Discrete and Continuous Dynamical Systems

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NONLINEARITIES AND SINGULARITIES IN ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. This articles is concerned with the interaction between nonlinearities and the spectrum of linear differential operators in elliptic boundary value problems. The focus is on how solution multiplicity arises from spectral interaction and how singularity theory provides a framework for understanding this phenomenon. Starting with the classic Ambrosetti–Prodi theorem as a paradigm of a global fold, we progress to cusps, swallowtails, and butterfly singularities that appear when convexity assumptions are relaxed and when different types of nonlinearities are considered. The classical results in singularity theory of H. Whitney, R. Thom, V.I. Arnold, and J. Mather are presented, as well as their extension to the infinite-dimensional setting in Banach spaces. Detailed examples, including cubic nonlinearities and models with local eigenvalue crossing, illustrate the rich geometric structure of solution sets. Several open problems and recent developments are discussed.

Contents

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1. **Interaction between nonlinearity and spectrum.** An important principle in the study of nonlinear differential equations is that **multiplicity of solutions often arises from the interaction between the nonlinearity and the spectrum of the associated linear operator**. This observation has deep roots in bifurcation theory, singularity theory, and nonlinear functional analysis.

Consider the elliptic boundary value problem on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary:

$$\begin{cases} -\Delta u = f(u) + h(x) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where f is a smooth function and $h \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$. We can formulate this as a nonlinear operator equation:

$$\Phi(u) = h, \quad \Phi(u) := -\Delta u - f(u).$$

The operator Φ acts between appropriate Banach spaces, typically $\Phi : E = C_0^{2,\alpha}(\Omega) \to F = C^\alpha(\Omega)$ or between Sobolev spaces $W_0^{2,p}(\Omega) \to L^p(\Omega)$.

1.1. **Spectral avoidance vs. Spectral interaction.** Let $\sigma(-\Delta) = \{\lambda_i\}_{i\in\mathbb{N}}$ denote the pure point spectrum of the Dirichlet Laplacian on Ω, with $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$.

Case 1. No spectral interaction. If the derivative f' avoids the spectrum, such as when

- $f'(s) < \lambda_1 - \delta$ for all $s \in \mathbb{R}$ (Hammerstein, 1930), and
- $\lambda_k + \delta < f'(s) < \lambda_{k+1} - \delta$ for all $s \in \mathbb{R}$ (Dolph, 1944),

then $\Phi'(u)$ is an isomorphism for every $u \in E$. By the Hadamard–Caccioppoli global inversion theorem (also called the Hadamard–Lévy theorem), Φ is a global diffeomorphism. Consequently, (1) has a unique solution for every $h \in F$.

Case 2. Spectral interaction. When $f'(\mathbb{R}) \cap \sigma(-\Delta) \neq \emptyset$, the operator Φ may have **singular points**, that is, points where $\Phi'(u)$ is not invertible. The **singular set**

$$\Sigma = \{u \in E : \ker \Phi'(u) \neq \{0\}\}$$

becomes non-empty, and the solution structure becomes richer and more complicated.

The aim of this article is to give a short introduction to the study of these singularities and their implications for solution multiplicity. Several open problems will be stated.

2. **The Ambrosetti–Prodi theorem: A paradigm of global folding.** The 1972 result of Ambrosetti and Prodi represents a cornerstone in the theory of nonlinear elliptic equations with spectral interaction. Consider problem (1) with $f \in C^2(\mathbb{R})$ satisfying:

(i) $f(0) = 0$,

(ii) $0 < f'(-\infty) < \lambda_1 < f'(+\infty) < \lambda_2$,

(iii) $f''(t) > 0$ for all $t \in \mathbb{R}$ (strict convexity),

where λ_1, λ_2 are the first two Dirichlet eigenvalues of $-\Delta$ on Ω.

2.1. **Precise statement.**

Theorem 2.1 (Ambrosetti–Prodi, 1972, [1]). *Under the above assumptions (i) - (iii), there exists a closed connected C^1-manifold $M \subset F$ of codimension 1 such that:*

- *$F \setminus M = A_0 \cup A_2$ has exactly two connected components;*
- *If $h \in A_0$, then (1) has no solution;*
- *If $h \in A_2$, then (1) has exactly two solutions;*
- *If $h \in M = \Phi(\Sigma)$, then (1) has exactly one solution.*

Moreover, $M = \Phi(\Sigma)$ and Σ is a C^1-manifold of codimension 1 in E.

Note that the theorem gives a **complete geometric characterization** of the solution structure of Equation (1) under assumptions (i)-(iii).

2.2. **Key ideas of the proof.** The proof proceeds through several geometric and analytical steps:

1. **Characterization of the singular set.** For $u \in E$, consider the linearized operator
$$L_u v := \Phi'(u)[v] = -\Delta v - f'(u)v.$$
Let $\mu_1(u) < \mu_2(u) \leq \cdots$ be the eigenvalues of L_u with eigenfunctions $v_i(u)$. Then,
$$u \in \Sigma \iff \mu_1(u) = 0.$$
2. **Transversality of the singular set.** For fixed $u_1 \in E$, consider the function
$$\psi(s) := \mu_1(u_1 + se_1), \quad s \in \mathbb{R},$$
where e_1 is the (normailized) first eigenfunction of $-\Delta$. Note that e_1 does not change sign in Ω, and can be chosen positive. The assumptions imply
$$\lim_{s\to-\infty} \psi(s) = \lambda_1 - f'(-\infty) > 0, \qquad \lim_{s\to+\infty} \psi(s) = \lambda_1 - f'(+\infty) < 0,$$
and
$$\psi'(s) = -\int_\Omega f''(u_1 + se_1)v_1(u)^2 e_1 \, dx < 0 \ , \quad \text{using} \quad \text{(iii)}\,.$$
Hence, each line $\{u_1 + se_1\}$ intersects Σ exactly once, and e_1 is transverse to Σ, that is, Σ is locally a codimension 1 manifold, with normal vector $\nu(u) = f''(u)v_1^2(u)$ in $u \in \Sigma$.

3. **Local structure near Σ.** For $u \in \Sigma$, if the eigenvector

   $v_1(u)$ is transverse to Σ , i.e. $\int_\Omega f''(u)v_1(u)^3 \neq 0$,

   then Φ behaves locally like a

   fold map: Φ folds E (locally) along Σ, mapping points on one side of Σ to one side of $\Phi(\Sigma)$ and points on the other side of Σ to the same side of $\Phi(\Sigma)$.
4. **Global structure.** Using the properness of Φ and topological degree arguments, one shows that $\Phi(\Sigma)$ divides F into exactly two components: $F = F_0 \cup \Phi(\Sigma) \cup F_2$, with constant solution number on each component, i.e. the 0 solution for $h \in F_0$, exactly 2 solutions for $h \in F_2$, and exactly 1 solution for $h \in \Phi(\Sigma)$.

These results can be generalized to large classes of operators, see Calanchi and Tomei [12].

2.3. **The global fold interpretation.** Berger and Church (1979) provided a powerful reinterpretation of the Ambrosetti–Prodi result through the lens of singularity theory.

Theorem 2.2. [4] *Under the Ambrosetti–Prodi assumptions, there exists a Banach space X and a homeomorphism identifying Φ with the global fold map:*

$$\varphi : \mathbb{R} \times X \to \mathbb{R} \times X, \quad \varphi(t, x) = (t^2, x).$$

This means that, up to coordinate changes, the Ambrosetti–Prodi operator is equivalent to squaring in one direction and the identity in the complementary directions. This global perspective explains why the solution structure is so clean: exactly 0, 1, or 2 solutions depending on the forcing term's position relative to $\Phi(\Sigma)$.

The statement of Theorem 2.2 is in the language of *singularity theory* as initiated by H. Whitney in 1955 in his seminal paper [25].

3. **Singularity theory: Mathematical background.** In this section, we put the above and the subsequent results in the historical context of *singularity theory.* We first provide a short overview of singularity theory (or catastrophe theory) in finite-dimensional spaces.

To motivate the theory, consider the local behavior of a smooth function $f : \mathbb{R} \to \mathbb{R}$. In a given point x_0, we have two possibilites:

i) $f'(x_0) \neq 0$ (x_0 regular point): then, $f \sim id$ (locally diffeomorphic)

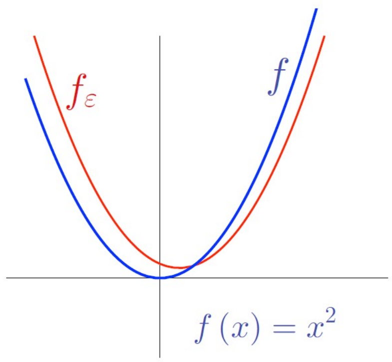
ii) $f'(x_0) = 0$ (x_0 critical, or singular): then, if $f''(x_0) \neq 0$ (i.e. x_0 non-degenerate), we can define the "normal form" map $x \to x^2$ (the "fold map"), and then, clearly,

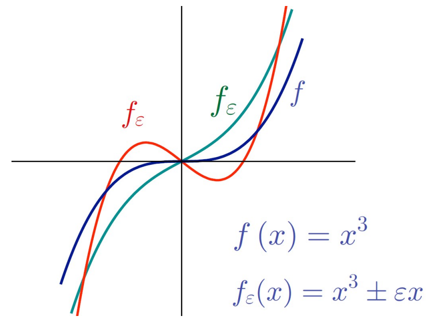
$f(x) \sim \pm x^2$ (locally diffeomorphic by stretching of the coordinates)

Main observation:

stable: If f has a non-degenerate critical (= singular) point, then any f_ε "nearby" has a non-degenerate critical point

dense: If f has a degenerate critical point, then $\exists\ f_\varepsilon$ arbitrarily close having only non-degenerate critical points





stable *dense*

So, in one dimension, we have just one stable singular situation, the fold point, and maps with "fold singularities" are dense.

The aim of singularity theory is to generalize this observation to arbitrary dimensions.

We begin with a general definition of *stable functions*.

3.1. **Stable functions.**

Definition 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth mapping.

(a) A point $u \in \mathbb{R}^n$ is a regular point for f if the rank of the Jacobian matrix $J_f(x) = (\partial f_i/\partial x_j)$ is maximal; the image $f(x)$ of a regular point x is a regular value.

(b) A point $u \in \mathbb{R}^n$ is a singular point (or critical point) for f if the rank of the Jacobian matrix $J_f(x) = (\partial f_i/\partial x_j)$ is less than maximal. The image $f(x)$ of a singular point x is a singular value (or critical value) of f. The set of singular points is the singular set, usually denoted by Σ.

Next, we introduce the notion of differential equivalence of smooth maps.

Definition 3.2. Let $U\,(x_0)$ denote a neighborhood of a point $x_0 \in \mathbb{R}^n$. A smooth mapping $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is locally diffeomorphic to a map $g : V \subset \mathbb{R}^n \to \mathbb{R}^m$ if there exist diffeomorphisms $\alpha : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ and $\beta : f(U) \subset \mathbb{R}^m \to g(V) \subset \mathbb{R}^m$ such that the diagram

$$\begin{array}{lcl} U \subset \mathbb{R}^n & \xrightarrow{f} & f(U) \subset \mathbb{R}^m \\ \downarrow \alpha & & \downarrow \beta \\ V \subset \mathbb{R}^n & \xrightarrow{g} & g(V) \subset \mathbb{R}^m \end{array}$$

commutes.

The connection with stability is then expressed in the following.

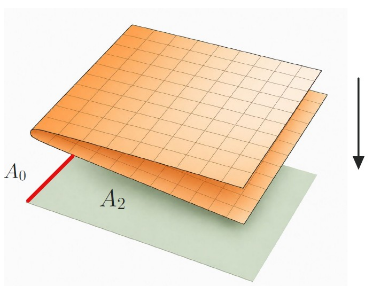
Definition 3.3. A smooth map $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is stable in $C^k\,(U, \mathbb{R}^m)$ if for some $\epsilon > 0$, every map $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ with $\|f - g\|_{C^k} \le \epsilon$ is locally diffeomorphic to f.

As we will see, necessary and sufficient conditions for a function to be diffeomorphic to a normal form are given by certain transversality conditions for f (the nontriviality of certain partial derivatives); hence, it is clear that if f is diffeomorphic to a stable normal form, then it is stable, since all nearby functions will also satisfy these transversality conditions.

3.2. **Whitney's foundations.** Hassler Whitney's 1955 paper "Mappings of the plane into the plane" laid the foundation for modern singularity theory. For smooth maps $f : \mathbb{R}^2 \to \mathbb{R}^2$, Whitney proved that the only *stable singularities* (those whose type persists under small perturbations) are:

i) Fold singularity

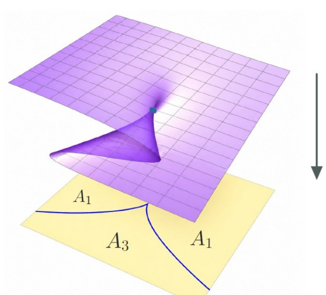
$$\boxed{\begin{array}{l}\varphi : \mathbb{R}^2 \to \mathbb{R}^2 \\ \varphi(x,y) = (x^2, y)\end{array}}$$



$\varphi(x,y) = z \in \mathbb{R}^2$
- component A_0 with 0 preimage
- component A_2 with 2 preimages

ii) Cusp singularity

$$\boxed{\begin{array}{l}\gamma : \mathbb{R}^2 \to \mathbb{R}^2 \\ \gamma(x,y) = (x^3 - xy, y)\end{array}} :$$



$\gamma(x,y) = z \in \mathbb{R}^2$
- component A_1 with 1 preimage
- component A_3 with 3 preimages

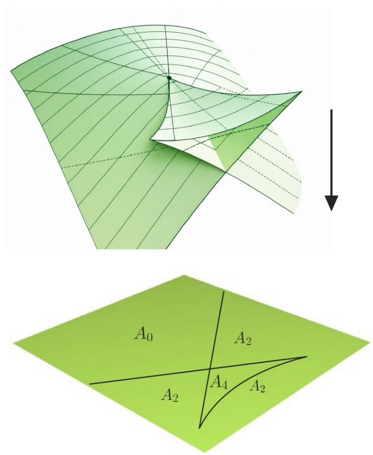
H. Whitney proved the following groundbreaking theorem.

Theorem 3.4. *(H. Whitney, 1955 [25])*
For smooth maps $f : \mathbb{R}^2 \to \mathbb{R}^2$, the fold and the cusp are the only stable singularities, and functions having just these two singularities are dense.

3.3. **Higher singularities.** For smooth maps $f : \mathbb{R}^3 \to \mathbb{R}^3$, an additional stable singularity occurs, the

Swallowtail singularity

$$\boxed{\begin{aligned} &\sigma : \mathbb{R}^3 \to \mathbb{R}^3 \\ &(s,t,r) \mapsto (s^4 + s^2 t + sr, t, r) \end{aligned}}$$



$\sigma(x, y, t) = z \in \mathbb{R}^3$

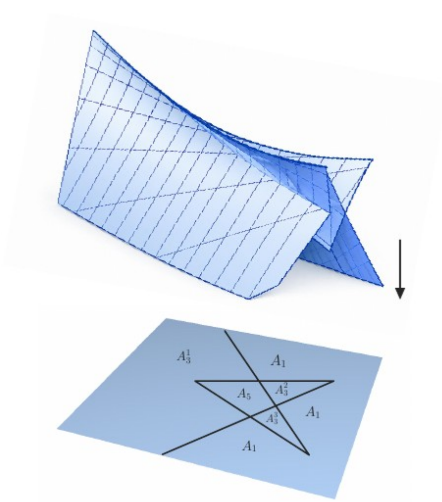
- component A_0 with 0 preimage
- component A_2 with 2 preimages
- component A_4 with 4 preimages

Of course, the fold and the cusp persist in $\mathbb{R}^3$ as trivial extensions

$$\varphi(x, y, z) = (x^2, y, z) \quad \text{and} \quad \gamma(x, y, z) = (x^3 - xy, y, z)$$

For $f : \mathbb{R}^4 \to \mathbb{R}^4$, again a new singularity of the same type appears, the
Butterfly singularity

$$\boxed{\begin{aligned} &\beta : \mathbb{R}^4 \to \mathbb{R}^4 \\ &(s,t,r,q) \mapsto (s^5 + s^3 t + s^2 r + sq, t, r, q) \end{aligned}}$$



$\beta(x, y) = z \in \mathbb{R}^4$
- 1 component A_1 with 1 preimage
- 3 components A_3^1, A_3^2, A_3^3 with 3 preimages
- 1 component A_5 with 5 preimages

For maps $\mathbb{R}^4 \to \mathbb{R}^4$, singularities of a different type also occur: In the singularities which we discussed up to now, the maps $f : \mathbb{R}^n \to \mathbb{R}^n$ (after change of variable) can be written as $f(x) = (f_1(x), x_2, \ldots, x_n)$; one calls such maps "codimension 1 singularities" (or $\mathcal{S}_1$ singularities). For maps in dimension $n = 4$, three "codimension 2 singularities" ($\mathcal{S}_2$ singularities) occur: the *elliptic, parabolic, and hyperbolic umbilic*.

3.4. **Catastrophe theory.** René Thom proved that these 7 singularities are *stable and dense*. He called these singularities the

seven elementary catastrophes

So, the seven elementary catastrophes of R. Thom are:

fold, cusp, swallowtail, and butterfly : codimension 1 singularities

elliptic, parabolic, and hyperbolic umbilic : codimension 2 singularities

Since in our applications to differential equations the codimension 2 singularities do not appear, we do not discuss them here.

In analogy to the *5 regular solids of Platon*, which in Platon's philosophy determine all forms, the *7 elementary catastrophes* determine (in Thom's theory [24]) the *dynamic changes in all phenomena*. Thom argued that these singularities model abrupt qualitative changes ("catastrophes") in physical, biological, and social systems.

3.5. **The Morin singularities.** The above list of $\mathcal{S}_1$ singularities can be extended to any order: The kth singularity of this type is a $\underbrace{S_{1,\ldots,1}}_{k \text{ times}}$ singularity, and is given by the

kth-order Morin singularities (see [19])

$$\boxed{\begin{aligned} &\kappa_k : \mathbb{R}^k \to \mathbb{R}^k \\ &(x_1, \ldots, x_k) \mapsto (x_1^{k+1} + x_1^{k-1} x_2 + \cdots + x_1 x_k, x_2, \ldots, x_k) \end{aligned}}$$

$\kappa_k(x_1, \ldots x_k) = z \in \mathbb{R}^k$

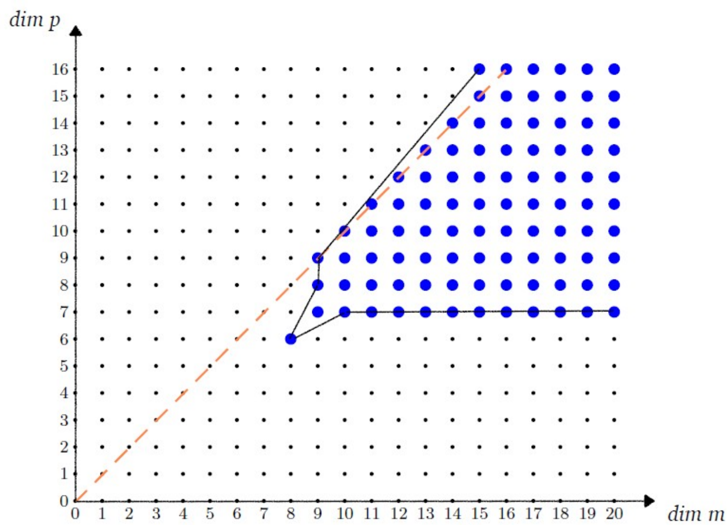
- (k odd): $\exists$ component A_1 with 1 preimage
- (k even): $\exists$ component A_0 with 0 preimage
- $\exists$ component A_k with k preimages

3.6. **Classification of singularities.** With rising dimensions, more and more stable singularities appear; for functions from $\mathbb{R}^m$ to $\mathbb{R}^p$, there is a whole "zoo" of stable singularities. These singularities have been completely classifed, starting with the work of Whitney [25] and Thom [24], and then B. Morin [19], J. Mather [16], and V.I. Arnold [3].

In the beginnings of singlarity theory, it was generally assumed that in all dimensions the stable singularities are dense. Surprisingly, Thom discovered that for maps $\mathbb{R}^{16} \to \mathbb{R}^{16}$, the **stable maps are not dense**. This opened a new research field to determine the dimensions m and p for which the stable maps are dense.

But, already in 1971, John Mather gave a complete answer for all dimensions: With a beautiful theory, he succeeded in determining all dimensions m and p for which the stable singlarities are dense. In his work on the stability of C^∞ mappings, he introduced the concept of **nice dimensions**: pairs (m, p) for which stable maps are dense in $C^\infty(\mathbb{R}^m, \mathbb{R}^p)$.

The boundary of the nice dimensions is given by a complicated algebraic condition. For large m and p, most dimensions are not nice, meaning that "typical" maps may have unstable singularities that cannot be removed by small perturbations.



The "nice" ($\cdot$) and the "not nice" ($\bullet$) dimensions

4. **From finite to infinite dimensions: Singularities in Banach spaces.** In this section, we give a description of singularity theory for nonlinear operators between Banach spaces: $\Phi : E \to F$. In the 1980's M.S. Berger, P.T. Church, J.G. Timourian, F. Lazzeri, and A.M. Micheletti [4, 5, 6, 7, 8, 15] laid the groundwork for this theory, which works for *proper Fredholm maps of index* 0. Indeed, under suitable assumptions on the nonlinear operators, one defines and finds singularities in a similar way as in finite dimensions.

We restrict our attention to the class $\mathcal{S}_1$, the codimension 1 singularities or Morin singularities, in which the singular set consists locally of an (infinite-dimensional) manifold of codimension 1. The singularities contained in this class have the following "stratified" structure; the first singularity, the fold, is characterized by the fact that the singular set Σ_1 is locally a codimension 1 manifold consisting entirely of fold points. The next singularity, the cusp, has the property that the singular set Σ_1 contains (locally) a submanifold $\Sigma_{1,1}$ of codimension 1, consisting of cusp singularities, while $\Sigma_1 \backslash \Sigma_{1,1}$ consists of fold singularities. This process can now be repeated ad infinitum: If we have a swallow tail singularity, then we find a codimension 1 manifold $\Sigma_{1,1,1} \subset \Sigma_{1,1}$ (i.e., $\Sigma_{1,1,1}$ is of codimension 3 with respect to the ambient space) such that $\Sigma_{1,1} \backslash \Sigma_{1,1,1}$ consists of cusps, and $\Sigma_1 \backslash \Sigma_{1,1}$ consists of fold singularities.

4.1. **Singularities for elliptic equations.** We describe here the applicaiton of singularity theory to elliptic boundary value problems of the form

$$\begin{cases} -\Delta u - f(u) = h \ , & \Omega \subset \mathbb{R}^n \quad \text{bounded} \\ u = 0 \ \left(\text{or } \frac{\partial u}{\partial n} = 0\right), & \partial\Omega \end{cases} \tag{1}$$

Associated with (1), consider the mapping

$$\Phi : E = \left\{u \in C^{2,\alpha}(\Omega), \text{b.c.}\right\} \to F = C^{\alpha}(\Omega), \alpha \in (0,1)$$
$$\Phi(u) = -\Delta u - f(u)$$

Since we are in an infinite-dimensional setting, stable singularities of any order may appear, and so we do not look for denseness results. In fact, our focus is as follows.

Aim. Specify conditions on the smooth nonlinearity $g(s)$ under which the singularities (the singular set Σ) of Φ can be classified, and then:

- determine the **geometry** of Σ
- **classify** all singularities
- characterize the **image** $\Phi(\Sigma)$

Consider the linearization of Φ in some point $u \in E$:

$$\Phi'(u)[v] = -\Delta v - f'(u)v \ , \ v \in E$$

The linear operator $\Phi'(u) : E \to F$ has a discrete spectrum; denote by $\mu_i(u), i \in \mathbb{N}$, the sequence of eigenvalues.

Definition. $u \in E$ is a **singular point** if $\Phi'(u)[v] = 0$ for some $v \in E \setminus \{0\}$. Denote by $\Sigma \subset E$ the singular set $\Sigma = \left\{u \in E : \ u \text{ singular point}\right\}$.

Note that $\Phi'(u)[v] = 0$ (for some $v \in E \setminus \{0\}$) if and only if 0 is an eigenvalue of $\Phi'(u)$, with corresponding eigenfunction $v(u)$, i.e.

$$\Sigma = \left\{u \in E; u \text{ singular point} \ ; \ \mu_i(u) = 0 \text{ for some } \ i \in \mathbb{N}\right\}$$

Assumption. We make throughout this article the assumption that

$$\boxed{\mu_i(u) \ \text{ is a simple eigenvalue}} \tag{2}$$

As a consequence of this assumption, only codimension 1 singularities of Morin type will appear.

Due to assumption (2), the corresponding eigenfunction $v_i(u)$ is (up to normalization and multiplication by -1) unique, and it depends smoothly on u.

From now on, we drop the index i for denoting the ith eigenvalue and eigenfunction.

We can now state the following condition.

Transversality condition. Let $u \in \Sigma$, i.e. $\mu(u) = 0$. Assume there exists a $w \in E$ such that

$$\mu'(u)[w] \neq 0 \ , \quad \text{i.e. } w \text{ is transversal to } \Sigma \text{ in } u$$

Denote $\Sigma_1 = \left\{u \in \Sigma \ ; \text{ transversality holds in } u \ \right\}$. Then,

$$\Sigma_1 : \text{ locally a codimension 1 manifold in } E$$

Definition. $u_0 \in \Sigma_1$ is a **fold singularity** for Φ if $v(u)$ is transversal to Σ_1, i.e.

$$\mu'(u)[v(u)] \neq 0$$

Note that this condition says that $v(u_0) \notin T_u\Sigma_1$, where $T_u\Sigma_1$ denotes the tangent space to Σ_1 in the point u.

The definition "fold singularity" is justified by the following proposition.

Proposition 4.1. *Suppose that u_0 is a fold singularity for Φ. Then, $\Phi(u)$ is locally diffeomorphic to the map*

$$\varphi : \mathbb{R} \times X \to \mathbb{R} \times X \ , \ \varphi(t,x) = (t^2, x) \ ,$$

i.e., there exist a Banach space X and diffeomorphisms $\alpha : U(x_0) \to \mathbb{R} \times X$ and $\beta : U(\Phi(x_0)) \to \mathbb{R} \times X$ such that $\Phi(u) = \alpha \circ \varphi \circ \beta^{-1}(u)$.

From now on, we write

$$\Phi(u) \sim \varphi(s,x)$$

for "locally diffeomorphic".

For remarks onthe proof, see below.

Suppose now that $u \in \Sigma_1$ is not a fold point: $\mu'(u)[v(u)] = 0$. Define

$$\Sigma_{1,1} = \{u \in \Sigma_1; \exists\, w \in T_u\Sigma_1 : (\mu'(u)[v(u)])'[w] \neq 0\}$$

Then, $\Sigma_{1,1} \subset \Sigma_1$ is a codimension 1 manifold in Σ_1, respectively a codimension 2 submanifold in E.

Note that since $\mu'(u)[v(u)] = 0$, we have

$$v(u) \in T_u\Sigma_1(u)$$

Definition. $u \in \Sigma_{1,1}$ is a **cusp singularity** for Φ if

$$\left(\mu'(u)[v(u)]\right)'[v(u)] \neq 0$$

For abbreviation, we write

$$\mu_{vv}(u) := \left(\mu'(u)[v(u)]\right)'[v(u)]$$

and analogously for any higher derivative of this kind.

Proposition 4.2. *If u is a cusp singularity for Φ, then*

$$\Phi(u) \sim \quad \psi(s,t,x) := (s^3 + st, t, x)$$

$$\psi : \mathbb{R}^2 \times X \to \mathbb{R}^2 \times X$$

In the same manner, we have the following definition.

Definition. $u \in \Sigma_{1,1,1}$ is a **swallowtail singularity** for Φ, if $v(u) \in T_u\Sigma_{1,1}$ and

$$\mu_{vvv}(u) \neq 0$$

Proposition 4.3. *If u is a swallowtail singularity for Φ, then*

$$\Phi(u) \sim \quad \sigma(s,t,r,x) := (s^4 + s^2t + sr, t, r, x)$$

$$\sigma : \mathbb{R}^3 \times X \to \mathbb{R}^3 \times X$$

Definition. $u \in \Sigma_{1,1,1,1}$ is a **butterfly singularity**, if $v(u) \in T_u\Sigma_{1,1,1}$ and

$$\mu_{vvvv}(u) \neq 0$$

Proposition 4.4. *If u is a butterfly singularity for Φ, then*

$$\Phi(u) \sim \quad \beta(s,t,r,q,x) := \left(s^5 + s^3t + s^2r + st, r, q, x\right)$$

$$\beta : \mathbb{R}^4 \times X \to \mathbb{R}^4 \times X$$

This process continues ad infinitum.

Definition. $u \in \Sigma_{\underbrace{1,\dots,1}_{k \text{ times}}}$ is a **kth-order Morin singularity** for Φ if

$$\mu_{\underbrace{v\dots v}_{k \text{ times}}}(u) \neq 0$$

Proposition 4.5. *Suppose that u is a kth-order Morin singularity for Φ. Then,*

$$\begin{aligned}\Phi(u) \sim \quad & m_k(t, s_1, \dots, s_{k-1}, x) = (t^{k+1} + t^{k-1}s_1 + t^{k-2}s_2 + \cdots + ts_{k-1}, s_1, \dots, s_{k-1}, x) \\ & m_k : \mathbb{R}^k \times X \to \mathbb{R}^k \times X\end{aligned}$$

where m_k is the normal form of the kth Morin singularity.

The proof of these propositions is based (as in the finite-dimensional case) on the following consequence of the *generalized Malgrange preparation theorem* (in the Banach space setting). In the version suitable for our purposes, it states the following.

Theorem 4.6. *Let X be a Banach space, U a neighborhood of the origin $(0, \mathbf{0})$ of $\mathbb{R} \times X$, and $f : U \to \mathbb{R}$ a smooth function such that*

$$f(0, \mathbf{0}) = \frac{\partial}{\partial t} f(0, \mathbf{0}) = \cdots = \frac{\partial^{k-1}}{\partial t^{k-1}} f(0, \mathbf{0}) = 0 \ , \ \frac{\partial^k}{\partial t^k} f(0, \mathbf{0}) \neq 0 \ .$$

Then, there exist smooth functions $a_j : U \to \mathbb{R}$, $j = 0, \dots, k-1$, vanishing at $(0, \mathbf{0})$ such that

$$t^k = \sum_{j=0}^{k-1} a_j(f(t, \mathbf{z}), \mathbf{z}) \ t^j$$

For more details, see [22].

5. **Appearance of higher singularities.** In the theorem of Ambrosetti-Prodi [1], the convexity assumption $f'' > 0$ is essential. It is actually almost necessary for obtaining a pure fold structure. When convexity is relaxed, more complex singularities emerge.

5.1. **Beyond convexity in the Ambrosetti-Prodi problem.** Calanchi, Tomei, and Zaccur (2018) [13] studied the equation

$$-\Delta u = f(u) + h(x), \quad u|_{\partial\Omega} = 0, \tag{3}$$

with f satisfying assumptions (i) and (ii) of Ambrosetti–Prodi, but replacing (iii) by

(iii') there exists t_0 with $f''(t_0) < 0$.

Theorem 5.1 (Calanchi–Tomei–Zaccur, 2018)**.**
Assume (i) and (ii) of the theorem of Ambrosetti-Prodi 2.1 and the non-convexity assumption (iii').
Then, there exists $h \in F$ such that Equation (3) admits at least four solutions.
Moreover, assume that the following genericity condition holds:

(iv) The functions $f' - \lambda$, f'', and f''' have no common zero.
Then, the operator Φ exhibits (locally) a cusp singularity.

Open problems.

1) Show that in the situation of Theorem 5.1 there occur *swallowtail* singularities.

2) Can one find conditions on $f(s)$ such that Φ is a *globally diffeomorphic to a swallowtail*?

5.2. **The cubic nonlinearity: A global cusp.** Consider the model problem with cubic nonlinearity

$$\begin{cases} -\Delta u + u^3 - \lambda u = h(x) \ , \ \text{in} \ \ \Omega \subset \mathbb{R}^n \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4}$$

For $\lambda < \lambda_1$, the linear part $-\Delta - \lambda I$ is positive definite, so $\Sigma = \emptyset$ and the solutions to (4) are unique for every $h \in C(\Omega)$. As λ crosses eigenvalues, singularities appear. Strikingly, a global description up to homeomorphism can be proved.

Theorem 5.2. *(Church, Dancer, Timourian [14]) There exists $\delta > 0$ such that for $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$, the family*

$$(\Phi_\lambda, \lambda) : E \times \mathbb{R} \to F \times \mathbb{R}, \quad \Phi_\lambda(u) = -\Delta u + u^3 - \lambda u$$

is globally homeomorphic to the cusp map:

$$(t, s, x) \mapsto (t^3 - st, s, x), \quad (t, s, x) \in \mathbb{R} \times \mathbb{R} \times X.$$

Thus, as λ varies through λ_1, the solution structure undergoes a cusp catastrophe. We emphasize that in the construction of the homeomorphism, λ appears as an additional parameter.

In a different approach (see [20]), the parameter λ is fixed. Then, the aim consists of identifying and classifying all singularities that appear, and to arrive at a global decscription of the geometry of the solution structure.

5.3. **A geometric characterization.** For Neumann conditions, the first eigenvalue is $\lambda_1 = 0$ and the first eigenfunction is constant, say $e_1 = 1$. For $\lambda > 0$ (and near 0), a complete *geometric description* of the solution structure of the equation

$$\begin{cases} -\Delta u + u^3 - \lambda u = h(x), \\ \partial_\nu u\big|_{\partial\Omega} = 0. \end{cases} \tag{5}$$

can be given.

Theorem 5.3. *(Ruf, 1990 [20]) For $0 = \lambda_1 < \lambda < \frac{1}{12}\lambda_2$, the operator Φ_λ in (5) has only fold and cusp singularities. Moreover:*

- *The singular set Σ $(= \Sigma_1)$ is a codmension 1 manifold, radially diffeomorphic to the unit sphere in E.*
- *Σ contains a codimension 1 submanifold $\mathcal{C}$ (the "equator", with codimension 2 with respect to E) consisting of cusp singularities. The upper hemispheres Σ^+ and lower hemisphere Σ^- consist of fold singularities.*
- *The image $\Phi_\lambda(\Sigma)$ divides the space F into two components:*

  $$F \setminus \Phi_\lambda(\Sigma) = F_1 \cup F_3$$

  with exactly 1 solution for $h \in F_1$ and exactly 3 solutions for $h \in F_3$. For $h \in \Phi(\Sigma \setminus \mathcal{C})$, there exist exactly two solutions.

5.4. **The occurrence of butterfly singularities.** It turns out that in Theorem 5.3, one cannot avoid a restriction from above for λ. The following theorem shows that for λ near λ_2 (but below λ_2), higher than cusp singularities appear.

Theorem 5.4. *(Ruf, 1995, [21]) For $\Omega = (0,1)$ and λ below and near λ_2, the operator Φ_λ in (3) exhibits **butterfly singularities**. Consequently, there exist forcing terms h such that (3) has at least five solutions.*

The appearance of higher singularities for λ near λ_2 is surprising. We conjecture more.

Open problem.

3) For λ approaching λ_2 (from below), there occur all Morin singularities. As a consequence, for every $n \in \mathbb{N}$ there exist $h_n \in F$ with n solutions.

6. Polynomial nonlinearites of odd degree.

6.1. **Bound on the number of solutions.** In this section, we consider the semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u + a_n(x)u^n + a_{n-1}(x)u^{n-1} + \cdots + a_1(x)u = h(x) & \text{in } \Omega \\ \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega \end{cases} \tag{6}$$

with a polynomial nonlinearity of odd degree $n \geq 3$. Here, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$. Under suitable assumptions on the coefficients $a_i(x)$, the number of solutions to Equation (6) can be bounded independently of the right-hand side h. More precisely, if $a_n(x) \geq 1$ and all the other coefficients are sufficiently small (in absolute value), then (6) admits at most n solutions for arbitrary $h \in C^{0,\alpha}(\Omega)$ ($C^{0,\alpha}(\Omega)$ denotes the space of α-Hölder continuous functions for some $\alpha \in (0,1)$).

Let us now give the precise statement of the result. On the coefficients a_i, we make the following assumptions:

(i) $a_i \in C^{0,\alpha}(\Omega, \mathbb{R}), i = 1, \ldots, n$, with n odd;
(ii) $a_n(x) \geq 1, \forall x \in \Omega$;
(iii) $|a_i(x)| \leq \delta, i = 1, \ldots, n-1, \forall x \in \Omega$, for some $\delta > 0$.

With these assumptions, we proved the following theorem.

Theorem 6.1. *(Ruf, 1999, [23]) Suppose that conditions (i)-(iii) are satisfied. If $\delta > 0$ in condition (iii) is sufficiently small, then Equation (6) has, for any forcing term $h \in C^{0,\alpha}(\Omega)$, at most n solutions.*

We remark that similar estimates have been used in Calanchi-Ruf [9, 10] to estimate the number of closed solutions for first-order ODE's with polynomial nonlinearities. This constitutes a special case of Hilbert's 16th problem. In an interesting paper by Malta, Saldanha, and Tomei [17], an example of such an ODE is discussed with a polynomial nonlinearity of degree 4 having 6 periodic solutions.

Corollary 6.2. *Theorem 6.1 implies that no higher singularity than the nth Morin singularity can occur.*

Theorem 6.1 is optimal in the following sense:

For any $\delta > 0$, one can specify (constant) coefficients a_i, $i = 1, \ldots, n-1$, with $|a_i| \leq \delta$ and a function $h \in F$, such that Equation (6) has n solutions. This suggests that Morin singularities of order n do occur.

6.2. **A conjecture.** In view of the estimate of the number of solutions in Theorem 6.1 and the characterization Theorem 5.2 for the cubic equation, we made the following conjecture in [23].

Open problem.

4. **Conjecture** (Ruf [23]). *Let* Φ *denote the mapping*

$$\Phi : E \times \mathbb{R}^{n-2} \to F, \quad (u, a_1, \dots, a_{n-2}) \mapsto -\Delta u + u^n + \sum_{j=1}^{n-2} a_j u^j,$$

where E *and* F *are as in (6.1).*

Suppose n *is odd and* $a_j \in \mathbb{R}$ *with* $|a_j| \le \delta$ $(j = 1, 2, \dots, n-2)$*. Then, for* $\delta > 0$ *sufficiently small, the mapping*

$$\begin{aligned} \Psi :& E \times (-\infty, \delta)^{n-2} \to F \times (-\infty, \delta)^{n-2}, \\ & (u, a_1, \dots, a_{n-2}) \mapsto (\Phi(u, a_1, \dots, a_{n-2}), a_1, \dots, a_{n-2}) \end{aligned}$$

is globally C^∞ *equivalent (1.3) to the Morin singularity* m_{n-1}*.*

Church and Timourian were able to provide a first step towards solving the conjecture: they proved the result locally near the origin.

Theorem 6.3. *(Church-Timourian, 2002)*
Define the $\mathcal{C}^\omega$ *map* $\Psi : E \times \mathbb{R}^{n-2} \to F \times \mathbb{R}^{n-2}$ *by* $\Psi(u, a) = (\Phi(u, a), a)$*, where* $a = (a_1, a_2, \dots, a_{n-2})$ *and* Φ *is given in (6.8). Then, there is a* $\delta > 0$ *sufficiently small that the map germ* $\Psi_{(0,0)}$ *is* $\mathcal{C}^\omega$ *equivalent (1,3) to the Morin map germ* m_{n-1} *at 0.*

Open problem.

5) Specify parameters for which one can give a global geometric characterization of the singular set. We expect it to consist of stratified loci of singular sets, with the following structure:

The singular set Σ_1 is radially diffeomorphic to the unit sphere in E. Σ_1 contains a codimension 1 manifold $\Sigma_{1,1}$ of higher singularities than folds:

$\Sigma_1 \setminus \Sigma_{1,1}$ consists of fold points

$\Sigma_{1,1}$ contains a codimension 1 set $\Sigma_{1,1,1}$ of higher singularities than cusps, and

$\Sigma_{1,1} \setminus \Sigma_{1,1,1}$ consists of cusps points.

And so on, until reaching

$\Sigma_{\underbrace{1,\dots,1}_{k \text{ times}}}$ consisting of kth-order Morin singularities, and $\Sigma_{\underbrace{1,\dots,1}_{k-1 \text{ times}}} \setminus \Sigma_{\underbrace{1,\dots,1}_{k \text{ times}}}$ consisting

of $(k-1)$-th Morin singularities.

All the above examples consider interactions of the nonlinearity with the *first eigenvalue* λ_1 of the Laplacian. In the following section, we give an example of a nonlinearity which interacts with a (simple) eigenvalue λ_k.

7. **Interaction with higher eigenvalues.** In a forthcoming paper, Calanchi and Ruf [11] study a model where the nonlinearity interacts with an arbitrary simple eigenvalue λ_k. We outline here the result in 1 dimension; the higher-dimensional case is work in progress.

Let λ_k be a simple Dirichlet eigenvalue and let erf(s) denote the well-known Gauss error function. Consider

$$f_{a,b}(s) = as - b\,\mathrm{erf}(s), \quad \mathrm{erf}(s) := \frac{2}{\sqrt{\pi}} \int_0^s e^{-t^2} dt,$$

with $0 < a < b$ small. For the equation

$$\begin{cases} -u'' - \lambda_k u + f_{a,b}(u) = h(x) \\ \quad u(0) = u(1) = 0 \end{cases} \tag{7}$$

a complete characterization of the solution structure can be given.

7.1. **Geometric structure of the singular set.** Let

$$\Phi(u) = -\tfrac{d^2}{dx^2} u - \lambda_k u + f_{a,b}(u)$$
$$\Phi : E := \{\mathcal{C}^2(0,1), u(0) = u(1) = 0\} \to F := \mathcal{C}(0,1)$$

The singular set is

$$\Sigma = \{u \in E : \mu_k(u) = 0\},$$

where $\mu_k(u)$ is the kth eigenvalue of $\Phi'(u) = -\frac{d^2}{dx^2} - \lambda_k + a - b\,e^{-u^2}$.

Theorem 7.1 (Calanchi–Ruf [11])**.** *For $b > a > 0$ sufficiently small:*

1. *Σ is a star-shaped C^∞-manifold of codimension 1 in E.*
2. *Σ contains a codimension-1 submanifold $C \subset \Sigma$ consisting of cusp points.*
3. *$\Sigma \setminus C$ consists of fold points.*
4. *$F \setminus \Phi(\Sigma) = F_1 \cup F_3$ has two components: $h \in F_1$ gives exactly 1 solution, $h \in F_3$ gives exactly 3 solutions.*

7.2. **Key technical steps.**

1. **Radial analysis:** For fixed direction $u \in E$ with $\|u\| = 1$, consider $\psi(\rho) := \mu_k(\rho u)$. This gives:
   - $\psi(0) = a - b < 0$,
   - $\psi(\rho) \to a > 0$ as $\rho \to \infty$,
   - $\psi'(\rho) > 0$ for $\rho > 0$.

   Hence, for each normalized $u \in E$, the ray $\psi(\rho)$ meets Σ exactly one, say in $\rho(u)$. Furthermore,

   $$\psi'(\rho) = \mu'(\rho u)[u] = \int_\Omega 2b\,(\rho u)^2 e^{-(\rho u)^2} dx > 0\ ,$$

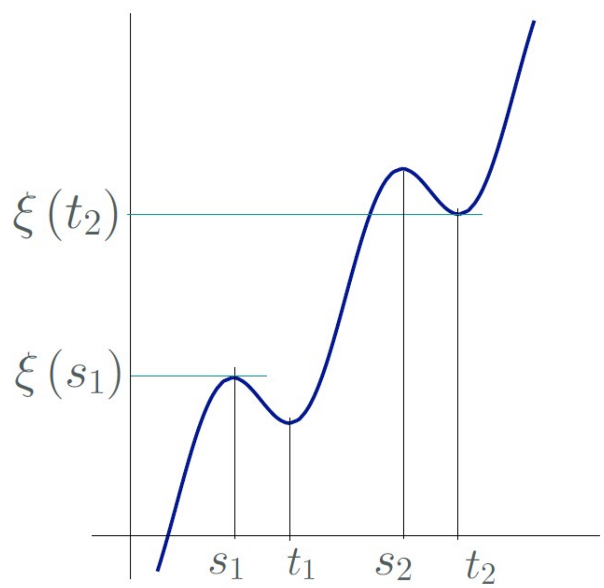
   that is, u is transversal to E in $\rho(u)u$, i.e. $\Sigma = \Sigma_1$, $\Sigma \cong \partial B_1(0)$ is star-shaped, and $E \setminus \Sigma$ consists of exactly two components $E_1 \cup E_3$, where E_3 is bounded and E_1 is unbounded in E.
2. **Detection of non-fold points:** For $z \in \Sigma$ with $\int_\Omega z\,e_k = 0$, consider the paths $\Gamma_z \subset \Sigma$ connecting ρe_k to $-\rho e_k$, where e_k is the kth eigenfunction. The function

   $$\gamma(\alpha) = \mu_k'(\Gamma_z(\alpha))[v_k(\Gamma_z(\alpha))]\ ,\ \alpha \in [-1,1]\ ,\ \gamma_{\pm 1} = \pm \rho e_k$$

   changes sign. By the intermediate value theorem, there exists α_0 with $\gamma(\alpha_0) = 0$, corresponding to a point where v_k is tangent to Σ; that is, $\gamma(\alpha_0)$ is a potential cusp point.

   To make sure that each such point is indeed a cusp, we need to verify the following.

3. **Verification of the cusp condition:** At points where $\mu_k'(u)[v_k(u)] = 0$, one computes the second derivative
$$(\mu_k'(u)[v_k(u)])'[v_k(u)] =$$
$$= 2b\left[\int_\Omega (1-2u^2)e^{-u^2}v_k^4 - 6b\int_\Omega ue^{-u^2}v_k^2\, L^{-1}(ue^{-u^2}v_k^2)\right],$$
where $L = -\frac{d^2}{dx^2} - (\lambda_k - a) - be^{-u^2}$. For small a, b, this expression is positive, confirming the cusp condition.
4. **Φ is injective on Σ:** To prove injectivity, one performs a Lyapunov-Schmidt reduction: Let $u = se_k + u_k \in E_k \oplus [e_k] = E$, where $E_k = P_k E$ (P_k orthogonal projection), and consider the system
$$\begin{cases} -\frac{d^2}{dx^2}u_k - (\lambda_k - a)u_k - bP_k\mathrm{erf}(se_k + u_k) = h_k =: P_k h \\ a\, se_k - b\left(\displaystyle\int_\Omega \mathrm{erf}(se_k + u_k)e_k dx\right) e_k = \left(\displaystyle\int_\Omega he_k\right) e_k \end{cases}$$
The first equation has for fixed $h_k \in E_k$ and any $s \in \mathbb{R}$ a unique solution $u_k(s, h_k)$. It then suffices to study the function
$$\xi_{h_k}(s) := as - b\int_\Omega \mathrm{erf}\big(se_k + u_k(s, h_k)\big)e_k dx$$
The singular set is then given by $\Sigma = \{(s, u_k(s, h_k)) : \xi_{h_k}'(s) = 0\,,\ h_k \in E_k\}$. By similar (but refined) estimates as in 3, one shows that for $0 < a < b$ sufficiently small:
   - If $\xi_{h_k}(s)$ enters and exits E_3 only once, say in $s_1 < t_1$, then $\xi_{h_k}(s_1) > \xi_{h_k}(t_1)$ (since $\xi'(s) < 0$ for $s_1 < s < t_1$), and we are done.
   - If $\xi_{h_k}(s)$ enters and exits E_3 several times, say in $[s_i, t_i]$, $i = 1, \dots k$, then one shows that $\xi_{h_k}(t_{i+1}) > \xi_{h_k}(s_i)$, for all i, and again we are done.

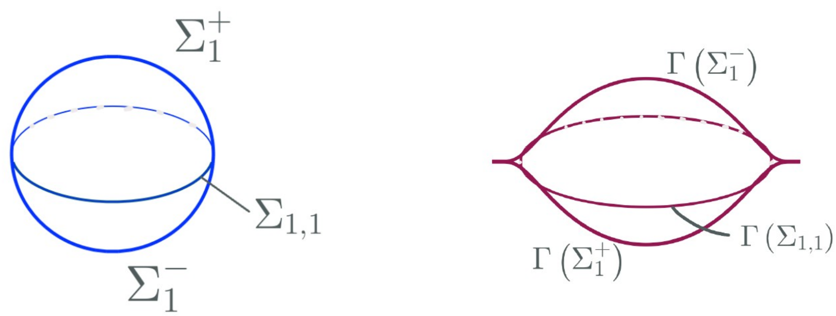


One then concludes as in the Ambrosetti-Prodi case that $F \setminus \Phi(\Sigma) = F_1 \cup F_3$ has exactly two components, and then the result follows since the number of solutions on each component is constant.

Open problem.

5. We conjecture that the map Φ is **globally diffeomorphic** to the explicit map $\Gamma : \mathbb{R}^2 \times X \to \mathbb{R}^2 \times X$ given by

$$\boxed{\Gamma(s,t,x) = \big(s^3 + (|x| - t)s, t, x\big)}$$



8. **Outlook.** The solution structure of nonlinear equations can be very complicated and many solutions may exist. Indeed, in recent years much research efforts were dedicated to find ever more solutions for nonlinear equations, using degree theories, variational methods, Lyapunov-Schmidt reduction, etc.

Our aim here is different: Using singularity theory, we try to find *model equations* which have the minimal number of solutions for the considered type of intersection with the spectrum of the linear operator. We presented some examples in which the *global solution structure* of a nonlinear elliptic equation can be completely characterized by elementary singularities (or "catastrophes"). In particular, one obtains the *exact number* of solutions for forcing terms lying in the various components of the image space. The most complete results give a diffeomorphic equivalence of the nonlinear operators to explicit "normal form" mappings; see the Ambrosetti-Prodi fold, the Church-Dancer-Timourian result on the global cusp, and the conjectured equivalence in *open problem* 5.

These geometric insights not only deepen our theoretical understanding of nonlinear differential equations, but they can also guide numerical computation and provide indications for the solution structure in applications in physics, biology, and engineering.

The extension of finite-dimensional singularity theory to infinite-dimensional operators is an area with connections to bifurcation theory, dynamical systems, and geometric analysis. As tools from algebraic geometry and computational mathematics continue to develop, we can expect further progress in classifying and understanding the singularities that govern solution multiplicity in nonlinear PDEs.

The study of nonlinearities and singularities in elliptic boundary value problems reveals a remarkable interplay between analysis, geometry, and topology. The Ambrosetti–Prodi theorem provides a canonical example where spectral interaction leads to a global fold structure. Relaxing convexity in the Ambrosetti-Prodi problem, or moving a parameter close to the next eigenvalue introduces cusps, swallowtails, and even butterfly singularities, precisely those classified by Thom in finite dimensions.

We close with some further indications on future directions.

Open research directions:

- *Nonlinear mappings which interact with more than one eigenvalue* of the linearized operator: In this situation, several singular surfaces appear, see e.g. [22]. McKean and Scovel [18] studied the equation $-u'' + \frac{1}{2}u^2 = h \text{ in } (0,1), u(0) = u(1) = 0$, and proved the existence of infinitely many singular surfaces. The first singular surface (corresponding to the interaction with the first eigenvalue λ_1) consists of fold singularities. In a remarkable result, Ardila-Tomei [2] showed that the entire critical set consists only of Morin singularities and that all types of Morin singularities occur, thus giving a positive answer to a conjecture by McKean-Scovel. A further characterization of the structure of the singularities on the other surfaces is an open problem.

- *Classification of stable singularities in Banach spaces:* While folds and cusps are well-understood, the full classification of stable singularities for Fredholm maps of index 0 between Banach spaces remains incomplete.

- *Singularities with symmetry:* Many PDEs have symmetry (e.g., radial symmetry, periodicity). How do singularities interact with symmetry-breaking bifurcations?

- Infinite-codimension singularities: What is the solution structure near degenerate singularities that are not stable? These occur generically in problems with parameters.

- *Stochastic perturbations:* How do singularities and solution multiplicities behave under random perturbations of the equation?

REFERENCES

[1] A. Ambrosetti and G. Prodi, On the inversion of some differentiable mappings with singularities between Banach spaces, *Annali di Matematica Pura ed Applicata*, **93** (1972), 231-246.

[2] L. A. Ardila and C. Tomei, *Deep Morin Singularities of the McKean-Scovel Operator*, PhD-thesis by L.A. Ardila, PUC, Brazil, 2021.

[3] V. I. Arnol'd, Singularities of smooth mappings, *Russian Math. Surveys*, **23** (1968), 3-44.

[4] M. S. Berger and P. T. Church, Complete integrability and perturbation of a nonlinear Dirichlet problem. I, *Indiana Univ. Math. J.*, **28** (1979), 935-952.

[5] M. S. Berger and P. T. Church, Complete integrability and perturbation of a nonlinear Dirichlet problem. II, *Indiana Univ, Math. J.*, **29** (1980), 715-735.

[6] M. S. Berger, P. T. Church and J. G. Timourian, An application of singularity theory to nonlinear elliptic partial differential equations, in *Singularities (Arcata, Calif., 1981)*, Proc. Sympos. Pure Math., 40. Amer. Math. Soc., Providence, R.I., 1983, 119-126.

[7] M. S. Berger, P. T. Church and J. G. Timourian, Folds and cusps in Banach spaces, with applications to nonlinear partial differential equations. I, *Indiana Univ. Math. J.*, **34** (1985), 1-19.

[8] M. S. Berger, P. T. Church and J. G. Timourian, Folds and cusps in Banach spaces, with applications to nonlinear partial differential equations. II, *Trans. Amer. Math. Soc.*, **307** (1988), 225-244.

[9] M. Calanchi and B. Ruf, On the number of closed solutions for polynomial ODE's and a special case of Hilbert's 16th problem, *Adv. Differential Equations*, **7** (2002), 197-216.

[10] M. Calanchi and B. Ruf, Hilbert Type Numbers for Polynomial ODE's, *Progress in Nonlinear Differential Equations and Their Applications*, Progr. Nonlinear Differential Equations Appl., 54. Birkhäuser Verlag, Basel, 2003, 53-60.

[11] M. Calanchi and B. Ruf, Cusp formation in elliptic problems with local eigenvalue crossing, Preprint.

[12] M. Calanchi and C. Tomei, Positive eigenvectors and simple nonlinear maps, *Journal of Functional Analysis*, **280** (2021), Paper No. 108823, 35 pp.

[13] M. Calanchi, C. Tomei and A. Zaccur, Cusps and a converse to the Ambrosetti-Prodi Theorem, *Annali della Scuola Normale Superiore di Pisa Classe di Scienze*, **18** (2018), 483-507.

[14] P. T. Church, E. N. Dancer and J. G. Timourian, The structure of a nonlinear elliptic operator, *Transactions of the American Mathematical Society*, **338** (1993), 1-42.

[15] F. Lazzeri and A. M. Micheletti, An application of singularity theory to nonlinear differentiable mappings between Banach spaces, *Nonlinear Analysis: TMA*, **11** (1987), 795-808.

[16] J. N. Mather, Stability of C^∞ mappings: VI. The nice dimensions, *Proceedings of Liverpool Singularities-Symposium I*, Lecture Notes in Math., Vol. 192. Springer-Verlag, Berlin-New York, 1971, 207-253.

[17] T. Malta, N. Saldanha and C. Tomei, Morin singularities and global geometry in a class of ordinary differential operators, *Top. Meth. Nonlin. Analysis*, **10** (1997), 137-169.

[18] H. P. Mc Kean and J. C. Scovel, Geometry of some simple nonlinear differential operators, *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, **13** (1986), 299-346.

[19] B. Morin, Formes canoniques des singularités d'une application différentiable, *C. R. Acad. Sci. Paris*, **260** (1965), 5662-5665.

[20] B. Ruf, Singularity theory and the geometry of a nonlinear elliptic equation, *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, **17** (1990), 1-33.

[21] B. Ruf, Higher singularities and forced secondary bifurcation, *SIAM J. Math. Analysis*, **26** (1995), 1342-1360.

[22] B. Ruf, Singularity theory and bifurcation phenomena in differential equations, *Topological Nonlinear Analysis II*, Progr. Nonlinear Differential Equations Appl., 27. Birkhäuser Boston, Inc., Boston, MA, 1997, 315-395.

[23] B. Ruf, Bounds on the number of solutions for elliptic equations with polynomial nonlinearities, *J. Diff. Eqns.*, **151** (1999), 111-133.

[24] R. Thom, *Structural Stability and Morphogenesis*, W. A. Benjamin, Inc., Reading, Mass.-London-Amsterdam, 1976.

[25] H. Whitney, On singularities of mappings of euclidean spaces. I. Mappings of the plane into the plane, *Annals of Mathematics*, **62** (1955), 374-410.

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