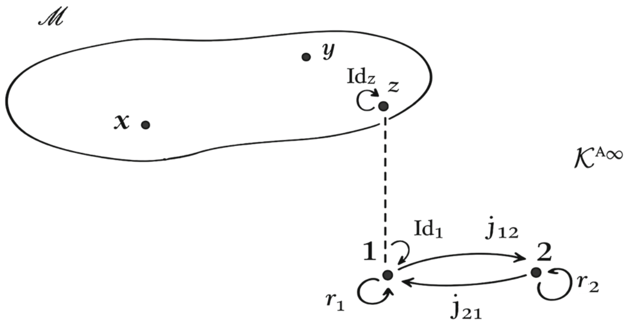
1. 2. 3. Straightforward.
4. First we note that I-inj $= \mathscr{W} \cap J$-inj $\subset J$-inj. It implies that J-cof $\subset I$-cof and J-cell $\subset J$-cof $\subset I$-cof. So it remains to prove that, for every A_∞-category $\mathscr{M}$ and $\star \in J'$, the strict A_∞-functor inc, fitting the pushout diagram

$$\begin{array}{ccc} & \longrightarrow & \mathscr{M} \\ {\scriptstyle \star}\downarrow & & \downarrow{\scriptstyle \mathrm{inc}} \\ & \longrightarrow & \mathscr{P}, \end{array} \tag{17}$$

is a quasi-equivalence.
We consider the push-out diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{N} & \mathscr{M} \\ {\scriptstyle \mathsf{F}}\downarrow & & \downarrow{\scriptstyle \mathrm{inc}} \\ \mathcal{K}^{A\infty} & \xrightarrow{N'} & \mathscr{P} \end{array} \tag{18}$$

We denote $N(A) \in \mathscr{M}$ by z. The push-out $\mathscr{P}$ is obtained by taking the disjoint union of $\mathscr{M}$ and $\mathcal{K}^{A\infty}$ and gluing the object z of $\mathscr{M}$ with the object 1 of $\mathcal{K}^{A\infty}$. See the following picture:



We have:

$$\mathscr{P}(x, y) := \bigoplus_{m \geq 0} \mathscr{P}^{(m)}(x, y), \tag{19}$$

where

$$\mathscr{P}^{(m)}(x, y) = \underbrace{\mathscr{M}(z, y) \otimes \overline{\mathcal{K}}^{A\infty}(z, z) \otimes \mathscr{M}(z, z) \otimes \ldots \otimes \overline{\mathcal{K}}^{A\infty}(z, z) \otimes \mathscr{M}(x, z)}_{m \text{ factors } \overline{\mathcal{K}}^{A\infty}}.$$

Here $\overline{\mathcal{K}}^{A\infty}(z, z)$ is the chain complex:

$$\overline{\mathcal{K}}^{A\infty}(z, z) := \mathcal{K}^{A\infty}(1, 1)/R \cdot 1_1. \tag{20}$$

Note that we take the quotient complex (20) because we "glue" the identity of the object $z \in \mathscr{M}$ with the identity of the object $1 \in \mathcal{K}^{A\infty}$. Note that the chain complex $(\mathcal{K}^{A\infty}(1, 1), m^1_{\mathcal{K}^{A\infty}})$ is homotopy equivalent to $(R, 0)$ and $\overline{\mathcal{K}}^{A\infty}(z, z)$ is contractible. So

$$\mathrm{inc} : \mathscr{M}(x, y) \to \mathscr{P}(x, y)$$

is a quasi-isomorphism. It is also clear that $H^0(\mathrm{inc})$ is essentially surjective since $N'(2)$ is quasi-isomorphic to $N'(1) = N(A) = z$.
On the other hand, we consider the push-out diagram:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{T} & \mathscr{M} \\ {\scriptstyle \mathsf{R}(n)}\downarrow & & \downarrow{\scriptstyle \mathrm{inc}} \\ \mathcal{P}(n) & \longrightarrow & \mathscr{P} \end{array} \tag{21}$$

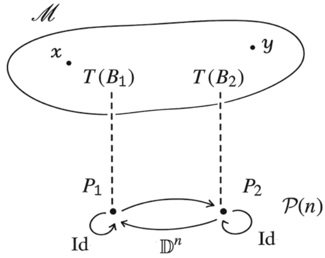


Fig. 1 The push-out $\mathscr{P}$

The category $\mathscr{P}$ is given by the disjoint union of $\mathscr{M}$ and $\mathcal{P}(n)$, and by gluing the object $T(B_1)$ with the object P_1, and the object $T(B_2)$ with the object P_2. See Fig. 1.
We have:

$$\mathscr{P}(x, y) = \bigoplus_{m \geq 0} \mathscr{P}^{(m)}(x, y),$$

where

$$\mathscr{P}^{(m)}(x, y) = \underbrace{\mathscr{M}(T(B_2), y) \otimes \mathbb{D}^n \otimes \mathscr{M}(T(B_2), T(B_1)) \otimes \ldots \otimes \mathbb{D}^n \otimes \mathscr{M}(x, T(B_1))}_{m \text{ factors } \mathbb{D}^n},$$

Since $\mathbb{D}^n$ is contractile then inc is a quasi-equivalence and we are done.

5. 6. We claim that Surj $= I$-inj $= J$-inj $\cap \mathscr{W}$. To prove of Surj $= I$-inj we can use Lemma 3.2 item F1$'$) and F3). Note that it is the same of [19, Lemme 1.11], since the set of generating cofibrations I is the same of Example 2.2. Now we want to prove J-inj $\cap \mathscr{W}$ = Surj. If $f \in J$-inj $\cap \mathscr{W}$ then f has right lifting property with respect to $\mathsf{R}(n)$ so, by Lemma 3.2 item F1), it is surjective on the morphisms. For the item F3) of the same Lemma it is an isofibration. Since it is a quasi-equivalence it is surjective on the objects.
On the other hand if $f \in$ Surj, then $\mathsf{R}(n)$ has the right lifting property with respect to f (see item F1) of Lemma 3.2). Moreover F has the right lifting property with respect to f since $\mathcal{K}^{A\infty}$ is semi-free (see [14, Lemma 6.6]) and we are done.

□

Corollary 3.3 *The fibrations are the isofibrations* F *such that* F^1 *are degreewise surjective. Every* A_∞*-category is fibrant.*

Proof It follows directly from Lemma 3.2, every A_∞-category is fibrant since the terminal object of $A_\infty Cat_{strict}$ is the category with one object and one (trivial) morphism $\mathscr{A}$ (defined Example 2.2). □

Theorem 3.4 *If* $\mathscr{A}$ *is a cofibrant* A_∞*-category then* $\mathscr{A}(x, y)$ *is a cofibrant object in Ch(R).*

Proof The proof is the same as in the case of DG-categories (see [23, Proposition 2.3 (3)]) since they have the same set of generating cofibrations I of Example 2.2. □

As in the case of DG-categories not all the h-projective A_∞-categories are cofibrant object in this model structure. For example the category $\mathcal{C}(1) \otimes \mathcal{C}(1)$ is not a cofibrant object. It is not hard to prove that the map $\emptyset \to \mathcal{C}(1) \otimes \mathcal{C}(1)$ does not have the right lifting property with respect to the trivial fibrations (see [24, Exercise 14 4.]).

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Declarations

Ethical Approval Not applicable.

Competing interests The authors declare no competing interests.

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