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# On a phase-field model of damage for hybrid laminates with cohesive interface

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In this paper, we investigate a rate-independent model for hybrid laminates described by a damage phase-field approach on two layers coupled with a cohesive law governing the behaviour of their interface in a one-dimensional set-up. For the analysis, we adopt the notion of energetic evolution, based on global minimisation of the involved energy. Due to the presence of the cohesive zone, as already emerged in literature, compactness issues lead to the introduction of a fictitious variable replacing the physical one which represents the maximal opening of the interface displacement discontinuity reached during the evolution. A new strategy which allows to recover the equivalence between the fictitious and the real variable under general loading–unloading regimes is illustrated. The argument is based on time regularity of energetic evolutions. This regularity is achieved by means of a careful balance between the convexity of the elastic energy of the layers and the natural concavity of the cohesive energy of the interface.

**KEYWORDS**
cohesive interface, damage phase-field model, energetic evolutions, time regularity

**MSC CLASSIFICATION**
49J45; 49S05; 74A45; 74C15

## 1 | INTRODUCTION

Composite fibre-reinforced materials are increasingly finding applications in the manufacturing industry due to their capacity of offering high strength and stiffness with low mass density. Their only mechanical weakness is the brittleness. Indeed, rapid failure occurs without sufficient warning, due to the intrinsic nature of the adopted materials. A possible strategy to provide a ductile failure response is to consider novel composite architectures where fibres of different stiffness and ultimate strain values are combined through cohesive interfaces (hybridisation). In this case, complex rupture processes occur with diffuse crack pattern (fragmentation) and/or delamination. A deep analytical comprehension of the failure mechanisms of these kind of materials is thus needed in order to predict and control the appearance and the evolution of the cracks.

Among the mathematical community, the variational approach to fracture, as formulated by Bourdin et al. and Francfort and Marigo,[1,2] is one of the most adopted viewpoints to deal with crack problems. It is based on the Griffith's idea[3] that the crack growth is governed by a reciprocal competition between the internal elastic energy of the body and the energy spent to increase the crack length. In the original theory, the energy associated with the fracture is proportional to the measure of the fracture itself, while in the cohesive case (Barenblatt[4]), where the process is more gradual, the energy depends on the opening of the crack.

Due to the complexity of the phenomenon and the technical difficulties of the related mathematical analysis, especially from the numerical point of view, in the last 20 years, a phase-field damage approach has been developed to overcome the aforementioned issues. Nowadays, it is a well-established and consolidated method to approximate both brittle (see several studies[5–7]) and cohesive fractures (see Bonacini et al. and Iurlano[8,9]). It consists in the introduction of an internal variable taking values in $[0, 1]$ and representing the damage state of the material. Usually, values 0 and 1 mean a completely sound and a completely broken state, respectively, while a value in between represents the case of a partial damage. The presence of a fracture is thus ideally replaced by those parts of the body whose damage variable has reached the value 1.

In this work, a rigorous mathematical analysis is carried out for a one-dimensional model for hybrid laminates, which was previously introduced and numerically investigated in Alessi and Freddi.[10] Its description is given by coupling the damage phase-field approach, which models the elastic–brittle behaviour of the layers, with a cohesive law in the interface connecting the materials. The investigation is restricted to the case of incomplete damage in the sense that a reservoir of elastic material stiffness is always maintained, even if the damage variable reaches the maximum value 1. This situation can be concretely justified by considering materials formed by different components from which only a part can undergo a damage (for instance, in composite materials obtained with a matrix and a reinforcement) and delamination may take place; on the other side, it can be seen as a mathematical approximation of the complete damage setting in which the material goes through full rupture. We refer, for instance, to Bouchitté et al. and Mielke and Roubíček[11,12] for an analysis of complete damage between two viscoelastic bodies, or to Bonetti et al[13] for a complete damage model in elastic materials, while we postpone the inspection of this model to future works, due to high mathematical difficulties related to the cohesive zone.

Here, the model we want to analyse describes the evolution of a unidirectional hybrid laminate in hard device: A prescribed time-dependent displacement $\bar{u}(t)$ is applied on one side of the bar, whereas the other is fixed. We restrict our attention to slow prescribed displacements, so that inertial effects can be neglected and the analysis can be included in a quasi-static and rate-independent regime. For the sake of simplicity, we consider a bar composed by only two layers with thickness $\rho_1$ and $\rho_2$, respectively, bonded together along the entire length by a cohesive interface. The thickness of the interface is very thin compared with $\rho_1$ and $\rho_2$, which in turn are way smaller than the length of the laminate $L > 0$. Thus, the model can be considered as one-dimensional.

As already mentioned, the brittle behaviour of the two elastic layers is modelled by a phase-field damage approach. It suits with the rate-independent framework we are considering. For the reader interested instead in dynamic and rate-dependent damage models, we refer, for instance, to Bonetti and Schimperna and Frémond et al.[14,15] The unknowns of the problem are thus the displacements of the two layers, denoted by $u_1$ and $u_2$, and their damage variables $\alpha_1$ and $\alpha_2$, which are irreversible in time.

Despite its apparent simplicity, the model has considerable application potential for the study and analysis of different failure phenomena in thin multilayered materials subject to membranal mechanical regime such as composite materials. In Alessi and Freddi,[10,16] it has been adopted to investigate the complex failure modes of hybrid laminates. The experimental evidences have been successfully replicated in 1D and 2D settings. The model, suitably extended to anisotropic materials and/or curvilinear geometries, can be an extremely powerful tool to analyse craquelure phenomena in artworks as preliminarily highlighted in Negri.[17]

In the quasi-static setting, a huge variety of notions of solution can be considered (see, for instance, the monograph[18]). In this paper, we focus our attention on the concept of energetic evolution, based on two ingredients: at every time the solution is a global minimiser of the involved total energy, and the sum of internal and dissipated energy balances the work done by the external prescribed displacement. The same kind of evolution in an analogous cohesive fracture model between two elastic bodies is studied in Cagnetti and Toader and Dal Maso and Zanini[19,20]; other notions based on stationary points of the energy, always in the framework of cohesive fractures, are instead analysed in Negri and Scala and Negri and Vitali.[21,22]

The choice of working with energetic evolutions is motivated by the future aim of analysing the complete damage situation, for which the main tool usually adopted (see Bouchitté et al. and Mielke and Roubíček[11,12]) is given by Γ-convergence,[23] notion which fits well with global minimisers.

The total energy we consider is composed by a first part taking into account elastic responses of the layers and dissipation due to damage and a second part reflecting the cohesive behaviour of the interface. The cohesive interface is governed by the slip between the two layers $\delta = |u_1 - u_2|$ and its irreversible counterpart $\delta_h$ which represents the maximal slip achieved during the evolution. The presence of an irreversible history variable can be also found in different models than cohesive fracture: We mention, for instance, the notion of fatigue, investigated in Alessi et al. and Crismale et al.[24,25]

The expression of the energy in the model under consideration is hence given by

$$\sum_{i=1}^{2}\rho_i\left(\underbrace{\frac{1}{2}\int_0^L E_i(\alpha_i(x))(u_i'(x))^2\,\mathrm{d}x}_{\text{elastic energy of the i-th layer}}+\underbrace{\frac{1}{2}\int_0^L(\alpha_i'(x))^2\,\mathrm{d}x}_{\substack{\text{internal energy of}\\ \text{the i-th damage variable}}}+\underbrace{\int_0^L w_i(\alpha_i(x))\,\mathrm{d}x}_{\substack{\text{energy dissipated by}\\ \text{damage in the i-th layer}}}\right)+\underbrace{\int_0^L\varphi(\delta(x),\delta_h(x))\,\mathrm{d}x}_{\substack{\text{internal and dissipated}\\ \text{energy in the interface}}},$$

where the symbol prime $'$ denotes the one-dimensional spatial derivative, $E_i:[0,1]\to(0,+\infty)$ is the elastic Young modulus of the $i$-th layer (which is strictly positive since we are in the incomplete damage framework), $w_i:[0,1]\to[0,+\infty)$ is a dissipation density and $\varphi:\{(y,z)\in\mathbb{R}^2|z\geq y\geq 0\}\to[0,+\infty)$ is the loading–unloading density of the cohesive interface.

As usual in the context of energetic evolutions, we follow a time-discretisation algorithm to show existence of solutions. More precisely, we consider a fine partition of the time interval $[0,T]$, and at each time step, we select a global minimiser of the total energy; we then recover the time-continuous evolution by sending to zero the discretisation parameter. Due to compactness issues regarding the maximal slip $\delta_h$, the time-discretisation process leads to the introduction of a weaker notion of solution where a fictitious history variable $\gamma$ replaces the concrete one $\delta_h$. We point out that this auxiliary variable only appears when dealing with global minima of the energy; indeed, it can be found in Cagnetti and Toader and Dal Maso and Zanini,[19,20] but not in Negri and Scala and Negri and Vitali[21,22] where stationary points are considered. The issue has been partially overcome in Cagnetti and Toader and Dal Maso and Zanini[19,20] with different approaches, but assuming the hypothesis of constant unloading response, namely, when the loading–unloading density $\varphi$ depends only on the second variable $z$.

Here, an original strategy based on time-regularity properties of energetic evolutions in order to recover the equivalence between the fictitious variable $\gamma$ and the proper one $\delta_h$ under reasonable assumptions on the density $\varphi$ is developed and illustrated. In particular, we are able to cover all the general cases of density $\varphi$ considered in Negri and Scala.[21] We point out, however, that the arguments here adopted steadily exploit the 1D setting since they rely on the embedding of the Sobolev space $H^1$ into the Hölder space $C^{\frac{1}{2}}$. Hence, the proposed approach fits well with the one-dimensional model under consideration; nevertheless, we believe that with suitable adjustments, the main idea may be also adapted to more general situations and/or higher dimensions.

An alternative strategy to deal with cohesive problems can be found in literature, where adhesion is treated with the introduction of a damage variable that macroscopically defines the bond state between two solids. Detachment corresponds to full damage state. The problem has been investigated theoretically in several studies[26–29] and numerically in Freddi and Frémond and Freddi and Iurlano.[30,31]

The paper is organised as follows. In Section 2, we introduce in a rigorous way the variational problem, presenting the global and history variables: the displacement field $u_i$, the damage variables $\alpha_i$, the slip $\delta$ and the history slip $\delta_h$. Subsequently, details of the involved energies are given together with a precise notion of energetic evolution and of its weak counterpart, here named generalised energetic evolution, including the fictitious variable $\gamma$.

Section 3 is devoted to the proof of existence of generalised energetic evolutions under very mild assumptions on the loading–unloading cohesive density $\varphi$. We first introduce the time-discretisation algorithm based on global minimisation of the energy, and we provide uniform bounds on the sequence of discrete minimisers. Thanks to these bounds and by means of a suitable version of Helly's selection theorem, we are able to extract convergent subsequences as the time step vanishes. After the introduction of the fictitious history variable $\gamma$ and by exploiting the fact that the discrete functions selected by the algorithm are global minima of the total energy, we finally deduce that the previously obtained limit functions actually are a generalised energetic evolution.

In Section 4, attention is focused on the equations that a generalised energetic evolution must satisfy; they are a by-product of the global minimality condition together with the energy balance. It turns out that the displacements fulfil a system of equations in divergence form (see (4.1a)), while the damage variables satisfy a Karush–Kuhn–Tucker condition (see (4.1b)), assuming a priori certain regularity in time. Of course, these equations have to be meant in a weak sense. The results of this third section are a first step in order to obtain the equivalence between $\gamma$ and the concrete history variable $\delta_h$.

Section 5 illustrates the main result of the paper. We first adapt a convexity argument introduced in Mielke and Thomas[32] to our setting in which a cohesive energy (concave by nature) is present, in order to gain regularity in time (absolute continuity) of generalised energetic evolutions. Once this regularity is achieved, we exploit the Euler–Lagrange equations of Section 4 together with the monotonicity (in time) of $\gamma$ and $\delta_h$ to deduce their equivalence under reasonable assumptions on $\varphi$. We thus obtain as a by-product that the generalised energetic evolution found in Section 3 is actually an energetic evolution, since $\gamma$ coincides with $\delta_h$. We however stress that, even though we need to work in a convex setting to complete the argument, our techniques do not necessarily imply uniqueness of energetic evolutions, which still remains an open issue.

At the end of the work, we attach Appendix A in which we gather some definitions and properties we need throughout the paper about absolutely continuous and bounded variation functions with values in Banach spaces.

## 2 | SETTING OF THE PROBLEM

In this section, we present the variational formulation of the one-dimensional continuum model described in Section 1 of two layers bonded together by a cohesive interface in a hard device set-up. We list all the main assumptions we need throughout the paper. We also introduce the two notions of energetic evolution and generalised energetic evolution in our context (see Definitions 2.5 and 2.8).

For the sake of clarity, in this work, every function in the Sobolev space $H^1(a,b)$ is always identified with its continuous representative. The prime symbol $'$ is used to denote spatial derivatives, while the dot symbol $\dot{}$ is used to denote time derivatives. In the case of a function $f:[0,T]\to H^1(a,b)$, which thus depends on both time and space, we write $f(t)'$ to denote the (weak) spatial derivative of $f(t)\in H^1(a,b)$, and with a little abuse of notation, we write $f'(t,x)$ to denote its value at a.e. $x\in[a,b]$. If $f$ is sufficiently regular in time, for instance, in $C^1([0,T];H^1(a,b))$, for the time derivative, we instead adopt the scripts $\dot f$, $\dot f(t)$ and $\dot f(t,x)$, with the obvious meanings: $\dot f$ is the function from $[0,T]$ to $H^1(a,b)$, $\dot f(t)$ is its value as a function in $H^1(a,b)$, once $t\in[0,T]$ is fixed, and $\dot f(t,x)$ is its value (as a real number) at $x\in[a,b]$. By $a\vee b$ and $a\wedge b$, we finally mean the maximum and the minimum between two extended real numbers $a$ and $b$ in $[-\infty,+\infty]$.

We fix a time $T>0$ and the length of the laminate $L>0$. We also normalise the thickness of the two layers $\rho_1$ and $\rho_2$ to 1, since this does not affect the results.

### 2.1 | The variables

To describe the evolution of the system, for $i=1,2$, we introduce the function $u_i : [0,T]\times[0,L]\to\mathbb{R}$, where $u_i(t,x)$ denotes the displacement at time $t$ of the point $x$ of the $i$-th layer; here, $\boldsymbol{u}(t,x)$ represents the vector in $\mathbb{R}^2$ with components $u_1(t,x)$ and $u_2(t,x)$. For the structure of the model itself, at every time $t\in[0,T]$, the displacement $u_i(t)$ will belong to the space $H^1(0,L)$. The function $\delta:[0,T]\times[0,L]\to[0,+\infty)$ defined as

$$\delta(t,x)=\delta[\boldsymbol{u}](t,x) := |u_1(t,x)-u_2(t,x)|, \tag{2.1a}$$

instead denotes the displacement slip on the interface between the two layers. Then, we introduce the non-decreasing function $\delta_h:[0,T]\times[0,L]\to[0,+\infty)$ as

$$\delta_h(t,x) := \sup_{\tau\in[0,t]} \delta(\tau,x), \tag{2.1b}$$

namely, the history variable which records the maximal slip reached at the point $x$ in the interface till the time $t$. Internal constraints, such as unilateral conditions (see Bonetti et al.[27,28]), are not necessary on the kinematics as this only permits displacement slips between the two solids and interpenetration is prevented a priori.

Finally, for $i=1,2$, we consider the function $\alpha_i:[0,T]\times[0,L]\to[0,1]$, where $\alpha_i(t,x)$ represents the amount of damage at time $t$ of the point $x$ of the $i$-th layer. It is non-decreasing in time with values in $[0,1]$. The value 0 means completely sound material, whereas the value 1 represents fully damaged state. We however point out that we confine ourselves to the incomplete damage setting; namely, the fully damaged state does not describe the rupture of the layer, whose stiffness indeed never vanishes; this will be clear in (2.3), in which we assume a strictly positive elastic modulus for both layers. As for the displacement, the damage variable $\alpha_i(t)$ will be in $H^1(0,L)$ for every $t\in[0,T]$. In analogy with the previous setting, $\boldsymbol{\alpha}(t,x)$ denotes the vector in $\mathbb{R}^2$ with components $\alpha_1(t,x)$ and $\alpha_2(t,x)$.

## 2.2 | The energies

We now present the energies involved in our model. Given a pair $(\boldsymbol{u}, \boldsymbol{\alpha})$ belonging to $[H^1(0,L)]^2 \times [H^1(0,L)]^2$ and representing an admissible displacement and damage, the stored elastic energy of the two layers is given by

$$\mathcal{E}[\boldsymbol{u}, \boldsymbol{\alpha}] := \sum_{i=1}^{2} \frac{1}{2} \int_0^L E_i(\alpha_i(x))(u_i'(x))^2 \mathrm{d}x, \tag{2.2}$$

where, for $i = 1, 2$, we assume that the elastic Young moduli $E_i$ satisfy

$$E_i \in C^0([0,1]) \text{ such that } E_i(y) \geq \min_{\tilde{y} \in [0,1]} E_i(\tilde{y}) =: \varepsilon_i > 0, \text{ for every } y \in [0,1]. \tag{2.3}$$

We define

$$\varepsilon := \varepsilon_1 \wedge \varepsilon_2 > 0, \tag{2.4}$$

which is strictly positive by (2.3). This feature reflects the fact that we are considering the incomplete damage framework, and it will be used to gain coercivity of $\mathcal{E}$. This property of the energy is indeed missing in the complete damage setting where the functions $E_i$ can vanish, and a completely different notion of solution and strategy must be adopted. We refer to Bouchitté et al. and Mielke and Roubíček[11,12] or to Bonetti et al[13] for the interested reader.

We can now introduce for $i = 1, 2$ the stress $\sigma_i : [0,T] \times [0,L] \to \mathbb{R}$, defined as

$$\sigma_i(t,x) = \sigma_i[u_i, \alpha_i](t,x) := E_i(\alpha_i(t,x))u_i'(t,x). \tag{2.5}$$

As before, by $\boldsymbol{\sigma}(t,x)$, we mean the vector with components $\sigma_1(t,x)$ and $\sigma_2(t,x)$.

Another energy term appearing in the model is the sum of the stored and the dissipated energy of the phase-field variable $\boldsymbol{\alpha} \in [H^1(0,L)]^2$ during the damaging process and expressed by

$$\mathcal{D}[\boldsymbol{\alpha}] := \sum_{i=1}^{2} \left( \frac{1}{2} \int_0^L (\alpha_i'(x))^2 \mathrm{d}x + \int_0^L w_i(\alpha_i(x)) \mathrm{d}x \right). \tag{2.6}$$

In literature, there are very different choices of dissipation functions $w_i$ (see, for instance, several studies[10,16,33–36]). As elementary examples, we can consider $w_i(y) = \frac{y^2+y}{2}$ or $w_i(y) = y$.

In this work, we permit quite general assumptions on $w_i$ as follows:

$$w_i \in C^0([0,1]) \text{ such that } w_i(y) \geq 0 \text{ for every } y \in [0,1]. \tag{2.7}$$

*Remark* 2.1. The dissipated damage density, usually a process dependent function (i.e., depending on the time derivative of the damage variable $\dot{\alpha}(t)$), is here treated as a state function due to the underlying gradient damage model (see also Alessi and Freddi[10,16] and Mielke and Roubíček[12]).

We finally introduce the cohesive energy in the interface between the two layers:

$$\mathcal{K}[\delta, \gamma] := \int_0^L \varphi(\delta(x), \gamma(x)) \mathrm{d}x, \tag{2.8}$$

where $\delta$ and $\gamma$ are two non-negative functions in $[0,L]$ such that $\gamma \geq \delta$ and representing, respectively, the slip and the history slip of the displacement at a given instant. The non-negative function

$$\varphi : \mathcal{T} \to [0, +\infty), \text{ where } \mathcal{T} = \{(y,z) \in \mathbb{R}^2 \mid z \geq y \geq 0\}, \tag{2.9}$$

is the loading–unloading density of the cohesive interface; the variable $y$ governs the unloading regime (usually convex), while $z$ the loading regime (usually concave).

Since several assumptions on $\varphi$ will be needed throughout the paper, we prefer listing them here. The first set of assumptions, very mild, will be used in Section 3 to prove existence of (generalised) energetic evolutions (see Definitions 2.5 and 2.8):

($\varphi$1) $\varphi$ is lower semicontinuous;
($\varphi$2) $\varphi(0,\cdot)$ is bounded in $[0,+\infty)$;
($\varphi$3) $\varphi(y,\cdot)$ is continuous and non-decreasing in $[y,+\infty)$, for every $y \geq 0$.

We also present here the specific—but often not suited for physical applications—assumption which has been used in Cagnetti and Toader[19] and Dal Maso and Zanini[20] to deal with the fictitious variable $\gamma$ (see Definition 2.8) and which in this work we are able to avoid. We however include it in the list because we make use of it in Theorem 3.12, where we employ the argument of Cagnetti and Toader[19] in our context:

($\varphi$4) there exist two functions $\varphi_1, \varphi_2 : [0,+\infty) \to [0,+\infty)$ such that $\varphi_1$ is lower semicontinuous, $\varphi_2$ is bounded, non-decreasing and concave, and $\varphi(y,z) = \varphi_1(y) + \varphi_2(z)$.

We notice that ($\varphi$4) implies ($\varphi$1)–($\varphi$3).

*Remark* 2.2. Actually, in Cagnetti and Toader and Dal Maso and Zanini,[19,20] the function $\varphi_1$ appearing above is chosen identically 0, so that the cohesive density $\varphi$ depends only on the second variable $z$ (constant unloading regime). However, their argument can be easily adapted to the case depicted in ($\varphi$4), where $\varphi_1$ may also be different from the null function.

To overcome the necessity of ($\varphi$4) in recovering the equality $\gamma = \delta_h$ (see and compare Definitions 2.5 and 2.8), in Sections 4 and 5, we develop an alternative and new argument based on time regularity of solutions (we however point out that condition ($\varphi$4) is completely unrelated with time regularity and in general does not imply it). The assumptions we need to perform the whole strategy are listed just below. For the sake of brevity, given a non-negative function $\varphi$ with domain $\mathcal{T}$, for $z \in [0,+\infty)$, we define

$$\psi(z) := \varphi(z,z),$$

namely, the restriction of $\varphi$ on the diagonal. The function $\psi$ governs the loading regime. Moreover, we introduce the constant

$$\bar{\delta} := \inf\{z > 0 \mid \psi \text{ is constant in } [z,+\infty)\}, \tag{2.10}$$

with the convention $\inf\{\emptyset\} = +\infty$; it represents the limit slip which triggers complete delamination. Indeed, according to Alessi and Freddi,[10,16] complete delamination may occur for finite or infinite slip value (see Remark 2.3).

We then set

$$\mathcal{T}_{\bar{\delta}} := \{(y,z) \in \mathcal{T} \mid z < \bar{\delta}\}.$$

We thus require

($\varphi$5) the function $\psi \in C^1([0,+\infty))$ is $\lambda$-convex for some $\lambda > 0$, namely, for every $\theta \in [0,1]$ and $z^a, z^b \in [0,+\infty)$, it holds

$$\psi(\theta z^a + (1-\theta)z^b) \leq \theta\psi(z^a) + (1-\theta)\psi(z^b) + \frac{\lambda}{2}\theta(1-\theta)|z^a - z^b|^2;$$

($\varphi$6) for every $z \in (0,+\infty)$, the map $\varphi(\cdot,z) \in C^1([0,z])$ is non-decreasing and convex;
($\varphi$7) for every $z \in (0,+\infty)$, there holds $\partial_y\varphi(z,z) = \psi'(z)$ and $\partial_y\varphi(0,z) = 0$;
($\varphi$8) the partial derivative $\partial_y\varphi$ belongs to $C^0(\mathcal{T}\backslash(0,0))$, and it is bounded in $\mathcal{T}$;
($\varphi$9) for every $y \in [0,\bar{\delta})$, the map $\varphi(y,\cdot)$ is differentiable in $[y,\bar{\delta})$ and the partial derivative $\partial_z\varphi$ is continuous and strictly positive on $\mathcal{T}_{\bar{\delta}}\backslash\{(z,z) \in \mathbb{R}^2 \mid z \geq 0\}$.

Condition ($\varphi$9) will be actually weakened in Section 5, where only a uniform strict monotonicity with respect to $z$ will be needed (see (5.17)).

We want to point out that this set of assumptions includes a huge variety of mechanically meaningful loading–unloading densities $\varphi$, as precised in the next remark. We also notice that these conditions are similar to the one considered in Negri and Scala.[21]

*Remark* 2.3 (Main example)**.** The prototypical example of a physically meaningful loading–unloading density is obtained reasoning in the opposite way of what we presented before, namely, firstly a function $\psi$ is given and then the density $\varphi$ is built from $\psi$. As regards $\psi$, which governs the loading regime, natural assumptions arising from applications are the following: $\psi \in C^2([0,\bar{\delta})) \cap C^1([0,+\infty))$ is a non-decreasing, concave and bounded function such that $\psi(0) = 0$, $\psi' > 0$ and $\psi''$ is bounded from below in $[0,\bar{\delta})$. In particular, ($\varphi$5) is satisfied with $\lambda = \sup\limits_{z\in[0,\bar{\delta})} |\psi''(z)|$. For instance, one can consider

$$\psi(z) = \begin{cases} cz(2k-z), & \text{if } z \in [0,k), \\ ck^2, & \text{if } z \in [k,+\infty), \end{cases} \quad \text{or } \psi(z) = c\left(1-e^{-kz}\right), \text{ for } c,k>0.$$

In the first example, $\bar{\delta} = k < +\infty$, while in the second one, $\bar{\delta} = +\infty$.

The function $\varphi$ is then defined by considering a quadratic unloading regime:

$$\varphi(y,z) := \begin{cases} \frac{1}{2}\frac{\psi'(z)}{z}y^2 + \psi(z) - \frac{1}{2}z\psi'(z), & \text{if } (y,z) \in \mathcal{T}\backslash(0,0), \\ 0, & \text{if } (y,z) = (0,0). \end{cases} \tag{2.11}$$

We refer to Figure 1 for the graphs of $\varphi$. By construction, $\varphi$ is continuous on $\mathcal{T}$ and ($\varphi$6), ($\varphi$7) and ($\varphi$8) are satisfied. To verify also ($\varphi$9), we notice that it holds

$$\partial_z\varphi(y,z) = \frac{\psi'(z) - z\psi''(z)}{2}\left(1 - \frac{y^2}{z^2}\right), \text{ for every } (y,z) \in \mathcal{T}_{\bar{\delta}}.$$

Thus, we deduce that $\partial_z\varphi$ is continuous in $\mathcal{T}_{\bar{\delta}}\backslash(0,0)$; if moreover $z > y$, since $\psi'(z)$ is strictly positive in $[0,\bar{\delta})$, we get that $\partial_z\varphi(y,z) > 0$, and so ($\varphi$9) is fulfilled.

We finally observe that by the boundedness of $\psi$, we also obtain ($\varphi$2).

We now present a very simple lemma regarding the behaviour of $\varphi$ in the case $\bar{\delta} < +\infty$.

**Lemma 2.4.** *Assume that $\varphi$ satisfies ($\varphi$5)–($\varphi$7), and assume that $\bar{\delta}$ is finite. Then $\varphi$ is constant in $\mathcal{T}\backslash\mathcal{T}_{\bar{\delta}}$, and in particular:*

$$\varphi(y,z) = \psi(\bar{\delta}), \textit{for every } (y,z) \in \mathcal{T}\backslash\mathcal{T}_{\bar{\delta}}.$$

*Proof.* Since $\psi$ is $C^1[0,+\infty)$, then by definition of $\bar{\delta}$, it holds $\psi'(z) = 0$ for every $z \in [\bar{\delta},+\infty)$. We now fix $z \in [\bar{\delta},+\infty)$; by ($\varphi$7), we deduce that $\partial_y\varphi(z,z) = 0$. Condition ($\varphi$6) thus yields $\partial_y\varphi(y,z) = 0$ for every $y \in [0,z]$, and hence, we conclude. □

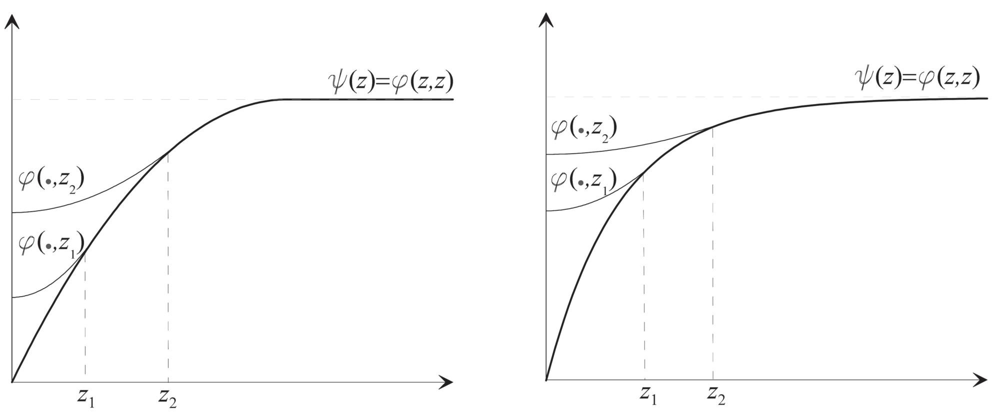


**FIGURE 1** The loading–unloading density $\varphi$ of example (2.11) in the cases $\bar{\delta} < +\infty$ (left) and $\bar{\delta} = +\infty$ (right)

We finally introduce the function $\varphi_{\bar\delta}$, defined as

$$\varphi_{\bar\delta}(y,z) := \varphi(y\wedge\bar\delta, z\wedge\bar\delta), \text{ for every } (y,z)\in\mathcal{T}.$$

Thanks to previous lemma, it is easy to deduce that if conditions ($\varphi$5), ($\varphi$6) and ($\varphi$7) are fulfilled, then actually $\varphi$ and $\varphi_{\bar\delta}$ coincide; namely, it holds

$$\varphi(y,z) = \varphi_{\bar\delta}(y,z), \text{ for every } (y,z)\in\mathcal{T}. \tag{2.12}$$

This last equality will be widely exploited in Section 5.

## 2.3 | Energetic evolutions

We are now in a position to introduce the notion of solution we want to investigate in this work. Before presenting it, we need to consider the prescribed displacement acting on the boundary $x = L$ of the laminate, namely, a function $\bar u \in AC([0,T])$; we also need to consider initial data for the displacements and damage variables, namely, functions $u_i^0, \alpha_i^0$, which must satisfy, for $i = 1, 2$, the following regularity and compatibility conditions:

$$u_i^0, \alpha_i^0 \in H^1(0,L), \tag{2.13a}$$

$$u_1^0(0) = u_2^0(0) = 0,\ u_1^0(L) = u_2^0(L) = \bar u(0), \tag{2.13b}$$

$$0 \le \alpha_i^0(x) \le 1, \text{ for every } x\in[0,L]. \tag{2.13c}$$

Once the initial displacements are given, we define the initial slip

$$\delta^0 := |u_1^0 - u_2^0|.$$

For $t\in[0,T]$, we denote by $H^1_{0,\bar u(t)}(0,L)$ the set of functions $v\in H^1(0,L)$ attaining the boundary values $v(0) = 0$ and $v(L) = \bar u(t)$. We instead denote by $H^1_{[0,1]}(0,L)$ the set of functions $v\in H^1(0,L)$ such that $0\le v(x)\le 1$ for every $x\in[0,L]$.

**Definition 2.5.** Given a prescribed displacement $\bar u \in AC([0,T])$ and initial data $\boldsymbol{u}^0$, $\boldsymbol{\alpha}^0$ satisfying (2.13), we say that a bounded pair $(\boldsymbol{u},\boldsymbol{\alpha}) : [0,T]\times[0,L]\to\mathbb{R}^2\times\mathbb{R}^2$ is an **energetic evolution** if

(CO) $\boldsymbol{u}(t)\in[H^1_{0,\bar u(t)}(0,L)]^2$, $\boldsymbol{\alpha}(t)\in[H^1_{[0,1]}(0,L)]^2$, for every $t\in[0,T]$;
(ID) $\boldsymbol{u}(0) = \boldsymbol{u}^0$, $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0$;
(IR) for $i = 1, 2$ the damage function $\alpha_i$ is non-decreasing in time, namely,

$$\text{for every } 0\le s\le t\le T, \text{ it holds: } \alpha_i(s,x)\le\alpha_i(t,x), \text{ for every } x\in[0,L];$$

(GS) for every $t\in[0,T]$, for every $\tilde{\boldsymbol{u}}\in[H^1_{0,\bar u(t)}(0,L)]^2$ and for every $\tilde{\boldsymbol{\alpha}}\in[H^1(0,L)]^2$ such that $\alpha_i(t)\le\tilde\alpha_i\le 1$ in $[0,L]$, $i = 1, 2$, one has

$$\mathcal{E}[\boldsymbol{u}(t),\boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t),\delta_h(t)] \le \mathcal{E}\left[\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\alpha}}\right] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}\left[\tilde\delta, \delta_h(t)\vee\tilde\delta\right];$$

here, we mean $\tilde\delta = |\tilde u_1 - \tilde u_2|$; and

(EB) the function $\tau\mapsto\frac{\dot{\bar u}(\tau)}{L}\int_0^L\sum_{i=1}^2\sigma_i(\tau,x)\mathrm{d}x$ belongs to $L^1(0,T)$, and for every $t\in[0,T]$, it holds

$$\mathcal{E}[\boldsymbol{u}(t),\boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t),\delta_h(t)] = \mathcal{E}\left[\boldsymbol{u}^0,\boldsymbol{\alpha}^0\right] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}\left[\delta^0,\delta^0\right] + \mathcal{W}[\boldsymbol{u},\boldsymbol{\alpha}](t),$$

where

$$\mathcal{W}[\boldsymbol{u},\boldsymbol{\alpha}](t) := \int_0^t\frac{\dot{\bar u}(\tau)}{L}\int_0^L\sum_{i=1}^2\sigma_i(\tau,x)\mathrm{d}x\ d\tau, \tag{2.14}$$

is the work done by the external prescribed displacement.

In the above definition, (CO) stands for compatibility, (ID) for initial data and (IR) for irreversibility (of the damage variables); the main conditions which characterise this sort of solution are of course the global stability (GS) and the energy balance (EB).

We notice that, by (GS), a necessary condition for the existence of such an evolution is the global minimality of the initial data at time $t = 0$, namely:

$$\mathcal{E}[\boldsymbol{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] \le \mathcal{E}\left[\widetilde{\boldsymbol{u}}, \tilde{\boldsymbol{\alpha}}\right] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}\left[\widetilde{\delta}, \delta^0 \vee \widetilde{\delta}\right], \tag{2.15}$$

for every $\widetilde{\boldsymbol{u}} \in [H^1_{0,\bar{u}(0)}(0, L)]^2$ and for every $\tilde{\boldsymbol{\alpha}} \in [H^1(0, L)]^2$ such that $\alpha_i^0 \le \tilde{\alpha}_i \le 1$ in $[0, L]$, $i = 1, 2$.

We also observe that the definition yields some very weak time regularity on the solution, namely, $\boldsymbol{u}$ and $\boldsymbol{\alpha}$ are bounded in time with values in $[H^1(0, L)]^2$, as stated in the next proposition. As a by-product, we also obtain both time and space regularities on the history variable $\delta_h$, which actually is bounded in time with values in $C_0^{1/2}([0, L])$, namely, the space of Hölder-continuous functions with exponent 1/2 vanishing at $x = 0$ and $x = L$.

**Proposition 2.6.** *Assume that $E_i$ satisfies (2.3), $w_i$ satisfies (2.7) and $\varphi$ satisfies ($\varphi$2), and let $(\boldsymbol{u}, \boldsymbol{\alpha})$ be an energetic evolution. Then there exists a positive constant $C$ such that*

$$\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|_{[H^1(0,L)]^2} \le \frac{C}{\sqrt{\varepsilon}}, \text{ and } \sup_{t\in[0,T]} \|\boldsymbol{\alpha}(t)\|_{[H^1(0,L)]^2} \le C, \tag{2.16a}$$

*where $\varepsilon > 0$ has been introduced in (2.4). In particular, $\delta_h(t)$ belongs to $C_0^{1/2}([0, L])$ for every $t \in [0, T]$, and the following estimate holds true:*

$$\sup_{t\in[0,T]} |\delta_h(t, x) - \delta_h(t, y)| \le \frac{C}{\sqrt{\varepsilon}} \sqrt{|x - y|}, \text{ for every } x, y \in [0, L]. \tag{2.16b}$$

*Proof.* Choosing as competitors in (GS), the functions

$$\tilde{u}_i(x) = \frac{\bar{u}(t)}{L} x, \ \widetilde{\alpha}_i \equiv 1, \text{ for } i = 1, 2,$$

and exploiting (2.3), (2.7) and ($\varphi$2), we deduce that

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{2} \left(\varepsilon \|u_i(t)'\|^2_{L^2(0,L)} + \|\alpha_i(t)'\|^2_{L^2(0,L)}\right) &\le \mathcal{E}[\boldsymbol{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \delta_h(t)] \\ &\le \mathcal{E}[(\tilde{u}_1, \tilde{u}_2), (1, 1)] + \mathcal{D}[(1, 1)] + \mathcal{K}[0, \delta_h(t)] \\ &\le C_1 \left(\|\bar{u}\|^2_{C^0([0,T])} + 1\right),\end{aligned}$$

for every $t \in [0, T]$, where $C_1$ is a suitable positive constant independent of $t$. Since $u_i(t, 0) = 0$ and $0 \le \alpha_i(t, x) \le 1$, we deduce (2.16a).

By (2.16a) and Sobolev embedding theorems, we now know that $u_i(t)$ are uniformly Hölder-continuous with exponent 1/2, for every $t \in [0, T]$. We thus fix $t \in [0, T]$ and $x, y \in [0, L]$; by definition of $\delta_h(t, x)$, for every $\eta > 0$, there exists $\tau_\eta \in [0, t]$ such that

$$\delta_h(t, x) - \eta \le |u_1(\tau_\eta, x) - u_2(\tau_\eta, x)|.$$

Hence, we can estimate

$$\begin{aligned}\delta_h(t, x) - \eta &\le |u_1(\tau_\eta, x) - u_1(\tau_\eta, y)| + |u_1(\tau_\eta, y) - u_2(\tau_\eta, y)| + |u_2(\tau_\eta, y) - u_2(\tau_\eta, x)| \\ &\le \frac{C}{\sqrt{\varepsilon}} \sqrt{|x - y|} + \delta_h(t, y),\end{aligned}$$

for any $t \in [0, T]$ and $x, y \in [0, L]$. By the arbitrariness of $\eta$ and reverting the role of $x$ and $y$, we deduce that $\delta_h(t)$ is Hölder-continuous with exponent 1/2 and (2.16b) holds true. Trivially, $\delta_h(t, 0) = \delta_h(t, L) = 0$, and so we conclude. □

*Remark* 2.7. In the previous proposition, we stressed the dependence on $\varepsilon > 0$ to point out the importance of assumption (2.3), which ensures the coerciveness of the elastic energy. In the complete damage setting, where $E_i$ can vanish, one needs to consider the sequence of functions $E_i + \varepsilon$, fulfilling (2.3), and then to perform an analysis of the limit $\varepsilon \to 0^+$, usually via $\Gamma$-convergence.[23] We refer, for instance, to Bouchitté et al. and Mielke and Roubíček[11,12] for a model of contact between two viscoelastic bodies, or to Bonetti et al.[13]

As we said in Section 1, the common procedure used to prove existence of energetic evolutions (and which we will perform in Section 3) is based on a time discretisation algorithm and then on a limit passage as the time step goes to 0. Due to lack of compactness for the history variable $\delta_h$, one needs to weaken the notion of energetic evolution and to introduce a fictitious variable $\gamma$ replacing $\delta_h$ (see also Cagnetti and Toader and Dal Maso and Zanini[19,20]). Thanks to Proposition 2.6, we however expect that $\gamma(t)$ should be at least continuous in $[0, L]$; we are thus led to the following definition:

**Definition 2.8.** Given a prescribed displacement $\bar{u} \in AC([0, T])$ and initial data $\boldsymbol{u}^0$, $\boldsymbol{\alpha}^0$ satisfying (2.13), we say that a triple $(\boldsymbol{u}, \boldsymbol{\alpha}, \gamma) : [0, T] \times [0, L] \to \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ is a **generalised energetic evolution** if

(CO′) $\boldsymbol{u}(t) \in [H^1_{0,\bar{u}(t)}(0, L)]^2$, $\boldsymbol{\alpha}(t) \in [H^1_{[0,1]}(0, L)]^2$, $\gamma(t) \in C^0([0, L])$, for every $t \in [0, T]$;

(ID′) $\boldsymbol{u}(0) = \boldsymbol{u}^0$, $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0$, $\gamma(0) = \delta^0$;

(IR′) for $i = 1, 2$, the damage function $\alpha_i$ and the generalised history variable $\gamma$ are non-decreasing in time, namely,

$$\text{for every } \ 0 \le s \le t \le T, \text{ it holds: } \ \alpha_i(s, x) \le \alpha_i(t, x), \text{ for every } \ x \in [0, L];$$

$$\text{for every } \ 0 \le s \le t \le T, \text{ it holds: } \ \gamma(s, x) \le \gamma(t, x), \text{ for every } \ x \in [0, L];$$

(GS′) for every $t \in [0, T]$, one has $\gamma(t) \ge \delta(t)$ in $[0, L]$ and

$$\mathcal{E}[\boldsymbol{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \le \mathcal{E}\left[\widetilde{\boldsymbol{u}}, \tilde{\boldsymbol{\alpha}}\right] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}\left[\widetilde{\delta}, \gamma(t) \vee \widetilde{\delta}\right],$$

for every $\widetilde{\boldsymbol{u}} \in [H^1_{0,\bar{u}(t)}(0, L)]^2$ and for every $\tilde{\boldsymbol{\alpha}} \in [H^1(0, L)]^2$ such that $\alpha_i(t) \le \tilde{\alpha}_i \le 1$ in $[0, L]$ for $i = 1, 2$; and

(EB′) the function $\tau \mapsto \frac{\dot{\bar{u}}(\tau)}{L} \int_0^L \sum_{i=1}^{2} \sigma_i(\tau, x) \mathrm{d}x$ belongs to $L^1(0, T)$, and for every $t \in [0, T]$, it holds

$$\mathcal{E}[\boldsymbol{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] = \mathcal{E}[\boldsymbol{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \mathcal{W}[\boldsymbol{u}, \boldsymbol{\alpha}](t),$$

where $\mathcal{W}[\boldsymbol{u}, \boldsymbol{\alpha}](t)$ is defined as in (2.14).

*Remark* 2.9. If conditions ($\varphi$5), ($\varphi$6) and ($\varphi$7) are satisfied, then equality (2.12) allows us to replace the function $\varphi$ in the functional $\mathcal{K}$ (see (2.8)) by $\varphi_{\bar{\delta}}$. This means that the functions which actually play a role in the cohesive energy are $\delta \wedge \bar{\delta}$, $\delta_h \wedge \bar{\delta}$ and $\gamma \wedge \bar{\delta}$. This observation will be useful in Section 5.

From the very definition, it is easy to see that a pair $(\boldsymbol{u}, \boldsymbol{\alpha})$ is an energetic evolution if and only if the triple $(\boldsymbol{u}, \boldsymbol{\alpha}, \delta_h)$ is a generalised energetic evolution. It is also easy to see that given a generalised energetic evolution $(\boldsymbol{u}, \boldsymbol{\alpha}, \gamma)$, it necessarily holds $\gamma(t, x) \ge \delta_h(t, x)$, for every $(t, x) \in [0, T] \times [0, L]$. Unfortunately, there are no easy arguments which ensure that $\gamma = \delta_h$ in a general case. This will be the topic of Section 5 and the main outcome of the paper.

We finally notice that the same argument used to prove Proposition 2.6 leads to the bound (2.16a) also for a generalised energetic evolution. However, (2.16b) only holds for $\delta_h$ due to its explicit definition (2.1b), and nothing can be said, in general, about the generalised history variable $\gamma$.

## 3 | EXISTENCE RESULT

In this section, we show existence of generalised energetic evolutions under very weak assumptions on the data, especially on the density $\varphi$. We indeed require (2.3), (2.7) and only ($\varphi$1)–($\varphi$3) (see Theorem 3.11). Of course, we always assume that the prescribed displacement $\bar{u}$ belongs to $AC([0, T])$. We then prove the existence of an energetic evolution assuming the specific assumption ($\varphi$4), following the same approach of Cagnetti and Toader[19] (see Theorem 3.12). We will overcome the necessity of ($\varphi$4) in Section 5, recovering the existence of energetic evolutions in meaningful mechanical situations (viz., assuming ($\varphi$5)–($\varphi$9); see also Remark 2.3) and thus obtaining our main result, Theorem 5.8.

The classical tool used to prove existence of energetic evolutions is a time-discretisation procedure. Here, we combine the ideas of Mielke and Roubíček[12] to deal with the irreversible damage variables and of Cagnetti and Toader and Dal Maso and Zanini[19,20] to handle the history variable.

## 3.1 | Time discretisation

We consider a sequence of partition $0 = t_0^n < t_1^n < \ldots < t_n^n = T$ such that

$$\lim_{n\to+\infty} \max_{k=1,\ldots,n} \left(t_k^n - t_{k-1}^n\right) = 0, \tag{3.1}$$

and for $k = 1, \ldots, n$, we perform the following implicit Euler scheme: Given $(\boldsymbol{u}^{k-1}, \boldsymbol{\alpha}^{k-1}, \delta_h^{k-1})$, we first select $(\boldsymbol{u}^k, \boldsymbol{\alpha}^k)$ by minimising the total energy among suitable natural competitors:

$$(\boldsymbol{u}^k, \boldsymbol{\alpha}^k) \in \underset{\substack{\widetilde{\boldsymbol{u}}\in[H^1_{0,\bar{u}(t_k^n)}(0,L)]^2,\\ \widetilde{\boldsymbol{\alpha}}\in[H^1(0,L)]^2 \text{ s.t. } \alpha_i^{k-1}\le\widetilde{\alpha}_i\le 1}}{\operatorname{argmin}} \left\{\mathcal{E}\left[\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}\right] + \mathcal{D}\left[\widetilde{\boldsymbol{\alpha}}\right] + \mathcal{K}[\tilde{\delta}, \delta_h^{k-1} \vee \tilde{\delta}]\right\}. \tag{3.2a}$$

Here, we want to recall that we mean $\tilde{\delta} = |\tilde{u}_1 - \tilde{u}_2|$.

We then define $\delta_h^k$ as

$$\delta_h^k := \delta_h^{k-1} \vee |u_1^k - u_2^k| = \delta_h^{k-1} \vee \delta^k. \tag{3.2b}$$

The initial values in the minimisation algorithm are functions $(\boldsymbol{u}^0, \boldsymbol{\alpha}^0)$ satisfying the compatibility conditions (2.13); moreover, we set $\delta_h^0 := \delta^0 = |u_1^0 - u_2^0|$.

**Proposition 3.1.** *Assume that $E_i$ satisfies (2.3), $w_i$ satisfies (2.7) and $\varphi$ satisfies ($\varphi$1). Then there exists a solution to the minimisation algorithm (3.2a).*

*Proof.* We fix $n \in \mathbb{N}$, and for every $k = 1, \ldots, n$, we prove the existence of a minimum by means of the direct method of calculus of variations. For the sake of clarity, we denote by $\mathcal{F}^{k-1}$ the functional we want to minimise, namely,

$$\mathcal{F}^{k-1}\left[\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}\right] = \mathcal{E}\left[\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}\right] + \mathcal{D}\left[\widetilde{\boldsymbol{\alpha}}\right] + \mathcal{K}[\tilde{\delta}, \delta_h^{k-1} \vee \tilde{\delta}] + \chi_{A^{k-1}}\left[\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}\right], \tag{3.3}$$

where $\chi_{A^{k-1}}$ denotes the indicator function of the set of constraints $A^{k-1}$, which is given by

$$A^{k-1} := \left\{ \left(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}\right) \in \left[H^1_{0,\bar{u}(t_k^n)}(0,L)\right]^2 \times [H^1(0,L)]^2 | \alpha_i^{k-1}(x) \le \widetilde{\alpha}_i(x) \le 1 \text{ for every } x \in [0,L] \right\}.$$

Weak (sequential) compactness in $[H^1(0,L)]^4$ for a minimising sequence for $\mathcal{F}^{k-1}$ follows by means of uniform bounds which can be obtained by reasoning as in the proof of Proposition 2.6.

As regards the (sequential) lower semicontinuity of $\mathcal{F}^{k-1}$ with respect the considered topology, we exploit the compact embedding $H^1(0,L) \subset\subset C^0(0,L)$. By ($\varphi$1) and Fatou's lemma, we thus deduce that $\mathcal{K}$ is lower semicontinuous; the same holds true for $\mathcal{D}$ by using again Fatou's lemma together with weak lower semicontinuity of the norm. To prove lower semicontinuity of $\mathcal{E}$, it is enough to show that, given weakly convergent sequences $\tilde{u}_i^j \rightharpoonup \tilde{u}_i$, $\tilde{\alpha}_i^j \rightharpoonup \tilde{\alpha}_i$ in $H^1(0,L)$, we have that $\sqrt{E_i(\tilde{\alpha}_i^j)}(\tilde{u}_i^j)'$ weakly converges to $\sqrt{E_i(\tilde{\alpha}_i)}\tilde{u}_i'$ in $L^2(0,L)$ as $j \to +\infty$, for $i = 1, 2$. To prove it, we fix $\phi \in L^2(0,L)$ and we estimate by exploiting (2.3):

$$
\begin{aligned}
&\left|\int_0^L \sqrt{E_i(\tilde{\alpha}_i^j(x))}(\tilde{u}_i^j)'(x)\phi(x)\mathrm{d}x - \int_0^L \sqrt{E_i(\tilde{\alpha}_i(x))}\tilde{u}_i'(x)\phi(x)\mathrm{d}x\right| \\
&\leq \left\|\sqrt{E_i(\tilde{\alpha}_i^j)} - \sqrt{E_i(\tilde{\alpha}_i)}\right\|_{C^0([0,L])} \|\tilde{u}_i^j\|_{H^1(0,L)}\|\phi\|_{L^2(0,L)} \\
&+ \left|\int_0^L (\tilde{u}_i^j)'(x)\sqrt{E_i(\tilde{\alpha}_i(x))}\phi(x)\mathrm{d}x - \int_0^L \tilde{u}_i'(x)\sqrt{E_i(\tilde{\alpha}_i(x))}\phi(x)\mathrm{d}x\right|.
\end{aligned}
$$

The first term goes to zero as $j \to +\infty$ since $\tilde{\alpha}_i^j$ uniformly converges to $\tilde{\alpha}_i$ as $j \to +\infty$ and the function $E_i$ is continuous. The second term vanishes too as $j \to +\infty$ since $\sqrt{E_i(\tilde{\alpha}_i)}\phi$ belongs to $L^2(0,L)$ by the boundedness of $E_i$.

We conclude by noticing that, exploiting again the compactness of the embedding $H^1(0,L) \subset\subset C^0(0,L)$, the set $A^{k-1}$ is (sequentially) closed with respect to the considered topology, and thus, its indicator function $\chi_{A^{k-1}}$ is lower semicontinuous as well. □

To pass from discrete to continuous evolutions, we now introduce the (right-continuous) piecewise constant interpolants $(\boldsymbol{u}^n, \boldsymbol{\alpha}^n)$ of the discrete displacement and damage variables, and the piecewise constant interpolant $\delta_h^n$ of the discrete history variable, namely:

$$
\begin{cases}
\boldsymbol{u}^n(t) := \boldsymbol{u}^k, \ \ \boldsymbol{\alpha}^n(t) := \boldsymbol{\alpha}^k, \ \ \delta_h^n(t) := \delta_h^k, \ \text{if} \ \ t \in [t_k^n, t_{k+1}^n), \\
\boldsymbol{u}^n(T) := \boldsymbol{u}^n, \ \ \boldsymbol{\alpha}^n(T) := \boldsymbol{\alpha}^n, \ \ \delta_h^n(T) := \delta_h^n.
\end{cases}
\tag{3.4a}
$$

Of course, in the following, by the expression $\delta^n$, we mean the piecewise constant slip, namely,

$$
\delta^n(t,x) = |u_1^n(t,x) - u_2^n(t,x)|. \tag{3.4b}
$$

Analogously, we consider a piecewise constant version $\bar{u}^n$ of the prescribed displacement:

$$
\begin{cases}
\bar{u}^n(t) := \bar{u}(t_k^n), \text{ if } \ t \in [t_k^n, t_{k+1}^n), \\
\bar{u}^n(T) := \bar{u}(T).
\end{cases}
\tag{3.4c}
$$

We also adopt the following notation:

$$
\tau^n(t) := \max\left\{t_k^n | t_k^n \leq t\right\}. \tag{3.4d}
$$

The next proposition provides useful uniform bounds on the just introduced piecewise constant interpolants. It is the analogue of Proposition 2.6 in this discrete setting.

**Proposition 3.2.** *Assume that $E_i$ satisfies (2.3), $w_i$ satisfies (2.7) and $\varphi$ satisfies ($\varphi$1) and ($\varphi$2). Then there exists a positive constant $C$ independent of $n$ such that*

$$
\max_{t\in[0,T]} \|\boldsymbol{u}^n(t)\|_{[H^1(0,L)]^2} \leq \frac{C}{\sqrt{\varepsilon}}, \ \ \max_{t\in[0,T]} \|\boldsymbol{\alpha}^n(t)\|_{[H^1(0,L)]^2} \leq C, \tag{3.5a}
$$

$$
\max_{t\in[0,T]}\left(\sup_{x,y\in[0,L],x\neq y} \frac{|\delta_h^n(t,x) - \delta_h^n(t,y)|}{\sqrt{|x-y|}}\right) \leq \frac{C}{\sqrt{\varepsilon}}, \tag{3.5b}
$$

*where $\varepsilon > 0$ has been introduced in (2.4).*

*Proof.* The result follows by using exactly the same argument of Proposition 2.6. We only notice that here, we need to choose as competitors for $(\boldsymbol{u}^k, \boldsymbol{\alpha}^k)$ in (3.2a) the functions

$$
\tilde{u}_i(x) = \frac{\bar{u}(t_k^n)}{L}x, \ \widetilde{\alpha}_i \equiv 1, \text{ for } \ i = 1, 2,
$$

and then we argue in the same way. □

Since the piecewise constant interpolants are built starting from the minimisation algorithm (3.2a), they automatically fulfil the following inequality, which is related to the energy balance (EB):

**Lemma 3.3** (Discrete energy inequality). *Assume that $E_i$ satisfies (2.3), $w_i$ satisfies (2.7) and $\varphi$ satisfies ($\varphi$1) and ($\varphi$2). Then there exists a vanishing sequence of positive real numbers $R^n$ such that for every $t \in [0, T]$ and for every $n \in \mathbb{N}$, the following inequality holds true:*

$$\begin{aligned} &\mathcal{E}[\boldsymbol{u}^n(t), \boldsymbol{\alpha}^n(t)] + \mathcal{D}[\boldsymbol{\alpha}^n(t)] + \mathcal{K}[\delta^n(t), \delta^n_h(t)] \\ &\leq \mathcal{E}[\boldsymbol{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \int_0^t W^n(\tau)d\tau + R^n, \end{aligned}$$

*where* $W^n(\tau) := \frac{\dot{\bar{u}}(\tau)}{L} \sum_{i=1}^{2} \int_0^L E_i(\alpha_i^n(\tau, x))(u_i^n)'(\tau, x)\mathrm{d}x.$

*Proof.* We fix $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$; for $j = 1, \dots, k$, we then choose as competitors for $(\boldsymbol{u}^j, \boldsymbol{\alpha}^j)$ in (3.2a) the functions $\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\alpha}}$, with components

$$\tilde{u}_i(x) = u_i^{j-1}(x) + (\bar{u}(t_j^n) - \bar{u}(t_{j-1}^n))x/L, \text{ and } \widetilde{\alpha}_i = \alpha_i^{j-1}, \text{ for } i = 1, 2.$$

We thus obtain

$$\mathcal{E}[\boldsymbol{u}^j, \boldsymbol{\alpha}^j] + \mathcal{D}[\boldsymbol{\alpha}^j] + \mathcal{K}[\delta^j, \delta^j_h] \leq \mathcal{E}[\boldsymbol{u}^{j-1} + \boldsymbol{v}^{j-1}, \boldsymbol{\alpha}^{j-1}] + \mathcal{D}[\boldsymbol{\alpha}^{j-1}] + \mathcal{K}[\delta^{j-1}, \delta^{j-1}_h],$$

where we denoted by $\boldsymbol{v}^{j-1}(x)$ the vector in $\mathbb{R}^2$ with both components equal to $(\bar{u}(t_j^n) - \bar{u}(t_{j-1}^n))\frac{x}{L}$. From the above inequality, we now get

$$\begin{aligned} &\mathcal{E}[\boldsymbol{u}^j, \boldsymbol{\alpha}^j] + \mathcal{D}[\boldsymbol{\alpha}^j] + \mathcal{K}[\delta^j, \delta^j_h] - \mathcal{E}[\boldsymbol{u}^{j-1}, \boldsymbol{\alpha}^{j-1}] - \mathcal{D}[\boldsymbol{\alpha}^{j-1}] - \mathcal{K}[\delta^{j-1}, \delta^{j-1}_h] \\ &\leq \mathcal{E}[\boldsymbol{u}^{j-1} + \boldsymbol{v}^{j-1}, \boldsymbol{\alpha}^{j-1}] - \mathcal{E}[\boldsymbol{u}^{j-1}, \boldsymbol{\alpha}^{j-1}] \\ &= \int_{t_{j-1}^n}^{t_j^n} \frac{\dot{\bar{u}}(\tau)}{L} \sum_{i=1}^{2} \int_0^L E_i(\alpha_i^{j-1}(x)) \left( (u_i^{j-1})'(x) + \frac{\bar{u}(\tau) - \bar{u}(t_{j-1}^n)}{L} \right) \mathrm{d}x\mathrm{d}\tau. \end{aligned}$$

Summing the obtained inequality from $j = 1$ to $j = k$, we hence deduce

$$\begin{aligned} &\mathcal{E}[\boldsymbol{u}^k, \boldsymbol{\alpha}^k] + \mathcal{D}[\boldsymbol{\alpha}^k] + \mathcal{K}[\delta^k, \delta^k_h] - \mathcal{E}[\boldsymbol{u}^0, \boldsymbol{\alpha}^0] - \mathcal{D}[\boldsymbol{\alpha}^0] - \mathcal{K}[\delta^0, \delta^0] \\ &\leq \sum_{j=1}^{k} \left( \int_{t_{j-1}^n}^{t_j^n} W^n(\tau)\mathrm{d}\tau + \int_{t_{j-1}^n}^{t_j^n} \frac{\dot{\bar{u}}(\tau)}{L} \frac{\bar{u}(\tau) - \bar{u}^n(\tau)}{L} \sum_{i=1}^{2} \int_0^L E_i(\alpha_i^n(\tau, x))\mathrm{d}x\mathrm{d}\tau \right) \\ &= \int_0^{t_k^n} W^n(\tau)\mathrm{d}\tau + \int_0^{t_k^n} \frac{\dot{\bar{u}}(\tau)}{L} \frac{\bar{u}(\tau) - \bar{u}^n(\tau)}{L} \sum_{i=1}^{2} \int_0^L E_i(\alpha_i^n(\tau, x))\mathrm{d}x\mathrm{d}\tau. \end{aligned}$$

Recalling the definition of the interpolants $\boldsymbol{u}^n$, $\boldsymbol{\alpha}^n$ and $\tau^n$ (see (3.4)), by the arbitrariness of $k$, we finally obtain for every $t \in [0, T]$:

$$\begin{aligned} &\mathcal{E}[\boldsymbol{u}^n(t), \boldsymbol{\alpha}^n(t)] + \mathcal{D}[\boldsymbol{\alpha}^n(t)] + \mathcal{K}[\delta^n(t), \delta^n_h(t)] \\ &\leq \mathcal{E}[\boldsymbol{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \int_0^t W^n(\tau)d\tau \\ &+ \int_0^{\tau^n(t)} \frac{\dot{\bar{u}}(\tau)}{L} \frac{\bar{u}(\tau) - \bar{u}^n(\tau)}{L} \sum_{i=1}^{2} \int_0^L E_i(\alpha_i^n(\tau, x))\mathrm{d}x\mathrm{d}\tau - \int_{\tau^n(t)}^t W^n(\tau)\mathrm{d}\tau. \end{aligned}$$

We thus conclude by defining

$$R^n := \int_0^T \frac{|\dot{\bar{u}}(\tau)|}{L} \frac{|\bar{u}(\tau) - \bar{u}^n(\tau)|}{L} \sum_{i=1}^{2} \int_0^L E_i(\alpha_i^n(\tau,x))\mathrm{d}x\mathrm{d}\tau + \sup_{t\in[0,T]} \int_{\tau^n(t)}^t |W^n(\tau)|\mathrm{d}\tau. \tag{3.6}$$

Indeed, we now show that $\lim\limits_{n\to+\infty} R^n = 0$. First of all by the very definition of $W^n$ and exploiting (3.5a), it is easy to see that $|W^n(\tau)| \leq C|\dot{\bar{u}}(\tau)|$, with $C>0$ independent of $n$; hence, by the absolute continuity of the integral, the second term in (3.6) vanishes as $n\to+\infty$ (we recall that by assumption, the sequence of partitions satisfies (3.1)). Then we notice that the first term is bounded by

$$C\|\dot{\bar{u}}\|_{L^1(0,T)} \sup_{t\in[0,T]} |\bar{u}(t) - \bar{u}^n(t)|,$$

which vanishes since $\bar{u}$ is absolutely continuous and the sequence of partitions satisfies (3.1). □

## 3.2 | Extraction of convergent subsequences

By the uniform bounds obtained in Proposition 3.2, we are able to deduce the existence of convergent subsequences of the piecewise constant interpolants $\boldsymbol{u}^n$, $\boldsymbol{\alpha}^n$ and $\delta_h^n$. We first need the following Helly-type compactness result:

**Lemma 3.4** (Helly). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of non-decreasing functions from [0, T] to $C^0([0,L])$, meaning that for every $0\leq s\leq t\leq T$, it holds $f_n(s,x)\leq f_n(t,x)$ for all $x\in[0,L]$, such that*

- *the families $\{f_n(0)\}_{n\in\mathbb{N}}$ and $\{f_n(T)\}_{n\in\mathbb{N}}$ are equibounded, and*
- *the family $\{f_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous uniformly with respect to $t\in[0,T]$.*

*Then there exist a subsequence (not relabelled) and a function $f:[0,T]\to C^0([0,L])$ such that $f_n(t)$ converges uniformly to $f(t)$ as $n\to+\infty$ for every $t\in[0,T]$, and $f$ is non-decreasing in time, in the above sense.*

*Moreover, for every $t\in[0,T]$, the right and left limits $f^{\pm}(t)$, which are well-defined pointwise by monotonicity, actually belong to $C^0([0,L])$, and it holds*

$$f^{\pm}(t) = \lim_{h\to0^{\pm}} f(t+h), \textit{ uniformly in } [0,L]. \tag{3.7}$$

*Proof.* The proof follows exactly the same lines of Lemma 4.6 of Dal Maso and Zanini[20]; we only stress two differences. Here, the topology is the one inherited by uniform convergence and compactness is ensured by the Ascoli–Arzelá theorem, thanks to the equiboundedness and equicontinuity assumptions. The additional requirement of uniform equicontinuity with respect to $t\in[0,T]$ is finally used to deduce that the limit family $\{f(t)\}_{t\in[0,T]}$ is equicontinuous as well, thus yielding (3.7). □

**Proposition 3.5.** *Assume that $E_i$ satisfies (2.3), $w_i$ satisfies (2.7) and $\varphi$ satisfies ($\varphi$1) and ($\varphi$2). Consider the sequences of functions $\boldsymbol{u}^n$, $\boldsymbol{\alpha}^n$, $\delta_h^n$ introduced in (3.4a). Then there exist a subsequence $n_j$ and, for every $t\in[0,T]$, a further subsequence $n_j(t)$ (depending on time) such that*

*(a) $\boldsymbol{u}^{n_j(t)}(t) \rightharpoonup \boldsymbol{u}(t)$ in $[H^1(0,L)]^2$ as $n_j(t)\to+\infty$;*
*(b) $\boldsymbol{\alpha}^{n_j(t)}(t) \rightharpoonup \boldsymbol{\alpha}(t)$ in $[H^1(0,L)]^2$ as $n_j(t)\to+\infty$;*
*(c) $\delta_h^{n_j}(t) \to \gamma(t)$ uniformly in $[0,L]$ as $n_j\to+\infty$.*

*Moreover, the limit functions satisfy*

*(1) $\boldsymbol{u}(t) \in [H^1_{0,\bar{u}(t)}(0,L)]^2$, $\boldsymbol{\alpha}(t) \in [H^1_{[0,1]}(0,L)]^2$ and $\gamma(t) \in C_0^{1/2}([0,L])$ for every $t\in[0,T]$;*
*(2) $\boldsymbol{u}(0) = \boldsymbol{u}^0$, $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0$ and $\gamma(0) = \delta^0$;*
*(3) $\alpha_i$ and $\gamma$ are non-decreasing in time;*
*(4) $\gamma(t) \geq \delta_h(t) = \sup\limits_{\tau\in[0,t]} |u_1(\tau) - u_2(\tau)|$ for every $t\in[0,T]$; and*
*(5) the family $\{\gamma(t)\}_{t\in[0,T]}$ is equicontinuous.*

*Remark* 3.6. We want to point out that also the subsequence of the damage variable in (b) could be chosen independent of time, since each term of the sequence is non-decreasing in time. This follows by means of a suitable version of Helly's selection theorem (see, for instance, Mielke and Roubíček,[18] Theorem B.5.13) and arguing as in Proposition 3.2 of Mielke and Roubíček.[12] However, both for the sake of simplicity and since for (a) the same cannot be done, we prefer to consider a time-dependent subsequence; this will be enough for our purposes.

The fact that the subsequence in (c) does not depend on time is instead crucial for the validity of (4), as the reader can check from the proof.

*Remark* 3.7. For the sake of clarity, in order to avoid too heavy notations, from now on, we prefer not to stress the occurrence of the subsequence via the subscript $j$; namely, we still write $n$ instead of $n_j$ and $n(t)$ instead of $n_j(t)$.

*Proof of Proposition 3.5.* The validity of (c) and (5), the Hölder-continuity of exponent 1/2 of the limit function $\gamma(t)$ and the fact that $\gamma(t,0) = \gamma(t,L) = 0$ are a by-product of (3.5b) and Lemma 3.4; (a) and (b) instead follow by (3.5a) together with the weak sequential compactness of the unit ball in $H^1(0,L)$. Since $H^1(0,L) \subset\subset C^0([0,L])$, we also deduce (1)–(3).

We only need to prove (4). So let us assume by contradiction that there exists a pair $(t,x) \in [0,T] \times [0,L]$ such that

$$\delta_h(t,x) > \gamma(t,x) = \lim_{n\to+\infty} \delta_h^n(t,x). \tag{3.8}$$

By (3.8) and the definition of $\delta_h$, there exists a time $\tau_t \in [0,t]$ for which $|u_1(\tau_t,x) - u_2(\tau_t,x)| > \gamma(t,x)$; thus, we infer

$$\begin{aligned}
|u_1(\tau_t,x) - u_2(\tau_t,x)| &> \lim_{n\to+\infty} \delta_h^n(t,x) \geq \lim_{n\to+\infty} \delta_h^n(\tau_t,x) \\
&\geq \limsup_{n\to+\infty} |u_1^n(\tau_t,x) - u_2^n(\tau_t,x)| \\
&\geq \lim_{n(\tau_t)\to+\infty} |u_1^{n(\tau_t)}(\tau_t,x) - u_2^{n(\tau_t)}(\tau_t,x)| = |u_1(\tau_t,x) - u_2(\tau_t,x)|,
\end{aligned}$$

which is a contradiction. □

## 3.3 | Existence of generalised energetic evolutions

The aim of this subsection is proving that the limit functions obtained in Proposition 3.5 are actually a generalised energetic evolution. We only need to show that global stability (GS′) and energy balance (EB′) hold true, being the other conditions automatically satisfied due to Lemma 3.5. This first proposition deals with the global stability:

**Proposition 3.8.** *Assume that $E_i$ satisfies (2.3), $w_i$ satisfies (2.7) and $\varphi$ satisfies ($\varphi$1)–($\varphi$3). Assume that the initial data $\boldsymbol{u}^0$, $\boldsymbol{\alpha}^0$ fulfil the stability condition (2.15). Then the limit functions $\boldsymbol{u}$, $\boldsymbol{\alpha}$ and $\gamma$ obtained in Proposition 3.5 satisfy (GS′).*

*Proof.* If $t = 0$ there is nothing to prove, so we consider $t \in (0,T]$, and we first notice that by (4) in Proposition 3.5, we know $\gamma(t) \geq \delta(t)$. Then we fix $\widetilde{\boldsymbol{u}} \in [H^1_{0,\bar{u}(t)}(0,L)]^2$ and $\widetilde{\boldsymbol{\alpha}} \in [H^1(0,L)]^2$ such that $\alpha_i(t) \leq \tilde{\alpha}_i \leq 1$ for $i = 1,2$.

By weak lower semicontinuity of the energy, taking the subsequence $n(t)$ obtained in Proposition 3.5 (see also Remark 3.7), we get

$$\begin{aligned}
&\mathcal{E}[\boldsymbol{u}(t),\boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t),\gamma(t)] \\
&\leq \liminf_{n(t)\to+\infty} \left( \mathcal{E}[\boldsymbol{u}^{n(t)}(t),\boldsymbol{\alpha}^{n(t)}(t)] + \mathcal{D}[\boldsymbol{\alpha}^{n(t)}(t)] + \mathcal{K}[\delta^{n(t)}(t),\delta_h^{n(t)}(t)] \right) =: (\star).
\end{aligned}$$

Now, we can use the minimality properties of the discrete functions, considering as competitors the functions $\widehat{\boldsymbol{u}}^{n(t)}$ and $\widehat{\boldsymbol{\alpha}}^{n(t)}$ whose components are

$$\widehat{u}_i^{n(t)}(x) := \tilde{u}_i(x) - (\bar{u}(t) - \bar{u}(\tau^{n(t)}(t)))\frac{x}{L}, \quad \widehat{\alpha}_i^{n(t)} := \min\left\{ \tilde{\alpha}_i + \max_{[0,L]} \left| \alpha_i^{n(t)}(t) - \alpha_i(t) \right|, 1 \right\}.$$

It is easy to see that they are admissible; moreover, since $\tau^{n(t)}(t) \to t$ and $\alpha_i^{n(t)}(t) \to \alpha_i(t)$ uniformly as $n(t) \to +\infty$, they strongly converge to $\widetilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{\alpha}}$ in $[H^1(0,L)]^2$ (see also Mielke and Roubíček,[12 Lemma 3.5]).

By minimality, going back to the previous estimate, we obtain

$$\begin{aligned}(\star) &\le \liminf_{n(t)\to+\infty}\left(\mathcal{E}[\hat{\boldsymbol{u}}^{n(t)},\hat{\boldsymbol{\alpha}}^{n(t)}]+\mathcal{D}[\hat{\boldsymbol{\alpha}}^{n(t)}]+\mathcal{K}\left[\widetilde{\delta},\delta_h^{n(t)}(t)\vee\widetilde{\delta}\right]\right)\\ &=\mathcal{E}\left[\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{\alpha}}\right]+\mathcal{D}[\widetilde{\boldsymbol{\alpha}}]+\mathcal{K}\left[\widetilde{\delta},\gamma(t)\vee\widetilde{\delta}\right],\end{aligned}$$

where in the last equality, we exploited the strong convergence of $\hat{\boldsymbol{u}}^{n(t)}$ and $\hat{\boldsymbol{\alpha}}^{n(t)}$ towards $\widetilde{\boldsymbol{u}}$ and $\widetilde{\boldsymbol{\alpha}}$, plus assumption ($\varphi$3). Thus, we conclude. □

To show the validity of (EB′), we prove separately the two inequalities. The first one follows from the discrete energy inequality presented in Lemma 3.3:

**Proposition 3.9** (Upper energy estimate)**.** *Assume that $E_i$ satisfies (2.3), $w_i$ satisfies (2.7) and $\varphi$ satisfies ($\varphi$1) and ($\varphi$2). Then for every $t \in [0, T]$, the limit functions $\boldsymbol{u}$, $\boldsymbol{\alpha}$ and $\gamma$ obtained in Proposition 3.5 satisfy the following inequality:*

$$\mathcal{E}[\boldsymbol{u}(t),\boldsymbol{\alpha}(t)]+\mathcal{D}[\boldsymbol{\alpha}(t)]+\mathcal{K}[\delta(t),\gamma(t)]\le\mathcal{E}[\boldsymbol{u}^0,\boldsymbol{\alpha}^0]+\mathcal{D}[\boldsymbol{\alpha}^0]+\mathcal{K}[\delta^0,\delta^0]+\mathcal{W}[\boldsymbol{u},\boldsymbol{\alpha}](t).$$

*Proof.* We fix $t \in [0, T]$, and we again consider the subsequence $n(t)$ obtained in Proposition 3.5 (see also Remark 3.7); by lower semicontinuity of the energy and Lemma 3.3, we deduce

$$\begin{aligned}&\mathcal{E}[\boldsymbol{u}(t),\boldsymbol{\alpha}(t)]+\mathcal{D}[\boldsymbol{\alpha}(t)]+\mathcal{K}[\delta(t),\gamma(t)]\\ &\le\liminf_{n(t)\to+\infty}\left(\mathcal{E}[\boldsymbol{u}^{n(t)}(t),\boldsymbol{\alpha}^{n(t)}(t)]+\mathcal{D}[\boldsymbol{\alpha}^{n(t)}(t)]+\mathcal{K}[\delta^{n(t)}(t),\delta_h^{n(t)}(t)]\right)\\ &\le\mathcal{E}[\boldsymbol{u}^0,\boldsymbol{\alpha}^0]+\mathcal{D}[\boldsymbol{\alpha}^0]+\mathcal{K}[\delta^0,\delta^0]+\limsup_{n(t)\to+\infty}\int_0^t W^{n(t)}(\tau)\mathrm{d}\tau.\end{aligned}$$

By means of the reverse Fatou's lemma (we recall that the whole sequence $W^n$ is bounded from above by $C|\dot{\bar{u}}(\tau)|$), we thus get

$$\limsup_{n(t)\to+\infty}\int_0^t W^{n(t)}(\tau)\mathrm{d}\tau\le\int_0^t\limsup_{n(t)\to+\infty}W^{n(t)}(\tau)\mathrm{d}\tau=:(*).$$

In order to deal with $(*)$, we argue as follows (see also Cagnetti and Toader[19, Section 4]). We consider the subsequence $n$ (independent of time) obtained in Proposition 3.5 (see also Remark 3.7), and for every, $\tau \in [0, T]$, we first set

$$W(\tau):=\limsup_{n\to+\infty}W^n(\tau),\tag{3.9}$$

which belongs to $L^1(0, T)$ since we recall that $|W^n(\tau)| \le C|\dot{\bar{u}}(\tau)|$. Without loss of generality, we can assume that the time-dependent subsequences further obtained in Proposition 3.5 also satisfy

$$W(\tau)=\lim_{n(\tau)\to+\infty}W^{n(\tau)}(\tau),\ \text{for every}\ \ \tau\in[0,T].$$

Thus, exploiting (a) and (b) in Proposition 3.5 for a.e. $\tau \in [0, T]$, we obtain

$$\begin{aligned}W(\tau)=\lim_{n(\tau)\to+\infty}W^{n(\tau)}(\tau)&=\lim_{n(\tau)\to+\infty}\frac{\dot{\bar{u}}(\tau)}{L}\sum_{i=1}^2\int_0^L E_i(\alpha_i^{n(\tau)}(\tau,x))(u_i^{n(\tau)})'(\tau,x)\mathrm{d}x\\ &=\frac{\dot{\bar{u}}(\tau)}{L}\sum_{i=1}^2\int_0^L E_i(\alpha_i(\tau,x))(u_i)'(\tau,x)\mathrm{d}x.\end{aligned}\tag{3.10}$$

Combining (3.9) and (3.10), we finally get

$$(*) \leq \int_0^t W(\tau)\mathrm{d}\tau = \mathcal{W}[\boldsymbol{u}, \boldsymbol{\alpha}](t),$$

and we conclude. □

The opposite inequality is instead a by-product of the global stability condition we proved in Proposition 3.8:

**Proposition 3.10** (Lower energy estimate). *Assume that $E_i$ satisfies (2.3), $w_i$ satisfies (2.7) and $\varphi$ satisfies ($\varphi$1)–($\varphi$3). Assume that the initial data $\boldsymbol{u}^0$, $\boldsymbol{\alpha}^0$ fulfil the stability condition (2.15). Then for every $t \in [0, T]$, the limit functions $\boldsymbol{u}$, $\boldsymbol{\alpha}$ and $\gamma$ obtained in Proposition 3.5 satisfy*

$$\mathcal{E}[\boldsymbol{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \geq \mathcal{E}[\boldsymbol{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \mathcal{W}[\boldsymbol{u}, \boldsymbol{\alpha}](t).$$

*Proof.* If $t = 0$, the inequality is trivial, so we fix $t \in (0, T]$, and we consider a sequence of partitions of $[0, t]$ of the form $0 = s_0^n < s_1^n < \ldots < s_n^n = t$ (we stress that this sequence of partitions is completely unrelated with the one considered at the beginning of Section 3.1 and used to perform the time-discretisation argument) satisfying

(i) $\lim_{n\to+\infty} \max_{k=1,\ldots,n} \left|s_k^n - s_{k-1}^n\right| = 0$;

(ii) $\lim_{n\to+\infty} \sum_{k=1}^{n} \left|(s_k^n - s_{k-1}^n)\dot{\bar{u}}(s_k^n) - \int_{s_{k-1}^n}^{s_k^n} \dot{\bar{u}}(\tau)\mathrm{d}\tau\right| = 0$;

(iii) $\lim_{n\to+\infty} \sum_{k=1}^{n} (s_k^n - s_{k-1}^n)W(s_k^n) = \mathcal{W}[\boldsymbol{u}, \boldsymbol{\alpha}](t)$, where $W$ is the function introduced in (3.9) and (3.10). The existence of such a sequence of partitions follows from Lemma 4.5 of Francfort and Mielke,[37] since both $\dot{\bar{u}}$ and $W$ belong to $L^1(0, T)$. In particular, by (i) and the absolute continuity of the integral, we can assume without loss of generality that

(iv) for every $n \in \mathbb{N}$, it holds $\int_{s_{k-1}^n}^{s_k^n} |\dot{\bar{u}}(\tau)|\mathrm{d}\tau \leq \frac{1}{n}$ for every $k = 1, \ldots, n$.

For a given partition, we fix $k = 1, \ldots, n$, and recalling Proposition 3.8, we choose as competitors for $\boldsymbol{u}\left(s_{k-1}^n\right)$, $\boldsymbol{\alpha}\left(s_{k-1}^n\right)$ and $\gamma\left(s_{k-1}^n\right)$ in (GS′) the functions $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{\alpha}}$, with components

$$\tilde{u}_i(x) = u_i(s_k^n, x) + \left(\bar{u}\left(s_{k-1}^n\right) - \bar{u}\left(s_k^n\right)\right)\frac{x}{L}, \quad \tilde{\alpha}_i = \alpha_i(s_k^n), \ \text{ for } \ i = 1, 2.$$

Recalling that $\gamma(s_{k-1}^n) \vee \delta(s_k^n) \leq \gamma(s_k^n)$, and hence, $\mathcal{K}[\delta(s_k^n), \gamma(s_{k-1}^n) \vee \delta(s_k^n)] \leq \mathcal{K}[\delta(s_k^n), \gamma(s_k^n)]$ by ($\varphi$3), arguing as in the proof of Lemma 3.3, we thus deduce

$$\begin{aligned}
&\mathcal{E}[\boldsymbol{u}(s_{k-1}^n), \boldsymbol{\alpha}(s_{k-1}^n)] + \mathcal{D}[\boldsymbol{\alpha}(s_{k-1}^n)] + \mathcal{K}[\delta(s_{k-1}^n), \gamma(s_{k-1}^n)] - \mathcal{E}[\boldsymbol{u}(s_k^n), \boldsymbol{\alpha}(s_k^n)] - \mathcal{D}[\boldsymbol{\alpha}(s_k^n)] - \mathcal{K}[\delta(s_k^n), \gamma(s_k^n)] \\
&\leq -\int_{s_{k-1}^n}^{s_k^n} \frac{\dot{\bar{u}}(\tau)}{L} \sum_{i=1}^{2} \int_0^L E_i(\alpha_i(s_k^n, x)) \left(u_i'(s_k^n, x) + \frac{\bar{u}(\tau) - \bar{u}(s_k^n)}{L}\right) \mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Summing the above inequality from $k = 1$ to $k = n$, we obtain

$$\begin{aligned}
&\mathcal{E}[\boldsymbol{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] - \mathcal{E}[\boldsymbol{u}^0, \boldsymbol{\alpha}^0] - \mathcal{D}[\boldsymbol{\alpha}^0] - \mathcal{K}[\delta^0, \delta^0] \\
&\geq \sum_{k=1}^{n} \int_{s_{k-1}^n}^{s_k^n} \frac{\dot{\bar{u}}(\tau)}{L} \int_0^L \sum_{i=1}^{2} E_i(\alpha_i(s_k^n, x)) \left(u_i'(s_k^n, x) + \frac{\bar{u}(\tau) - \bar{u}(s_k^n)}{L}\right) \mathrm{d}x\mathrm{d}\tau =: J_n
\end{aligned}$$

Now, we easily notice that $J_n$ can be written as

$$\begin{aligned} J_n =& \sum_{k=1}^{n}(s_k^n - s_{k-1}^n)W(s_k^n) \\ &+ \sum_{k=1}^{n}\int_{s_{k-1}^n}^{s_k^n}\frac{\dot{\bar{u}}(\tau)-\dot{\bar{u}}(s_k^n)}{L}\mathrm{d}\tau\int_0^L\sum_{i=1}^{2}E_i(\alpha_i(s_k^n,x))u_i'(s_k^n,x)\mathrm{d}x \\ &+ \sum_{k=1}^{n}\int_{s_{k-1}^n}^{s_k^n}\frac{\dot{\bar{u}}(\tau)}{L}\frac{\bar{u}(\tau)-\bar{u}(s_k^n)}{L}\mathrm{d}\tau\int_0^L\sum_{i=1}^{2}E_i(\alpha_i(s_k^n,x))\mathrm{d}x =: J_n^1 + J_n^2 + J_n^3. \end{aligned}$$

By (iii), we know that $\lim_{n\to+\infty} J_n^1 = \mathcal{W}[\boldsymbol{u},\boldsymbol{\alpha}](t)$, so we conclude if we prove that $\lim_{n\to+\infty} J_n^2 = \lim_{n\to+\infty} J_n^3 = 0$. With this aim, we estimate

$$\begin{aligned} |J_n^2| &\le C\sum_{k=1}^{n}\left|\int_{s_{k-1}^n}^{s_k^n}(\dot{\bar{u}}(\tau)-\dot{\bar{u}}(s_k^n))\mathrm{d}\tau\right|\left(\sum_{i=1}^{2}\|u_i(s_k^n)\|_{H^1(0,L)}\right) \\ &\le C\sum_{k=1}^{n}\left|(s_k^n - s_{k-1}^n)\dot{\bar{u}}(s_k^n) - \int_{s_{k-1}^n}^{s_k^n}\dot{\bar{u}}(\tau)\mathrm{d}\tau\right|, \end{aligned}$$

which goes to 0 by (ii). As regards $J_n^3$, by using (iv), we get

$$\begin{aligned} |J_n^3| &\le C\sum_{k=1}^{n}\int_{s_{k-1}^n}^{s_k^n}|\dot{\bar{u}}(\tau)||\bar{u}(\tau)-\bar{u}(s_k^n)|\mathrm{d}\tau = C\sum_{k=1}^{n}\int_{s_{k-1}^n}^{s_k^n}|\dot{\bar{u}}(\tau)|\left|\int_{\tau}^{s_k^n}\dot{\bar{u}}(s)\mathrm{d}s\right|\mathrm{d}\tau \\ &\le C\sum_{k=1}^{n}\left(\int_{s_{k-1}^n}^{s_k^n}|\dot{\bar{u}}(\tau)|\mathrm{d}\tau\right)^2 \le \frac{C}{n}\sum_{k=1}^{n}\int_{s_{k-1}^n}^{s_k^n}|\dot{\bar{u}}(\tau)|\mathrm{d}\tau = \frac{C}{n}\|\dot{\bar{u}}\|_{L^1(0,t)}, \end{aligned}$$

and the proof is complete. □

Putting together what we obtained in this section, we infer our first result of existence of generalised energetic evolutions:

**Theorem 3.11** (Existence of generalised energetic evolutions)**.** *Let the prescribed displacement $\bar{u}$ belong to AC([0, T]) and the initial data $\boldsymbol{u}^0$, $\boldsymbol{\alpha}^0$ fulfil (2.13) together with the stability condition (2.15). Assume that $E_i$ satisfies (2.3), $w_i$ satisfies (2.7) and $\varphi$ satisfies ($\varphi$1)–($\varphi$3). Then the triplet composed by the functions $\boldsymbol{u}$, $\boldsymbol{\alpha}$ and $\gamma$ obtained in Proposition 3.5 is a generalised energetic evolution.*

We conclude this section by showing that, assuming in addition the specific condition ($\varphi$4), which we rewrite also here for the sake of clarity:

($\varphi$4) there exist two functions $\varphi_1, \varphi_2 : [0,+\infty) \to [0,+\infty)$ such that $\varphi_1$ is lower semicontinuous, $\varphi_2$ is bounded, non-decreasing and concave, and $\varphi(y,z) = \varphi_1(y) + \varphi_2(z)$,

the functions $\boldsymbol{u}$ and $\boldsymbol{\alpha}$ obtained in Proposition 3.5 are actually an energetic evolution. The approach is exactly the same of Cagnetti and Toader.[19] We recall that ($\varphi$4) implies ($\varphi$1)–($\varphi$3).

We however point out again that ($\varphi$4) does not include most of the cases of loading–unloading cohesive densities $\varphi$ usually arising and adopted in real-world applications, like, for instance, the one presented in Remark 2.3. The analogous result of Theorem 3.12 for more realistic densities from the physical point of view is obtained in our main result, contained in Theorem 5.8, via an alternative strategy developed in the forthcoming sections.

**Theorem 3.12.** *Let the prescribed displacement $\bar{u}$ belong to AC([0, T]) and the initial data $\boldsymbol{u}^0$, $\boldsymbol{\alpha}^0$ fulfil (2.13) together with the stability condition (2.15). Assume that $E_i$ satisfies (2.3), $w_i$ satisfies (2.7) and $\varphi$ satisfies ($\varphi$4). Then the pair ($\boldsymbol{u}$, $\boldsymbol{\alpha}$) obtained in Proposition 3.5 is an energetic evolution.*

*If in addition $\varphi_2$ is strictly increasing, then the function $\gamma$ obtained in Proposition 3.5 coincides with the history variable $\delta_h$.*

*Proof.* Thanks to Theorem 3.11, we only need to show the validity of (GS) and (EB) in Definition 2.5. We first focus on (GS); so we fix $t\in[0,T]$ and two functions $\widetilde{\boldsymbol{u}} \in [H^1_{0,\bar{u}(t)}(0,L)]^2$, $\tilde{\boldsymbol{\alpha}} \in [H^1(0,L)]^2$ such that $\alpha_i(t) \le \tilde{\alpha}_i \le 1$ in $[0,L]$ for $i=1,2$. Since the triplet $(\boldsymbol{u},\boldsymbol{\alpha},\gamma)$ satisfies (GS′) we know that

$$\mathcal{E}[\boldsymbol{u}(t),\boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] \le \mathcal{E}\left[\widetilde{\boldsymbol{u}},\tilde{\boldsymbol{\alpha}}\right] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}\left[\widetilde{\delta},\gamma(t)\vee\widetilde{\delta}\right] - \mathcal{K}[\delta(t),\gamma(t)];$$

thus, we conclude if we prove

$$\mathcal{K}\left[\widetilde{\delta},\gamma(t)\vee\widetilde{\delta}\right] - \mathcal{K}[\delta(t),\gamma(t)] \le \mathcal{K}\left[\widetilde{\delta},\delta_h(t)\vee\widetilde{\delta}\right] - \mathcal{K}[\delta(t),\delta_h(t)]. \tag{3.11}$$

With this aim, exploiting ($\varphi$4), in particular the monotonicity and concavity of $\varphi_2$, and recalling that $\gamma(t)\ge\delta_h(t)$, we get

$$\begin{aligned}\varphi_2(\gamma(t)\vee\tilde{\delta}) &= \varphi_2(\gamma(t)+[\tilde{\delta}-\gamma(t)]^+) \le \varphi_2(\gamma(t)+[\tilde{\delta}-\delta_h(t)]^+)\\ &\le \varphi_2(\gamma(t)) + \varphi_2(\delta_h(t)+[\tilde{\delta}-\delta_h(t)]^+) - \varphi_2(\delta_h(t))\\ &= \varphi_2(\gamma(t)) + \varphi_2(\delta_h(t)\vee\tilde{\delta}) - \varphi_2(\delta_h(t)).\end{aligned}$$

The above inequality implies

$$\begin{aligned}\mathcal{K}\left[\widetilde{\delta},\gamma(t)\vee\widetilde{\delta}\right] - \mathcal{K}\left[\widetilde{\delta},\delta_h(t)\vee\tilde{\delta}\right] &= \int_0^L \left(\varphi_2(\gamma(t,x)\vee\widetilde{\delta}(x)) - \varphi_2(\delta_h(t,x)\vee\widetilde{\delta}(x))\right)\mathrm{d}x\\ &\le \int_0^L (\varphi_2(\gamma(t,x)) - \varphi_2(\delta_h(t,x)))\,\mathrm{d}x\\ &= \mathcal{K}[\delta(t),\gamma(t)] - \mathcal{K}[\delta(t),\delta_h(t)],\end{aligned}$$

which is equivalent to (3.11).

We now prove (EB). Since the triplet $(\boldsymbol{u},\boldsymbol{\alpha},\gamma)$ satisfies (EB′), it is enough to prove

$$\mathcal{K}[\delta(t),\gamma(t)] = \mathcal{K}[\delta(t),\delta_h(t)], \text{ for every } t\in[0,T]. \tag{3.12}$$

Since $\gamma(t)\ge\delta_h(t)$, we easily deduce $\mathcal{K}[\delta(t),\gamma(t)] \ge \mathcal{K}[\delta(t),\delta_h(t)]$. To get the other inequality we first observe that arguing exactly as in the proof of Proposition 3.10, but replacing $\gamma$ with $\delta_h$ (indeed we have just proved (GS)) we get:

$$\mathcal{E}[\boldsymbol{u}(t),\boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t),\delta_h(t)] \ge \mathcal{E}[\boldsymbol{u}^0,\boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0,\delta^0] + \mathcal{W}[\boldsymbol{u},\boldsymbol{\alpha}](t).$$

Combining the above inequality with (EB′), we finally obtain

$$\mathcal{K}[\delta(t),\delta_h(t)] \ge \mathcal{K}[\delta(t),\gamma(t)];$$

hence, (3.12) holds true.

If in addition $\varphi_2$ is strictly increasing, then (3.12) implies $\gamma(t)=\delta_h(t)$ since both functions are continuous in $[0,L]$. Thus, we conclude. □

## 4 | PDE FORM OF ENERGETIC EVOLUTIONS

In this section, we compute the Euler–Lagrange equations coming from the global stability condition (GS′). More precisely, we prove that any generalised energetic evolution $(\boldsymbol{u},\boldsymbol{\alpha},\gamma)$ must satisfy, in a suitable weak formulation, the following system of equilibrium equations governing the stresses $\sigma_i$ (see Proposition 4.3):

$$\begin{cases} -\sigma_1(t)' + \partial_y\varphi(\delta(t),\gamma(t))sgn(u_1(t)-u_2(t)) = 0, & \text{in } [0,L], \\ -\sigma_2(t)' - \partial_y\varphi(\delta(t),\gamma(t))sgn(u_1(t)-u_2(t)) = 0, & \text{in } [0,L], \end{cases} \quad \text{for every } t \in [0,T], \tag{4.1a}$$

where $sgn(\cdot)$ denotes the signum function, together with a Karush–Kuhn–Tucker condition describing the evolution of the damage variables (if regular in time, see Propositions 4.4 and 4.5):

$$\begin{cases} \dot\alpha_i(t) \geq 0, & \text{in } [0,L], \\ -\alpha_i(t)'' + \frac{1}{2}E_i'(\alpha_i(t))(u_i(t)')^2 + w_i'(\alpha_i(t)) \geq 0, & \text{in } [0,L], \\ \left[-\alpha_i(t)'' + \frac{1}{2}E_i'(\alpha_i(t))(u_i(t)')^2 + w_i'(\alpha_i(t))\right]\dot\alpha_i(t) = 0, & \text{in } [0,L], \end{cases} \quad \text{for a.e. } t \in [0,T]. \tag{4.1b}$$

The results of this section will be crucial for the achievement of our goal, namely, the equivalence between the fictitious history variable $\gamma$ and the concrete one $\delta_h$, under meaningful assumptions on $\varphi$. The argument based on time regularity of generalised energetic evolutions will be developed in Section 5.

We recall that, given the loading–unloading density $\varphi : \mathcal{T} \to [0,+\infty)$, we denote by $\psi$ its restriction to the diagonal, namely, $\psi(z) = \varphi(z,z)$, for $z \in [0,+\infty)$. Throughout the section, the main assumptions on $\varphi$ (and $\psi$) are

$$\text{the function } \psi \text{ belongs to } C^1([0,+\infty)); \tag{4.2a}$$

$$\text{for every } z \in (0,+\infty), \text{ the map } \varphi(\cdot,z) \text{ belongs to } C^1([0,z]); \tag{4.2b}$$

$$\text{for every } z \in (0,+\infty), \text{ there holds } \partial_y\varphi(z,z) = \psi'(z) \text{ and } \partial_y\varphi(0,z) = 0; \tag{4.2c}$$

$$\text{the partial derivative } \partial_y\varphi \text{ belongs to } C^0(\mathcal{T}\setminus(0,0)), \text{ and it is bounded in } \mathcal{T}. \tag{4.2d}$$

We notice that the above conditions are slightly more general than properties ($\varphi$5)–($\varphi$8) listed in Section 2, since we do not require any convexity assumption (which will be instead employed in Section 5).

We start the analysis with a simple but useful lemma.

**Lemma 4.1.** *Let $f, g \in \mathbb{R}$ such that $f \geq |g|$, and assume that the function $\varphi : \mathcal{T} \to [0,+\infty)$ satisfies*

$$\textit{the function } z \mapsto \varphi(z,z) =: \psi(z) \textit{ is differentiable in } [0,+\infty); \tag{4.3a}$$

$$\textit{for every } z \in (0,+\infty), \textit{ the map } \varphi(\cdot,z) \textit{ is differentiable in } [0,z]; \tag{4.3b}$$

$$\textit{for every } z \in (0,+\infty), \textit{ there holds } \partial_y\varphi(z,z) = \psi'(z). \tag{4.3c}$$

*Then for every $v \in \mathbb{R}$, one has*

$$\lim_{h\to 0^+} \frac{\varphi(|g+hv|, f \vee |g+hv|) - \varphi(|g|,f)}{h} = \begin{cases} \partial_y\varphi(|g|,f)sgn(g)v, & \text{if } f > |g| > 0, \\ \psi'(|g|)sgn(g)v, & \text{if } f = |g| > 0, \\ \partial_y\varphi(0,f)|v|, & \text{if } f > |g| = 0, \\ \psi'(0)|v|, & \text{if } f = |g| = 0. \end{cases}$$

*Proof.* We denote by $I$ the limit we want to compute, and we distinguish among all the different cases. We first assume that $f > |g|$, so we get

- if $g = 0$, then $I = \lim\limits_{h\to 0^+} \frac{\varphi(h|v|,f)-\varphi(0,f)}{h} = \partial_y\varphi(0,f)|v|$;
- if $g > 0$, then $I = \lim\limits_{h\to 0^+} \frac{\varphi(g+hv,f)-\varphi(g,f)}{h} = \partial_y\varphi(g,f)v = \partial_y\varphi(|g|,f)sgn(g)v$;
- if $g < 0$, then $I = \lim\limits_{h\to 0^+} \frac{\varphi(|g|-hv,f)-\varphi(|g|,f)}{h} = -\partial_y\varphi(|g|,f)v = \partial_y\varphi(|g|,f)sgn(g)v$.

If instead $f = |g|$, we have

- if $g = 0$, then $I = \lim_{h\to 0^+} \frac{\varphi(h|v|,h|v|)-\varphi(0,0)}{h} = \lim_{h\to 0^+} \frac{\psi(h|v|)-\psi(0)}{h} = \psi'(0)|v|$;
- if $g > 0$ and $v \geq 0$, then $I = \lim_{h\to 0^+} \frac{\varphi(g+hv,g+hv)-\varphi(g,g)}{h} = \psi'(g)v = \psi'(|g|)sgn(g)v$;
- if $g > 0$ and $v < 0$, then $I = \lim_{h\to 0^+} \frac{\varphi(g+hv,g)-\varphi(g,g)}{h} = \partial_y\varphi(g,g)v = \psi'(g)v = \psi'(|g|)sgn(g)v$;
- if $g < 0$ and $v \geq 0$, then $I = \lim_{h\to 0^+} \frac{\varphi(|g|-hv,|g|)-\varphi(|g|,|g|)}{h} = -\partial_y\varphi(|g|,|g|)v = \psi'(|g|)sgn(g)v$;
- if $g < 0$ and $v < 0$, then $I = \lim_{h\to 0^+} \frac{\psi(|g|-hv)-\psi(|g|)}{h} = -\psi'(|g|)v = \psi'(|g|)sgn(g)v$.

So we conclude. □

As an immediate corollary, we deduce:

**Corollary 4.2.** *Let f, g be two measurable functions such that $f \in L^\infty(0, L)$ and $f \geq |g|$ a.e. in [0, L], and assume that $\varphi$ satisfies (4.2a)–(4.2c). Then for every $v \in L^\infty(0, L)$, it holds*

$$\lim_{h\to 0^+} \frac{\mathcal{K}[|g + hv|, f \vee |g + hv|] - \mathcal{K}[|g|, f]}{h} = \int_{\{|g|>0\}} \partial_y\varphi(|g(x)|, f(x))sgn(g(x))v(x)\mathrm{d}x + \psi'(0)\int_{\{f=0\}} |v(x)|\mathrm{d}x.$$

*Proof.* We notice that, by the explicit expression of $\mathcal{K}$ given by (2.8), the limit we want to compute can be written as

$$\lim_{h\to 0^+} \int_0^L \frac{\varphi(|g(x) + hv(x)|, f(x) \vee |g(x) + hv(x)|) - \varphi(|g(x)|, f(x))}{h}\mathrm{d}x.$$

Assumptions (4.2a) and (4.2b) allow us to pass to the limit inside the integral; thus, we conclude by means of Lemma 4.1 and exploiting (4.2c). □

We are now in a position to state and prove the first result of this section, namely, a weak form of the Euler–Lagrange equation for the displacement $\boldsymbol{u}$, or better for the stress $\boldsymbol{\sigma}$.

**Proposition 4.3.** *Let $E_i \in C^0([0, 1])$, and assume that $\varphi$ satisfies (4.2a)–(4.2c). Let $(\boldsymbol{u}, \boldsymbol{\alpha}, \gamma)$ satisfy (CO′) and (GS′) of Definition 2.8. Then for every $t \in [0, T]$ and for every $\boldsymbol{v} \in [H_0^1(0, L)]^2$, it holds*

$$\left|\int_0^L \sum_{i=1}^2 \sigma_i(t)v_i'\mathrm{d}x + \int_{\{\delta(t)>0\}} \left[\partial_y\varphi(\delta(t), \gamma(t))sgn(u_1(t) - u_2(t))\right](v_1 - v_2)\mathrm{d}x\right| \leq \psi'(0)\int_{\{\gamma(t)=0\}} |v_1 - v_2|\mathrm{d}x, \tag{4.4}$$

*where the stresses $\sigma_i$ have been introduced in (2.5).*

*In particular, for every $t \in [0, T]$, the sum of the stresses $\sum_{i=1}^2 \sigma_i(t)$ is constant in [0, L].*

*Proof.* We fix $t \in [0, T]$, and by choosing $\widetilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}(t)$ in (GS′), we get for every $h > 0$ and $\boldsymbol{v} \in [H_0^1(0, L)]^2$:

$$\begin{aligned}&\mathcal{E}[\boldsymbol{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)]\\ &\leq \mathcal{E}[\boldsymbol{u}(t) + h\boldsymbol{v}, \boldsymbol{\alpha}(t)] + \mathcal{K}[|u_1(t) - u_2(t) + h(v_1 - v_2)|, \gamma(t) \vee |u_1(t) - u_2(t) + h(v_1 - v_2)|].\end{aligned}$$

Letting $h \to 0^+$, we thus deduce

$$\begin{aligned}0 \leq &\lim_{h\to 0^+} \frac{\mathcal{E}[\boldsymbol{u}(t) + h\boldsymbol{v}, \boldsymbol{\alpha}(t)] - \mathcal{E}[\boldsymbol{u}(t), \boldsymbol{\alpha}(t)]}{h}\\ &+ \lim_{h\to 0^+} \frac{\mathcal{K}[|u_1(t) - u_2(t) + h(v_1 - v_2)|, \gamma(t) \vee |u_1(t) - u_2(t) + h(v_1 - v_2)|] - K[\delta(t), \gamma(t)]}{h}.\end{aligned}$$

The first limit is trivially equal to $\int_0^L \sum_{i=1}^2 \sigma_i(t)v_i' \mathrm{d}x$, while for the second one, we employ Corollary 4.2, and we finally obtain

$$0 \leq \int_0^L \sum_{i=1}^2 \sigma_i(t)v_i' \mathrm{d}x + \int_{\{\delta(t)>0\}} \left[\partial_y \varphi(\delta(t),\gamma(t)) sgn(u_1(t)-u_2(t))\right](v_1-v_2)\mathrm{d}x$$
$$+ \psi'(0)\int_{\{\gamma(t)=0\}} |v_1-v_2|\mathrm{d}x.$$

By following the same argument with $-\boldsymbol{v}$, we prove (4.4).

In particular, if $v_1 = v_2 =: v$, we deduce that

$$\int_0^L \left(\sum_{i=1}^2 \sigma_i(t)\right) v' \mathrm{d}x = 0, \text{ for every } v \in H_0^1(0,L),$$

and so $\sum_{i=1}^2 \sigma_i(t)$ is constant in $[0,L]$. □

We want to point out that if $\psi'(0)$ were equal to 0 (usually false in a cohesive setting, in which $\psi$ is concave and strictly increasing; see Remark 2.3), then inequality (4.4) would actually be equivalent to system (4.1a). The simplifications brought by the assumption $\psi'(0) = 0$ can be also found in Alessi and Freddi,[10] where it has been used for numerical reasons, and in Negri and Vitali,[22] where it has been exploited to perform an approximation argument.

In our work, however, we do not need that additional (and not mechanically justified) assumption; indeed, inequality (4.4) will be enough for our purposes.

The next proposition deals with the damage variable $\boldsymbol{\alpha}$:

**Proposition 4.4.** *Assume that $E_i, w_i \in C^1([0,1])$, and let $(\boldsymbol{u},\boldsymbol{\alpha},\gamma)$ satisfy (CO′) and (GS′) of Definition 2.8. Then, for every $t \in [0,T]$ and for every $\boldsymbol{\beta} \in [H^1(0,L)]^2$ such that $\beta_i \geq 0$ for $i = 1,2$, it holds*

$$\sum_{i=1}^2 \left(\frac{1}{2}\int_{\{\alpha_i(t)<1\}} E_i'(\alpha_i(t))(u_i(t)')^2\beta_i \mathrm{d}x + \int_{\{\alpha_i(t)<1\}} w_i'(\alpha_i(t))\beta_i \mathrm{d}x + \int_{\{\alpha_i(t)<1\}} \alpha_i(t)'\beta_i' \mathrm{d}x\right) \geq 0.$$

*Proof.* We fix $t \in [0,T]$, and by choosing $\widetilde{\boldsymbol{u}} = \boldsymbol{u}(t)$ in (GS′), we get

$$\mathcal{E}[\boldsymbol{u}(t),\boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] \leq \mathcal{E}\left[\boldsymbol{u}(t),\widetilde{\boldsymbol{\alpha}}\right] + \mathcal{D}\left[\widetilde{\boldsymbol{\alpha}}\right], \text{ for every } \widetilde{\boldsymbol{\alpha}} \in [H^1(0,L)]^2 \text{ s.t. } \alpha_i(t) \leq \alpha_i \leq 1. \tag{4.5}$$

We now fix $\boldsymbol{\beta} \in [H^1(0,L)]^2$ such that $\beta_i \geq 0$, and given $h>0$, we define $\widetilde{\boldsymbol{\alpha}}^h(t,x)$ as the vector in $\mathbb{R}^2$ whose components are $(\alpha_i(t,x)+h\beta_i(x)) \wedge 1$. By plugging $\widetilde{\boldsymbol{\alpha}}^h(t)$ in (4.5) as a test function and letting $h \to 0^+$, we thus deduce

$$\begin{aligned}
0 &\leq \liminf_{h\to 0^+} \frac{\mathcal{E}\left[\boldsymbol{u}(t),\widetilde{\boldsymbol{\alpha}}^h(t)\right] - \mathcal{E}[\boldsymbol{u}(t),\boldsymbol{\alpha}(t)] + \mathcal{D}\left[\widetilde{\boldsymbol{\alpha}}^h(t)\right] - \mathcal{D}[\boldsymbol{\alpha}(t)]}{h} \\
&= \liminf_{h\to 0^+} \sum_{i=1}^2 \left(\frac{1}{2}\int_0^L \frac{E_i\left(\widetilde{\alpha}_i^h(t)\right) - E_i(\alpha_i(t))}{h}(u_i(t)')^2 \mathrm{d}x + \int_0^L \frac{w_i\left(\widetilde{\alpha}_i^h(t)\right) - w_i(\alpha_i(t))}{h}\mathrm{d}x\right. \\
&\quad \left. + \frac{1}{2}\int_0^L \frac{\left(\widetilde{\alpha}_i^h(t)'\right)^2 - (\alpha_i(t)')^2}{h}\mathrm{d}x\right) = \liminf_{h\to 0^+}(I_h + II_h + III_h).
\end{aligned} \tag{4.6}$$

We study the limits of $I_h$, $II_h$ and $III_h$ separately. Since $E_i, w_i$ are in $C^1([0,1])$, we can pass the limit inside the integral in both $I_h$ and $II_h$. We also notice that given $f \in C^1([0,1])$, $a \in [0,1]$ and $b \geq 0$, one has

$$\lim_{h\to 0^+} \frac{f((a+hb)\wedge 1) - f(a)}{h} = \begin{cases} f'(a)b, & \text{if } a \in [0,1), \\ 0, & \text{if } a = 1. \end{cases}$$

Thus, we deduce that

$$\lim_{h\to 0^+} I_h = \sum_{i=1}^{2} \frac{1}{2}\int_{\{\alpha_i(t)<1\}} E_i'(\alpha_i(t))(u_i(t)')^2\beta_i \mathrm{d}x, \text{ and } \lim_{h\to 0^+} II_h = \sum_{i=1}^{2}\int_{\{\alpha_i(t)<1\}} w_i'(\alpha_i(t))\beta_i \mathrm{d}x. \tag{4.7}$$

To deal with $III_h$, we first observe that $\widetilde{\alpha}_i^h(t)' = \begin{cases} \alpha_i(t)' + h\beta_i', & \text{a.e. in } \{\alpha_i(t)+h\beta_i<1\}, \\ 0, & \text{a.e. in} \{\alpha_i(t)+h\beta_i \geq 1\}, \end{cases}$ and so

$$\begin{aligned} III_h &= \sum_{i=1}^{2}\left(\int_{\{\alpha_i(t)+h\beta_i<1\}} \alpha_i(t)'\beta_i' \mathrm{d}x + \frac{h}{2}\int_{\{\alpha_i(t)+h\beta_i<1\}} (\beta_i')^2 \mathrm{d}x - \frac{1}{2h}\int_{\{\alpha_i(t)+h\beta_i\geq 1\}} (\alpha_i(t)')^2 \mathrm{d}x\right) \\ &\leq \sum_{i=1}^{2}\left(\int_{\{\alpha_i(t)+h\beta_i<1\}} \alpha_i(t)'\beta_i' \mathrm{d}x + \frac{h}{2}\int_0^L (\beta_i')^2 \mathrm{d}x\right). \end{aligned}$$

By an easy application of dominated convergence theorem, we hence obtain

$$\limsup_{h\to 0^+} III_h \leq \sum_{i=1}^{2}\int_{\{\alpha_i(t)<1\}} \alpha_i(t)'\beta_i' \mathrm{d}x, \tag{4.8}$$

and collecting (4.6)–(4.8), we conclude. □

The last result of the section is a by-product of the energy balance (EB′), assuming a priori that a generalised energetic evolution possesses a certain regularity in time. This kind of regularity will be in fact proved in Section 5 under suitable convexity assumptions on the data; thus, this a priori requirement is not restrictive.

We refer to Appendix A for the definition and the main properties of absolutely continuous functions in Banach spaces, concepts we use in the next proposition.

**Proposition 4.5.** *Assume that $E_i, w_i \in C^1([0,1])$ and that $\varphi$ satisfies (4.2a)–(4.2d) and ($\varphi$3). Let $(\boldsymbol{u}, \boldsymbol{\alpha}, \gamma)$ be a generalised energetic evolution such that*

$$\boldsymbol{u}, \boldsymbol{\alpha} \in AC([0,T];[H^1(0,L)]^2), \text{ and } \gamma \in C^0([0,T], C^0([0,L])).$$

*Then for a.e. $t \in [0,T]$, one has*

$$\begin{aligned} &\bullet \frac{1}{2}\int_0^L E_i'(\alpha_i(t))(u_i(t)')^2\dot{\alpha}_i(t)\mathrm{d}x + \int_0^L w_i'(\alpha_i(t))\dot{\alpha}_i(t)\mathrm{d}x + \int_0^L \alpha_i(t)'\dot{\alpha}_i(t)'\mathrm{d}x = 0, \textit{ for } i = 1,2; \\ &\bullet \lim_{h\to 0}\int_0^L \frac{\varphi(\delta(t),\gamma(t+h)) - \varphi(\delta(t),\gamma(t))}{h}\mathrm{d}x = 0. \end{aligned} \tag{4.9}$$

*Proof.* First of all, we notice that the time regularity we are assuming on $\boldsymbol{u}$ and $\boldsymbol{\alpha}$ ensures that the maps $t \mapsto \mathcal{E}[\boldsymbol{u}(t),\boldsymbol{\alpha}(t)]$ and $t \mapsto \mathcal{D}[\boldsymbol{\alpha}(t)]$ are absolutely continuous in $[0,T]$. Moreover, for almost every time $t \in [0,T]$, the following expressions for their derivatives can be easily obtained:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[\boldsymbol{u}(t),\boldsymbol{\alpha}(t)] = \sum_{i=1}^{2}\left(\frac{1}{2}\int_0^L E_i'(\alpha_i(t))(u_i(t)')^2\dot{\alpha}_i(t)\mathrm{d}x + \int_0^L E_i(\alpha_i(t))u_i(t)'\dot{u}_i(t)'\mathrm{d}x\right); \tag{4.10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}[\boldsymbol{\alpha}(t)] = \sum_{i=1}^{2}\left(\int_0^L w_i'(\alpha_i(t))\dot{\alpha}_i(t)\mathrm{d}x + \int_0^L \alpha_i(t)'\dot{\alpha}_i(t)'\mathrm{d}x\right). \tag{4.10b}$$

By (EB′), since the work of the prescribed displacement $\mathcal{W}[\boldsymbol{u},\boldsymbol{\alpha}]$ is absolutely continuous by definition, we now deduce that also the map $t \mapsto \mathcal{K}[\delta(t),\gamma(t)]$ is absolutely continuous in $[0,T]$. Moreover, we know that $\delta$ belongs to $AC([0,T];H^1_0(0,L))$; indeed, both $u_1$ and $u_2$ are absolutely continuous with values in $H^1(0,L)$ by assumption. Thus, for almost every $t\in[0,T]$, there exists the derivative of $\mathcal{K}[\delta(t),\gamma(t)]$, and we can compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}[\delta(t),\gamma(t)] &= \lim_{h\to 0}\int_0^L \frac{\varphi(\delta(t+h),\gamma(t+h))-\varphi(\delta(t),\gamma(t))}{h}\mathrm{d}x\\ &= \int_0^L \partial_y\varphi(\delta(t),\gamma(t))\dot{\delta}(t)\mathrm{d}x + \lim_{h\to 0}\int_0^L \frac{\varphi(\delta(t),\gamma(t+h))-\varphi(\delta(t),\gamma(t))}{h}\mathrm{d}x,\end{aligned} \tag{4.11}$$

where we exploited the continuity assumption of both $\partial_y\varphi$ and $\gamma$.

Differentiating (EB′), using (4.10) and (4.11), and recalling that the sum of the stresses $\sigma_i$ is constant in $[0,L]$ by Proposition 4.3, we deduce, for almost every $t\in[0,T]$,

$$\begin{aligned}0 = &\sum_{i=1}^{2}\left(\frac{1}{2}\int_0^L E_i'(\alpha_i(t))(u_i(t)')^2\dot{\alpha}_i(t)\mathrm{d}x + \int_0^L w_i'(\alpha_i(t))\dot{\alpha}_i(t)\mathrm{d}x + \int_0^L \alpha_i(t)'\dot{\alpha}_i(t)'\mathrm{d}x\right)\\ &+\int_0^L\sum_{i=1}^{2}E_i(\alpha_i(t))u_i(t)'\dot{u}_i(t)'\mathrm{d}x + \int_0^L \partial_y\varphi(\delta(t),\gamma(t))\dot{\delta}(t)\mathrm{d}x - \dot{\bar{u}}(t)\sum_{i=1}^{2}\sigma_i(t,0)\\ &+\lim_{h\to 0}\int_0^L \frac{\varphi(\delta(t),\gamma(t+h))-\varphi(\delta(t),\gamma(t))}{h}\mathrm{d}x.\end{aligned} \tag{4.12}$$

The term in the third line of (4.12) is non-negative by means of ($\varphi$3) and the fact that $\gamma$ is non-decreasing (in time). We thus conclude if we show that also the sum of the terms in the second line and each of the two terms (for $i=1,2$) in the sum in the first line are non-negative.

We first focus on the first line. We notice that for $i=1,2$, the function $\dot{\alpha}_i(t)\in H^1(0,L)$ is non-negative and vanishes on the set $\{\alpha_i(t)=1\}$; indeed, $\alpha_i$ is non-decreasing in time, and it is always less or equal than 1. This means that we can use it as a test function in Proposition 4.4, getting for a.e. $t\in[0,T]$:

$$\frac{1}{2}\int_0^L E_i'(\alpha_i(t))(u_i(t)')^2\dot{\alpha}_i(t)\mathrm{d}x + \int_0^L w_i'(\alpha_i(t))\dot{\alpha}_i(t)\mathrm{d}x + \int_0^L \alpha_i(t)'\dot{\alpha}_i(t)'\mathrm{d}x \geq 0.$$

As regards the sum of the terms in the second line in (4.12), we actually prove it is equal to zero. To this aim, we make use of Proposition 4.3 choosing as test functions $v_i(x)=\dot{u}_i(t,x)-\dot{\bar{u}}(t)x/L\in H^1_0(0,L)$, so that $|v_1-v_2|=|\dot{u}_1(t)-\dot{u}_2(t)|$. We indeed notice that $|v_1-v_2|=0$ on the set $\{\gamma(t)=0\}$: If $x$ belongs to that set, then $u_1(\tau,x)=u_2(\tau,x)$ for every $\tau\in[0,t]$, and thus, $\dot{u}_1(t,x)=\dot{u}_2(t,x)$. So we deduce for a.e. $t\in[0,T]$:

$$\begin{aligned}0 &= \int_0^L\sum_{i=1}^{2}\sigma_i(t)\left(\dot{u}_i(t)'-\frac{\dot{\bar{u}}(t)}{L}\right)\mathrm{d}x + \int_{\{\delta(t)>0\}}\left[\partial_y\varphi(\delta(t),\gamma(t))sgn(u_1(t)-u_2(t))\right](\dot{u}_1(t)-\dot{u}_2(t))\mathrm{d}x\\ &= \int_0^L\sum_{i=1}^{2}E_i(\alpha_i(t))u_i(t)'\dot{u}_i(t)'\mathrm{d}x + \int_0^L \partial_y\varphi(\delta(t),\gamma(t))\dot{\delta}(t)\mathrm{d}x - \dot{\bar{u}}(t)\sum_{i=1}^{2}\sigma_i(t,0).\end{aligned}$$

In the above equality, we first used the fact that by definition,

$$\dot{\delta}(t) = (\dot{u}_1(t)-\dot{u}_2(t))sgn(u_1(t)-u_2(t)), \text{ in } \{\delta(t)>0\},$$

and then we exploited the assumption $\partial_y\varphi(0,z)=0$ for $z>0$.

So the proof is complete. □

## 5 | TIME REGULARITY AND EQUIVALENCE BETWEEN $\gamma$ AND $\delta_h$

In this section, we finally develop the strategy which will allow to show that the fictitious history variable $\gamma$ actually coincides with the concrete one $\delta_h$ in some meaningful cases (see Theorems 5.7 and 5.8). The argument, which exploits the results of Section 4, is based on the regularity in time of generalised energetic evolutions; this feature, as noticed in Mielke and Thomas,[32] is a peculiarity of systems governed by convex energies. For this reason, in this section, we need to strengthen the assumptions on the data, requiring for $i = 1, 2$:

$$E_i \in C^2([0,1]) \text{ is convex and satisfies } \frac{1}{2}E_i''(y)E_i(y) - E_i'(y)^2 > 0 \text{ for every } y \in [0,1]; \tag{5.1}$$

$$\begin{gathered} w_i \in C^1([0,1]) \text{ satisfies (1.7) and is uniformly convex with parameter } \mu_i > 0, \text{ namely,} \\ w_i(\theta y^a + (1-\theta)y^b) \leq \theta w_i(y^a) + (1-\theta)w_i(y^b) - \frac{\mu_i}{2}\theta(1-\theta)|y^a - y^b|^2, \text{ for every } \theta, y^a, y^b \in [0,1]. \end{gathered} \tag{5.2}$$

We notice that (5.1) implies (2.3), while (5.2) is trivially satisfied, for instance, by the simple example $w_i(y) = \frac{y^2+y}{2}$. We also define

$$M_i := \max_{y\in[0,1]} E_i''(y), \quad m_i := \min_{y\in[0,1]} \left(\frac{1}{2}E_i''(y)E_i(y) - E_i'(y)^2\right), \tag{5.3}$$

which are strictly positive by (5.1), and we finally denote by $\mu$ the minimum between $\mu_1$ and $\mu_2$, namely,

$$\mu := \mu_1 \wedge \mu_2 > 0. \tag{5.4}$$

*Remark* 5.1 (Hardening materials). Condition (5.1) is a characteristic of the so-called hardening materials, namely, those materials for which the compliance $S(y) := E(y)^{-1}$ is strictly concave. Indeed, by simple calculations, one has

$$S''(y) = -\frac{2}{E(y)^3}\left(\frac{1}{2}E''(y)E(y) - E'(y)^2\right),$$

from which $S'' < 0$ if and only if (5.1) is satisfied. Time regularity of evolutions is expected only for this kind of materials; indeed, in the opposite framework of softening materials (with convex compliance $S$), discontinuous evolutions are common due to snapback phenomena (see also the analysis of Pham et al.[35]).

Of course, we also need some sort of convexity for the loading–unloading density $\varphi$. However, we recall that usually it originates from a concave function $\psi$ (see Remark 2.3); thus, in order to keep that crucial property, we only require a weak form of convexity assumption on $\psi$, already adopted in Negri and Scala[21]:

$$\begin{gathered} \text{the function } \psi \text{ is } \lambda\text{-convex for some } \lambda > 0, \text{ namely, for every } \theta \in [0,1] \text{ and } z^a, z^b \in [0,+\infty) \\ \psi(\theta z^a + (1-\theta)z^b) \leq \theta\psi(z^a) + (1-\theta)\psi(z^b) + \frac{\lambda}{2}\theta(1-\theta)|z^a - z^b|^2, \end{gathered} \tag{5.5a}$$

while for $\varphi$ itself, in addition to (4.2a)–(4.2d), we assume

$$\text{for every } z \in (0,+\infty), \text{ the map } \varphi(\cdot, z) \text{ is non-decreasing and convex.} \tag{5.5b}$$

*Remark* 5.2. Coupling (4.2a)–(4.2d) with (5.5), we have thus recovered the assumptions ($\varphi$5)–($\varphi$8) listed in Section 2. We point out again that they are satisfied by the prototypical example of loading–unloading density $\varphi$ given by (2.11).

A crucial condition on the involved parameters will be given by

$$\frac{m_1}{M_1} \wedge \frac{m_2}{M_2} > \lambda\frac{L^2}{\pi^2}. \tag{5.6}$$

It morally says that the convexity of the internal energy $\mathcal{E}$, represented by $\frac{m_1}{M_1} \wedge \frac{m_2}{M_2}$, is stronger than the concavity of $\mathcal{K}$, represented by $\lambda$, and thus, the overall behaviour is the one of a convex energy.

*Remark* 5.3. As already observed in Mielke and Thomas and Pham et al,[32,35] a simple example of functions satisfying (5.1) is given by

$$E_i(y) = \frac{a_i}{(1+y)^{b_i}}, \text{ with } a_i > 0 \text{ and } b_i \in (0,1).$$

In this case, indeed, it holds

$$\frac{1}{2}E_i''(y)E_i(y) - E_i'(y)^2 = \frac{a_i^2}{2}\frac{b_i(1-b_i)}{(1+y)^{2(1+b_i)}} \geq \frac{a_i^2}{2}\frac{b_i(1-b_i)}{4^{1+b_i}} = m_i.$$

Moreover, $M_i = \max\limits_{y\in[0,1]} E_i''(y) = a_i b_i(1+b_i)$, so that

$$\frac{m_i}{M_i} = \frac{a_i}{2}\frac{1-b_i}{1+b_i}\frac{1}{4^{1+b_i}}.$$

In the particular case in which $a_1 = a_2 =: a$ and $b_1 = b_2 = 1/2$, we get $\frac{m_1}{M_1} = \frac{m_2}{M_2} = \frac{a}{48}$, and so condition (5.6) can be written as

$$\frac{a}{\lambda L^2} > \frac{48}{\pi^2},$$

and can be achieved by increasing the parameter $a$ or by decreasing $\lambda$ or the length of the bar $L$.

For convenience, in this section, we also introduce the notation of the 'shifted' energy (see also Mielke and Roubíček,[12] Remark 3.2). For $t \in [0,T]$ and $x \in [0,L]$, we define the function $\bar{\boldsymbol{u}}_D(t,x) := \left(\frac{\bar{u}(t)}{L}x, \frac{\bar{u}(t)}{L}x\right)$, and we present the shifted variable $\boldsymbol{v}(t) = \boldsymbol{u}(t) - \bar{\boldsymbol{u}}_D(t)$, which has zero boundary conditions, and hence, it belongs to $[H_0^1(0,L)]^2$. We finally introduce the shifted energy:

$$\mathcal{E}_D[t, \boldsymbol{v}(t), \boldsymbol{\alpha}(t)] := \mathcal{E}[\boldsymbol{v}(t) + \bar{\boldsymbol{u}}_D(t), \boldsymbol{\alpha}(t)] = \mathcal{E}[\boldsymbol{u}(t), \boldsymbol{\alpha}(t)],$$

and we want to highlight its explicit dependence on time given by the prescribed displacement and encoded by the function $\bar{\boldsymbol{u}}_D$. Written in this form, the energy allows us to recast the work of the external prescribed displacement (2.14) in the following way:

$$\mathcal{W}[\boldsymbol{u}, \boldsymbol{\alpha}](t) = \int_0^t \partial_t \mathcal{E}_D[\tau, \boldsymbol{v}(\tau), \boldsymbol{\alpha}(\tau)] \mathrm{d}\tau. \tag{5.7}$$

Moreover, by simple computations, it is easy to see that for almost every time $\tau \in [0,T]$ and for every $t \in [0,T]$, the following inequality holds true:

$$|\partial_t \mathcal{E}_D[\tau, \boldsymbol{v}(\tau), \boldsymbol{\alpha}(\tau)] - \partial_t \mathcal{E}_D[\tau, \boldsymbol{v}(t), \boldsymbol{\alpha}(t)]| \leq C|\dot{\bar{u}}(\tau)| \left( \|\boldsymbol{\alpha}(\tau) - \boldsymbol{\alpha}(t)\|^2_{[H^1(0,L)]^2} + \|\boldsymbol{v}(\tau)' - \boldsymbol{v}(t)'\|^2_{[L^2(0,L)]^2} \right)^{\frac{1}{2}}, \tag{5.8}$$

where $C > 0$ is a suitable positive constant.

Furthermore, we also notice that the global stability condition (GS′) of Definition 2.8 can be rewritten as

for every $t \in [0,T]$, one has $\gamma(t) \geq \delta(t)$ in $[0,L]$, and

$$\mathcal{E}_D[t, \boldsymbol{v}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \leq \mathcal{E}_D\left[t, \widetilde{\boldsymbol{v}}, \tilde{\boldsymbol{\alpha}}\right] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}\left[\widetilde{\delta}, \gamma(t) \vee \widetilde{\delta}\right],$$

for every $\widetilde{\boldsymbol{v}} \in [H_0^1(0,L)]^2$ and for every $\tilde{\boldsymbol{\alpha}} \in [H^1(0,L)]^2$ such that $\alpha_i(t) \leq \tilde{\alpha}_i \leq 1$ in $[0,L]$ for $i = 1,2$.

We finally have all the ingredients to start the analysis regarding the time regularity of generalised energetic evolutions. We first state a useful lemma, whose simple proof can be found, for instance, in Lemma 5.6 of Gidoni and Riva[38] or in Lemma 4.3 of Heida and Mielke.[39]

**Lemma 5.4.** *Let $(X, \|\cdot\|)$ be a normed space, and let $f\colon [a, b] \to X$ be a bounded measurable function such that*

$$\|f(t) - f(s)\|^2 \leq \int_s^t \|f(t) - f(\tau)\| g(\tau) d\tau, \text{ for every } a \leq s \leq t \leq b,$$

*for some non-negative $g \in L^1(a, b)$. Then actually it holds*

$$\|f(t) - f(s)\| \leq \int_s^t g(\tau) d\tau, \text{ for every } a \leq s \leq t \leq b.$$

We are now in a position to state and prove the first result of this section, which yields time regularity of generalised energetic evolutions under the convexity assumptions we stated before. The argument is based on the ideas of Mielke and Thomas,[32] adapted to our setting where also a cohesive energy (concave by nature) is taken into account.

**Proposition 5.5** (Time regularity). *Assume that $E_i$ satisfies (5.1), $w_i$ satisfies (5.2) and $\varphi \in C^0(\mathcal{T})$ satisfies ($\varphi$3) and ($\varphi$5)–($\varphi$8). Let $(\boldsymbol{u}, \boldsymbol{\alpha}, \gamma)$ be a generalised energetic evolution related to the prescribed displacement $\bar{u} \in AC([0, T])$. If condition (5.6) on the parameters is satisfied, then both the displacements $\boldsymbol{u}$ and the damage variables $\boldsymbol{\alpha}$ belong to $AC([0, T]; [H^1(0, L)]^2)$, and so one also has $\delta \in AC([0, T]; H^1(0, L))$ and $\delta_h \in AC([0, T]; C^0([0, L]))$.*

*If in addition the family $\{\gamma(t) \wedge \bar{\delta}\}_{t \in [0,T]}$, with $\bar{\delta}$ introduced in (2.10), is equicontinuous and for every $y \in [0, \bar{\delta})$ the map $\varphi(y, \cdot)$ is strictly increasing in $[y, \bar{\delta})$, then the function $\gamma \wedge \bar{\delta}$ belongs to $C^0([0, T]; C^0([0, L])$.*

*Remark* 5.6. We want to point out that the additional requirement of equicontinuity of the family $\{\gamma(t) \wedge \bar{\delta}\}_{t \in [0,T]}$, although cannot be derived directly from the Definition 2.8 of generalised energetic evolutions, is automatically satisfied by the limit function $\gamma$ obtained in Proposition 3.5. Thus, it is not restrictive.

*Proof of Proposition 5.5.* We first consider, for $i = 1, 2$, the Hessian matrix of the function $[0, 1] \times \mathbb{R} \ni (\alpha, v) \mapsto \frac{1}{2} E_i(\alpha) v^2$, denoted by $H_i(\alpha, v)$, and its quadratic form, namely, the map:

$$(x, y) \mapsto \langle (x, y), H_i(\alpha, v)(x, y) \rangle = \frac{1}{2} E_i''(\alpha) v^2 x^2 + 2 E_i'(\alpha) v x y + E_i(\alpha) y^2.$$

By (5.1), it must be $E_i''(\alpha) > 0$ for every $\alpha \in [0, 1]$, and so we can write

$$\begin{aligned} \langle (x, y), H_i(\alpha, v)(x, y) \rangle &= \frac{2}{E_i''(\alpha)} \left[ \left( \frac{1}{2} E_i''(\alpha) v x + E_i'(\alpha) y \right)^2 + \left( \frac{1}{2} E_i''(\alpha) E_i(\alpha) - E_i'(\alpha)^2 \right) y^2 \right] \\ &\geq 2 \frac{m_i}{M_i} y^2. \end{aligned}$$

Thanks to this estimate on the Hessian matrix, it is easy to infer that for every $t \in [0, T]$, for every $\theta \in [0, 1]$ and for every $\boldsymbol{v}^a, \boldsymbol{v}^b \in [H^1_0(0, L)]^2$ and $\boldsymbol{\alpha}^a, \boldsymbol{\alpha}^b \in [H^1_{[0,1]}(0, L)]^2$, it holds

$$\begin{aligned} &\mathcal{E}_D[t, \theta \boldsymbol{v}^a + (1 - \theta) \boldsymbol{v}^b, \theta \boldsymbol{\alpha}^a + (1 - \theta) \boldsymbol{\alpha}^b] \\ &\leq \theta \mathcal{E}_D[t, \boldsymbol{v}^a, \boldsymbol{\alpha}^a] + (1 - \theta) \mathcal{E}_D[t, \boldsymbol{v}^b, \boldsymbol{\alpha}^b] - \frac{m_1}{M_1} \wedge \frac{m_2}{M_2} \theta (1 - \theta) \|(\boldsymbol{v}^a)' - (\boldsymbol{v}^b)'\|^2_{[L^2(0,L)]^2}. \end{aligned} \tag{5.9}$$

By means of (5.2), we also deduce that for every $t \in [0, T]$, for every $\theta \in [0, 1]$ and for every $\boldsymbol{\alpha}^a, \boldsymbol{\alpha}^b \in [H^1_{[0,1]}(0, L)]^2$, we have

$$\mathcal{D}[\theta \boldsymbol{\alpha}^a + (1 - \theta) \boldsymbol{\alpha}^b] \leq \theta \mathcal{D}[\boldsymbol{\alpha}^a] + (1 - \theta) \mathcal{D}[\boldsymbol{\alpha}^b] - \frac{\mu \wedge 1}{2} \theta (1 - \theta) \|\boldsymbol{\alpha}^a - \boldsymbol{\alpha}^b\|^2_{[H^1(0,L)]^2}. \tag{5.10}$$

Finally, by (4.2c), (5.5a) and (5.5b) (which are implied by ($\varphi$5)–($\varphi$8)), we deduce that for every $z \in [0, +\infty)$, the function $y \mapsto \varphi(y, z \vee y)$ is $\lambda$-convex in $[0, +\infty)$; thus, for every $t \in [0, T]$, for every $\theta \in [0, 1]$ and for every non-negative $\delta^a, \delta^b \in H^1_0(0, L)$, it holds

$$
\begin{aligned}
&\mathcal{K}[\theta\delta^a + (1-\theta)\delta^b, \gamma(t) \vee (\theta\delta^a + (1-\theta)\delta^b)] \\
&\le \theta\mathcal{K}[\delta^a, \gamma(t) \vee \delta^a] + (1-\theta)\mathcal{K}[\delta^b, \gamma(t) \vee \delta^b] + \frac{\lambda}{2}\theta(1-\theta)\|\delta^a - \delta^b\|^2_{L^2(0,L)}.
\end{aligned} \tag{5.11}
$$

We now fix $t \in [0,T]$, $\theta \in (0,1)$, $\widetilde{\boldsymbol{v}} \in [H^1_0(0,L)]^2$, $\widetilde{\boldsymbol{\alpha}} \in [H^1(0,L)]^2$ such that $\alpha_i(t) \le \widetilde{\alpha}_i \le 1$ for $i = 1,2$, and we consider as competitors in (GS′) the functions $\theta\widetilde{\boldsymbol{v}} + (1-\theta)\boldsymbol{v}(t)$ and $\theta\widetilde{\boldsymbol{\alpha}} + (1-\theta)\boldsymbol{\alpha}(t)$; by means of (5.9)–(5.11), together with ($\varphi$3) and (5.5b), we thus get

$$
\begin{aligned}
&\mathcal{E}_D\,[t, \boldsymbol{v}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \\
&\le \mathcal{E}_D\left[t, \theta\widetilde{\boldsymbol{v}} + (1-\theta)\boldsymbol{v}(t), \theta\widetilde{\boldsymbol{\alpha}} + (1-\theta)\boldsymbol{\alpha}(t)\right] + \mathcal{D}\left[\theta\widetilde{\boldsymbol{\alpha}} + (1-\theta)\boldsymbol{\alpha}(t)\right] \\
&\quad + \mathcal{K}[|\theta(\tilde{v}_1 - \tilde{v}_2) + (1-\theta)(v_1(t) - v_2(t))|, \gamma(t) \vee |\theta(\tilde{v}_1 - \tilde{v}_2) + (1-\theta)(v_1(t) - v_2(t))|] \\
&\le \theta\mathcal{E}_D\left[t, \widetilde{\boldsymbol{v}}, \widetilde{\boldsymbol{\alpha}}\right] + (1-\theta)\mathcal{E}_D[t, \boldsymbol{v}(t), \boldsymbol{\alpha}(t)] - \frac{m_1}{M_1} \wedge \frac{m_2}{M_2}\theta(1-\theta)\|\left(\widetilde{\boldsymbol{v}}\right)' - (\boldsymbol{v}(t))'\|^2_{[L^2(0,L)]^2} \\
&\quad + \theta\mathcal{D}\left[\widetilde{\boldsymbol{\alpha}}\right] + (1-\theta)\mathcal{D}[\boldsymbol{\alpha}(t)] - \frac{\mu \wedge 1}{2}\theta(1-\theta)\|\widetilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}(t)\|^2_{[H^1(0,L)]^2} \\
&\quad + \theta\mathcal{K}\left[\widetilde{\delta}, \gamma(t) \vee \widetilde{\delta}\right] + (1-\theta)\mathcal{K}[\delta(t), \gamma(t)] + \frac{\lambda}{2}\theta(1-\theta)\|\widetilde{\delta} - \delta(t)\|^2_{L^2(0,L)}.
\end{aligned} \tag{5.12}
$$

We now exploit the well-known sharp Poincaré inequality:

$$
\int_a^b f(x)^2 \mathrm{d}x \le \frac{(b-a)^2}{\pi^2} \int_a^b f'(x)^2 \mathrm{d}x, \text{ for every } f \in H^1_0(a,b),
$$

to deduce that

$$
\|\widetilde{\delta} - \delta(t)\|^2_{L^2(0,L)} \le 2\frac{L^2}{\pi^2}\|\left(\widetilde{\boldsymbol{v}}\right)' - (\boldsymbol{v}(t))'\|^2_{[L^2(0,L)]^2}. \tag{5.13}
$$

By plugging (5.13) in (5.12), dividing by $\theta$ and then letting $\theta \to 0^+$, we finally deduce

$$
\begin{aligned}
&\left(\frac{m_1}{M_1} \wedge \frac{m_2}{M_2} - \lambda\frac{L^2}{\pi^2}\right)\|\left(\widetilde{\boldsymbol{v}}\right)' - (\boldsymbol{v}(t))'\|^2_{[L^2(0,L)]^2} + \frac{\mu \wedge 1}{2}\|\widetilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}(t)\|^2_{[H^1(0,L)]^2} \\
&+ \mathcal{E}_D[t, \boldsymbol{v}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \\
&\le \mathcal{E}_D\left[t, \widetilde{\boldsymbol{v}}, \widetilde{\boldsymbol{\alpha}}\right] + \mathcal{D}\left[\widetilde{\boldsymbol{\alpha}}\right] + \mathcal{K}\left[\widetilde{\delta}, \gamma(t) \vee \widetilde{\delta}\right].
\end{aligned} \tag{5.14}
$$

For the sake of simplicity, we denote by $c$ the minimum between $\frac{m_1}{M_1} \wedge \frac{m_2}{M_2} - \lambda\frac{L^2}{\pi^2}$ and $\frac{\mu\wedge 1}{2}$, and we notice that $c$ is strictly positive by (5.6). We now fix two times $0 \le s \le t \le T$. Exploiting (5.14) at time $s$ with $\widetilde{\boldsymbol{v}} = \boldsymbol{v}(t)$ and $\widetilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}(t)$, and recalling (EB′) and (5.7), we obtain

$$
\begin{aligned}
&c\left(\|\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(s)\|^2_{[H^1(0,L)]^2} + \|\boldsymbol{v}(t)' - \boldsymbol{v}(s)'\|^2_{[L^2(0,L)]^2}\right) \\
&\le \mathcal{E}_D[s, \boldsymbol{v}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(s) \vee \delta(t)] - \mathcal{E}_D[s, \boldsymbol{v}(s), \boldsymbol{\alpha}(s)] - \mathcal{D}[\boldsymbol{\alpha}(s)] - \mathcal{K}[\delta(s), \gamma(s)] \\
&\le \mathcal{E}_D[s, \boldsymbol{v}(t), \boldsymbol{\alpha}(t)] - \mathcal{E}_D[t, \boldsymbol{v}(t), \boldsymbol{\alpha}(t)] + \mathcal{W}[\boldsymbol{u}, \boldsymbol{\alpha}](t) - \mathcal{W}[\boldsymbol{u}, \boldsymbol{\alpha}](s) \\
&\le \int_s^t |\partial_t\mathcal{E}_D[\tau, \boldsymbol{v}(\tau), \boldsymbol{\alpha}(\tau)] - \partial_t\mathcal{E}_D[\tau, \boldsymbol{v}(t), \boldsymbol{\alpha}(t)]|\mathrm{d}\tau.
\end{aligned}
$$

By using (5.8), we thus deduce

$$
\begin{aligned}
&\|\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(s)\|^2_{[H^1(0,L)]^2} + \|\boldsymbol{v}(t)' - \boldsymbol{v}(s)'\|^2_{[L^2(0,L)]^2} \\
&\le \frac{C}{c}\int_s^t |\dot{\bar{u}}(\tau)|\left(\|\boldsymbol{\alpha}(\tau) - \boldsymbol{\alpha}(t)\|^2_{[H^1(0,L)]^2} + \|\boldsymbol{v}(\tau)' - \boldsymbol{v}(t)'\|^2_{[L^2(0,L)]^2}\right)^{\frac{1}{2}}\mathrm{d}\tau.
\end{aligned}
$$

By means of (2.16a), we can apply Lemma 5.4 getting

$$\left(\|\boldsymbol{\alpha}(t)-\boldsymbol{\alpha}(s)\|^2_{[H^1(0,L)]^2}+\|\boldsymbol{v}(t)'-\boldsymbol{v}(s)'\|^2_{[L^2(0,L)]^2}\right)^{\frac{1}{2}}\le\frac{C}{c}\int_s^t|\dot{\bar u}(\tau)|\mathrm{d}\tau,$$

and so we infer that $\boldsymbol{\alpha}$ belongs to $AC([0,T];[H^1(0,L)]^2)$ and $\boldsymbol{v}$ belongs to $AC([0,T];[H^1_0(0,L)]^2)$. By construction, we also have

$$\begin{aligned}\|\boldsymbol{u}(t)-\boldsymbol{u}(s)\|_{[H^1(0,L)]^2}&\le\|\boldsymbol{v}(t)-\boldsymbol{v}(s)\|_{[H^1(0,L)]^2}+\|\boldsymbol{u}_D(t)-\boldsymbol{u}_D(s)\|_{[H^1(0,L)]^2}\\&\le\|\boldsymbol{v}(t)-\boldsymbol{v}(s)\|_{[H^1(0,L)]^2}+C|\bar u(t)-\bar u(s)|,\end{aligned}$$

so also $\boldsymbol{u}$ belongs to $AC([0,T];[H^1(0,L)]^2)$, and as a simple by-product, we obtain that $\delta$ is $AC([0,T];H^1(0,L))$.

Since $H^1(0,L)\subseteq C^0([0,L])$, in particular, there exists a non-negative function $\phi\in L^1(0,T)$ such that

$$\|\delta(t)-\delta(s)\|_{C^0([0,L])}\le\int_s^t\phi(\tau)\mathrm{d}\tau,\ \text{for every}\ \ 0\le s\le t\le T. \tag{5.15}$$

We now show that the same inequality holds true for $\delta_h$ in place of $\delta$. We thus fix $0\le s\le t\le T$ and $x\in[0,L]$. If $\delta_h(t,x)=\delta_h(s,x)$, there is nothing to prove, so let us assume $\delta_h(t,x)>\delta_h(s,x)$. By definition of $\delta_h$ and since now we know that $\delta$ is continuous both in time and space, we deduce that

$$\delta_h(t,x)=\max_{\tau\in[0,t]}\delta(\tau,x)=\delta(t_x,x),\ \text{ for some }\ t_x\in[s,t].$$

So we have

$$\delta_h(t,x)-\delta_h(s,x)\le\delta(t_x,x)-\delta(s,x)\le\int_s^{t_x}\phi(\tau)\mathrm{d}\tau\le\int_s^t\phi(\tau)\mathrm{d}\tau.$$

We have thus proved the validity of (5.15) with $\delta_h$ in place of $\delta$, and hence, $\delta_h$ belongs to $AC([0,T];C^0([0,L]))$.

We only need to prove that $\gamma\wedge\bar\delta\in C^0([0,T];C^0([0,L])$ under the additional assumptions that $\{\gamma(t)\wedge\bar\delta\}_{t\in[0,T]}$ is an equicontinuous family and $\varphi(y,\cdot)$ is strictly increasing in $[y,\bar\delta)$ for any given $y\in[0,\bar\delta)$. For the sake of clarity, we prove it only in the case $\bar\delta=+\infty$; in the other situation, the result can be obtained arguing in the same way and recalling equality (2.12). To this aim, we observe that, by equicontinuity, for every $t\in[0,T]$, the right and the left limits $\gamma^+(t)$ and $\gamma^-(t)$ are continuous in $[0,L]$. By monotonicity and using classical Dini's theorem, we hence obtain

$$\gamma^\pm(t)=\lim_{h\to0^\pm}\gamma(t+h),\ \text{ uniformly in }\ [0,L]. \tag{5.16}$$

So we conclude if we prove that $\gamma^+(t)=\gamma^-(t)$.

Arguing as in the proof of Proposition 4.5, since $\boldsymbol{u}$ and $\boldsymbol{\alpha}$ are in $AC([0,T];[H^1(0,L)]^2)$, we deduce by (EB′) that the map $t\mapsto\mathcal{K}[\delta(t),\gamma(t)]$ is continuous in $[0,T]$, and thus, for every $t\in[0,T]$, we have

$$\lim_{h\to0^+}\mathcal{K}[\delta(t+h),\gamma(t+h)]=\lim_{h\to0^-}\mathcal{K}[\delta(t+h),\gamma(t+h)].$$

By using (5.16), we can pass to the limit inside the integral getting

$$\int_0^L\varphi(\delta(t),\gamma^+(t))\mathrm{d}x=\int_0^L\varphi(\delta(t),\gamma^-(t))\mathrm{d}x.$$

Since $\varphi(y,\cdot)$ is strictly increasing, we conclude. □

Thanks to the time regularity obtained in the previous proposition, we are able to prove our main results. The first theorem ensures the equality between $\gamma$ and $\delta_h$ (actually between $\gamma\wedge\bar\delta$ and $\delta_h\wedge\bar\delta$, which however are the meaningful ones; see Remark 2.9) assuming a priori equicontinuity on the family $\{\gamma(t)\}_{t\in[0,T]}$, which is however not restrictive due to Remark 5.6; a similar argument to the one adopted here, but in an easier setting, can be found in Proposition 2.7

of Riva.[40] The second theorem states that the generalised energetic evolution obtained in Section 3 as limit of discrete minimisers is actually an energetic evolution. We thus reach our goal, avoiding the assumption ($\varphi$4) and considering the list of reasonable assumptions ($\varphi$5)–($\varphi$9) (actually, we replace ($\varphi$9) by the weaker (5.17)) which, for instance, are satisfied by the example provided in Remark 2.3.

We finally point out that, even under the convexity assumptions of this section, the issue of uniqueness for energetic evolutions cannot be easily inferred from our techniques, due to the presence of the history slip and of the cohesive energy in the interface.

**Theorem 5.7** (Equivalence between $\gamma$ and $\delta_h$). *Let the prescribed displacement $\bar{u}$ belong to the space $AC([0,T])$. Assume that $E_i$ satisfies (5.1), $w_i$ satisfies (5.2) and $\varphi \in C^0(\mathcal{T})$ satisfies ($\varphi$5)–($\varphi$8), plus the following uniform strict monotonicity with respect to $z$:*

$$\begin{aligned} &\textit{for every compact set } K \in \{z > y \geq 0\} \cap \bar{\mathcal{T}}_{\bar\delta}, \textit{ there exists a positive constant } C_K > 0 \textit{ such that} \\ &\qquad\qquad \varphi(y,z_2) - \varphi(y,z_1) \geq C_K(z_2 - z_1) \\ &\qquad\qquad \textit{for every } (z_2,y),(z_1,y) \in K \textit{ satisfying } z_2 \geq z_1. \end{aligned} \tag{5.17}$$

*Assume also condition (5.6) on the parameters. Then, given a generalised energetic evolution $(\boldsymbol{u},\boldsymbol{\alpha},\gamma)$ such that the family $\{\gamma(t)\wedge\bar{\delta}\}_{t\in[0,T]}$ is equicontinuous, the function $\gamma\wedge\bar{\delta}$ coincides with $\delta_h\wedge\bar{\delta}$.*

*Proof.* For the sake of clarity, we prove the result only in the case $\bar{\delta} = +\infty$, being the other situation analogous by (2.12).

We know that $\gamma \geq \delta_h$ and that $\gamma(0) = \delta_h(0) = \delta^0$ and $\gamma(t,0) = \gamma(t,L) = \delta_h(t,0) = \delta_h(t,L) = 0$ for every $t\in[0,T]$. Moreover, by Proposition 5.5, we know that both $\gamma$ and $\delta_h$ are continuous on $[0,T]\times[0,L]$.

We thus assume by contradiction that there exists $(\bar{t},\bar{x}) \in (0,T]\times(0,L)$ for which $\gamma(\bar{t},\bar{x}) > \delta_h(\bar{t},\bar{x})$; by continuity, we thus deduce that there exists $\eta > 0$ such that

$$\gamma(t,x) > \delta_h(t,x) \geq \delta(t,x), \text{ for every } (t,x) \in [\bar{t}-\eta,\bar{t}]\times[\bar{x}-\eta,\bar{x}+\eta].$$

By assumption (5.17), we hence infer the existence of constant $c_\eta > 0$ for which

$$\begin{aligned} &\varphi(\delta(s,x),\gamma(t,x)) - \varphi(\delta(s,x),\gamma(s,x)) \geq c_\eta(\gamma(t,x)-\gamma(s,x)), \\ &\qquad \text{for every } \bar{t}-\eta \leq s \leq t \leq \bar{t} \text{ and } x \in [\bar{x}-\eta,\bar{x}+\eta]. \end{aligned} \tag{5.18}$$

We now recall that by Proposition 5.5, we know the map $t \mapsto \mathcal{K}[\delta(t),\gamma(t)]$ is absolutely continuous in $[0,T]$. So for every $0\leq s\leq t\leq T$, we can estimate

$$\begin{aligned} &\int_0^L (\varphi(\delta(s),\gamma(t)) - \varphi(\delta(s),\gamma(s)))\mathrm{d}x \\ &= \mathcal{K}[\delta(t),\gamma(t)] - \mathcal{K}[\delta(s),\gamma(s)] + \int_0^L (\varphi(\delta(s),\gamma(t)) - \varphi(\delta(t),\gamma(t)))\mathrm{d}x \\ &\leq \int_s^t \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}[\delta(\tau),\gamma(\tau)]\mathrm{d}\tau + C\|\delta(t)-\delta(s)\|_{C^0([0,L])} \leq \int_s^t \phi(\tau)\mathrm{d}\tau, \end{aligned} \tag{5.19}$$

where $\phi\in L^1(0,T)$ is a suitable non-negative function.

Combining (5.18) and (5.19), we now obtain

$$c_\eta \int_{\bar{x}-\eta}^{\bar{x}+\eta} (\gamma(t)-\gamma(s))\mathrm{d}x \leq \int_s^t \phi(\tau)\mathrm{d}\tau, \text{ for every } \bar{t}-\eta \leq s \leq t \leq \bar{t};$$

hence, $\gamma \in AC([\bar{t}-\eta,\bar{t}];L^1(\bar{x}-\eta,\bar{x}+\eta))$.

By means of (4.9), we now deduce that for a.e. $t \in [\bar{t}-\eta, \bar{t}]$, we have

$$0 = \lim_{h\to 0}\int_0^L \frac{\varphi(\delta(t),\gamma(t+h)) - \varphi(\delta(t),\gamma(t))}{h}\mathrm{d}x \geq c_\eta \limsup_{h\to 0}\int_{\bar{x}-\eta}^{\bar{x}+\eta}\frac{\gamma(t+h)-\gamma(t)}{h}\mathrm{d}x \geq 0,$$

namely, for almost every $t \in [\bar{t}-\eta, \bar{t}]$, the function $\gamma$ is strongly differentiable in $L^1(\bar{x}-\eta, \bar{x}+\eta)$ and $\dot{\gamma}(t) = 0$. By Proposition A.3, we now obtain

$$\gamma(t) = \gamma(\bar{t}-\eta) + \int_{\bar{t}-\eta}^{t}\dot{\gamma}(\tau)\mathrm{d}\tau = \gamma(\bar{t}-\eta), \text{ for every } t \in [\bar{t}-\eta, \bar{t}], \text{ as an equality in } L^1(\bar{x}-\eta, \bar{x}+\eta).$$

In particular, since $\gamma$ is continuous, we deduce that $\gamma(\bar{t}, \bar{x}) = \gamma(\bar{t}-\eta, \bar{x})$.

We now show that, since $\delta_h$ is non-decreasing, we can iterate the previous argument getting $\gamma(\bar{t}, \bar{x}) = \gamma(0, \bar{x})$. This would lead to a contradiction, since it would imply

$$\delta^0(\bar{x}) = \gamma(0,\bar{x}) = \gamma(\bar{t},\bar{x}) > \delta_h(\bar{t},\bar{x}) \geq \delta_h(0,\bar{x}) = \delta^0(\bar{x}).$$

To check that $\gamma(\bar{t},\bar{x}) = \gamma(0,\bar{x})$, we consider $t_0(\bar{x}) := \min\left\{t \in [0,\bar{t}] | \gamma(t,\bar{x}) = \gamma(\bar{t},\bar{x})\right\}$, which is well defined by the continuity of $\gamma(\cdot,\bar{x})$. Notice that there holds

$$\gamma(t_0(\bar{x}),\bar{x}) = \gamma(\bar{t},\bar{x}) > \delta_h(\bar{t},\bar{x}) \geq \delta_h(t_0(\bar{x}),\bar{x}).$$

Hence, if $t_0(\bar{x}) > 0$, by arguing exactly as before, we can find $\widetilde{\eta} > 0$ such that $\gamma(t_0(\bar{x}) - \widetilde{\eta}, \bar{x}) = \gamma(\bar{t},\bar{x})$, contradicting the minimality of $t_0(\bar{x})$. Thus, $t_0(\bar{x}) = 0$, and we conclude. □

**Theorem 5.8** (Existence of energetic evolutions)**.** *Let the prescribed displacement $\bar{u}$ belong to the space AC([0, T]) and the initial data $\boldsymbol{u}^0$, $\boldsymbol{\alpha}^0$ fulfil (2.13) together with stability condition (2.15). Assume that $E_i$ satisfies (5.1), $w_i$ satisfies (5.2) and $\varphi \in C^0(\mathcal{T})$ satisfies ($\varphi$2), ($\varphi$5)–($\varphi$8) and (5.17). Assume also condition (5.6) on the parameters. Then the pair $(\boldsymbol{u}, \boldsymbol{\alpha})$ composed by the functions obtained in Proposition 3.5 is an energetic evolution, since it holds $\gamma \wedge \bar{\delta} = \delta_h \wedge \bar{\delta}$, with $\bar{\delta}$ introduced in (2.10).*

*Moreover, $\boldsymbol{u}$ and $\boldsymbol{\alpha}$ belong to $AC([0, T]; [H^1(0, L)]^2)$, and so in particular, the history slip $\delta_h$ is in $AC([0, T]; C^0([0, L]))$.*

*Proof.* The result is a simple by-product of Theorem 3.11 together with Proposition 5.5 and Theorem 5.7 (we also recall (2.12)). We indeed notice that the equicontinuity assumption on the family $\{\gamma(t) \wedge \bar{\delta}\}_{t\in[0,T]}$ (actually on the whole $\{\gamma(t)\}_{t\in[0,T]}$) is automatically satisfied by the limit function $\gamma$ obtained in Proposition 3.5. □

## 6 | CONCLUSIONS

The obtained results offer new insights for further investigations. The 2D numerical investigations presented in Alessi and Freddi,[16] where the complex failure modes of hybrid laminates are consistently reproduced, suggest to extend the theoretical investigation to higher dimensional settings whereby the introduction of the anisotropic behaviour of materials allows the analysis of problems of interest for the conservation of cultural heritage[17,41] and other microcracking phenomena such.[42] A second line of exploration could also be the analysis of the problem in case of complete damage, meant as complete loss of material stiffness.

Moreover, it would be interesting to extend the proposed approach to classical problems of cohesive fracture mechanics. In this case, dissipation combined with irreversible effects introduces difficulties, at least when dealing with global minimisers of the energy, in considering loading–unloading cohesive laws that reflect the real behaviour of materials rather than hypotheses dictated by mere mathematical assumptions. The main difference provided by cohesive fracture models with respect to the considered problem of cohesive interface relies in the reduced dimension of the fracture, which is a $(d-1)$-dimensional object in a $d$-dimensional material. This feature involves the use of weaker topologies, which cannot be directly treated following our argument, and thus requires further adaptations in order to transfer our results.

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## CONFLICT OF INTERESTS

This work does not have any conflicts of interest.

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## APPENDIX A: ABSOLUTELY CONTINUOUS AND BV VECTOR-VALUED FUNCTIONS

In this appendix, we briefly present the main definitions and properties of vector-valued absolutely continuous functions and functions of bounded variation we used throughout the paper. A deeper and more detailed analysis can be found in the Appendix of Brezis,[43] to which we refer for all the proofs and examples. Here, $(X, \|\cdot\|)$ will denote a Banach space, and by $X^*$, we mean its topological dual. The duality product between $w \in X^*$ and $x \in X$ is finally denoted by $\langle w, x \rangle$.

**Definition A.1.** A function $f : [0, T] \to X$ is said to be

- a function of bounded variation ($BV([0, T]; X)$) if

$$V_X(f; 0, T) := \sup_{\substack{\text{finite partitions} \\ \text{of } [0,T]}} \sum \|f(t_k) - f(t_{k-1})\| < +\infty;$$

- absolutely continuous ($AC([0, T]; X)$) if there exists a non-negative function $\phi \in L^1(0, T)$ such that

$$\|f(t) - f(s)\| \leq \int_s^t \phi(\tau) \mathrm{d}\tau, \text{ for every } 0 \leq s \leq t \leq T;$$

- in the space $\tilde{W}^{1,p}(0, T; X)$, $p \in [1, +\infty]$, if there exists a non-negative function $\phi \in L^p(0, T)$ such that

$$\|f(t) - f(s)\| \leq \int_s^t \phi(\tau) \mathrm{d}\tau, \text{ for every } 0 \leq s \leq t \leq T;$$

- in the Sobolev space $W^{1,p}(0,T;X)$, $p\in[1,+\infty]$, if there exists a function $g\in L^p(0,T;X)$ such that

$$f(t)=f(0)+\int_0^t g(\tau)\mathrm{d}\tau,\ \text{ for every }\ t\in[0,T].$$

As in the classical case $X=\mathbb{R}$ any function of bounded variation belongs to $L^\infty(0,T;X)$, it admits right and left (strong) limits at every $t\in[0,T]$ and the set of its discontinuity points is at most countable. To gain the well-known property of almost everywhere differentiability also in the vector-valued framework, it is instead crucial to require $X$ to be reflexive (see the examples in Brezis[43]).

**Proposition A.2.** *If $X$ is reflexive, then any function $f$ belonging to $BV([0,T];X)$ is weakly differentiable almost everywhere in $[0,T]$. Moreover, $\|\dot f(t)\|\le\frac{\mathrm{d}}{\mathrm{d}t}V_X(f;0,t)$ for a.e. $t\in[0,T]$ and in particular $\dot f\in L^1(0,T;X)$.*

We now focus our attention on absolutely continuous and Sobolev functions. By the very definition, it is easy to see that any absolutely continuous function is also of bounded variation; furthermore, the spaces $AC([0,T];X)$ and $\widetilde{W}^{1,1}(0,T;X)$ coincide, while $\tilde W^{1,\infty}(0,T;X)$ is the space of Lipschitz functions from $[0,T]$ to $X$. Moreover, for every $p\in[1,+\infty]$, the inclusion $W^{1,p}(0,T;X)\subseteq\widetilde{W}^{1,p}(0,T;X)$ always holds but in general is strict.

The next proposition states that the Sobolev space $W^{1,p}(0,T;X)$ is actually characterised by the strong differentiability of its elements.

**Proposition A.3.** *Let $p\in[1,+\infty]$, and let $f$ be a function from $[0,T]$ to $X$. Then the following are equivalent:*

*(i)* $f\in W^{1,p}(0,T;X)$;
*(ii)* $f\in\widetilde{W}^{1,p}(0,T;X)$, *and it is strongly differentiable for a.e.* $t\in[0,T]$; *and*
*(iii)* *for every* $w\in X^*$, *the map* $t\mapsto\langle w,f(t)\rangle$ *is absolutely continuous in* $[0,T]$, $f$ *is weakly differentiable for a.e.* $t\in[0,T]$ *and* $\dot f\in L^p(0,T;X)$.

*If one of the above condition holds, then one has*

$$f(t)=f(0)+\int_0^t \dot f(\tau)\mathrm{d}\tau,\ \textit{ for every }\ t\in[0,T]. \tag{A1}$$

In the reflexive case, as in Proposition A.2, we gain differentiability of absolutely continuous functions, and so we deduce the equivalence between the two spaces $\widetilde{W}^{1,p}(0,T;X)$ and $W^{1,p}(0,T;X)$.

**Proposition A.4.** *If $X$ is reflexive, then for every $p\in[1,+\infty]$, the Sobolev space $W^{1,p}(0,T;X)$ coincides with $\widetilde{W}^{1,p}(0,T;X)$.*