Metrology of weak quantum perturbations

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We consider quantum systems with a Hamiltonian containing a weak perturbation i.e. $\boldsymbol{H} = \boldsymbol{H_0} + \boldsymbol{\lambda} \cdot \boldsymbol{\tilde{H}}$, $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, ...\}$, $\boldsymbol{\tilde{H}} = \{H_1, H_2, ...\}$, $|\boldsymbol{\lambda}| \ll 1$, and address situations where $\boldsymbol{\tilde{H}}$ is known but the values of the couplings $\boldsymbol{\lambda}$ are unknown, and should be determined by performing measurements on the system. We consider two scenarios: in the first one we assume that measurements are performed on a given stationary state of the system, e.g., the ground state, whereas in the second one an initial state is prepared and then measured after evolution. In both cases, we look for the optimal measurements to estimate the couplings and evaluate the ultimate limits to precision. In particular, we derive general results for one and two couplings, and analyze in details some specific qubit models. Our results indicates that dynamical estimation schemes may provide enhanced precision upon a suitable choice of the initial preparation and the interaction time.

I. INTRODUCTION

It is often the case that relevant physical phenomena correspond to weak perturbations to a stable unperturbed situation. This happens in a wide range of disciplines, ranging from applied mathematics [1], biology [2] to chemistry [3] and physics [4]. In these situations, the nature of the perturbations is usually known, whereas the strenghts of the perturbations are the quantities of interest. The Hamiltonian of those systems may be generally written as

$$\boldsymbol{H} = \boldsymbol{H_0} + \boldsymbol{\lambda} \cdot \boldsymbol{\tilde{H}}, \qquad (1)$$

where $\boldsymbol{H_0}$ and $\boldsymbol{\tilde{H}} = \{H_1, H_2, ...\}$ are known Hamiltonian operators and $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, ...\}$ with $|\boldsymbol{\lambda}| \ll 1$ is a vector of *small* unknown coupling parameters, whose values are unknown, and should be determined by performing measurements on the system. To achieve this goal, there are two paradigmatic approaches, which will be referred to as *static* and *dynamical* estimation schemes throughout the paper. In the first one, the system may be prepared in a given stationary state, usually the ground state, which is measured to gain information about the value of the parameters. In a dynamical scenario, the system is instead prepared in a certain state, left to evolve for a given interaction time and finally measured. In a dynamical estimation scheme, the initial state, as well as interaction time, may be optimized and thus the overall precision may be enhanced compared to a static scheme, though the practical implementation may be more challenging. Besides the case of small perturbations to a given Hamiltonian $\boldsymbol{H_0}$, the Hamiltonian in Eq. (1) may also describe systems where the couplings have some target values $\boldsymbol{\overline{\lambda}_0}$ and the scope of the measurement is to monitor the system [5–7], i.e. to estimate possible deviations $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 - \boldsymbol{\overline{\lambda}_0}$ from those values.

A convenient framework to investigate the precision achievable by static and dynamical estimation schemes is that of quantum estimation theory [8–13], which provides a set of tools to determine the measurement that has to be performed on the system, i.e. to find the observable that is most sensitive to tiny variations of the parameters [14–25], and to optimize the initial preparation of the probe [26–35].

In particular, if the value of a single parameter is encoded in the family of quantum states $\{|\psi_\lambda\rangle\}$ (usually referred to as the *quantum statistical model*), one may prove that the ultimate precision achievable in estimating λ is obtained by measuring the observable L_λ, known as symmetric logarithmic derivative (SLD), which is the self-adjoint operator given by

$$L_\lambda = 2 \big[|\partial_\lambda \psi_\lambda\rangle \langle \psi_\lambda| + |\psi_\lambda\rangle \langle \partial_\lambda \psi_\lambda| \big]. \qquad (2)$$

Upon collecting the result of M repeated measurements on identical preparations of the system and suitably processing data (e.g. by maxlik [36–39] or Bayesian analysis [40, 41]) the uncertainty in the determination of λ, i.e., the precision of the estimation scheme, is given by

$$\mathrm{Var}\lambda \simeq \frac{1}{M\, H(\lambda)} \qquad (3)$$

where $H(\lambda)$ is the so-called Quantum Fisher information of the quantum statistical model $\{|\psi_\lambda\rangle\}$, i.e.,

$$Q_\lambda = 4 \left[\langle \partial_\lambda \psi_\lambda | \partial_\lambda \psi_\lambda \rangle - |\langle \partial_\lambda \psi_\lambda | \psi_\lambda \rangle|^2 \right], \qquad (4)$$

(notice that $\langle \psi_\lambda | \partial_\lambda \psi_\lambda \rangle$ is a purely imaginary c-number, i.e., $\langle \psi_\lambda | \partial_\lambda \psi_\lambda \rangle^* = \langle \partial_\lambda \psi_\lambda | \psi_\lambda \rangle = - \langle \psi_\lambda | \partial_\lambda \psi_\lambda \rangle$).

The generalization to the estimation of more than one parameter can be obtained by introducing the so-called quantum

Fisher information matrix (QFIM) $\boldsymbol{Q}$, which is a real symmetric matrix with entries

$$Q_{\mu\nu} = 4\left[\text{Re}\,\langle\partial_\mu\psi|\partial_\nu\psi\rangle + \langle\partial_\mu\psi|\psi\rangle\,\langle\partial_\nu\psi|\psi\rangle\right] \qquad (5)$$

The QFIM provides a bound on the covariance matrix (CM) of the estimates

$$\text{Cov}(\boldsymbol{\lambda}) \geq \frac{1}{M}\,\boldsymbol{Q}^{-1}\,. \qquad (6)$$

This is a matrix inequality, and it cannot, in general, be saturated. Physically, this corresponds to the unavoidable quantum noise that originate when the SLDs corresponding to different parameters do not commute [42]. In those cases, the total variance $\sum_\mu V(\lambda_\mu)$ (or a weighted combination of the CM elements) is a more interesting quantity to study, and since the μth diagonal entry of the covariance matrix is just the variance of the parameter λ_μ, the bound on the total variance is given as

$$\sum_\mu V(\lambda_\mu) \geq \frac{B}{M} \qquad B = \text{Tr}\left[\boldsymbol{Q}^{-1}\right]\,. \qquad (7)$$

The incompatibility between the parameters can be quantified by the so-called *asymptotic incompatibility* [43–46], also referred to as the *quantumness* of the quantum statistical model. This is defined as

$$R := \|i\,\boldsymbol{Q}^{-1}\boldsymbol{D}\|_\infty \qquad (8)$$

where $\|\boldsymbol{A}\|_\infty$ is the largest eigenvalue of the matrix $\boldsymbol{A}$ and

$$D_{\mu\nu} = -\frac{i}{2}\,\langle\psi_{\boldsymbol{\lambda}}|[L_\mu, L_\nu]|\psi_{\boldsymbol{\lambda}}\rangle = 4\,\text{Im}\,\langle\partial_\mu\psi|\partial_\nu\psi\rangle \qquad (9)$$

is the Uhlmann curvature of the statistical model. The quantity R is a real number in the range $0 \leq R \leq 1$ with the equality $R = 0$ satisfied for compatible parameters, i.e., when $\langle\psi_{\boldsymbol{\lambda}}|[L_\mu, L_\nu]|\psi_{\boldsymbol{\lambda}}\rangle = 0$. For just two parameters, one may write

$$R = \sqrt{\frac{\det \boldsymbol{D}}{\det \boldsymbol{Q}}}\,. \qquad (10)$$

A tighter scalar bound, known as the Holevo-Cramer-Rao bound $\sum_\mu V(\lambda_\mu) \geq C_H/M$, with $C_H \geq B$, may also be derived (see [47] for details), and the quantumness R provides a bound to the normalized difference between C_H and B as follows

$$\frac{C_H - B}{B} \leq R\,. \qquad (11)$$

In the following Sections, we aim at finding general formulas for B and R for estimation problems involving the parameters of weakly perturbed systems in both the static and the dynamical estimation scenarios. We also analyze some specific models involving qubits, qutrits and harmonic oscillators, and where the Holevo-Cramer-Rao bound is known analitically, we check whether the inequality in Eq. (11) is

tight. More precisely, Section II is devoted to static estimation schemes, with Section II A reporting general results and Sections II B, II C, and II D devoted to specific models involving qubit, qutrit, and oscillatory systems, respectively. Section III is devoted to dynamical estimation schemes, with Section III A reporting general results and Sections III B, III C, and III D discussing specific results for qubit, qutrit, and oscillatory systems, also comparing the performance of dynamical schemes to that of the corresponding static ones. Section IV closes the paper with some concluding remarks.

II. STATIC ESTIMATION OF WEAK PERTURBATIONS

In this Section, we address estimation of weak perturbations in systems descibed by one- and two-parameter (time-independent) Hamiltonians of the form $H = H_0 + \lambda H_1$ and $H = H_0 + \lambda_1 H_1 + \lambda_2 H_2$. In particular, we assume that the system may be prepared in a given state (e.g, the ground state) and that repeated measurements may be performed on the system. We derive general expressions for the QFI Q and the quantumness R and discuss specific models involving qubit, qutrit and oscillator systems.

A. General results for one and two parameters

Let us consider a system with Hamiltonian $H = H_0 + \lambda H_1$ where $\lambda \ll 1$. The n-th eigenstate $|\psi_n\rangle$ of H may be obtained perturbatively to first-order in λ as follows

$$|\psi_n\rangle = |\psi_n^0\rangle + \lambda |\psi_n^1\rangle + O(\lambda^2), \qquad (12)$$

where $|\psi_n^0\rangle$ are eigenstates of H_0 and

$$|\psi_n^1\rangle = \sum_{m\neq n} \frac{\langle\psi_m^0| H_1 |\psi_n^0\rangle}{E_n^0 - E_m^0} |\psi_m^0\rangle$$

is the first-order correction to the n-th eigenstate. In general $\langle\psi^1|\psi^1\rangle = N \neq 1$, and it is thus convenient to introduce the state $|\psi^1\rangle = \sqrt{N} |\phi^1\rangle$ and write the first-order corrected eigenstate $|\psi\rangle$ as a combination of two orthonormal states $|\psi^0\rangle$ and $|\phi^1\rangle$. The subscript n is omitted to simplify notation. The perturbed state and its derivative are thus given by

$$|\psi\rangle = |\psi^0\rangle + \lambda\sqrt{N} |\phi^1\rangle \qquad (13a)$$
$$|\partial_\lambda\psi\rangle = \sqrt{N} |\phi^1\rangle\,. \qquad (13b)$$

According to Eq. (2), the SLD of this general model may be written, up to first order in λ as

$$L_\lambda = 2\sqrt{N}\left[|\psi^0\rangle\langle\phi^1| + |\phi^1\rangle\langle\psi^0| + 2\lambda\sqrt{N} |\phi^1\rangle\langle\phi^1|\right], \qquad (14)$$

and the corresponding QFI as

$$Q(\lambda) = 4N + O(\lambda^2)\,. \qquad (15)$$

The QFI is independent on the perturbation (up to second-order) and proportional to the norm of the first-order correction $|\psi^1\rangle$. This is a remarkably intuitive results, linking the estimability of a perturbation to its physical effect on the system. The same result may be also obtained expressing the QFI in terms of fidelity [48] [49]. Notice also that the λ-dependent term in the SLD leads to negligible (second order) contributions to the QFI and may be dropped. The optimal measurement is thus given by

$$L = 2\sqrt{N} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{16}$$

This expression makes it clear that the optimal measurements set coincides with the Pauli matrix σ_x over the basis $\{|\psi^0\rangle, |\phi^1\rangle\}$, i.e. a detection scheme that senses the coherence of the perturbed state in that basis.

Let us now address the case of systems with Hamiltonian of the form $H = H_0 + \lambda_1 H_1 + \lambda_2 H_2$ where H_1 and H_2 are in general non commuting operators, $[H_1, H_2] \neq 0$. In this case the pertubations depend in a non trivial way on two different parameters λ_1 and λ_2, which should be jointly estimated. For weak perturbations, the the n-th eigenstate of H $|\psi_n\rangle$ may be written, in terms of the eigenbasis of H_0, as follows

$$|\psi_n\rangle = |\psi_n^0\rangle + \lambda_1 \sum_{m \neq n} \frac{\langle \psi_m^0 | H_1 |\psi_n^0\rangle}{E_n^0 - E_m^0} |\psi_m^0\rangle$$

$$+ \lambda_2 \sum_{l \neq n} \frac{\langle \psi_l^0 | H_2 |\psi_n^0\rangle}{E_n^0 - E_l^0} |\psi_l^0\rangle \tag{17}$$

$$= |\psi_n^0\rangle + \lambda_1 \sqrt{N_1} |\phi_{n,1}^1\rangle + \lambda_2 \sqrt{N_2} |\phi_{n,2}^1\rangle$$

where $|\phi_{n,\mu}^1\rangle$ are states, i.e. the normalized version of the first-order corrections $|\psi_{n,\mu}^1\rangle = \sqrt{N_\mu} |\phi_{n,\mu}^1\rangle$ having squared norms N_j (with $\mu = 1, 2$ the index of the parameter λ_μ). As we have done before, we drop the index n in order to simplify the notation. These states are not orthogonal one to each other but both are orthogonal to the unperturbed eigenspace of H_0, hence, we can express the perturbed state $|\psi\rangle$ in an orthonormal basis spanned by the triplet $\{|\psi^0\rangle, |j\rangle, |k\rangle\}$, with $\langle j|k\rangle = \delta_{jk}$. Upon writing the states $|\phi^l\rangle\}$ as

$$|\phi^1\rangle = \sin \frac{\theta_1}{2} |j\rangle + e^{i\gamma} \cos \frac{\theta_1}{2} |k\rangle$$

$$|\phi^2\rangle = \sin \frac{\theta_2}{2} |j\rangle + e^{i(\gamma+\varphi)} \cos \frac{\theta_2}{2} |k\rangle ,$$

the perturbed state and its derivatives $|\partial_{\lambda_\mu} \psi\rangle = |\psi_\mu^1\rangle$ may be written as

$$|\psi\rangle = |\psi^0\rangle + \left(\lambda_1 \sqrt{N_1} \cos \frac{\theta_1}{2} + \lambda_2 \sqrt{N_2} e^{i\gamma} \cos \frac{\theta_2}{2} \right) |j\rangle$$

$$+ \left(\lambda_1 \sqrt{N_1} \sin \frac{\theta_1}{2} + \lambda_2 \sqrt{N_2} e^{i(\gamma+\varphi)} \sin \frac{\theta_2}{2} \right) |k\rangle , \tag{18}$$

$$|\partial_{\lambda_1} \psi\rangle = \sqrt{N_1} \left(\cos \frac{\theta_1}{2} |j\rangle + \sin \frac{\theta_1}{2} |k\rangle \right) , \tag{19}$$

$$|\partial_{\lambda_2} \psi\rangle = \sqrt{N_2} e^{i\gamma} \left(\cos \frac{\theta_2}{2} |j\rangle + e^{i\varphi} \sin \frac{\theta_2}{2} |k\rangle \right) . \tag{20}$$

In order to quantify the orthogonality between the two perturbations, we consider the overlap $\omega = \langle \phi_1^1 | \phi_2^1 \rangle$ between the two first-order corrections, i.e.,

$$\omega = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\gamma} + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\gamma+\varphi)} \tag{21}$$

The SLD operators L_1 and L_2 for the two parameters λ_1 and λ_2 may be calculated according to Eq. (2). The explicit expressions are reported in Appendix A. The corresponding QFIM Q and Uhlmann curvature D are given by

$$Q = \begin{pmatrix} 4N_1 & 4\sqrt{N_1 N_2} \operatorname{Re} \omega \\ 4\sqrt{N_1 N_2} \operatorname{Re} \omega & 4N_2 \end{pmatrix} , \tag{22}$$

$$D = \begin{pmatrix} 0 & 4\sqrt{N_1 N_2} \operatorname{Im} \omega \\ -4\sqrt{N_1 N_2} \operatorname{Im} \omega & 0 \end{pmatrix} . \tag{23}$$

The ultimate bound B and the quantumness R thus read as follows

$$B = \frac{N_1 + N_2}{4 N_1 N_2 \left[1 - \operatorname{Re}^2 \omega \right]} , \tag{24}$$

$$R = \sqrt{\frac{\operatorname{Im}^2 \omega}{1 - \operatorname{Re}^2 \omega}} . \tag{25}$$

As expected, the overlap between the perturbations is involved in all the quantities of interest. In particular, a real overlap ($\operatorname{Im} \omega = 0$) always provides maximum compatibility ($R = 0$) between the parameters to estimate. Moreover, if the overlap is zero (both $\operatorname{Re} \omega = 0$ and $\operatorname{Im} \omega = 0$), i.e. perturbations are orthogonal, the QFI matrix is diagonal, meaning that parameters are uncorrelated. On the other hand, if the overlap is a just a phase factor, we have $\operatorname{Re}^2 \omega + \operatorname{Im}^2 \omega = 1$, and thus $R = 1$, i.e. maximal incompatibility between the parameters. This may happen also when the dimension of the probing system is insufficient to estimate a certain number of parameters, as it will be illustrated in the next Section by means of a qubit statistical model.

B. Qubit models

Let us consider a qubit system described by the orthonormal basis states $\{|0\rangle, |1\rangle\}$ of the unperturbed Hamiltonian $H_0 = \sigma_z$ with eigenenergies $E_0 = 1$ and $E_1 = -1$. The perturbed Hamiltonian is given by $H = \sigma_z + \lambda \sigma_x$, where σ_z and σ_x are standard Pauli matrices, and λ is the small perturbation parameter that we want to estimate. The first-order perturbed ground state is given by

$$|\psi\rangle = |0\rangle + \frac{\lambda}{2} |1\rangle , \tag{26}$$

and the first-order corrected state is $|\psi^1\rangle = \frac{1}{2} |1\rangle$ with (squared) norm $N = 1/4$. The corresponding SLD is $L_\lambda = \sigma_x$ and the QFI is given by

$$Q = 1 + O(\lambda^2) \tag{27}$$

confirming the general results in Eqs. (15) and (2).

Let us now consider the more interesting case of a two-parameter perturbation, which highlights the issues arising from using an under-dimensioned (compared to the number of parameters) probe system. The perturbed Hamiltonian is

$$H = \sigma_z + \lambda_1 \sigma_x + \lambda_2(\cos \alpha \, \sigma_x + \sin \alpha \, \sigma_y),$$

where λ_i (with $i = 1, 2$) are the perturbation parameters and α denotes a mixing angle which governs the orthogonality of the two perturbations. The first-order perturbed ground state of the system is given by

$$|\psi\rangle = |0\rangle + \frac{1}{2} \left(\lambda_1 + \lambda_2 \, e^{i\alpha} \right) |1\rangle \,. \tag{28}$$

Looking at the above equation, it is clear that the two perturbations cannot, in general, generate two orthogonal states where information about the two parameters is encoded [44]. In fact, the first-order corrected states corresponding to λ_1 and λ_2 are the same state except for a phase factor. In other words, the two perturbations lead to two degenerate states proportional to $|1\rangle$. Referring to the Bloch sphere representation introduced above, we have $\theta_1 = \theta_2 = 0$ and $\gamma = \alpha$. The overlap in (21) is given by $\omega = e^{i\alpha}$ and the QFIM displays off-diagonal elements. In order to make the two parameters compatible, a probe system with larger dimension should be necessarily employed (see the next Section).

C. Qutrit models

Let us consider a three-dimensional spin-1 system with a perturbed Hamiltonian given by $H = S_z + \lambda_1 S_x + \lambda_2(\cos \alpha \, S_x + \sin \alpha \, S_y)$, where $\{S_z, S_x, S_y\}$ denote the irreducible representation of spin-1 operators in the z-basis:

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \tag{29}$$

with $|m_s\rangle$, $m_1 = \{1, 0, -1\}$ being the standard eigenvectors and eigenvalues of S_z. This Hamiltonian is the direct generalization of that considered in the previous Section, and a comparison will reveal the role of system dimension.

For the eigenstate $|\psi^0\rangle = |0\rangle$, the first-order corrections are given by

$$|\psi_1^1\rangle = \frac{|1, -1\rangle - |1, 1\rangle}{\sqrt{2}} = |\phi_1^1\rangle \tag{30a}$$

$$|\psi_2^1\rangle = \frac{e^{i\alpha} |1, -1\rangle - e^{-i\alpha} |1, 1\rangle}{\sqrt{2}} = |\phi_2^1\rangle \,, \tag{30b}$$

with squared norms given by $N_1 = N_2 = 1$. It is easy to see that these perturbation states live in two-level subsystem spanned by $|j\rangle = |1\rangle$ and $|k\rangle = |-1\rangle$, and that they may

be expressed as in Eq. (18) by setting $\theta_1 = \theta_2 = 3\pi/2$, $\gamma = -\alpha$ and $\varphi = 2\alpha$. The resulting overlap is real and given by $\omega = \cos \alpha$. In this case, the resulting QFI matrix $\boldsymbol{Q}$ and the ultimate bound B are

$$\boldsymbol{Q} = 4 \begin{pmatrix} 1 & \cos \alpha \\ \cos \alpha & 1 \end{pmatrix} \qquad B = \frac{\csc^2 \alpha}{2} \,, \tag{31a}$$

whereas the mean Uhlmann curvature is vanishing and the quantumness is zero $R = 0$. Moreover, the two perturbed states become orthogonal for $\alpha = \pi/2$, which corresponds to apply non-overlapping perturbations. The QFI matrix becomes diagonal, meaning that the two parameters to be estimated are uncorrelated, and the ultimate bound $B = 1/2$ is minimal and coincides with the Holevo bound C_H. Notice that perturbing a different eigenvector, say $|1, 1\rangle$, the situation is dramatically different, since he two perturbations S_x and S_y generate the same first-order perturbed state $|1, 0\rangle$ and the resulting overlap is $\omega = e^{i\alpha}$.

Summarizing, a two-parameter perturbation cannot be suitably characterized (with maximum precision and compatibility) using a qubit system, whereas the use of a qutrit system allows one to achieve the ultimate limits to precision, via a proper choice of the encoding Hamiltonian terms and of the initial unperturbed state.

D. A quantum anharmonic oscillator model

An relevant class of models that may be treated in our formalism is that of a quantum oscillator weakly perturbed by anharmonic terms. Here the anharmonic couplings are the quantities to be estimated, e.g., because they may represent a resource [50]. The Hilbert space of the system is infinite dimensional and may offer an ideal playground to encode as much information as needed.

For the sake of simplicity, we choose natural units ($\hbar = 1$) and set the frequency and the mass of the oscillator to one $m = \omega = 1$. The position and momentum operators can be expressed in terms of ladder operators as $x = (a + a^\dagger)/\sqrt{2}$ and $p = i(a^\dagger - a)/\sqrt{2}$ and hence the unperturbed hamiltonian can be expressed as $H_0 = (p^2 + x^2)/2 = a^\dagger a + 1/2$ with corresponding eigenenergies $E_n^0 = n + 1/2$. We consider anharmonic perturbations to the harmonic potential such that the perturbed Hamiltonians read

$$H = \frac{1}{2} \left(p^2 + x^2 \right) + \epsilon_1 x^3 + \epsilon_2 x^4, \tag{32}$$

where we introduced the two anharmonic couplings ϵ_1 and ϵ_2 as the unknown parameters to be estimated. The perturbed ground state of the system may be obtained as in Eq. (17), where the two first order corrections are given by

$$|\psi_1^1\rangle = -\frac{1}{2} \left(\frac{3}{\sqrt{2}} |1\rangle + \frac{1}{\sqrt{3}} |3\rangle \right) \tag{33a}$$

$$|\psi_2^1\rangle = -\frac{1}{2} \left(\frac{3}{\sqrt{2}} |2\rangle + \frac{1}{2} \sqrt{\frac{3}{2}} |4\rangle \right) \,, \tag{33b}$$

with squared norms $N_1 = \frac{29}{24}$ and $N_2 = \frac{39}{32}$. The two perturbed states (33) are orthogonal and the same happens for any eigenstate of the perturbed Hamiltonian. The QFI (22) matrix reads

$$\boldsymbol{Q} = 4 \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}, \qquad (34)$$

leading to $B = 466/1131 \simeq 0.41$. The Uhlmann curvature (23) vanishes, corresponding to a zero quantumness (25). In conclusion, preparing a quantum oscillator in its vacuum state, is an efficient strategy to precisely sense the amplitude of anharmonic perturbations.

III. SENSING PERTURBATIONS BY DYNAMICAL PROBES

In this Section, we address detection of weak perturbations by performing measurements on an evolved state $|\psi(t)\rangle = e^{-iHt}|\psi^0\rangle$ where $|\psi^0\rangle$ is a (non stationary) given initial state and H is the perturbed Hamiltonian under investigation.

A. General results for one and two parameters

Let us start with a perturbation described by a single parameter Hamiltonian. In order to obtain the QFI it is convenient to move into the interaction picture (with respect to the unperturbed Hamiltonian H_0), where the state vector is given by the unitary transformation $|\psi_I(t)\rangle = U_0^\dagger(t)|\psi(t)\rangle$, being $U_0(t) = e^{-iH_0 t}$. The whole time evolution is expressed by

$$|\psi_I(t)\rangle = U_I(t)|\psi^0\rangle, \qquad (35a)$$

$$U_I(t) = \mathcal{T}\left[\exp\{-i\lambda K(t)\}\right], \qquad (35b)$$

$$K(t) = \int_0^t ds\, U_0^\dagger(s) H_1 U_0(s), \qquad (35c)$$

where $\mathbb{I}$ denotes the identity matrix, $\mathcal{T}[...]$ denotes time-ordering and the operator $K(t) = K^\dagger(t)$ is hermitian. Up to first order in λ we have

$$U_I(t) \simeq \mathbb{I} - i\lambda K(t), \qquad (36)$$

Going back to the Schröedinger picture the evolved state and its derivative with respect to the unknown parameter may be written as

$$|\psi_\lambda(t)\rangle = U_0(t)\left[\mathbb{I} - i\lambda K(t)\right]|\psi^0\rangle \qquad (37a)$$

$$|\partial_\lambda \psi_\lambda(t)\rangle = -iU_0(t)K(t)|\psi^0\rangle. \qquad (37b)$$

The leading-order behaviour corresponds to a λ-independent (zero-th order) expression of the QFI

$$Q(t) = 4\left[\langle\psi^0|K^2(t)|\psi^0\rangle - \langle\psi^0|K(t)|\psi^0\rangle^2\right]. \qquad (38)$$

Despite it may appear as a rough approximation, this expression of the QFI allows us to grab the main features of the dynamical case and to made a comparison to the static one. The

QFI in Eq. (38) depends on time and is independent on λ. In other words, for weak perturbations the evolution introduces a time dependence, whereas it does not affect the covariant nature of the estimation problem.

Analogously, in the case of a two-parameter Hamiltonian $H = H_0 + \lambda_1 H_1 + \lambda_2 H_2$, the time evolution operator in the interaction picture can be approximated at first-order as $U_I(t) \simeq \mathbb{I} - i\int_0^t ds\, U_0^\dagger(s)(\lambda_1 H_1 + \lambda_2 H_2) U_0$. Upon introducing the operators

$$K_1(t) = \int_0^t ds\, U_0^\dagger(s) H_1 U_0(s), \qquad (39)$$

$$K_2(t) = \int_0^t ds\, U_0^\dagger(s) H_2 U_0(s), \qquad (40)$$

the leading order of the elements of the QFI matrix may be evaluated as follows

$$Q_{11} = 4\left[\langle\psi^0|K_1^2|\psi^0\rangle - \langle\psi^0|K_1|\psi^0\rangle^2\right]$$
$$Q_{12} = 4\left[\text{Re}\,\langle\psi^0|K_1 K_2|\psi^0\rangle - \langle\psi^0|K_1|\psi^0\rangle\langle\psi^0|K_2|\psi^0\rangle\right]$$
$$Q_{21} = 4\left[\text{Re}\,\langle\psi^0|K_2 K_1|\psi^0\rangle - \langle\psi^0|K_2|\psi^0\rangle\langle\psi^0|K_1|\psi^0\rangle\right]$$
$$Q_{22} = 4\left[\langle\psi^0|K_2^2|\psi^0\rangle - \langle\psi^0|K_2|\psi^0\rangle^2\right], \qquad (41)$$

where we omitted the time dependence. The matrix elements of the Uhlmann curvature are given by

$$D_{12} = 4\,\text{Im}\,\langle\psi^0|K_1 K_2|\psi^0\rangle = -D_{21} \qquad (42)$$

and the quantumness parameter R reads as follows

$$R = 4\frac{\left|\text{Im}\,\langle\psi^0|K_1 K_2|\psi^0\rangle\right|}{\sqrt{\det \boldsymbol{Q}}}. \qquad (43)$$

Now that the general framework has been set, in the following we re-examine some of the examples of the previous Sections in order to compare the performance of static and dynamical estimation schemes.

B. Qubit models

Let us consider a single qubit, initially prepared in generic state $|\psi^0\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle$. Given the results of Section II B, we consider a single-parameter perturbation. The system evolves according to the unitary $U = \exp(-itH)$ where t is the interaction time and $H = \sigma_3 + \lambda\sigma_1$ is the perturbed Hamiltonian with λ small. Using Eqs. (35c) and (38) we have

$$K(t) = e^{it}\sin t\,|0\rangle\langle 1| + e^{-it}\sin t\,|1\rangle\langle 0| \qquad (44a)$$

$$Q(t) = 4\sin^2 t\left[1 - \cos^2(t+\phi)\sin^2(\theta)\right]. \qquad (44b)$$

In order to compare this result with the QFI obtained in the static case, we set $|0\rangle$ as the initial (unperturbed) state at $t = 0$, i.e. $\theta = 0$. The dynamical QFI is given by $Q(t) = 4\sin^2 t$, and achieves a maximum at $t = \pi/2$, where it is four times greater than the corresponding static QFI.

C. Qutrit models

We consider the same spin-1 system as in Section II C and the same Hamiltonian. In order to compare results with the static scenario, we set the initial state to $\left|\psi^0\right\rangle = |1,0\rangle$, the QFI and the bound B reads:

$$\boldsymbol{Q} = 16 \sin^2 \frac{t}{2} \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}, \qquad \boldsymbol{D} = \boldsymbol{0} \qquad (45)$$

$$\boldsymbol{B} = \left(8 \sin^2 t/2 \sin^2 \alpha\right)^{-1} \qquad \boldsymbol{R} = 0. \qquad (46)$$

The D matrix and the R parameter vanish, i.e. we have compatibility between the two parameters. The bound B is minimized for orthogonal perturbations $\alpha = \pi/2$, and for $t = \pi$. As it happens with qubits, in the dynamical scenario the bound is improved by a factor four.

D. Anharmonic oscillator

We consider the same system of Section II D, prepare the oscillator in the unperturbed ground state and let it evolve according to the perturbed Hamiltonian. Using Eqs. (39) and (40) we evaluate the QFIM, which is a diagonal matrix with entries (see Appendix B for details)

$$Q_{11} = \frac{29}{3} - 9\cos t - \frac{2}{3}\cos 3t \qquad (47)$$

$$Q_{22} = 3\left(7 + \cos 2t\right)\sin^2 t \qquad (48)$$

$$Q_{12} = Q_{21} = 0, \qquad (49)$$

whereas the quantumness R vanishes.

In Fig. 1 we show the bound B as a function of time (B is a periodic function) compared to the static bound. As it is apparent from the plot, the dynamical scheme beats the static one in the range $t \in (0.721, 2.79)$. The absolute minimum is obtained for $t \simeq 2.0$, where we have $B \simeq 0.14$, clearly lower than the corresponding static value.
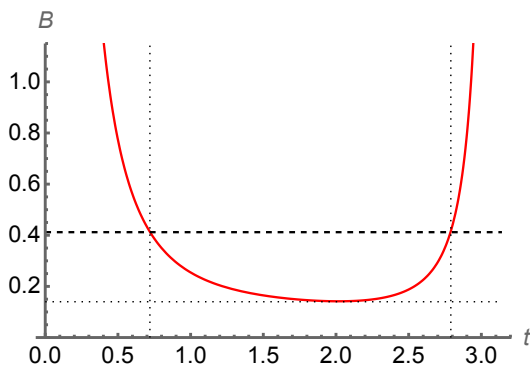

Figure 1. B bound on the total variance for the joint estimation of the anharmonicity parameters as a function of the interaction time. The red solid line is the dynamical bound and the dashed black line denotes the static one.

We conclude that preparing the oscillator in the unperturbed ground state and performing measurements after a relatively short interaction time is an effective way to reveal the presence of anharmonic perturbations and to estimate their amplitudes.

IV. CONCLUSIONS

In this paper, we have addressed the estimation of weak quantum perturbations analyzing two estimation scenarios: a static one, where the parameters are inferred by performing measurements on a stationary state, and a dynamical one, where the system is prepared in a suitably optmized initial state and measurements are performed after a given interaction time, which itself may be optimized to enhance precision.

We have found general formulas for the relevant quantities to assess precision (i.e. the SLD, the QFIM, the scalar bound B on the total variance, and the quantumness R) up to the leading order in the perturbation parameters, and analyzed in some details few quantum statistical models involving qubit, qutrit and oscillatory systems.

Our results indicate that dynamical estimation schemes generally improve precision, although only for specific preparations of the system and values of the interaction time. Ultimately, the choice between one scheme and the other does depend on the specific features of the involved system, and on the experimental difficulties related to the preparation of the initial state and the modulation of the interaction time. Our results provide solid tools to compare the two approaches in a generic situations.

Appendix A: Explicit expressions of the SLDs for a two-parameter perturbations

Starting from the perturbed state in Eq. (18) and its derivatives in Eqs. (19-20), the matrix elements $\alpha_{jk} = [L_1]_{jk}$ of the SLD operator relative to the parameter λ_1 reads:

$$\alpha_{11} = 0 \qquad (A1)$$

$$\alpha_{22} = 4\left(\lambda_1 N_1 c_1^2 + \lambda_2\sqrt{N_1 N_2}\,c_1 c_2 \cos\gamma\right) \qquad (A2)$$

$$\alpha_{33} = 4\left(\lambda_1 N_1 s_1^2 + \lambda_2\sqrt{N_1 N_2}\,s_1 s_2 \cos(\gamma+\varphi)\right) \qquad (A3)$$

$$\alpha_{12} = \alpha_{21} = 2\sqrt{N_1}\,c_1 \qquad (A4)$$

$$\alpha_{13} = \alpha_{31} = 2\sqrt{N_1}\,s_1 \qquad (A5)$$

$$\alpha_{23} = \alpha_{32}^* = 4\lambda_1 N_1 c_1 s_1 \qquad (A6)$$

$$+ 2\lambda_2\sqrt{N_1 N_2}\left(c_1 s_2 e^{-i(\gamma+\varphi)} + c_2 s_1 e^{i\gamma}\right),$$

whereas $\beta_{jk} = [L_2]_{jk}$, i.e. those of the SLD operator relative to the parameter λ_2 are given by

$$\beta_{11} = 0 \tag{A7}$$

$$\beta_{22} = 4\left(\lambda_2 N_2 c_2^2 + \lambda_1 \sqrt{N_1 N_2} c_1 c_2 \cos\gamma\right) \tag{A8}$$

$$\beta_{33} = 4\left(\lambda_2 N_2 s_2^2 + \lambda_1 \sqrt{N_1 N_2} s_1 s_2 \cos(\gamma+\varphi)\right) \tag{A9}$$

$$\beta_{12} = \beta_{21} = 2\sqrt{N_2} c_2 e^{-i\gamma} \tag{A10}$$

$$\beta_{13} = \beta_{31}^* = 2\sqrt{N_2} s_2 e^{-i(\gamma+\varphi)} \tag{A11}$$

$$\beta_{23} = \beta_{32}^* = 4\lambda_2 N_2 c_2 s_2 e^{-i\varphi} \tag{A12}$$
$$+ 2\lambda_1 \sqrt{N_1 N_2}\left(c_1 s_2 e^{-i(\gamma+\varphi)} + c_2 s_1 e^{i\gamma}\right)$$

where N_j is the squared norm of the perturbation vector $\left|\psi_{n,j}^1\right\rangle$, $c_j = \cos\frac{\theta_j}{2}$, and $s_j = \sin\frac{\theta_j}{2}$, with $j = 1, 2$.

Appendix B: K_1 and K_2 for the anharmonic oscillator

In this Section, we present the explicit expressions of K_1 and K_2 in Eqs. (39) and (40) and their use in evaluating the elements of the QFIM. The calculations are tedious but straightforward, upon writing the nonlinear Hamiltonians in normal order as follows [51–53]

$$x^n = \frac{1}{2^{n/2}}(a + a^\dagger)^n$$
$$= \frac{n!}{2^{n/2}} \sum_{k=0}^{[n/2]} \sum_{l=0}^{n-2k} \frac{a^{\dagger l} a^{n-2k-l}}{2^k \, k! \, l! \, (n-2k-l)!}, \tag{B1}$$

where $[n]$ denotes the integer part of n. We also use the fact that for a generic function $f(a, a^\dagger)$ of the bosonic operators one has

$$e^{iya^\dagger a} f(a, a^\dagger) e^{-iya^\dagger a} = f(ae^{-iy}, a^\dagger e^{iy}) \tag{B2}$$

We thus have

$$K_1 = \sum_{k=0}^{1} \sum_{l=0}^{3-2k} \int_0^t dy \, e^{-iy(3-2k-2l)}$$
$$\times \frac{3!}{2^{3/2}} \frac{a^{\dagger l} a^{3-2k-l}}{2^k \, k! \, l! \, (3-2k-l)!}, \tag{B3}$$

$$= t \sum_{k=0}^{1} \sum_{l=0}^{3-2k} e^{-i\frac{t}{2}(2k+2l-3)} \text{sinc}\left[\frac{t}{2}(2k+2l-3)\right]$$
$$\times \frac{3!}{2^{3/2}} \frac{a^{\dagger l} a^{3-2k-l}}{2^k \, k! \, l! \, (3-2k-l)!}, \tag{B4}$$

and

$$K_2 = \sum_{k=0}^{2} \sum_{l=0}^{4-2k} \int_0^t dy \, e^{-iy(4-2k-2l)}$$
$$\times \frac{4!}{2^2} \frac{a^{\dagger l} a^{4-2k-l}}{2^k \, k! \, l! \, (4-2k-l)!}, \tag{B5}$$

$$= t \sum_{k=0}^{2} \sum_{l=0}^{4-2k} e^{-i\frac{t}{2}(2k+2l-4)} \text{sinc}\left[\frac{t}{2}(2k+2l-4)\right]$$
$$\times \frac{4!}{2^2} \frac{a^{\dagger l} a^{4-2k-l}}{2^k \, k! \, l! \, (4-2k-l)!}. \tag{B6}$$

If we take the unperturbed ground states (the vacuum state of the harmonic oscillator) we have $\langle 0|K_1|0\rangle = 0$ and $\langle 0|K_2|0\rangle = \frac{3}{4}t$.

In order to calculate the expectation values $\langle 0|K_1^2|0\rangle$, $\langle 0|K_2^2|0\rangle$, and $\langle 0|K_1 K_2|0\rangle$ and evaluate the QFIM using Eqs. (41) we need to calculate expectations values of the form $\langle 0|a^{\dagger l'} a^{n'-2k'-l'} a^{\dagger l} a^{n-2k-l}|0\rangle$. In particular, in order to calculate $\langle 0|K_1 K_2|0\rangle$, we need

$$\langle 0|a^{\dagger l'} a^{n'-2k'-l'} a^{\dagger l} a^{n-2k-l}|0\rangle$$
$$= \delta_{l',0}\delta_{l,n-2k}\langle 0|a^{n'-2k'} a^{\dagger n-2k}|0\rangle$$
$$= \delta_{l',0}\delta_{l,n-2k}\delta_{k',k+\frac{n'-n}{2}} \sqrt{(n'-2k')!(n-2k)!}$$
$$= 0 \quad \text{if } n' = n \pm 1. \tag{B7}$$

We conclude that $\langle 0|K_1 K_2|0\rangle = 0$ and the same happens for the quantumness R. To calculate the diagonal elements of the QFIM we use

$$\langle 0|a^{\dagger l'} a^{n-2k'-l'} a^{\dagger l} a^{n-2k-l}|0\rangle = (n-2k)!\,\delta_{l',0}\delta_{l,n-2k}\delta_{k,k'},$$

such that

$$Q_{11} = 4\langle 0|K_1^2|0\rangle$$
$$= 4\left(\frac{3!}{2^{3/2-1}}\right)^2 \sum_{k=0}^{1} \frac{\sin^2[\frac{t}{2}(3-2k)]}{(3-2k)^2 \, 2^{2k} \, (k!)^2 \, (3-2k)!}$$
$$= \frac{29}{3} - 9\cos t - \frac{2}{3}\cos 3t \tag{B8}$$

and

$$Q_{22} = 4\left(\langle 0|K_2^2|0\rangle - \langle 0|K_2|0\rangle^2\right)$$
$$= 4\left\{\left(\frac{4!}{2^{4/2-1}}\right)^2 \left[\sum_{k=0}^{1} \frac{\sin^2[\frac{t}{2}(4-2k)]}{(4-2k)^2 \, 2^{2k} \, (k!)^2 \, (4-2k)!}\right.\right.$$
$$\left.\left. + \lim_{k\to 2} \frac{\sin^2[\frac{t}{2}(4-2k)]}{(4-2k)^2 \, 2^{2k} \, (k!)^2 \, (4-2k)!}\right] - \left(\frac{3}{4}t\right)^2\right\}$$
$$= 3\left(7 + \cos 2t\right)\sin^2 t \tag{B9}$$

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