SUSLIN HOMOLOGY VIA CYCLES WITH MODULUS AND APPLICATIONS

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Abstract. We show that for a smooth projective variety X over a field k and a reduced effective Cartier divisor $D \subset X$, the Chow group of 0-cycles with modulus $\mathrm{CH}_0(X|D)$ coincides with the Suslin homology $H_0^S(X \setminus D)$ under some necessary conditions on k and D. We derive several consequences, and we answer to a question of Barbieri-Viale and Kahn.

Contents

1. Introduction

The theory of Chow groups with modulus is presently an active area of research whose primary goal is to provide a cycle theoretic description of the relative K-theory of smooth varieties and of the (ordinary) K-theory of singular varieties. As such, this is a non-$\mathbb{A}^1$-homotopy invariant cohomology theory: this poses a major hurdle while dealing with Chow groups with modulus. The question then arises whether one can isolate a number of special cases in which the Chow groups with modulus behave like a homotopy invariant cohomology theory. This note is an attempt to answer this question.

More specifically, we exhibit a phenomenon which justifies the belief that the Chow groups with modulus associated to a normal crossing divisor on a smooth scheme over a field should behave like a homotopy invariant theory. The precise result that we prove is the following.

1.1. **Main result.** Let k be a field and X a smooth projective scheme of pure dimension $d \geq 0$ over k. Let $D \subset X$ be a reduced effective Cartier divisor (possibly empty) on X with complement U. In this case, we shall say that (X, D) is a reduced modulus pair. Let $\mathrm{CH}_0(X|D)$ be the Chow group of 0-cycles on X with modulus D (see [32]). Let $H_0^S(U)$ denote the (zeroth) Suslin homology of U (see [43, Defn. 10.8], where it is called the algebraic singular homology). If k admits resolution of singularities, then $H_0^S(U)$ coincides with the Suslin-Voevodsky motivic cohomology with compact support $H_c^{2d}(U, \mathbb{Z}(d))$. There is a canonical surjection (e.g., using [50, Thm. 5.1])

$$\phi_{X|D}\colon \mathrm{CH}_0(X|D) \twoheadrightarrow H_0^S(U). \tag{1.1}$$

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This map is clearly an isomorphism if $d \le 1$. However, it is known (see [7, Thm. 1.1]) that $\phi_{X|D}$ may not be an isomorphism if $d \ge 2$ (even if k is algebraically closed). The goal of this paper is to prove the following result.

Theorem 1.1. *Assume that one of the following conditions holds.*

1. *D is a simple normal crossing divisor on X.*
2. *k is perfect, $d \le 2$ and D is seminormal.*
3. *k is algebraically closed of positive characteristic.*
4. *$k \subseteq \overline{\mathbb{Q}}$.*

Then the map

$$\phi_{X|D}\colon \mathrm{CH}_0(X|D) \to H_0^S(U)$$

is an isomorphism.

The case (3) of Theorem 1.1 is known by [7, Thm. 1.1] (we included it in the theorem only for completeness). Hence, the new results are (1), (2) and (4). We expect the condition $d \le 2$ in (2) to be unnecessary, but do not know how to show it. However, the condition on $\mathrm{char}(k)$ in (3) and algebraicity of k over $\mathbb{Q}$ in (4) can not be relaxed (see [7, Thm. 4.4]). We also remark that $\phi_{X|D}$ is almost never an isomorphism if D is not reduced.

1.2. **Applications.** As Theorem 1.1 identifies an a priori non-homotopy invariant theory with a homotopy invariant one, we expect it to have many consequences. We list some in this paper.

1.2.1. *Class field theory of Kerz-Saito.* The goal of geometric class field theory is to describe the abelian fundamental group of a variety (say, defined over a finite field) in terms of certain groups of algebraic cycles. The modern perspective on the problem is given by the work of Kerz and Saito [32] (see also [8]), where the class groups used to describe the abelian fundamental group of a variety X with bounded ramification along a divisor D is precisely the Chow group of zero-cycles with modulus.

By a clever induction argument on the ramification index, the proof of the main theorem of [32] uses as key ingredient the existence of a reciprocity isomorphism

$$\rho_{X|D}\colon \mathrm{CH}_0(X|D)_0 \xrightarrow{\cong} \pi_1^{\mathrm{ab}}(X,D)_0 \tag{1.2}$$

for a reduced simple normal crossing divisor D on a smooth projective surface X over a finite field, where $\pi_1^{\mathrm{ab}}(X,D)_0$ is (the degree zero part of) the abelian fundamental group of X with modulus D, a quotient of the usual étale fundamental group $\pi_1^{\mathrm{ab}}(U)$, where $U = X \setminus D$. For the proof of this fact, Kerz and Saito refer to a result of Kerz-Schmidt (see [33, Thm. 8.3]) that, reformulated in an appropriate way, affirms the existence of an isomorphism

$$\rho_U^t\colon H_0^S(U)_0 \xrightarrow{\cong} \pi_1^{t,\mathrm{ab}}(U)_0, \tag{1.3}$$

where $\pi_1^{t,\mathrm{ab}}(U)$ is the tame fundamental group of U (classifying tame finite étale coverings of U), a further quotient of $\pi_1^{\mathrm{ab}}(U)$. The comparison between (1.2) and (1.3) is very indirect, and passes through non-trivial results in ramification theory.

An immediate application of Theorem 1.1 is that the isomorphism (1.2) is in fact a direct corollary of (1.3), and holds in any dimension. It also follows immediately from Theorem 1.1 and Kerz-Schmidt theorem that for any smooth projective variety X over a finite field and a reduced simple normal crossing divisor $D \subset X$ with complement U, the canonical map

$$\pi_1^{\mathrm{ab}}(X,D) \to \pi_1^{t,\mathrm{ab}}(U)$$

is an isomorphism of topological groups.

1.2.2. *Reciprocity for Russell's relative Chow group.* An independent theory of relative Chow groups with modulus was introduced and extensively studied by Russell (see [48]). For $D \subset X$ an effective Cartier divisor, let us denote Russell's relative Chow group of 0-cycles by $\mathrm{CH}_0^{\mathrm{Rus}}(X|D)$. It is clear from the definition of this group (see op. cit.) and [50, Thm. 5.1] that it coincides with $H_0^S(U)$ when D is reduced. We thus immediately get the following.

Corollary 1.2. *Under the hypotheses of Theorem 1.1, the canonical map*

$$\phi_{X|D}: \mathrm{CH}_0(X|D) \to \mathrm{CH}_0^{\mathrm{Rus}}(X|D) \tag{1.4}$$

is an isomorphism.

Combining (1.4) with the main results of [8], [32] and [23, Thm. 1.4], we obtain the following reciprocity theorem for Russell's relative Chow group. Let $\pi_1^{\mathrm{abk}}(X, D)$ be the log version (see [24, Defn. 9.6] or [4, Defn. 7.2]) of the non-log abelian fundamental group with modulus $\pi_1^{\mathrm{adiv}}(X, D)$ (see [22, Defn. 7.5]). The latter group coincides with the fundamental group with modulus $\pi_1^{\mathrm{ab}}(X, D)$ used in [8] and [32].

Theorem 1.3. *Assume that k is finite and (X, D) is a reduced modulus pair over k such that one of the following conditions holds.*

1. *D is a simple normal crossing divisor.*
2. *$d \le 2$ and D is seminormal.*

Then the Frobenius substitution at the closed points of $X \smallsetminus D$ gives rise to a reciprocity isomorphism

$$\rho_{X|D}: \mathrm{CH}_0^{\mathrm{Rus}}(X|D)_0 \xrightarrow{\cong} \pi_1^{\mathrm{abk}}(X, D)_0$$

of finite groups.

When $\mathrm{char}(k) \neq 2$ in case (1), the theorem was claimed by Barrientos (see [4, Thm. 7.3]). For the proof, Barrientos only refers to the (highly intricate) arguments of Kerz-Saito [32] in the non-log case. Note that the canonical map $\pi_1^{\mathrm{adiv}}(X, D) \to \pi_1^{\mathrm{abk}}(X, D)$ is an isomorphism under the assumption of the corollary. This follows directly from definitions.

1.2.3. *Roitman's theorem for Suslin homology.* Assume that k is algebraically closed. Let (X, D) be a reduced modulus pair over k with $U = X \smallsetminus D$. Let $\mathrm{Alb}(U)$ denote the generalized Albanese variety of U, introduced by Serre [52]. This is universal for morphisms from U to semi-abelian varieties. There is an Albanese homomorphism $\mathrm{alb}_U: H_0^S(U)_0 \to \mathrm{Alb}(U)(k)$, where $H_0^S(U)_0$ is the kernel of the push-forward map $H_0^S(U)_0 \to H_0^S(\pi_0(U))$. A famous theorem of Roitman [47] says that if U is projective (i.e., $D = \varnothing$ in our set-up), then alb_U induces an isomorphism between the torsion subgroups, away from $\mathrm{char}(k)$. The latter condition was subsequently removed by Milne [41].

Spieß and Szamuely [53] showed that, away from $\mathrm{char}(k)$, alb_U induces an isomorphism between the torsion subgroups even if $D \neq \varnothing$. Geisser [16, Thm. 1.1] showed that the condition imposed by Spieß-Szamuely could be removed if one assumed resolution of singularities. Recently, Ghosh-Krishna [19, Thm. 1.7] showed that Geisser's condition could be eliminated. But their proof is long and intricate. Using Theorem 1.1, we can give a very quick proof (see § 3.6) of (the unconditional version of) the torsion theorem of Spieß-Szamuely in positive characteristic. The result is the following.

Theorem 1.4. *Let (X, D) be a reduced modulus pair over k and $U = X \smallsetminus D$. Then the Albanese map for U induces an isomorphism*

$$\mathrm{alb}_U: H_0^S(U)_{\mathrm{tor}} \xrightarrow{\cong} \mathrm{Alb}(U)(k)_{\mathrm{tor}}.$$

1.2.4. *Motivic cohomology of normal crossing schemes.* Let k be a field and let X be a reduced quasi-projective k-scheme. Let $H^m(X, \mathbb{Z}(n))$ denote the Friedlander-Voevodsky motivic cohomology of X (see § 4.2). This is an abstractly defined cohomology theory for X which is homotopy-invariant. If X is smooth over k of pure dimension d, then it is well known that there is a canonical isomorphism $\mathrm{CH}_0(X) \xrightarrow{\cong} H^{2d}(X, \mathbb{Z}(d))$. This is a special case of a more general result of Voevodsky, that identifies the motivic cohomology groups of smooth schemes over any field with the higher Chow groups as defined by Bloch. See [57, Corollary 2].

As one knows, the cohomological analogue of $\mathrm{CH}_0(X)$ is the Levine-Weibel Chow group $\mathrm{CH}_0^{LW}(X)$ when X is singular. We let Λ be a commutative ring which is $\mathbb{Z}$ if k admits resolution of singularities and is any $\mathbb{Z}[\frac{1}{p}]$-algebra if $\mathrm{char}(k) = p > 0$. The following is an open question in the theory of algebraic cycles.

Question 1.5. Let X be a seminormal[1] quasi-projective k-scheme of pure dimension d. Is there a canonical isomorphism

$$\mathrm{CH}_0^{LW}(X)_\Lambda \to H^{2d}(X, \Lambda(d))?$$

We do not know if this question may have a positive answer. We can however prove the following result using Theorem 1.1.

Let $\mathrm{CH}_0^{\mathrm{l.c.i.}}(X)$ denote the lci version of the Levine-Weibel Chow group of X as defined in [5, § 3] (see § 3.2). This is a modified form of $\mathrm{CH}_0^{LW}(X)$ with better functorial properties. As another application of Theorem 1.1, we can prove the following result with regard to the above question.

Theorem 1.6. *Let k be any field and X a reduced quasi-projective scheme of pure dimension d over k. Then the following hold.*

1. *There exists a canonical homomorphism*
$$\lambda_X \colon \mathrm{CH}_0^{\mathrm{l.c.i.}}(X) \to H^{2d}(X, \mathbb{Z}(d)).$$
2. *λ_X is surjective with Λ-coefficients if X is projective and the regular locus of X is smooth over k.*
3. *λ_X is an isomorphism with Λ-coefficients if X is a projective normal crossing scheme over k.*

The last part of Theorem 1.6 was earlier shown in [7, Thm. 1.6] if one assumes that $\mathrm{CH}_0^{LW}(X)_\Lambda \cong \mathrm{CH}_0^{\mathrm{l.c.i.}}(X)_\Lambda$ and one of the following holds.

1. k is infinite and perfect of positive characteristic.
2. $\mathrm{char}(k) = 0$ and $\Lambda = \mathbb{Z}/m, m \neq 0$.

The assumption $\mathrm{CH}_0^{LW}(X)_\Lambda \cong \mathrm{CH}_0^{\mathrm{l.c.i.}}(X)_\Lambda$ is usually very hard to check, even though it is unavoidable in [7]. Note that it is automatically satisfied if e.g. $\mathrm{char}(k) = 0$ and k is algebraically closed. We refer the reader to [7, Lemma 8.1] and the references in *loc. cit.* for a more detailed comparison.

1.2.5. *A question of Barbieri-Viale and Kahn.* Let k be an algebraically closed field of characteristic zero. As an application of the comparison between the Levine-Weibel Chow group of zero cycles and the $(2d, d)$ motivic cohomology group, we can give a positive answer to a question posed by Barbieri-Viale and Kahn in [3]. This can be interpreted as a comparison between the Roitman theorem for the cdh-motivic cohomology, proved in [3], and the more classical Roitman theorem for singular projective varieties in characteristic zero, proved in [9].

[1]If one wants to replace Λ by $\mathbb{Z}$, then one should replace seminormal by weakly normal.

Theorem 1.7. *Let k be an algebraically closed field of characteristic zero and let X be a reduced projective k-scheme of pure dimension d. Then the morphism*

$$\lambda_X : \mathrm{CH}_0^{LW}(X) \to H^{2d}(X, \mathbb{Z}(d))$$

is surjective with uniquely divisible kernel, and there is a commutative diagram

$$\begin{tikzcd} \mathrm{CH}_0^{LW}(X)_{\mathrm{tors}} \arrow[r, "\lambda_X"] \arrow[d, "a^+"] & H^{2d}(X, \mathbb{Z}(d))_{\mathrm{tors}} \arrow[d, "u"] \\ \mathrm{Alb}^+(X)(k)_{\mathrm{tors}} \arrow[r] & \mathbf{L}_1\mathrm{Alb}^*(X)(k)_{\mathrm{tors}}, \end{tikzcd}$$

where all the arrows are isomorphisms. Here, $\mathrm{Alb}^+(X)(k)$ is the universal semi-abelian regular quotient of $\mathrm{CH}_0^{\mathrm{l.c.i.}}(X)$, and $\mathbf{L}_1\mathrm{Alb}^*(X)(k)$ is the semi-abelian part of the 1-motive $\mathrm{LAlb}(M(X)^*(d)[2d])$.

We end the discussion of our main result and its application with the following larger question. Let Λ be as in § 1.2.4.

Question 1.8. Let k be any field. Let X be a smooth projective k-scheme of pure dimension d and let $D \subset X$ be a reduced simple normal crossing divisor. Let $U = X \setminus D$. Is there a canonical isomorphism

$$\mathrm{CH}^m(X|D, n)_\Lambda \xrightarrow{\cong} H_c^{2m-n}(U, \Lambda(m))?$$

1.3. **Overview of proofs.** We prove Theorem 1.1 by induction on $\dim(X)$. This reduces the proof to the case when X is a surface. The case of surfaces is the most delicate one and the main work goes into proving this case. The main steps are as follows.

We use the decomposition theorem of [25] as first of the key tools. This result provides an injective homomorphism $p_* : \mathrm{CH}_0(X|D) \to \mathrm{CH}_0^{\mathrm{l.c.i.}}(S_X)$, where S_X is the double of X along D. The proof of Theorem 1.1 is then essentially equivalent to showing that p_* factors through the quotient $\mathrm{CH}_0(X|D) \twoheadrightarrow H_0^S(U)$. The second step is to show that if we compose p_* with the pull-back $\mathrm{CH}_0^{\mathrm{l.c.i.}}(S_X) \to \mathrm{CH}_0^{\mathrm{l.c.i.}}(S_X^{sn})$, then p_* does factor through $H_0^S(U)$, where S_X^{sn} is the seminormalization of S_X. The third step is to show that this pull-back map is an isomorphism (under the given assumptions on k and D). To show the latter, we prove some results that compare Quillen's algebraic K-theory and Weibel's homotopy KH-theory for certain types of curves and surfaces.

Most of the applications given above are immediate consequences of Theorem 1.1, with the exception of Theorem 1.6. To prove Theorem 1.6, we proceed as follows. We first construct the map λ_X using the Gysin maps for Chow group and motivic cohomology. This reduces the construction to dimension one case which we deduce using the slice spectral sequence for singular schemes from [37]. The key idea then is to replace $\mathrm{CH}_0^{\mathrm{l.c.i.}}(X)$ with a cycle group $\mathrm{CH}_0^{EKW}(X)$, introduced by Esnault-Kerz-Wittenberg [13]. This is possible, thanks to Theorem 1.1. We then use a result of Cisinski-Déglise [11] on the perfection properties of various cycle groups to pass to a perfect base field. Theorem 1.6 then follows.

In § 2, we collect the K-theoretic results that we need to prove Theorem 1.1 for surfaces. In § 3, we prove the key factorization lemma which allows us to conclude the proof. We also prove Theorem 1.4 in this section. We prove Theorem 1.6 in § 4. Finally, § 5 is dedicated to the proof of Theorem 1.7.

1.4. **Notations.** Throughout this note, we fix a field k. A k-scheme will mean a separated and essentially of finite type k-scheme. We shall denote the category of such schemes by $\mathbf{Sch}_k$. We shall let $\mathbf{Sm}_k$ be the subcategory of $\mathbf{Sch}_k$ consisting of smooth schemes over k. If $X \in \mathbf{Sch}_k$ is reduced, we shall let X^n (resp. X^{sn}) denote the normalization (resp. seminormalization) of X.

Recall that for A a reduced commutative Noetherian ring and B a subring of the integral closure of A in its ring of total quotients, which is finite as A-module, we say that an ideal $I \subset A$ is a conducting ideal for the inclusion $A \subset B$ if $I = IB$. More generally, if $f: X' \to X$ is a finite birational map, a closed subscheme Y of X is called a conducting subscheme for f if the sheaf of ideals $\mathcal{I}_Y \subset \mathcal{O}_X$ is a sheaf of conducting ideals for the inclusion of sheaves of rings $\mathcal{O}_X \to f_*(\mathcal{O}_{X'})$. We shall let $k(X)$ denote the total ring of quotients of X. For a morphism $f: X' \to X$ of k-schemes and $D \subset X$ a subscheme, we shall write $D \times_X X'$ as $f^*(D)$. If $Y, Z \subset X$ are two closed subschemes, then $Y \cap Z$ will mean the scheme theoretic intersection $Y \times_X Z$ unless we say otherwise. We shall let $\mathcal{Z}_0(X)$ denote the free abelian group on the set of closed points on X.

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2. Algebraic and homotopy K-groups of a double

The goal of this section is to show that if S_X is the double of a regular surface X over a field along a reduced Cartier divisor, then $SK_0(S_X)$ coincides with an analogous subgroup of $KH_0(S_X)$ under some necessary conditions on k and D. We shall begin by recollecting necessary concepts. We shall then prove some preliminary K-theoretic results before reaching the goal. We fix a field k throughout this section.

2.1. **Review of double along a divisor.** Let $X \in \mathbf{Sch}_k$ be a regular scheme and let $D \subset X$ be an effective Cartier divisor. Recall from [5, § 2.1] that the double of X along D is the push-out $S_X := X \amalg_D X$. One knows that

$$\begin{array}{ccc} D & \xrightarrow{\iota} & X_+ \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota_+} \\ X_- & \xrightarrow{\iota_-} & S_X \end{array} \tag{2.1}$$

is a bi-Cartesian square. Moreover, there is a finite and flat morphism $\Delta: S_X \to X$ whose composition with $\iota_\pm$ are identity. S_X is a reduced Cohen-Macaulay scheme with two irreducible components $X_\pm$ and its normalization S^n_X is canonically isomorphic to $X_+ \amalg X_-$ (see again [5, Prop. 2.4]). For the normalization morphism $\pi: S^n_X \to S_X$, the smallest conductor subscheme inside S_X is D whose inverse image in S^n_X is $D \amalg D$. It follows that S_X is a seminormal scheme if D is reduced (see [35, Prop. 4.2]).

If $C_\pm$ are two closed subschemes of X not contained in D such that $C_+ \cap D = C_- \cap D$ as closed subschemes, then the join $C_+ \amalg_D C_-$ along $D \cap C_\pm$ is canonically a closed subscheme of S_X. If $\nu: C \hookrightarrow X$ is a regular closed immersion whose image is not contained in D, then the double of C along $C \cap D$ (which we shall also denote by S_C) has the property that the inclusion $\nu': S_C \hookrightarrow S_X$ is also a regular closed immersion (again by [5, Prop. 2.4(5)]). We shall use this fact often in this paper.

2.2. **Review of homotopy K-theory.** Recall (e.g., see [38, § 5]) that the homotopy K-theory spectrum (introduced by Weibel [59]) of a k-scheme is defined as the homotopy colimit spectrum $KH(X) = \mathrm{hocolim}_n K(X \times \Delta_n)$, where $\Delta_\bullet$ is the standard cosimplicial scheme defined by setting $\Delta_n = \mathrm{Spec}\,(k[t_0, \dots, t_n]/(\sum_i t_i - 1))$. There is a natural transformation between the presheaves of S^1-spectra $K(X) \to KH(X)$ on $\mathbf{Sch}_k$, which is a weak equivalence if X is regular. Furthermore, if $f: X' \to X$ is a proper local complete intersection morphism (or, more generally, a morphism of finite Tor-dimension), then so is $f \times \mathrm{id}: X' \times \Delta_\bullet \to X \times \Delta_\bullet$. It follows from [56, Prop. 3.18] that there is a push-forward map between the simplicial spectra $f_*: K(X' \times \Delta_\bullet) \to K(X \times \Delta_\bullet)$. Taking the homotopy colimits, we see that there is a push-forward map $f_*: KH(X') \to KH(X)$. This map satisfies usual properties such as the composition law and commutativity with pull-back.

Let $\tau: (\mathbf{Sch}_k)_{\mathrm{cdh}} \to (\mathbf{Sch}_k)_{\mathrm{zar}}$ be the canonical morphism of sites, where $(\mathbf{Sch}_k)_{\mathrm{cdh}}$ denotes the category $\mathbf{Sch}_k$ equipped with the cdh topology (e.g., see [43, Chap. 12]). Since $KH(X)$ is homotopy equivalent to the cdh-fibrant replacement of the spectrum $K(X)$ (see [10], [26]), there is a commutative diagram of strongly convergent spectral sequences

$$\begin{array}{ccc} E_2^{p,q} = H^p_{\mathrm{zar}}(X, \mathcal{K}_{q,X}) & \Rightarrow & K_{q-p}(X) \\ \big\downarrow & & \big\downarrow \\ {'E}_2^{p,q} = H^p_{\mathrm{cdh}}(X, \mathcal{K}_{q,X}) & \Rightarrow & KH_{q-p}(X), \end{array} \tag{2.2}$$

where the top one is the Zariski descent spectral sequence due to Thomason-Trobaugh [56, Thm. 10.3].

Let us describe the edge homomorphisms of these spectral sequences in low degrees. First, there is a natural map $\mathrm{rk}: KH_0(X) \to H^0(X, \mathbb{Z})$ whose composition with $K_0(X) \to KH_0(X)$ is the (classically defined) rank map. We let $\widetilde{K}_0(X)$ and $\widetilde{KH}_0(X)$ denote the respective kernels. Using the above spectral sequences again, we get a natural map $\det: \widetilde{KH}_0(X) \to H^1_{\mathrm{cdh}}(X, \mathcal{O}_X^\times)$, which is surjective if $\dim(X) \le 2$. We let $SKH_0(X)$ denote its kernel. We let $SK_0(X)$ be the kernel of the (surjective) determinant map $\det: \widetilde{K}_0(X) \twoheadrightarrow H^1_{\mathrm{zar}}(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X)$ ([58, Thm. II.8.1]).

Applying the above spectral sequences to $KH_1(X)$, we get an edge map $KH_1(X) \to H^0_{\mathrm{cdh}}(X, \mathcal{O}_X^\times)$. We let $SKH_1(X)$ denote its kernel. Similarly, we let $SK_1(X)$ denote the kernel of the edge map $K_1(X) \to H^0_{\mathrm{zar}}(X, \mathcal{O}_X^\times)$. Let $X^{sn} \to X$ denote the seminormalization morphism when X is reduced (see [35, § 4.1]).

Lemma 2.1. *Let $X \in \mathbf{Sch}_k$ be a reduced scheme. Then we have the following.*

(1) *The canonical map $H^0_{\mathrm{zar}}(X, \mathcal{O}_X^\times) \to H^0_{\mathrm{cdh}}(X, \mathcal{O}_X^\times)$ has a factorization*

$$\begin{tikzcd} H^0_{\mathrm{zar}}(X, \mathcal{O}_X^\times) \arrow[r] \arrow[d] & H^0_{\mathrm{zar}}(X^{sn}, \mathcal{O}_{X^{sn}}^\times) \arrow[d, "\cong"] \arrow[dl, dotted] \\ H^0_{\mathrm{cdh}}(X, \mathcal{O}_X^\times) \arrow[r, "\cong"] & H^0_{\mathrm{cdh}}(X^{sn}, \mathcal{O}_{X^{sn}}^\times), \end{tikzcd}$$

where the horizontal arrows are induced by the projection $X^{sn} \to X$ and the vertical arrows are induced by the change of topology. Moreover, the bottom horizontal and the right vertical arrows are isomorphisms.

(2) *There is a commutative diagram of short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & SK_1(X) \arrow[r] \arrow[d] & K_1(X) \arrow[r] \arrow[d] & H^0_{\mathrm{zar}}(X, \mathcal{O}^{\times}_X) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & SKH_1(X) \arrow[r] & KH_1(X) \arrow[r] & H^0_{\mathrm{cdh}}(X, \mathcal{O}^{\times}_X) \arrow[r] & 0, \end{tikzcd} \tag{2.3}$$

which split functorially in X.

Proof. The first part is well known (e.g., apply [28, Prop. 6.14] with $Y = \mathbb{G}_m$). It is shown in [34, Lem. 2.1] that the top sequence is split exact such that the splitting is functorial in X. The bottom sequence is left exact by definition. We now show that it is actually split exact.

Using [59, Prop. 3.2] and Zariski descent for KH-theory, it follows that the canonical map $KH(X) \to KH(X^{sn})$ is a weak equivalence. Hence, we can assume X to be seminormal to prove the split exactness of the bottom sequence in (2.3). Using the split exact property of the top sequence, it suffices to show that the change of topology map $H^0_{\mathrm{zar}}(X, \mathcal{O}^{\times}_X) \to H^0_{\mathrm{cdh}}(X, \mathcal{O}^{\times}_X)$ is an isomorphism. But this is the first part of the lemma. ☐

The spectral sequences of (2.2) imply that there is a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{zar}}(X, \mathcal{K}_{i+1,X}) \arrow[r] \arrow[d] & K_i(X) \arrow[r] \arrow[d] & H^0_{\mathrm{zar}}(X, \mathcal{K}_{i,X}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^1_{\mathrm{cdh}}(X, \mathcal{K}_{i+1,X}) \arrow[r] & KH_i(X) \arrow[r] & H^0_{\mathrm{cdh}}(X, \mathcal{K}_{i,X}) \arrow[r] & 0 \end{tikzcd} \tag{2.4}$$

for every $i \geq 0$ if $\dim(X) \leq 1$. Combining Lemma 2.1 and (2.4), we get

$$SK_1(X) \cong H^1_{\mathrm{zar}}(X, \mathcal{K}_{2,X}) \quad \text{and} \quad SKH_1(X) \cong H^1_{\mathrm{cdh}}(X, \mathcal{K}_{2,X}) \tag{2.5}$$

if $\dim(X) \leq 1$.

For a closed immersion $W \subset Z$ in $\mathbf{Sch}_k$, we let $K(Z, W)$ be the relative homotopy K-theory spectrum of the pair (Z, W). It is defined as the homotopy fiber of the restriction map of spectra $K(Z) \to K(W)$. If $f: Z' \to Z$ is a morphism of k-schemes such that $W' = W \times_Z Z'$, we let $K(Z, Z', W)$ (the double relative K-theory spectrum) denote the homotopy fiber of the canonical pull-back map $f^*: K(Z, W) \to K(Z', W')$. We define $KH(Z, W)$ and $KH(Z, Z', W)$ in analogous fashion.

2.3. **Algebraic and homotopy K_2-groups of normal crossing curves.** We shall now compare $K_2(X)$ and $KH_2(X)$ when X is a normal crossing curve. We first recall the definition of normal crossing schemes that we shall use in this paper.

Let $X \in \mathbf{Sch}_k$ be a reduced scheme of pure dimension $d \geq 0$. Let $\{X_1, \ldots, X_n\}$ be the set of irreducible components of X. We shall say that X is a normal crossing k-scheme if for every nonempty subset $J \subset [1, n]$, the scheme theoretic intersection $X_J := \bigcap\limits_{i \in J} X_i$ is either empty or a smooth k-scheme of pure dimension $d + 1 - |J|$. Recall that $X \in \mathbf{Sch}_k$ is called K_i-regular if the map $K_i(X) \to K_i(X \times \Delta_n)$, induced by the projection, is an isomorphism for all $n \geq 0$.

Lemma 2.2. *Let $X \in \mathbf{Sch}_k$ be a normal crossing curve. Then X is K_i-regular for $i \leq 1$.*

Proof. It is well known (e.g., use the Bass fundamental exact sequence) that the lemma is equivalent to the assertion that X is K_1-regular. We first assume that X is affine. We let $\mu(X)$ denote the number of irreducible components of X and write $X(n) = X \times \Delta_n$. We shall prove K_1-regularity of X by induction on $\mu(X)$. The case $\mu(X) = 1$ is trivial because X is then smooth, and one knows that smooth (more generally, regular) schemes are K_i-regular for all i. Let us now assume that $\mu(X) > 1$. We let X_1 be an irreducible component of X and let X_2 be the scheme theoretic closure of $X \setminus X_1$ in X. We let $Y = X_1 \cap X_2$. Then X_2 is a normal crossing curve such that $\mu(X_2) = \mu(X) - 1$ and Y is a 0-dimensional smooth k-scheme. We have a commutative square

$$\begin{tikzcd} Y(n) \arrow[r] \arrow[d] & X_1(n) \arrow[d] \\ X_2(n) \arrow[r] & X(n) \end{tikzcd} \tag{2.6}$$

of affine schemes for every $n \geq 0$, which is Cartesian as well as co-Cartesian and in which all arrows are closed immersions.

By [42, Thm. 6.4], there exists a commutative diagram of exact sequences

$$\begin{tikzcd} K_2(Y) \arrow[r] \arrow[d] & K_1(X) \arrow[r] \arrow[d, "p_X^*"] & K_1(X_1) \oplus K_1(X_2) \arrow[r] \arrow[d] & K_1(Y) \arrow[d] \\ K_2(Y(n)) \arrow[r] & K_1(X(n)) \arrow[r] & K_1(X_1(n)) \oplus K_1(X_2(n)) \arrow[r] & K_1(Y(n)), \end{tikzcd} \tag{2.7}$$

in which the vertical arrows are induced by the projection maps. The left-most and the right-most vertical arrows are isomorphisms because Y is smooth. The vertical arrow involving X_1 and X_2 is an isomorphism by induction on $\mu(X)$. It follows via a diagram chase that p_X^* is surjective. As this map is always (split) injective, the affine case of the lemma follows.

If X is not necessarily affine, we choose a dense open affine subscheme $U \subset X$ such that $X_{\mathrm{sing}} \subset U$ and let $Y = X \setminus U$ with the reduced induced closed subscheme structure. Then Y is a regular k-scheme. Using the Thomason-Trobaugh localization sequence ([56, Thm. 7.4])

$$K^{Y(n)}(X(n)) \to K(X(n)) \to K(U(n))$$

and the weak equivalence $K(Y(n)) \xrightarrow{\sim} K^{Y(n)}(X(n))$ (this uses excision and the fact that $Y = Y_{\mathrm{reg}} \subset X_{\mathrm{reg}}$), we get a commutative diagram of exact sequence of homotopy groups

$$\begin{tikzcd} K_1(Y) \arrow[r] \arrow[d] & K_1(X) \arrow[r] \arrow[d, "p_X^*"] & K_1(U) \arrow[r] \arrow[d, "p_U^*"] & K_0(Y) \arrow[d] \\ K_1(Y(n)) \arrow[r] & K_1(X(n)) \arrow[r] & K_1(U(n)) \arrow[r] & K_0(Y(n)). \end{tikzcd} \tag{2.8}$$

The left-most and the right-most vertical arrows are isomorphisms because Y is regular. The arrow p_U^* is an isomorphism because U is affine. It follows via a diagram chase that p_X^* is surjective. As this map is always (split) injective, the lemma follows. □

Lemma 2.3. *Let $X \in \mathbf{Sch}_k$ be a normal crossing curve. Then the canonical map $K_2(X) \to KH_2(X)$ is surjective. In particular, the map $H^0_{\mathrm{zar}}(X, \mathcal{K}_{2,X}) \to H^0_{\mathrm{cdh}}(X, \mathcal{K}_{2,X})$ is surjective.*

Proof. It follows from the spectral sequences (2.2) that the edge maps $K_2(X) \to H^0_{\mathrm{zar}}(X, \mathcal{K}_{2,X})$ and $KH_2(X) \to H^0_{\mathrm{cdh}}(X, \mathcal{K}_{2,X})$ are surjective. Hence, we only need to prove the first assertion of the lemma. In view of Lemma 2.2, the spectral sequence

$$E_1^{p,q} = K_q(X \times \Delta_p) \Rightarrow KH_{p+q}(X),$$

degenerates to an exact sequence

$$K_2(X \times \Delta_1) \xrightarrow{\partial_1^* - \partial_0^*} K_2(X) \to KH_2(X) \to 0. \tag{2.9}$$

This finishes the proof. □

2.4. **Algebraic and homotopy K_1-groups of curves.** For the rest of § 2, we shall work with the following set-up. We let X be a regular integral quasi-projective surface over k and $D \subset X$ a reduced effective Cartier divisor. Recall that S_X denotes the double of X along D. The goal of this subsection is to prove the following two lemmas.

Lemma 2.4. *The map $SK_1(D) \to SKH_1(D)$ is an isomorphism under any of the following conditions.*

(1) *k is perfect and D is seminormal.*
(2) *$k \subseteq \overline{\mathbb{Q}}$.*

Proof. Let $\pi: D^n \to D$ be the normalization morphism. Let $E \subset D$ be a conducting closed subscheme for π and let $E' = \pi^*(E)$. We then have an exact sequence of relative and double relative K-groups:

$$\cdots \to K_i(D, D^n, E) \to K_i(D, E) \to K_i(D^n, E') \to K_{i-1}(D, D^n, E) \to \cdots. \tag{2.10}$$

Since $\dim(D) = 1$ and D is reduced, the conducting subscheme E is supported on a finite set of closed points (hence it is affine). It follows from [17, Thm. 0.2] and the Thomason-Trobaugh descent spectral sequence [56] that $K_0(D, D^n, E) = 0$ and $K_1(D, D^n, E) \cong \mathcal{I}_E/\mathcal{I}_E^2 \otimes_{E'} \Omega^1_{E'/E}$, where $\mathcal{I}_E$ is the Zariski sheaf of ideals on D defining E.

Assume that (1) holds. Since D is seminormal, we can choose our conducting subschemes E and E' to be reduced (in fact, E can be chosen to be $V(\mathcal{I})$ where $\mathcal{I}$ is the largest conducting ideal for the map $\mathcal{O}_D \subset \pi_* \mathcal{O}_{D^n}$, see [35, Prop. 4.2(1)]). It follows that the coordinate rings of E and E' are finite products of finite and separable extensions of k, and that the extension E'/E is separable (this uses the perfectness hypothesis). It follows that $\Omega^1_{E'/E} = 0$, hence $K_1(D, D^n, E) = 0$. This gives a Mayer-Vietoris exact sequence

$$K_2(D^n) \oplus K_2(E) \to K_2(E') \to SK_1(D) \to SK_1(D^n) \to 0, \tag{2.11}$$

where E' is reduced (note that $SK_1(E) = SK_1(E') = 0$ since E and E' are semilocal).

If D is not seminormal, the conducting subscheme E cannot be chosen to be reduced. However, for m sufficiently large, we have that $E \subset mE_{\mathrm{red}}$ and $E' \subset mE'_{\mathrm{red}}$. It follows from the above expression of $K_1(D, D^n, E)$ that there is some conducting closed subscheme $E \subset D$, having the same support as that of the maximal conducting subscheme such that one has an exact Mayer-Vietoris sequence

$$K_2(D^n) \oplus K_2(E) \to K_2(E') \to SK_1(D) \to SK_1(D^n) \to 0. \tag{2.12}$$

We are interested in estimating the term $K_2(E')$. We claim that if (2) holds then $K_2(E') = K_2(E'_{\mathrm{red}})$. Since D^n is a normal curve, the coordinate ring A of E' is a finite product $A = \prod_{i=1}^r A_i$ of Artinian k-algebras A_i, each of which is isomorphic to a truncated polynomial ring of the form $A_i = k_i[t]/(t^{n_i})$, where k_i/k is a finite field extension and $n_i \geq 1$ is an integer (this follows, for example, from Cohen's structure theorem). In order to prove the claim, we can clearly assume that $r = 1$. Let k' be the residue field of A.

Let J be the kernel of the augmentation ideal $A \to k'$ and write $\Omega^1_{(A,J)}$ for the kernel of the map $\Omega^1_{A/\mathbb{Z}} \to \Omega^1_{k'/\mathbb{Z}}$. If k is a $\mathbb{Q}$-algebra, a result of Bloch (see, e.g., [39, Thm. 4.1]) gives an isomorphism

$$K_2(A, J) \cong HC_1(A, J) \cong \Omega^1_{(A,J)}/d(J).$$

By computing these groups for truncated polynomial rings, we conclude that if $k \subset \overline{\mathbb{Q}}$, then $K_2(A, J) = 0$. Since the map $K_2(A) \to K_2(A/J) = K_2(k')$ is anyway surjective, we conclude that $K_2(A) = K_2(A/J) = K_2(k')$ in this case. This proves the claim. It follows from the claim that (2.12) is of the form

$$K_2(D^n) \oplus K_2(E) \to K_2(E'_{\mathrm{red}}) \to SK_1(D) \to SK_1(D^n) \to 0. \tag{2.13}$$

We now compare the sequences (2.11) and (2.13) with the corresponding ones for KH. Since KH-theory satisfies *cdh*-descent (see [10], [26]), we always have an exact sequence (see [58, Cor. IV. 12.6])

$$KH_2(D^n) \oplus KH_2(E_{\mathrm{red}}) \to KH_2(E'_{\mathrm{red}}) \to SKH_1(D) \to SKH_1(D^n) \to 0. \tag{2.14}$$

Putting things together, we get a commutative diagram (in both cases (1) and (2)) with exact rows

$$\begin{tikzcd} K_2(D^n) \oplus K_2(E) \arrow[r] \arrow[d] & K_2(E'_{\mathrm{red}}) \arrow[r] \arrow[d] & SK_1(D) \arrow[r] \arrow[d] & SK_1(D^n) \arrow[r] \arrow[d] & 0 \\ KH_2(D^n) \oplus KH_2(E_{\mathrm{red}}) \arrow[r] & KH_2(E'_{\mathrm{red}}) \arrow[r] & SKH_1(D) \arrow[r] & SKH_1(D^n) \arrow[r] & 0. \end{tikzcd} \tag{2.15}$$

The left vertical arrow is surjective, since $K_2(D^n) = KH_2(D^n)$ (because D^n is regular) and $K_2(E) \to K_2(E_{\mathrm{red}}) = KH_2(E_{\mathrm{red}})$ is surjective because E is semi-local. The second vertical arrow (from left) is an isomorphism, since E'_{red} is regular. Similarly, the right vertical arrow is an isomorphism. A diagram chase now finishes the proof. □

Lemma 2.5. *The map $SK_1(D) \to SKH_1(D)$ is an isomorphism if k is infinite and D is a normal crossing curve.*

Proof. By (2.5), the lemma is equivalent to showing that the map

$$H^1_{\mathrm{zar}}(D, \mathcal{K}_{2,D}) \to H^1_{\mathrm{cdh}}(D, \mathcal{K}_{2,D}) \tag{2.16}$$

is an isomorphism. Let $\mu(D)$ denote the number of irreducible components of D. We shall prove the above isomorphism of cohomology groups by induction on $\mu(D)$. If $\mu(D) = 1$, then D is regular in which case (2.16) is clear, since $K_1(D) = KH_1(D)$. Otherwise, we let D_1 be an irreducible component of D and let D_2 be the scheme theoretic closure of $D \setminus D_1$ in D. Then D_2 is a normal crossing curve such that $\mu(D_2) < \mu(D)$ and $D_3 := D_1 \cap D_2$ is a 0-dimensional smooth k-scheme.

We now consider the commutative diagram of Zariski sheaves

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_{2,(D,D_2)} \arrow[r] \arrow[d] & \mathcal{K}_{2,D} \arrow[r] \arrow[d] & \mathcal{K}_{2,D_2} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{K}_{2,(D_1,D_3)} \arrow[r] & \mathcal{K}_{2,D_1} \arrow[r] & \mathcal{K}_{2,D_3} \arrow[r] & 0, \end{tikzcd} \tag{2.17}$$

where the terms $\mathcal{K}_{2,(D,D_2)}$ and $\mathcal{K}_{2,(D_1,D_3)}$ in the first and the second row are defined to be the kernels of the right horizontal arrows. Since k is infinite, the Quillen $\mathcal{K}_2$-sheaf coincides with the Milnor $\mathcal{K}_2^M$-sheaf on the big Zariski site of $\mathbf{Sch}_k$ (see [31, Prop. 2]). In particular, sections of $\mathcal{K}_2$ are given by symbols, thus the two right horizontal arrows of (2.17) are indeed surjective and thus both rows in (2.17) are exact.

By [31, Prop. 2] again, the terms $\mathcal{K}_{2,(D,D_2)}$ and $\mathcal{K}_{2,(D_1,D_3)}$ coincide with the relative Milnor K-sheaves of Kato-Saito [29, 1.3]. In particular, the map $\mathcal{K}^M_{2,(D,D_2)} \to \mathcal{K}^M_{2,(D_1,D_3)}$ is surjective by [29, Lem. 1.3.1]. Furthermore, the kernel of this map is supported on D_3. It follows that the left vertical arrow in the above diagram is surjective whose kernel is supported on D_3. In particular, the induced map between the first Zariski cohomology groups is an isomorphism.

Using a diagram chase of the cohomology groups induced by (2.17), we therefore get a Mayer-Vietoris exact sequence for the Zariski cohomology in low degrees:

$$\begin{matrix} H^0_{\mathrm{zar}}(D_1, \mathcal{K}_{2,D_1}) \\ \oplus \\ H^0_{\mathrm{zar}}(D_2, \mathcal{K}_{2,D_2}) \end{matrix} \to H^0_{\mathrm{zar}}(D_3, \mathcal{K}_{2,D_3}) \to H^1_{\mathrm{zar}}(D, \mathcal{K}_{2,D}) \to \begin{matrix} H^1_{\mathrm{zar}}(D_1, \mathcal{K}_{2,D_1}) \\ \oplus \\ H^1_{\mathrm{zar}}(D_2, \mathcal{K}_{2,D_2}) \end{matrix} \to 0. \tag{2.18}$$

We next observe that the square

$$\begin{tikzcd} D_3 \arrow[r] \arrow[d] & D_1 \arrow[d] \\ D_2 \arrow[r] & D \end{tikzcd} \tag{2.19}$$

defines a cdh cover $\{D_1 \amalg D_2 \to D\}$ of D, so that we have an associated exact sequence of cdh cohomology groups similar to (2.18). Comparing these two sequences, we get a commutative diagram of exact sequences

$$\begin{tikzcd} \begin{matrix} H^0_{\mathrm{zar}}(D_1, \mathcal{K}_{2,D_1}) \\ \oplus \\ H^0_{\mathrm{zar}}(D_2, \mathcal{K}_{2,D_2}) \end{matrix} \arrow[r] \arrow[d] & H^0_{\mathrm{zar}}(D_3, \mathcal{K}_{2,D_3}) \arrow[r] \arrow[d] & H^1_{\mathrm{zar}}(D, \mathcal{K}_{2,D}) \arrow[r] \arrow[d] & \begin{matrix} H^1_{\mathrm{zar}}(D_1, \mathcal{K}_{2,D_1}) \\ \oplus \\ H^1_{\mathrm{zar}}(D_2, \mathcal{K}_{2,D_2}) \end{matrix} \arrow[r] \arrow[d] & 0 \\ \begin{matrix} H^0_{\mathrm{cdh}}(D_1, \mathcal{K}_{2,D_1}) \\ \oplus \\ H^0_{\mathrm{cdh}}(D_2, \mathcal{K}_{2,D_2}) \end{matrix} \arrow[r] & H^0_{\mathrm{cdh}}(D_3, \mathcal{K}_{2,D_3}) \arrow[r] & H^1_{\mathrm{cdh}}(D, \mathcal{K}_{2,D}) \arrow[r] & \begin{matrix} H^1_{\mathrm{cdh}}(D_1, \mathcal{K}_{2,D_1}) \\ \oplus \\ H^1_{\mathrm{cdh}}(D_2, \mathcal{K}_{2,D_2}) \end{matrix} \arrow[r] & 0. \end{tikzcd} \tag{2.20}$$

The left vertical arrow is surjective by Lemma 2.3. The second vertical arrow (from left) is an isomorphism because D_3 is smooth. The right vertical arrow is an isomorphism by induction on $\mu(D)$. By a diagram chase, it follows that (2.16) is an isomorphism. This concludes the proof of the lemma. □

2.5. **Algebraic and homotopy K_0-groups of the double.** We continue with the set-up described in § 2.4. In this subsection, we shall compare $SK_0(S_X)$ with the analogous subgroup of $KH_0(S_X)$. Let $\pi\colon S^n_X \to S_X$ denote the normalization map.

Lemma 2.6. *There exists an exact sequence*

$$0 \to \frac{SK_1(S^n_X)}{SK_1(S_X)} \to SK_1(D) \to SK_0(S_X) \xrightarrow{\pi^*} SK_0(S^n_X) \to 0.$$

Proof. This is a consequence of (the proof of) [2, Thm. 3.3] or [23, Prop. 2.7], noting that we can choose Y (in the notation of op. cit.) to be D. The claimed exact sequence exists if excision holds for the K_0. The obstruction for this excision is controlled by $\mathcal{I}_D/\mathcal{I}^2_D \otimes_{D'} \Omega^1_{D'/D}$. As $D' = D \amalg D$, this term vanishes. □

Remark 2.7. It is worth noting that the proof of Lemma 2.6 did not use our assumption that D is reduced. Hence, the lemma remains valid for any effective Cartier divisor D.

The analogue of Lemma 2.6 also holds for the KH-groups by the cdh-descent, as we show now. We consider the abstract blow-up square

$$
\begin{tikzcd}
D \amalg D \arrow[r] \arrow[d] & S_X^n \arrow[r, "\cong"] \arrow[d, "\pi"] & X_+ \amalg X_- \arrow[dl] \\
D \arrow[r] & S_X &
\end{tikzcd}
\tag{2.21}
$$

Applying the spectral sequence (2.2) and the cdh-excision [58, IV.12.6] to this abstract blow-up square, we get a commutative diagram of exact sequences

(2.22)

$$
\begin{tikzcd}
KH_1(S_X) \arrow[r] \arrow[d] & KH_1(S_X^n) \arrow[r] \arrow[d] & KH_1(D) \arrow[r] \arrow[d] & \widetilde{KH}_0(S_X) \arrow[r] \arrow[d] & \widetilde{KH}_0(S_X^n) \arrow[d] \\
H^0_{\mathrm{cdh}}(S_X, \mathcal{O}^\times_{S_X}) \arrow[r] & H^0_{\mathrm{cdh}}(S_X^n, \mathcal{O}^\times_{S_X^n}) \arrow[r] & H^0_{\mathrm{cdh}}(D, \mathcal{O}^\times_{D}) \arrow[r] & H^1_{\mathrm{cdh}}(S_X, \mathcal{O}^\times_{S_X}) \arrow[r] & H^1_{\mathrm{cdh}}(S_X^n, \mathcal{O}^\times_{S_X^n}).
\end{tikzcd}
$$

It follows from Lemma 2.1 that the first three vertical arrows from left in (2.22) are surjective. The map $H^0_{\mathrm{cdh}}(S_X, \mathcal{O}^\times_{S_X}) \to H^0_{\mathrm{cdh}}(S_X^n, \mathcal{O}^\times_{S_X^n})$ is clearly injective because S_X is reduced and $S_X^n \to S_X$ is a cdh cover. Using a diagram chase and taking the kernels of the vertical arrows, we get an exact sequence

$$
0 \to \frac{SKH_1(S_X^n)}{SKH_1(S_X)} \to SKH_1(D) \to SKH_0(S_X) \to SKH_0(S_X^n). \tag{2.23}
$$

The main result of § 2 is the following.

Proposition 2.8. *The map $SK_0(S_X) \to SKH_0(S_X)$ is an isomorphism under any of the following conditions.*

1. *k is perfect and D is seminormal.*
2. *$k \subseteq \overline{\mathbb{Q}}$.*
3. *k is infinite and D is a normal crossing curve.*

Proof. A comparison of (2.23) with the exact sequence of Lemma 2.6 gives rise to a commutative diagram of exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & \frac{SK_1(S_X^n)}{SK_1(S_X)} \arrow[r] \arrow[d, two heads] & SK_1(D) \arrow[r] \arrow[d, "\cong"] & SK_0(S_X) \arrow[r, "\pi^*"] \arrow[d] & SK_0(S_X^n) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \frac{SKH_1(S_X^n)}{SKH_1(S_X)} \arrow[r] & SKH_1(D) \arrow[r] & SKH_0(S_X) \arrow[r, "\pi^*"] & SKH_0(S_X^n) \arrow[r] & 0.
\end{tikzcd}
\tag{2.24}
$$

Note that π^* on the bottom is surjective because the same holds for the corresponding arrow on the top and the right vertical arrow is an isomorphism by the regularity of the scheme S_X^n. The latter also implies that the left vertical arrow is surjective. The vertical arrow involving D is an isomorphism by Lemmas 2.4 and 2.5. The desired assertion now follows by a diagram chase. □

3. Proof of the main result

In this section, we shall prove our main result Theorem 1.1. We shall also give proofs of some of its applications. We begin by recalling the definitions of the Chow group of 0-cycles with modulus and Suslin homology. To prove Theorem 1.1, we shall use two other 0-cycle groups, namely, the Levine-Weibel Chow group and its modified version called the lci Chow group of the double. We shall recall these too. We fix a field k.

3.1. **Review of Chow group with modulus and Suslin homology.** Let X be an integral quasi-projective k-scheme of dimension $d \geq 1$ and let $D \subset X$ be an effective Cartier divisor. Let $j: U \hookrightarrow X$ be the inclusion of the complement of D in X. Assume that U is regular. Recall from [32, § 1] that the Chow group of 0-cycles on X with modulus D is the quotient of $\mathcal{Z}_0(U)$ by the subgroup $\mathcal{R}_0(X|D)$ generated by $\nu_*(\mathrm{div}(f))$, where $\nu: C \to X$ is a finite (and birational to its image) morphism from an integral normal curve C whose image is not contained in D and $f \in \mathrm{Ker}(\mathcal{O}^\times_{C,\nu^{-1}(D)} \twoheadrightarrow \mathcal{O}^\times_{\nu^*(D)})$. This group is denoted by $\mathrm{CH}_0(X|D)$.

Recall that the Suslin-Voevodsky singular homology $H^S_n(U)$ of U (also called Suslin homology in the literature) is defined as the n-th homology of a certain explicit complex of algebraic cycles, introduced by Suslin and Voevodsky [54]. We do not need to recall this complex. Instead, we shall use the following equivalent definition of $H^S_0(U)$ in this paper. This equivalence was shown by Schmidt [50, Thm. 5.1].

Lemma 3.1. *Assume that X is projective. Then $H^S_0(U)$ is canonically isomorphic to the quotient of $\mathcal{Z}_0(U)$ by the subgroup $\mathcal{R}^S_0(U)$ generated by $\nu_*(\mathrm{div}(f))$, where $\nu: C \to X$ is a finite (and birational to its image) morphism from an integral normal curve C whose image is not contained in D and $f \in \mathrm{Ker}(\mathcal{O}^\times_{C,\nu^{-1}(D)} \twoheadrightarrow \mathcal{O}^\times_{\nu^*(D)_{\mathrm{red}}})$.*

It is clear that there is a canonical surjection

$$\phi_{X|D}: \mathrm{CH}_0(X|D) \twoheadrightarrow H^S_0(U). \tag{3.1}$$

From the definition one gets immediately that $\phi_{X|D}$ is an isomorphism if X is of dimension 1 and D is reduced. It was shown in [7, Thm. 4.4] that $\phi_{X|D}$ may have a non-trivial kernel, even if D is reduced and k is algebraically closed as soon as $\dim(X) \geq 2$, so that the relationship between the two objects is quite subtle. This relationship is the main object of study in this paper.

3.2. **Review of Levine-Weibel and lci Chow groups.** Let X be an equidimensional reduced quasi-projective k-scheme of dimension $d \geq 1$. Let X_{sing} denote the singular locus of X with reduced closed subscheme structure and let X_{reg} denote the complement of X_{sing} in X. Let $C \subset X$ be a curve (i.e., an equidimensional one-dimensional k-scheme). Recall (see [40, § 1]) that C is called a Cartier curve on X if no component of C lies in X_{sing}, no embedded point of C lies away from X_{sing}, $\mathcal{O}_{C,\eta}$ is a field if $\overline{\{\eta\}}$ is a component of C disjoint from X_{sing} and, C is defined by a regular sequence at every point of $C \cap X_{\mathrm{sing}}$. We let $k(C, C \cap X_{\mathrm{sing}})^\times$ be the image of the natural map $\mathcal{O}^\times_{C,S} \to \overset{s}{\underset{i=1}{\oplus}} \mathcal{O}^\times_{C,\eta_i}$, where $\{\eta_1, \dots, \eta_s\}$ is the set of generic points of C and S is the union of the closed subset $C \cap X_{\mathrm{sing}}$ and the set of generic points η_i of C such that $\overline{\{\eta_i\}}$ is disjoint from X_{sing}.

For $f \in k(C, C \cap X_{\mathrm{sing}})^\times$, we let $\mathrm{div}(f) = \sum_{i=1}^{s} \mathrm{div}(f_i)$, where f_i is the projection of f onto $\mathcal{O}^\times_{C,\eta_i}$, and $\mathrm{div}(f_i)$ is the divisor of the restriction of f_i to the maximal Cohen-Macaulay subscheme C_i of C supporting η_i. If C is reduced, then $k(C, C \cap X_{\mathrm{sing}})^\times = \mathcal{O}^\times_{C,S}$ and for $f \in \mathcal{O}^\times_{C,S}$, $\mathrm{div}(f)$ is the sum of $\mathrm{div}(f_i)$, where the sum runs through the divisors (in the classical sense, see [15, Chap. 1]) of the restrictions of f to the components of C. The Levine-Weibel Chow group $\mathrm{CH}^{LW}_0(X)$ is the quotient of $\mathcal{Z}_0(X_{\mathrm{reg}})$ by the subgroup $\mathcal{R}^{LW}_0(X)$ generated by $\mathrm{div}(f)$, where $f \in k(C, C \cap X)_{\mathrm{sing}})^\times$ for a Cartier curve C on X.

One says that C a good curve (relative to X_{sing}) if it is reduced and there is a finite local complete intersection (lci) morphism $\nu: C \to X$ such that $\nu^{-1}(X_{\mathrm{sing}})$ is nowhere dense in

C. The lci Chow group of 0-cycles $\mathrm{CH}_0^{\mathrm{l.c.i.}}(X)$ is the quotient of $\mathcal{Z}_0(X_{\mathrm{reg}})$ by the subgroup $\mathcal{R}_0^{\mathrm{l.c.i.}}(X)$ generated by $\nu_*(\mathrm{div}(f))$, where $\nu: C \to X$ is a good curve and $f \in k(C, \nu^{-1}(X_{\mathrm{sing}}))^\times$. We let $\mathrm{CH}_0^F(X)$ denote the classical homological Chow group of 0-cycles on X as defined in [15, Chap. 1]. Clearly, there are canonical maps $\mathrm{CH}_0^{LW}(X) \twoheadrightarrow \mathrm{CH}_0^{\mathrm{l.c.i.}}(X) \to \mathrm{CH}_0^F(X)$.

3.3. **The fundamental exact sequence.** Assume that X is a regular quasi-projective scheme and $D \subset X$ is an effective Cartier divisor with complement U. To prove Theorem 1.1, we shall use the following fundamental exact sequence (see [5, Thm. 1.9] when k is perfect, [8, Thm. 2.11] if $\dim(X) = 2$ and [25, Thm. 1.1] in the general case).

Theorem 3.2. *There is a split short exact sequence*

$$0 \to \mathrm{CH}_0(X|D) \xrightarrow{p_*} \mathrm{CH}_0^{\mathrm{l.c.i.}}(S_X) \xrightarrow{\iota^*} \mathrm{CH}_0^F(X) \to 0.$$

In this sequence, p_* takes a 0-cycle on U identically onto U_+ and ι^* takes a 0-cycle on $U_+ \amalg U_-$ onto U_- via projection.

In order to prove our main result, we shall modify slightly the set of relations used to define the Kerz-Saito Chow group of zero-cycles with modulus in the spirit of the Levine-Weibel Chow group. We proceed as follows. We let $\mathrm{CH}_0^{LW}(X|D)$ be the quotient of $\mathcal{Z}_0(U)$ by the subgroup $\mathcal{R}_0^{LW}(X|D)$ generated by $\mathrm{div}(f)$, where

(1) $C \subset X$ is an integral curve with the property that $C \not\subset D$;
(2) C is regular at every point of $E := C \cap D$;
(3) $f \in \mathrm{Ker}(\mathcal{O}_{C,E}^\times \twoheadrightarrow \mathcal{O}_E^\times)$.

Clearly, the difference between $\mathrm{CH}_0^{LW}(X|D)$ and $\mathrm{CH}_0(X|D)$ is in the requirement that the the curves giving the rational equivalence (that we see here as embedded in X) are regular in a neighborhood of every point of intersection with the divisor D. By taking normalizations, each such curve gives rise to a curve allowed in the definition of $\mathcal{R}_0(X|D)$, hence there is a canonical surjection

$$\mathrm{CH}_0^{LW}(X|D) \twoheadrightarrow \mathrm{CH}_0(X|D).$$

Note also that the inclusion $\mathcal{Z}_0(U_+) \hookrightarrow \mathcal{Z}_0((S_X)_{\mathrm{reg}}) = \mathcal{Z}_0(U_+) \oplus \mathcal{Z}_0(U_-)$ induces a push-forward map $p_*: \mathrm{CH}_0^{LW}(X|D) \to \mathrm{CH}_0^{LW}(S_X)$ (see the proof of [5, Prop. 5.9]).

3.4. **The factorization lemma.** We now fix an integral and smooth projective k-scheme X of dimension $d \geq 1$. Let $D \subset X$ be a reduced effective Cartier divisor. Let U denote the complement of D. The key step in the proof of Theorem 1.1 is the following factorization lemma.

Lemma 3.3. *Assume that k is infinite and one of the conditions (1), (2) and (4) of Theorem 1.1 holds. Then the (injective) map $\mathrm{CH}_0(X|D) \xrightarrow{p_*} \mathrm{CH}_0^{\mathrm{l.c.i.}}(S_X)$ has a factorization*

$$\mathrm{CH}_0(X|D) \xrightarrow{\phi_{X|D}} H_0^S(U) \xrightarrow{\widetilde{p}_*} \mathrm{CH}_0^{\mathrm{l.c.i.}}(S_X).$$

Proof. We let C be an integral normal curve and let $\nu: C \to X$ be a finite morphism whose image is not contained in D such that ν is birational to its image. We let $E = \nu^*(D)$ and let $f \in \mathrm{Ker}(\mathcal{O}_{C,E}^\times \twoheadrightarrow \mathcal{O}_{E_{\mathrm{red}}}^\times)$. We need to show that $p_*(\nu_*(\mathrm{div}(f))) = 0$. We do it in few steps. We write $V = C \setminus E$.

Step 1. We can factorize ν as

$$
\begin{tikzcd}
C \arrow[r, "\nu'"] \arrow[dr, "\nu"'] & \mathbb{P}^n_X \arrow[d, "\pi"] \\
& X
\end{tikzcd}
\tag{3.2}
$$

for some $n \geq 0$ such that ν' is a regular closed immersion and π is the canonical projection. We let $X' = \mathbb{P}^n_X$. Then note that $S_{X'} \cong \mathbb{P}^n_{S_X}$ by [5, Prop. 2.3(7)]. We let $D' = \pi^*(D) = \mathbb{P}^n_D$ and $U' = X' \setminus D' = \mathbb{P}^n_U$. Note that D' is a reduced divisor on X'. Furthermore, if D satisfies any of the conditions given in the statement of Theorem 1.1, then so does D'. This is a consequence of the smoothness of π. Let $\pi' : S_{X'} \to S_X$ be the projection map.

Suppose that the image of $\operatorname{div}(f)$ under the composite map

$$
\mathcal{Z}_0(V) \xrightarrow{\nu'_*} \mathcal{Z}_0(U') \twoheadrightarrow \mathrm{CH}_0^{LW}(X'|D') \xrightarrow{p_*} \mathrm{CH}_0^{LW}(S_{X'}) \tag{3.3}
$$

is zero. By composing further with the canonical surjection $\mathrm{CH}_0^{LW}(S_{X'}) \twoheadrightarrow \mathrm{CH}_0^{\mathrm{l.c.i.}}(S_{X'})$, we see that $\operatorname{div}(f)$ dies in $\mathrm{CH}_0^{\mathrm{l.c.i.}}(S_{X'})$ under p_*. Let $\pi'_* : \mathrm{CH}_0^{\mathrm{l.c.i.}}(S_{X'}) \to \mathrm{CH}_0^{\mathrm{l.c.i.}}(S_X)$ be the push-forward map, which exists by [5, Prop. 3.18]. It is then clear that

$$
p_* \circ \nu_*(\operatorname{div}(f)) = \pi'_* \circ p_* \circ \nu'_*(\operatorname{div}(f)) = 0.
$$

We thus need to show that if $\nu : C \hookrightarrow X$ is a regular closed immersion, then the image of $\operatorname{div}(f)$ under the composite map $\mathcal{Z}_0(V) \hookrightarrow \mathcal{Z}_0(U) \twoheadrightarrow \mathrm{CH}_0^{LW}(X|D) \xrightarrow{p_*} \mathrm{CH}_0^{LW}(S_X)$ is zero.

Step 2. If X is a curve, then $\mathrm{CH}_0^{LW}(X|D) = H_0^S(U)$ and there is nothing to prove. We now assume that X is a surface. Let $\mathrm{cyc}_{S_X} : \mathrm{CH}_0^{LW}(S_X) \to K_0(S_X)$ denote the cycle class map which takes the class $[x]$ of a closed point $x \in (S_X)_{\mathrm{reg}}$ to the class $[\mathcal{O}_x] \in K_0(S_X)$ (see [5, Lem. 3.13]). It is shown in [8, Thm. 7.7] (based on the original result due to Levine) that cyc_{S_X} is injective and its image is $SK_0(S_X)$. It suffices therefore to show that $\mathrm{cyc}_{S_X} \circ p_*(\operatorname{div}(f)) = 0$ in $K_0(S_X)$. By Proposition 2.8 (which can be applied if one of the conditions (1), (2) or (4) of Theorem 1.1 hold), it suffices to show that $\mathrm{cyc}_{S_X} \circ p_*(\nu_* \operatorname{div}(f))$ dies in $KH_0(S_X)$.

We let S_C and S_C^{sn} be the doubles of C along E and E_{red}, respectively. It is then easy to see that the canonical map $\psi : S_C^{sn} \to S_C$ is the seminormalization morphism (for example, one can use [35, Prop. 4.2(1)] noting that the conductor subscheme of $S_C^n = C \amalg C \to S_X^{sn}$ is reduced and that S_C^{sn} is Cohen-Macaulay, hence S_2). We let $h \in \mathcal{O}^{\times}_{S_C^{sn}, E}$ be the rational function on S_C^{sn} such that $h|_{C_+} = f$ and $h|_{C_-} = 1$. Note that the condition $f \in \operatorname{Ker}(\mathcal{O}^{\times}_{C,E} \twoheadrightarrow \mathcal{O}^{\times}_{E_{\mathrm{red}}})$ and the exact sequence (e.g., see the proof of [5, Lem. 2.2])

$$
0 \to \mathcal{O}^{\times}_{S_C^{sn}, E} \to \mathcal{O}^{\times}_{C,E} \times \mathcal{O}^{\times}_{C,E} \to \mathcal{O}^{\times}_{E_{\mathrm{red}}} \to 0
$$

imply that h is well defined. Note that h is also a rational function on S_C but may not lie in $\mathcal{O}^{\times}_{S_C, E}$.

To simplify the notation, let us write p_* also for the (injective) maps

$$
p_* : \mathrm{CH}_0(C|E) \to \mathrm{CH}_0^{\mathrm{l.c.i.}}(S_C), \quad p_* : \mathrm{CH}_0(C|E_{\mathrm{red}}) \to \mathrm{CH}_0^{\mathrm{l.c.i}}(S_C^{sn})
$$

given by the fundamental sequence applied to the pairs (C, E) and (C, E_{red}) respectively. It follows from the above discussion that $p_*(\operatorname{div}_C(f)) = \operatorname{div}_{S_C^{sn}}(h) = 0$ in $\mathrm{CH}_0^{\mathrm{l.c.i}}(S_C^{sn}) \cong \mathrm{CH}_0^{LW}(S_C^{sn})$ (the latter isomorphism holds for any 1-dimensional reduced scheme [5, Lem. 3.12]). By the same token, we get that $\mathrm{cyc}_{S_C^{sn}}(p_*(\operatorname{div}_C(f))) = 0$ in $K_0(S_C^{sn})$, and a fortiori that $p_*(\operatorname{div}(f))$ dies in $KH_0(S_C^{sn})$. On the other hand, it easy to see using the excision sequence for KH-theory [58, IV.12.6] that the canonical map $KH_0(S_C) \xrightarrow{\psi^*} KH_0(S_C^{sn})$ is an isomorphism.

We conclude that the image of $\mathrm{div}(f)$ under the composite map $\mathcal{Z}_0(C \setminus E) \twoheadrightarrow \mathrm{CH}_0(C|E) \to K_0(S_C) \to KH_0(S_C)$ is zero.

Since $\nu' : S_C \hookrightarrow S_X$ is a regular closed immersion (see § 2.1), there is a push-forward map $\nu'_* : KH(S_C) \to KH(S_X)$ (see § 2.2). We now consider the commutative diagram

$$\begin{tikzcd} \mathcal{Z}_0(C \setminus E) \arrow[r, two heads] \arrow[d] & \mathrm{CH}_0(C|E) \arrow[r, "\mathrm{cyc}_{S_C} \circ p_*"] \arrow[d, "\nu_*"] & KH_0(S_C) \arrow[d, "\nu'_*"] \\ \mathcal{Z}_0(U) \arrow[r, two heads] & \mathrm{CH}_0(X|D) \arrow[r, "\mathrm{cyc}_{S_X} \circ p_*"] & KH_0(S_X). \end{tikzcd}$$

Using this, we get

$$\mathrm{cyc}_{S_X} \circ p_* \circ \nu_*(\mathrm{div}(f)) = \nu'_* \circ \mathrm{cyc}_{S_C} \circ p_*(\mathrm{div}(f)) = 0.$$

This concludes the proof of the lemma when X is a surface (in particular, this covers the case where (2) holds).

Step 3. We now assume $d \geq 3$ and fix a closed embedding $X \hookrightarrow \mathbb{P}^n_k$. Let $\{D_1, \dots, D_r\}$ be the set of all irreducible components of D. Let $\{E_1, \dots, E_s\}$ be the set of irreducible components of D_{sing}. We let $\Delta(X) \subset X$ be the set defined in such a way that $x \in \Delta(X)$ if and only if x is a generic point of one of the schemes X, D and D_{sing}. We assume that we are in the case (1), namely, D is a normal crossing k-scheme. In particular, each D_i is smooth over k of dimension $d-1$ and each E_j is smooth over k of dimension $d-2 \geq 1$. Since C is regular and not contained in any of the D_i's, it follows that the scheme theoretic intersection $C \cap D_i$ is a finite closed subscheme of C. Since the dimension of the closure of each of the points of $\Delta(X)$ is at least one, it follows that $\Delta(X) \cap C = \varnothing$.

Given the above arrangement of X, D_i, E_j and C in $\mathbb{P}^n_k$, we can apply either the Bertini theorem of Altman-Kleiman [1, Thm. 7] or of Ghosh-Krishna [18, Thm. 3.9]) to find a complete intersection hypersurface $H = H_1 \cap \cdots \cap H_{d-2}$ in $\mathbb{P}^n_k$ of large enough degrees containing C such that $Y = X \cap H$ satisfies the following.

(1) Y is a smooth k-scheme of pure dimension two.
(2) Each $Y \cap D_i$ is a smooth k-scheme of dimension one.
(3) Each $Y \cap E_j$ is a smooth k-scheme of dimension zero.
(4) $Y \cap D_J = Y \cap (\bigcap_{i \in J} D_i) = \varnothing$ if $|J| \geq 3$.

Since $\dim(D_i) \geq 2$, it follows (for instance, from [27, Cor. 6.2]) that Y and well as each $Y \cap D_i$ is connected, hence integral. Furthermore, $Y \cap D$ is a curve which is reduced away from C by [18, Thm. 3.2]. Since $C \cap D$ is finite, Y is regular and $Y \cap D$ is a Cartier divisor on Y, it follows that $Y \cap D$ is a Cohen-Macaulay curve which is generically reduced. This implies that $Y \cap D$ must be reduced (see [21, Prop. 14.124]). We let $F = Y \cap D$. Then we conclude from (1), (2) and (3) above that Y is a complete intersection smooth integral surface inside X which contains C and $F = Y \cap D$ is a normal crossing curve on Y.

If condition (4) holds, i.e., if $k \subset \overline{\mathbb{Q}}$, then we can repeat the above argument to find a complete intersection smooth integral surface $Y \subset X$ which contains C and $F = Y \cap D$ is a reduced Cartier divisor on Y. The only difference is that we can not no longer guarantee that the irreducible components of F are regular.

In any case, let (Y, F) be the pair constructed above, and let $\tau : Y \hookrightarrow X$ be the inclusion map. Let $W = Y \setminus F = U \cap Y$. Let $\tau' : S_Y \hookrightarrow S_X$ denote the inclusion map, where S_Y is the double of Y along F. Then τ' is a regular closed embedding by [5, Prop. 2.4] (see § 2.1). It follows from Step 2 that the image of $\mathrm{div}(f)$ under the composite map $\mathcal{Z}_0(V) \hookrightarrow \mathcal{Z}_0(W) \twoheadrightarrow$

$\mathrm{CH}_0^{LW}(Y|F) \xrightarrow{p_*} \mathrm{CH}_0^{LW}(S_Y)$ is zero. We now consider the commutative diagram

$$\begin{tikzcd}
\mathcal{Z}_0(W) \arrow[r, two heads] \arrow[d] & \mathrm{CH}_0^{LW}(Y|F) \arrow[r, "p_*"] \arrow[d, "\tau_*"] & \mathrm{CH}_0^{LW}(S_Y) \arrow[d, "\tau'_*"] \\
\mathcal{Z}_0(U) \arrow[r, two heads] & \mathrm{CH}_0^{LW}(X|D) \arrow[r, "p_*"] & \mathrm{CH}_0^{LW}(S_X).
\end{tikzcd} \tag{3.4}$$

We note that τ'_* exists because τ' is a regular closed immersion and $S_Y \cap (S_X)_{\mathrm{reg}} = (S_Y)_{\mathrm{reg}}$. One easily checks that the push-forward map τ_* also exists and the diagram commutes. We thus get

$$p_*(\mathrm{div}(f)) = p_* \circ \tau_*(\mathrm{div}(f)) = \tau'_* \circ p_*(\mathrm{div}(f)) = 0.$$

This concludes the proof of the lemma. □

To take care of the case when k is a finite field, we shall need the following result. For any $X \in \mathbf{Sch}_k$ and k'/k a field extension, we let $X_{k'} = X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(k')$ with projection $v\colon X_{k'} \to X$.

Lemma 3.4 ([50], p.191)**.** *Let X be a smooth quasi-projective k-scheme and let k'/k be an algebraic field extension. Then the flat pull-back on 0-cycles induces a homomorphism*

$$v^*\colon H_0^S(X) \to H_0^S(X_{k'}).$$

If k'/k is finite, then the push-forward on 0-cycles induces a homomorphism

$$v_*\colon H_0^S(X_{k'}) \to H_0^S(X)$$

such that $v_* \circ v^*$ is multiplication by $[k' : k]$.

We remark that the pull-back map v^* is defined in [50] for finite field extensions. But this implies the case of arbitrary algebraic extensions by an easy limit argument.

3.5. **Proof of Theorem 1.1.** We let X, D and U be as in Theorem 1.1. We first assume that k is infinite and one of the conditions (1), (2) and (4) of Theorem 1.1 holds. In this case, the map $p_*\colon \mathrm{CH}_0(X|D) \to \mathrm{CH}_0^{\mathrm{l.c.i.}}(S_X)$ is injective by Theorem 3.2. Combining this injectivity with Lemma 3.3, one immediately concludes that $\phi_{X|D}$ must be an isomorphism.

We now assume that k is finite and one of the conditions (1), (2) and (4) of Theorem 1.1 holds. We only have to show that $\phi_{X|D}$ is injective. Let $\alpha \in \mathrm{CH}_0(X|D)$ be a class such that $\phi_{X|D}(\alpha) = 0$. Let $\ell_1 \neq \ell_2$ be two primes different from $\mathrm{char}(k)$. Let k_i/k be the pro-ℓ_i field extension of k for $i = 1, 2$.

Using [23, Prop. 8.5], we have a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_0(X|D) \arrow[r] \arrow[d] & H_0^S(U) \arrow[d] \\
\mathrm{CH}_0(X_{k_i}|D_{k_i}) \arrow[r] & H_0^S(U_{k_i}),
\end{tikzcd}$$

where the vertical arrows are the base change maps. The right vertical arrow exists by Lemma 3.4. Using the case of infinite fields, it follows that α dies in $\mathrm{CH}_0(X_{k_i}|D_{k_i})$. In particular, it dies in $\mathrm{CH}_0(X_{k'_i}|D_{k'_i})$ for a finite extension k'_i whose degree is a power of ℓ_i for each $i = 1, 2$. Using the projection formula for Chow groups with modulus (see [23, Prop. 8.5]), we conclude that $\ell_1^{n_1}\alpha = \ell_2^{n_2}\alpha = 0$ in $\mathrm{CH}_0(X|D)$ for some $n_1, n_2 \geq 1$. It follows that $\alpha = 0$. This concludes the proof of Theorem 1.1 under the conditions (1), (2) and (4). The remaining case (3) is already shown in [7, Thm. 1.1]. This concludes the proof of Theorem 1.1. □

3.6. **Proofs of some applications.** In this subsection, we shall give the proofs of the some of the applications of Theorem 1.1 mentioned in § 1. Corollary 1.2 and Theorem 1.3 are immediate from Theorem 1.1 using the references given before their statements. We shall therefore prove Theorem 1.4.

Proof of Theorem 1.4: Since the theorem is already known for torsion away from the characteristic by [53], we shall assume that k is algebraically closed of positive characteristic. We consider the diagram

$$\begin{array}{ccc} \mathrm{CH}_0(X|D)_0 & \xrightarrow{\phi_{X|D}} & H_0^S(U)_0 \\ {\scriptstyle \mathrm{alb}_{X|D}}\downarrow & & \downarrow{\scriptstyle \mathrm{alb}_U} \\ J^d(X|D)(k) & \xrightarrow{\lambda_{X|D}} & \mathrm{Alb}(U)(k), \end{array} \tag{3.5}$$

where $J^d(X|D)$ is the semi-abelian Albanese variety with modulus, constructed in [5, § 11.1]. This diagram is commutative and the bottom horizontal arrow is an isomorphism by [7, Thm. 3.2]. The left vertical arrow is an isomorphism on the torsion subgroups by [35, Thm. 6.7]. The top horizontal arrow is an isomorphism by Theorem 1.1. The desired assertion follows. □

4. Motivic cohomology of normal crossing schemes

The goal of this section is to prove Theorem 1.6. We shall need few ingredients in order to achieve this. The first is a perfection property of the cycle groups which we recall below.

4.1. **Perfection property of cycle groups.** We let k be a field of exponential characteristic p. We let Λ be a commutative ring which we assume to be $\mathbb{Z}$ if $\mathrm{char}(k) = 0$ or any $\mathbb{Z}[\frac{1}{p}]$-algebra if $\mathrm{char}(k) = p > 0$. We begin with a short recap about motivic cohomology of k-schemes, and related motivic invariants. Recall our notation that for a field extension k'/k and $X \in \mathbf{Sch}_k$, we write $X_{k'}$ for the base change of X by k' over k and $v: X_{k'} \to X$ denotes the projection map.

4.2. **Motivic homology and cohomology of singular schemes.** Let $X \in \mathbf{Sch}_k$ with the structure map $f: X \to \mathrm{Spec}\,(k)$ and let $m, n \in \mathbb{Z}$.

Definition 4.1. The motivic cohomology groups of X are defined as

$$H^m(X, \Lambda(n)) = \mathrm{Hom}_{\mathbf{DM}(k,\Lambda)}(M(X), \Lambda(n)[m]),$$

where $\mathbf{DM}(k, \Lambda)$ is Voevodsky's non-effective category of motives for the cdh-topology (also known as the 'big' category of motives) with Λ-coefficients, $\Lambda(n)$ is the motivic complex, and $M(X)$ is the motive of X (see [55] or [11, § 1]).

Let $\mathcal{SH}(X)$ be the monoidal stable homotopy category of smooth schemes over X and $\mathcal{SH}_{\mathrm{cdh}}(k)$ the stable homotopy category of $\mathbf{Sch}_k$ with respect to the cdh topology (e.g., see [37, § 2]). There is an adjoint pair of functors $(\psi_X, \phi_X): \mathcal{SH}(X) \to \mathbf{DM}(k, \Lambda)$. By [11, Thm. 5.1] and [37, Thm. 2.14], these functors give rise to functorial isomorphisms

$$\begin{aligned} H^p(X, \Lambda(q)) &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{SH}(X)}(\mathbb{S}_X, \Sigma^{p,q}(H\Lambda_X)) \\ &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}(k)}(\Sigma_T^{\infty} X_+, \Sigma^{p,q} H\Lambda), \end{aligned} \tag{4.1}$$

where $\mathbb{S}_X$ is the sphere spectrum (the unit object) of $\mathcal{SH}(X)$, $H\Lambda$ is the motivic Eilenberg-MacLane spectrum in $\mathcal{SH}(k)$, and $H\Lambda_X = \mathbf{L}f^*(H\Lambda)$ for the structure map $f: X \to \mathrm{Spec}\,(k)$. We refer to, e.g., [37, § 2,3] for the definitions of the suspension operators Σ_T^{∞} and $\Sigma^{p,q}$.

In a similar fashion, one can define motivic cohomology groups with compact support and motivic homology as follows.

$$\begin{aligned} H^m_c(X, \Lambda(n)) &= \mathrm{Hom}_{\mathbf{DM}(k,\Lambda)}(M_c(X), \Lambda(n)[m]) \\ H_m(X, \Lambda(n)) &= \mathrm{Hom}_{\mathbf{DM}(k,\Lambda)}(\Lambda(n)[m], M(X)), \end{aligned}$$

where $M_c(X)$ is the motive of X with compact support [43, Defn. 16.13]. In particular, there is a canonical isomorphism [43, Prop. 14.18]:

$$H_n(X, \Lambda(0)) \xrightarrow{\simeq} H^S_n(X)_\Lambda,$$

where the right-hand side is the n-th Suslin homology group of X recalled in 3.1.

We recall the following result of Cisinski-Déglise [11, Prop. 8.1].

Theorem 4.2. *Let k'/k be a purely inseparable field extension and let $v: \mathrm{Spec}\,(k') \to \mathrm{Spec}\,(k)$ be the projection map. Then the pull-back functor*

$$u^*: \mathbf{DM}(k, \Lambda) \to \mathbf{DM}(k', \Lambda)$$

is an equivalence of triangulated categories.

Using Theorem 4.2 and the description of various groups above, we get the following result which we shall use in our proofs.

Corollary 4.3. *Let k'/k be a purely inseparable field extension and let $u: \mathrm{Spec}\,(k') \to \mathrm{Spec}\,(k)$ be the projection map. Then for any $X \in \mathbf{Sch}_k$, the pull-back maps*

$$\begin{aligned} v^*: H^m(X, \Lambda(n)) &\longrightarrow H^m(X_{k'}, \Lambda(n)) \\ v^*: H^m_c(X, \Lambda(n)) &\longrightarrow H^m_c(X_{k'}, \Lambda(n)) \end{aligned}$$

are isomorphisms.

Assume now that X is smooth of pure dimension d over k. Duality in motivic homotopy theory makes it possible to identify motivic cohomology and homology groups (as well as their compactly supported version) with the appropriate twist and shift. We shall need an explicit description, in the bi-degree $(2d, d)$, of the map realizing such duality isomorphism for later applications. We quickly recall its construction. For every closed point $x \in X$, the inclusion $\mathrm{Spec}\,(k(x)) \hookrightarrow X$ gives a Gysin map $M_c(X) \to M_{\{x\}}(X) \xrightarrow{\cong} M(k(x))(d)[2d]$. Taking cohomology, we get

$$\mathbb{Z} \xrightarrow{\simeq} H^0(k(x), \mathbb{Z}(0)) \to H^{2d}_c(X, \mathbb{Z}(d)), \tag{4.2}$$

and extending (4.2) by linearity, we get $\phi_X: \mathcal{Z}_0(X) \to H^{2d}_c(X, \mathbb{Z}(d))$.

Lemma 4.4. *The map ϕ_X descends to an isomorphism*

$$\phi_X: H^S_0(X)_\Lambda \xrightarrow{\cong} H^{2d}_c(X, \Lambda(d)).$$

Proof. Let k' be a perfect closure of k and consider the commutative diagram

$$\begin{tikzcd} H^S_0(X)_\Lambda \arrow[r, "\phi_X"] \arrow[d, "v^*"'] & H^{2d}_c(X, \Lambda(d)) \arrow[d, "v^*"] \\ H^S_0(X_{k'})_\Lambda \arrow[r, "\phi_{X_{k'}}"] & H^{2d}_c(X_{k'}, \Lambda(d)). \end{tikzcd} \tag{4.3}$$

The left vertical arrow is an isomorphism by Lemma 3.4 using a limit argument and the right vertical arrow is an isomorphism by Corollary 4.3. The bottom horizontal arrow is an isomorphism (e.g., by [30, Thm. 5.5.14]). The lemma now follows. □

4.3. **The snc subcurves.** We fix a normal crossing k-scheme X of dimension $d \geq 1$ and let $\{X_1, \dots, X_n\}$ be the set of irreducible components of X. A snc subcurve $C \subset X$ (see [13, § 2.1]) is a reduced closed subscheme of pure dimension one such that the scheme theoretic intersection of C with each irreducible component X_i of X is either empty or smooth of pure dimension one, its intersections with $X_i \cap X_j$ (for all $i \neq j$) are either empty or smooth and 0-dimensional, and its intersections with $X_i \cap X_j \cap X_l$ (for all $i \neq j \neq l \neq i$) are empty.

Remark 4.5. We note that the above definition of snc subcurves is more restrictive than the one given in [13, § 2.1] because the latter only requires the intersections $C \cap X_i \cap X_j$ (for $i \neq j$) to be reduced (not necessarily smooth) and 0-dimensional. The stronger assumption allows us to prove the following result. But this distinction disappears if k is perfect.

Lemma 4.6. *Let $X \in \mathbf{Sch}_k$ be a normal crossing scheme and let $C \subset X$ be a snc subcurve. Let k'/k be a finite purely inseparable field extension. Then $X_{k'}$ is a normal crossing k'-scheme and $C_{k'} \subset X_{k'}$ is a snc subcurve.*

Proof. Let $v: X_{k'} \to X$ be the base change morphism. Then v is a universal homeomorphism. In particular, there is a bijective correspondence between the irreducible components of X and $X_{k'}$. We let $X'_i = (X_i)_{k'}$ for $1 \leq i \leq n$. Since $X_i \in \mathbf{Sm}_k$, it follows each X'_i is integral and smooth over k'. In turn, this implies that $X_{k'}$ is generically reduced (i.e., $X_{k'}$ satisfies Serre's R_0-condition). Since v is finite and flat, and X satisfies Serre's S_1-condition (because it is reduced), it follows that $X_{k'}$ also satisfies Serre's S_1-condition. It follows that $X_{k'}$ is reduced. By the same token, for every nonempty subset $J \subset [1, n]$, the scheme theoretic intersection $X'_J := \bigcap_{i \in J} X'_i$ is a smooth k'-scheme (unless empty) of pure dimension $d + 1 - |J|$. In other words, $X_{k'}$ is a normal crossing k'-scheme. An identical proof shows that $C_{k'} \subset X_{k'}$ is a snc subcurve. □

4.4. **The cycle group $\mathrm{CH}_0^{EKW}(X)$.** Let $X \in \mathbf{Sch}_k$ be a normal crossing scheme as above. The cycle group $\mathrm{CH}_0^{EKW}(X)$ is the quotient of $\mathcal{Z}_0(X_{\mathrm{reg}})$ by the subgroup $\mathcal{R}_0^{EKW}(X)$ generated by $\mathrm{div}(f)$, where $f \in k(C)^\times$ is a rational function on a curve $C \subset X$ such that the pair (C, f) satisfies either of the conditions (1) and (2) below.

(1) C is an integral curve not contained in X_{sing} with normalization $\nu: C^n \to C \hookrightarrow X$ and $f \in \mathcal{O}^\times_{C^n, \nu^*(X_{\mathrm{sing}})}$ such that $f(x) = 1$ for all $x \in \nu^*(X_{\mathrm{sing}})$.
(2) $C \subset X$ is a snc subcurve and $f \in \mathcal{O}^\times_{C, (C \cap X_{\mathrm{sing}})}$.

Let $Y = X_{\mathrm{sing}}$ for the normal crossing scheme X. The inclusion $\mathrm{Spec}\,(k(x)) \hookrightarrow Y_{\mathrm{reg}}$ gives a Gysin homomorphism $k(x)^\times \xrightarrow{\cong} H^1(k(x), \mathbb{Z}(1)) \to H^{2d-1}(Y, \mathbb{Z}(d))$ for every closed point $x \in Y_{\mathrm{reg}}$. Note that $Y_{\mathrm{reg}} \in \mathbf{Sm}_k$ since X is a normal crossing k-scheme. Hence, we get the global Gysin map $\bigoplus_{x \in Y_{\mathrm{reg}}^{(d-1)}} k(x)^\times \to H^{2d-1}(Y, \mathbb{Z}(d))$, where $Y_{\mathrm{reg}}^{(d-1)}$ is the set of closed points of Y_{reg}.

Lemma 4.7. *The map*

$$\alpha_Y: \bigoplus_{x \in Y_{\mathrm{reg}}^{(d-1)}} k(x)^\times \otimes_{\mathbb{Z}} \Lambda \to H^{2d-1}(Y, \Lambda(d))$$

is surjective.

Proof. This is [13, Prop. 6.4] if k is perfect. The general case follows from the perfect one using Corollary 4.3 in order to identify $H^{2d-1}(Y, \Lambda(d))$ with $H^{2d-1}(Y_{k'}, \Lambda(d))$, where k' is a perfect closure of k. □

Let us now assume that X is a projective normal crossing k-scheme. The next step for proving Theorem 1.6 is the description of the motivic cohomology groups $H^{2d}(X, \Lambda(d))$ for normal crossing varieties discussed in [13]. We have seen in (4.2) that there is a canonical map $\phi_{X_{\mathrm{reg}}}: \mathcal{Z}_0(X_{\mathrm{reg}}) \to H^{2d}_c(X_{\mathrm{reg}}, \mathbb{Z}(d))$. Composing with the map $H^{2d}_c(X_{\mathrm{reg}}, \mathbb{Z}(d)) \to H^{2d}(X, \mathbb{Z}(d))$, we get $\widetilde{\lambda}_X: \mathcal{Z}_0(X_{\mathrm{reg}}) \to H^{2d}(X, \mathbb{Z}(d))$.

Proposition 4.8. *The map $\widetilde{\lambda}_X$ induces an isomorphism*

$$\widetilde{\lambda}_X: \mathrm{CH}_0^{EKW}(X)_\Lambda \xrightarrow{\cong} H^{2d}(X, \Lambda(d)).$$

Proof. This is [13, Thm. 7.1] when k is perfect. The same proof works in the general case, using Lemma 4.7 instead of [13, Prop. 6.4] and passing to a perfect closure of k. The latter is achieved using Corollary 4.3 and Lemma 4.6. The vanishing $H^{2d}(Y, \Lambda(d)) = 0$ for $Y = X_{\mathrm{sing}}$, that is also used in the proof of [13, Thm. 7.1], can be deduced from [37, Thm. 5.1] using again Corollary 4.3. □

4.5. **Proof of Theorem 1.6(1).** Let k be any field and let $X \in \mathbf{Sch}_k$ be as in part (1) of Theorem 1.6. In other words, X is a reduced quasi-projective k-scheme of pure dimension d. To construct the map $\lambda_X: \mathrm{CH}_0^{\mathrm{l.c.i.}}(X) \to H^{2d}(X, \mathbb{Z}(d))$, we proceed as follows.

Using (4.1) and [45, Defn. 2.30, Thm. 2.31], we have a Gysin map $\tau_x: \mathbb{Z} \cong H^0(k(x), \mathbb{Z}(0)) \to H^{2d}(X, \mathbb{Z}(d))$ for any closed point $x \in X_{\mathrm{reg}}$. Extending this linearly, we get a homomorphism $\lambda_X: \mathcal{Z}_0(X_{\mathrm{reg}}) \to H^{2d}(X, \mathbb{Z}(d))$.

When $d = 1$, it is shown in the proof of [37, Lem. 7.12] that λ_X factors through the Chow group (this uses the slice spectral sequence for singular schemes). For $d \geq 2$, we let $\nu: C \to X$ be a good curve and let $f \in \mathcal{O}^{\times}_{C,S}$, where $S = \nu^{-1}(X_{\mathrm{sing}}) \cup C_{\mathrm{sing}}$. By [5, Lem. 3.4], we can assume that ν is a lci morphism. In particular, there is a Gysin homomorphism $\nu_*: H^2(C, \mathbb{Z}(1)) \to H^{2d}(X, \mathbb{Z}(d))$ by [45, Defn. 2.31, Thm. 2.31]. We now consider the diagram

$$\begin{array}{ccc} \mathcal{Z}_0(C \setminus S) & \xrightarrow{\lambda_C} & H^2(C, \mathbb{Z}(1)) \\ \downarrow{\scriptstyle \nu_*} & & \downarrow{\scriptstyle \nu_*} \\ \mathcal{Z}_0(X_{\mathrm{reg}}) & \xrightarrow{\lambda_X} & H^{2d}(X, \mathbb{Z}(d)). \end{array} \tag{4.4}$$

It is immediate from the construction of λ_X and Gysin maps that this diagram is commutative. By the curve case, we have that $\lambda_C(\mathrm{div}(f)) = 0$. It follows that

$$\lambda_X(\mathrm{div}(f)) = \lambda_X \circ \nu_*(\mathrm{div}(f)) = \nu_* \circ \lambda_C(\mathrm{div}(f)) = 0.$$

This shows that λ_X factors through a homomorphism $\lambda_X: \mathrm{CH}_0^{\mathrm{l.c.i.}}(X) \to H^{2d}(X, \mathbb{Z}(d))$.

By construction, for a regular closed immersion $f: X' \hookrightarrow X$ of equidimensional schemes such that $\dim(X') = d'$ and $f^{-1}(X_{\mathrm{sing}}) \subset X'_{\mathrm{sing}}$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_0^{\mathrm{l.c.i.}}(X') & \xrightarrow{\lambda_{X'}} & H^{2d'}(X', \mathbb{Z}(d')) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{CH}_0^{\mathrm{l.c.i.}}(X) & \xrightarrow{\lambda_X} & H^{2d}(X, \mathbb{Z}(d)), \end{array} \tag{4.5}$$

in which the left and the right vertical arrows are the Gysin homomorphisms of [5, Prop. 3.18] and [45, Defn. 2.30, Thm. 2.31], respectively.

4.6. **A key lemma.** We shall need the following key result about the lci Chow group of normal crossing schemes. Let X be a normal crossing k-scheme of dimension d as above and let Y be an irreducible component of X. We let $Z \subset X$ be the scheme theoretic closure of $X \setminus Y$ and let $E = Y \cap Z$. Then X_{sing} and E are normal crossing k-schemes of dimension $d-1$ and E is a simple normal crossing divisor on Y. We let $V = Y \setminus E$ so that there is an inclusion of the 0-cycle groups $\iota_*: \mathcal{Z}_0(V) \hookrightarrow \mathcal{Z}_0(X_{\mathrm{reg}})$, where $\iota: Y \hookrightarrow X$ is the inclusion.

Lemma 4.9. *Assume that k is infinite. Then the map ι_* descends to a homomorphism*
$$\iota_*: \mathrm{CH}_0(Y|E) \to \mathrm{CH}_0^{\mathrm{l.c.i.}}(X).$$

Proof. This is shown in [7, Thm. 8.3] when either k is algebraically closed or $d \le 2$. We shall closely follow that proof. We can assume that $d \ge 3$ and that the lemma holds in smaller dimensions. We fix a locally closed embedding $X \hookrightarrow \mathbb{P}^N_k$. Let $C \subset Y$ be an integral curve not contained in E and let $f \in \mathrm{Ker}(\mathcal{O}^\times_{C^n, \nu^*(E)} \to \mathcal{O}^\times_{\nu^*(E)})$, where $\nu : C^n \to Y \hookrightarrow X$ is the canonical map from the normalization of C. We need to show that $\mathrm{div}(f)$ dies in $\mathrm{CH}_0^{\mathrm{l.c.i.}}(X)$.

We can write ν as the composition of two maps $C^n \xrightarrow{\nu'} \mathbb{P}^m_X \xrightarrow{\pi} X$ for some integer $m \ge 0$, where ν' is a regular closed immersion and π is the projection. Note that ν' factors through $\mathbb{P}^m_Y$. We now note that $\mathbb{P}^m_X$ is a normal crossing k-scheme of dimension $d \ge 3$ and $\mathbb{P}^m_Y$ is an irreducible component of $\mathbb{P}^m_X$. Using the push-forward map $\pi_*: \mathrm{CH}_0^{\mathrm{l.c.i.}}(\mathbb{P}^m_X) \to \mathrm{CH}_0^{\mathrm{l.c.i.}}(X)$ (see [5, Prop. 3.18]) and the canonical map $\mathrm{CH}_0^{LW}(\mathbb{P}^m_X) \to \mathrm{CH}_0^{\mathrm{l.c.i.}}(\mathbb{P}^m_X)$, it suffices to show that $\nu_*(\mathrm{div}(f))$ dies in $\mathrm{CH}_0^{LW}(\mathbb{P}^m_X)$. We can therefore replace $\mathrm{CH}_0^{\mathrm{l.c.i.}}(X)$ with $\mathrm{CH}_0^{LW}(X)$ and assume that C is normal. Note that the map $\mathrm{CH}_0^{LW}(X) \to \mathrm{CH}_0^{\mathrm{l.c.i.}}(X)$ is an isomorphism for $d \le 2$ by [8, Thm. 8.1] (see also [5]). Hence, the base case of the induction holds for the modified problem too.

We can now repeat the argument of the proof of [7, Thm. 8.3] (without using any blow-up) to find a hypersurface section $X' \subset X$ inside $\mathbb{P}^N_k$ containing C such that X' is a $(d-1)$-dimensional normal crossing k-scheme, $X'_{\mathrm{reg}} = X_{\mathrm{reg}} \cap H$ and $Y' = X' \cap Y = H \cap Y$ is a smooth irreducible component of X'. It follows by induction that $\nu_*(\mathrm{div}(f))$ dies in $\mathrm{CH}_0^{LW}(X')$. In particular, it dies in $\mathrm{CH}_0^{LW}(X)$ via the push-forward map $\mathrm{CH}_0^{LW}(X') \to \mathrm{CH}_0^{LW}(X)$, induced by the regular closed immersion $X' \hookrightarrow X$. This concludes the proof. $\square$

4.7. **Proof of Theorem 1.6(2,3).** If X is projective and X_{reg} is smooth over k, then the surjectivity of λ_X (as asserted in part (2)) follows from the surjection $H^{2d}_c(U, \Lambda(d)) \twoheadrightarrow H^{2d}(X, \Lambda(d))$ and Lemma 4.4. We now prove the last part of the theorem. We are given that X is a projective normal crossing k-scheme and need to show that λ_X is an isomorphism. To prove this, we first assume that k is infinite and look at the diagram

$$\begin{tikzcd}
\mathcal{Z}_0(U) \arrow[r, two heads] \arrow[dr] & \mathrm{CH}_0^{EKW}(X)_\Lambda \arrow[d, dotted, "\psi_X"] \arrow[dr, "\widetilde{\lambda}_X"] & \\
& \mathrm{CH}_0^{\mathrm{l.c.i.}}(X)_\Lambda \arrow[r, "\lambda_X"] & H^{2d}(X, \Lambda(d)).
\end{tikzcd} \tag{4.6}$$

It suffices to show that ψ_X exists such that the resulting left triangle commutes. This is shown in the proof of [7, Thm. 8.4]. But we do not need Λ-coefficient for constructing ψ_X, thanks to Theorem 1.1. We sketch the steps. We let $Y = X_{\mathrm{sing}}$.

We let $C \subset X$ be a reduced curve and $f \in k(C)^\times$ a rational function, where $k(C)$ is the ring of total quotients for C. We now observe that if the pair (C, f) is of type (1) in the definition of $\mathrm{CH}_0^{EKW}(X)$ in § 4.4, then C must be integral. In particular, it must be contained in one and only one irreducible component X' of X. Moreover, for this component X', the intersection

$E = Z \cap Y$ must be a simple normal crossing divisor on X', where Z is the scheme theoretic closure of $X \setminus X'$. We now conclude from Theorem 1.1 that $\mathrm{div}(f)$ dies in $\mathrm{CH}_0(X'|E)$. Hence, it dies in $\mathrm{CH}_0^{\mathrm{l.c.i.}}(X)$ by Lemma 4.9. If (C, f) is of type (2) in the definition of $\mathrm{CH}_0^{EKW}(X)$ in § 4.4, then $C \subset X$ is a Cartier curve (see § 3.2) by [7, Lem. 7.6] and hence, $\mathrm{div}(f)$ already dies in $\mathrm{CH}_0^{LW}(X)$. This concludes the proof of part (3) when k is infinite.

We now assume that k is finite. Since we already showed surjectivity of λ_X above, we only need to show that it is injective. We let k' be the pro-p-extension of k where $\mathrm{char}(k) = p$. We then get a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_0^{\mathrm{l.c.i.}}(X)_\Lambda & \xrightarrow{\lambda_X} & H^{2d}(X, \Lambda(d)) \\ \Big\downarrow{\scriptstyle v^*} & & \Big\downarrow{\scriptstyle v^*} \\ \mathrm{CH}_0^{\mathrm{l.c.i.}}(X_{k'})_\Lambda & \xrightarrow{\lambda_{X_{k'}}} & H^{2d}(X_{k'}, \Lambda(d)), \end{array} \tag{4.7}$$

where $v: X_{k'} \to X$ is the base change map. The left vertical arrow exists and is injective by [5, Prop. 6.1]. Since k' is infinite, the bottom horizontal arrow is injective. It follows that λ_X must be injective too. This concludes the proof of Theorem 1.6. ☐

5. A question of Barbieri-Viale and Kahn

Let k be an algebraically closed field of characteristic zero and let X be a projective and reduced k-scheme of pure dimension d. We shall now prove our application of the existence of the map λ_X given by Theorem 1.6.

In [3, 13.7.6], the authors refer that in a private correspondence, Marc Levine outlined the construction of a cycle map $c\ell$ from $\mathrm{CH}_0^{LW}(X)$ to $H^{2d}(X, \mathbb{Z}(d))$ inducing, in particular, a morphism

$$c\ell_{\mathrm{tors}}: \mathrm{CH}_0^{LW}(X)_{\mathrm{tors}} \longrightarrow H^{2d}(X, \mathbb{Z}(d))_{\mathrm{tors}}$$

that they conjecture to satisfy a number of properties. We can now give a positive answer to their conjecture.

We shall verify the expectations of Barbieri-Viale and Kahn by working with the modified version $\mathrm{CH}_0^{\mathrm{l.c.i.}}(X)$ instead of $\mathrm{CH}_0^{LW}(X)$, keeping in mind that the two Chow groups actually agree under the above assumption on k, by [5, Thm. 3.17].

First, let $J^d(X)$ be the universal regular semi-abelian variety quotient of $\mathrm{CH}_0^{LW}(X)_{\deg 0}$, constructed in [9]. This is universal for regular homomorphisms (see op. cit. for the definition of a regular homomorphism) from $\mathrm{CH}_0^{LW}(X)$ to semi-abelian varieties. It was shown in [5, Prop. 9.7] that $J^d(X)$ is also the universal regular semi-abelian variety quotient of $\mathrm{CH}_0^{\mathrm{l.c.i.}}(X)_{\deg 0}$.

Next, let $\mathbf{L}_1\mathrm{Alb}^*(X)$ be the 1-motive

$$\mathbf{L}_1\mathrm{Alb}^*(X) = H_1^t(\mathrm{LAlb}(M(X)^*(d)[2d])),$$

where $M(X)^*$ is the dual of $M(X)$ in $\mathbf{DM}(k)$, the homology $H_1^t(-)$ denotes the $H_1(-)$ homology with respect to the t-structure (introduced in [3, 3.1]) on Deligne's category of 1-motives $D^b(\mathcal{M}_1)$, and finally $\mathrm{LAlb}(-)$ denotes the integrally defined derived Albanese functor

$$\mathrm{LAlb}: \mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \longrightarrow D^b(\mathcal{M}_1),$$

introduced in [3, Def. 2.1.1] (note that $M(X)^*(d)[2d]$ is effective, so that the definition makes sense, and that we are working in characteristic zero). In particular, $\mathbf{L}_1\mathrm{Alb}^*(X)$ is a semi-abelian variety. By [3, (13.7.1)], there is a canonical map

$$u: H^{2d}(X, \mathbb{Z}(d)) \longrightarrow \mathbf{L}_1\mathrm{Alb}^*(X)(k) \tag{5.1}$$

that is an isomorphism on the torsion subgroups by [3, Corollary 13.7.4].

We now have all the ingredients to state and prove the following result. This verifies all expectations of Barbieri-Viale and Kahn.

Theorem 5.1. *Let X and k be as above. Then the morphism*

$$\lambda_X\colon \mathrm{CH}_0^{\mathrm{l.c.i.}}(X) \to H^{2d}(X,\mathbb{Z}(d))$$

is surjective with uniquely divisible kernel, and there is a commutative diagram

$$\begin{tikzcd} \mathrm{CH}_0^{\mathrm{l.c.i.}}(X)_{\mathrm{tors}} \arrow[r, "\lambda_X"] \arrow[d, "a^+"] & H^{2d}(X,\mathbb{Z}(d))_{\mathrm{tors}} \arrow[d, "u"] \\ \mathrm{Alb}^+(X)(k)_{\mathrm{tors}} \arrow[r] & \mathbf{L}_1\mathrm{Alb}^*(X)(k)_{\mathrm{tors}} \end{tikzcd} \tag{5.2}$$

where all the arrows are isomorphisms.

Proof. The existence and explicit construction of λ_X was shown in Theorem 1.6. To check that (5.2) commutes, it suffices to check it for the cycle class of a closed point $x \in X_{\mathrm{reg}}$. This reduces to checking the commutativity for points where this is well known.

Now, the left vertical arrow in (5.2) is an isomorphism by the main result of [9]. The right vertical arrow is an isomorphism by [3, 13.7.5]. The bottom horizontal arrow is an isomorphism by [3, Thm. 12.12.6]. Thus every arrow in (5.2) is an isomorphism. Finally, recall (see, e.g., [6, Lemma 5.1]) that since k is algebraically closed, the subgroup $\mathrm{CH}_0^{\mathrm{l.c.i.}}(X)_{\deg 0}$ is divisible. Since λ_X is an isomorphism on torsion by the above discussion, an easy diagram chase implies that the kernel of λ_X is uniquely divisible, completing the proof of the theorem. □

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