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A Doctrinal View Of Logic

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Introduction

Given a theory $\mathcal{T}$ in a first-order language $\mathcal{L}$, consider for each set of variables the set of well-formed formulae written with at most those variables. That set can be ordered by provable consequence in the theory $\mathcal{T}$. By that we mean that the formula α is less than or equal to the formula β if the consequence $\alpha \vdash_{\mathcal{T}} \beta$ holds. The logical operations of conjunction, disjunction, implication, negation, true and false give this set the structure of a Boolean algebra. And the assignment $\vec{x} \mapsto \mathrm{WWF}(\vec{x})$ of the Boolean algebra of formulae to a list (i.e. a context) of distinct variables can be extended to a functor $\mathrm{WWF}\colon \mathbb{C}\mathrm{tx}^{\mathrm{op}} \to \mathbf{BA}$ from the opposite category of contexts and terms to the category of Boolean algebras and homomorphisms.

This can be considered the motivating example at the basis of the notion of hyperdoctrine which was introduced by Lawvere in 1969 in a series of seminal papers [Law69a, Law69b, Law70]. It is a categorical tool that allows the analysis of both syntax and semantics of logical theories through the same mathematical structure. One of the main intuitions of Lawvere was to recognize that quantifiers in logic are instances of adjunctions between the posets of formulae.

As we have seen, the logical operations and operators provide an extremely abundant structure. In order to understand such a complex array, it is often useful to restrict one's view to a particular side of it. A doctrine is possibly the basic fabric of Lawvere's hyperdoctrine: just a functor $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$, from a category $\mathbb{C}$ with finite products into the category $\mathbf{Pos}$ of partially ordered sets and monotone functions. Doctrines naturally organize into a 2-category, and they are the main subject of the thesis.

The aim of this work is to offer an interpretation of some classical results in first-order logic and in universal algebra via doctrines. The first is performed in the context of existential implicational doctrines, while the second in the context of elementary doctrines. We then show how the classical results are actually instances of more general results that live in the context of doctrines.

The first goal is the analysis of Henkin's Theorem for first-order logic [Hen49], formulated as follows:

> *Every consistent theory has a model.*

The key points in the proof of the original theorem are adding a suitable amount of constants to the starting language, and then adding some axioms of the extended language to the starting theory. In Chapter 2 we start our investigation by extending to doctrines the construction of adding a constant to a language. And we also extend the construction of forcing a new axiom for primary doctrines, which are doctrines where all orders are inf-semilattices and reindexing preserves them—essentially, what it amounts to the ability to interpret conjunctions of formulae. Actually we do both constructions in one step, using a Kleisli object for a convenient comonad on the original doctrine seen as an indexed poset. The existence of Kleisli objects for comonads in indexed posets is proven in Proposition 1.2.5. Given a primary doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$, a fixed object X, and an element $\varphi \in P(X)$, we construct a homomorphism of doctrines $P \to P_{(X,\varphi)}$. In keeping a reasonable parallel with the logical intuition, think of the primary doctrine P as the syntactic consequences of a theory—not just formulae of a language, but that is already a good intuition—; think of an object X as a list of fresh variables, and think of φ as a formula in the fresh variables. Staying with the parallel, the doctrine $P_{(X,\varphi)}$ acts like the extension of the theory with new constant symbols and with the new axiom φ evaluated in those constants. The homomorphism of doctrines acts like a translation of the original theory in the new one.

The construction has a universal property: any other morphism of primary doctrines $P \to R$ such that the interpretation of φ evaluated in some constant in R is true factors through $P \to P_{(X,\varphi)}$, essentially in a unique way. This is, in broad terms, the statement of Theorem 2.4.2. Moreover, the result is extended to 2-arrows in Theorem 2.4.4. We also show in Theorem 2.4.3 that the construction $P \to P_{(X,\varphi)}$ preserves many additional structures and properties that the original doctrine may already enjoy.

The construction includes the two constructions we discussed at the beginning: adding no axiom has a structural parallel in adding $\top$ to the axioms, while adding no constant corresponds to performing the construction picking the terminal object for X. We can clearly decide to add just a constant of type X, without adding any axiom to the theory, and obtain a homomorphism $P \to P_X$. Similarly we can decide to add just an axiom φ to a theory, without adding any constant symbol, and obtain the homomorphism $P \to P_\varphi$.

We then proceed in Chapter 3 with the interpretation of Henkin's proof by adding a suitable amount of new constant symbols. To do this, we first prove in Proposition 1.3.1 how to compute colimits of directed diagrams in the category of doctrines $\mathbf{Dct}$. To have an insight into this process, once we know how to add one constant symbol, we can iterate the construction to add a finite number of constant symbols. Then, taking the colimit over a convenient directed diagram $D\colon J \to \mathbf{Dct}$ in which every image $D(j)$ for $j \in J$ is a doctrine with a finite number of constants added, we can add an infinite amount of constants. This construction gives a homomorphism $P \to \underline{P}$ from the original doctrine into the colimit $\underline{P}$. The next step is to add new axioms to the new doctrine $\underline{P}$. To do this, we work with implicational existential doctrines—i.e. doctrines in which we can interpret implication of formulae and the existential quantifier. In this setting, for any formula $\varphi(x)$, we make true a formula of the kind $\exists x \varphi(x) \to \varphi(c)$ for some suitable constant c. Since there is an infinite number of axioms that we have to add, we use a similar technique to the one seen above: we define a directed diagram $\boldsymbol{\Delta}\colon I \to \mathbf{Dct}$ in which every image $\boldsymbol{\Delta}(i)$ for $i \in I$ is a doctrine with a finite number of axioms added, so the colimit adds all the needed axioms. This construction gives another homomorphism $\underline{P} \to \underrightarrow{P}$ into the colimit $\underrightarrow{P}$, and in particular a homomorphism $P \to \underrightarrow{P}$. The doctrine $\underrightarrow{P}$ is rich: for each formula $\varphi(x)$ there exists a constant c such that $\varphi(c)$ and $\exists x\varphi(x)$ have the same truth-value.

When the starting doctrine P is also bounded—i.e. a doctrine in which we can interpret also the false—, we find the properties for P in order to have that the construction $P \to \underrightarrow{P}$ preserves coherence, since we obviously do not want the doctrine $\underrightarrow{P}$ to be such that the true constant and the false collapse in the same formula. Section 3.5 and Section 3.7 collect all these results: initially Proposition 3.5.6 establishes the consistency of $\underrightarrow{P}$ in the Boolean case, then Proposition 3.7.1 shows consistency in the implicational setting. Proposition 3.7.1 follows from Proposition 3.5.6 itself and on the existence of a suitable notion of Boolean completion for bounded implicational doctrines, provided in Section 1.4.

Finally, we prove in Proposition 3.8.1 that a bounded consistent implicational existential rich doctrine has a homomorphism to the doctrine of subsets, the "standard" model. Applying this proposition to the rich doctrine $\underrightarrow{P}$, we obtain Theorem 3.8.5:

> *Let P be a bounded existential implicational doctrine, with non-trivial fibers and with a small base category. Then there exists a bounded existential implicational model of P in the doctrine of subsets $\mathscr{P}_*\colon \mathrm{Set}_*^{\mathrm{op}} \to \mathbf{Pos}$.*

The last chapter proposes a further analysis of the procedure for adding structure and axioms to a theory, this time in the context of elementary doctrines—i.e. a doctrine in which we can interpret equality of terms. It is well known in universal algebra that adding structure and equational axioms generates forgetful functors between varieties, and such functors all have left adjoints. From a categorical point of view, every variety is equivalent to a category of homomorphisms of elementary doctrines $\mathbf{ED}(\mathtt{HF}^{\Sigma}_{\mathbb{T}}, \mathscr{P}_*)$ between a doctrine of formulae and the subsets doctrine. Moreover, adding structure and equational axioms translates to a doctrine homomorphism $(E, \mathfrak{e})\colon \mathtt{HF}^{\Sigma}_{\mathbb{T}} \to \mathtt{HF}^{\Sigma'}_{\mathbb{T}'}$. Precomposition with this homomorphism induces a functor $- \circ (E, \mathfrak{e})\colon \mathbf{ED}(\mathtt{HF}^{\Sigma'}_{\mathbb{T}'}, \mathscr{P}_*) \to \mathbf{ED}(\mathtt{HF}^{\Sigma}_{\mathbb{T}}, \mathscr{P}_*)$, and it represents the forgetful functor between the correspondent varieties, hence it has a left adjoint. If we start from any elementary homomorphism $(F, \mathfrak{f})\colon P \to R$ we can again define the precomposition functor $- \circ (F, \mathfrak{f})\colon \mathbf{ED}(R, \mathrm{Sub}) \to \mathbf{ED}(P, \mathrm{Sub})$, where Sub is the subobject doctrine of a Grothendieck topos. The whole chapter is dedicated to the proof that also in this case the functor $- \circ (F, \mathfrak{f})$ has a left adjoint, showing how the existence of free functors in universal algebra follows from a more general result that lives in the theory of elementary doctrines.

Chapter 1

Preliminaries and initial results

In this chapter, we lay the groundwork for the thesis by introducing the language of doctrines and establishing their key properties. We then prove several general results that will serve as essential tools in later chapters. Specifically, we compute the Eilenberg–Moore and Kleisli objects for comonads in the 2-category of indexed posets, which will be instrumental in Chapter 2. We also demonstrate the existence of directed colimits in the category of doctrines, which will enable us to construct our main argument in Chapter 3. Finally, we define the Boolean completion of an implicational doctrine with a bottom element and review some key findings about filters, which will be crucial in the latter part of Chapter 3.
Most of the notions and results concerning category theory used in this thesis are standard, and we refer to any textbook, for instance [Bor94, Joh02, Mac71].

1.1 Doctrines

In this section, we define the 2-category of doctrines and show some relevant examples. Then we will gradually add more structure in order to be able to interpret symbols of first-order logic—such as connectives and quantifiers—in the context of doctrines.

Definition 1.1.1. Let $\mathbb{C}$ be a category with finite products and let **Pos** be the category of partially-ordered sets and monotone functions. A *doctrine* is a functor $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$. The category $\mathbb{C}$ is called *base category of* P, each poset $P(X)$ for an object $X \in \mathbb{C}$ is called *fiber*, the function $P(f)$ for an arrow f in $\mathbb{C}$ is called *reindexing.*

By viewing doctrines as a broad generalization of doctrines of well-formed formulae, we can interpret the objects of category $\mathbb{C}$ as lists of variables, the arrows as terms, the fibers as sets of formulae, and reindexing as substitutions, providing an intuitive understanding of the strucure.

Example 1.1.2. We propose the following examples.

(a) The functor $\mathscr{P}\colon \mathrm{Set}^{\mathrm{op}} \to \mathbf{Pos}$, sending each set in the poset of its subsets, ordered by inclusion, and each function $f\colon A \to B$ to the inverse image $f^{-1}\colon \mathscr{P}(B) \to \mathscr{P}(A)$ is a doctrine.

(b) For a given category $\mathbb{C}$ with finite limits, the functor $\mathrm{Sub}_{\mathbb{C}}\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ sending each object to the poset of its subobjects in $\mathbb{C}$ and each arrow $f\colon A \to B$ to the pullback function $f^{*}\colon \mathrm{Sub}_{\mathbb{C}}(B) \to \mathrm{Sub}_{\mathbb{C}}(A)$, is a doctrine.

(c) For a given theory $\mathcal{T}$ on a one-sorted first-order language $\mathcal{L}$, define the category $\mathbb{C}\mathrm{tx}_{\mathcal{L}}$ of contexts: an object is a finite list of distinct variables and an arrow between two lists $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_m)$ is

$$(t_1(\vec{x}), \ldots, t_m(\vec{x}))\colon (x_1, \ldots, x_n) \to (y_1, \ldots, y_m)$$

an m-tuple of terms in the context $\vec{x}$. The empty list $()$ is the terminal object in $\mathbb{C}\mathrm{tx}_{\mathcal{L}}$, the product of two lists $\vec{x}$ and $\vec{y}$ in $\mathbb{C}\mathrm{tx}_{\mathcal{L}}$ is given by any list whose length is the sum of the length of $\vec{x}$ and $\vec{y}$—if the variables in the two lists are all distinct, their product can be written as the juxtaposition $\langle \vec{x}; \vec{y}\rangle = (x_1 \ldots, x_n, y_1, \ldots, y_m)$. The functor $\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}\colon \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \to \mathbf{Pos}$ sends each list of variables to the poset reflection of well-formed formulae written with at most those variables ordered by provable consequence in $\mathcal{T}$; moreover, $\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}\colon \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \to \mathbf{Pos}$ sends an arrow $\vec{t}(\vec{x})\colon \vec{x} \to \vec{y}$ into the substitution $[\vec{t}(\vec{x})/\vec{y}]$, that maps the equivalence class of a formula $\alpha(\vec{y})$ to the equivalence class of the formula $\alpha(\vec{t}(\vec{x})/\vec{y})$. We refer to any standard textbook about first-order logic for definitions of concepts including language, variables, theory, terms, substitution, formulae, see for instance [TZ12].

(d) For a given category $\mathbb{D}$ with finite products and weak pullbacks, the functor of weak subobjects $\mathbf{\Psi}_{\mathbb{D}}\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ sending each object A to the poset reflection of the comma category $\mathbb{D}/A$ is a doctrine: for each arrow $f\colon A \to B$, $\mathbf{\Psi}_{\mathbb{D}}(f)$ sends the equivalence class of an arrow $\alpha\colon \mathrm{dom}\,\alpha \to B$ to the equivalence class of the projection π_1 of a chosen weak pullback of α along f—see Example 2.9 in [MR13] for more details.

$$\begin{tikzcd}
W \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathrm{dom}\,\alpha \arrow[d, "\alpha"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Definition 1.1.3. A *doctrine homomorphism*—1-cell or 1-arrow—between $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ and $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ is a pair $(F, \mathfrak{f})$ where $F\colon \mathbb{C} \to \mathbb{D}$ is a functor that preserves finite products and $\mathfrak{f}\colon P \stackrel{\cdot}{\to} R \circ F^{\mathrm{op}}$ is a natural transformation. Sometimes a morphism between P and R will be called a model of P in R. A *2-cell* between $(F, \mathfrak{f})$ and $(G, \mathfrak{g})$ from P to R is a natural transformation $\theta\colon F \stackrel{\cdot}{\to} G$ such that $\mathfrak{f}_A(\alpha) \leq R(\theta_A)(\mathfrak{g}_A(\alpha))$ for any object A in $\mathbb{C}$ and $\alpha \in P(A)$. Doctrine, doctrine morphisms with 2-cells defined here form a 2-category, that will be denoted **Dct**.

$$\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}"] \arrow[dr, "P"', ""{name=P}] & & \mathbb{D}^{\mathrm{op}} \arrow[dl, "R", ""'{name=R}] \\
& \mathbf{Pos} &
\arrow[from=P, to=R, bend left, "\mathfrak{f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \arrow[rr, bend left, "F^{\mathrm{op}}", ""'{name=F}] \arrow[rr, bend right, "G^{\mathrm{op}}"', ""{name=G}] \arrow[dr, "P"', ""{name=P}] & & \mathbb{D}^{\mathrm{op}} \arrow[dl, "R", ""'{name=R}] \\
& \mathbf{Pos} &
\arrow[Rightarrow, from=G, to=F, "\theta^{\mathrm{op}}"]
\arrow[from=P, to=R, bend left, "\mathfrak{f}"]
\arrow[from=P, to=R, bend right, "\mathfrak{g}"']
\end{tikzcd}$$

By definition of doctrine, the fibers are simply posets. However, we can define specific doctrines by imposing additional structure on these posets or by requiring the existence of adjoints to certain reindexing. To work in a setting that interprets the conjunction of formulae and the true constant, primary doctrines are necessary.

Definition 1.1.4. A *primary doctrine* $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is a doctrine such that for each object A in $\mathbb{C}$, the poset $P(A)$ has finite meets, and the related operations $\wedge\colon P \times P \stackrel{\cdot}{\to} P$ and $\top\colon \mathbf{1} \stackrel{\cdot}{\to} P$ yield natural transformations.

Example 1.1.5. Examples seen in 1.1.2 are primary doctrines:

(a) For any set A, intersection of two subsets is their meet, A is the top element.

(b) For any object A in $\mathbb{C}$, the pullback of a subobject along another defines their meet.

$$\begin{tikzcd}
\operatorname{dom}(\alpha \wedge \beta) \arrow[r, tail, "\pi_2"] \arrow[d, tail, "\pi_1"'] \arrow[dr, tail, "\alpha\wedge\beta" description] & \operatorname{dom}\alpha \arrow[d, tail, "\alpha"] \\
\operatorname{dom}\beta \arrow[r, tail, "\beta"'] & A
\end{tikzcd}$$

The arrow id_A is the top element.

(c) For any list $\vec{x}$, the conjunction of two formulae is their binary meet, the true constant $\top$ is the top element.

(d) For any object A in $\mathbb{D}$, a choice of a weak pullback of a representative of a weak subobject along another defines their meet,

$$\begin{tikzcd}
\operatorname{dom}(\alpha \wedge \beta) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "\alpha\wedge\beta" description] & \operatorname{dom}\alpha \arrow[d, "\alpha"] \\
\operatorname{dom}\beta \arrow[r, "\beta"'] & A
\end{tikzcd}$$

the class of id_A is the top element.

In order to interpret equality, we define elementary doctrines. The following definition is taken from unpublished notes by G. Rosolini.

Definition 1.1.6. A primary doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is *elementary* if for any pair of objects B, C of $\mathbb{C}$, the map $P(\mathrm{id}_C \times \Delta_B)\colon P(C \times B \times B) \to P(C \times B)$ has a left adjoint

$$\text{Æ}_C^B\colon P(C \times B) \to P(C \times B \times B),$$

which is natural in C; moreover, the adjunction $\text{Æ}_C^B \dashv P(\mathrm{id}_C \times \Delta_B)$ satisfies the Frobenius reciprocity, i.e. for any $\alpha \in P(C \times B)$ and $\beta \in P(C \times B \times B)$ the inequality

$$\text{Æ}_C^B(\alpha \wedge P(\mathrm{id}_C \times \Delta_B)(\beta)) \leq \text{Æ}_C^B(\alpha) \wedge \beta$$

given by properties of the adjunction is an equality.

Remark 1.1.7. In the Definition above, naturality in C is usually known as Beck-Chevalley condition with respect to any pullback diagram of the form:

$$\begin{tikzcd}
C \times B \arrow[r, "\mathrm{id}_C \times \Delta_B"] \arrow[d, "f \times \mathrm{id}_B"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \times B \times B \arrow[d, "f \times \mathrm{id}_{B \times B}"] \\
C' \times B \arrow[r, "\mathrm{id}_{C'} \times \Delta_B"'] & C' \times B \times B
\end{tikzcd}$$

An equivalent way to define elementary doctrines can be found in Proposition 2.5 of [EPR20]:

Definition 1.1.8. A primary doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is *elementary* if for any object A in $\mathbb{C}$ there exists an element $\delta_A \in P(A \times A)$ such that:

1. $\top_A \leq P(\Delta_A)(\delta_A)$;
2. $P(A) = \mathscr{D}es_{\delta_A} := \{\alpha \in P(A) \mid P(\mathrm{pr}_1)(\alpha) \wedge \delta_A \leq P(\mathrm{pr}_2)(\alpha)\}$;
3. $\delta_A \boxtimes \delta_B \leq \delta_{A \times B}$, where $\delta_A \boxtimes \delta_B = P(\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle)(\delta_A) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_4 \rangle)(\delta_B)$.

In 2., pr_1 and pr_2 are the projections from $A \times A$ in A; in 3., the projections are from $A \times B \times A \times B$. The element δ_A will be called *fibered equality* on A.

Proposition 1.1.9. Definition 1.1.6 and Definition 1.1.8 are equivalent.

Proof. Suppose $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be an elementary doctrine with respect to Definition 1.1.6. Define for each object A in the base category, $\delta_A = \text{Æ}_{\mathbf{t}}^A(\top_A) \in P(A \times A)$, where $\text{Æ}_{\mathbf{t}}^A \dashv P(\Delta_A)$. The first condition of Definition 1.1.8 holds using the unit of the adjunction. Concerning the second one, take any $\alpha \in P(A)$ and use Frobenius reciprocity to get

$$P(\mathrm{pr}_1)(\alpha) \wedge \text{Æ}_{\mathbf{t}}^A(\top_A) = \text{Æ}_{\mathbf{t}}^A(P(\Delta_A)P(\mathrm{pr}_1)(\alpha) \wedge \top_A) = \text{Æ}_{\mathbf{t}}^A(\alpha)$$

but $\text{Æ}_{\mathbf{t}}^A(\alpha) \leq P(\mathrm{pr}_2)(\alpha)$ if and only if $\alpha \leq P(\Delta_A)P(\mathrm{pr}_2)(\alpha) = \alpha$, hence also 2. holds. To conclude, compute

$$P(\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle)\text{Æ}^A_{\mathbf{t}}(\top_A) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_4\rangle)\text{Æ}^B_{\mathbf{t}}(\top_B) \leq_{A\times B\times A\times B} \text{Æ}^{A\times B}_{\mathbf{t}}(\top_{A\times B}) \quad \text{if and only if}$$
$$P(\langle \mathrm{pr}_3, \mathrm{pr}_4\rangle)\text{Æ}^A_{\mathbf{t}}(\top_A) \wedge P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)\text{Æ}^B_{\mathbf{t}}(\top_B) \leq_{B\times B\times A\times A} P(\langle \mathrm{pr}_3, \mathrm{pr}_1, \mathrm{pr}_4, \mathrm{pr}_2\rangle)\text{Æ}^{A\times B}_{\mathbf{t}}(\top_{A\times B})$$

applying the isomorphism $P(\langle \mathrm{pr}_3, \mathrm{pr}_1, \mathrm{pr}_4, \mathrm{pr}_2\rangle)$, but using naturality of Æ^A with respect to the arrow $!_{B\times B}\colon B \times B \to \mathbf{t}$, we know that $P(\langle \mathrm{pr}_3, \mathrm{pr}_4\rangle)\text{Æ}^A_{\mathbf{t}} = \text{Æ}^A_{B\times B}P(\mathrm{pr}_3)$, so by Frobenius reciprocity we need

$$\text{Æ}^A_{B\times B}(\top_{B\times B\times A} \wedge P(\mathrm{id}_{B\times B} \times \Delta_A)P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)\text{Æ}^B_{\mathbf{t}}(\top_B)) \leq_{B\times B\times A\times A}$$
$$P(\langle \mathrm{pr}_3, \mathrm{pr}_1, \mathrm{pr}_4, \mathrm{pr}_2\rangle)\text{Æ}^{A\times B}_{\mathbf{t}}(\top_{A\times B})$$

if and only if

$$P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)\text{Æ}^B_{\mathbf{t}}(\top_B) \leq_{B\times B\times A} P(\langle \mathrm{pr}_3, \mathrm{pr}_1, \mathrm{pr}_3, \mathrm{pr}_2\rangle)\text{Æ}^{A\times B}_{\mathbf{t}}(\top_{A\times B})$$

if and only if

$$P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)\text{Æ}^B_{\mathbf{t}}(\top_B) \leq_{A\times B\times B} P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3\rangle)\text{Æ}^{A\times B}_{\mathbf{t}}(\top_{A\times B})$$

applying the isomorphism $P(\langle \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_1\rangle)$. Now as before, use naturality of Æ^B with respect to the arrow $!_A\colon A \to \mathbf{t}$, so $P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)\text{Æ}^B_{\mathbf{t}} = \text{Æ}^B_A P(\mathrm{pr}_2)$, hence by the adjunction $\text{Æ}^B_A P(\mathrm{id}_A \times \Delta_B)$ we want

$$\top_{A\times B} \leq_{A\times B} P(\mathrm{id}_A \times \Delta_B)P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3\rangle)\text{Æ}^{A\times B}_{\mathbf{t}}(\top_{A\times B}) = P(\Delta_{A\times B})\text{Æ}^{A\times B}_{\mathbf{t}}(\top_{A\times B})$$

hence 3. holds, as claimed. Conversely, define for any pair of objects B, C,

$$\text{Æ}^B_C(\alpha) = P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)(\alpha) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)(\delta_B).$$

This defines a left adjoint of $P(\mathrm{id}_C \times \Delta_B)$, and the proof of this is in Proposition 2.5 of [EPR20]. Now take any $f\colon C' \to C$, we check that $P(f \times \mathrm{id}_{B\times B})\text{Æ}^B_C = \text{Æ}^B_{C'}P(f \times \mathrm{id}_B)$:

$$P(f \times \mathrm{id}_{B\times B})\text{Æ}^B_C(\alpha) = P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)P(f \times \mathrm{id}_B)(\alpha) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)(\delta_B) = \text{Æ}^B_{C'}P(f \times \mathrm{id}_B).$$

At last, we show Frobenius reciprocity. First of all compute:

$$\text{Æ}^B_C(\alpha \wedge P(\mathrm{id}_C \times \Delta_B)(\beta)) = P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)(\alpha \wedge P(\mathrm{id}_C \times \Delta_B)(\beta)) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)(\delta_B) =$$
$$P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)(\alpha) \wedge P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2\rangle)(\beta) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)(\delta_B)$$

then,

$$\begin{aligned}
&Æ_C^B(\alpha) \wedge \beta \leq Æ_C^B(\alpha \wedge P(\mathrm{id}_C \times \Delta_B)(\beta)) \quad \text{if and only if} \\
&\qquad\qquad\qquad P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)(\alpha) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\delta_B) \wedge \beta \leq \\
&\qquad P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)(\alpha) \wedge P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle)(\beta) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\delta_B) \quad \text{if and only if} \\
&\qquad\qquad\qquad P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)(\alpha) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\delta_B) \wedge \beta \leq P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle)(\beta)
\end{aligned}$$

so it is enough to show $P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\delta_B) \wedge \beta \leq P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle)(\beta)$. To see this, observe that in $P(C \times B \times B \times C \times B \times B)$ we have that both inequalities

$$\begin{aligned}
P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \rangle)\beta \wedge \delta_{C\times B\times B} &\leq P(\langle \mathrm{pr}_4, \mathrm{pr}_5, \mathrm{pr}_6 \rangle)\beta \quad \text{(using 2.)} \\
P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_4, \mathrm{pr}_5 \rangle)\delta_{C\times B} \wedge P(\langle \mathrm{pr}_3, \mathrm{pr}_6 \rangle)\delta_B &\leq \delta_{C\times B\times B} \quad \text{(using 3.)}
\end{aligned}$$

hold, so that

$$P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \rangle)\beta \wedge P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_4, \mathrm{pr}_5 \rangle)\delta_{C\times B} \wedge P(\langle \mathrm{pr}_3, \mathrm{pr}_6 \rangle)\delta_B \leq P(\langle \mathrm{pr}_4, \mathrm{pr}_5, \mathrm{pr}_6 \rangle)\beta.$$

Apply $P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle)$ so that in $P(C \times B \times B)$ we have:

$$\beta \wedge P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_2 \rangle)\delta_{C\times B} \wedge P(\langle \mathrm{pr}_3, \mathrm{pr}_2 \rangle)\delta_B \leq P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle)\beta.$$

However, $P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_2 \rangle)\delta_{C\times B} = P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)P(\Delta_{C\times B})\delta_{C\times B} = \top_{C\times B\times B}$. Moreover, observe that in $P(B \times B)$, we have $\delta_B \leq P(\langle \mathrm{pr}_2, \mathrm{pr}_1 \rangle)\delta_B$. Indeed, in $P(B \times B \times B \times B)$ we have using 3. and 2. that

$$\begin{aligned}
&P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)\delta_B \wedge P(\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\delta_B \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_4 \rangle)\delta_B \leq \\
&\qquad\qquad\qquad\qquad P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)\delta_B \wedge \delta_{B\times B} \leq P(\langle \mathrm{pr}_3, \mathrm{pr}_4 \rangle)\delta_B.
\end{aligned}$$

Taking the reindexing along $P(\langle \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_1 \rangle)$, we obtain in $P(B \times B)$ (using 1.) that $\delta_B \leq P(\langle \mathrm{pr}_2, \mathrm{pr}_1 \rangle)\delta_B$. Now take the reindexing of this last inequality along $P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)$, so that in $P(C \times B \times B)$ we have $P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)\delta_B \leq P(\langle \mathrm{pr}_3, \mathrm{pr}_2 \rangle)\delta_B$. We conclude by observing:

$$P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\delta_B) \wedge \beta \leq P(\langle \mathrm{pr}_3, \mathrm{pr}_2 \rangle)(\delta_B) \wedge \beta \leq P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle)(\beta),$$

so Frobenius reciprocity holds, and the two stated definitions are equivalent. ☐

Example 1.1.10. Examples seen in 1.1.2 are elementary doctrines:

(a) For any set A, the subset $\Delta_A \subseteq A \times A$ is the fibered equality on A.

(b) For any object A in $\mathbb{C}$, the map $\Delta_A \colon A \rightarrowtail A \times A$ is the fibered equality on A—see in [MR12] the Example 2.4.a.

(c) For any list $\vec{x}$, the formula $\big(x_1 = x_1' \wedge \cdots \wedge x_n = x_n'\big)$ in $\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}(\vec{x};\vec{x}')$ is the fibered equality on $\vec{x}$.

(d) For any object A in $\mathbb{D}$, the equivalence class of the map $\Delta_A\colon A \rightarrowtail A\times A$ is the fibered equality on A.

We now generalize the existential and universal quantifier, which are defined as adjoint to some reindexing.

Definition 1.1.11. A primary doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is *existential* if for any pair of objects B, C of $\mathbb{C}$, the map $P(\mathrm{pr}_1)\colon P(C) \to P(C\times B)$ has a left adjoint

$$\exists^B_C\colon P(C\times B) \to P(C),$$

which is natural in C; moreover, the adjunction $\exists^B_C \dashv P(\mathrm{pr}_1)$ satisfies the Frobenius reciprocity, i.e. for any $\alpha \in P(C\times B)$ and $\beta \in P(C)$ the inequality $\exists^B_C(\alpha \wedge P(\mathrm{pr}_1)(\beta)) \leq \exists^B_C(\alpha)\wedge\beta$ given by properties of the adjunction is an equality.

Remark 1.1.12. In an elementary existential doctrine, every arrow $f\colon A\to B$ in $\mathbb{C}$ the reindexing $P(f)\colon P(B)\to P(A)$ has a left adjoint $\exists_f\colon P(A)\to P(B)$, computed as follows:

$$\exists_f(\alpha) \colon= \exists^A_B\big(P(\mathrm{pr}_2)(\alpha) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_1\rangle)P(f\times \mathrm{id}_B)(\delta_B)\big),$$

for any α in $P(A)$ and where $\mathrm{pr}_1, \mathrm{pr}_2$ are the projections from $A\times B$. This fact is stated in Remark 2.13 of [MR13]. Since we could not find a reference for the proof, we provide it here for the sake of completeness. For any $\alpha \in P(A)$ and any $\beta\in P(B)$, we need to show that $\exists_f(\alpha)\leq\beta$ in $P(B)$ if and only if $\alpha \leq P(f)\beta$ in $P(A)$. We obtain:

$$\begin{aligned}
&\exists_f(\alpha) \leq_B \beta \quad \text{if and only if}\\
&\qquad P(\mathrm{pr}_2)(\alpha)\wedge P(\langle \mathrm{pr}_2,\mathrm{pr}_1\rangle)P(f\times\mathrm{id}_B)(\delta_B) \leq_{B\times A} P(\mathrm{pr}_1)\beta \quad \text{if and only if}\\
&\qquad\qquad P(\mathrm{pr}_1)(\alpha)\wedge P(f\times\mathrm{id}_B)(\delta_B) \leq_{A\times B} P(\mathrm{pr}_2)\beta
\end{aligned}$$

using at first the adjunction $\exists^A_B \dashv P(\mathrm{pr}_1)$ and then applying the isomorphism $P(\langle \mathrm{pr}_2,\mathrm{pr}_1\rangle)$ between $P(B\times A)$ and $P(A\times B)$. Now, applying $P(\Delta_A)P(\mathrm{id}_A \times f)$ to the last inequality, we get

$$\alpha \wedge P(\Delta_A)P(\mathrm{id}_A \times f)P(f\times\mathrm{id}_B)(\delta_B) \leq_A P(f)\beta$$

but since $(f\times\mathrm{id}_B)(\mathrm{id}_A \times f)(\Delta_A) = \Delta_B f$ and $P(f)P(\Delta_B)\delta_B = P(f)(\top_B) = \top_A$ we get that $\alpha \leq P(f)\beta$ in $P(A)$, as claimed. Conversely, apply $P(f\times\mathrm{id}_B)$ on both sides of the inequality $P(\mathrm{pr}_1)(\beta)\wedge\delta_B \leq P(\mathrm{pr}_2)(\beta)$ in $P(B\times B)$ to get $P(\mathrm{pr}_1)P(f)(\beta)\wedge P(f\times\mathrm{id}_B)\delta_B \leq P(\mathrm{pr}_2)(\beta)$ in

$P(A \times B)$. If we assume $\alpha \leq_A P(f)\beta$, we then get $P(\mathrm{pr}_1)(\alpha) \wedge P(f \times \mathrm{id}_B)(\delta_B) \leq_{A\times B} P(\mathrm{pr}_2)\beta$, which is equivalent to $\exists_f \alpha \leq_B \beta$. This concludes the proof.

Definition 1.1.13. A doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is *universal* if for any pair of objects B, C of $\mathbb{C}$, the map $P(\mathrm{pr}_1)\colon P(C) \to P(C \times B)$ has a right adjoint $\forall^B_C\colon P(C \times B) \to P(C)$, which is natural in C.

Remark 1.1.14. In both Definition 1.1.11 and Definition 1.1.13, naturality in C is usually known as Beck-Chevalley condition with respect to any pullback diagram of the form:

$$\begin{tikzcd}
C \times B \arrow[r, "\mathrm{pr}_1"] \arrow[d, "f\times \mathrm{id}_B"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
C' \times B \arrow[r, "\mathrm{pr}_1"'] & C'
\end{tikzcd}$$

Definition 1.1.15. A doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$:

- is *implicational* if for any object A, the poset $P(A)$ is cartesian closed, and the related operations $\wedge\colon P \times P \dot{\to} P$, $\top\colon \mathbf{1} \dot{\to} P$, $\to\colon P^{\mathrm{op}} \times P \dot{\to} P$ yield natural transformations–in particular it is a primary doctrine;
- *has bottom element* if for any object A, the poset $P(A)$ has a bottom element, and the related operation, $\bot\colon \mathbf{1} \dot{\to} P$ yields a natural transformation;
- *is bounded* if for any object A, the poset $P(A)$ has a top and a bottom element, and the related operation, $\top\colon \mathbf{1} \dot{\to} P$ and $\bot\colon \mathbf{1} \dot{\to} P$ yield natural transformations;
- *has finite joins* if for any object A, the poset $P(A)$ has finite joins, and the related operations $\vee\colon P \times P \dot{\to} P$, $\bot\colon \mathbf{1} \dot{\to} P$ yield natural transformations;
- is *Horn* if it is implicational and universal;
- is *Heyting* if for any object A, the poset $P(A)$ is an Heyting algebra, and the related operations $\wedge\colon P \times P \dot{\to} P$, $\top\colon \mathbf{1} \dot{\to} P$, $\to\colon P^{\mathrm{op}} \times P \dot{\to} P$, $\vee\colon P \times P \dot{\to} P$, $\bot\colon \mathbf{1} \dot{\to} P$ yield natural transformations;
- is *Boolean* if it is Heyting and the operation $\neg(-)\colon = (-) \to \bot\colon P^{\mathrm{op}} \dot{\to} P$ is an isomorphism.

Example 1.1.16. The doctrine $\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}\colon \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \to \mathbf{Pos}$ is Boolean elementary existential universal: in addition to the structure mentioned above, the implication of two formulae gives the implicational structure, the disjunction of two formulae is their join, the false is the bottom element, existential and universal quantifier define the left and the right adjoint to the inclusions of formulae $\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}(\vec{x}) \subseteq \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}(\vec{x}; \vec{y})$ for any pair $\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_m)$:

$$\begin{tikzcd}
\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}(\vec{x}; \vec{y}) \arrow[r, shift left=3, "\exists y_1 \dots \exists y_m"] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"'] \arrow[r, shift right=3, "\forall y_1 \dots \forall y_m"'] & \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}(\vec{x}) \arrow[l, hook']
\end{tikzcd}$$

Definition 1.1.17. Any homomorphism $(F, \mathfrak{f}) \colon P \to R$ from $P \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ to $R \colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ is called respectively *primary, elementary, existential, universal, implicational, bounded, Horn, Heyting, Boolean* if both P and R are, and $\mathfrak{f}$ preserves the said structure. For example an elementary homomorphism is such that for any object A in $\mathbb{C}$, and any $\alpha, \alpha' \in P(A)$:

$$\mathfrak{f}_A(\alpha \wedge_A \alpha') = \mathfrak{f}_A(\alpha) \wedge_{FA} \mathfrak{f}(\alpha'); \qquad \mathfrak{f}_A(\top_A) = \top_{FA}; \qquad \mathfrak{f}_{A \times A}(\delta_A) = \delta_{FA};$$

while an universal homomorphism is such that for any pair of objects B, C in $\mathbb{C}$, and any element $\alpha \in P(C \times B)$:

$$\mathfrak{f}_C \forall_C^B(\alpha) = \forall_{FC}^{FB} \mathfrak{f}_{C \times B}(\alpha).$$

Example 1.1.18. For a given category $\mathbb{C}$ with finite limits, the inclusion of $\mathrm{Sub}_{\mathbb{C}}(A)$ into the poset reflection of $\mathbb{C}/A$ yields a natural transformation $\mathrm{Sub}_{\mathbb{C}} \to \mathbf{\Psi}_{\mathbb{C}}$; pairing it with the identity on the base category $\mathbb{C}$, this defines a 1-arrow in **ED**.

Notation 1.1.19. We will write some 2-full 2-subcategories of **Dct** as follows:

- **PD** for the category of primary doctrines and primary homomorphisms;
- **ED** for the category of elementary doctrines and elementary homomorphisms;
- **Bool** for the category of Boolean doctrines and Boolean homomorphisms.

1.2 Eilenberg–Moore and Kleisli constructions in the 2-category of indexed posets

This section is devoted to show the existence of Eilenberg–Moore and Kleisli objects for comonads in the 2-category of indexed posets. We will use them in Chapter 2 to prove a universal property of a construction we will introduce later. Before we delve into the details, we will provide a brief overview of the relevant definitions and concepts. In the 2-category **IdxPos** of indexed posets the cells are defined as follows:

- a *0-cell* is a functor $P \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$;
- a *1-cell* between $P \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ and $R \colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ is a pair $(F, \mathfrak{f})$ where $F \colon \mathbb{C} \to \mathbb{D}$ is a functor and $\mathfrak{f} \colon P \dot{\to} R \circ F^{\mathrm{op}}$ is a natural transformation;
- a *2-cell* between $(F, \mathfrak{f}), (G, \mathfrak{g}) \colon P \to R$ is a natural transformation $\theta \colon F \dot{\to} G$ such that $\mathfrak{f}_A(\alpha) \leq R(\theta_A)(\mathfrak{g}_A(\alpha))$ for any object A in $\mathbb{C}$ and $\alpha \in P(A)$.

Remark 1.2.1. The 2-category **Dct** is a 2-full 2-subcategory of **IdxPos**.

In the following section, definitions of comonads, Eilenberg–Moore and Kleisli objects in a 2-category are taken from [PW02] (see also [Str72]).

A *comonad* in the 2-category of indexed posets is a list $(P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}, (K, \mathfrak{k}), \gamma, \varepsilon)$ where P is a indexed poset, $(K, \mathfrak{k})$ is a 1-arrow, γ and ε are 2-arrows, and (K, γ, ε) is a comonad in $\mathbb{C}$. In particular, the following diagrams commute:

$$\begin{tikzcd}
K \arrow[r, "\gamma"] \arrow[d, "\gamma"] & K^2 \arrow[d, "K(\gamma)"] \\
K^2 \arrow[r, "\gamma_K"] & K^3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & K \arrow[dl, "\gamma"'] \arrow[d, equal] \arrow[dr, "\gamma"] & \\
K^2 \arrow[r, "K(\varepsilon)"'] & K & K^2 \arrow[l, "\varepsilon_K"]
\end{tikzcd}$$

Moreover, since γ and ε are 2-arrows, the following inequalities hold:

$$\mathfrak{k}_A(\alpha) \leq P(\gamma_A)\mathfrak{k}_{KA}\mathfrak{k}_A(\alpha); \qquad \mathfrak{k}_A(\alpha) \leq P(\varepsilon_A)(\alpha).$$

1.2.1 Eilenberg–Moore construction

We now define the 2-category Cmd(**IdxPos**).

- a *0-cell* is a comonad $(P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}, (K, \mathfrak{k}), \gamma, \varepsilon)$;
- a *1-cell* from the comonad $(P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}, (K, \mathfrak{k}), \gamma, \varepsilon)$ to $(P'\colon \mathbb{C}'^{\mathrm{op}} \to \mathbf{Pos}, (K', \mathfrak{k}'), \gamma', \varepsilon')$ is a lax morphism of comonads, i.e. a pair $((F, \mathfrak{f}), \mathfrak{j})$ where the first entry $(F, \mathfrak{f})\colon P \to P'$ is a 1-arrow in **IdxPos** and the second one $\mathfrak{j}\colon (F, \mathfrak{f})(K, \mathfrak{k}) \to (K', \mathfrak{k}')(F, \mathfrak{f})$ is a 2-arrow, i.e. $\mathfrak{j}\colon FK \dot{\to} K'F$ such that $\mathfrak{f}_{KA}\mathfrak{k}_A(\alpha) \leq P'(\mathfrak{j}_A)\mathfrak{k}'_{FA}\mathfrak{f}_A(\alpha)$, satisfying the coherence diagrams below;

$$\begin{tikzcd}
FK \arrow[r, "\mathfrak{j}"] \arrow[d, "F(\gamma)"] & K'F \arrow[dr, bend left, "\gamma'_F"] & \\
FK^2 \arrow[r, "\mathfrak{j}_K"] & K'FK \arrow[r, "K'(\mathfrak{j})"] & K'^2F
\end{tikzcd}
\qquad
\begin{tikzcd}
FK \arrow[r, "\mathfrak{j}"] \arrow[dr, bend right, "F(\varepsilon)"] & K'F \arrow[d, "\varepsilon'_F"] \\
 & F
\end{tikzcd}$$

- a *2-cell* between $((F, \mathfrak{f}), \mathfrak{j})$ and $((G, \mathfrak{g}), \mathfrak{h})$ is a 2-arrow $\eta\colon (F, \mathfrak{f}) \to (G, \mathfrak{g})$ in **IdxPos**, i.e. $\eta\colon F \dot{\to} G$ such that $\mathfrak{f}_A(\alpha) \leq P'(\eta_A)\mathfrak{g}_A(\alpha)$, satisfying the coherence diagram below.

$$\begin{tikzcd}
FK \arrow[r, "\mathfrak{j}"] \arrow[d, "\eta_K"] & K'F \arrow[d, "K'(\eta)"] \\
GK \arrow[r, "\mathfrak{h}"] & K'G
\end{tikzcd}$$

Definition 1.2.2. A 2-category χ has *Eilenberg–Moore object for comonads* if the 2-functor Inc: $\chi \to \mathrm{Cmd}(\chi)$, which associates to every object the identity comonad, has a right 2-adjoint

$$(-)\text{-Coalg}\colon \mathrm{Cmd}(\chi) \to \chi.$$

We will dedicate the whole section to prove the following:

Proposition 1.2.3. The 2-category **IdxPos** has Eilenberg–Moore object.

Although this result can be also found in [DR21], for the sake of completeness we decided to display this proof, since we adapted it to show that **IdxPos** has also Kleisli objects.

Proof. In order to prove the statement, we shall explicitly construct the right 2-adjoint

$$(-)\text{-Coalg}\colon \mathrm{Cmd}(\mathbf{IdxPos}) \to \mathbf{IdxPos}.$$

We obviously begin with

0-cells: Fix a comonad $(P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}, (K, \mathfrak{k}), \gamma, \varepsilon)$, and consider the functor $P^K\colon {\mathbb{C}^K}^{\mathrm{op}} \to \mathbf{Pos}$. Let $\mathbb{C}^K$ be the category of coalgebras for the comonad $(\mathbb{C}, K, \gamma, \varepsilon)$ in **Cat**: its objects are pairs (A, c) where A is an object of $\mathbb{C}$ and $c\colon A \to KA$ is a $\mathbb{C}$-arrow such that

$$\begin{tikzcd}
A \arrow[r, "c"] \arrow[d, "c"] & KA \arrow[d, "K(c)"] \\
KA \arrow[r, "\gamma_A"] & K^2A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "c"] \arrow[dr, bend right, "\mathrm{id}_A"] & KA \arrow[d, "\varepsilon_A"] \\
& A
\end{tikzcd}$$

while an arrow between (A, c) and (B, d) is a $\mathbb{C}$-arrow $f\colon A \to B$ such that

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "c"] & B \arrow[d, "d"] \\
KA \arrow[r, "K(f)"] & KB
\end{tikzcd}$$

Let $P^K(A, c)$ be $\{\alpha \in P(A) \mid \alpha \le P(c)\mathfrak{k}_A(\alpha)\}$ and $P^K(f)$ be $P(f)$. This restriction is well defined: take $\beta \in P(B)$ such that $\beta \le P(d)\mathfrak{k}_B(\beta)$, so that $P(f)(\beta) \le P(f)P(d)\mathfrak{k}_B(\beta) = P(c)PK(f)\mathfrak{k}_B(\beta) = P(c)\mathfrak{k}_A P(f)(\beta)$, i.e. $P(f)(\beta) \in P^K(A, c)$. Moreover, P^K is a functor because P is.

1-cells: Consider $((F, \mathfrak{f}), \mathfrak{j})\colon (P, (K, \mathfrak{k}), \gamma, \varepsilon) \to (P'\colon {\mathbb{C}'}^{\mathrm{op}} \to \mathbf{Pos}, (K', \mathfrak{k}'), \gamma', \varepsilon')$ in $\mathrm{Cmd}(\mathbf{IdxPos})$. The corresponding 1-cell in **IdxPos** will be $(F', \mathfrak{f}')$ from P^K to ${P'}^{K'}$.

$$\begin{tikzcd}
{\mathbb{C}^K}^{\mathrm{op}} \arrow[rr, "{F'}^{\mathrm{op}}"] \arrow[dr, "P^K"', ""{name=L}] & & {\mathbb{C}'^{K'}}^{\mathrm{op}} \arrow[dl, "{P'}^{K'}", ""'{name=R}] \\
& \mathbf{Pos} &
\arrow[from=L, to=R, bend left, "\mathfrak{f}'", "\cdot"']
\end{tikzcd}$$

Define $F'(A, c) \colon= (FA, \mathfrak{j}_A F(c))$. Using naturality and coherence of $\mathfrak{j}$ and properties of c, we prove that this is indeed a K'-algebra:

$$\begin{aligned}
K'(\mathfrak{j}_A F(c))\mathfrak{j}_A F(c) &= K'(\mathfrak{j}_A)K'F(c)\mathfrak{j}_A F(c) = K'(\mathfrak{j}_A)\mathfrak{j}_{KA} FK(c)F(c) \\
&= K'(\mathfrak{j}_A)\mathfrak{j}_{KA} F(\gamma_A)F(c) = \gamma'_{FA}\mathfrak{j}_A F(c); \\
\varepsilon'_{FA}\mathfrak{j}_A F(c) &= F(\varepsilon_A)F(c) = F(\mathrm{id}_A) = \mathrm{id}_{FA}\,.
\end{aligned}$$

Then, take a morphism between K-coalgebras $f\colon (A, c) \to (B, d)$, $F'(f) \colon= F(f)$ is a morphism of K'-coalgebras:

$$\mathfrak{j}_B F(d)F(f) = \mathfrak{j}_B FK(f)F(c) = K'F(f)\mathfrak{j}_A F(c)$$

because of naturality of $\mathfrak{j}$ and properties of f. Consider that $\mathfrak{f}'\colon P^K \dot{\to} P'^{K'}F'^{\mathrm{op}}$, i.e. for any (A,c)

$$\mathfrak{f}'_{(A,c)}\colon \{\alpha \in P(A) \mid \alpha \leq P(c)\mathfrak{k}_A(\alpha)\} \to \{\alpha' \in P'F(A) \mid \alpha' \leq P'(\mathfrak{j}_A F(c))\mathfrak{k}'_{FA}(\alpha')\}.$$

So define $\mathfrak{f}'_{(A,c)}(\alpha)\colon= \mathfrak{f}_A(\alpha)$: it is well defined since

$$\mathfrak{f}_A(\alpha) \leq \mathfrak{f}_A P(c)\mathfrak{k}_A(\alpha) = P'F(c)\mathfrak{f}_{KA}\mathfrak{k}_A(\alpha) \leq P'F(c)P'(\mathfrak{j}_A)\mathfrak{k}'_{FA}\mathfrak{f}_A(\alpha)$$

from naturality of $\mathfrak{f}$ and inequality property of $\mathfrak{j}$. Naturality of $\mathfrak{f}'$ follows trivially from the naturality of $\mathfrak{f}$.

2-cells: Take a 2-cell $\eta\colon ((F,\mathfrak{f}),\mathfrak{j}) \to ((G,\mathfrak{g}),\mathfrak{h})$, and define $\eta'_{(A,c)}\colon (FA, \mathfrak{j}_A F(c)) \to (GA, \mathfrak{h}_A G(c))$, $\eta'_{(A,c)}\colon= \eta_A$. Clearly η' is a morphism of coalgebras:

$$\mathfrak{h}_A G(c)\eta_A = \mathfrak{h}_A \eta_{KA} F(c) = K'(\eta_A)\mathfrak{j}_A F(c)$$

because of naturality and coherence of η. Naturality of η' follows trivially from the naturality of η. Finally, η' is indeed a 2-arrow since the definition of η implies the following:

$$\mathfrak{f}'_{(A,c)}(\alpha) = \mathfrak{f}_A(\alpha) \leq P'(\eta_A)\mathfrak{g}_A(\alpha) = P'^{K'}(\eta'_{(A,c)})(\mathfrak{g}'_{(A,c)}(\alpha)).$$

Universal property: In order to prove that $(-)$-Coalg: $\mathrm{Cmd}(\mathbf{IdxPos}) \to \mathbf{IdxPos}$ is indeed a right adjoint, we have to find for each comonad $(P,(K,\mathfrak{k}),\gamma,\varepsilon)$ a universal arrow

$$(P^K,(\mathrm{id},\mathrm{id}),\mathrm{id},\mathrm{id}) \longrightarrow (P,(K,\mathfrak{k}),\gamma,\varepsilon)$$

i.e. a 1-arrow $((U^K,\mathfrak{u}),\nu)$ such that, for any indexed poset $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ and any arrow $((F,\mathfrak{f}),\mathfrak{j})\colon \mathrm{Inc}(R) \to (P,(K,\mathfrak{k}),\gamma,\varepsilon)$, there exists a unique morphism $\overline{((F,\mathfrak{f}),\mathfrak{j})}$ between the indexed posets R and P^K such that $((U^K,\mathfrak{u}),\nu) \circ (\overline{((F,\mathfrak{f}),\mathfrak{j})},\mathrm{id}) = ((F,\mathfrak{f}),\mathfrak{j})$.

$$\begin{tikzcd}
(P^K,(\mathrm{id},\mathrm{id}),\mathrm{id},\mathrm{id}) \arrow[rr, "((U^K,\mathfrak{u}),\nu)"] & & (P,(K,\mathfrak{k}),\gamma,\varepsilon) \\
& (R,(\mathrm{id},\mathrm{id}),\mathrm{id},\mathrm{id}) \arrow[ul, dashed] \arrow[ur, bend right, "((F,\mathfrak{f}),\mathfrak{j})"'] &
\end{tikzcd}$$

Define $(U^K,\mathfrak{u})\colon P^K \to P$, where $U^K\colon \mathbb{C}^K \to \mathbb{C}$ is the forgetful functor and the natural transformation $\mathfrak{u}\colon P^K \dot{\to} PU^{K^{\mathrm{op}}}$ is the inclusion on each component $\mathfrak{u}_{(A,c)}\colon P^K(A,c) \hookrightarrow PA$. Finally, define the 2-arrow $\nu\colon (U^K,\mathfrak{u}) \to (K,\mathfrak{k})(U^K,\mathfrak{u})$ to be $\nu_{(A,c)}\colon= \big(c\colon U^K(A,c) = A \to KU^K(A,c) = KA\big)$. This is indeed a 2-arrow, since $P^K(A,c) \ni \alpha \leq P(c)(\mathfrak{k}_{U^K(A,c)}(\alpha)) = P(c)(\mathfrak{k}_A(\alpha))$ by definition of $P^K(A,c)$.

Now consider $((F,\mathfrak{f}),\mathfrak{j})$; by definition of 1-cells in $\mathrm{Cmd}(\mathbf{IdxPos})$, we know that $(F,\mathfrak{f})$ is a 1-arrow from R to P, and $\mathfrak{j}\colon F \overset{\cdot}{\to} KF$ is such that $\mathfrak{f}_X(x) \leq P(\mathfrak{j}_X)\mathfrak{k}_{FX}\mathfrak{f}_X(x)$, and the coherence diagrams become:

$$\begin{tikzcd}
F \arrow[r, "\mathfrak{j}"] \arrow[d, "\mathfrak{j}"] & KF \arrow[d, "\gamma_F"] & & F \arrow[r, "\mathfrak{j}"] \arrow[dr, bend right, "\mathrm{id}"] & KF \arrow[d, "\varepsilon_F"] \\
KF \arrow[r, "K(\mathfrak{j})"] & K^2F & & & F
\end{tikzcd}$$

which exactly means that for any $\mathbb{D}$-object X, the pair $(FX, \mathfrak{j}_X)$ is a K-coalgebra. Moreover, for any $\mathbb{D}$-arrow $f\colon X \to Y$, naturality of $\mathfrak{j}$ is the diagram that proves that $F(f)$ is a morphism between coalgebras; to sum up, we can define a functor $(F(-), \mathfrak{j}_{(-)})\colon \mathbb{D} \to \mathbb{C}^K$ that sends a morphism $f\colon X \to Y$ to $F(f)\colon (FX, \mathfrak{j}_X) \to (FY, \mathfrak{j}_Y)$.

To conclude the definition of $\overline{((F, \mathfrak{f}), \mathfrak{j})}$, we have to find a natural transformation

$$\mathfrak{f}'\colon R \overset{\cdot}{\to} P^K(F(-), \mathfrak{j}_{(-)})^{\mathrm{op}}.$$

Set $\mathfrak{f}'_X\colon= \mathfrak{f}_X$: this is well defined since, taking $x \in R(X)$, and recalling that $P^K(FX, \mathfrak{j}_X) = \{\alpha \in P(FX) \mid \alpha \leq P(\mathfrak{j}_X)(\mathfrak{k}_{FX}(\alpha))\}$, we have that $\mathfrak{f}_X(x) \in P^K(FX, \mathfrak{j}_X)$ if and only if $\mathfrak{f}_X(x) \leq P(\mathfrak{j}_X)\mathfrak{k}_{FX}\mathfrak{f}_X(x)$, which follows from the definition of $\mathfrak{j}$. Naturality follows trivially from naturality of $\mathfrak{f}$.

We now prove that $((U^K, \mathfrak{u}), \nu) \circ (((F(-), \mathfrak{j}_{(-)}), \mathfrak{f}'), \mathrm{id}) = ((F, \mathfrak{f}), \mathfrak{j})$

$$\begin{tikzcd}
\mathbb{D}^{\mathrm{op}} \arrow[r, "(F{,}\mathfrak{j})^{\mathrm{op}}"] \arrow[dr, bend right, "R"', ""{name=R}] & \mathbb{C}^{K\,\mathrm{op}} \arrow[r, "U^{K\,\mathrm{op}}"] \arrow[d, "P^K", ""{name=PK}] & \mathbb{C}^{\mathrm{op}} \arrow[dl, bend left, "P", ""'{name=P}] \\
& \mathbf{Pos} &
\arrow[from=R, to=PK, bend left, "\mathfrak{f}'", "\cdot"']
\arrow[from=PK, to=P, bend left, "\mathfrak{u}", "\cdot"']
\end{tikzcd}$$

The composition of the functors is indeed F:

$$\Big(f\colon X \to Y\Big) \longmapsto \Big(F(f)\colon (FX, \mathfrak{j}_X) \to (FY, \mathfrak{j}_Y)\Big) \longmapsto \Big(F(f)\colon FX \to FY\Big),$$

while the composition of the natural transformations is $\mathfrak{f}$ because $\mathfrak{u}$ is the inclusion on every component.

The composition $(\nu \circ \mathrm{id})_X$ is $\nu_{(F,\mathfrak{j})X} = \nu_{(FX,\mathfrak{j}_X)} = \mathfrak{j}_X$.

Finally, suppose that also $(G, \mathfrak{g})$ is such that $((U^K, \mathfrak{u}), \nu) \circ ((G, \mathfrak{g}), \mathrm{id}) = ((F, \mathfrak{f}), \mathfrak{j})$. Then in particular $U^K G = F$, so that $GX = (FX, *)$; moreover $\mathfrak{j}_X = \nu_{GX} = \nu_{(FX,*)} = *$, so that the coalgebra structure on FX must be $\mathfrak{j}_X$, i.e. $G = (F, \mathfrak{j})$. To conclude, since $\mathfrak{g}$ post-composed with the inclusion must be equal to $\mathfrak{f}$, we deduce that $\mathfrak{g} = \mathfrak{f} = \mathfrak{f}'$, so that $((F, \mathfrak{j}), \mathfrak{f}')$ is indeed unique and $((U^K, \mathfrak{u}), \nu)$ is a universal arrow.

The isomorphism between the Hom-categories: The adjunction we proved above induces a bijection on objects of the Hom-categories below for any indexed poset R and any comonad $(P, (K, \mathfrak{k}), \gamma, \varepsilon) \in \mathrm{Cmd}(\mathbf{IdxPos})$. We need to extend it on 2-arrows and prove it is an isomorphism of categories.

$$\begin{tikzcd}
\mathbf{IdxPos} \arrow[r, shift left, "\mathrm{Inc}"] \arrow[r, phantom, "\perp"] & \mathrm{Cmd}(\mathbf{IdxPos}) \arrow[l, shift left, "(-)\text{-Coalg}"]
\end{tikzcd}$$

$$\mathrm{Cmd}(\mathbf{IdxPos})[\mathrm{Inc}(R), (P, (K, \mathfrak{k}), \gamma, \varepsilon)] \quad \cong \quad \mathbf{IdxPos}[R, P^K]$$

$$\begin{tikzcd}
\big((F, \mathfrak{f}), \mathfrak{j}\big) \arrow[d, Rightarrow, "\eta"] & & \big(\langle F(-), \mathfrak{j}_{(-)}\rangle, \mathfrak{f}\big) \arrow[d, Rightarrow, "\eta"] \\
\big((G, \mathfrak{g}), \mathfrak{h}\big) \arrow[u, phantom, ""] & & \big(\langle G(-), \mathfrak{h}_{(-)}\rangle, \mathfrak{g}\big)
\arrow[from=1-1, to=2-3, phantom]
\end{tikzcd}$$

$$\longmapsto$$

Take $\eta\colon \big((F, \mathfrak{f}), \mathfrak{j}\big) \to \big((G, \mathfrak{g}), \mathfrak{h}\big)$, i.e.

1. $\eta\colon F \overset{\cdot}{\to} G$ is a natural transformation;
2. $\mathfrak{f}_A(\alpha) \leq P(\eta_A)\mathfrak{g}_A(\alpha)$ for any object A in $\mathbb{D}$ and $\alpha \in RA$;
3. $\begin{tikzcd} F \arrow[r, "\mathfrak{j}"] \arrow[d, "\eta"'] & KF \arrow[d, "K\eta"] \\ G \arrow[r, "\mathfrak{h}"'] & KG \end{tikzcd}$ is commutative.

We prove that η in also a 2-arrow between the correspondent indexed posets. Each component is a well defined $\mathbb{C}^K$-arrow: $\eta_X\colon (FX, \mathfrak{j}_X) \to (GX, \mathfrak{h}_X)$ following from 3. It is a natural transformation between the functors $\langle F, \mathfrak{j}\rangle, \langle G, \mathfrak{h}\rangle\colon \mathbb{D} \to \mathbb{C}^K$ following from 1. Finally, $\mathfrak{f}_X(\alpha) \leq P^K(\eta_X)\mathfrak{g}_X(\alpha)$ follows from 2, since $P^K(\eta_X) = P(\eta_X)$. These three condition we proved are actually equivalent to 1,2 and 3, so the association is full. Faithfulness follows by definition, and it is essentially surjective because of the properties of adjunction. It is clear that the quasi-inverse is actually an inverse. □

1.2.2 Kleisli construction

We now define the 2-category $\mathrm{Cmd}^*(\mathbf{IdxPos})\colon = \mathrm{Cmd}(\mathbf{IdxPos}^{\mathrm{op}})^{\mathrm{op}}$.

- a *0-cell* is a comonad $(P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}, (K, \mathfrak{k}), \gamma, \varepsilon)$;
- a *1-cell* from the comonad $(P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}, (K, \mathfrak{k}), \gamma, \varepsilon)$ to $(P'\colon \mathbb{C}'^{\mathrm{op}} \to \mathbf{Pos}, (K', \mathfrak{k}'), \gamma', \varepsilon')$ is an oplax morphism of comonads, i.e. a pair $((F, \mathfrak{f}), \mathfrak{j})$ where the first entry $(F, \mathfrak{f})\colon P \to P'$ is a 1-arrow in **IdxPos** and the second one $\mathfrak{j}\colon (K', \mathfrak{k}')(F, \mathfrak{f}) \to (F, \mathfrak{f})(K, \mathfrak{k})$ is a 2-arrow, i.e. $\mathfrak{j}\colon K'F \overset{\cdot}{\to} FK$ such that $\mathfrak{k}'_{FA}\mathfrak{f}_A(\alpha) \leq P'(\mathfrak{j}_A)\mathfrak{f}_{KA}\mathfrak{k}_A(\alpha)$, satisfying the coherence diagrams below;

$$\begin{tikzcd}
K'F \arrow[r, "\mathfrak{j}"] \arrow[d, "\gamma'_F"] & FK \arrow[rd, bend left, "F(\gamma)"] & \\
K'^2F \arrow[r, "K'(\mathfrak{j})"] & K'FK \arrow[r, "\mathfrak{j}_K"] & FK^2
\end{tikzcd}
\qquad
\begin{tikzcd}
K'F \arrow[r, "\mathfrak{j}"] \arrow[rd, bend right, "\varepsilon'_F"] & FK \arrow[d, "F(\varepsilon)"] \\
& F
\end{tikzcd}$$

- a *2-cell* between $((F, \mathfrak{f}), \mathfrak{j})$ and $((G, \mathfrak{g}), \mathfrak{h})$ is a 2-arrow $\eta\colon (F, \mathfrak{f}) \to (G, \mathfrak{g})$ in **IdxPos**, i.e. $\eta\colon F \overset{\cdot}{\to} G$ such that $\mathfrak{f}_A(\alpha) \leq P'(\eta_A)\mathfrak{g}_A(\alpha)$, satisfying the coherence diagram below.

$$\begin{tikzcd}
K'F \arrow[r, "\mathfrak{j}"] \arrow[d, "K'(\eta)"] & FK \arrow[d, "\eta_K"] \\
K'G \arrow[r, "\mathfrak{h}"] & GK
\end{tikzcd}$$

Definition 1.2.4. A 2-category χ has *Kleisli object for comonads* if the 2-functor associating to every object the identity comonad $\mathrm{Inc}\colon \chi \to \mathrm{Cmd}^*(\chi)$, has a left 2-adjoint

$$(-)\text{-coKl}\colon \mathrm{Cmd}^*(\chi) \to \chi.$$

We will devote the whole section to prove the following:

Proposition 1.2.5. The 2-category **IdxPos** has Kleisli object.

Proof. In order to prove the statement, we shall explicitly construct the left 2-adjoint

$$(-)\text{-coKl}\colon \mathrm{Cmd}^*(\mathbf{IdxPos}) \to \mathbf{IdxPos}.$$

We obviously begin with
0-cells: Fix a comonad $(P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}, (K, \mathfrak{k}), \gamma, \varepsilon)$, and consider the functor $P_K\colon {\mathbb{C}_K}^{\mathrm{op}} \to \mathbf{Pos}$. Let $\mathbb{C}_K$ be the category of free coalgebras for the comonad $(\mathbb{C}, K, \gamma, \varepsilon)$ in **Cat**: it is the full subcategory of $\mathbb{C}^K$ whose objects are pairs (KA, γ_A) where A is an object of $\mathbb{C}$. Let $P_K(KA, \gamma_A)$ be $P^K(KA, \gamma_A) = \{\alpha \in P(KA) \mid \alpha \leq P(\gamma_A)\mathfrak{k}_{KA}(\alpha)\}$ and $P_K(f)$ be $P^K(f) = P(f)$. This restriction is well defined because P^K is.

Remark 1.2.6. The category $\mathbb{C}_K$ is isomorphic to the category $\overline{\mathbb{C}_K}$ whose objects are the same as $\mathbb{C}$, and a $\overline{\mathbb{C}_K}$-arrow $A \rightsquigarrow B$ is a $\mathbb{C}$-arrow $KA \to B$; composition between $g\colon A \rightsquigarrow B$ and $h\colon B \rightsquigarrow C$ is computed as

$$KA \xrightarrow{\gamma_A} K^2A \xrightarrow{K(g)} KB \xrightarrow{h} C;$$

the identity of A is given by ε_A.
The functor $\mathbb{C}_K \to \overline{\mathbb{C}_K}$ sends $f\colon (KA, \gamma_A) \to (KB, \gamma_B)$ to $\varepsilon_B f\colon A \rightsquigarrow B$: it trivially respects identity; concerning composition, we have to prove that given $f\colon (KA, \gamma_A) \to (KB, \gamma_B)$ and $f'\colon (KB, \gamma_B) \to (KC, \gamma_C)$ we have

$$\varepsilon_C f' K(\varepsilon_B f)\gamma_A = \varepsilon_C f' f,$$

but $K(f)\gamma_A = \gamma_B f$ by definition of f, and $K(\varepsilon_B)\gamma_B$ is the identity, so the equality holds. The inverse $\overline{\mathbb{C}_K} \to \mathbb{C}_K$ sends $g\colon A \rightsquigarrow B$ to $K(g)\gamma_A$. This is well defined since

$$K(K(g)\gamma_A)\gamma_A = K^2(g)\gamma_{KA}\gamma_A = \gamma_B K(g)\gamma_A.$$

Identity is trivially preserved; concerning composition, we need to show that

$$K(h)\gamma_B K(g)\gamma_A = K(hK(g)\gamma_A)\gamma_A,$$

however $K(h)K^2(g)K(\gamma_A)\gamma_A = K(h)K^2(g)\gamma_{KA}\gamma_A = K(h)\gamma_B K(g)\gamma_A$, as claimed. Now take $f\colon (KA,\gamma_A) \to (KB,\gamma_B)$, map it to $\varepsilon_B f$ and then into $K(\varepsilon_B f)\gamma_A = K(\varepsilon_B)\gamma_B f = f$; conversely take $g\colon A \rightsquigarrow B$, map it to $K(g)\gamma_A$ and then into $\varepsilon_B K(g)\gamma_A = g\varepsilon_{KA}\gamma_A = g$. So the two functors are one the inverse of the other.

We now resume the proof of Proposition 1.2.5.
1-cells: Consider $((F,\mathfrak{f}),\mathfrak{j})\colon (P,(K,\mathfrak{k}),\gamma,\varepsilon) \to (P'\colon \mathbb{C}'^{\mathrm{op}} \to \mathbf{Pos},(K',\mathfrak{k}'),\gamma',\varepsilon')$ in $\mathrm{Cmd}^*(\mathbf{IdxPos})$. The corresponding 1-cell in **IdxPos** will be $(F',\mathfrak{f}')$ from P_K to $P'_{K'}$.

$$\begin{tikzcd}
{\mathbb{C}_K}^{\mathrm{op}} \arrow[rr, "F'^{\mathrm{op}}"] \arrow[dr, "P_K"'] & {} \arrow[d, phantom, "\overset{\mathfrak{f}'}{\dot{\longrightarrow}}"] & {\mathbb{C}'_{K'}}^{\mathrm{op}} \arrow[dl, "{P'}_{K'}"] \\
& \mathbf{Pos} &
\end{tikzcd}$$

Define $F'(KA,\gamma_A) := (K'FA,\gamma'_{FA})$, which is by definition a free K'-coalgebra.
Then, take a morphism between free K-coalgebras $f\colon (KA,\gamma_A) \to (KB,\gamma_B)$, and let $F'(f)$ be $K'(F(\varepsilon_B f)\mathfrak{j}_A)\gamma'_{FA}$.

$$K'FA \xrightarrow{\mathfrak{j}_A} FKA \xrightarrow{F(f)} FKB \xrightarrow{F(\varepsilon_B)} FB$$

$$F'(f) := \left(K'FA \xrightarrow{\gamma'_{FA}} K'^2FA \xrightarrow{K'(F(\varepsilon_B f)\mathfrak{j}_A)} K'FB\right)$$

This is a morphism of K'-coalgebras if and only if the following diagram commutes:

$$\begin{tikzcd}
K'FA \arrow[r, "F'(f)"] \arrow[d, "\gamma'_{FA}"] & K'FB \arrow[d, "\gamma'_{FB}"] \\
K'^2FA \arrow[r, "K'F'(f)"] & K'^2FB
\end{tikzcd}$$

$$\gamma'_{FB}K'(F(\varepsilon_B f)\mathfrak{j}_A)\gamma'_{FA} = K'^2(F(\varepsilon_B f)\mathfrak{j}_A)\gamma'_{K'FA}\gamma'_{FA} = K'^2(F(\varepsilon_B f)\mathfrak{j}_A)K'(\gamma'_{FA})\gamma'_{FA},$$

which holds using naturality of γ' and its comultiplication property.
Consider that $\mathfrak{f}'\colon P_K \dot{\to} {P'}_{K'}F'^{\mathrm{op}}$, i.e. for any (KA,γ_A)

$$\mathfrak{f}'_{(KA,\gamma_A)}\colon P_K(KA,\gamma_A) \to P'_{K'}(K'FA,\gamma'_{FA}),$$

where $P_K(KA,\gamma_A) \subseteq PKA$ and $P'_{K'}(K'FA,\gamma'_{FA}) \subseteq P'K'FA$. So define $\mathfrak{f}'_{(KA,\gamma_A)}$ to be the restriction of the following composition:

$$PKA \xrightarrow{\mathfrak{f}_{KA}} P'FKA \xrightarrow{P'(\mathfrak{j}_A)} P'K'FA \xrightarrow{\mathfrak{k}'_{K'FA}} P'K'^2FA \xrightarrow{P'(\gamma'_{FA})} P'K'FA.$$

To prove that the restriction is well defined, take $\alpha \in P_K(KA, \gamma_A)$, i.e. $\alpha \in PKA$ such that $\alpha \leq P(\gamma_A)\mathfrak{k}_{KA}(\alpha)$, we want to check that

$$\mathfrak{f}'_{(KA,\gamma_A)}(\alpha) \leq P'(\gamma'_{FA})\mathfrak{k}'_{K'FA}(\mathfrak{f}'_{(KA,\gamma_A)}(\alpha)).$$

However we compute:

$$\begin{aligned}
P'(\gamma'_{FA})\mathfrak{k}'_{K'FA}P'(\mathfrak{j}_A)\mathfrak{f}_{KA}(\alpha) &\leq P'(\gamma'_{FA})P'(\gamma'_{K'FA})\mathfrak{k}'_{K'^2FA}\mathfrak{k}'_{K'FA}P'(\mathfrak{j}_A)\mathfrak{f}_{KA}(\alpha)\\
&= P'(\gamma'_{FA})P'K'(\gamma'_{FA})\mathfrak{k}'_{K'^2FA}\mathfrak{k}'_{K'FA}P'(\mathfrak{j}_A)\mathfrak{f}_{KA}(\alpha)\\
&= P'(\gamma'_{FA})\mathfrak{k}'_{K'FA}P'(\gamma'_{FA})P'K'(\mathfrak{j}_A)\mathfrak{k}'_{FKA}\mathfrak{f}_{KA}(\alpha)\\
&= P'(\gamma'_{FA})\mathfrak{k}'_{K'FA}P'(\gamma'_{FA})\mathfrak{k}'_{K'FA}P'(\mathfrak{j}_A)\mathfrak{f}_{KA}(\alpha)\\
&= P'(\gamma'_{FA})\mathfrak{k}'_{K'FA}(\mathfrak{f}'_{(KA,\gamma_A)}(\alpha)),
\end{aligned}$$

since $\gamma' \colon (K', \mathfrak{k}') \to (K', \mathfrak{k}')^2$ is a 2-arrow, by comultiplication property of γ' and by naturality of $\mathfrak{k}'$.

Naturality of $\mathfrak{f}'$ means we have to prove that for any $f \colon (KA, \gamma_A) \to (KB, \gamma_B)$, the following diagram commutes:

$$\begin{tikzcd}
P_K(KB,\gamma_B) \arrow[r, "\mathfrak{f}'_{(KB,\gamma_B)}"] \arrow[d, "P(f)"] & P'_{K'}(K'FB,\gamma'_{FB}) \arrow[d, "P'(K'(F(\varepsilon_B f)\mathfrak{j}_A)\gamma'_{FA})"] \\
P_K(KA,\gamma_A) \arrow[r, "\mathfrak{f}'_{(KA,\gamma_A)}"] & P'_{K'}(K'FA,\gamma'_{FA})
\end{tikzcd}$$

Observe that we can decompose the diagram above as follows:

$$\begin{tikzcd}
PKB \arrow[r, "\mathfrak{f}_{KB}"] \arrow[d, "P(f)"] & P'FKB \arrow[r, "\mathfrak{k}'_{FKB}"] \arrow[d, "P'F(f)"] & P'K'FKB \arrow[r, "P'K'(\mathfrak{j}_B)"] \arrow[d, "P'K'F(f)"] & P'K'^2FB \arrow[r, "P'(\gamma'_{FB})"] & P'K'FB \arrow[d, "P'(K'(F(\varepsilon_B f)\mathfrak{j}_A)\gamma'_{FA})"] \\
PKA \arrow[r, "\mathfrak{f}_{KA}"] & P'FKA \arrow[r, "\mathfrak{k}'_{FKA}"] & P'K'FKA \arrow[r, "P'K'(\mathfrak{j}_A)"] & P'K'^2FA \arrow[r, "P'(\gamma'_{FA})"] & P'K'FA
\end{tikzcd}$$

The first two square commute because they are naturality squares of $\mathfrak{f}$ and $\mathfrak{k}$ respectively. To prove commutativity of the third square, it is enough to prove that the following square commutes:

$$\begin{tikzcd}
K'FA \arrow[r, "\gamma'_{FA}"] \arrow[d, "K'(F(\varepsilon_B f)\mathfrak{j}_A)\gamma'_{FA}"] & K'^2FA \arrow[r, "K'(\mathfrak{j}_A)"] & K'FKA \arrow[d, "K'F(f)"] \\
K'FB \arrow[r, "\gamma'_{FB}"] & K'^2FB \arrow[r, "K'(\mathfrak{j}_B)"] & K'FKB
\end{tikzcd}$$

$$\begin{aligned}
&K'(\mathfrak{j}_B)\gamma'_{FB}K'(F(\varepsilon_B f)\mathfrak{j}_A)\gamma'_{FA} = K'(\mathfrak{j}_B)K'^2(F(\varepsilon_B f)\mathfrak{j}_A)\gamma'_{K'FA}\gamma'_{FA}\\
&\quad = K'(\mathfrak{j}_B K'F(\varepsilon_B f))K'^2(\mathfrak{j}_A)\gamma'_{K'FA}\gamma'_{FA} = K'(FK(\varepsilon_B f)\mathfrak{j}_{KA})K'^2(\mathfrak{j}_A)K'(\gamma'_{FA})\gamma'_{FA}\\
&\quad\quad = K'FK(\varepsilon_B f)K'(\mathfrak{j}_{KA}K'(\mathfrak{j}_A)\gamma'_{FA})\gamma'_{FA} = K'FK(\varepsilon_B f)K'(F(\gamma_A)\mathfrak{j}_A)\gamma'_{FA}\\
&\quad\quad\quad = K'F(K(\varepsilon_B f)\gamma_A)K'(\mathfrak{j}_A)\gamma'_{FA}.
\end{aligned}$$

from naturality of γ', naturality of $\mathfrak{j}$, γ' comultiplication property, $\mathfrak{j}$ coherence.
To conclude, observe that $K(\varepsilon_B f)\gamma_A = K(\varepsilon_B)\gamma_B f = f$ because of the definition of morphism between coalgebras and a property of ε, so that the diagram above commutes, and $\mathfrak{f}'$ is indeed a natural transformation.
2-cells: Take a 2-cell $\eta\colon((F,\mathfrak{f}),\mathfrak{j}) \to ((G,\mathfrak{g}),\mathfrak{h})$, and look for $\eta'\colon F' \dot{\to} G'$ such that

$$\mathfrak{f}'_{(KA,\gamma_A)}(\alpha) \leq P'_{K'}(\eta'_{(KA,\gamma_A)})(\mathfrak{g}'_{(KA,\gamma_A)}(\alpha)).$$

Define $\eta'_{(KA,\gamma_A)}\colon (K'FA,\gamma'_{FA}) \to (K'GA,\gamma'_{GA})$, $\eta'_{(KA,\gamma_A)} := K'(\eta_A)$.
Naturality diagram of γ' applied to η_A proves that η' is a morphism of coalgebras.
To prove naturality of η' we have to check that for any $f\colon(KA,\gamma_A) \to (KB,\gamma_B)$ the following diagram commutes:

$$\begin{tikzcd}
(K'FA,\gamma'_{FA}) \arrow[r, "K'(\eta_A)"] \arrow[d, "K'(F(\varepsilon_B f)\mathfrak{j}_A)\gamma'_{FA}"] & (K'GA,\gamma'_{GA}) \arrow[d, "K'(G(\varepsilon_B f)\mathfrak{h}_A)\gamma'_{GA}"] \\
(K'FB,\gamma'_{FB}) \arrow[r, "K'(\eta_B)"] & (K'GB,\gamma'_{GB})
\end{tikzcd}$$

$$\begin{aligned}
K'(G(\varepsilon_B f)\mathfrak{h}_A)\gamma'_{GA}K'(\eta_A) &= K'G(\varepsilon_B f)K'(\mathfrak{h}_A)K'^2(\eta_A)\gamma'_{FA} \\
&= K'(G(\varepsilon_B f)\eta_{KA}\mathfrak{j}_A)\gamma'_{FA} = K'(\eta_B F(\varepsilon_B f))K'(\mathfrak{j}_A)\gamma'_{FA},
\end{aligned}$$

from naturality of γ', coherence diagram of η and its naturality.
Finally, η' is indeed a 2-arrow:

$$\begin{aligned}
\mathfrak{f}'_{(KA,\gamma_A)}(\alpha) &= P'(\gamma'_{FA})\mathfrak{k}'_{K'FA}P'(\mathfrak{j}_A)\mathfrak{f}_{KA}(\alpha) \\
&\leq P'(\gamma'_{FA})\mathfrak{k}'_{K'FA}P'(\mathfrak{j}_A)P'(\eta_{KA})(\mathfrak{g}_{KA}(\alpha)) = P'(\gamma'_{FA})\mathfrak{k}'_{K'FA}P'K'(\eta_A)P'(\mathfrak{h}_A)\mathfrak{g}_{KA}(\alpha) \\
&= P'(\gamma'_{FA})P'K'^2(\eta_A)\mathfrak{k}'_{K'GA}P'(\mathfrak{h}_A)\mathfrak{g}_{KA}(\alpha) = P'(K'^2(\eta_A)\gamma'_{FA})\mathfrak{k}'_{K'GA}P'(\mathfrak{h}_A)\mathfrak{g}_{KA}(\alpha) \\
&= P'(\gamma'_{GA}K'(\eta_A))\mathfrak{k}'_{K'GA}P'(\mathfrak{h}_A)\mathfrak{g}_{KA}(\alpha) = P'(K'(\eta_A))(P'(\gamma'_{GA})\mathfrak{k}'_{K'GA}P'(\mathfrak{h}_A)\mathfrak{g}_{KA}(\alpha)) \\
&= P'_{K'}(\eta'_{(KA,\gamma_A)})(\mathfrak{g}'_{(KA,\gamma_A)}(\alpha))
\end{aligned}$$

since η is a 2-arrow, its coherence, naturality of $\mathfrak{k}$ and naturality of γ'.
Universal property: In order to prove that $(-)$-coKl: $\mathrm{Cmd}^*(\mathbf{IdxPos}) \to \mathbf{IdxPos}$ is indeed a left adjoint, we have to find for each comonad $(P,(K,\mathfrak{k}),\gamma,\varepsilon)$ a universal arrow

$$(P,(K,\mathfrak{k}),\gamma,\varepsilon) \longrightarrow (P_K,(\mathrm{id},\mathrm{id}),\mathrm{id},\mathrm{id})$$

i.e. a 1-arrow $((F_K,\mathfrak{k}'),\mathfrak{j}')$ such that, for any indexed poset $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ and any arrow $((F,\mathfrak{f}),\mathfrak{j})\colon (P,(K,\mathfrak{k}),\gamma,\varepsilon) \to \mathrm{Inc}(R)$, there exists a unique morphism $\overline{((F,\mathfrak{k}),\mathfrak{j})}$ between the indexed posets P_K and R such that $(\overline{((F,\mathfrak{k}),\mathfrak{j})},\mathrm{id}) \circ ((F_K,\mathfrak{k}'),\mathfrak{j}') = ((F,\mathfrak{f}),\mathfrak{j})$.

$$\begin{tikzcd}
(P,(K,\mathfrak{k}),\gamma,\varepsilon) \arrow[rr, "((F_K,\mathfrak{k}'),\mathfrak{j}')"] \arrow[dr, bend right, "((F,\mathfrak{f}),\mathfrak{j})"'] & & (P_K,(\mathrm{id},\mathrm{id}),\mathrm{id},\mathrm{id}) \arrow[dl, dashed, bend left] \\
& (R,(\mathrm{id},\mathrm{id}),\mathrm{id},\mathrm{id}) &
\end{tikzcd}$$

Define $(F_K,\mathfrak{k}')\colon P \to P_K$, where $F_K\colon \mathbb{C} \to \mathbb{C}_K$ is the cofree functor

$$\Big(f\colon A \to B\Big) \longmapsto \Big(K(f)\colon (KA,\gamma_A) \to (KB,\gamma_B)\Big)$$

and the natural transformation $\mathfrak{k}'\colon P \stackrel{\cdot}{\to} P_K {F_K}^{\mathrm{op}}$ is computed as $\mathfrak{k}$: this is well defined since, recalling that γ is a 2-arrow, we know that $\mathfrak{k}_A(\alpha) \leq P(\gamma_A)\mathfrak{k}_{KA}\mathfrak{k}_A(\alpha)$, i.e. $\mathfrak{k}_A(\alpha) \in P_K F_K A$. Naturality of $\mathfrak{k}'$ follows from naturality of $\mathfrak{k}$.

Finally, define the 2-arrow $\mathfrak{j}'\colon (F_K,\mathfrak{k}) \to (F_K,\mathfrak{k})(K,\mathfrak{k})$ to be $\mathfrak{j}'_A := \gamma_A$. This natural and a 2-arrow because γ is.

Now consider $((F,\mathfrak{f}),\mathfrak{j})$; by definition of 1-cells in $\mathrm{Cmd}^*(\mathbf{IdxPos})$, we know that $(F,\mathfrak{f})$ is a 1-arrow from P to R, and $\mathfrak{j}\colon F \stackrel{\cdot}{\to} FK$ is such that $\mathfrak{f}_A(\alpha) \leq R(\mathfrak{j}_A)\mathfrak{f}_{KA}\mathfrak{k}_A(\alpha)$, and the coherence diagrams become:

$$\begin{tikzcd}
F \arrow[r, "\mathfrak{j}"] \arrow[d, "\mathfrak{j}"] & FK \arrow[d, "F(\gamma)"] \\
FK \arrow[r, "\mathfrak{j}_K"] & FK^2
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, "\mathfrak{j}"] \arrow[dr, bend right, "\mathrm{id}"] & FK \arrow[d, "F(\varepsilon)"] \\
& F
\end{tikzcd}$$

Define the functor $F'\colon \mathbb{C}_K \to \mathbb{D}$ to be the one that maps $f\colon (KA,\gamma_A) \to (KB,\gamma_B)$ to the composition $F(\varepsilon_B f)\mathfrak{j}_A\colon FA \to FB$. We check that this is indeed a functor, proving that $F'(gf) = F'(g)F'(f)$ for any pair of composable arrows $f\colon KA \to KB$, $g\colon KB \to KC$ between free coalgebras.

$$\begin{aligned}
F'(g)F'(f) = F(\varepsilon_C g)\mathfrak{j}_B F(\varepsilon_B f)\mathfrak{j}_A &= F(\varepsilon_C g)FK(\varepsilon_B f)\mathfrak{j}_{KA}\mathfrak{j}_A \\
= F(\varepsilon_C g K(\varepsilon_B))FK(f)F(\gamma_A)\mathfrak{j}_A &= F(\varepsilon_C g K(\varepsilon_B)\gamma_B f)\mathfrak{j}_A \\
&= F(\varepsilon_C g f)\mathfrak{j}_A = F'(gf)
\end{aligned}$$

from naturality of $\mathfrak{j}$, its coherence, definition of morphism between coalgebras and property of the counit.

To conclude the definition of $\overline{((F,\mathfrak{f}),\mathfrak{j})}$, we have to find a natural transformation $\mathfrak{f}'\colon P_K \stackrel{\cdot}{\to} RF'^{\mathrm{op}}$. Define $\mathfrak{f}'_{(KA,\gamma_A)}\colon P_K(KA,\gamma_A) \to RFA$ to be the restriction of $R(\mathfrak{j}_A)\mathfrak{f}_{KA}$. To show $\mathfrak{f}'$ is natural we need for any $f\colon KA \to KB$ between free coalgebras that the following diagram commutes:

$$\begin{tikzcd}
P_K(KB,\gamma_B) \arrow[r, "\mathfrak{f}'_{(KB,\gamma_B)}"] \arrow[d, "P(f)"] & RF'(KB,\gamma_B) \arrow[d, "RF'(f)"] \\
P_K(KA,\gamma_A) \arrow[r, "\mathfrak{f}'_{(KA,\gamma_A)}"] & RF'(KA,\gamma_A)
\end{tikzcd}$$

To see this, observe that it is enough to prove the commutativity of the second square of

$$\begin{tikzcd}
PKB \arrow[r, "\mathfrak{f}_{KB}"] \arrow[d, "P(f)"] & RFKB \arrow[r, "R(\mathfrak{j}_B)"] \arrow[d, "RF(f)"] & RFB \arrow[d, "R(F(\varepsilon_B f)\mathfrak{j}_A)"] \\
PKA \arrow[r, "\mathfrak{f}_{KA}"] & RFKA \arrow[r, "R(\mathfrak{j}_A)"] & RFA
\end{tikzcd}$$

since commutativity of the first square follows from naturality of $\mathfrak{f}$, but again it is enough to prove:

$$\begin{tikzcd}
FA \arrow[r, "\mathfrak{j}_A"] \arrow[d, "F(\varepsilon_B f)\mathfrak{j}_A"] & FKA \arrow[d, "F(f)"] \\
FB \arrow[r, "\mathfrak{j}_B"] & FKB
\end{tikzcd}$$

$$\begin{aligned}
\mathfrak{j}_B F(\varepsilon_B f)\mathfrak{j}_A &= FK(\varepsilon_B f)\mathfrak{j}_{KA}\mathfrak{j}_A = FK(\varepsilon_B f)F(\gamma_A)\mathfrak{j}_A \\
&= F(K(\varepsilon_B)\gamma_B f)\mathfrak{j}_A = F(f)\mathfrak{j}_A
\end{aligned}$$

from naturality of $\mathfrak{j}$, its coherence, definition of morphism between coalgebras and counit property; so $\mathfrak{f}'$ is indeed a natural transformation.
We now prove that $((F', \mathfrak{f}'), \mathrm{id}) \circ ((F_K, \mathfrak{k}), \gamma) = ((F, \mathfrak{f}), \mathfrak{j})$.

$$\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \arrow[r, "{F_K}^{\mathrm{op}}"] \arrow[dr, bend right=50, "P"'] & {\mathbb{C}_K}^{\mathrm{op}} \arrow[r, "{F'}^{\mathrm{op}}"] \arrow[d, "P_K"] \arrow[d, phantom, shift right=30, "\overset{\mathfrak{k}}{\xrightarrow{\cdot}}"] \arrow[d, phantom, shift left=30, "\overset{\mathfrak{f}'}{\xrightarrow{\cdot}}"] & \mathbb{D}^{\mathrm{op}} \arrow[dl, bend left=50, "R"] \\
 & \mathbf{Pos} &
\end{tikzcd}$$

The composition of the functors is indeed F:

$$\begin{aligned}
\Big(f\colon A \to B\Big) &\longmapsto \Big(K(f)\colon F_K A \to F_K B\Big) \\
&= \Big(K(f)\colon (KA, \gamma_A) \to (KB, \gamma_B)\Big) \longmapsto \Big(F(\varepsilon_B K(f))\mathfrak{j}_A\colon FA \to FB\Big),
\end{aligned}$$

but $F(\varepsilon_B K(f))\mathfrak{j}_A = F(f\varepsilon_A)\mathfrak{j}_A = F(f)$ from naturality of ε and coherence of $\mathfrak{j}$.
Concerning the composition of the natural transformations, we need to check the equality $R(\mathfrak{j}_A)\mathfrak{f}_{KA}\mathfrak{k}_A(\alpha) = \mathfrak{f}_A(\alpha)$. The direction $(\geq)$ follows from the definition of $\mathfrak{j}$. To prove the converse, recall that coherence of $\mathfrak{j}$ implies that $F(\varepsilon_A)\mathfrak{j}_A$ is the identity, so

$$\mathfrak{f}_A = R(\mathfrak{j}_A)RF(\varepsilon_A)\mathfrak{f}_A = R(\mathfrak{j}_A)\mathfrak{f}_{KA}P(\varepsilon_A)$$

from naturality of $\mathfrak{f}$. Moreover, since ε is a 2-arrow, we know that $\mathfrak{k}_A(\alpha) \leq P(\varepsilon_A)(\alpha)$, so that $R(\mathfrak{j}_A)\mathfrak{f}_{KA}\mathfrak{k}_A(\alpha) \leq R(\mathfrak{j}_A)\mathfrak{f}_{KA}P(\varepsilon_A)(\alpha) = \mathfrak{f}_A(\alpha)$, i.e. $(\leq)$ holds.
The composition $(\mathrm{id} \circ \gamma)_A$ is $F'(\gamma_A) = F(\varepsilon_{KA}\gamma_A)\mathfrak{j}_A = \mathfrak{j}_A$.
Finally, suppose that also $(G, \mathfrak{g})$ is such that $((G, \mathfrak{g}), \mathrm{id}) \circ ((F_K, \mathfrak{k}), \gamma) = ((F, \mathfrak{f}), \mathfrak{j})$. Then in particular $GF_K = F$, so that $G = F'$ on objects; moreover, $\mathrm{id} \circ \gamma = \mathfrak{j}$ means that $G(\gamma_A) = \mathfrak{j}_A$. Observe also that, given a $\mathbb{C}$-arrow g, $GK(g) = GF_K(g) = F(g)$. We claim that for any morphism

$f\colon KA \to KB$ between free coalgebras, $G(f) = F'(f)$, i.e. $G(f) = F(\varepsilon_B f)\mathfrak{j}_A$. First of all, coherence of $\mathfrak{j}$ proves that

$$G(f) = F(\varepsilon_B)\mathfrak{j}_B G(f) F(\varepsilon_A)\mathfrak{j}_A.$$

However,

$$\begin{aligned}\mathfrak{j}_B G(f) F(\varepsilon_A) &= G(\gamma_B) G(f) GK(\varepsilon_A) = G(\gamma_B) G(fK(\varepsilon_A)) \\ &= G(K(f)\gamma_A K(\varepsilon_A)) = F(f)\mathfrak{j}_A F(\varepsilon_A)\end{aligned}$$

using the properties of G described above, definition of morphism between coalgebras; so we obtain $G(f) = F(\varepsilon_B)F(f)\mathfrak{j}_A F(\varepsilon_A)\mathfrak{j}_A = F(\varepsilon_B f)\mathfrak{j}_A = F'(f)$, i.e. the functor G is indeed the functor F'.

To conclude, we have to prove that $\mathfrak{g} = \mathfrak{f}'$, i.e. $\mathfrak{g}_{(KA,\gamma_A)} = R(\mathfrak{j}_A)\mathfrak{f}_{KA}$. We know that $\mathfrak{g}\mathfrak{k} = \mathfrak{f}$, i.e. $\mathfrak{g}_{(KA,\gamma_A)}\mathfrak{k}_A = \mathfrak{f}_A$, where $\mathfrak{g}_{(KA,\gamma_A)}\colon P_K(KA,\gamma_A) \to RFA$.

Note that

$$\begin{aligned}\mathfrak{f}'_{(KA,\gamma_A)} &= R(\mathfrak{j}_A)\mathfrak{f}_{KA} = R(\mathfrak{j}_A)\mathfrak{g}_{(K^2A,\gamma_{KA})}\mathfrak{k}_{KA} \\ &= RG(\gamma_A)\mathfrak{g}_{(K^2A,\gamma_{KA})}\mathfrak{k}_{KA} = \mathfrak{g}_{(KA,\gamma_A)} P(\gamma_A)\mathfrak{k}_{KA},\end{aligned}$$

because of the property of the composition of $\mathfrak{k}$ and $\mathfrak{g}$ described above and naturality of $\mathfrak{g}$. We only need to prove that the composition $P(\gamma_A)\mathfrak{k}_{KA}$ acts like the identity on $P_K(KA,\gamma_A)$. So take $\alpha \in PKA$ such that $\alpha \leq P(\gamma_A)\mathfrak{k}_{KA}(\alpha)$; we claim that $\alpha = P(\gamma_A)\mathfrak{k}_{KA}(\alpha)$. Clearly $(\leq)$ holds by definition, so we prove the converse. Recall that ε is a 2-arrow, so $\mathfrak{k}_{KA}(\alpha) \leq P(\varepsilon_{KA})(\alpha)$, and apply $P(\gamma_A)$:

$$P(\gamma_A)\mathfrak{k}_{KA}(\alpha) \leq P(\gamma_A)P(\varepsilon_{KA})(\alpha) = \alpha$$

so we proved that $\mathfrak{g} = \mathfrak{f}'$ and $((F_K,\mathfrak{k}),\gamma)$ is a universal arrow.

The isomorphism between the Hom-categories: The adjunction we proved above induces a bijection on objects of the Hom-categories below for any indexed poset R and any comonad $(P,(K,\mathfrak{k}),\gamma,\varepsilon) \in \mathrm{Cmd}^*(\mathbf{IdxPos})$. We need to extend it on 2-arrows and prove it is an isomorphism of categories.

$$\begin{tikzcd}\mathbf{IdxPos} \arrow[r, shift left=1.5, "\mathrm{Inc}"] \arrow[r, phantom, "\top"] & \mathrm{Cmd}^*(\mathbf{IdxPos}) \arrow[l, shift left=1.5, "(-)\text{-coKl}"]\end{tikzcd}$$

$$\mathrm{Cmd}^*(\mathbf{IdxPos})[(P,(K,\mathfrak{k}),\gamma,\varepsilon),\mathrm{Inc}(R)] \quad\cong\quad \mathbf{IdxPos}[P_K,R]$$

$$\begin{tikzcd}\big((F,\mathfrak{f}),\mathfrak{j}\big) \arrow[d, Rightarrow, "\eta"] & & (F',R(\mathfrak{j})\mathfrak{f}_K) \arrow[d, Rightarrow, "\eta"] \\ \big((G,\mathfrak{g}),\mathfrak{h}\big) \arrow[r, phantom, ""] & & (G',R(\mathfrak{h})\mathfrak{g}_K)\end{tikzcd} \qquad \longmapsto$$

where $F'\big(f\colon (KA,\gamma_A) \to (KB,\gamma_B)\big) = \big(F(\varepsilon_B f)\mathfrak{j}_A\colon FA \to FB\big)$, and similarly for G'.
Take $\eta\colon \big((F,\mathfrak{f}),\mathfrak{j}\big) \to \big((G,\mathfrak{g}),\mathfrak{h}\big)$, i.e.

1. $\eta\colon F \overset{\cdot}{\to} G$ is a natural transformation;
2. $\mathfrak{f}_A(\alpha) \leq P(\eta_A)\mathfrak{g}_A(\alpha)$ for any object A in $\mathbb{C}$ and $\alpha \in PA$;
3. $$\begin{tikzcd} F \arrow[r, "\mathfrak{j}"] \arrow[d, "\eta"'] & KF \arrow[d, "\eta_K"] \\ G \arrow[r, "\mathfrak{h}"'] & KG \end{tikzcd}$$ is commutative.

We prove that η in also a 2-arrow between the correspondent indexed posets, defining $\eta_{(KA,\gamma_A)} = \eta_A\colon FA \to GA$. It is a natural transformation between the functors F' and G' if for any $f\colon (KA,\gamma_A) \to (KB,\gamma_B)$, we have $\eta_B F(\varepsilon_B f)\mathfrak{j}_A = G(\varepsilon_B f)\mathfrak{h}_A\eta_A$. However:

$$\eta_B F(\varepsilon_B f)\mathfrak{j}_A = G(\varepsilon_B f)\eta_{KA}\mathfrak{j}_A = G(\varepsilon_B f)\mathfrak{h}_A\eta_A$$

using 1. and 3. Then we need $R(\mathfrak{j}_A)\mathfrak{f}_{KA}(\alpha) \leq R(\eta_{(KA,\gamma_A)})R(\mathfrak{h}_A)\mathfrak{g}_{KA}(\alpha)$, but we know that $R(\eta_A)R(\mathfrak{h}_A)\mathfrak{g}_{KA}(\alpha) = R(\mathfrak{j}_A)R(\eta_{KA})\mathfrak{g}_{KA}(\alpha)$, so the inequality follows from 2.
To show this functor is full, take a natural transformation $\eta\colon F' \to G'$ such that $R(\mathfrak{j}_A)\mathfrak{f}_{KA}(\alpha) \leq R(\eta_{(KA,\gamma_A)})R(\mathfrak{h}_A)\mathfrak{g}_{KA}(\alpha)$ for any $\alpha \in P(KA)$ satisfying $\alpha \leq P(\gamma_A)\mathfrak{k}_{KA}(\alpha)$, and we prove that 1., 2. and 3. hold. Take any $f\colon A \to B$, so that $K(f)\colon (KA,\gamma_A) \to (KB,\gamma_B)$ is a $\mathbb{C}_K$ arrow; apply naturality to $K(f)$ so $\eta_{(KB,\gamma_B)}F'(Kf) = G'(Kf)\eta_{(KA,\gamma_A)}$. However, $F'(Kf) = F(\varepsilon_B)FK(f)\mathfrak{j}_A = F(\varepsilon_B)\mathfrak{j}_B F(f) = F(f)$, using naturality and coherence of $\mathfrak{j}$; similarly $G'(Kf) = G(f)$, so η is a natural transformation from F to G. To show 2., take any $\beta \in PB$, we want $\mathfrak{f}_B(\beta) \leq R(\eta_B)\mathfrak{g}_B(\beta)$, but by some computation we did above, we know that $\mathfrak{f}_B = R(\mathfrak{j}_B)\mathfrak{f}_{KB}\mathfrak{k}_B$—and similarly $\mathfrak{g}_B = R(\mathfrak{h}_B)\mathfrak{g}_{KB}\mathfrak{k}_B$—, so $\mathfrak{f}_B(\beta) = R(\mathfrak{j}_B)\mathfrak{f}_{KB}\mathfrak{k}_B(\beta) \leq R(\eta_B)R(\mathfrak{h}_B)\mathfrak{g}_{KB}\mathfrak{k}_B(\beta) = R(\eta_B)\mathfrak{g}_B(\beta)$, using the fact that $\mathfrak{k}_B(\beta) \in P_K(B)$ and definition of η. Finally, observe that by definition of γ, we have that $\gamma_A\colon (KA,\gamma_A) \to (K^2A,\gamma_{K^2A})$ is a $\mathbb{C}_K$-arrow, so apply naturality of η with respect to γ_A to obtain $G'(\gamma_A)\eta_{(KA,\gamma_A)} = \eta_{(K^2A,\gamma_{K^2A})}F'(\gamma_A)$, i.e. $G(\varepsilon_{KA}\gamma_A)\mathfrak{h}_A\eta_A = \eta_{KA}F(\varepsilon_{KA}\gamma_A)\mathfrak{j}_A$, hence $\mathfrak{h}_A\eta_A = \eta_{KA}\mathfrak{j}_A$, so that also 3. holds. Again, faithfulness follows by definition, and it is essentially surjective because of the properties of adjunction. It is clear that the quasi-inverse is actually an inverse. ☐

1.3 Existence of directed colimits in Dct

This section is devoted to the construction of direct colimits in the category **Dct**. We will demonstrate that this construction preserves many properties, which will be crucial for our later work in Sections 3.1 and 3.3. Specifically, we will use these results to verify that two constructions we introduce respect all the needed structure of the starting doctrine.
While some of the results in this section are well known, such as how directed colimits are

computed in categories like **Cat** or **Pos**, we present them here in detail in order to compute how additional structure is preserved.

Proposition 1.3.1. The category **Dct** has colimits over directed preorders.

Proof. We begin by considering a directed preorder I, so that for each $i, j \in I$ there exists a $k \in I$ such that $k \geq i, j$. Then suppose to have a diagram over this preorder, i.e. a functor $D\colon I \to \mathbf{Dct}$. In particular, for all $i \in I$ we have $P_i := D(i)\colon \mathbb{C}_i^{\mathrm{op}} \to \mathbf{Pos}$, and for all $i \leq k$ a morphism $(F_{ik}, \mathfrak{f}_{ik})\colon P_i \to P_k$ where $F_{ik}\colon \mathbb{C}_i \to \mathbb{C}_k$ is a functor preserving finite products and $\mathfrak{f}_{ik}\colon P_i \stackrel{\cdot}{\to} P_k F_{ik}^{\mathrm{op}}$ is a natural transformation. Moreover, we ask for $(F_{ii}, \mathfrak{f}_{ii})$ to be the identity on P_i, and for $(F_{jk}, \mathfrak{f}_{jk}) \circ (F_{ij}, \mathfrak{f}_{ij}) = (F_{ik}, \mathfrak{f}_{ik})$ whenever $i \leq j \leq k$.
Our goal is to define a suitable doctrine $P_\bullet\colon \mathbb{C}_\bullet^{\mathrm{op}} \to \mathbf{Pos}$, and then show that it is the colimit over I.
The base category $\mathbb{C}_\bullet$: The base category $\mathbb{C}_\bullet$ is the colimit over I in **Cat** of the diagram given by $\mathbb{C}_i$'s and F_{ij}'s: objects are classes of objects from any $\mathbb{C}_i$, identified as follows.

$$\mathrm{ob}\mathbb{C}_\bullet = \bigsqcup_{i\in I} \mathbb{C}_i \big/ \!\sim,$$

where two objects $A_{(i)}, B_{(j)}$ in $\mathbb{C}_i$ and $\mathbb{C}_j$ respectively are such that $A_{(i)} \sim B_{(j)}$ if and only if there exists $k \geq i, j$ such that $F_{ik}A_{(i)} = F_{jk}B_{(j)}$ in $\mathbb{C}_k$. Then for any pair of objects $[A_{(i)}], [B_{(j)}]$ we have as morphisms:

$$\mathrm{Hom}_{\mathbb{C}_\bullet}\big([A_{(i)}], [B_{(j)}]\big) = \bigsqcup_{k \geq i,j} \mathrm{Hom}_{\mathbb{C}_k}\big(F_{ik}A_{(i)}, F_{jk}B_{(j)}\big)\big/\!\sim$$

where $(f_k\colon F_{ik}A_{(i)} \to F_{jk}B_{(j)}) \sim (f_{k'}\colon F_{ik'}A_{(i)} \to F_{jk'}B_{(j)})$ if and only if there exists $h \geq k, k'$ such that $F_{kh}f_k = F_{k'h}f_{k'}$ in $\mathbb{C}_h$. This is well defined: suppose $i \leq l$ and $j \leq m$, so that $[A_{(i)}] = [F_{il}A_{(i)}]$ and $[B_{(j)}] = [F_{jm}B_{(j)}]$, we want to show that the inclusion

$$\bigsqcup_{n \geq l,m} \mathrm{Hom}_{\mathbb{C}_n}\big(F_{ln}F_{il}A_{(i)}, F_{mn}F_{jm}B_{(j)}\big) \hookrightarrow \bigsqcup_{k \geq i,j} \mathrm{Hom}_{\mathbb{C}_k}\big(F_{ik}A_{(i)}, F_{jk}B_{(j)}\big)$$

becomes a bijection on the corresponding quotients:

$$\begin{tikzcd}
\bigsqcup_{n\geq l,m} \mathrm{Hom}_{\mathbb{C}_n}\big(F_{in}A_{(i)}, F_{jn}B_{(j)}\big) \arrow[r, hook] \arrow[d] & \bigsqcup_{k\geq i,j} \mathrm{Hom}_{\mathbb{C}_k}\big(F_{ik}A_{(i)}, F_{jk}B_{(j)}\big) \arrow[d] \\
\bigsqcup_{n\geq l,m} \mathrm{Hom}_{\mathbb{C}_n}\big(F_{in}A_{(i)}, F_{jn}B_{(j)}\big)\big/\!\sim \arrow[r, dotted] & \bigsqcup_{k\geq i,j} \mathrm{Hom}_{\mathbb{C}_k}\big(F_{ik}A_{(i)}, F_{jk}B_{(j)}\big)\big/\!\sim
\end{tikzcd}$$

Take f_{n_1}, f_{n_2}, with $n_s \geq l, m$, and $f_{n_s}\colon F_{in_s}A_{(i)} \to F_{jn_s}B_{(j)}$ for $s = 1, 2$. It follows from the definition that $f_{n_1} \sim f_{n_2}$ as arrows seen in the union on the left if and only if $f_{n_1} \sim f_{n_2}$ seen in the union on the right, so that the dotted arrow is both well defined and injective. This arrow is also surjective: consider $f_k\colon F_{ik}A_{(i)} \to F_{jk}B_{(j)}$ for some $k \geq i, l$ and take $n \geq k, l, m$; then clearly

$[f_k] = [F_{kn} f_k]$, with $F_{kn} f_k$ belonging to the union on the left. To conclude, since the preorder is directed one can show the isomorphism between such quotients of unions also in the general case $i \not\leq l$ or $j \not\leq m$.

Composition in $\mathbb{C}_\bullet$ between two composable arrows

$$[A_{(i)}] \xrightarrow{[f_k]} [B_{(j)}] \xrightarrow{[f_{k'}]} [C_{(l)}],$$

where $f_k \colon F_{ik} A_{(i)} \to F_{jk} B_{(j)}$ and $f_{k'} \colon F_{jk'} B_{(j)} \to F_{lk'} C_{(l)}$, is $[f_{k'}] \circ [f_k] = [F_{k'h} f_{k'} \circ F_{kh} f_k]$ for a given $h \geq k, k'$. This is clearly well defined on the choice of h, and on the representative of f_k and $f_{k'}$.

Finite products in $\mathbb{C}_\bullet$: The category $\mathbb{C}_\bullet$ has binary products, defined in the obvious way: take objects $[A_{(i)}], [B_{(j)}]$ and call $[A_{(i)}] \underset{\to}{\times} [B_{(j)}] \coloneqq [F_{ik} A_{(i)} \times F_{jk} B_{(j)}]$, having as projections the classes of projections from $F_{ik} A_{(i)} \times F_{jk} B_{(j)}$ in $\mathbb{C}_k$ for some $k \geq i, l$—note that $[F_{ik} A_{(i)}] = [A_{(i)}]$ and similarly for the other object, so the codomains of projections make sense in the diagram below. Such class of objects is well defined because the $F_{\star\star}$'s preserve products. To see that it is indeed a product consider the diagram:

$$\begin{tikzcd}
 & & {[V_{(h)}]} \arrow[d, dashed] \arrow[lldd, bend right, "{[\alpha_s]}"'] \arrow[rrdd, bend left, "{[\beta_t]}"] & & \\
 & & {[F_{ik} A_{(i)} \times F_{jk} B_{(j)}]} \arrow[lldd, "{[\mathrm{pr}_1]}"] \arrow[rrdd, "{[\mathrm{pr}_2]}"'] & & \\
{[A_{(i)}]} \arrow[d, phantom, "||"] & & & & {[B_{(j)}]} \arrow[d, phantom, "||"] \\
{[F_{ik} A_{(i)}]} & & & & {[F_{jk} B_{(j)}]}
\end{tikzcd}$$

where $\alpha_s \colon F_{hs} V_{(h)} \to F_{is} A_{(i)}$, $\beta_t \colon F_{ht} V_{(h)} \to F_{jt} B_{(j)}$, for some $s \geq h, i$ and $t \geq h, j$. Now let $m \geq i, j, k, h, s, t$ and consider the diagram in $\mathbb{C}_m$:

$$\begin{tikzcd}
 & & F_{hm} V_{(h)} \arrow[d, dashed, "{\langle F_{sm}(\alpha_s), F_{tm}(\beta_t) \rangle}"] \arrow[lldd, bend right, "F_{sm}(\alpha_s)"'] \arrow[rrdd, bend left, "F_{tm}(\beta_t)"] & & \\
 & & F_{im} A_{(i)} \times F_{jm} B_{(j)} \arrow[lld, "q_1"] \arrow[rrd, "q_2"'] & & \\
F_{im} A_{(i)} & & & & F_{jm} B_{(j)}
\end{tikzcd}$$

Clearly $[\langle F_{sm}(\alpha_s), F_{tm}(\beta_t) \rangle]$ makes the diagram in $\mathbb{C}_\bullet$ commute. Now, to prove uniqueness, take $\psi_n = \langle \psi_{n1}, \psi_{n2} \rangle \colon F_{hn} V_{(h)} \to F_{in} A_{(i)} \times F_{jn} B_{(j)}$ for some $n \geq h, k$, such that $[\psi_{n1}] = [\alpha_s]$ and $[\psi_{n2}] = [\beta_t]$. Then, there exists $r \geq n, s, t$ such that $F_{nr}(\psi_{n1}) = F_{sr}(\alpha_s)$ and $F_{nr}(\psi_{n2}) = F_{tr}(\beta_t)$; in particular

$$[\psi_n] = [\langle F_{nr}(\psi_{n1}), F_{nr}(\psi_{n2}) \rangle].$$

Finally, take $u \geq r, m$:

$$\begin{aligned} F_{mu}(\langle F_{sm}(\alpha_s), F_{tm}(\beta_t)\rangle) &= \langle F_{su}(\alpha_s), F_{tu}(\beta_t)\rangle \\ &= \langle F_{ru}F_{sr}(\alpha_s), F_{ru}F_{tr}(\beta_t)\rangle \\ &= F_{ru}\langle F_{nr}(\psi_{n1}), F_{nr}(\psi_{n2})\rangle, \end{aligned}$$

i.e. $[\langle F_{sm}(\alpha_s), F_{tm}(\beta_t)\rangle] = [\psi_n]$.
In order to conclude the argument about existence of finite products, observe that if $\mathbf{t}_i$ is a terminal object in $\mathbb{C}_i$, then $[\mathbf{t}_i]$ is a terminal object in $\mathbb{C}_\bullet$: take an object $[B_{(j)}]$, $k \geq i, j$ and consider the unique map $!_{F_{jk}B_{(j)}}\colon F_{jk}B_{(j)} \to \mathbf{t}_k$ in $\mathbb{C}_k$. Then $[!_{F_{jk}B_{(j)}}]$ is a map from $[B_{(j)}]$ to $[\mathbf{t}_i] = [F_{ik}\mathbf{t}_i] = [\mathbf{t}_k]$. We show uniqueness by considering a map $[u_h]\colon [B_{(j)}] \to [\mathbf{t}_i]$ for some $u_h\colon F_{jh}B_{(j)} \to F_{ih}\mathbf{t}_i$: then $u_h = !_{F_{jh}B_{(j)}}$ in $\mathbb{C}_h$. Taking $l \geq k, h$ we get $[!_{F_{jk}B_{(j)}}] = [F_{kl}(!_{F_{jk}B_{(j)}})] = [!_{F_{jl}B_{(j)}}] = [F_{hl}(!_{F_{jh}B_{(j)}})] = [!_{F_{jh}B_{(j)}}] = [u_h]$.
$P_\bullet$ on objects: Now that we built a suitable base category with finite products, we define the doctrine $P_\bullet$. For an object $[A_{(i)}]$, we take:

$$P_\bullet([A_{(i)}]) = \bigsqcup_{k \geq i} P_k(F_{ik}A_{(i)})\big/\!\sim$$

where $a_{k_1} \sim a_{k_2}$, with $a_{k_s} \in P_{k_s}(F_{ik_s}A_{(i)})$ for $s = 1, 2$, if and only if there exists $j \geq k_1, k_2$ such that

$$\big(\mathfrak{f}_{k_1 j}\big)_{F_{ik_1}A_{(i)}}(a_{k_1}) = \big(\mathfrak{f}_{k_2 j}\big)_{F_{ik_2}A_{(i)}}(a_{k_2}) \text{ in } P_j(F_{ij}A_{(i)}).$$

This is well defined on the choice of the representative of $[A_{(i)}]$: in a similar way to what we did above defining arrows in $\mathbb{C}_\bullet$, we prove that the dotted arrow induced by the inclusion is bijective, in the case $l \geq i$.

$$\begin{tikzcd} \bigsqcup_{k \geq l} P_k(F_{ik}A_{(i)}) \arrow[r, hook] \arrow[d] & \bigsqcup_{n \geq i} P_n(F_{in}A_{(i)}) \arrow[d] \\ \bigsqcup_{k \geq l} P_k(F_{ik}A_{(i)})\big/\!\sim \arrow[r, dotted] & \bigsqcup_{n \geq i} P_n(F_{in}A_{(i)})\big/\!\sim \end{tikzcd}$$

Take a_{h_1}, a_{h_2} for $h_1, h_2 \geq l$, then $a_{h_1} \sim a_{h_2}$ on the left if and only if they are equivalent on the right, hence well-definition and injectivity of the function follows. Surjectivity also follows easily: take $[b_m]$ for some $m \geq i$, and let $u \geq m, l$. Then $b_m \sim \big(\mathfrak{f}_{mu}\big)_{F_{im}A_{(i)}}(b_m) \in P_u(F_{iu}A_{(i)})$ as wanted. If we fix $A_{(i)}$, we observe that $P_\bullet([A_{(i)}])$ is a directed colimit in **Pos** on the diagram defined over elements of I greater or equal to i. An element $j \geq i$ is sent to $P_j(F_{ij}A_{(i)})$, and for any $j \leq k$ we have the monotone function $\big(\mathfrak{f}_{jk}\big)_{F_{ij}A_{(i)}}\colon P_j(F_{ij}A_{(i)}) \to P_k(F_{ik}A_{(i)})$. Hence we defined a poset for each object of $\mathbb{C}_\bullet$.
$P_\bullet$ on arrows: Take a $\mathbb{C}_\bullet$-arrow $[f]\colon [A_{(i)}] \to [B_{(j)}]$ for some $f\colon F_{ik}A_{(i)} \to F_{jk}B_{(j)} \in \operatorname{arr}\mathbb{C}_k, k \geq i, j$.

$$\begin{tikzcd}
\mathbb{C}_\bullet^{\mathrm{op}} \arrow[r] & \mathbf{Pos} & & \\
{[B_{(j)}]} & P_\bullet([B_{(j)}]) \arrow[d, "P_\bullet({[f]})"] & = & P_\bullet([F_{jk}B_{(j)}]) \\
{[A_{(i)}]} \arrow[u, "{[f]}"] & P_\bullet([A_{(i)}]) & = & P_\bullet([F_{ik}A_{(i)}])
\end{tikzcd}$$

For any given $l \geq k$ we have $F_{kl}(f)\colon F_{il}A_{(i)} \to F_{jl}B_{(j)} \in \operatorname{arr}\mathbb{C}_l$ and

$$P_l(F_{kl}(f))\colon P_l(F_{jl}B_{(j)}) \to P_l(F_{il}A_{(i)}).$$

Since $P_\bullet([F_{jk}B_{(j)}]) = \bigsqcup_{l\geq k} P_l(F_{jl}B_{(j)})\big/\sim$, we prove that the map

$$\bigsqcup_{l\geq k} P_l(F_{jl}B_{(j)}) \longrightarrow \bigsqcup_{m\geq k} P_m(F_{im}A_{(i)})\big/\sim$$

sending any β_l in $[P_l(F_{kl}(f))\beta_l]$ is well defined on the quotient, hence defining a map from $P_\bullet([B_{(j)}])$ to $P_\bullet([A_{(i)}])$. Take $l' \geq l$—then, the statement for any $h \geq k$ follows—, so that $\beta_l \sim (\mathfrak{f}_{ll'})_{F_{jl}B_{(j)}}\beta_l \in P_{l'}(F_{jl'}B_{(j)})$ and

$$(\mathfrak{f}_{ll'})_{F_{jl}B_{(j)}}\beta_l \mapsto [P_{l'}(F_{kl'}(f))(\mathfrak{f}_{ll'})_{F_{jl}B_{(j)}}\beta_l].$$

We now use the naturality of $\mathfrak{f}_{ll'}$ and get:

$$[P_{l'}(F_{kl'}(f))(\mathfrak{f}_{ll'})_{F_{jl}B_{(j)}}\beta_l] = [(\mathfrak{f}_{ll'})_{F_{il}A_{(i)}}P_l(F_{kl}(f))\beta_l] = [P_l(F_{kl}(f))\beta_l]$$

as claimed.

The following step is to prove that the definition of $P_\bullet([f])$ does not depend on the representative of $[f]$. Take $k' \geq k$, then $[f] = [F_{kk'}(f)]$, with $F_{kk'}(f)\colon F_{ik'}A_{(i)} \to F_{jk'}B_{(j)}$. Hence we have for any $\beta_{l'} \in P_{l'}(F_{jl'}B_{(j)})$, $l' \geq k'$

$$[\beta_{l'}] \mapsto [P_{l'}(F_{k'l'}F_{kk'}(f))\beta_{l'}]$$

but $F_{k'l'}F_{kk'} = F_{kl'}$, the two maps act in the same way from $P_\bullet([B_{(j)}])$ to $P_\bullet([A_{(i)}])$.

It follows from the fact that $P_\bullet([f])$ is defined on any suitable $k' \geq k$ and that both $[-]$—in any $P_\bullet([C_h])$—and $P_{k'}(F_{kk'}(f))$ preserve the order, that $P_\bullet([f])$ preserves the order; moreover, also functoriality comes easily. Hence $P_\bullet\colon \mathbb{C}_\bullet^{\mathrm{op}} \to \mathbf{Pos}$ is indeed a doctrine.

A universal cocone into $P_\bullet$: Now, for any $i \in I$, define the 1-cell $(F_i, \mathfrak{f}_i)\colon P_i \to P_\bullet$ in **Dct** as follows:

$$\begin{tikzcd}
\mathbb{C}_i^{\mathrm{op}} \arrow[rr, "{F_i}^{\mathrm{op}}"] \arrow[dr, "P_i"'] \arrow[rr, phantom, shift right=3, "\overset{\mathfrak{f}_i}{\dot{\longrightarrow}}"'] & & {\mathbb{C}_\bullet}^{\mathrm{op}} \arrow[dl, "P_\bullet"] \\
& \mathbf{Pos} &
\end{tikzcd}$$

The functor F_i is the quotient map, sending $f\colon A_{(i)} \to B_{(i)}$ to $[f]\colon [A_{(i)}] \to [B_{(i)}]$; observe that by construction such functors preserve finite products. Similarly $\mathfrak{f}_i\colon P_i \dot{\to} P_\bullet F_i^{\mathrm{op}}$ is the quotient map on every object of $\mathbb{C}_i$:

$$\big(\mathfrak{f}_i\big)_{A_{(i)}}\colon P_i(A_{(i)}) \to P_\bullet([A_{(i)}]) \text{ is defined by the assignment } \alpha_i \mapsto [\alpha_i].$$

Such functions are clearly order preserving. It follows trivially from the definition of $P_\bullet$ on arrows that $\mathfrak{f}_i$ is a natural transformation. Now, to check that it is indeed a cocone, take $i \leq k$: we want $(F_k, \mathfrak{f}_k) \circ (F_{ik}, \mathfrak{f}_{ik}) = (F_i, \mathfrak{f}_i)$.

$$\begin{tikzcd}
\mathbb{C}_i^{\mathrm{op}} \arrow[r, "{F_{ik}}^{\mathrm{op}}"] \arrow[dr, bend right, "P_i"'] \arrow[dr, phantom, "\overset{\mathfrak{f}_{ik}}{\dot{\longrightarrow}}"] & {\mathbb{C}_k}^{\mathrm{op}} \arrow[r, "{F_k}^{\mathrm{op}}"] \arrow[d, "P_k"] \arrow[dr, phantom, "\overset{\mathfrak{f}_k}{\dot{\longrightarrow}}"'] & \mathbb{C}_\bullet^{\mathrm{op}} \arrow[dl, bend left, "P_\bullet"] \\
& \mathbf{Pos} &
\end{tikzcd}$$

Concerning the functors between the base categories, observe that the composition

$$\begin{tikzcd}
A_{(i)} \arrow[dd, "f"] & F_{ik}A_{(i)} \arrow[dd, "F_{ik}(f)"] & {[F_{ik}A_{(i)}]} \arrow[dd, "{[F_{ik}(f)]\ = [f]}"] & = & {[A_{(i)}]} \arrow[dd] \\
{} \arrow[r, mapsto] & {} \arrow[r, mapsto] & {} & & \\
B_{(i)} & F_{ik}B_{(i)} & {[F_{ik}B_{(i)}]} & = & {[B_{(i)}]}
\end{tikzcd}$$

is indeed F_i. Then, for any $\alpha_i \in P_i(A_{(i)})$, we have:

$$\big(\mathfrak{f}_k \circ \mathfrak{f}_{ik}\big)_{A_{(i)}}\alpha_i = \big(\mathfrak{f}_k\big)_{F_{ik}A_{(i)}}\big(\mathfrak{f}_{ik}\big)_{A_{(i)}}\alpha_i = [\big(\mathfrak{f}_{ik}\big)_{A_{(i)}}\alpha_i] = [\alpha_i] = \big(\mathfrak{f}_i\big)_{A_{(i)}}\alpha_i,$$

so that $\mathfrak{f}_k \circ \mathfrak{f}_{ik} = \mathfrak{f}_i$.

Suppose we have another cocone, i.e. any doctrine $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ that comes with a family of 1-cells $\{(G_i, \mathfrak{g}_i)\colon P_i \to R\}_{i \in I}$ such that for any $i \leq k$ one has $(G_k, \mathfrak{g}_k) \circ (F_{ik}, \mathfrak{f}_{ik}) = (G_i, \mathfrak{g}_i)$ we look for a unique 1-cell $(G, \mathfrak{g})\colon P_\bullet \to R$ such that $(G, \mathfrak{g}) \circ (F_i, \mathfrak{f}_i) = (G_i, \mathfrak{g}_i)$ for all $i \in I$.

In order to define $G\colon \mathbb{C}_\bullet \to \mathbb{D}$, take any $[f]\colon [A_{(i)}] \to [B_{(j)}]$ with $f\colon F_{ik}A_{(i)} \to F_{jk}B_{(j)}$ for some $k \geq i, j$ and send it to $G_k(f)\colon G_iA_{(i)} \to G_jB_{(j)}$. This is well defined because of the commutativity properties of the cocone. Similarly we define $\mathfrak{g}\colon P_\bullet \dot{\to} RG^{\mathrm{op}}$: for a given object $[A_{(i)}]$, we take

$$\mathfrak{g}_{[A_{(i)}]}\colon P_\bullet([A_{(i)}]) \to RG_iA_{(i)}, \text{ such that } [\alpha_k] \mapsto \big(\mathfrak{g}_k\big)_{F_{ik}A_{(i)}}\alpha_k$$

for any $\alpha_k \in P_k(F_{ik}A_{(i)}), k \geq i$. This is well defined on both $[\alpha_k]$ and $[A_{(i)}]$ again from the properties of the cocone. Naturality of $\mathfrak{g}$ is also easy to see: given an arrow $[f]\colon [A_{(i)}] \to [B_{(j)}]$

we compute both $RG([f])\mathfrak{g}_{[B_{(j)}]}$ and $\mathfrak{g}_{[A_{(i)}]}P_\bullet([f])$ on a given $[\beta_l] \in P_\bullet([B_{(j)}])$:

$$\begin{aligned} RG([f])\mathfrak{g}_{[B_{(j)}]}[\beta_l] &= RG_k(f)\big(\mathfrak{g}_l\big)_{F_{jl}B_{(j)}}\beta_l = RG_l(F_{kl}(f))\big(\mathfrak{g}_l\big)_{F_{jl}B_{(j)}}\beta_l \\ &= \big(\mathfrak{g}_l\big)_{F_{il}A_{(i)}}P_l(F_{kl}(f))\beta_l = \mathfrak{g}_{[A_{(i)}]}[P_l(F_{kl}(f))\beta_l] = \mathfrak{g}_{[A_{(i)}]}P_\bullet([f])[\beta_l]. \end{aligned}$$

Uniqueness is given by the fact that all triangles like the one below must commute.

$$\begin{tikzcd} P_i \arrow[rr, "{(G_i,\mathfrak{g}_i)}"] \arrow[dr, "{(F_i,\mathfrak{f}_i)}"'] & & R \\ & P_\bullet \arrow[ur, "{(G,\mathfrak{g})}"'] & \end{tikzcd}$$

☐

1.3.1 Additional structure

We now show that many properties are preserved by a directed colimit.

Proposition 1.3.2. Let I be a directed preorder, $D\colon I \to \mathbf{Dct}$ be a diagram, $D(i \le j) = [(F_{ij}, \mathfrak{f}_{ij})\colon P_i \to P_j]$ for any $i, j \in I$, and let $(P_\bullet, \{(F_i, \mathfrak{f}_i)\}_{i\in I})$ be the colimit of D. Suppose that for every $i, j \in I$, the doctrine P_i and the morphism $(F_{ij}, \mathfrak{f}_{ij})$ are primary. Then the doctrine $P_\bullet$ is a primary doctrine, and for every $i \in I$ the morphism $(F_i, \mathfrak{f}_i)$ is primary. Moreover, if for any cocone $(R, \{(G_i, \mathfrak{g}_i)\}_{i\in I})$, R and $(G_i, \mathfrak{g}_i)$ are primary, then the unique arrow $(G, \mathfrak{g})\colon P_\bullet \to R$ defined by the universal property of the colimit is primary. The same statement holds if we write respectively bounded, with binary joins, implicational, elementary, existential, universal, Horn, Heyting, Boolean instead of primary.

Proof. **Algebraic properties:** It is a well known fact that directed colimit of algebraic structures exists, hence if for all $i \in I$, P_i is endowed with equational structure such as $\wedge, \top$ or $\vee, \bot$, then these operation are defined also in $P_\bullet$, preserved by $\mathfrak{f}_i$ for all $i \in I$. Such properties are also preserved by reindexing: this can be shown using naturality of $\mathfrak{f}_{ij}$ and the fact that they are preserved by reindexing in each P_i . Moreover, since $\mathfrak{g}$ is defined through $\mathfrak{g}_i$'s, which preserve operations, also $\mathfrak{g}$ preserves them.

Implication: We define for each pair of elements $[\alpha_k], [\beta_{k'}] \in P_\bullet[A_{(i)}]$, with $\alpha_k \in F_{ik}A_{(i)}$ and $\beta_{k'} \in F_{ik'}A_{(i)}$ for some $k, k' \ge i$

$$[\alpha_k] \to [\beta_{k'}] := [\big(\mathfrak{f}_{kh}\big)_{F_{ik}A_{(i)}}\alpha_k \to \big(\mathfrak{f}_{k'h}\big)_{F_{ik'}A_{(i)}}\beta_{k'}]$$

for some $h \ge k, k'$. This is well defined because every function in $\{\mathfrak{f}_{ij}\}_{i,j\in I}$ preserves implications. Moreover, this is indeed a right adjoint to the binary meet operation:

$$[\gamma_{\overline{k}}] \le [\alpha_k] \to [\beta_{k'}] \text{ in } P_\bullet[A_{(i)}] \tag{1.1}$$

if and only if there exists $s \geq \overline{k}, k, k'$ such that in $P_s(F_{is}A_{(i)})$

$$\big(\mathfrak{f}_{\overline{k}s}\big)_{F_{i\overline{k}}A_{(i)}}\gamma_{\overline{k}} \leq \big(\mathfrak{f}_{ks}\big)_{F_{ik}A_{(i)}}\alpha_k \to \big(\mathfrak{f}_{k's}\big)_{F_{ik'}A_{(i)}}\beta_{k'},$$

but this inequality holds if and only if

$$\big(\mathfrak{f}_{\overline{k}s}\big)_{F_{i\overline{k}}A_{(i)}}\gamma_{\overline{k}} \wedge \big(\mathfrak{f}_{ks}\big)_{F_{ik}A_{(i)}}\alpha_k \leq \big(\mathfrak{f}_{k's}\big)_{F_{ik'}A_{(i)}}\beta_{k'}$$

so (1.1) holds if and only if

$$[\gamma_{\overline{k}}] \wedge [\alpha_k] \leq [\beta_{k'}].$$

Now, since $[\alpha_k] \to [\beta_{k'}]$ is computed in a common poset, as in the case of algebraic properties, implication is preserved by reindexings, $\{\mathfrak{f}_i\}_{i\in I}$ and $\mathfrak{g}$.

Elementarity: For a given pair of objects $[C_{(i)}], [B_{(j)}]$, take the reindexing over $\mathrm{id}_{[C_{(i)}]} \underset{\to}{\times} \Delta_{[B_{(j)}]}$ computed as

$$P_\bullet([\mathrm{id}_{F_{ik}C_{(i)}} \times \Delta_{F_{jk}B_{(j)}}])\colon P_\bullet([C_{(i)}] \underset{\to}{\times} [B_{(j)}] \underset{\to}{\times} [B_{(j)}]) \to P_\bullet([C_{(i)}] \underset{\to}{\times} [B_{(j)}])$$

for any $k \geq i, j$. We define $\text{Æ}_{\bullet}{}_{[C_{(i)}]}^{[B_{(j)}]}$ as the function sending $[\alpha_h]$ to

$$\big[\text{Æ}_k{}_{F_{ik}C_{(i)}}^{F_{jk}B_{(j)}}\big(\mathfrak{f}_{hk}\big)_{F_{ih}C_{(i)}\times F_{jh}B_{(j)}}\alpha_h\big]$$

for some $k \geq i, j, h$, where we write Æ_k for the left adjoint to $P_k(\mathrm{id} \times \Delta)$ (see Definition 1.1.6). This is well defined since every map in $\{\mathfrak{f}_{ij}\}_{i,j\in I}$ preserves the structure. Again, one can prove that it is indeed the left adjoint of the reindexing above, naturality in $[C_{(i)}]$ and Frobenius reciprocity—it follows from Frobenius reciprocity for any Æ_k. Moreover, $\{\mathfrak{f}_i\}_{i\in I}$ and $\mathfrak{g}$ preserve the structure.

Existentiality and Universality: In a similar way to what we did to define the elementary structure, we build the existential and the universal quantifier. Take $[C_{(i)}], [B_{(j)}]$, consider $[\mathrm{pr}_1]\colon [C_{(i)}] \underset{\to}{\times} [B_{(j)}] \to [C_{(i)}]$, where we call $\mathrm{pr}_1\colon F_{ik}C_{(i)} \times F_{jk}B_{(j)} \to F_{ik}C_{(i)}$ the projection in $\mathbb{C}_k$ for any $k \geq i, j$. Then consider

$$P_\bullet([q_1])\colon P_\bullet([C_{(i)}]) \to P_\bullet([C_{(i)}] \underset{\to}{\times} [B_{(j)}])$$

and define

$$\exists_{\bullet}{}_{[C_{(i)}]}^{[B_{(j)}]}[\beta_l] := [\exists_l{}_{F_{il}C_{(i)}}^{F_{jl}B_{(j)}}\beta_l] \text{ and } \forall_{\bullet}{}_{[C_{(i)}]}^{[B_{(j)}]}[\beta_l] := [\forall_l{}_{F_{il}C_{(i)}}^{F_{jl}B_{(j)}}\beta_l]$$

for $\beta_l \in P_l(F_{il}C_{(i)} \times F_{jl}B_{(j)})$.

This is well defined since every map in $\{\mathfrak{f}_{ij}\}_{i,j\in I}$ preserves the structure, with similar arguments to the above. Moreover, one can prove that $\exists_{\bullet}{}_{[C_{(i)}]}^{[B_{(j)}]}$ and $\forall_{\bullet}{}_{[C_{(i)}]}^{[B_{(j)}]}$ define respectively the left adjoint and the right adjoint to $P_\bullet([q_1])$, that they are both natural in $[C_{(i)}]$ and Frobenius reciprocity for the existential quantifier holds. Furthermore, $\{\mathfrak{f}_i\}_{i\in I}$ and $\mathfrak{g}$ preserve the structures. $\square$

1.4 Boolean completion

In this section, we focus on constructing the Boolean completion of an implicational doctrine P with bottom element. In other words, we seek for a universal way to associate P with a Boolean doctrine. To construct the Boolean completion of P, we begin by associating to each fiber of P, which is a bounded implicative inf-semilattice, the set of its closed elements. We then show that this construction respects reindexing and other additional structure that P might have. All general results about bounded implicative inf-semilattices can be found in [Fri62] and [Nem65]. Ultimately, our goal is to use the Boolean completion to improve upon a result, specifically Proposition 3.5.6, by deriving a new proposition, Proposition 3.7.1, that has weaker assumptions.

Given an implicational bounded doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$, we define its boolean completion as follows: for each object $X \in \mathrm{ob}\mathbb{C}$,

$$P_{\neg\neg}(X) \coloneqq \{\alpha \in P(X) \mid \neg\neg\alpha \leq \alpha\} \subseteq P(X),$$

where $\neg\alpha \coloneqq \alpha \to \bot$. The order is induced by the order of $P(X)$.

The reindexing is defined as the restriction of the reindexing in $P(X)$ and it is well defined since P preserves the negation—in fact, it preserves both implication and bottom element. We have a doctrine homomorphism $(\mathrm{id}_{\mathbb{C}}, \neg\neg)\colon P \to P_{\neg\neg}$, where $(\neg\neg)_X\colon P(X) \to P_{\neg\neg}(X)$, sends $\alpha \mapsto \neg\neg\alpha$ for all $X \in \mathrm{ob}\mathbb{C}$ and for all $\alpha \in P(X)$.

Before we go on, we recall some auxiliary properties. The proofs are trivial, and can be found in [Nem65].

Lemma 1.4.1. Let $\mathcal{P}$ be a bounded implicative inf-semilattices. Then for any $\alpha, \beta \in \mathcal{P}$:

(i) $\neg\top = \bot$, $\neg\bot = \top$;

(ii) $\alpha \leq \neg\neg\alpha$;

(iii) if $\alpha \leq \beta$, then $\neg\beta \leq \neg\alpha$;

(iv) $\neg\neg\neg\alpha = \neg\alpha$;

(v) $\neg(\alpha \wedge \beta) = \alpha \to \neg\beta$;

(vi) $\neg(\alpha \to \beta) = \neg\neg\alpha \wedge \neg\beta$;

(vii) $\neg\neg(\alpha \wedge \beta) = \neg\neg\alpha \wedge \neg\neg\beta$.

From [Fri62] we get that each fiber $P_{\neg\neg}(X)$ is a Boolean algebra with top element, meets and implication computed as in $P(X)$, and the join of a pair $\alpha, \beta \in P_{\neg\neg}(X)$ is defined as $\neg(\neg\alpha \wedge \neg\beta)$; moreover from [Nem65] the map preserves the structure of bounded implicative inf-semilattices.

Theorem 1.4.2. Let $\mathbf{Dct}_{\wedge,\top,\to,\bot}$ be the 2-full 2-subcategory of $\mathbf{Dct}$ whose objects are implicational doctrines with bottom element, and 1-morphism are the one that preserve the said structure, and **Bool** 2-full 2-subcategory of Boolean doctrines and Boolean morphism. Then, for any $P \in \mathbf{Dct}_{\wedge,\top,\to,\bot}$, precomposition with $(\mathrm{id}_{\mathbb{C}}, \neg\neg)$ in $\mathbf{Dct}_{\wedge,\top,\to,\bot}$ induces an essential equivalence of categories

$$- \circ (\mathrm{id}_{\mathbb{C}}, \neg\neg)\colon \mathbf{Bool}(P_{\neg\neg}, R) \to \mathbf{Dct}_{\wedge,\top,\to,\bot}(P, R)$$

for every R in **Bool**.

Proof. Precomposition sends $\theta\colon (K, \mathfrak{k}) \to (K', \mathfrak{k}')$ to $\theta\colon (K, \mathfrak{k}\neg\neg) \to (K', \mathfrak{k}'\neg\neg)$, so the functor is trivially faithfull. Also fullness is easy: suppose to have a $\theta\colon K \mathrel{\dot{\to}} K'$ such that for any object X and any $\beta \in P(X)$ it holds that $\mathfrak{k}_X(\neg\neg\beta) \leq R(\theta_X)(\mathfrak{k}'(\neg\neg\beta))$, but then for any $\alpha \in P_{\neg\neg}(X)$, we have $\alpha = \neg\neg\alpha$, so $\mathfrak{k}_X(\alpha) \leq R(\theta_X)(\mathfrak{k}'(\alpha))$, i.e. θ is a 2-arrow between $(K, \mathfrak{k})$ and $(K', \mathfrak{k}')$, which is sent to θ. To conclude, take a 1-arrow $(H, \mathfrak{h})\colon P \to R$ in $\mathbf{Dct}_{\wedge,\top,\to,\bot}$. We look for a 1-arrow from $P_{\neg\neg} \to R$, such that it is equal to $(H, \mathfrak{h})$ when precomposed with $(\mathrm{id}_{\mathbb{C}}, \neg\neg)$. The functor between the base categories is necessarily $H\colon \mathbb{C} \to \mathbb{D}$; define $\mathfrak{k}\colon P_{\neg\neg} \to RH^{\mathrm{op}}$ to be on each component the restriction of $\mathfrak{h}$ to $P_{\neg\neg}$. It preserves all the operations since in $P_{\neg\neg}$ they are computed as in P, and are preserved by $\mathfrak{h}$. Now take the composition

$$P(X) \xrightarrow{\neg\neg} P_{\neg\neg}(X) \xrightarrow{\mathfrak{k}_X} R(HX)$$

which must be equal to $\mathfrak{h}_X$. However, for any $\alpha \in P(X)$, $\mathfrak{k}_X(\neg\neg\alpha) = \mathfrak{h}_X(\neg\neg\alpha) = \neg\neg\mathfrak{h}_X(\alpha) = \mathfrak{h}_X(\alpha)$, since $\mathfrak{h}$ preserves operations and $R(X)$ is a Boolean algebra. Uniqueness of the functor between base categories is trivial. Moreover, suppose $\mathfrak{k}'$ such that $(H, \mathfrak{k}')(\mathrm{id}_{\mathbb{C}}, \neg\neg) = (H, \mathfrak{h})$, then $\mathfrak{k}'_X(\beta) = \mathfrak{k}'_X(\neg\neg\beta) = \mathfrak{h}_X(\beta)$, hence $\mathfrak{k}'$ must be the restriction of $\mathfrak{h}$. ☐

Proposition 1.4.3. Let P be a bounded implicational doctrine and $(\mathrm{id}_{\mathbb{C}}, \neg\neg)\colon P \to P_{\neg\neg}$ be its Boolean completion. If P is elementary, then $P_{\neg\neg}$ and $(\mathrm{id}_{\mathbb{C}}, \neg\neg)$ are elementary.

Proof. Take $\delta_X \in P(X \times X)$ and define $\bar{\delta}_X = \neg\neg\delta_X \in P_{\neg\neg}(X \times X)$. First of all we prove $\top_X \leq P_{\neg\neg}(\Delta_X)(\bar{\delta}_X)$, but $P_{\neg\neg}(\Delta_X)(\bar{\delta}_X) = P(\Delta_X)(\neg\neg\delta_X) = \neg\neg P(\Delta_X)(\delta_X) = \neg\neg\top_X = \top_X$. Then we show that for any element $\alpha \in P_{\neg\neg}(X)$ we have $P_{\neg\neg}(\mathrm{pr}_1)(\alpha) \wedge \bar{\delta}_X \leq P_{\neg\neg}(\mathrm{pr}_2)(\alpha)$, i.e. $P(\mathrm{pr}_1)(\alpha) \wedge \neg\neg\delta_X \leq P(\mathrm{pr}_2)(\alpha)$ but $\alpha = \neg\neg\alpha$, so:

$$\begin{aligned} P(\mathrm{pr}_1)(\alpha) \wedge \neg\neg\delta_X &= \neg\neg P(\mathrm{pr}_1)(\alpha) \wedge \neg\neg\delta_X \\ &= \neg\neg(P(\mathrm{pr}_1)(\alpha) \wedge \delta_X) \leq \neg\neg P(\mathrm{pr}_2)(\alpha) = P(\mathrm{pr}_2)(\alpha). \end{aligned}$$

To conclude, we show that $\bar{\delta}_X \boxtimes \bar{\delta}_Y \leq \bar{\delta}_{X \times Y}$, i.e.

$$P_{\neg\neg}(\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle)(\neg\neg\delta_X) \wedge P_{\neg\neg}(\langle \mathrm{pr}_2, \mathrm{pr}_4 \rangle)(\neg\neg\delta_Y) \leq \neg\neg\delta_{X \times Y};$$

however

$$\neg\neg P(\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle)(\delta_X) \wedge \neg\neg P(\langle \mathrm{pr}_2, \mathrm{pr}_4\rangle)(\delta_Y) = \neg\neg(\delta_X \boxtimes \delta_Y) \leq \neg\neg\delta_{X\times Y},$$

so $P_{\neg\neg}$ is indeed an elementary doctrine. Moreover, by definition the fibered equality is preserved by the 1-arrow $(\mathrm{id}_{\mathbb{C}}, \neg\neg)$. ☐

Corollary 1.4.4. The equivalence of Theorem 1.4.2 restricts to an equivalence between the subcategories of correspondent elementary doctrines.

Proof. It is enough to add some details to the prof of the fact that precomposition is essentially surjective. Given $(H, \mathfrak{h})\colon P \to R$, where $\mathfrak{h}$ preserves also the fibered equality, we show that its restriction $\mathfrak{k}$ to $P_{\neg\neg}$ preserves the fibered equality:

$$\mathfrak{k}_{X\times X}(\bar{\delta}_X) = \mathfrak{h}_{X\times X}(\neg\neg\delta_X) = \neg\neg\mathfrak{h}_{X\times X}(\delta_X) = \mathfrak{h}_{X\times X}(\delta_X) = \delta_{HX}.$$ ☐

Proposition 1.4.5. Let P be a bounded implicational doctrine and $(\mathrm{id}_{\mathbb{C}}, \neg\neg)\colon P \to P_{\neg\neg}$ be its Boolean completion. If P is existential, then $P_{\neg\neg}$ and $(\mathrm{id}_{\mathbb{C}}, \neg\neg)$ are existential.

Proof. Recall the adjunction in P:

$$\begin{tikzcd} P(C\times B) \arrow[r, bend left, "\exists^B_C"] \arrow[r, phantom, "\perp"] & P(C) \arrow[l, bend left, "P(\mathrm{pr}_1)"] \end{tikzcd}$$

Define $\exists^B_C\colon P_{\neg\neg}(C\times B) \to P_{\neg\neg}(C)$, $\exists^B_C\alpha := \neg\neg\exists^B_C\alpha$; we show that it is the left adjoint of the restriction of $P(\mathrm{pr}_1)$: take $\alpha \in P_{\neg\neg}(C\times B)$ and $\beta \in P_{\neg\neg}(C)$. Suppose $\exists^B_C\alpha \leq \beta$, so $\exists^B_C\alpha \leq \neg\neg\exists^B_C\alpha \leq \beta$, but then $\alpha \leq P(\mathrm{pr}_1)\beta$. For the converse, if $\alpha \leq P(\mathrm{pr}_1)\beta$, then $\exists^B_C\alpha \leq \beta$, so $\neg\neg\exists^B_C\alpha \leq \neg\neg\beta = \beta$, as claimed. Concerning naturality, take a $\mathbb{C}$-arrow $f\colon C \to A$ and $\gamma \in P_{\neg\neg}(A\times B)$:

$$\exists^B_C P(f\times \mathrm{id}_B)\gamma = \neg\neg\exists^B_C P(f\times \mathrm{id}_B)\gamma = \neg\neg P(f)\exists^B_A\gamma = P(f)\neg\neg\exists^B_A\gamma = P(f)\exists^B_A\gamma.$$

Lastly, we show Frobenius reciprocity: given $\alpha \in P_{\neg\neg}(C\times B)$ and $\beta \in P_{\neg\neg}(C)$ we have

$$\begin{aligned}\exists^B_C\alpha \wedge \beta = \neg\neg\exists^B_C\alpha \wedge \beta = \neg\neg\exists^B_C\alpha \wedge \neg\neg\beta = \neg\neg(\exists^B_C\alpha \wedge \beta)\\ = \neg\neg(\exists^B_C(\alpha \wedge P(\mathrm{pr}_1)\beta) = \exists^B_C(\alpha \wedge P(\mathrm{pr}_1)\beta).\end{aligned}$$

So $P_{\neg\neg}$ is indeed an existential doctrine.
Moreover, $(\mathrm{id}, \neg\neg)$ preserves the existential quantifier, i.e for any $\alpha \in P(C\times B)$, we have $\neg\neg\exists^B_C\alpha = \exists^B_C\neg\neg\alpha$. Since $\alpha \leq \neg\neg\alpha$, clearly $\neg\neg\exists^B_C\alpha \leq \neg\neg\exists^B_C\neg\neg\alpha = \exists^B_C\neg\neg\alpha$. Conversely, start from $\alpha \leq P(\mathrm{pr}_1)\exists^B_C\alpha$, so that $\top \leq \alpha \to P(\mathrm{pr}_1)\exists^B_C\alpha$; then apply Lemmas 1.4.1.(iii) and 1.4.1.(vi) to get $\neg\neg\alpha \wedge \neg P(\mathrm{pr}_1)\exists^B_C\alpha = \neg(\alpha \to P(\mathrm{pr}_1)\exists^B_C\alpha) \leq \neg\top = \bot = P(\mathrm{pr}_1)\bot$; using the definition

of existential, Frobenius reciprocity and the fact that $P(\mathrm{pr}_1)$ preserves $\neg$, we equivalently get in $P(C)$ the following:

$$\exists^B_C \neg\neg\alpha \wedge \neg\exists^B_C \alpha = \exists^B_C(\neg\neg\alpha \wedge \neg P(\mathrm{pr}_1)\exists^B_C \alpha) \leq \bot.$$

From here we get $\neg\exists^B_C \alpha \leq \neg\exists^B_C \neg\neg\alpha$: then, applying Lemma 1.4.1.(iii), we get $\exists^B_C \neg\neg\alpha = \neg\neg\exists^B_C \neg\neg\alpha \leq \neg\neg\exists^B_C \alpha$ as claimed. □

Corollary 1.4.6. The equivalence of Theorem 1.4.2 restricts to an equivalence between the subcategories of correspondent existential doctrines.

Proof. It is enough to add some details to the prof of the fact that precomposition is essentially surjective. Given $(H, \mathfrak{h})\colon P \to R$, where $\mathfrak{h}$ preserves also the existential quantifier, we show that its restriction $\mathfrak{k}$ to $P_{\neg\neg}$ preserves it: for any $\alpha \in P_{\neg\neg}(C \times B)$

$$\begin{aligned} \exists^{HB}_{HC} \mathfrak{k}_{C\times B}(\alpha) &= \neg\neg\exists^{HB}_{HC} \mathfrak{k}_{C\times B}(\alpha) = \neg\neg\exists^{HB}_{HC} \mathfrak{h}_{C\times B}(\alpha) \\ &= \neg\neg\mathfrak{h}_C \exists^B_C(\alpha) = \mathfrak{h}_C(\neg\neg\exists^B_C(\alpha)) = \mathfrak{k}_C(\exists^B_C(\alpha)). \end{aligned}$$

□

1.5 Filters, ultrafilters, quotients

Filters play a significant role in lattice theory, particularly in the study of Boolean algebra. In this section, we present some essential findings concerning filters and ultrafilters in bounded implicative inf-semilattices. While these proofs are already established in the context of Boolean algebras—see for example [Mon89] or [BS81]—, we demonstrate their adaptability in this weaker framework.

Then, for a given primary doctrine P, we will define the quotient of the doctrine over a filter in the fiber of the terminal object, and prove that the quotient map preserves many properties of P itself.

The quotient of a doctrine over some suitable ultrafilter will be a key point in the proof of the existence of a model in Section 3.8.

Definition 1.5.1. Let A be an inf-semilattice. A subset $\nabla \subseteq A$ is a *filter* if the following properties hold:

- $\top \in \nabla$;
- if $a \in \nabla$ and $a \leq b$, then $b \in \nabla$;
- if $a, b \in \nabla$, then $a \wedge b \in \nabla$.

A filter ∇ is *proper* if $\nabla \neq A$

Remark 1.5.2. In a bounded inf-semilattice, a filter ∇ is proper if and only if $\bot \notin \nabla$.

Definition 1.5.3. Let A be a bounded implicative inf-semilattice and $\nabla \subseteq A$ a filter.

- ∇ is an *ultrafilter* if for all $a \in A$, either $a \in \nabla$ or $\neg a \in \nabla$, where $\neg a \coloneqq a \to \bot$.
- ∇ is a *maximal filter* if it is maximal with respect to the inclusion, meaning that $\nabla \neq A$ and, whenever $\nabla \subsetneqq \nabla'$ where ∇' is a filter, then $\nabla' = A$.

Lemma 1.5.4. Let A be an inf-semilattice and $E \subseteq A$. Consider the set

$$F = \{y \in A \mid \text{ there exist } x_1, \dots, x_n \in E \text{ such that } x_1 \wedge \cdots \wedge x_n \leq y\} \cup \{\top\},$$

Then $\langle E \rangle = F$, where $\langle E \rangle$ is the filter generated by E.

Proof. First of all, observe that F is a filter:

- $\top \in F$;
- let $y \in F$ and $z \in A$, $y \leq z$. If $y = \top$, then $z = \top \in F$. Otherwise, take $x_1, \dots, x_n \in E$ such that $x_1 \wedge \cdots \wedge x_n \leq y \leq z$, then also $z \in F$;
- take $y, z \in F$. If $y = \top$ then $y \wedge z = z \in F$; similarly if $z = \top$. Otherwise $x_1 \wedge \cdots \wedge x_n \leq y$, $w_1 \wedge \cdots \wedge w_m \leq z$ with $x_1, \dots, x_n, w_1, \dots, w_m \in E$; then $x_1 \wedge \cdots \wedge x_n \wedge w_1 \wedge \cdots \wedge w_m \leq y \wedge z$, so that $y \wedge z \in F$.

Then $E \subseteq F$: take $x \in E$, since $x \leq x$, we have $x \in F$. In particular $\langle E \rangle \subseteq F$. To conclude, take $y \in F$. If $y = \top$, then $y \in \langle E \rangle$; otherwise, take $x_1 \wedge \cdots \wedge x_n \leq y$ for some $x_1, \dots, x_n \in E$. Any filter $G \supseteq E$ is such that $x_1 \wedge \cdots \wedge x_n \in G$ and since $x_1 \wedge \cdots \wedge x_n \leq y$, also $y \in G$. Hence $y \in \langle E \rangle$, as claimed. □

Lemma 1.5.5. Let A be a bounded implicative inf-semilattice and $\nabla \subseteq A$ a filter. Then ∇ is a maximal filter if and only if ∇ is an ultrafilter.

Proof. Suppose ∇ is an ultrafilter. Since $\top \in \nabla$, then $\nabla \not\ni \neg\top = \top \to \bot = \top \wedge (\top \to \bot) = \bot$, so $\nabla \neq A$. So take another filter $\nabla \subsetneqq \nabla'$, in particular there exists $y \in \nabla'$ such that $y \notin \nabla$. By assumption $y \to \bot \in \nabla$ and also $y \to \bot \in \nabla'$. Then, since $y \wedge (y \to \bot) \leq \bot$, $\bot \in \nabla'$, so that $\nabla' = A$. For the converse, suppose ∇ is a maximal filter. In particular, given $x \in A$, it cannot be the case that both $x, x \to \bot \in \nabla$—otherwise we would have also $\bot \in \nabla$, which would give $\nabla = A$. Suppose that $x \notin \nabla$, we claim that $\neg x = x \to \bot \in \nabla$. Consider $E = \nabla \cup \{x\}$ and take $\langle E \rangle$. Clearly $\langle E \rangle \supsetneqq \nabla$, since $x \in E$ but $x \notin \nabla$. Hence by assumption $\langle E \rangle = A$. In particular $\neg x \in A = \langle E \rangle$. If $\neg x = \top$, then we have $\neg x \in \nabla$. Otherwise there exist $x_1, \dots, x_n \in \nabla \cup \{x\}$ such that $x_1 \wedge \cdots \wedge x_n \leq \neg x$. Now, if every x_i's belong to the filter ∇, we get $\neg x \in \nabla$. Instead, if some x_i's are actually x, we can rewrite the inequality as $x \wedge y \leq \neg x$ for some $y \in \nabla$. But $x \wedge y \leq x \to \bot$ if and only if $x \wedge y \leq \bot$ if and only if $y \leq x \to \bot$, hence again $\neg x \in \nabla$, as claimed. □

Lemma 1.5.6. Given a proper filter ∇ of a bounded implicative inf-semilattice A, there exists an ultrafilter $U \supseteq \nabla$.

Proof. We use Zorn's Lemma. Take $\mathscr{F}$ the set of all proper filters that contain ∇, ordered by inclusion. Clearly $\nabla \in \mathscr{F}$. The upper bound of a chain $\nabla \subseteq \nabla_1 \subseteq \cdots \subseteq \nabla_n \dots$ is given by the union $\cup_{i \in \mathbb{N}} \nabla_i$. So take U a maximal element in $\mathscr{F}$. This is a maximal filter: let W be a proper filter containing U, in particular it contains ∇, so $W = U$. □

Moreover, given a bounded implicative inf-semilattice A and an ultrafilter $\nabla \subseteq A$, define the function $\alpha \colon A \to \mathbb{2}$, where $\mathbb{2} = \{\bot < \top\}$ is the two-element boolean algebra, as follows:

$$\alpha(x) = \begin{cases} \top & \text{if} \, x \in \nabla \\ \bot & \text{if} \, x \notin \nabla. \end{cases}$$

This function preserves the structure of a bounded implicative inf-semilattice.

- $\top \in \nabla$, $\top \mapsto \top$;
- $\bot \notin \nabla$, $\bot \mapsto \bot$;
- take $a, b \in A$, then:
  - if $a, b \in \nabla$, then $a \wedge b \in \nabla$, so $\alpha(a \wedge b) = \top = \alpha(a) \wedge \alpha(b)$;
  - if $a \notin \nabla$, then $a \wedge b \notin \nabla$—indeed: $a \wedge b \leq a$—, so $\alpha(a \wedge b) = \bot = \bot \wedge \alpha(b) = \alpha(a) \wedge \alpha(b)$;
  - if $b \notin \nabla$ same proof as above;

  so α preserves the meet;
- take $a, b \in A$, then:
  - if $b \in \nabla$, since $b \leq a \to b$, then $a \to b \in \nabla$, so compute $\alpha(a \to b) = \top = \alpha(a) \to \top = \alpha(a) \to \alpha(b)$;
  - if $b \notin \nabla$ and $a \in \nabla$, then $a \to b \notin \nabla$—indeed: $a \wedge (a \to b) \leq b$—, so $\alpha(a \to b) = \bot = \top \to \bot = \alpha(a) \to \alpha(b)$;
  - if $b \notin \nabla$ and $a \notin \nabla$, then $\neg b, \neg a \in \nabla$. Observe that $\neg a \wedge \neg b \leq a \to b$ since $a \wedge \neg a \wedge \neg b = \bot \leq b$, so $a \to b \in \nabla$. Hence $\alpha(a \to b) = \top = \bot \to \bot = \alpha(a) \to \alpha(b)$;

  so α preserves the implication.

In particular we use this fact to prove the following characterization.

Lemma 1.5.7. Let A be a bounded implicative inf-semilattice and $\nabla \subseteq A$ a filter. Then ∇ is an ultrafilter if and only if $\nabla = \alpha^{-1}(\top)$ for some morphism $\alpha \colon A \to \mathbb{2}$ of bounded implicational inf-semilattice.

Proof. If ∇ is an ultrafilter, we define α as above, and it is indeed a morphism. For the converse, take a morphism $\alpha \colon A \to \mathbb{2}$, we prove that $\alpha^{-1}(\top)$ is an ultrafilter. It is clearly a filter: $\alpha(\top) = \top$;

if $\alpha(a) = \top$ and $a \leq b$, then $\alpha(b) = \top$; if $\alpha(a) = \top$ and $\alpha(b) = \top$, then $\alpha(a \wedge b) = \top$. Now take any $a \in A$: it cannot be the case that both $a, \neg a \in \alpha^{-1}(\top)$; suppose then $\alpha(a) \neq \top$, hence $\alpha(a) = \bot$, so that $\alpha(a \to \bot) = \bot \to \bot = \top$. $\square$

1.5.1 The quotient of a doctrine over a filter

Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $\nabla \subseteq P(\mathbf{t})$ be a filter in the fiber of the terminal object $\mathbf{t}$. Define, in each $X \in \mathrm{ob}\mathbb{C}$ the following preorder: $\alpha \sqsubseteq_\nabla \beta$ if and only if there exists a $\theta \in \nabla$ such that $P(!_X)\theta \wedge \alpha \leq \beta$ in $P(X)$. This is clearly reflexive; it is also transitive: take $P(!_X)\theta_1 \wedge \alpha \leq \beta$ and $P(!_X)\theta_2 \wedge \beta \leq \gamma$ for some $\theta_1, \theta_2 \in \nabla$, then $P(!_X)(\theta_1 \wedge \theta_2) \wedge \alpha \leq \beta$.
Now define $P/\nabla\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ as follows: for each object X, $P/\nabla(X)$ is the poset reflection of the preorder defined above. In particular we have $[\alpha] = [\beta]$ if and only if there exists $\theta \in \nabla$ such that $P(!_X)\theta \wedge \alpha = P(!_X)\theta \wedge \beta$: indeed, suppose $\theta_1, \theta_2 \in \nabla$ such that $P(!_X)\theta_1 \wedge \alpha \leq \beta$ and $P(!_X)\theta_2 \wedge \beta \leq \alpha$, then $P(!_X)(\theta_1 \wedge \theta_2) \wedge \alpha = P(!_X)(\theta_1 \wedge \theta_2) \wedge \beta$, with $\theta_1 \wedge \theta_2 \in \nabla$. Then, take a $\mathbb{C}$-arrow $f\colon X \to Y$ and define $P/\nabla(f)[\alpha] = [P(f)\alpha]$ for a given $[\alpha] \in P/\nabla(X)$. If this is monotone with respect to the preorder $\sqsubseteq_\nabla$, it is well defined on equivalence classes: take $\alpha \sqsubseteq_\nabla \beta$ for $\alpha, \beta \in P(Y)$, hence there exists $\theta \in \nabla$ such that $P(!_Y)\theta \wedge \alpha \leq \beta$ in $P(Y)$; applying $P(f)$ we get $P(!_X)\theta \wedge P(f)\alpha \leq P(f)\beta$, hence $P(f)\alpha \sqsubseteq_\nabla P(f)\beta$. So P/∇ is indeed a doctrine, since composition and identities are clearly preserved.
Note that the quotient map of each $P(X)$ is a monotone function: if $\alpha \leq \beta$, also $\alpha \sqsubseteq_\nabla \beta$ by taking $\theta = \top \in \nabla$. Call for each object Y, $\mathfrak{q}_Y$ the quotient map: $\mathfrak{q}_Y(\alpha) = [\alpha] \in P/\nabla(Y)$ for a given $\alpha \in P(Y)$; then $(\mathrm{id}_\mathbb{C}, \mathfrak{q})$ is a morphism of doctrines. Indeed, to prove that $\mathfrak{q}$ is a natural transformation, take $f\colon X \to Y$ and observe that:

$$\mathfrak{q}_X P(f)\alpha = [P(f)\alpha] = P/\nabla(f)[\alpha] = P/\nabla(f)\mathfrak{q}_Y(\alpha).$$

Moreover, P/∇ is primary, with top and meet of $P/\nabla(X)$ computed as in $P(X)$: given two elements $[\alpha], [\beta] \in P(X)$, clearly $[\alpha \wedge \beta] \leq [\alpha], [\beta]$; then take $[\gamma] \leq [\alpha], [\beta]$, i.e. $P(!_X)\theta_1 \wedge \gamma \leq \alpha$ and $P(!_X)\theta_2 \wedge \gamma \leq \beta$, for some $\theta_1, \theta_2 \in \nabla$. Then $P(!_X)(\theta_1 \wedge \theta_2) \wedge \gamma \leq \alpha \wedge \beta$ and $\theta_1 \wedge \theta_2 \in \nabla$, hence $[\gamma] \leq [\alpha \wedge \beta]$. So in $P/\nabla(X)$ we have $[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$, as claimed. Naturality of the meet in P/∇ follows from naturality of $\mathfrak{q}$ and of the meet in P.
Then, observe that $[\top_X]$ is the top element in $P/\nabla(X)$: take any $[\alpha] \in P/\nabla(X)$ and note that $P(!_X)\top_\mathbf{t} \wedge \alpha = \alpha \leq \top_X$. Again, the top element is trivially preserved by reindexing. In particular the quotient $(\mathrm{id}_\mathbb{C}, \mathfrak{q})$ is a morphism of primary doctrines.

Proposition 1.5.8. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $\nabla \subseteq P(\mathbf{t})$ be a filter. The 1-arrow $(\mathrm{id}_\mathbb{C}, \mathfrak{q})\colon P \to P/\nabla$ is such that $\top \leq \mathfrak{q}_\mathbf{t}(\theta)$ in $P/\nabla(\mathbf{t})$ for all $\theta \in \nabla$, and it is universal with respect to this property, i.e. for any primary 1-arrow $(G, \mathfrak{g})\colon P \to R$, where $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ is a primary doctrine, such that $\top \leq \mathfrak{g}_\mathbf{t}(\theta)$ in $R(\mathbf{t}_\mathbb{D})$ for all $\theta \in \nabla$, there exists a unique up to a unique natural isomorphism primary 1-arrow $(G', \mathfrak{g}')\colon P/\nabla \to R$ such that $(G', \mathfrak{g}') \circ (\mathrm{id}_\mathbb{C}, \mathfrak{q}) = (G, \mathfrak{g})$.

Proof. At first, observe that any $\theta \in \nabla$ is sent to the top element of $P/\nabla(\mathbf{t})$: indeed, consider $\theta \in \nabla$ itself to observe that $\theta \wedge \top \leq \theta$, to that $[\top_t] \leq [\theta]$. We now show the universal property. First of all, since $G' \,\mathrm{id}_{\mathbb{C}} = G$, we observe that $G' = G\colon \mathbb{C} \to \mathbb{D}$. Then we show that fo any fixed $\mathbb{C}$-object X, the function $\mathfrak{g}_X\colon P(X) \to RGX$ factors through the quotient $\mathfrak{q}_X$, defining $\mathfrak{g}'_X([\alpha]) = \mathfrak{g}_X(\alpha)$. To prove that this is well-defined, take $\alpha \sqsubseteq_\nabla \beta$ in $P(X)$, i.e. $P(!_X)(\theta) \wedge \alpha \leq \beta$. Then apply $\mathfrak{g}_X$ to get $\mathfrak{g}_X P(!_X)(\theta) \wedge \mathfrak{g}_X \alpha \leq \mathfrak{g}_X \beta$ in $R(GX)$. However $\mathfrak{g}_X P(!_X)(\theta) = R(!_{GX})\mathfrak{g}_{\mathbf{t}}(\theta) = \top_{GX}$, hence $\mathfrak{g}_X(\alpha) \leq \mathfrak{g}_X(\beta)$. As a result we obtain a well-defined monotone function $\mathfrak{g}'_X\colon P/\nabla(X) \to R(GX)$ such that $\mathfrak{g}'_X \mathfrak{q}_X = \mathfrak{g}_X$—and it is also unique. Since $\mathfrak{g}_X$ preserves finite meets, and finite meets in P/∇ are computed as in P, it follows that $\mathfrak{g}'_X$ preserves finte meets. Moreover, we can use naturality of $\mathfrak{g}$ to show that $\mathfrak{g}'\colon P/\nabla \to RG^{\mathrm{op}}$ defines a natural transformation. In particular $(G'\mathfrak{g}')$ is a primary 1-arrow such that $(G', \mathfrak{g}') \circ (\mathrm{id}_{\mathbb{C}}, \mathfrak{q}) = (G, \mathfrak{g})$, and it is unique with respect to this property, as claimed. ☐

In the following Lemma we show that if P has some additional structure, then P/∇ has them as well, and the structure is preserved by the quotient morphism.

Lemma 1.5.9. Let P be a primary doctrine, $\nabla \subseteq P(\mathbf{t})$ be a filter and P/∇ be the quotient.

(i) If P is bounded, then the doctrine P/∇ and the 1-arrow $(\mathrm{id}_{\mathbb{C}}, \mathfrak{q})$ are bounded.

(ii) If P is implicational, then the doctrine P/∇ and the 1-arrow $(\mathrm{id}_{\mathbb{C}}, \mathfrak{q})$ are implicational.

(iii) If P is elementary, then the doctrine P/∇ and the 1-arrow $(\mathrm{id}_{\mathbb{C}}, \mathfrak{q})$ are elementary.

(iv) If P is existential, then the doctrine P/∇ and the 1-arrow $(\mathrm{id}_{\mathbb{C}}, \mathfrak{q})$ are existential.

(v) If P is universal, then the doctrine P/∇ and the 1-arrow $(\mathrm{id}_{\mathbb{C}}, \mathfrak{q})$ are universal.

Proof. (i) We show that $\mathfrak{q}_X(\bot_X) = [\bot_X] \leq [\alpha]$ in $P/\nabla(X)$ for all $[\alpha] \in P/\nabla(X)$, but this holds since $P(!_X)\top_{\mathbf{t}} \wedge \bot_X = \bot_X \leq \alpha$ in $P(X)$. Naturality of the bottom element follows from naturality of $\mathfrak{q}$ and of the bottom in P. The quotient $(\mathrm{id}_{\mathbb{C}}, \mathfrak{q})$ trivially preserves the bottom element.

(ii) We show that $\mathfrak{q}_X(\beta \to \gamma) = [\beta \to \gamma] = [\beta] \to [\gamma]$ in $P/\nabla(X)$ for all $[\beta], [\gamma] \in P/\nabla(X)$. Suppose $[\alpha] \wedge [\beta] \leq [\gamma]$, if and only if there exists $\theta \in \nabla$ such that $P(!_X)\theta \wedge \alpha \wedge \beta \leq \gamma$ in $P(X)$, if and only if there exists $\theta \in \nabla$ such that $P(!_X)\theta \wedge \alpha \leq \beta \to \gamma$ in $P(X)$, if and only if $[\alpha] \leq [\beta \to \gamma]$, i.e. $[\beta \to \gamma] = [\beta] \to [\gamma]$ in $P/\nabla(X)$. Naturality again follows from naturality of $\mathfrak{q}$ and of the bottom in P. The quotient $(\mathrm{id}_{\mathbb{C}}, \mathfrak{q})$ preserves implication.

(iii) Consider the elementary doctrine P, with left adjoint $\text{Æ}^B_A \dashv P(\mathrm{id}_A \times \Delta_B)$ for any arrow of the kind $\mathrm{id}_A \times \Delta_B\colon A \times B \to A \times B \times B$ in $\mathbb{C}$.
We show that $\mathfrak{q}_{A\times B}(\text{Æ}^B_A \alpha) = [\text{Æ}^B_A \alpha] = \underline{\text{Æ}}^B_A[\alpha]$ in $P/\nabla(A \times B \times B)$ for all $[\alpha] \in P/\nabla(A \times B)$. To show that $\underline{\text{Æ}}^B_A$ is well defined on the quotients, suppose $\alpha \sqsubseteq_\nabla \beta$, for some pair $\alpha, \beta \in P(A \times B)$, i.e. there exists $\theta \in \nabla$ such that $P(!_{A\times B})\theta \wedge \alpha \leq \beta$ in $P(A \times B)$; then $\text{Æ}^B_A(P(!_{A\times B})\theta \wedge \alpha) = \text{Æ}^B_A(P(\mathrm{id}_A \times \Delta_B)P(!_{A\times B\times B})\theta \wedge \alpha) = \text{Æ}^B_A \alpha \wedge P(!_{A\times B\times B})\theta \leq \text{Æ}^B_A \beta$ in

$P(A\times B\times B)$ by using Frobenius reciprocity, i.e. $[\text{Æ}^B_A\alpha]\leq[\text{Æ}^B_A\beta]$, so $\underline{\text{Æ}}^B_A[\alpha]=[\text{Æ}^B_A\alpha]$ is well defined.
This is the left adjoint to the reindexing along $\mathrm{id}_A\times\Delta_B$: indeed, take $[\alpha]\in P/\nabla(A\times B)$ and $[\gamma]\in P/\nabla(A\times B\times B)$, then $\underline{\text{Æ}}^B_A[\alpha]\leq[\gamma]$ if and only if there exists en element $\theta\in\nabla$ such that $P(!_{A\times B\times B})\theta\wedge\text{Æ}^B_A\alpha\leq\gamma$ in $P(A\times B\times B)$, but by Frobenius reciprocity we have $P(!_{A\times B\times B})\theta\wedge\text{Æ}^B_A\alpha=\text{Æ}^B_A(\alpha\wedge P(\mathrm{id}_A\times\Delta_B)P(!_{A\times B\times B})\theta)=\text{Æ}^B_A(\alpha\wedge P(!_{A\times B})\theta)$, hence if and only if there exists $\theta\in\nabla$ such that $\alpha\wedge P(!_{A\times B})\theta\leq P(\mathrm{id}_A\times\Delta_B)\gamma$ in $P(A\times B)$, if and only if $[\alpha]\leq P/\nabla(\mathrm{id}_A\times\Delta_B)[\gamma]$, as claimed.
Naturality of $\underline{\text{Æ}}^B_{(-)}$ and Frobenious reciprocity follow from the same properties of $\text{Æ}^B_{(-)}$ in P.
So the doctrine P/∇ is elementary, and the quotient is a morphism of primary elementary doctrines.

(iv) Consider the existential doctrine P, with left adjoint $\exists^B_A\dashv P(\mathrm{pr}_1)$ for any projection $\mathrm{pr}_1\colon A\times B\to A$ in $\mathbb{C}$.
We show that $\mathfrak{q}_A(\exists^B_A\alpha)=[\exists^B_A\alpha]=\underline{\exists}^B_A[\alpha]$ in $P/\nabla(A)$ for all $[\alpha]\in P/\nabla(A\times B)$. To show that $\underline{\exists}^B_A$ is well defined on the quotients, suppose $\alpha\sqsubseteq_\nabla\beta$, for some $\alpha,\beta\in P(A\times B)$, i.e. there exists $\theta\in\nabla$ such that $P(!_{A\times B})\theta\wedge\alpha\leq\beta$ in $P(A\times B)$; then $\exists^B_A(P(!_{A\times B})\theta\wedge\alpha)=\exists^B_A(P(\mathrm{pr}_1)P(!_A)\theta\wedge\alpha)=\exists^B_A\alpha\wedge P(!_A)\theta\leq\exists^B_A\beta$ in $P(A)$ by using Frobenius reciprocity, i.e. $[\exists^B_A\alpha]\leq[\exists^B_A\beta]$, so $\underline{\exists}^B_A[\alpha]=[\exists^B_A\alpha]$ is well defined. This is still the left adjoint to the reindexing along the first projection: take $[\alpha]\in P/\nabla(A\times B)$ and $[\gamma]\in P/\nabla(A)$, then $\underline{\exists}^B_A[\alpha]\leq[\gamma]$ if and only if there exists $\theta\in\nabla$ such that $P(!_A)\theta\wedge\exists^B_A\alpha\leq\gamma$ in $P(A)$, but $P(!_A)\theta\wedge\exists^B_A\alpha=\exists^B_A(\alpha\wedge P(\mathrm{pr}_1)P(!_A)\theta)=\exists^B_A(\alpha\wedge P(!_{A\times B})\theta)$ by Frobenius reciprocity, hence if and only if there exists $\theta\in\nabla$ such that $\alpha\wedge P(!_{A\times B})\theta\leq P(\mathrm{pr}_1)\gamma$ in $P(A\times B)$, if and only if $[\alpha]\leq P/\nabla(\mathrm{pr}_1)[\gamma]$, as claimed.
Naturality of $\underline{\exists}^B_{(-)}$ and Frobenious reciprocity follow from the same properties of $\exists^B_{(-)}$ in P.
So the doctrine P/∇ is existential, and the quotient is a morphism of existential primary doctrines.

(v) Consider the universal doctrine P, with right adjoint $P(\mathrm{pr}_1)\dashv\forall^B_A$ for any projection $\mathrm{pr}_1\colon A\times B\to A$ in $\mathbb{C}$.
We show that $\mathfrak{q}_A(\forall^B_A\alpha)=[\forall^B_A\alpha]=\underline{\forall}^B_A[\alpha]$ in $P/\nabla(A)$ for all $[\alpha]\in P/\nabla(A\times B)$. To show that $\underline{\forall}^B_A$ is well defined on the quotients, suppose $\alpha\sqsubseteq_\nabla\beta$, for some $\alpha,\beta\in P(A\times B)$, i.e. there exists $\theta\in\nabla$ such that $P(!_{A\times B})\theta\wedge\alpha\leq\beta$ in $P(A\times B)$; then $P(!_A)\theta\wedge\forall^B_A\alpha\leq\forall^B_AP(\mathrm{pr}_1)P(!_A)\theta\wedge\forall^B_A\alpha=\forall^B_A(P(!_{A\times B})\theta\wedge\alpha)\leq\forall^B_A\beta$ in $P(A)$ by using the unity of the adjunction and the fact that right adjoint preserve limits—hence meets too—, i.e. $[\forall^B_A\alpha]\leq[\forall^B_A\beta]$, so $\underline{\forall}^B_A[\alpha]=[\forall^B_A\alpha]$ is well defined. This is still the right adjoint to the reindexing along the first projection: take $[\alpha]\in P/\nabla(A\times B)$ and $[\gamma]\in P/\nabla(A)$, then $[\gamma]\leq\underline{\forall}^B_A[\alpha]$ if and only if there exists $\theta\in\nabla$ such that $P(!_A)\theta\wedge\gamma\leq\forall^B_A\alpha$ in $P(A)$, if and only if there exists $\theta\in\nabla$ such that $P(!_{A\times B})\theta\wedge P(\mathrm{pr}_1)\gamma\leq\alpha$ in $P(A\times B)$, if and only if $P/\nabla(\mathrm{pr}_1)[\gamma]\leq[\alpha]$,

as claimed.

Naturality of $\underline{\forall}^B_{(-)}$ follows from the same property of $\forall^B_{(-)}$ in P.

So the doctrine P/∇ is universal, and the quotient is a morphism of universal primary doctrines. $\square$

Chapter 2

Adding a constant and an axiom to a doctrine

In this chapter, we explore how to translate into the language of doctrines, seen as a generalization of the doctrine of well-formed formulae, the process of adding a constant of some fixed sort and adding a sentence to a theory. Although these may seem like separate processes, we show that they can be computed simultaneously.

Given a formula φ of some sort X, we add a constant of sort X to our language and require that it satisfies φ. If we simply wish to add a constant, we choose φ to be the true constant so that the new constant automatically satisfies φ. Conversely, if we add a constant of the empty sort, we are not adding anything new, but rather making φ true. Notably, in this case, φ is a sentence and does not depend on any variable.

For the whole chapter, $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is a fixed primary doctrine, unless otherwise specified.

2.1 A comonad on the indexed poset P

Fix an object X in the base category $\mathbb{C}$, and an element $\varphi \in P(X)$.

$$\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \arrow[rr, "(X\times -)^{\mathrm{op}}"] \arrow[dr, "P"', ""{name=L}] & & \mathbb{C}^{\mathrm{op}} \arrow[dl, "P", ""'{name=R}] \\
& \mathbf{Pos} &
\arrow[from=L, to=R, bend left, "\mathfrak{f}", "\cdot"']
\end{tikzcd}$$

Consider the product functor $X \times -\colon \mathbb{C} \to \mathbb{C}$ sending $(A \xrightarrow{f} B)$ to $(X \times A \xrightarrow{\mathrm{id}_X \times f} X \times B)$, and define each component of the natural transformation $\mathfrak{f}\colon P \overset{\cdot}{\to} P \circ (X \times -)^{\mathrm{op}}$ as follows:

$$\mathfrak{f}_A\colon P(A) \to P(X \times A),\ \alpha \mapsto P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha)$$

where pr_1 and pr_2 are the projection from $X \times A$ to X and A respectively.
Note that $\mathfrak{f}$ is monotone, and is indeed a natural transformation: to prove the first part, take $\alpha \leq \alpha'$ in $P(A)$, so $P(\mathrm{pr}_2)(\alpha) \leq P(\mathrm{pr}_2)(\alpha')$, and then $P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha) \leq P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha')$, i.e. $\mathfrak{f}_A(\alpha) \leq \mathfrak{f}_A(\alpha')$; to prove the second one, take $\beta \in P(B)$:

$$\begin{tikzcd}
B & P(B) \arrow[r, "\mathfrak{f}_B"] \arrow[d, "P(f)"] & P(X \times B) \arrow[d, "P(\mathrm{id} \times f)"] \\
A \arrow[u, "f"] & P(A) \arrow[r, "\mathfrak{f}_A"] & P(X \times A)
\end{tikzcd}$$

$$\begin{aligned}
P(\mathrm{id} \times f)\mathfrak{f}_B(\beta) &= P(\mathrm{id} \times f)\left(P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\beta)\right) = P(\mathrm{id} \times f)P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{id} \times f)P(\mathrm{pr}_2)(\beta) \\
&= P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)P(f)(\beta) = \mathfrak{f}_A P(f)(\beta),
\end{aligned}$$

where the projections are from $X \times A$ and $X \times B$.
Then, $(X \times -, \mathfrak{f})$ is a 1-cell between P and itself in the category **IdxPos**.
We now prove that the 1-arrow $(X \times -, \mathfrak{f})$ is part of a 2-comonad on P; to do this, we have to find two 2-arrows $\varepsilon\colon (X \times -, \mathfrak{f}) \overset{\cdot}{\to} \mathrm{id}_P$ and $\gamma\colon (X \times -, \mathfrak{f}) \overset{\cdot}{\to} (X \times -, \mathfrak{f})^2$ satisfying the proper diagrams. We adapt the comonad on the functor $X \times -$ (also known as the *reader comonad*) to indexed posets.
Define $\varepsilon_A\colon X \times A \to A$ to be the second projection $\varepsilon_A := \mathrm{pr}_2$, which is clearly natural, and is indeed a 2-arrow since $\mathfrak{f}_A(\alpha) = P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha) \leq P(\mathrm{pr}_2)(\alpha) = P(\varepsilon_A)(\mathrm{id}_A(\alpha))$ for any $\alpha \in P(A)$. Then, define $\gamma_A := \Delta_X \times \mathrm{id}_A\colon X \times A \to X \times X \times A$, which is again natural; it is a 2-arrow if and only if $\mathfrak{f}_A(\alpha) \leq P(\Delta \times \mathrm{id})(\mathfrak{f}_{X \times A}\mathfrak{f}_A(\alpha))$, however

$$\begin{aligned}
P(\Delta \times \mathrm{id})(\mathfrak{f}_{X \times A}\mathfrak{f}_A(\alpha)) &= P(\langle \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_2 \rangle)\big(P(\mathrm{pr}_1)\varphi \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(P(\mathrm{pr}_1)\varphi \wedge P(\mathrm{pr}_2)\alpha)\big) \\
&= P(\langle \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_2 \rangle)\big(P(\mathrm{pr}_1)\varphi \wedge P(\mathrm{pr}_2)\varphi \wedge P(\mathrm{pr}_3)\alpha\big) \\
&= P(\mathrm{pr}_1)\varphi \wedge P(\mathrm{pr}_2)\alpha = \mathfrak{f}_A(\alpha).
\end{aligned}$$

Finally, we check that the following diagrams commute:

$$\begin{tikzcd}
& X \times - \arrow[dl, equal] \arrow[d, "\gamma"] \arrow[dr, equal] & \\
X \times - & (X \times -)^2 \arrow[l, "\varepsilon_{X \times -}"'] \arrow[r, "\mathrm{id}_X \times \varepsilon"'] & X \times -
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times - \arrow[r, "\gamma"] \arrow[d, "\gamma"] & (X \times -)^2 \arrow[d, "\mathrm{id}_X \times \gamma"] \\
(X \times -)^2 \arrow[r, "\gamma_{X \times -}"] & (X \times -)^3
\end{tikzcd}$$

- $\varepsilon_{X \times A} \circ \gamma_A = \langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle \circ (\Delta_X \times \mathrm{id}_A) = \mathrm{id}_{X \times A}$;
- $(\mathrm{id}_X \times \mathrm{pr}_2) \circ \gamma_A = \mathrm{id}_{X \times A}$;
- $(\mathrm{id}_X \times \Delta_X \times \mathrm{id}_A) \circ (\Delta_X \times \mathrm{id}_A) = \langle \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_2 \rangle = (\Delta_X \times \mathrm{id}_{X \times A}) \circ (\Delta_X \times \mathrm{id}_A)$.

Proposition 2.1.1. With the notation defined above, $(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon)$ is a comonad in **IdxPos**.

Remark 2.1.2. We are interested in finding a distributive law between two different comonads, both of the form seen in Proposition 2.1.1 on the same primary doctrine P seen as an indexed poset: for two objects X, Y and two elements $\varphi \in P(X), \psi \in P(Y)$, the first comonad relies on the 1-cell $(X \times -, \mathfrak{f})$, where $\mathfrak{f} = P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(-)$, while the second one relies on the 1-cell $(Y \times -, \mathfrak{g})$, where $\mathfrak{g} = P(\mathrm{pr}_1)(\psi) \wedge P(\mathrm{pr}_2)(-)$.

Dualizing distributive laws for monads in [Bec69], recall that in general, for two given comonads $(P, (K, \mathfrak{f}), \gamma, \epsilon)$ and $(P, (C, \mathfrak{g}), \lambda, \delta)$ in **IdxPos** on the same indexed poset $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$, a distributive law between two comonads is a 2-cell $\ell\colon (K, \mathfrak{f}) \circ (C, \mathfrak{g}) \to (C, \mathfrak{g}) \circ (K, \mathfrak{f})$ such that $((K, \mathfrak{f}), \ell)$ is a lax morphism of comonads and $((C, \mathfrak{g}), \ell)$ is an oplax morphism of comonads. In details, in **IdxPos**, we ask for $\ell\colon KC \overset{\cdot}{\to} CK$ to be a natural transformation, such that $\mathfrak{f}_{CA}\mathfrak{g}_A(\alpha) \leq P(\ell_A)\mathfrak{g}_{KA}\mathfrak{f}_A(\alpha)$ for any object A in $\mathbb{C}$ and $\alpha \in P(A)$ and such that the following diagram commute:

$$\begin{tikzcd}
KC \arrow[rr, "\ell"] \arrow[d, "K(\lambda)"'] & & CK \arrow[d, "\lambda_K"] \\
KC^2 \arrow[r, "\ell_C"'] & CKC \arrow[r, "C(\ell)"'] & C^2K
\end{tikzcd}
\qquad
\begin{tikzcd}
KC \arrow[rr, "\ell"] \arrow[d, "\gamma_C"'] & & CK \arrow[d, "C(\gamma)"] \\
K^2C \arrow[r, "K(\ell)"'] & KCK \arrow[r, "\ell_K"'] & CK^2
\end{tikzcd}$$

$$\begin{tikzcd}
KC \arrow[rr, "K(\delta)"] \arrow[dr, "\ell"'] & & K \\
& CK \arrow[ur, "\delta_K"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
KC \arrow[rr, "\epsilon_C"] \arrow[dr, "\ell"'] & & C \\
& CK \arrow[ur, "C(\epsilon)"'] &
\end{tikzcd}$$

In our case, define for each object A in $\mathbb{C}$,

$$\ell_A := \langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3 \rangle\colon X \times Y \times A \to Y \times X \times A$$

This is trivially a natural transformation. Moreover, recall that the comultiplication is given by $\Delta \times \mathrm{id}$, and the counit by the projection on the second component. With these definitions, the diagrams clearly commute:

$$\begin{tikzcd}
X \times Y \times A \arrow[rr, "\langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3 \rangle"] \arrow[d, "\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2, \mathrm{pr}_3 \rangle"'] & & Y \times X \times A \arrow[d, "\langle \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \rangle"] \\
X \times Y \times Y \times A \arrow[r, "\langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3, \mathrm{pr}_4 \rangle"'] & Y \times X \times Y \times A \arrow[r, "\langle \mathrm{pr}_1, \mathrm{pr}_3, \mathrm{pr}_2, \mathrm{pr}_4 \rangle"'] & Y \times Y \times X \times A
\end{tikzcd}$$

$$\begin{tikzcd}
X \times Y \times A \arrow[rr, "\langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3 \rangle"] \arrow[d, "\langle \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \rangle"'] & & Y \times X \times A \arrow[d, "\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2, \mathrm{pr}_3 \rangle"] \\
X \times X \times Y \times A \arrow[r, "\langle \mathrm{pr}_1, \mathrm{pr}_3, \mathrm{pr}_2, \mathrm{pr}_4 \rangle"'] & X \times Y \times X \times A \arrow[r, "\langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3, \mathrm{pr}_4 \rangle"'] & Y \times X \times X \times A
\end{tikzcd}$$

$$\begin{tikzcd}
X \times Y \times A \arrow[r, "{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}"] \arrow[d, "{\langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3 \rangle}"'] & X \times A \\
Y \times X \times A \arrow[ur, "{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times Y \times A \arrow[r, "{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}"] \arrow[d, "{\langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3 \rangle}"'] & Y \times A \\
Y \times X \times A \arrow[ur, "{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}"'] &
\end{tikzcd}$$

Now we need to prove that ℓ is indeed a 2-cell, i.e.

$$\begin{tikzcd}
P(A) \arrow[r, "\mathfrak{f}_A"] \arrow[d, "\mathfrak{g}_A"'] & P(X \times A) \arrow[r, "\mathfrak{g}_{KA}"] \arrow[d, phantom, "\leq"] & P(Y \times X \times A) \arrow[d, "P(\ell_A)"] \\
P(Y \times A) \arrow[rr, "\mathfrak{f}_{CA}"'] & & P(X \times Y \times A)
\end{tikzcd}$$

Compute:

$$\begin{aligned}
\mathfrak{f}_{CA}\mathfrak{g}_A(\alpha) &= \mathfrak{f}_{CA}\big(P(\mathrm{pr}_1)\psi \wedge P(\mathrm{pr}_2)\alpha\big) = P(\mathrm{pr}_1)\varphi \wedge P(\mathrm{pr}_2)\psi \wedge P(\mathrm{pr}_3)\alpha \\
&= P(\ell_A)\big(P(\mathrm{pr}_1)\psi \wedge P(\mathrm{pr}_2)\varphi \wedge P(\mathrm{pr}_3)\alpha\big) \\
&= P(\ell_A)\mathfrak{g}_{KA}\big(P(\mathrm{pr}_1)\varphi \wedge P(\mathrm{pr}_2)\alpha\big) = P(\ell_A)\mathfrak{g}_{KA}\mathfrak{f}_A(\alpha).
\end{aligned}$$

In this particular case ℓ is actually an isomorphism.
By looking at the commutative triangles, we observe that ℓ is unique:

$$\begin{aligned}
\mathrm{pr}_1\ell_A &= \mathrm{pr}_1\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle\ell_A = \mathrm{pr}_1\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle = \mathrm{pr}_2; \\
\mathrm{pr}_2\ell_A &= \mathrm{pr}_1\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle\ell_A = \mathrm{pr}_1\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle = \mathrm{pr}_1; \\
\mathrm{pr}_3\ell_A &= \mathrm{pr}_2\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle\ell_A = \mathrm{pr}_2\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle = \mathrm{pr}_3.
\end{aligned}$$

We conclude that the distributive law ℓ induces a composite comonad, having the 1-cell computed as $(X \times -, \mathfrak{f}) \circ (Y \times -, \mathfrak{g}) = (X \times Y \times -, \mathfrak{f} \circ \mathfrak{g})$, where

$$(\mathfrak{f} \circ \mathfrak{g})_A = \mathfrak{f}_{Y \times A}\mathfrak{g}_A \colon P(A) \to P(X \times Y \times A), \quad \alpha \mapsto P(\mathrm{pr}_1)\varphi \wedge P(\mathrm{pr}_2)\psi \wedge P(\mathrm{pr}_3)\alpha.$$

Moreover, the composite comonad induced by the distributive law is again of the form seen in Proposition 2.1.1, defined with respect to the object $X \times Y$ and the element $P(\mathrm{pr}_1)\varphi \wedge P(\mathrm{pr}_2)\psi$ in $P(X \times Y)$.

2.2 The Eilenberg–Moore and the Kleisli construction for the comonad $(X \times -, \mathfrak{f})$

We now study the Eilenberg–Moore and the Kleisli construction of the comonad described before, applying the results shown in Section 1.2 to this particular case. Recall that the Eilenberg–Moore category $\mathbb{C}^{X \times -}$ has as objects pairs (A, c), where $c \colon A \to X \times A$ is an arrow in $\mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & A \arrow[r, "c"] \arrow[d, "c"] \arrow[dl, equal] & X\times A \arrow[d, "\gamma_A=\Delta\times\mathrm{id}"] \\
A & X\times A \arrow[l, "\varepsilon_A=\mathrm{pr}_2"] \arrow[r, "\mathrm{id}\times c"'] & X\times X\times A
\end{tikzcd}$$

so that the second component of $c=\langle c_1,c_2\rangle$ must be the identity $c_2=\mathrm{id}_A$, while the first one can be any map $c_1\colon A\to X$. Moreover, an arrow $f\colon(A,c)\to(B,d)$ in $\mathbb{C}^{X\times -}$ is an arrow $f\colon A\to B$ in $\mathbb{C}$ such that the diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\langle c_1,\mathrm{id}_A\rangle"'] & B \arrow[d, "\langle d_1,\mathrm{id}_B\rangle"] \\
X\times A \arrow[r, "\mathrm{id}\times f"] & X\times B
\end{tikzcd}$$

i.e. we ask for the $\mathbb{C}$-arrow f to satisfy $d_1f=c_1$. From now on, we will write $\mathbb{C}^X$ instead of $\mathbb{C}^{X\times -}$. By looking at the description of objects and arrows of the category $\mathbb{C}^X$, it is easy to observe $\mathbb{C}^X$ is isomorphic to the slice category $\mathbb{C}/X$: they are both categories of coalgebras of the reader comonad $X\times -$.

As seen in Proposition 1.2.3, the induced indexed poset $P^{(X,\varphi)}\colon\mathbb{C}^{X^{\mathrm{op}}}\to\mathbf{Pos}$ is defined as follows:

$$\begin{tikzcd}
 & (B,d) & & \{\beta\in P(B)\mid \beta\le P(d)(\mathfrak{f}_B(\beta))\} \arrow[dd, "P(f)"] \\
\text{For} & & \text{the reindexing is} & \\
 & (A,c) \arrow[uu, "f"] & & \{\alpha\in P(A)\mid \alpha\le P(c)(\mathfrak{f}_A(\alpha))\}
\end{tikzcd}$$

with the order of the subsets given by $P(B)$ and $P(A)$ respectively. Since by definition $\mathfrak{f}_A(\alpha)=P(\mathrm{pr}_1)(\varphi)\wedge P(\mathrm{pr}_2)(\alpha)$, we can write

$$\begin{aligned} P^{(X,\varphi)}(A,c) &= \{\alpha\in P(A)\mid \alpha\le P(\langle c_1,\mathrm{id}\rangle)(P(\mathrm{pr}_1)(\varphi)\wedge P(\mathrm{pr}_2)(\alpha))\} \\ &= \{\alpha\in P(A)\mid \alpha\le P(c_1)(\varphi)\}=P(A)_{\downarrow P(c_1)(\varphi)}. \end{aligned}$$

Note that there is an adjunction in the 2-category of indexed posets between P and $P^{(X,\varphi)}$, that is a pair of 1-arrows $(U^X,\mathfrak{u})\colon P^{(X,\varphi)}\to P$, $(F^X,\mathfrak{f})\colon P\to P^{(X,\varphi)}$ and two 2-arrows $\eta\colon\mathrm{id}\stackrel{\cdot}{\to}F^XU^X$ and $\varepsilon\colon U^XF^K\stackrel{\cdot}{\to}\mathrm{id}$ such that $(\mathbb{C}^X,\mathbb{C},U^X,F^X,\eta,\varepsilon)$ is an adjunction in **Cat**.

- The functor U^X is the forgetful functor from the Eilenberg–Moore category $\mathbb{C}^X$ in $\mathbb{C}$;
- the natural transformation $\mathfrak{u}\colon P^{(X,\varphi)}\stackrel{\cdot}{\to}PU^{X^{\mathrm{op}}}$ is the inclusion on every component

$$\mathfrak{u}_{(A,c)}\colon P^{(X,\varphi)}(A,c)=\{\alpha\in P(A)\mid \alpha\le P(c)(\mathfrak{f}_A(\alpha))\}\hookrightarrow P(A);$$

- the functor F^X is the co-free functor that sends $A\xrightarrow{f}B$ to $(X\times A,\gamma_A)\xrightarrow{\mathrm{id}\times f}(X\times B,\gamma_B)$;
- the natural transformation $\mathfrak{f}'\colon P\stackrel{\cdot}{\to}P^{(X,\varphi)}F^{X^{\mathrm{op}}}$ is defined as $\mathfrak{f}$ on each component—we will write $\mathfrak{f}$ instead of $\mathfrak{f}'$—: indeed, $\mathfrak{f}'{}_A$ has image in $P^{(X,\varphi)}(X\times A,\gamma_A)=P(X\times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$; hence we define $\mathfrak{f}'{}_A=\mathfrak{f}_A\colon P(A)\to P^{(X,\varphi)}(X\times A,\gamma_A)=\{\delta\in P(X\times A)\mid \delta\le P(\mathrm{pr}_1)(\varphi)\}$, where $\alpha\mapsto P(\mathrm{pr}_1)(\varphi)\wedge P(\mathrm{pr}_2)(\alpha)$;

- define the natural transformation $\eta\colon \mathrm{id} \xrightarrow{\cdot} F^X U^X$ to be $\eta_{(A,c)} \coloneqq c$. This is indeed a map from (A,c) to $(X \times A, \gamma_A = \langle \mathrm{pr}_1, \mathrm{id}_{X\times A}\rangle)$, since $\mathrm{pr}_1 c = c_1$; moreover, it is a natural transformation, since the definition of maps in $\mathbb{C}^X$ is exactly the naturality diagram for η. In conclusion, to check that η is a 2-arrow, we prove that for any $\alpha \in P^{(X,\varphi)}(A,c)$, the inequality $\alpha \leq P^{(X,\varphi)}(\eta_{(A,c)})((\mathfrak{f}\circ\mathfrak{u})_{(A,c)}(\alpha))$ holds: take $\alpha \in P(A), \alpha \leq P(c_1)(\varphi)$, then $P(c)((\mathfrak{f}\circ\mathfrak{u})_{(A,c)}(\alpha)) = P(c)(\mathfrak{f}_A\mathfrak{u}_{(A,c)}(\alpha)) = P(c)(P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha)) = P(c_1)(\varphi) \wedge \alpha \geq \alpha$;
- the natural transformation ε is defined as before as the second projection on every component $\varepsilon_A = \mathrm{pr}_2\colon X \times A \to A$, so it is clearly natural and is again a 2-arrow, since the inequality $(\mathfrak{u}\circ\mathfrak{f})_A(\alpha) \leq P(\varepsilon_A)(\mathrm{id}_A(\alpha))$, i.e. $P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha) \leq P(\mathrm{pr}_2)(\alpha)$, trivially holds;
- in conclusion, we check the triangular identities for η and ε, so that $U^X \dashv F^X$ is indeed an adjunction: first of all $\varepsilon_{U^X(A,c)} U^X(\eta_{(A,c)}) = \varepsilon_A c = \mathrm{pr}_2 c = \mathrm{id}_A$ for any co-algebra (A,c) in $\mathbb{C}^K$, and moreover $F^X(\varepsilon_A)\eta_{F^X(A)} = (\mathrm{id} \times \mathrm{pr}_2)(\Delta_X \times \mathrm{id}_A) = \mathrm{id}_{(X\times A,\gamma_A)}$ for any object A of $\mathbb{C}$.

Now, consider the Kleisli category $\mathbb{C}_{X\times -}$, i.e. the full subcategory of $\mathbb{C}^X$ whose objects are the co-free algebras. From now on, we will write $\mathbb{C}_X$ instead on $\mathbb{C}_{X\times -}$. Observe that an arrow $f = \langle f_1, f_2\rangle\colon (X \times A) \to (X \times B)$ has to satisfy $\mathrm{pr}_1 f = \mathrm{pr}_1$, so f_1 must be the first projection pr_1 and the map f is uniquely determined by its second component $f_2\colon X \times A \to B$. For this reason, we use the equivalent description of $\mathbb{C}_X$, that has as objects the same as $\mathbb{C}$, and as map $g\colon A \rightsquigarrow B$ is a $\mathbb{C}$-arrow $g\colon X \times A \to B$—see Remark 1.2.6 for more details; moreover, the composition between two arrows $g\colon A \rightsquigarrow B$ and $h\colon B \rightsquigarrow C$ is the arrow $h\langle \mathrm{pr}_1, g\rangle\colon A \rightsquigarrow C$. A new indexed poset $P_{(X,\varphi)}$ is trivially induced on the Kleisli category by simply taking the restriction of $P^{(X,\varphi)}$ on $\mathbb{C}_X{}^{\mathrm{op}}$, so that $P_{(X,\varphi)}\colon \mathbb{C}_X{}^{\mathrm{op}} \to \mathbf{Pos}$ is defined as follows:

$$\begin{tikzcd} & (X \times B, \gamma_B) & & P(X \times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \arrow[d, "P(\langle \mathrm{pr}_1, g\rangle)"] \\ \text{For} & & \text{the reindexing is} & \\ & (X \times A, \gamma_A) \arrow[uu, "\langle \mathrm{pr}_1, g\rangle"] & & P(X \times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \end{tikzcd}$$

Translating this in the equivalent description of the Kleisli category defined above instead, we can write $P_{(X,\varphi)}(A) \coloneqq P(X \times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$ and, given $g\colon A \rightsquigarrow B$, define $P_{(X,\varphi)}(g) \coloneqq P(\langle \mathrm{pr}_1, g\rangle)$. We can now define in the obvious way the 1-arrow $(U_X, \mathfrak{u})\colon P_{(X,\varphi)} \to P$, the restriction of $(U^X, \mathfrak{u})$, and $(F_X, \mathfrak{f})\colon P \to P_{(X,\varphi)}$ the restriction on the image of $(F^X, \mathfrak{f})$; moreover, this is clearly part of a 2-adjunction between $P_{(X,\varphi)}$ and P, with the same unit and co-unit as before.

2.3 The doctrine $P_{(X,\varphi)}$ and its inherited properties

We now want to study if some properties of P can be translated to $P_{(X,\varphi)}$, and when so, if they are preserved by the 1-arrow $(F_X, \mathfrak{f})$. We will use again the description of $\mathbb{C}_X$ with the same objects as $\mathbb{C}$ and arrows $A \rightsquigarrow B$ that are actually $\mathbb{C}$-arrows $X \times A \to B$; so the functor F_X sends $f\colon A \to B$ to its precomposition with the second projection $f\mathrm{pr}_2\colon A \rightsquigarrow B$. First of all, we check that $\mathbb{C}_X$ has finite products, preserved by F_X, so that both P and $P_{(X,\varphi)}$ are doctrines and $(F_X, \mathfrak{f})$ is a 1-arrow in **Dct**. Then we study a few other properties of $\mathbb{C}_X$ that can be inherited from $\mathbb{C}$; then we will take a look to various properties of P.

$$\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \arrow[rr, "{F_X}^{\mathrm{op}}"] \arrow[dr, "P"', ""{name=L}] & & {\mathbb{C}_X}^{\mathrm{op}} \arrow[dl, "P_{(X,\varphi)}", ""'{name=R}] \\
& \mathbf{Pos} &
\arrow[from=L, to=R, bend left, "\mathfrak{f}", "\cdot"']
\end{tikzcd}$$

We begin by collecting some elementary results regarding the category $\mathbb{C}_X$, as we could not find precise references.

Proposition 2.3.1. Let $\mathbb{C}$ be a category with finite products, and $\mathbb{C}_X$ be the Kleisli category of the comonad $(\mathbb{C}, X \times -, \Delta_X \times \mathrm{id}, \mathrm{pr}_2)$. Then the category $\mathbb{C}_X$ has finite products and the co-free functor F_X preserves them.

Proof. For a given pair of object A, B in $\mathbb{C}$, consider the following diagram in $\mathbb{C}_X$:

$$\begin{tikzcd}
& V \arrow[d, dashed] \arrow[ddl, rightsquigarrow, bend right, "\alpha"'] \arrow[ddr, rightsquigarrow, bend left, "\beta"] & \\
& A \times B \arrow[dl, rightsquigarrow, "\mathrm{pr}_2"] \arrow[dr, rightsquigarrow, "\mathrm{pr}_3"'] & \\
A & & B
\end{tikzcd}$$

where $\alpha\colon X \times V \to A$, $\beta\colon X \times V \to B$ are arrows in $\mathbb{C}$, and the projections are from $X \times A \times B$. Since $\mathbb{C}$ has binary products, there exists a unique $\psi\colon X \times V \to A \times B$ such that

$$\begin{tikzcd}
& X \times V \arrow[d, dashed, "\psi"] \arrow[ddl, bend right, "\alpha"'] \arrow[ddr, bend left, "\beta"] & \\
& A \times B \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\
A & & B
\end{tikzcd}$$

commutes, so that it is easy to check that $\psi\colon V \rightsquigarrow A \times B$ is the unique arrow that makes the product diagram in $\mathbb{C}_X$ commute.

The category $\mathbb{C}_X$ also has a terminal object, it being $\mathbf{t}$, the terminal object of $\mathbb{C}$.

To conclude, observe that F_X trivially preserves finite products, since F_X is part of the adjunction $U_X \dashv F_X$ and right adjoints preserve all limits. ☐

Now that we proved that $P_{(X,\varphi)}$ is indeed a doctrine, we check some structure concerning the base category, that is inherited by $\mathbb{C}_X$.

2.3.1 Closedness

Proposition 2.3.2. Let $\mathbb{C}$ be a category with finite products, and $\mathbb{C}_X$ be the Kleisli category of the comonad $(\mathbb{C}, X \times -, \Delta_X \times \mathrm{id}, \mathrm{pr}_2)$. If the category $\mathbb{C}$ is closed, then $\mathbb{C}_X$ is closed and the co-free functor F_X preserves the exponential.

Proof. Suppose that for any object Y there is a natural bijection

$$\underset{\mathbb{C}}{\mathrm{Hom}}(- \times Y, =) \cong \underset{\mathbb{C}}{\mathrm{Hom}}(-, (=)^Y).$$

We sum up the naturality in the two components with the following diagrams:

$$\begin{array}{c} A \times Y \xrightarrow{f\times \mathrm{id}} B \times Y \xrightarrow{\quad \widetilde{h} \quad} Z \\ \hline A \xrightarrow{\qquad f \qquad} B \xrightarrow{\quad h \quad} Z^Y \end{array} \qquad\qquad \begin{array}{c} B \times Y \xrightarrow{\quad k \quad} Z \xrightarrow{\quad g \quad} S \\ \hline B \xrightarrow{\quad \widehat{k} \quad} Z^Y \xrightarrow{\quad g^Y \quad} S^Y \end{array}$$

Consider the functor $- \times Y \colon \mathbb{C}_X \to \mathbb{C}_X$, that maps $f\colon A \rightsquigarrow B$, i.e. $f\colon X \times A \to B$, to the arrow $f \times \mathrm{id}\colon A \times Y \rightsquigarrow B \times Y$, i.e. $f \times \mathrm{id}\colon X \times A \times Y \to B \times Y$. Such functor is a left adjoint, since for each object A, there exists an object A^Y—which we will prove to be the exponential in $\mathbb{C}$—and a $\mathbb{C}_X$-arrow $\varepsilon_A\colon A^Y \times Y \rightsquigarrow A$, i.e. $\varepsilon_A\colon X \times A^Y \times Y \to A$ such that, for any object B and arrow $f\colon B \times Y \rightsquigarrow A$, there exists a unique $\widehat{f}\colon B \rightsquigarrow A^Y$—which we will prove to be the same hatted arrow in $\mathbb{C}$—such that

$$\begin{tikzcd} B \times Y \arrow[r, squiggly, "\widehat{f}\times \mathrm{id}"] \arrow[rr, squiggly, bend right, "f"'] & A^Y \times Y \arrow[r, squiggly, "\varepsilon_A"] & A \end{tikzcd}$$

Define $\varepsilon_A \coloneqq \widetilde{\mathrm{pr}_2}\colon X \times A^Y \times Y \to A$, the $\mathbb{C}$-map corresponding to $\mathrm{pr}_2\colon X \times A^Y \to A^Y$, so that ε_A is indeed a $\mathbb{C}_X$-map $A^Y \times Y \rightsquigarrow A$. We only have to check that the composition of $\mathbb{C}_X$-arrows above equals to f, i.e. in $\mathbb{C}$

$$\widetilde{\mathrm{pr}_2} \circ \langle \mathrm{pr}_1, \widehat{f} \times \mathrm{id}\rangle = \widetilde{\mathrm{pr}_2} \circ (\langle \mathrm{pr}_1, \widehat{f}\rangle \times \mathrm{id}) = \mathrm{pr}_2 \widetilde{\circ \langle \mathrm{pr}_1}, \widehat{f}\rangle = \widetilde{\widehat{f}} = f$$

At last, to prove the uniqueness of $\widehat{f}$, suppose f' such that $\widetilde{\mathrm{pr}_2} \circ \langle q_1, f' \times \mathrm{id}\rangle = f$, but the left-hand side is equal to $\widetilde{f}'$, so $\widehat{\widetilde{f}'} = \widehat{f}$, i.e. $f' = \widehat{f}$.
To conclude, take a $\mathbb{C}$-arrow $g\colon A \times Y \to B$, and its corresponding map $\widehat{g}\colon A \to B^Y$, we want to prove that $\widehat{F_X(g)} = F_X(\widehat{g})$.

$$F_X(g)\colon X \times A \times Y \xrightarrow{\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle} A \times Y \xrightarrow{g} B$$

$$F_X(\widehat{g})\colon X \times A \xrightarrow{\mathrm{pr}_2} A \xrightarrow{\widehat{g}} B^Y$$

By naturality we have:

$$\begin{array}{c} X\times A\times Y \xrightarrow{\mathrm{pr}_2\times\mathrm{id}} A\times Y \xrightarrow{\quad g\quad} B \\ \hline X\times A \xrightarrow{\quad\mathrm{pr}_2\quad} B \xrightarrow{\quad\widehat{g}\quad} B^Y \end{array}$$

i.e. $\widetilde{\widehat{g}\mathrm{pr}_2} = g(\mathrm{pr}_2\times\mathrm{id}) = g\langle\mathrm{pr}_2,\mathrm{pr}_3\rangle$, so that $\widehat{g}\mathrm{pr}_2 = g\widehat{\langle\mathrm{pr}_2,\mathrm{pr}_3\rangle}$ $\square$

2.3.2 Finite coproducts

Proposition 2.3.3. Let $\mathbb{C}$ be a category with finite products, and $\mathbb{C}_X$ be the Kleisli category of the comonad $(\mathbb{C}, X\times -, \Delta_X\times\mathrm{id}, \mathrm{pr}_2)$. Moreover, suppose that $\mathbb{C}$ has binary coproducts. The endofunctor $X\times -\colon\mathbb{C}\to\mathbb{C}$ preserves binary coproducts if and only if $\mathbb{C}_X$ has binary coproducts preserved by the co-free functor F_X.

Proof. Consider the coproduct diagram in $\mathbb{C}$, its image in $\mathbb{C}_X$ through F_X and a pair of arrows $\alpha\colon X\times A\to V$ and $\beta\colon X\times B\to V$:

$$\begin{tikzcd} A \arrow[dr, "\iota_A"] & & B \arrow[dl, "\iota_B"'] \\ & A+B & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[dr, rightsquigarrow, "\iota_A\mathrm{pr}_2"] \arrow[ddr, rightsquigarrow, bend right, "\alpha"] & & B \arrow[dl, rightsquigarrow, "\iota_B q_2"'] \arrow[ddl, rightsquigarrow, bend left, "\beta"'] \\ & A+B & \\ & V & \end{tikzcd}$$

Define the map $A+B \rightsquigarrow V$ to be the composition $\binom{\alpha}{\beta}\psi\colon X\times(A+B) \underset{\sim}{\xrightarrow{\psi}} (X\times A)+(X\times B) \xrightarrow{\binom{\alpha}{\beta}} V$, where ψ is the inverse of the canonical arrow below:

$$\begin{tikzcd} X\times A \arrow[dr, "\iota_{X\times A}"] & & X\times B \arrow[dl, "\iota_{X\times B}"] \\ & (X\times A)+(X\times B) \arrow[d, shift left, "\langle\binom{\mathrm{pr}_1}{\mathrm{pr}_1},\binom{\iota_A\mathrm{pr}_2}{\iota_B\mathrm{pr}_2}\rangle=\binom{\langle\mathrm{pr}_1,\iota_A\mathrm{pr}_2\rangle}{\langle\mathrm{pr}_1,\iota_B\mathrm{pr}_2\rangle}"] & \\ & X\times(A+B) \arrow[u, bend left, "\psi"] \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"] & \\ X & & (A+B) \end{tikzcd}$$

In particular,

$$\begin{pmatrix}\iota_{X\times A}\\ \iota_{X\times B}\end{pmatrix} = \mathrm{id}_{(X\times A)+(X\times B)} = \psi\begin{pmatrix}\langle\mathrm{pr}_1,\iota_A\mathrm{pr}_2\rangle\\ \langle\mathrm{pr}_1,\iota_B\mathrm{pr}_2\rangle\end{pmatrix} = \begin{pmatrix}\psi\langle\mathrm{pr}_1,\iota_A\mathrm{pr}_2\rangle\\ \psi\langle\mathrm{pr}_1,\iota_B\mathrm{pr}_2\rangle\end{pmatrix}.$$

So the composition $X\times A \xrightarrow{\langle\mathrm{pr}_1,\iota_A\mathrm{pr}_2\rangle} X\times(A+B) \xrightarrow{\binom{\alpha}{\beta}\psi} V$ is equal to $\binom{\alpha}{\beta}\psi\langle\mathrm{pr}_1,\iota_A\mathrm{pr}_2\rangle = \binom{\alpha}{\beta}\iota_{X\times A} = \alpha$; similarly, for β. Hence, $\binom{\alpha}{\beta}\psi$ makes the diagram commute, and it is clearly unique, so $\mathbb{C}_X$ has coproducts, preserved by F_X by constrution.

Conversely, suppose that $\mathbb{C}_X$ has coproducts, preserved by F_X, our claim is that in $\mathbb{C}$, $X\times -$ distribute over $+$. So in $\mathbb{C}_X$ take A, B and their coproduct

$$\begin{tikzcd} A \arrow[r, rightsquigarrow, "\iota_A \mathrm{pr}_2"] & (A+B) & B. \arrow[l, leftsquigarrow, "\iota_B \mathrm{pr}_2"'] \end{tikzcd}$$

Recall that $U_X\colon \mathbb{C}_X \to \mathbb{C}$, that maps $g\colon C \rightsquigarrow D$ to $\langle \mathrm{pr}_1, g\rangle\colon X \times C \to X \times D$ is a left adjoint, so it preserves all colimits, and in particular $X \times (A+B) = (X \times A) + (X \times B)$, as claimed. ☐

Remark 2.3.4. Is it important to observe that the hypothesis about F_X preserving coproducts is necessary for the equivalence just described. Indeed, suppose $\mathbb{C}$ to be a bounded lattice $(R, \wedge, \vee)$, and fix $x \in R$; so $\mathbb{C}_X = \overline{R} \coloneqq (|R|, \sqsubset)$ where $a \sqsubset a'$ if and only if $x \wedge a \leq a'$. The poset $\overline{R}$ has coproducts: indeed, $a \overline{\vee} b \coloneqq (x \wedge a) \vee (x \wedge b)$. Clearly $a \sqsubset a \overline{\vee} b$ and $b \sqsubset a \overline{\vee} b$; moreover, take $a, b \sqsubset y$, i.e. $x \wedge a \leq y$ and $x \wedge b \leq y$, then $a \overline{\vee} b \sqsubset y$ if and only if

$$x \wedge ((x \wedge a) \vee (x \wedge b)) \leq y,$$

which holds, so that $\overline{R}$ has indeed coproducts. However, if $x \wedge -$ does not distribute over $\vee$, coproducts are not preserved—$x \wedge (a \vee b) \neq (x \wedge a) \vee (x \wedge b)$.

Proposition 2.3.5. Let $\mathbb{C}$ be a category with finite products, and $\mathbb{C}_X$ be the Kleisli category of the comonad $(\mathbb{C}, X \times -, \Delta_X \times \mathrm{id}, \mathrm{pr}_2)$. Moreover, suppose that $\mathbb{C}$ has initial object I. The endofunctor $X \times -\colon \mathbb{C} \to \mathbb{C}$ preserves the initial object if and only if $\mathbb{C}_X$ has initial object preserved by the co-free functor F_X.

Proof. Consider any object A, we show that $F_X I = I$ is initial in $\mathbb{C}_X$: we look for a unique arrow $I \rightsquigarrow A$, i.e. a unique arrow $X \times I \to A$, but $X \times I = I$ by assumption. Conversely, suppose that $\mathbb{C}_X$ has initial object, preserved by F_X. Use again that $U_X\colon \mathbb{C}_X \to \mathbb{C}$, that acts $\big(g\colon C \rightsquigarrow D\big) \mapsto \big(\langle \mathrm{pr}_1, g\rangle\colon X \times C \to X \times D\big)$, is a left adjoint, so it preserves all colimits, and in particular $X \times I = I$, as claimed. ☐

We now study the structural properties of the fibers of P that are inherited by $P_{(X,\varphi)}$ and preserved by the morphism $(F_X, \mathfrak{f})$.

2.3.3 Finite meets

Proposition 2.3.6. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. Then $P_{(X,\varphi)}$ is a primary doctrine, and $(F_X, \mathfrak{f})$ is a primary homomorphism.

Proof. Recall that, by assumption, for any object A of $\mathbb{C}$, the poset $P(A)$ has finite meets. We want to check that $P_{(X,\varphi)}(A) = P(X \times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$ has finite meets too: for any two elements $\alpha, \beta \in P_{(X,\varphi)}(A)$, define $\alpha \sqcap \beta \coloneqq \alpha \wedge \beta$. The operation just described is clearly the meet, since the order of $P_{(X,\varphi)}(A)$ is given by its overset $P(X \times A)$, and $\sqcap$ is natural because $\wedge$ is. Moreover, the poset $P_{(X,\varphi)}(A)$ has the top element, which is $1_A \coloneqq P(\mathrm{pr}_1)(\varphi)$, and 1 is again natural.

Take $\mathfrak{f}_A \colon P(A) \to P_{(X,\varphi)}(A) = P(X \times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$. For any $\alpha, \beta \in P(A)$, one has $\mathfrak{f}_A(\alpha \wedge \beta) = P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha \wedge \beta) = P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha) \wedge P(\mathrm{pr}_2)(\beta) = \mathfrak{f}_A(\alpha) \sqcap \mathfrak{f}_A(\beta)$. Moreover, $\mathfrak{f}_A(\top_A) = P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\top_A) = P(\mathrm{pr}_1)(\varphi) \wedge \top_{X \times A} = P(\mathrm{pr}_1)(\varphi) = 1_A$. □

2.3.4 Elementarity

Proposition 2.3.7. Let $P \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P is an elementary doctrine, then $P_{(X,\varphi)}$ is an elementary doctrine, and $(F_X, \mathfrak{f})$ is an elementary homomorphism.

Proof. We already proved that, since P is a primary doctrine, $P_{(X,\varphi)}$ is a primary doctrine too. So now take objects B, C and consider the reindexing

$$P(\mathrm{id}_{X \times C} \times \Delta_B) \colon P(X \times C \times B \times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \to P(X \times C \times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)}.$$

Define its left adjoint $\text{Æ}_{X,\varphi}{}_C^B$ to be the restriction of $\text{Æ}_{X \times C}^B$. Such restriction is well defined: take $\beta \leq P(\mathrm{pr}_1)(\varphi)$ in $P(X \times C \times B)$, one has $\text{Æ}_{X \times C}^B(\beta) \leq P(\mathrm{pr}_1)(\varphi)$ if and only if $\beta \leq P(\mathrm{id}_{X \times C} \times \Delta_B) P(\mathrm{pr}_1)(\varphi) = P(\mathrm{pr}_1)(\varphi)$, which is true by assumption. Naturality in C and Frobenius reciprocity for $\text{Æ}_{X,\varphi}{}_C^B$ come easy from the same properties of $\text{Æ}_{X \times C}^B$.
Consider the following diagram:

$$\begin{tikzcd}
P(C \times B) \arrow[r, "\mathfrak{f}_{C \times B}"] \arrow[d, shift left, "\text{Æ}_C^B"] \arrow[d, phantom, "\vdash"] & P(X \times C \times B)_{\downarrow P(r_1)(\varphi)} \arrow[d, shift left, "\text{Æ}_{X,\varphi}{}_C^B = \text{Æ}_{X \times C}^B"] \arrow[d, phantom, "\vdash"] \\
P(C \times B \times B) \arrow[r, "\mathfrak{f}_{C \times B \times B}"] \arrow[u, shift left, "P(\mathrm{id}_C \times \Delta_B)"] & P(X \times C \times B \times B)_{\downarrow P(t_1)(\varphi)} \arrow[u, shift left, "P(\mathrm{id}_{X \times C} \times \Delta_B)"]
\end{tikzcd}$$

We want to prove that the square with arrows pointing down and right is commutative. To do this, recall that the following diagram is commutative because of the naturality of Æ^B:

$$\begin{tikzcd}
P(C \times B) \arrow[r, "\text{Æ}_C^B"] \arrow[d, "P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)"] & P(C \times B \times B) \arrow[d, "P(\langle \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_4 \rangle)"] \\
P(X \times C \times B) \arrow[r, "\text{Æ}_{X \times C}^B"] & P(X \times C \times B \times B)
\end{tikzcd}$$

So now take $\beta \in P(C \times B)$:

$$\begin{aligned}
\mathfrak{f}_{C \times B \times B} \text{Æ}_C^B(\beta) &= P(\mathrm{pr}_1)(\varphi) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_4 \rangle)(\text{Æ}_C^B(\beta)) \\
&= P(t_1)(\varphi) \wedge \text{Æ}_{X \times C}^B P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\beta) \\
&= \text{Æ}_{X \times C}^B (P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\beta) \wedge P(\mathrm{id}_{X \times C} \times \Delta_B) P(\mathrm{pr}_1)(\varphi)) \\
&= \text{Æ}_{X \times C}^B (P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\beta) \wedge P(\mathrm{pr}_1)(\varphi)) = \text{Æ}_C^B \mathfrak{f}_{X \times C}(\beta),
\end{aligned}$$

which proves our claim. □

2.3.5 Existential quantifier

Proposition 2.3.8. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P is an existential doctrine, then $P_{(X,\varphi)}$ is an existential doctrine, and $(F_X, \mathfrak{f})$ is an existential homomorphism.

Proof. We already proved that, since P is primary, $P_{(X,\varphi)}$ is primary as well. So now take objects B, C and consider $P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)\colon P(X \times C)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \to P(X \times C \times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$, and define its left adjoint $\exists_{X,\varphi}{}^B_C$ to be the restriction of $\exists^B_{X\times C}$. Such restriction is well defined: take $\beta \leq P(\mathrm{pr}_1)(\varphi)$ in $P(X \times C \times B)$, one has $\exists^B_{X\times C}(\beta) \leq P(\mathrm{pr}_1)(\varphi)$ if and only if $\beta \leq P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)P(\mathrm{pr}_1)(\varphi) = P(\mathrm{pr}_1)(\varphi)$, which is true by assumption. Naturality in C and Frobenius reciprocity for $\exists_{X,\varphi}{}^B_C$ come easy from the same properties of $\exists^B_{X\times C}$.
Consider the following diagram:

$$\begin{tikzcd}
P(C \times B) \arrow[r, "\mathfrak{f}_{C\times B}"] \arrow[d, shift left=1.5ex, "\exists^B_C"] \arrow[d, phantom, "\vdash"] & P(X \times C \times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \arrow[d, shift left=1.5ex, "\exists_{X,\varphi}{}^B_C = \exists^B_{X\times C}"] \arrow[d, phantom, "\vdash"] \\
P(C) \arrow[u, shift left=1.5ex, "P(\mathrm{pr}_1)"] \arrow[r, "\mathfrak{f}_C"] & P(X \times C)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \arrow[u, shift left=1.5ex, "P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)"]
\end{tikzcd}$$

We want to prove that the square with arrows pointing down and right is commutative. To do this, recall that the following diagram is commutative because of the naturality of $\exists^B$:

$$\begin{tikzcd}
P(C \times B) \arrow[r, "\exists^B_C"] \arrow[d, "P(\mathrm{pr}_2 \times \mathrm{id}_B)"] & P(C) \arrow[d, "P(\mathrm{pr}_2)"] \\
P(X \times C \times B) \arrow[r, "\exists^B_{X\times C}"] & P(X \times C)
\end{tikzcd}$$

So now take $\beta \in P(C \times B)$:

$$\begin{aligned}
\mathfrak{f}_C \exists^B_C(\beta) &= P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\exists^B_C(\beta)) \\
&= P(\mathrm{pr}_1)(\varphi) \wedge \exists^B_{X\times C} P(\mathrm{pr}_2 \times \mathrm{id})(\beta) \\
&= \exists^B_{X\times C}(P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)(\beta) \wedge P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)P(\mathrm{pr}_1)(\varphi)) = \exists^B_{X\times C}\mathfrak{f}_{X\times C}(\beta),
\end{aligned}$$

which proves our claim. ☐

2.3.6 Universal quantifier

Proposition 2.3.9. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P is a universal doctrine, then $P_{(X,\varphi)}$ is a universal doctrine, and $(F_X, \mathfrak{f})$ is a universal homomorphism.
Additionally, if the universal quantifier of P satisfies the Frobenius reciprocity, then also the universal quantifier of $P_{(X,\varphi)}$ does.

Proof. Take a pair of objects B, C and consider

$$P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)\colon P(X \times C)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \to P(X \times C \times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)}.$$

Define its right adjoint ${\forall_{X,\varphi}}^B_C(-) \coloneqq \forall^B_{X\times C}(-) \wedge P(\mathrm{pr}_1)(\varphi)$. To check that this yields indeed an adjunction, take $\gamma \leq P(\mathrm{pr}_1)(\varphi)$ in $P(X \times C)$ and $\beta \leq P(\mathrm{pr}_1)(\varphi)$ in $P(X \times C \times B)$, we want to prove that $P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)(\gamma) \leq \beta$ if and only if $\gamma \leq {\forall_{X,\varphi}}^B_C(\beta) = \forall^B_{X\times C}(\beta) \wedge P(\mathrm{pr}_1)(\varphi)$. First of all, suppose $P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)(\gamma) \leq \beta$, then it follows from the adjunction $P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle) \dashv \forall^B_{X\times C}(\beta)$ that $\gamma \leq \forall^B_{X\times C}(\beta)$; combining this with the assumption on γ, the inequality $\gamma \leq {\forall_{X,\varphi}}^B_C(\beta)$ holds. Conversely, suppose $\gamma \leq \forall^B_{X\times C}(\beta) \wedge P(\mathrm{pr}_1)(\varphi) \leq \forall^B_{X\times C}(\beta)$, then the claim holds again because of the adjunction.

We now want to prove the naturality of ${\forall_{X,\varphi}}^B$. Take an arrow $f_2\colon C' \rightsquigarrow C$, i.e. $f_2\colon X \times C' \to C$, and write $f \coloneqq \langle \mathrm{pr}_1, f_2\rangle\colon X \times C' \to X \times C$. Recall the naturality diagram for $\forall^B$:

$$\begin{tikzcd}
P(X \times C \times B) \arrow[r, "\forall^B_{X\times C}"] \arrow[d, "P(f\times \mathrm{id}_B)"] & P(X \times C) \arrow[d, "P(f)"] \\
P(X \times C' \times B) \arrow[r, "\forall^B_{X\times C'}"] & P(X \times C')
\end{tikzcd}$$

and use it to prove the naturality for ${\forall_{X,\varphi}}^B$:

$$\begin{tikzcd}
P(X \times C \times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \arrow[r, "{\forall_{X,\varphi}}^B_C"] \arrow[d, "P(f\times \mathrm{id}_B)"] & P(X \times C)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \arrow[d, "P(f)"] \\
P(X \times C' \times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \arrow[r, "{\forall_{X,\varphi}}^B_{C'}"] & P(X \times C')_{\downarrow P(\mathrm{pr}_1)(\varphi)}
\end{tikzcd}$$

So take $\beta \in P(X \times C \times B)$:

$$\begin{aligned}
P(f){\forall_{X,\varphi}}^B_C(\beta) &= P(f)(\forall^B_{X\times C}(\beta) \wedge P(\mathrm{pr}_1)(\varphi)) = P(f)\forall^B_{X\times C}(\beta) \wedge P(\mathrm{pr}_1)(\varphi) \\
&= \forall^B_{X\times C'} P(f \times \mathrm{id})(\beta) \wedge P(\mathrm{pr}_1)(\varphi) = {\forall_{X,\varphi}}^B_{C'} P(f \times \mathrm{id})(\beta).
\end{aligned}$$

It is worth mentioning that, if we ask in addition that the doctrine P satisfies Frobenius reciprocity for the adjunction $P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle) \dashv \forall^B_{X\times C}(\beta)$, then also the doctrine $P_{(X,\varphi)}$ satisfies Frobenius for the adjunction $P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle) \dashv {\forall_{X,\varphi}}^B_C$: for any $\gamma \leq P(\mathrm{pr}_1)(\varphi)$ and $\beta \leq P(\mathrm{pr}_1)(\varphi)$,

$$\begin{aligned}
P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)(\gamma \wedge {\forall_{X,\varphi}}^B_C(\beta)) &= P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)(\gamma \wedge \forall^B_{X\times C}(\beta) \wedge P(\mathrm{pr}_1)(\varphi)) \\
&= P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)(\gamma) \wedge \beta,
\end{aligned}$$

using $\gamma \wedge P(\mathrm{pr}_1)\varphi = \gamma$ and Frobenius reciprocity.

Consider the following diagram:

$$\begin{tikzcd}
P(C \times B) \arrow[r, "\mathfrak{f}_{C\times B}"] \arrow[d, shift left=1.5ex, "\forall_C^B"] & P(X \times C \times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \arrow[d, shift left=1.5ex, "{\forall_{X,\varphi}}_C^B"] \\
P(C) \arrow[u, shift left=1.5ex, "P(\mathrm{pr}_1)"] \arrow[u, phantom, shift left=0.75ex, "\dashv"] \arrow[r, "\mathfrak{f}_C"] & P(X \times C)_{\downarrow P(s_1)(\varphi)} \arrow[u, shift left=1.5ex, "P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)"] \arrow[u, phantom, shift left=0.75ex, "\dashv"]
\end{tikzcd}$$

We want to prove that the square with arrows pointing down and right is commutative. To do this, recall that the following diagram is commutative because of the naturality of $\forall^B$:

$$\begin{tikzcd}
P(C \times B) \arrow[r, "\forall_C^B"] \arrow[d, "P(\mathrm{pr}_2 \times \mathrm{id}_B)"] & P(C) \arrow[d, "P(\mathrm{pr}_2)"] \\
P(X \times C \times B) \arrow[r, "\forall_{X\times C}^B"] & P(X \times C)
\end{tikzcd}$$

So now take $\beta \in P(C \times B)$:

$$\begin{aligned}
{\forall_{X,\varphi}}_C^B \mathfrak{f}_{C\times B}(\beta) &= {\forall_{X,\varphi}}_C^B (P(\mathrm{pr}_1)(\varphi) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\beta)) \\
&= \forall_{X\times C}^B (P(\mathrm{pr}_1)(\varphi) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\beta)) \wedge P(\mathrm{pr}_1)(\varphi).
\end{aligned}$$

On the other hand

$$\mathfrak{f}_C \forall_C^B(\beta) = P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\forall_C^B(\beta)).$$

To prove ${\forall_{X,\varphi}}_C^B \mathfrak{f}_{C\times B}(\beta) \leq \mathfrak{f}_C \forall_C^B(\beta)$, note that if holds if and only if—since it is trivially smaller than $P(\mathrm{pr}_1)(\varphi)$—

$${\forall_{X,\varphi}}_C^B \mathfrak{f}_{C\times B}(\beta) \leq P(\mathrm{pr}_2) \forall_C^B(\beta)$$

but by naturality $P(\mathrm{pr}_2)\forall_C^B = \forall_{X\times C}^B P(\mathrm{pr}_2 \times \mathrm{id})$ and moreover $P(\mathrm{pr}_1)(\varphi) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\beta) = P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2 \times \mathrm{id})(\beta) \leq P(\mathrm{pr}_2 \times \mathrm{id})(\beta)$, so applying $\forall_{X\times C}^B$ to both sides of the last inequality the claim follows.

Conversely, $\mathfrak{f}_C \forall_C^B(\beta) \leq {\forall_{X,\varphi}}_C^B \mathfrak{f}_{C\times B}(\beta)$, if and only if $P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle) \mathfrak{f}_C \forall_C^B(\beta) \leq \mathfrak{f}_{C\times B}(\beta)$ but $P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle) = P_{(X,\varphi)} F_X(\mathrm{pr}_1)$, so equivalently $\mathfrak{f}_{C\times B} P(\mathrm{pr}_1) \forall_C^B(\beta) \leq \mathfrak{f}_{C\times B}(\beta)$. This proves the claim, by applying $\mathfrak{f}_{C\times B}$ to $P(\mathrm{pr}_1)\forall_C^B(\beta) \leq \beta$—which is the counit of the adjunction. □

2.3.7 Implication

Proposition 2.3.10. Let $P \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P is an implicational doctrine, then $P_{(X,\varphi)}$ is an implicational doctrine, and $(F_X, \mathfrak{f})$ is an implicational homomorphism.

Proof. Since we already know that $P_{(X,\varphi)}$ is primary, we check that $P_{(X,\varphi)}(A) = P(X \times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$ is cartesian closed too: for any $\beta, \gamma \in P_{(X,\varphi)}(A)$, define $\beta \Rightarrow \gamma \coloneqq (\beta \to \gamma) \wedge P(\mathrm{pr}_1)(\varphi)$.

This is indeed a natural transformation: take $f_2 \colon A \rightsquigarrow B$, i.e. $f_2 \colon X \times A \to B$, and write for convenience $f \coloneqq \langle \mathrm{pr}_1, f_2 \rangle \colon X \times A \to X \times B$.

$$\begin{tikzcd}
{P(X\times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)}}^{\mathrm{op}} \times P(X\times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \arrow[r, "\Rightarrow"] \arrow[d, "P(f)\times P(f)"] & P(X\times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \arrow[d, "P(f)"] \\
{P(X\times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}}^{\mathrm{op}} \times P(X\times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)} \arrow[r, "\Rightarrow"] & P(X\times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}
\end{tikzcd}$$

So, take a pair $\alpha, \alpha' \in P(X\times B)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$: on the one hand it is sent to $P(f)(\alpha \Rightarrow \alpha') = P(f)((\alpha \to \alpha') \wedge P(\mathrm{pr}_1)(\varphi)) = (P(f)(\alpha) \to P(f)(\alpha')) \wedge P(\mathrm{pr}_1)(\varphi)$; on the other hand to $P(f)(\alpha) \Rightarrow P(f)(\alpha') = (P(f)(\alpha) \to P(f)(\alpha')) \wedge P(\mathrm{pr}_1)(\varphi)$, so that $\Rightarrow$ is indeed a natural transformation.

Now, to check that $P_{(X,\varphi)}(A)$ endowed with this operation is cartesian closed, take three elements $\alpha, \beta, \gamma \in P(X\times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$, and we prove that $\alpha \wedge \beta \leq \gamma$ if and only if $\alpha \leq \beta \Rightarrow \gamma$. So, suppose $\alpha \wedge \beta \leq \gamma$, then from $\alpha \leq \beta \to \gamma$ combined with the assumption on α we obtain $\alpha \leq (\beta \to \gamma) \wedge P(\mathrm{pr}_1)(\gamma)$. Conversely, from $\alpha \leq (\beta \to \gamma) \wedge P(\mathrm{pr}_1)(\gamma) \leq \beta \to \gamma$, it follows that $\alpha \wedge \beta \leq \gamma$.

Take $\mathfrak{f}_A\colon P(A) \to P_{(X,\varphi)}(A) = P(X\times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$. For any $\alpha, \beta \in P(A)$, one has on the one side $\mathfrak{f}_A(\alpha \to \beta) = P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha \to \beta)$, and on the other hand $\mathfrak{f}_A(\alpha) \Rightarrow \mathfrak{f}_A(\beta) = (P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha)) \Rightarrow (P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\beta))$. So now we prove that in any cartesian closed poset,

$$((x \wedge a) \to (x \wedge b)) \wedge x = x \wedge (a \to b)$$

First of all, $x \wedge (a \to b) \leq ((x \wedge a) \to (x \wedge b)) \wedge x$ if and only if $x \wedge (a \to b) \leq (x \wedge a) \to (x \wedge b)$ if and only if $x \wedge (a \to b) \wedge x \wedge a \leq x \wedge b$.

Conversely, $((x \wedge a) \to (x \wedge b)) \wedge x \leq x \wedge (a \to b)$ if and only if $((x \wedge a) \to (x \wedge b)) \wedge x \leq a \to b$ if and only if $((x \wedge a) \to (x \wedge b)) \wedge x \wedge a \leq b$, but $((x \wedge a) \to (x \wedge b)) \wedge x \wedge a \leq x \wedge b \leq b$. ☐

Corollary 2.3.11. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X\times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P is a Horn doctrine, then $P_{(X,\varphi)}$ is a Horn doctrine, and $(F_X, \mathfrak{f})$ is a Horn homomorphism.

Proof. Both universal and implicational structure is preserved by the construction. ☐

2.3.8 Finite joins

Proposition 2.3.12. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X\times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P is bounded, then $P_{(X,\varphi)}$ is bounded, and $(F_X, \mathfrak{f})$ preserves the bottom element.

Proof. The poset $P_{(X,\varphi)}(A)$ has bottom element, which is $\mathsf{0}_A \coloneqq \bot_{X\times A}$, and $\mathsf{0}$ is natural.

Take $\mathfrak{f}_A\colon P(A) \to P_{(X,\varphi)}(A) = P(X\times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$. Compute $\mathfrak{f}_A(\bot_A) = P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\bot_A) = \bot_{X\times A} = \mathsf{0}_A$, so the bottom element is preserved. ☐

Proposition 2.3.13. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X\times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P has binary joins, then $P_{(X,\varphi)}$ has binary joins. If each fiber of P is a distributive lattice, then $(F_X, \mathfrak{f})$ preserves binary joins.

Proof. To check that $P_{(X,\varphi)}(A) = P(X \times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$ has binary joins, take any two elements $\alpha, \beta \in P_{(X,\varphi)}(A)$, and let $\alpha \sqcup \beta$ be $\alpha \vee \beta$. The operation is well defined and is clearly the join, since the order of $P_{(X,\varphi)}(A)$ is given by its overset $P(X \times A)$, and $\sqcup$ is natural because $\vee$ is.
Now, for any two elements $\alpha, \beta \in P(A)$, one has $\mathfrak{f}_A(\alpha \vee \beta) = P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha \vee \beta) = P(\mathrm{pr}_1)(\varphi) \wedge (P(\mathrm{pr}_2)(\alpha) \vee P(\mathrm{pr}_2)(\beta))$
On the other hand, $\mathfrak{f}_A(\alpha) \sqcup \mathfrak{f}_A(\beta) = (P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha)) \vee (P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\beta))$; in general this is not equal to $\mathfrak{f}_A(\alpha \vee \beta)$, computed above. However, the equality holds if we ask for $P(\mathrm{pr}_1)(\varphi) \wedge (-)$ to preserve joins, e.g. whenever the lattice is distributive. □

Corollary 2.3.14. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P is a Heyting doctrine, then $P_{(X,\varphi)}$ is a Heyting doctrine, and $(F_X, \mathfrak{f})$ is a Heyting homomorphism.

Proof. Finite meets, finite joins and implication are preserved by the construction. □

2.3.9 Booleanness

Proposition 2.3.15. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P is a Boolean doctrine, then $P_{(X,\varphi)}$ is a Boolean doctrine, and $(F_X, \mathfrak{f})$ is a Boolean homomorphism.

Proof. We want to check that $P_{(X,\varphi)}(A) = P(X \times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$ is a boolean algebra: for any element $\alpha \in P_{(X,\varphi)}(A)$, define $\urcorner\alpha := \alpha \Rightarrow \bot = (\alpha \to \bot) \wedge P(\mathrm{pr}_1)(\varphi) = \neg\alpha \wedge P(\mathrm{pr}_1)(\varphi)$. Since we already know that $P_{(X,\varphi)}(A)$ is a Heyting algebra, we only have to prove that $\urcorner\urcorner\alpha = \alpha$:

$$\begin{aligned}\urcorner\urcorner\alpha =& \urcorner(\neg\alpha \wedge P(\mathrm{pr}_1)(\varphi)) = \neg(\neg\alpha \wedge P(\mathrm{pr}_1)(\varphi)) \wedge P(\mathrm{pr}_1)(\varphi)\\ &= (\neg\neg\alpha \vee \neg P(\mathrm{pr}_1)(\varphi)) \wedge P(\mathrm{pr}_1)(\varphi) = (\alpha \vee P(\mathrm{pr}_1)(\neg\varphi)) \wedge P(\mathrm{pr}_1)(\varphi)\\ &= (\alpha \wedge P(\mathrm{pr}_1)(\varphi)) \vee P(\mathrm{pr}_1)(\neg\varphi \wedge \varphi) = \alpha \vee \bot = \alpha.\end{aligned}$$

To conclude, $(F_X, \mathfrak{f})$ is Boolean since the Heyting structure is preserved by $\mathfrak{f}$. □

2.3.10 Variations on negation

There are other ways to introduce negation in the context of inf-semilattices. Here we describe two examples and check that properties are again preserved.

Definition 2.3.16. A primary doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is $*$-*autonomous* if for every object A, the poset $P(A)$ is $*$-autonomous, that is: $P(A)$ is cartesian, endowed with operation $\neg$ such that

$\neg\neg a = a$ for every $a \in P(A)$ and such that $a \wedge b \leq \neg c$ if and only if $a \leq \neg(b \wedge c)$. Moreover the operation $\neg\colon P^{\mathrm{op}} \to P$ yields a natural transformation.
A primary doctrine homomorphism between two $*$-autonomous doctrines is *$*$-autonomous* if it preserves the negation.

Proposition 2.3.17. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P is a $*$-autonomous doctrine, then $P_{(X,\varphi)}$ is a $*$-autonomous doctrine, and $(F_X, \mathfrak{f})$ is a $*$-autonomous homomorphism.

Proof. For any $\alpha \in P_{(X,\varphi)}(A) = P(X \times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$, define $\rceil\alpha := \neg\alpha \wedge P(\mathrm{pr}_1)(\varphi)$. The operation extended on each fiber $\rceil\colon {P_{(X,\varphi)}}^{\mathrm{op}} \to P_{(X,\varphi)}$ is trivially a natural transformation. We call $x = P(\mathrm{pr}_1)(\varphi)$ for simplicity, and we prove $\rceil\rceil\alpha = \alpha$.

$$\alpha \leq \rceil\rceil\alpha \text{ if and only if } \alpha \leq \rceil(\neg\alpha \wedge\ x) \text{ if and only if } \alpha \leq \neg(\neg\alpha \wedge\ x) \wedge x$$
$$\text{if and only if } \alpha \leq \neg(\neg\alpha \wedge x) \text{ if and only if } \alpha \wedge x \leq \neg\neg\alpha = \alpha.$$

Conversely,

$$\rceil\rceil\alpha \leq \alpha \text{ if and only if } \neg(\neg\alpha \wedge\ x) \wedge x \leq \neg\neg\alpha \text{ if and only if } \neg(\neg\alpha \wedge\ x) \leq \neg(x \wedge \neg\alpha).$$

Now, to prove the equivalence, take $\alpha, \beta, \gamma \leq x$, then

$$\alpha \wedge \beta \leq \rceil\gamma \text{ if and only if } \alpha \wedge \beta \leq \neg\gamma \wedge x \text{ if and only if } \alpha \wedge \beta \leq \neg\gamma$$
$$\text{if and only if } \alpha \leq \neg(\beta \wedge \gamma) \text{ if and only if } \alpha \leq \neg(\beta \wedge \gamma) \wedge x = \rceil(\beta \wedge \gamma).$$

To conclude, we prove that $\mathfrak{f}_A\colon P(A) \to P_{(X,\varphi)}(A)$ preserves the negation. On the one hand $\mathfrak{f}_A(\neg\alpha) = P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\neg\alpha) = P(\mathrm{pr}_1)(\varphi) \wedge \neg P(\mathrm{pr}_2)(\alpha)$ and, on the other hand $\rceil\mathfrak{f}_A(\alpha) = \neg\mathfrak{f}_A(\alpha) \wedge P(\mathrm{pr}_1)(\varphi) = \neg(P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha)) \wedge P(\mathrm{pr}_1)(\varphi)$. To see this, it is enough to check that in a $*$-autonomous inf-semilattice we have $x \wedge \neg a = \neg(x \wedge a) \wedge x$, for any a, x. First of all,

$$x \wedge \neg a \leq \neg(x \wedge a) \wedge x \text{ if and only if } x \wedge \neg a \leq \neg(x \wedge a) \text{ if and only if } x \wedge \neg a \wedge x \leq \neg a;$$

conversely,

$$\neg(x \wedge a) \wedge x \leq x \wedge \neg a \text{ if and only if } \neg(x \wedge a) \wedge x \leq \neg a \text{ if and only if } \neg(x \wedge a) \leq \neg(x \wedge a). \quad \square$$

Definition 2.3.18. A primary doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ *has pseudo-complements* if for every object A, the poset $P(A)$ has pseudo-complements, that is: the poset $P(A)$ is cartesian, endowed with an operation $\neg$ and a bottom element $\bot$, where $\neg a = \max\{b \mid a \wedge b = \bot\}$; moreover the operations $\bot\colon \mathbf{1} \to P$, $\neg\colon P^{\mathrm{op}} \to P$ yield natural transformations .

A primary doctrine homomorphism between two doctrines with pseudo-complements *preserves pseudo-complements* if it is bounded and preserves the negation.

Proposition 2.3.19. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P has pseudo-complements, then $P_{(X,\varphi)}$ has pseudo-complements, and $(F_X, \mathfrak{f})$ preserves pseudo-complements.

Proof. For any $\alpha \in P_{(X,\varphi)}(A) = P(X \times A)_{\downarrow P(\mathrm{pr}_1)(\varphi)}$, define $\urcorner\alpha := \neg\alpha \wedge P(\mathrm{pr}_1)(\varphi)$; clearly $\urcorner\colon {P_{(X,\varphi)}}^{\mathrm{op}} \to P_{(X,\varphi)}$ is a natural transformation. First of all we observe that $\alpha \wedge \urcorner\alpha = \bot$, since $\alpha \wedge \neg\alpha \wedge P(\mathrm{pr}_1)(\varphi) = \bot$; then, suppose $\beta \leq P(\mathrm{pr}_1)(\varphi)$ such that $\alpha \wedge \beta = \bot$, but from $\alpha \wedge \beta = \bot$, $\beta \leq \neg\alpha$ follows in $P(X \times A)$, so $\beta \leq \neg\alpha \wedge P(\mathrm{pr}_1)(\varphi) = \urcorner\alpha$, hence $\urcorner\alpha = \max\{\beta \mid \alpha \wedge \beta = \bot\}$.
To conclude, we prove that $\mathfrak{f}_A\colon P(A) \to P_{(X,\varphi)}(A)$ preserves the negation. Take $\mathfrak{f}_A(\neg\alpha)$ and $\urcorner\mathfrak{f}_A(\alpha)$ computed as above—in the $*$-autonomous case—, so again we check that $x \wedge \neg a = \neg(x \wedge a) \wedge x$ for any a, x in a pseudo-complemented poset. First of all,

$$x \wedge \neg a \leq \neg(x \wedge a) \wedge x \text{ iff } x \wedge \neg a \leq \neg(x \wedge a), \text{ so it is sufficient } (x \wedge \neg a) \wedge (x \wedge a) = \bot.$$

Conversely,

$$\neg(x \wedge a) \wedge x \leq x \wedge \neg a \text{ iff } \neg(x \wedge a) \wedge x \leq \neg a, \text{ so it is sufficient } (\neg(x \wedge a) \wedge x) \wedge a = \bot. \qquad \square$$

2.3.11 Weak Power Objects

Recall from definition 4.9 in [Pas15] that a doctrine P has weak power objects if for every object A in the base category $\mathbb{C}$, there exists an object $\mathsf{P}(A)$ and an element $\in_A \in P(A \times \mathsf{P}(A))$ such that for any object B and $\phi \in P(A \times B)$ there exists an arrow $\{\phi\}\colon B \to \mathsf{P}(A)$ such that $\phi = P(\mathrm{id}_A \times \{\phi\})(\in_A)$.

Proposition 2.3.20. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and $P_{(X,\varphi)}$ be the Kleisli object of the comonad $(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon)$ defined by the pair $X \in \mathbb{C}$ and $\varphi \in P(X)$. If P is has weak power objects, then $P_{(X,\varphi)}$ has weak power objects.

Proof. Since $\mathbb{C}_X$ has the same objects as $\mathbb{C}$, for any object A consider $\mathsf{P}(A)$ and the element $\mathfrak{f}_{A\times\mathsf{P}(A)}(\in_A) \in P_{(X,\varphi)}(A \times \mathsf{P}(A))$, i.e. $P(\mathrm{pr}_1)\varphi \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)(\in_A) \in P(X \times A \times \mathsf{P}(A))_{\downarrow P(\mathrm{pr}_1)\varphi}$. We want to prove that $(\mathsf{P}(A), \mathfrak{f}_{A\times\mathsf{P}(A)}(\in_A))$ is a weak power object of A in the doctrine $P_{(X,\varphi)}$. To see this, we take any object C and $\psi \in P_{(X,\varphi)}(A \times C) = P(X \times A \times C)_{\downarrow P(\mathrm{pr}_1)\varphi}$ and look for an arrow $[\psi]\colon C \rightsquigarrow \mathsf{P}(A)$—i.e. a $\mathbb{C}$-arrow $X \times C \to \mathsf{P}(A)$—such that

$$\psi = P_{(X,\varphi)}(\mathrm{id}_A \times_{\mathbb{C}_X} [\psi])\mathfrak{f}_{A\times\mathsf{P}(A)}(\in_A).$$

Here, the product $\mathrm{id}_A \times_{\mathbb{C}_X} [\psi]$ is computed in $\mathbb{C}_X$, hence it is actually the $\mathbb{C}$-arrow

$$\langle \mathrm{pr}_2, [\psi]\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle\rangle\colon X \times A \times C \to A \times \mathsf{P}(A).$$

Using the fact that P has weak power object, we can take the object $X \times C$ and the element $P(\langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\psi \in P(A \times X \times C)$ and we know that there exists

$$[\psi] \coloneqq \{P(\langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\psi\} \colon X \times C \to \mathsf{P}(A)$$

such that

$$P(\langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\psi = P\left(\mathrm{id}_A \times \{P(\langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\psi\}\right)(\in_A) = P\left(\langle \mathrm{pr}_1, [\psi]\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle \rangle\right)(\in_A).$$

Now compute

$$\begin{aligned} P_{(X,\varphi)}(\mathrm{id}_A \times_{\mathbb{C}_X} [\psi])\mathfrak{f}_{A \times \mathsf{P}(A)}(\in_A) &= P(\langle \mathrm{pr}_1, \mathrm{pr}_2, [\psi]\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle \rangle)\big(P(\mathrm{pr}_1)\varphi \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)(\in_A)\big) \\ &= P(\mathrm{pr}_1)\varphi \wedge P\left(\langle \mathrm{pr}_2, [\psi]\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle \rangle\right)(\in_A) \end{aligned}$$

which is equal to ψ if and only if $P(\langle \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\psi = P(\mathrm{pr}_2)\varphi \wedge P\left(\langle \mathrm{pr}_1, [\psi]\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle \rangle\right)(\in_A)$, but this is true following from the definition of $[\psi]$ and the fact that $\psi \leq P(\mathrm{pr}_1)\varphi$. ☐

2.4 Universal properties of $P_{(X,\varphi)}$

Consider the following diagram for a primary doctrine P. In particular, $P_{(X,\varphi)}$ is primary too.

$$\begin{tikzcd} \mathbb{C}^{\mathrm{op}} \arrow[rr, "{F_X}^{\mathrm{op}}"] \arrow[rr, phantom, "\underset{\cdot}{\overset{\mathfrak{f}}{\longrightarrow}}"', shift right=3] \arrow[dr, "P"'] & & {\mathbb{C}_X}^{\mathrm{op}} \arrow[dl, "P_{(X,\varphi)}"] \\ & \mathbf{Pos} & \end{tikzcd}$$

We can interpret this 1-arrow as follows: we are adding a constant of sort X to the theory P, and making this constant verify φ. Indeed, take $\mathfrak{f}_X(\varphi) \in P_{(X,\varphi)}(X)$, which is the interpretation of φ in $P_{(X,\varphi)}$, and consider the constant $\mathrm{pr}_1 \colon \mathbf{t} \rightsquigarrow X$ in $\mathbb{C}_X$.

Notation 2.4.1. When there is no confusion, the terminal object of a given category will be simply called $\mathbf{t}$. Otherwise, a subscript will specify the category in which we are computing the terminal object.

This map is the $\mathbb{C}$-arrow $\mathrm{pr}_1 \colon X \times \mathbf{t} \to X$, which is a direction of the canonical isomorphism $X \times \mathbf{t} \cong X$, whose inverse is given by $\langle \mathrm{id}_X, !_X \rangle \colon X \to X \times \mathbf{t}$. This induces an isomorphism also between the corresponding fibers $P(X \times \mathbf{t}) \cong P(X)$, so that $P_{(X,\varphi)}(\mathbf{t}) = P(X \times \mathbf{t})_{\downarrow P(\mathrm{pr}_1)\varphi} \cong P(X)_{\downarrow\varphi}$. From now on we will write $\mathrm{id}_X \colon \mathbf{t} \rightsquigarrow X$ instead of $\mathrm{pr}_1 \colon \mathbf{t} \rightsquigarrow X$, and $P(X)_{\downarrow\varphi}$ instead of $P(X \times \mathbf{t})_{\downarrow P(\mathrm{pr}_1)\varphi}$. With this notation, we compute the reindexing of $\mathfrak{f}_X(\varphi)$ along the constant $\mathrm{id}_X \colon \mathbf{t} \rightsquigarrow X$, and we show that it is the top element in $P_{(X,\varphi)}(\mathbf{t}) = P(X)_{\downarrow\varphi}$. Indeed:

$$P_{(X,\varphi)}(\mathrm{id}_X)\mathfrak{f}_X(\varphi) = P(\Delta_X)\mathfrak{f}_X(\varphi) = P(\Delta_X)(P(\mathrm{pr}_1)\varphi \wedge P(\mathrm{pr}_2)\varphi) = \varphi \wedge \varphi = \varphi,$$

and φ is the top element of $P_{(X,\varphi)}(\mathbf{t}) = P(X)_{\downarrow\varphi}$.

Theorem 2.4.2. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine. Given an object X in the base category $\mathbb{C}$ and an element $\varphi \in P(X)$, the 1-arrow $(F_X, \mathfrak{f})\colon P \to P_{(X,\varphi)}$ and the $\mathbb{C}_X$-arrow $\mathrm{id}_X\colon \mathbf{t}_{\mathbb{C}_X} \rightsquigarrow X$ are such that $\top \leq P_{(X,\varphi)}(\mathrm{id}_X)\mathfrak{f}_X(\varphi)$ in $P_{(X,\varphi)}(\mathbf{t}_{\mathbb{C}_X})$, and they are universal with respect to this property, i.e. for any primary 1-arrow $(G, \mathfrak{g})\colon P \to R$, where $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ is a primary doctrine, and any $\mathbb{D}$-arrow $c\colon \mathbf{t}_{\mathbb{D}} \to G(X)$ such that $\top \leq R(c)\mathfrak{g}_X(\varphi)$ in $R(\mathbf{t}_{\mathbb{D}})$ there exists a unique up to a unique natural isomorphism primary 1-arrow $(G', \mathfrak{g}')\colon P_{(X,\varphi)} \to R$ such that $(G', \mathfrak{g}') \circ (F_X, \mathfrak{f}) = (G, \mathfrak{g})$ and $G'(\mathrm{id}_X) = c$.

Proof. Consider the diagram

$$\begin{tikzcd}
(P, (X \times -, \mathfrak{f}), \gamma, \varepsilon) \arrow[rr, "((F_X,\mathfrak{f}),\gamma)"] \arrow[dr, bend right, "((G,\mathfrak{g}),\mathfrak{j})"'] & & (P_{(X,\varphi)}, (\mathrm{id}, \mathrm{id}), \mathrm{id}, \mathrm{id}) \arrow[dl, dashed, "((G',\mathfrak{g}'),\mathrm{id})"] \\
& (R, (\mathrm{id}, \mathrm{id}), \mathrm{id}, \mathrm{id}) &
\end{tikzcd} \tag{2.1}$$

describing the universal property for the Kleisli construction for the comonad we are studying on P, see Proposition 1.2.5.

So, in order to construct $(G', \mathfrak{g}')\colon P_{(X,\varphi)} \to R$, we must define $\mathfrak{j}$ in such a way that $((G, \mathfrak{g}), \mathfrak{j})$ is an arrow in $\mathrm{Cmd}^*(\mathbf{IdxPos})$ as in (2.1), i.e. a natural transformation $\mathfrak{j}\colon G \overset{\cdot}{\to} G(X \times -)$, such that $\mathfrak{g}_A \leq R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}\mathfrak{f}_A$ and satisfying the coherence diagrams. Knowing that G preserves products, we need to define for every object A an arrow $\mathfrak{j}_A\colon GA \to GX \times GA$ take $\mathfrak{j}_A := \langle c{\cdot}!_{GA}, \mathrm{id}_{GA}\rangle$, where $!_{GA}\colon GA \to \mathbf{t}_{\mathbb{D}}$ is the unique arrow from GA to the teminal object.

This is a natural transformation:

$$\begin{tikzcd}
A \arrow[d, "f"] & GA \arrow[r, "\mathfrak{j}_A"] \arrow[d, "G(f)"] & GX \times GA \arrow[d, "G(\mathrm{id}\times f) = \mathrm{id}\times G(f)"] \\
B & GB \arrow[r, "\mathfrak{j}_B"] & GX \times GB
\end{tikzcd}$$

Indeed, for any $f\colon A \to B$, we have:

$$(\mathrm{id} \times G(f))\langle c{\cdot}!_{GA}, \mathrm{id}_{GA}\rangle = \langle c{\cdot}!_{GA}, G(f)\rangle = \langle c{\cdot}!_{GB}, \mathrm{id}_{GB}\rangle G(f).$$

Moreover, for any $\alpha \in P(A)$, we have

$$\begin{aligned}
R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}\mathfrak{f}_A(\alpha) &= R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}(P(\mathrm{pr}_1)(\varphi) \wedge P(\mathrm{pr}_2)(\alpha)) \\
&= R(\mathfrak{j}_A)\big(RG(\mathrm{pr}_1)\mathfrak{g}_X(\varphi) \wedge RG(\mathrm{pr}_2)\mathfrak{g}_A(\alpha)\big) = R(c{\cdot}!_{GA})\mathfrak{g}_X(\varphi) \wedge \mathfrak{g}_A(\alpha),
\end{aligned}$$

using naturality of $\mathfrak{g}$ and the fact that G preserves products.

So now observe that $\mathfrak{g}_A(\alpha) \leq R(c{\cdot}!_{GA})\mathfrak{g}_X(\varphi) \wedge \mathfrak{g}_A(\alpha)$ if and only if $\mathfrak{g}_A(\alpha) \leq R(c{\cdot}!_{GA})\mathfrak{g}_X(\varphi)$, but by assumption $R(c)\mathfrak{g}_X(\varphi) = \top$ in $R(\mathbf{t}_{\mathbb{D}})$, so $R(!_{GA})R(c)\mathfrak{g}_X(\varphi) = R(c{\cdot}!_{GA})\mathfrak{g}_X(\varphi) = \top$ in $R(GA)$, hence the inequality holds.

To conclude, we prove that the coherence diagrams commute:

$$\begin{tikzcd}
G \arrow[r, "\mathfrak{j}"] \arrow[d, "\mathfrak{j}"] & GX \times G- \arrow[d, "G(\gamma)"] \\
GX \times G- \arrow[r, "\mathfrak{j}_{X\times -}"] & GX \times GX \times G-
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\mathfrak{j}"] \arrow[dr, bend right, "\mathrm{id}"'] & GX \times G- \arrow[d, "G(\varepsilon)"] \\
& G
\end{tikzcd}$$

The first diagram commutes since

$$\begin{aligned}
G(\Delta_X \times \mathrm{id}_A)\mathfrak{j}_A = (\Delta_{GX} \times \mathrm{id}_{GA})\langle c{\cdot}!_{GA}, \mathrm{id}_{GA}\rangle &= \langle c{\cdot}!_{GA}, c{\cdot}!_{GA}, \mathrm{id}_{GA}\rangle \\
&= \langle c{\cdot}!_{G(X\times A)}, \mathrm{id}_{G(X\times A)}\rangle\langle c{\cdot}!_{GA}, \mathrm{id}_{GA}\rangle,
\end{aligned}$$

while the second one commutes since

$$G(\varepsilon_A)\mathfrak{j}_A = G(\mathrm{pr}_2)\mathfrak{j}_A = \mathrm{pr}_2'\mathfrak{j}_A = \mathrm{id}_{GA}\,.$$

So now we know from the universal property that there exists a unique $(G', \mathfrak{g}')\colon P_{(X,\varphi)} \to R$ such that $((G, \mathfrak{g}), \mathfrak{j}) = ((G', \mathfrak{g}'), \mathrm{id})((F_K, \mathfrak{f}), \gamma)$. In particular there is an arrow $(G', \mathfrak{g}')\colon P_{(X,\varphi)} \to R$ such that $(G', \mathfrak{g}') \circ (F_K, \mathfrak{f}) = (G, \mathfrak{g})$. Moreover, if we translate the universal property in Proposition 1.2.5 to the notation used here, we observe that $G'(g\colon A \rightsquigarrow B) = G(g)\mathfrak{j}_A$, where $g\colon X \times A \to B$ is an arrow in $\mathbb{C}$. In particular, taking $A = \mathfrak{t}_{\mathbb{C}_X}$ and $g = \mathrm{id}_X\colon X \to X$, we obtain $G'(\mathrm{id}_X) = \mathfrak{j}_{\mathfrak{t}_{\mathbb{C}}} = c$. Here we use the fact that G preserves the terminal object, and that the product of an object with the terminal object is the object itself.

We use the definition of G' on arrows to prove that G' preserves products; consider the following three diagrams: on the left there is the $\mathbb{C}$-diagram that mirrors a product diagram in $\mathbb{C}_X$—in the middle—, while on the right there is the image of such product through G'.

$$\begin{tikzcd}
& X \times A \times B \arrow[dl, bend right, "\mathrm{pr}_2"] \arrow[dr, bend left, "\mathrm{pr}_3"] & \\
A & & B
\end{tikzcd}
\begin{tikzcd}
& A \times B \arrow[dl, squiggly, "\mathrm{pr}_2"] \arrow[dr, squiggly, "\mathrm{pr}_3"] & \\
A & & B
\end{tikzcd}
\begin{tikzcd}
& GA \times GB \arrow[dl, bend right, "G(\mathrm{pr}_2)\mathfrak{j}_{A\times B}"] \arrow[dr, bend left, "G(\mathrm{pr}_3)\mathfrak{j}_{A\times B}"] & \\
GA & & GB
\end{tikzcd}$$

However, since G preserves products, $G(\mathrm{pr}_2)$ and $G(\mathrm{pr}_3)$ are respectively the second and third projections from $GX \times GA \times GB$, and these precomposed with $\mathfrak{j}_{A\times B}$ are precisely the first and second projections from $GA \times GB$, as claimed.

To show that $\mathfrak{g}'$ preserves infima and top element, recall from Proposition 1.2.5 that $\mathfrak{g}'_A$ is the restriction of $R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}$. Then notice that for any $\alpha, \beta \in P_{(X,\varphi)}(A)$ we have

$$\mathfrak{g}'_A(\alpha \wedge \beta) = R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}(\alpha \wedge \beta) = R(\mathfrak{j}_A)\big(\mathfrak{g}_{X\times A}(\alpha) \wedge \mathfrak{g}_{X\times A}(\beta)\big) = \mathfrak{g}'_A(\alpha) \wedge \mathfrak{g}'_A(\beta),$$

since by assumption $\mathfrak{g}$ respect the structures, while for the top element:

$$\begin{aligned}
\mathfrak{g}'_A(P(\mathrm{pr}_1)\varphi) = R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}P(\mathrm{pr}_1)\varphi &= R(\langle c{\cdot}!_{GA}, \mathrm{id}_{GA}\rangle)R(\mathrm{pr}_1)\mathfrak{g}_X\varphi \\
&= R(!_{GA})R(c)\mathfrak{g}_X\varphi = \top_{GA}.
\end{aligned}$$

So $(G', \mathfrak{g}')$ is indeed a primary 1-arrow.
Finally, suppose that $(\overline{G}, \overline{\mathfrak{g}})\colon P_{(X,\varphi)} \to R$ is another primary 1-arrow such that $(\overline{G}, \overline{\mathfrak{g}}) \circ (F_X, \mathfrak{f}) = (G, \mathfrak{g})$ and $\overline{G}(\mathrm{id}_X) = c$.
Then we can compute the composition $((G, \mathfrak{g}), \bar{\mathfrak{j}}) = ((\overline{G}, \overline{\mathfrak{g}}), \mathrm{id})((F_K, \mathfrak{f}), \gamma)$, where we define $\bar{\mathfrak{j}}_A \colon= (\mathrm{id} \circ \gamma)_A = \overline{G}(\varepsilon_{X \times A} \gamma_A) = \overline{G}(\mathrm{id}_{X \times A})$—see Remark 1.2.6 and Proposition 1.2.5. We claim that $\bar{\mathfrak{j}} = \mathfrak{j}$, so that by uniqueness given by the universal property, the equality $(G', \mathfrak{g}') = (\overline{G}, \overline{\mathfrak{g}})$ follows. In our notation, we have to think of $\mathrm{id}_{X \times A}$ as the map $A \rightsquigarrow X \times A$, and it is uniquely defined by its two components: the first one is $\mathrm{pr}_1 \colon A \rightsquigarrow X$, the second one is $\mathrm{pr}_2 \colon A \rightsquigarrow A$. Observe that pr_2 is the identity of A in $\mathbb{C}_X$, while $\mathrm{pr}_1 \colon A \rightsquigarrow X$ is the composition of the unique arrow $A \rightsquigarrow \mathbf{t}_{\mathbb{C}}$ and the constant $\mathrm{id}_X \colon \mathbf{t}_{\mathbb{C}_X} \rightsquigarrow X$. Since $\overline{G}$ preserves products, $\bar{\mathfrak{j}}_A = \overline{G}(\mathrm{id}_{X \times A}) \colon GA \to GX \times GA$ must be the identity of GA on the second component; in particular the second components of $\bar{\mathfrak{j}}_A$ and $\mathfrak{j}_A$ are the same. Concerning the first component we have $\overline{G}(\mathrm{pr}_1) = \overline{G}(\mathrm{id}_X)\overline{G}(!_{X \times A}) = c \cdot !_{GA}$, i.e. also the first component of $\bar{\mathfrak{j}}_A$ coincides with the first component of $\mathfrak{j}_A$, hence the two maps coincide as claimed. ☐

Theorem 2.4.3. Let P, $P_{(X,\varphi)}$, R, $(G, \mathfrak{g})\colon P \to R$ be the doctrines and a morphism with the same assumption of Theorem 2.4.2. Then

(i) if P, R and $(G, \mathfrak{g})$ are elementary, then $\mathfrak{g}'$ preserves the elementary structure;

(ii) if P, R and $(G, \mathfrak{g})$ are existential, then $\mathfrak{g}'$ preserves the existential quantifier;

(iii) if P, R and $(G, \mathfrak{g})$ are universal, then $\mathfrak{g}'$ preserves the universal quantifier;

(iv) if P, R and $(G, \mathfrak{g})$ are implicational, then $\mathfrak{g}'$ preserves the implication;

(v) if P, R are bounded, with top and bottom elements preserved by $\mathfrak{g}$, then $\mathfrak{g}'$ preserves them;

(vi) if P, R have binary joins, preserved by $\mathfrak{g}$, then $\mathfrak{g}'$ preserves binary joins;

(vii) if P, R and $(G, \mathfrak{g})$ are respectively Horn, Heyting or Boolean, then $\mathfrak{g}'$ preserves the corresponding structure.

Proof. (i) We need to check that $\text{Æ}^{GB}_{GC} \mathfrak{g}'_{C \times B} = \mathfrak{g}'_{C \times B \times B} \text{Æ}_{X, \varphi_C^{B}}$.

$$\begin{aligned} \text{Æ}^{GB}_{GC} \mathfrak{g}'_{C \times B} &= \text{Æ}^{GB}_{GC} R(\mathfrak{j}_{C \times B}) \mathfrak{g}_{X \times C \times B} = R(\mathfrak{j}_{C \times B \times B}) \text{Æ}^{GB}_{GX \times GC} \mathfrak{g}_{X \times C \times B} \\ &= R(\mathfrak{j}_{C \times B \times B}) \mathfrak{g}_{X \times C \times B \times B} \text{Æ}^{B}_{X \times C} = \mathfrak{g}'_{C \times B \times B} \text{Æ}_{X, \varphi_C^{B}}. \end{aligned}$$

(ii) We prove that $\exists^{GB}_{GC} \mathfrak{g}'_{C \times B} = \mathfrak{g}'_C \exists_{X, \varphi_C^{B}}$:

$$\begin{aligned} \exists^{GB}_{GC} \mathfrak{g}'_{C \times B} &= \exists^{GB}_{GC} R(\mathfrak{j}_{C \times B}) \mathfrak{g}_{X \times C \times B} = R(\mathfrak{j}_C) \exists^{GB}_{GX \times GC} \mathfrak{g}_{X \times C \times B} \\ &= R(\mathfrak{j}_C) \mathfrak{g}_{X \times C} \exists^{B}_{X \times C} = \mathfrak{g}'_C \exists_{X, \varphi_C^{B}}. \end{aligned}$$

(iii) The proof is similar to the one above, with a little alteration:

$$\begin{aligned}\forall^{GB}_{GC}\mathfrak{g}'{}_{C\times B} &= \forall^{GB}_{GC}R(\mathfrak{j}_{C\times B})\mathfrak{g}_{X\times C\times B}\\ &= R(\mathfrak{j}_C)\forall^{GB}_{GX\times GC}\mathfrak{g}_{X\times C\times B} = R(\mathfrak{j}_C)\mathfrak{g}_{X\times C}\forall^{B}_{X\times C}\end{aligned}$$

and also

$$\begin{aligned}\mathfrak{g}'_C\forall_{X,\varphi}{}^{B}_{C} &= R(\mathfrak{j}_C)\mathfrak{g}_{X\times C}\big(\forall^{B}_{X\times C}(-)\wedge P(\mathrm{pr}_1)\varphi\big)\\ &= R(\mathfrak{j}_C)\mathfrak{g}_{X\times C}\forall^{B}_{X\times C}(-)\wedge R(\mathfrak{j}_C)\mathfrak{g}_{X\times C}P(\mathrm{pr}_1)\varphi = R(\mathfrak{j}_C)\mathfrak{g}_{X\times C}\forall^{B}_{X\times C}.\end{aligned}$$

Note that $R(\mathfrak{j}_C)\mathfrak{g}_{X\times C}P(\mathrm{pr}_1)\varphi = \top_{GC}$ since $\mathfrak{g}'$ preserves the top element—see Theorem 2.4.2.

(iv) Take $\alpha,\beta\in P_{(X,\varphi)}(A)$:

$$\begin{aligned}\mathfrak{g}'_A(\alpha\Rightarrow\beta) &= R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}((\alpha\to\beta)\wedge P(\mathrm{pr}_1)\varphi)\\ &= \big(R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}(\alpha)\to R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}(\beta)\big)\wedge R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}P(\mathrm{pr}_1)\varphi = \mathfrak{g}'_A(\alpha)\to\mathfrak{g}'_A(\beta).\end{aligned}$$

(v) Compute $\mathfrak{g}'_A(0_A) = R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}(\bot_A) = \bot_{GA}$.

(vi) Take $\alpha,\beta\in P_{(X,\varphi)}(A)$, then

$$\mathfrak{g}'_A(\alpha\vee\beta) = R(\mathfrak{j}_A)\mathfrak{g}_{X\times A}(\alpha\vee\beta) = \mathfrak{g}'_A(\alpha)\vee\mathfrak{g}'_A(\beta).$$

Observe that in order to prove this point it was not necessary to ask for the condition that $P(\mathrm{pr}_1)\varphi\wedge(-)$ preserves finite joins in $P(X\times A)$, which was necessary for $\mathfrak{f}$ to preserve finite joins in Proposition 2.3.13: it is enough to ask $\mathfrak{g}$ to preserve them.

(vii) It follows trivially combining the previous properties. ☐

A stronger result for Theorem 2.4.2 holds. Let again $P\colon\mathbb{C}^{\mathrm{op}}\to\mathbf{Pos}$ be a primary doctrine; fix on object X in the base category, and an element $\varphi\in P(X)$. For any other primary doctrine $R\colon\mathbb{D}^{\mathrm{op}}\to\mathbf{Pos}$, define the category $\mathbf{PD}_{(X,\varphi)}(P,R)$ whose objects are pairs of the kind $\big((G,\mathfrak{g}),c\colon\mathbf{t}_{\mathbb{D}}\to GX\big)$, where $(G,\mathfrak{g})\in\mathbf{PD}(P,R)$, such that $\top\le R(c)\mathfrak{g}_X(\varphi)$ in $R(\mathbf{t}_{\mathbb{D}})$, and whose arrows are 2-arrows preserving the constant, meaning $\theta\colon\big((G,\mathfrak{g}),c\big)\to\big((H,\mathfrak{h}),d\big)$ is a 2-arrow $\theta\colon(G,\mathfrak{g})\to(H,\mathfrak{h})$ in $\mathbf{PD}$ such that $c\theta_X=d$. There is an obvious functor induced by precomposition with $(F_X,\mathfrak{f})$, from $\mathbf{PD}(P_{(X,\varphi)},R)$ to $\mathbf{PD}_{(X,\varphi)}(P,R)$: it maps any $\xi\colon(K,\mathfrak{k})\to(K',\mathfrak{k}')$ into $\xi_{F_X}\colon\big((K,\mathfrak{k})\circ(F_X,\mathfrak{f}),K(\mathrm{id}_X)\big)\to\big((K',\mathfrak{k}')\circ(F_X,\mathfrak{f}),K'(\mathrm{id}_X)\big)$. This is well defined on objects since $R(K(\mathrm{id}_X))(\mathfrak{k}_{F_XX}\mathfrak{f}_X\varphi)=\mathfrak{k}_{\mathbf{t}}P_{(X,\varphi)}(\mathrm{id}_X)\mathfrak{f}_X\varphi=\top$, and well defined on arrows since $K(\mathrm{id}_X)\xi_X=K'(\mathrm{id}_X)$ by naturality.

Theorem 2.4.4. Let P be a primary doctrine. Given an object X in the base category and an element $\varphi\in P(X)$, the functor $\mathbf{PD}(P_{(X,\varphi)},R)\to\mathbf{PD}_{(X,\varphi)}(P,R)$ induced by precomposition with $(F_X, \mathfrak{f})$ is an equivalence of categories for any primary doctrine R.

Proof. The functor is essentially surjective following from Theorem 2.4.2 and faithfulness is trivial since F_X is the identity on objects. To show that the functor is full, take any 2-arrow $\theta\colon \big((K,\mathfrak{k}) \circ (F_X,\mathfrak{f}), K(\mathrm{id}_X)\big) \to \big((K',\mathfrak{k}') \circ (F_X,\mathfrak{f}), K'(\mathrm{id}_X)\big)$ and prove that $\theta\colon (K,\mathfrak{k}) \to (K',\mathfrak{k}')$ is in $\mathbf{PD}(P_{(X,\varphi)}, R)$. First of all we check that it is a natural transformation $K \to K'$: take any $f\colon A \rightsquigarrow B$ in $\mathbb{C}_X$ and break it as the composition of $\mathrm{id}_{X\times A}\colon A \rightsquigarrow X \times A$ and $f\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle = F_X(f)\colon X \times A \rightsquigarrow B$; moreover observe that $\mathrm{id}_{X\times A}$ has as first projection the composition of the unique arrow $A \rightsquigarrow \mathbf{t}$ and the constant $\mathrm{id}_X\colon \mathbf{t} \rightsquigarrow X$, and as second projection the identity $\mathrm{pr}_2\colon A \rightsquigarrow A$—see the end of the proof of Theorem 2.4.2. So the naturality diagram becomes:

$$\begin{tikzcd}
A \arrow[d, squiggly, "\mathrm{id}_{X\times A}"] \arrow[dd, squiggly, bend right=60, "f"'] & KA \arrow[rr, "\theta_A"] \arrow[d, "\langle K(\mathrm{id}_X)!_{KA}, \mathrm{id}_{KA}\rangle"] \arrow[dd, bend right=60, "K(f)"'] & & K'A \arrow[d, "\langle K'(\mathrm{id}_X)!_{K'A}, \mathrm{id}_{K'A}\rangle"'] \arrow[dd, bend left=60, "K'(f)"] \\
X \times A \arrow[d, squiggly, "F_X(f)"] & KX \times KA \arrow[rr, "\theta_X \times \theta_A"] \arrow[d, "KF_X(f)"] & & K'X \times K'A \arrow[d, "K'F_X(f)"'] \\
B & KB \arrow[rr, "\theta_B"'] & & K'B
\end{tikzcd}$$

The lower square commutes since $\theta\colon KF_X \overset{\cdot}{\to} K'F_X$ by assumption, while the upper square commutes since $K(\mathrm{id}_X)\theta_X = K'(\mathrm{id}_X)$. To conclude, we need for any $\mathbb{C}$-object A and any $\alpha \in P_{(X,\varphi)}(A) = P(X \times A)_{\downarrow P(\mathrm{pr}_1)\varphi}$ the inequality $\mathfrak{k}_A(\alpha) \leq R(\theta_A)\mathfrak{k}'_A(\alpha)$ to hold. In particular $\alpha \in P(X \times A)$, so we can consider

$$\mathfrak{f}_{X\times A}\alpha = P(\mathrm{pr}_1)(\varphi) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)(\alpha) \in P_{(X,\varphi)}(X \times A) \subseteq P(X \times X \times A);$$

apply then naturality of $\mathfrak{k}$ to $\mathrm{id}_{X\times A}\colon A \rightsquigarrow X \times A$ to observe that

$$\begin{aligned}
R(\langle K(\mathrm{id}_X)!_{KA}, \mathrm{id}_{KA}\rangle)&\mathfrak{k}_{X\times A}\mathfrak{f}_{X\times A}\alpha = \mathfrak{k}_A P_{(X,\varphi)}(\mathrm{id}_{X\times A})\mathfrak{f}_{X\times A}\alpha \\
&= \mathfrak{k}_A P(\langle \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_2\rangle)\big(P(\mathrm{pr}_1)(\varphi) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)(\alpha)\big) = \mathfrak{k}_A(P(\mathrm{pr}_1)(\varphi) \wedge \alpha) = \mathfrak{k}_A(\alpha)
\end{aligned}$$

since $\alpha \leq P(\mathrm{pr}_1)(\varphi)$. Moreover, since in particular $\theta\colon (K,\mathfrak{k}) \circ (F_X,\mathfrak{f}) \to (K',\mathfrak{k}') \circ (F_X,\mathfrak{f})$, we know that $\mathfrak{k}_{X\times A}\mathfrak{f}_{X\times A}(\alpha) \leq R(\theta_{X\times A})\mathfrak{k}'_{X\times A}\mathfrak{f}_{X\times A}(\alpha)$. So we have:

$$\begin{aligned}
\mathfrak{k}_A(\alpha) &= R(\langle K(\mathrm{id}_X)!_{KA}, \mathrm{id}_{KA}\rangle)\mathfrak{k}_{X\times A}\mathfrak{f}_{X\times A}\alpha \\
&\leq R(\langle K(\mathrm{id}_X)!_{KA}, \mathrm{id}_{KA}\rangle)R(\theta_{X\times A})\mathfrak{k}'_{X\times A}\mathfrak{f}_{X\times A}(\alpha) \\
&= R(\theta_A)R(\langle K'(\mathrm{id}_X)!_{K'A}, \mathrm{id}_{K'A}\rangle)\mathfrak{k}'_{X\times A}\mathfrak{f}_{X\times A}(\alpha) = R(\theta_A)\mathfrak{k}'_A(\alpha).
\end{aligned}$$

□

The process studied above in Theorem 2.4.2 describes how to add a constant of sort X that verifies a formula φ in a universal way. Taking the particular case when $X = \mathbf{t}$ is the terminal object, we are not adding any constant—the functor $\mathbf{t} \times -\colon \mathbb{C} \to \mathbb{C}$ is essentially the identity—, and we are just requiring $\varphi \in P(\mathbf{t})$ to be true in the new doctrine—i.e. we are adding the axiom φ to the theory P—, in a universal way. In this case we write $P_\varphi \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$; for any given $\mathbb{C}$-arrow $f\colon A \to B$, we have $P_\varphi(f)\colon P_\varphi(B) \to P_\varphi(A)$, computed as

$$P(f)\colon P(B)_{\downarrow P(!_B)\varphi} \to P(A)_{\downarrow P(!_A)\varphi}.$$

The 1-arrow $P \to P_\varphi$ becomes $(\mathrm{id}_\mathbb{C}, \mathfrak{f})$, where $\mathfrak{f}_A\colon P(A) \to P_\varphi(A)$ maps an element $\alpha \in P(A)$ to $P(!_A)\varphi \wedge \alpha \in P_\varphi(A)$. All the additional properties of P described in Section 2.3 are clearly recovered by P_φ.

Corollary 2.4.5. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine. Given an element $\varphi \in P(\mathbf{t})$, the 1-arrow $(\mathrm{id}_\mathbb{C}, \mathfrak{f})\colon P \to P_\varphi$ is such that $\top \leq \mathfrak{f}_\mathbf{t}(\varphi)$ in $P_\varphi(\mathbf{t})$, and it is universal with respect to this property, i.e. for any primary 1-arrow $(G, \mathfrak{g})\colon P \to R$, where $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ is a primary doctrine, such that $\top \leq \mathfrak{g}_\mathbf{t}(\varphi)$ in $R(\mathbf{t}_\mathbb{D})$ there exists a unique up to a unique natural isomorphism primary 1-arrow $(G', \mathfrak{g}')\colon P_\varphi \to R$ such that $(G', \mathfrak{g}') \circ (\mathrm{id}_\mathbb{C}, \mathfrak{f}) = (G, \mathfrak{g})$.

Remark 2.4.6. In the corollary above, the universal property is the same seen in Proposition 1.5.8, taking the filter $\nabla =\uparrow \varphi = \{\alpha \in P(\mathbf{t}) \mid \alpha \geq \varphi\}$. It follows that there exists an isomorphism between the primary doctrines $P/\uparrow \varphi$ and P_φ.

The category corresponding to $\mathbf{PD}_{(X,\varphi)}(P, R)$ in Theorem 2.4.4 for some primary doctrine $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ in this case is called $\mathbf{PD}_\varphi(P, R)$: objects are morphisms $(G, \mathfrak{g}) \in \mathbf{PD}(P, R)$ such that $\top \leq \mathfrak{g}_\mathbf{t}(\varphi)$ in $R(\mathbf{t}_\mathbb{D})$ and arrows are 2-arrows of $\mathbf{PD}$. In particular $\mathbf{PD}_\varphi(P, R)$ is a full subcategory of $\mathbf{PD}(P, R)$. Precomposition with $(\mathrm{id}_\mathbb{C}, \mathfrak{f})$ is a functor from $\mathbf{PD}(P_\varphi, R)$ to $\mathbf{PD}(P, R)$, and has image in $\mathbf{PD}_\varphi(P, R)$: given $(K, \mathfrak{k})\colon P_\varphi \to R$, the composition $(K, \mathfrak{k})(\mathrm{id}_\mathbb{C}, \mathfrak{f})$ is such that $(\mathfrak{k}\mathfrak{f})_\mathbf{t}(\varphi) = \mathfrak{k}_\mathbf{t}\mathfrak{f}_\mathbf{t}(\varphi) = \mathfrak{k}_\mathbf{t}(\varphi) = \top$ in $R(\mathbf{t}_\mathbb{D})$, since φ is the top element in $P_\varphi(\mathbf{t})$.

Corollary 2.4.7. Let P be a primary doctrine. Given an element $\varphi \in P(\mathbf{t})$, precomposition with $(\mathrm{id}_\mathbb{C}, \mathfrak{f})$

$$- \circ (\mathrm{id}_\mathbb{C}, \mathfrak{f})\colon \mathbf{PD}(P_\varphi, R) \to \mathbf{PD}_\varphi(P, R)$$

is an equivalence of categories for any primary doctrine R.

Similarly, we can take the particular case when $\varphi = \top_X \in P(X)$ is the top element, so we are not making any formula true—the natural transformation $\mathfrak{f}_A\colon P(A) \to P(X \times A)$ represent the inclusion of formulae of sort A in the formulae of the same sort but in a language with a new constant—, and we are just adding a constant $\mathrm{id}_X\colon \mathbf{t} \rightsquigarrow X$, in a universal way. In this case we write $P_X\colon \mathbb{C}_X^{\mathrm{op}} \to \mathbf{Pos}$; for any given $\mathbb{C}_X$-arrow $f\colon A \rightsquigarrow B$, we have $P_X(f)\colon P_X(B) \to P_X(A)$ computed as

$$P(\langle \mathrm{pr}_1, f\rangle)\colon P(X \times B) \to P(X \times A).$$

The 1-arrow $P \to P_X$ becomes $(F_X, \mathfrak{f})$, where $\mathfrak{f}_A\colon P(A) \to P_X(A)$ maps an element $\alpha \in P(A)$ to $P(\mathrm{pr}_2)(\alpha) \in P_X(A)$. All the additional properties of P described in Section 2.3 are clearly recovered by P_X. Observe that for this construction, the assumption that the starting doctrine P is primary is not needed.

Corollary 2.4.8. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a doctrine. Given an object X in the base category, the 1-arrow $(F_X, \mathfrak{f})\colon P \to P_X$ and the $\mathbb{C}_X$-arrow $\mathrm{id}_X\colon \mathbf{t}_{\mathbb{C}_X} \rightsquigarrow X$ are universal, i.e. for any 1-arrow $(G, \mathfrak{g})\colon P \to R$, where $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ is a doctrine, and any $\mathbb{D}$-arrow $c\colon \mathbf{t}_{\mathbb{D}} \to G(X)$ there exists a unique up to a unique natural isomorphism 1-arrow $(G', \mathfrak{g}')\colon P_X \to R$ such that $(G', \mathfrak{g}') \circ (F_X, \mathfrak{f}) = (G, \mathfrak{g})$ and $G'(\mathrm{id}_X) = c$.

The category corresponding to $\mathbf{PD}_{(X,\varphi)}(P, R)$ in Theorem 2.4.4 can be defined for any doctrine $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$, and in this case is called $\mathbf{Dct}_X(P, R)$: objects are pairs $\big((G, \mathfrak{g}), c\colon \mathbf{t} \to GX\big)$ where $(G, \mathfrak{g}) \in \mathbf{Dct}(P, R)$ and arrows are 2-arrows of $\mathbf{Dct}$ preserving the constant. Precomposition with $(F_X, \mathfrak{f})$ induces a functor from $\mathbf{Dct}(P_X, R)$ to $\mathbf{Dct}_X(P, R)$: it maps any $\xi\colon (K, \mathfrak{k}) \to (K', \mathfrak{k}')$ into $\xi_{F_X}\colon \big((K, \mathfrak{k}) \circ (F_X, \mathfrak{f}), K(\mathrm{id}_X)\big) \to \big((K', \mathfrak{k}') \circ (F_X, \mathfrak{f}), K'(\mathrm{id}_X)\big)$. This is well defined on arrows since $K(\mathrm{id}_X)\xi_X = K'(\mathrm{id}_X)$ by naturality.

Corollary 2.4.9. Let P be a doctrine. Given an object X in the base category, the functor $\langle - \circ (F_X, \mathfrak{f}), -(\mathrm{id}_X)\rangle\colon \mathbf{Dct}(P_X, R) \to \mathbf{Dct}_X(P, R)$ induced by precomposition with $(F_X, \mathfrak{f})$ is an equivalence of categories for any doctrine R.

Remark 2.4.10. We showed how to obtain from the universal 1-arrow $(F_X, \mathfrak{f}_{(X,\varphi)})\colon P \to P_{(X,\varphi)}$ for fixed object X and element $\varphi \in P(X)$, both universal 1-arrows $(F_X, \mathfrak{f}_X)\colon P \to P_X$ in Corollary 2.4.8 for a fixed object X and $(\mathrm{id}_{\mathbb{C}}, \mathfrak{f}_\varphi)\colon P \to P_\varphi$ in Corollary 2.4.5 for a fixed element φ in $P(\mathbf{t})$ as particular cases. Note that we wrote some subscripts to avoid confusion between the constructions. We now show that we can recover the first 1-arrow from the other two. To do so, take a primary doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$, fix an object X and an element $\varphi \in P(X)$. Apply the construction that adds a constant to obtain $(F_X, \mathfrak{f}_X)\colon P \to P_X$. Now consider the primary doctrine $P_X\colon \mathbb{C}_X^{\mathrm{op}} \to \mathbf{Pos}$ and the element φ in the fiber over the terminal object $P_X(\mathbf{t}) = P(X)$. Apply the construction that adds an axiom to obtain $(\mathrm{id}_{\mathbb{C}_X}, \mathfrak{f}_\varphi)\colon P_X \to (P_X)_\varphi$.

$$\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \arrow[r, "F_X^{\mathrm{op}}"] \arrow[dr, "P"', ""{name=A}] & \mathbb{C}_X^{\mathrm{op}} \arrow[r, "\mathrm{id}_{\mathbb{C}_X}^{\mathrm{op}}"] \arrow[d, "P_X" description, ""{name=B}] & \mathbb{C}_X^{\mathrm{op}} \arrow[dl, "(P_X)_\varphi", ""'{name=C}] \\
& \mathbf{Pos} &
\arrow[from=A, to=B, Rightarrow, bend left, "\mathfrak{f}_X"', shorten=2mm]
\arrow[from=B, to=C, Rightarrow, bend left, "\mathfrak{f}_\varphi"', shorten=2mm]
\end{tikzcd}$$

Compute for each object A, the poset $(P_X)_\varphi(A) = P_X(A)_{\downarrow P_X(!_{X\times A})\varphi}$, where $!_{X\times A}\colon A \rightsquigarrow \mathbf{t}$ is the unique $\mathbb{C}_X$-arrow from A to $\mathbf{t}$. The reindexing along this arrow is $P_X(!_{X\times A})\colon P_X(\mathbf{t}) \to P_X(A)$, that maps φ to $P(\mathrm{pr}_1)\varphi$, so $(P_X)_\varphi(A) = P(X \times A)_{\downarrow P(\mathrm{pr}_1)\varphi}$ which is exactly how the fibers of $P_{(X,\varphi)}$ are computed. Then compute reindexing in $(P_X)_\varphi$: given $f\colon A \rightsquigarrow B$, we know that $(P_X)_\varphi(f)$ is defined as the restriction of $P_X(f)$, that is $P(\langle \mathrm{pr}_1, f\rangle)$, which is how reindexing are computed in $P_{(X,\varphi)}$. So the functor $(P_X)_\varphi$ is $P_{(X,\varphi)}$. Moreover, observe that the composition of the 1-arrows is $(\mathrm{id}_{\mathbb{C}_X}, \mathfrak{f}_\varphi)(F_X, \mathfrak{f}_X) = (F_X, \mathfrak{f}_{(X,\varphi)})$.

In the following, we apply separately the two constructions to a doctrine of well-formed formulae in some language $\mathcal{L}$ and theory $\mathcal{T}$. At first we apply the construction that adds a constant to a doctrine, and show that there is an isomorphism between this doctrine and the doctrine of well-formed formulae in the language with a new constant symbol. Then we apply the construction that adds an axiom to a doctrine, and show that there is an isomorphism between this doctrine and the doctrine of well-formed formulae where the theory has a new axiom.

Example 2.4.11. Let $\mathcal{L}$ be a first-order language and $\mathcal{T}$ be a theory. Consider the doctrine $\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}\colon \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \to \mathbf{Pos}$ and the fixed object (x) in the base category $\mathbb{C}\mathrm{tx}_{\mathcal{L}}$. On the one hand, consider the 1-arrow $(F_{(x)}, \mathfrak{f})\colon \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}} \to (\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{(x)}$, where $(\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{(x)}\colon (\mathbb{C}\mathrm{tx}_{\mathcal{L}})^{\mathrm{op}}_{(x)} \to \mathbf{Pos}$. Arrows in $(\mathbb{C}\mathrm{tx}_{\mathcal{L}})_{(x)}$ are of the form $\vec{t}(\,(x);\vec{z}\,)\colon \vec{z} \rightsquigarrow \vec{y}$, and the fibers $(\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{(x)}(\vec{z})$ are $\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}(\,(x);\vec{z}\,)$ for any list of variables $\vec{z}$. On the other hand consider the doctrine $\mathtt{LT}^{\mathcal{L}\cup\{c\}}_{\mathcal{T}}\colon \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}\cup\{c\}} \to \mathbf{Pos}$, where c is a constant symbol not appearing in $\mathcal{L}$. There is a trivial 1-arrow $(E, \mathfrak{e})\colon \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}} \to \mathtt{LT}^{\mathcal{L}\cup\{c\}}_{\mathcal{T}}$: the functor E is defined by the inclusion of terms in the extended language, the natural transformation $\mathfrak{e}$ is defined by the inclusion of formulae. The universal property of $(F_{(x)}, \mathfrak{f})$ defines a unique $(E', \mathfrak{e}')\colon (\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{(x)} \to \mathtt{LT}^{\mathcal{L}\cup\{c\}}_{\mathcal{T}}$ such that $(E', \mathfrak{e}')(F_{(x)}, \mathfrak{f}) = (E, \mathfrak{e})$ and such that $E'\big(\mathrm{id}_{(x)}\colon () \rightsquigarrow (x)\big) = \big(c\colon () \to (x)\big)$.

$$\begin{tikzcd}
\mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \arrow[rr, "E^{\mathrm{op}}"] \arrow[dr, "F^{\mathrm{op}}_{(x)}"] \arrow[ddr, bend right, "\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}"'] & & \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}\cup\{c\}} \arrow[ddl, bend left, "\mathtt{LT}^{\mathcal{L}\cup\{c\}}_{\mathcal{T}}"] \\
& (\mathbb{C}\mathrm{tx}_{\mathcal{L}})^{\mathrm{op}}_{(x)} \arrow[ur, "E'^{\mathrm{op}}"'] \arrow[d, "(\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{(x)}"] \arrow[d, phantom, bend right=40, "\mathfrak{f}"'] \arrow[d, phantom, bend left=40, "\mathfrak{e}'"] & \\
& \mathbf{Pos} &
\end{tikzcd}$$

$$\mathfrak{e}$$

The functor E' maps an arrow $\vec{t}(\,(x);\vec{z}\,)\colon \vec{z} \rightsquigarrow \vec{y}$ to the term $\vec{t}(\,[c/x];\vec{z}\,)\colon \vec{z} \to \vec{y}$ in $\mathbb{C}\mathrm{tx}_{\mathcal{L}\cup\{c\}}$. For a given pair $\vec{t}(\,(x);\vec{z}\,), \vec{s}(\,(x);\vec{z}\,)$ such that $\vec{t}(\,[c/x];\vec{z}\,) = \vec{s}(\,[c/x];\vec{z}\,)$, substitute again $[x/c]$ and get $\vec{t} = \vec{s}$, so E' is faithful. Then, for a given a term $u(\vec{z})$ in the language $\mathcal{L} \cup \{c\}$, we can consider c as a variable and substitute each occurrence of c with x, to obtain a term $u'(\,(x);\vec{z}\,)$ obviously written in the language $\mathcal{L}$: in particular $E'(u') = u'(\,[c/x];\vec{z}\,) = u(\vec{z})$, so E' is full. Moreover, since E' is the identity on objects, E' is an isomorphism.
Concerning formulae, a component of the natural transformation $\mathfrak{e}'_{(\vec{z})}$ sends a formula $\alpha(\,(x);\vec{z}\,)$ in $(\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{(x)}(\vec{z}) = \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}(x,\vec{z})$ to the formula $\alpha(\,[c/x];\vec{z}\,) \in \mathtt{LT}^{\mathcal{L}\cup\{c\}}_{\mathcal{T}}(\vec{z})$. A similar argument to the one that showed fullness of the functor E' proves that $\mathfrak{e}'$ is a natural isomorphism.
To conclude, we can say that the doctrine $(\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{(x)}$ is again a doctrine of well-formed formulae.

Example 2.4.12. Let $\mathcal{L}$ be a first-order language and $\mathcal{T}$ be a theory. Consider the doctrine $\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}\colon \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \to \mathbf{Pos}$ and the fixed $\mathcal{L}$-sentence $\varphi \in \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}()$. On the one hand, consider the 1-arrow

$(\mathrm{id}_{\mathbb{C}\mathrm{tx}_{\mathcal{L}}}, \mathfrak{f})\colon \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}} \to (\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{\varphi}$, where $(\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{\varphi}\colon \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \to \mathbf{Pos}$. Its fibers $(\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{\varphi}(\vec{z})$ are by definition $\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}(\vec{z})_{\downarrow\varphi}$ for any list of variables $\vec{z}$. On the other hand consider the doctrine $\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}\cup\{\varphi\}}\colon \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \to \mathbf{Pos}$. There is an obvious 1-arrow $(\mathrm{id}_{\mathbb{C}\mathrm{tx}_{\mathcal{L}}}, \mathfrak{e})\colon \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}} \to \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}\cup\{\varphi\}}$: the natural transformation $\mathfrak{e}$ is defined by the quotient of formulae with respect to the extended theory, meaning that for each component $\vec{x}$ it maps any $\mathcal{T}$-provable sequent $\alpha(\vec{x}) \vdash_{\mathcal{T}} \beta(\vec{x})$ into the $\mathcal{T} \cup \{\varphi\}$-provable sequent $\alpha(\vec{x}) \vdash_{\mathcal{T}\cup\{\varphi\}} \beta(\vec{x})$. To use the universal property of $(\mathrm{id}_{\mathbb{C}\mathrm{tx}_{\mathcal{L}}}, \mathfrak{f})$, we need to check that $\mathfrak{e}_{()}$ maps $\varphi \in \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}()$ to the top element of $\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}\cup\{\varphi\}}()$. However this is true since clearly $\top \vdash_{\mathcal{T}\cup\{\varphi\}} \varphi$. Consequently there exists a unique $(E', \mathfrak{e}')\colon (\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{\varphi} \to \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}\cup\{\varphi\}}$ such that $(E', \mathfrak{e}')(\mathrm{id}_{\mathbb{C}\mathrm{tx}_{\mathcal{L}}}, \mathfrak{f}) = (\mathrm{id}_{\mathbb{C}\mathrm{tx}_{\mathcal{L}}}, \mathfrak{e})$.

$$\begin{tikzcd}
\mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \arrow[rrrr, "\mathrm{id}^{\mathrm{op}}_{\mathbb{C}\mathrm{tx}_{\mathcal{L}}}"] \arrow[drr, "\mathrm{id}^{\mathrm{op}}_{\mathbb{C}\mathrm{tx}_{\mathcal{L}}}"] \arrow[ddddrr, bend right, "\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}"'] & & & & \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \arrow[ddddll, bend left, "\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}\cup\{\varphi\}}"] \\
& & \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \arrow[urr, "\mathrm{id}^{\mathrm{op}}_{\mathbb{C}\mathrm{tx}_{\mathcal{L}}}"'] \arrow[dd, dash] & & \\
& {} \arrow[rr, bend left, "\mathfrak{e}"] & & {} & \\
& & (\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{\varphi} \arrow[dd] & & \\
& {} \arrow[r, "\mathfrak{f}"] & {} \arrow[r, "\mathfrak{e}'"] & {} & \\
& & \mathbf{Pos} & &
\end{tikzcd}$$

The functor E' is the identity.

Concerning formulae, a component of the natural transformation $\mathfrak{e}'_{\vec{x}}$ sends a formula $\alpha(\vec{x})$ in $(\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{\varphi}(\vec{x}) = \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}}(\vec{x})_{\downarrow\varphi}$ to the formula $\alpha(\vec{x}) \in \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}\cup\{\varphi\}}(\vec{x})$. Define the inverse function: it maps $\beta(\vec{x})$ to $\beta(\vec{x}) \wedge \varphi$. This is well defined and monotone, since if we take $\alpha(\vec{x}) \vdash_{\mathcal{T}\cup\{\varphi\}} \beta(\vec{x})$, it easily follows that $\alpha(\vec{x}) \wedge \varphi \vdash_{\mathcal{T}} \beta(\vec{x}) \wedge \varphi$. On the one hand, take $\alpha(\vec{x})$ such that $\alpha(\vec{x}) \vdash_{\mathcal{T}} \varphi$, apply $\mathfrak{e}'_{\vec{x}}$ to get $\alpha(\vec{x})$, and then send it to $\alpha(\vec{x}) \wedge \varphi$, and observe that $\alpha(\vec{x}) \wedge \varphi \dashv\vdash_{\mathcal{T}} \alpha(\vec{x})$ using the initial assumption on $\alpha(\vec{x})$. Conversely, take $\beta(\vec{x}) \in \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}\cup\{\varphi\}}(\vec{x})$, send it to $\beta(\vec{x}) \wedge \varphi$, and then apply $\mathfrak{e}'_{\vec{x}}$ to get $\beta(\vec{x}) \wedge \varphi \in \mathtt{LT}^{\mathcal{L}}_{\mathcal{T}\cup\{\varphi\}}(\vec{x})$. Observe that $\beta(\vec{x}) \wedge \varphi \dashv\vdash_{\mathcal{T}\cup\{\varphi\}} \beta(\vec{x})$. So $\mathfrak{e}'$ is indeed a natural isomorphism.

To conclude, we can say that the doctrine $(\mathtt{LT}^{\mathcal{L}}_{\mathcal{T}})_{\varphi}$ is again a doctrine of well-formed formulae.

Chapter 3

Rich doctrines and Henkin's Theorem

In this chapter, we explore a generalization of Henkin's Theorem [Hen49], a crucial result in first-order logic that is used to prove the Completeness Theorem. This theorem asserts that any consistent theory has a model. To extend this result, we take inspiration from the various steps involved in the classical approach, but from the perspective of doctrines. We will gradually introduce the necessary properties that a doctrine ("theory") P must possess to establish the existence of a morphism into the subsets doctrine ("model"). A key element of Henkin's proof is to extend the language adding constant symbols, and extend the theory to a rich theory, meaning that every provable sentence of the form $\exists x\varphi(x)$ has a corresponding constant c that makes $\varphi(c)$ valid. He then proves that the set of constant in the extended language is a model of the rich theory, hence in particular it is a model of the original theory.

We will generalize these results by defining a new doctrine $\underset{\rightarrow}{P}$ ("extended rich theory") starting from the given doctrine P. The properties of $\underset{\rightarrow}{P}$ will be explored in Sections from 3.1 to 3.7. Section 3.8 will be dedicated to the definition of suitable models for rich doctrines in both the elementary and non-elementary cases. Finally, we will conclude our discussion with a possible statement of the "Henkin Theorem" in the language of doctrines, Theorem 3.8.5.

To begin, let us consider a doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$. We will not assume that the doctrine has any specific structure at this time, but we will add the necessary properties as we proceed through the chapter.

3.1 The construction of the directed colimit $\underline{P}$

The directed preorder J:

For the whole chapter, $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is a fixed doctrine, unless otherwise specified.

For a fixed cardinal $\Lambda \neq 0$, define J the set of finite lists with different entries with values in $\{(X,\lambda)\}_{X\in\mathrm{ob}\mathbb{C},\lambda\in\Lambda}$. We ask the empty list to belong to J. Define a preorder in J as follows:

$$\big((X_1,x_1),\dots,(X_n,x_n)\big) \leq \big((Y_1,y_1),\dots,(Y_m,y_m)\big)$$

if and only if

$$\big\{(X_1,x_1),\dots,(X_n,x_n)\big\} \subseteq \big\{(Y_1,y_1),\dots,(Y_m,y_m)\big\}.$$

Whenever we have $\bar{X} \leq \bar{Y}$ in J, there exists a unique function $\tau\colon \{1,\dots,n\} \to \{1,\dots,m\}$ induced by the inclusion such that $(X_i,x_i) = (Y_{\tau(i)}, y_{\tau(i)})$ for all $i = 1,\dots,n$.

Observe that J is actually a directed preorder: given $\bar{X},\bar{Y} \in J$, define the list $\bar{Z}$ to be the juxtaposition of $\bar{X}$ with all the entries of $\bar{Y}$ that do not appear in $\bar{X}$; then $\bar{X} \leq \bar{Z} \geq \bar{Y}$.

On a sidenote, we point out that we will not to study the case $J = \emptyset$, since this would imply the category $\mathbb{C}$ to have no object.

The diagram $D\colon J \to$ Dct: Define the following diagram on J:

$$\begin{tikzcd}
J \arrow[rr, "D"] & & \mathbf{Dct} \\
\emptyset \arrow[rr, mapsto] \arrow[d, phantom, "\rotatebox{-90}{$\leq$}"] & & P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos} \arrow[d, "(F_{\bar{X}}\text{,}\mathfrak{f}_{\bar{X}})"] \\
\bar{X} = \big((X_1,x_1),\dots,(X_n,x_n)\big) \arrow[rr, mapsto] \arrow[d, phantom, "\rotatebox{-90}{$\leq$}"] & & P_{\Pi_{a=1}^{n}X_a}\colon \mathbb{C}^{\mathrm{op}}_{\Pi_{a=1}^{n}X_a} \to \mathbf{Pos} \arrow[d, "(F_{\bar{X}\bar{Y}}\text{,}\mathfrak{f}_{\bar{X}\bar{Y}})"] \\
\bar{Y} = \big((Y_1,y_1),\dots,(Y_m,y_m)\big) \arrow[rr, mapsto] & & P_{\Pi_{b=1}^{m}Y_b}\colon \mathbb{C}^{\mathrm{op}}_{\Pi_{b=1}^{m}Y_b} \to \mathbf{Pos}
\end{tikzcd}$$

where:

- $\mathbb{C}_{\Pi_{a=1}^{n}X_a}$ has the same objects of $\mathbb{C}$ and an arrow from A to B is actually a $\mathbb{C}$-arrow $\prod_{a=1}^{n} X_a \times A \to B$;
- $P_{\Pi_{a=1}^{n}X_a}(A) = P(\prod_{a=1}^{n} X_a \times A)$, with trivial definition on arrows—it is the usual Kleisli contruction starting from P for the pair $(\prod_{a=1}^{n} X_a, \top)$;
- $F_{\bar{X}}\big(f\colon A \to B\big) = \big(f \circ \mathrm{pr}_A\colon A \rightsquigarrow B\big)$ seen as the composition $\prod_{a=1}^{n} X_a \times A \to A \to B$;
- $(\mathfrak{f}_{\bar{X}})_A\colon P(A) \to P_{\Pi_{a=1}^{n}X_a}(A) = P(\prod_{a=1}^{n} X_a \times A)$ is the reindexing along the projection over A;
- $F_{\bar{X}\bar{Y}}\big(f\colon A \rightsquigarrow B\big) = \big(f \circ (\langle \mathrm{pr}_{\tau(1)},\dots,\mathrm{pr}_{\tau(n)}\rangle \times \mathrm{id}_A)\colon A \rightsquigarrow B\big)$ seen as the following composition $\prod_{b=1}^{m} Y_b \times A \to \prod_{a=1}^{n} X_a \times A \to B$. Here $\langle \mathrm{pr}_{\tau(1)},\dots,\mathrm{pr}_{\tau(n)}\rangle$ is the projection on the corresponding components from $\prod_{b=1}^{m} Y_b$ to $\prod_{a=1}^{n} X_a$, since X_i appears as the $\tau(i)$-th component of $\bar{Y}$;

- $(\mathfrak{f}_{\bar{X}\bar{Y}})_A \colon P(\prod_{a=1}^{n} X_a \times A) \to P(\prod_{b=1}^{m} Y_b \times A)$ is defined as the reindexing along the map $\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A$.

For any $\emptyset \leq \bar{X} \leq \bar{Y}$ compute that the composition $(F_{\bar{X}\bar{Y}}, \mathfrak{f}_{\bar{X}\bar{Y}})(F_{\bar{X}}, \mathfrak{f}_{\bar{X}}) = (F_{\bar{Y}}, \mathfrak{f}_{\bar{Y}})$. Indeed, between the base categories we have:

$$F_{\bar{X}} \colon \Big(f \colon A \to B \Big) \mapsto \Big(f\mathrm{pr}_A \colon \prod_{a=1}^{n} X_a \times A \to B \Big)$$

and then

$$F_{\bar{X}\bar{Y}} \colon f\mathrm{pr}_A \mapsto \Big(f\mathrm{pr}_A \circ (\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A) \colon \prod_{b=1}^{m} Y_b \times A \to B \Big) = \Big(f\mathrm{pr}_A \colon \prod_{b=1}^{m} Y_b \times A \to B \Big),$$

so $F_{\bar{X}\bar{Y}} F_{\bar{X}} = F_{\bar{Y}}$. Moreover $(\mathfrak{f}_{\bar{X}\bar{Y}})_A (\mathfrak{f}_{\bar{X}})_A = P(\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A) P(\mathrm{pr}_A) = P(\mathrm{pr}_A) = (\mathfrak{f}_{\bar{Y}})_A$. Observe that both equalities follow from the fact that $\mathrm{pr}_A \circ (\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A) = \mathrm{pr}_A$.

Similarly, for any $\bar{X} \leq \bar{Y} \leq \bar{Z}$ with induced functions respectively $\tau \colon \{1, \dots, n\} \to \{1, \dots, m\}$ and $\tau' \colon \{1, \dots, m\} \to \{1, \dots, s\}$, we compute the composition $(F_{\bar{Y}\bar{Z}}, \mathfrak{f}_{\bar{Y}\bar{Z}})(F_{\bar{X}\bar{Y}}, \mathfrak{f}_{\bar{X}\bar{Y}}) = (F_{\bar{X}\bar{Z}}, \mathfrak{f}_{\bar{X}\bar{Z}})$ using the fact that

$$(\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A) \circ (\langle \mathrm{pr}_{\tau'(1)}, \dots, \mathrm{pr}_{\tau'(m)} \rangle \times \mathrm{id}_A) = (\langle \mathrm{pr}_{\tau'\tau(1)}, \dots, \mathrm{pr}_{\tau'\tau(n)} \rangle \times \mathrm{id}_A).$$

So $D \colon J \to \mathbf{Dct}$ is indeed a diagram.

The colimit of D: Take the colimit of D in $\mathbf{Dct}$, $\underline{P} \colon \underline{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Pos}$, computed as in Section 1.3. Objects in the base category are the same as $\mathbb{C}$, since $F_{\bar{X}\bar{Y}}$'s act like the identity on objects. An arrow $[f, \bar{X}]$ in $\mathrm{Hom}_{\underline{\mathbb{C}}}(A, B)$—we write $[f, \bar{X}] \colon A \dashrightarrow B$—is the equivalence class of an arrow $f \colon \prod_{a=1}^{n} X_a \times A \to B$ for some fixed $\bar{X} = \big((X_1, x_1), \dots, (X_n, x_n)\big) \in J$. Recall that one has $[f, \bar{X}] = [f', \bar{Y}]$, for some $f' \colon \prod_{b=1}^{m} Y_b \times A \to B$ with $\bar{Y} = \big((Y_1, y_1), \dots, (Y_m, y_m)\big) \in J$ if and only if there exists $\bar{Z} \in J$ such that $\bar{X} \leq \bar{Z} \geq \bar{Y}$ making the following diagram commute:

$$\begin{tikzcd}
 & \prod_{a=1}^{n} X_a \times A \arrow[dr, "f"] & \\
\prod_{c=1}^{s} Z_c \times A \arrow[ur, "\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A"] \arrow[dr, "\langle \mathrm{pr}_{\tau'(1)}, \dots, \mathrm{pr}_{\tau'(m)} \rangle \times \mathrm{id}_A"'] & & B \\
 & \prod_{b=1}^{m} Y_b \times A \arrow[ur, "f'"'] &
\end{tikzcd}$$

Here τ and τ' are induced by $\bar{X} \leq \bar{Z}$ and $\bar{Y} \leq \bar{Z}$ in J respectively.

For any object A, we have $\underline{P}(A) \ni [\varphi, \bar{X}]$ for some $\varphi \in P(\prod_{a=1}^{n} X_a \times A)$. Here $[\varphi, \bar{X}] = [\varphi', \bar{Y}]$, where $\varphi' \in P(\prod_{b=1}^{m} Y_b \times A)$ if and only if there exists $\bar{Z} \in J$ such that $\bar{X} \leq \bar{Z} \geq \bar{Y}$ with induced function τ and τ' such that $P(\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A)\varphi = P(\langle \mathrm{pr}_{\tau'(1)}, \dots, \mathrm{pr}_{\tau'(m)} \rangle \times \mathrm{id}_A)\varphi'$

in $P(\prod_{c=1}^{s} Z_c \times A)$. Reindexing is defined in a common list of J: if $\big[f, \bar{X}\big] : A \dashrightarrow B$ and $\big[\psi, \bar{Y}\big] \in \underline{P}(B)$, take $\bar{X} \leq \bar{Z} \geq \bar{Y}$; then

$$\begin{aligned}
&\underline{P}(\,[f, \bar{X}]\,)\,[\psi, \bar{Y}] \\
&\qquad = \underline{P}(\,\Big[f \circ (\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A), \bar{Z}\Big]\,)\,\Big[P(\langle \mathrm{pr}_{\tau'(1)}, \dots, \mathrm{pr}_{\tau'(m)} \rangle \times \mathrm{id}_B)\psi, \bar{Z}\Big] \\
&= \Big[P(\langle \mathrm{pr}_1, \dots, \mathrm{pr}_s, f \circ (\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A)\rangle)P(\langle \mathrm{pr}_{\tau'(1)}, \dots, \mathrm{pr}_{\tau'(m)} \rangle \times \mathrm{id}_B)\psi, \bar{Z}\Big] \\
&\qquad\qquad = \Big[P(\langle \mathrm{pr}_{\tau'(1)}, \dots, \mathrm{pr}_{\tau'(m)}, f \circ (\langle \mathrm{pr}_{\tau(1)}, \dots, \mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A)\rangle)\psi, \bar{Z}\Big] .
\end{aligned}$$

$$\begin{tikzcd}
\prod Z_c \times A \arrow[r, "\langle \mathrm{pr}_{\tau(1)},\dots,\mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A"] \arrow[d, "\langle \mathrm{pr}_1,\dots,\mathrm{pr}_s, f\circ(\langle \mathrm{pr}_{\tau(1)},\dots,\mathrm{pr}_{\tau(n)} \rangle \times \mathrm{id}_A)\rangle"] & \prod X_a \times A \arrow[r, "f"] & B \\
\prod Z_c \times B \arrow[r, "\langle \mathrm{pr}_{\tau'(1)},\dots,\mathrm{pr}_{\tau'(m)} \rangle \times \mathrm{id}_B"'] & \prod Y_b \times B &
\end{tikzcd}$$

Remark 3.1.1. Call $(\underline{F}, \underline{\mathfrak{f}})\colon P \to \underline{P}$ the map in the colimit starting from $D(\emptyset)$: the functor $\underline{F}$ maps a $\mathbb{C}$-arrow $f\colon A \to B$ into $[f, \emptyset]\colon A \dashrightarrow B$, a component of the natural transformation $\underline{\mathfrak{f}}_A$ sends $\alpha \in P(A)$ into $[\alpha, \emptyset] \in \underline{P}(A)$. Moreover, by the universal property of the Kleisli constructions, any morphism $D(\bar{X}) \to \underline{P}$ is uniquely determined by the homomorphism $(\underline{F}, \underline{\mathfrak{f}})\colon P \to \underline{P}$ and a choice of a constant $\mathbf{t} \to \prod_{a=1}^{n} X_a$. By definition of colimit, any doctrine homomorphism $(G, \mathfrak{g})\colon \underline{P} \to R$ is uniquely determined by its precompositions with $(\underline{F}, \underline{\mathfrak{f}})$ and a choice of a constant for any pair (X, λ) for every object X in $\mathbb{C}$ and any $\lambda \in \Lambda$.

Remark 3.1.2. Note that the same construction can be made if we change the cardinals over the objects: take for any object X a cardinal Λ_X, and call J the set of finite lists with values in $\{(X, \lambda)\}_{X \in \mathrm{ob}\mathbb{C}, \lambda \in \Lambda_X}$. In this case we just ask for the existence of at least one cardinal Λ_X different from 0.

3.2 Listing formulae and labelling new constants

For the whole chapter, $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is a fixed implicational existential doctrine, with a small base category, unless otherwise specified.

Call $\Lambda = \mathrm{card}\big(\bigsqcup_{X \in \mathrm{ob}\mathbb{C}} P(X)\big)$ and build the colimit doctrine $\underline{P}$ with respect to this cardinal. Since by Theorem 2.4.3 every doctrine and morphism that appear in the diagram D are implicational and existential, also $\underline{P}$ is implicational and existential, as seen in Proposition 1.3.2. First of all consider all objects of $\mathbb{C}$—hence also all objects of $\underline{\mathbb{C}}$—as $\mathrm{ob}\mathbb{C} = \{B\}_{B \in \mathrm{ob}\mathbb{C}}$. For any fixed B, we can surely list all elements of $\underline{P}(B)$ as $\Big\{\Big[\varphi_j^B, \bar{X}^{(B,j)}\Big]\Big\}_{j \in \Lambda}$ where we fix a representative

$$\varphi_j^B \in P(\prod_{a=1}^{n^{(B,j)}} X_a^{(B,j)} \times B)$$

for a given list $\bar{X}^{(B,j)} = \big((X_1^{(B,j)}, x_1^{(B,j)}), \dots, (X_{n^{(B,j)}}^{(B,j)}, x_{n^{(B,j)}}^{(B,j)})\big)$ in J. Now consider all formulae of the kind

$$\exists_{\mathbf{t}}^{B} \left[\varphi_j^B, \bar{X}^{(B,j)}\right], \text{ for all } j \in \Lambda.$$

Then we have in $\underline{P}(\mathbf{t})$

$$\exists_{\mathbf{t}}^{B} \left[\varphi_j^B, \bar{X}^{(B,j)}\right] = \left[\exists^B_{\Pi X_a^{(B,j)}} \varphi_j^B, \bar{X}^{(B,j)}\right],$$

where we recall the adjunction:

$$\begin{tikzcd} P(\prod X_a^{(B,j)} \times B) \arrow[r, bend left, "\exists^B_{\Pi X_a^{(B,j)}}"] \arrow[r, phantom, "\perp"] & P(\prod X_a^{(B,j)}) \arrow[l, bend left, "P(\mathrm{pr}_1)"] \end{tikzcd}$$

Here we write pr_1 meaning the first projection from the product $(\prod X_a^{(B,j)}) \times B$.
For any fixed $B \in \mathrm{ob}\mathbb{C}$ and for any $j \in \Lambda$ define d_j^B as follows:

- if $j = 0$, then d_0^B is the smallest ordinal such that
$$d_0^B > x_a^{(B,0)} \text{ for any } a = 1, \dots, n^{(B,0)};$$
- if j is a successor or a limit ordinal, then d_j^B is the smallest ordinal such that $d_j^B > d_h^B$ for all $h < j$ and such that
$$d_j^B > x_a^{(B,k)} \text{ for any } a = 1, \dots, n^{(B,k)} \text{ and } k \leq j.$$

Note that in particular for any $j \in \Lambda$:

$$(B, d_j^B) \notin \{(X_a^{(B,j)}, x_a^{(B,j)})\}_{a=1}^{n^{(B,j)}}.$$

Now, since

$$\varphi_j^B \in P(\prod_{a=1}^{n^{(B,j)}} X_a^{(B,j)} \times B)$$

we can take its equivalence class fixing $\bar{X}^{(B,j)} \in J$, hence we end up in $\underline{P}(B)$, or fixing the list $\bar{X}_\star^{(B,j)} = \big((X_1^{(B,j)}, x_1^{(B,j)}), \dots, (X_{n^{(B,j)}}^{(B,j)}, x_{n^{(B,j)}}^{(B,j)}), (B, d_j^B)\big)$—i.e. adding (B, d_j^B) to the list $\bar{X}^{(B,j)}$—, hence we end up in $\underline{P}(\mathbf{t})$. We compute in $\underline{P}(\mathbf{t})$:

$$\exists_{\mathbf{t}}^{B} \left[\varphi_j^B, \bar{X}^{(B,j)}\right] \longrightarrow \left[\varphi_j^B, \bar{X}_\star^{(B,j)}\right].$$

Define in $P(\prod X_a^{(B,j)} \times B)$

$$\psi_j^B := P(\mathrm{pr}_1) \exists^B_{\Pi X_a^{(B,j)}} \varphi_j^B \longrightarrow \varphi_j^B$$

so that taking its class fixing $\bar{X}_\star^{(B,j)}$ we get

$$\left[\psi_j^B, \bar{X}_\star^{(B,j)}\right] \in \underline{P}(\mathbf{t}), \text{ with } \left[\psi_j^B, \bar{X}_\star^{(B,j)}\right] = \exists_{\mathbf{t}}^B \left[\varphi_j^B, \bar{X}^{(B,j)}\right] \longrightarrow \left[\varphi_j^B, \bar{X}_\star^{(B,j)}\right]. \tag{3.1}$$

3.3 The construction of the directed colimit $\underset{\rightarrow}{\underline{P}}$

Starting from $\underline{P}$ defined in the last section, we do another construction.

The directed preorder I and the diagram $\mathbf{\Delta}\colon I \to$ Dct: Define the poset I of finite sets of pairs of the kind (B, j), where $B \in \operatorname{ob}\mathbb{C}$ and $j \in \Lambda$, ordered by inclusion. We want also the empty set to belong to I.

$$\begin{tikzcd}
I \arrow[r, "\mathbf{\Delta}"] & \mathbf{Dct} \\
\emptyset \arrow[r, mapsto] \arrow[d, phantom, sloped, "\subseteq"] & \underline{P}\colon \underline{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Pos} \arrow[d, "{(\mathrm{id}, \mathfrak{f}_{\mathcal{U}})}"] \\
\mathcal{U} = \{(B_1, j_1), \ldots, (B_n, j_n)\} \arrow[r, mapsto] \arrow[d, phantom, sloped, "\subseteq"] & \underline{P}^{\mathcal{U}}\colon \underline{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Pos} \arrow[d, "{(\mathrm{id}, \mathfrak{f}_{\mathcal{U}\mathcal{V}})}"] \\
\mathcal{V} = \{(B_1, j_1), \ldots, (B_{n+m}, j_{n+m})\} \arrow[r, mapsto] & \underline{P}^{\mathcal{V}}\colon \underline{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Pos}
\end{tikzcd}$$

where:

- $\underline{P}^{\mathcal{U}}(A) = \underline{P}(A)_{\downarrow \underline{P}(!) \bigwedge_{i=1}^n \left[\psi_{j_i}^{B_i}, \bar{X}_\star^{(B_i,j_i)}\right]}$, with trivial definition on arrows—it is the usual Kleisli contruction starting from $\underline{P}$ for the pair $(\mathbf{t}, \bigwedge_{i=1}^n \left[\psi_{j_i}^{B_i}, \bar{X}_\star^{(B_i,j_i)}\right])$;
- $(\mathfrak{f}_{\mathcal{U}})_A\colon \underline{P}(A) \to \underline{P}(A)_{\downarrow \underline{P}(!) \bigwedge_{i=1}^n \left[\psi_{j_i}^{B_i}, \bar{X}_\star^{(B_i,j_i)}\right]}$ is the assignment
$$\left[\alpha, \bar{Y}\right] \mapsto \left[\alpha, \bar{Y}\right] \wedge \underline{P}(!) \bigwedge_{i=1}^n \left[\psi_{j_i}^{B_i}, \bar{X}_\star^{(B_i,j_i)}\right];$$
- $(\mathfrak{f}_{\mathcal{U}\mathcal{V}})_A\colon \underline{P}(A)_{\downarrow \underline{P}(!) \bigwedge_{i=1}^n \left[\psi_{j_i}^{B_i}, \bar{X}_\star^{(B_i,j_i)}\right]} \to \underline{P}(A)_{\downarrow \underline{P}(!) \bigwedge_{i=1}^{n+m} \left[\psi_{j_i}^{B_i}, \bar{X}_\star^{(B_i,j_i)}\right]}$ is again the assignment $\left[\alpha, \bar{Y}\right] \mapsto \left[\alpha, \bar{Y}\right] \wedge \underline{P}(!) \bigwedge_{i=1}^{n+m} \left[\psi_{j_i}^{B_i}, \bar{X}_\star^{(B_i,j_i)}\right]$.

Use associativity and commutativity of conjunction to observe that this is a diagram.

The colimit of $\mathbf{\Delta}$: Take the colimit of $\mathbf{\Delta}$ in **Dct**, $\underset{\rightarrow}{\underline{P}}\colon \underline{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Pos}$. The base category is $\underline{\mathbb{C}}$, since all functors in 1-arrows of the diagram are identities.

We recall from Proposition 1.3.1 that the doctrine is defined as

$$\underset{\rightarrow}{\underline{P}}(C) = \bigsqcup_{\mathcal{U} \in I} \underline{P}^{\mathcal{U}}(C) \Big/ \sim,$$

where $[\left[\alpha, \bar{Y}\right], \mathcal{U}] \in \underset{\rightarrow}{\underline{P}}(C)$ for some $\left[\alpha, \bar{Y}\right] \in \underline{P}(C)$ such that $\left[\alpha, \bar{Y}\right] \leq \underline{P}(!) \bigwedge_{i=1}^n \left[\psi_{j_i}^{B_i}, \bar{X}_\star^{(B_i,j_i)}\right]$

with a fixed $\mathcal{U} = \{(B_1, j_1), \dots, (B_n, j_n)\} \in I$. Here $[\left[\alpha, \bar{Y}\right], \mathcal{U}] = [\left[\beta, \bar{Z}\right], \mathcal{V}]$, for $\left[\beta, \bar{Z}\right] \in \underline{P}(C)$, $\left[\beta, \bar{Z}\right] \leq \underline{P}(!) \bigwedge_{r=1}^{m} \left[\psi_{l_r}^{D_r}, \bar{X}_{\star}^{(D_r, l_r)}\right]$ with a fixed $\mathcal{V} = \{(D_1, l_1), \dots, (D_m, l_m)\} \in I$, if there exists a $\mathcal{W} = \{(A_1, q_1), \dots, (A_z, q_z)\} \supseteq \mathcal{U}, \mathcal{V}$ such that in $\underline{P}(C)$ we have

$$\left[\alpha, \bar{Y}\right] \wedge \underline{P}(!) \bigwedge_{k=1}^{z} \left[\psi_{q_k}^{A_k}, \bar{X}_{\star}^{(A_k, q_k)}\right] = \left[\beta, \bar{Z}\right] \wedge \underline{P}(!) \bigwedge_{k=1}^{z} \left[\psi_{q_k}^{A_k}, \bar{X}_{\star}^{(A_k, q_k)}\right].$$

This assignment appropriately extends to arrows in $\underline{\mathbb{C}}$.

Remark 3.3.1. We revise in a single diagram the two constructions we did above:

$$\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \arrow[r, "\underline{F}^{\mathrm{op}}"] \arrow[dr, bend right, "P"'] \arrow[dr, phantom, bend left=20, "\overset{\underline{\mathfrak{f}}}{\longrightarrow}"] & \underline{\mathbb{C}}^{\mathrm{op}} \arrow[r, "\mathrm{id}^{\mathrm{op}}"] \arrow[d, "\underline{P}"] & \underline{\mathbb{C}}^{\mathrm{op}} \arrow[dl, bend left, "\underrightarrow{P}"] \arrow[dl, phantom, bend right=20, "\overset{\underrightarrow{\mathfrak{f}}}{\longrightarrow}"'] \\
& \mathbf{Pos} &
\end{tikzcd}$$

The doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ has a small base category, and it is implicational and existential. Call the composition $(\mathrm{id}, \underrightarrow{\mathfrak{f}}) \circ (\underline{F}, \underline{\mathfrak{f}}) = (F, \mathfrak{f})$, so that both F and $\mathfrak{f}$ are the quotient map:

$$\begin{aligned} F\colon \mathbb{C} \to \underline{\mathbb{C}}, \qquad & (f\colon A \to B) \mapsto ([f, \emptyset]\colon A \dashrightarrow B) \\ \mathfrak{f}_A\colon P(A) \to \underrightarrow{P}(A), \qquad & \alpha \mapsto [[\alpha, \emptyset], \emptyset] \end{aligned}$$

This morphism preserves implicational and elementary structure.

3.4 $\underrightarrow{P}$ is rich

We extend the concept of richness for a theory to the language of doctrines.

Definition 3.4.1. Let $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ be an existential doctrine. Then R is *rich* if for all $A \in \mathrm{ob}\mathbb{D}$ and for all $\sigma \in R(A)$ there exists a $\mathbb{D}$-arrow $d\colon \mathbf{t} \to A$ such that $\exists_{\mathbf{t}}^{A}\sigma \leq R(d)\sigma$.

Remark 3.4.2. For every object A in the base category of a rich doctrine, there exists an arrow from the terminal object to A.

Example 3.4.3. The subsets doctrine $\mathscr{P}\colon \mathrm{Set}^{\mathrm{op}} \to \mathbf{Pos}$ is not rich, since there exists no arrow $\mathbf{t} \to \emptyset$. However, we can remove the empty set from the base category and consider the doctrine $\mathscr{P}_{*}\colon \mathrm{Set}_{*}^{\mathrm{op}} \to \mathbf{Pos}$, which is rich.

Theorem 3.4.4. Let P be an implicational existential doctrine with a small base category. Then the doctrine $\underrightarrow{P}$ is rich.

Proof. Now, given $[\left[\varphi, \bar{Y}\right], \mathcal{U}] \in \underrightarrow{P}(B)$, we will find an arrow $\left[c, \bar{Z}\right]\colon \mathbf{t} \dashrightarrow B$ such that

$$\underrightarrow{\exists}_{\mathbf{t}}^{B}[\left[\varphi, \bar{Y}\right], \mathcal{U}] \leq \underrightarrow{P}(\left[c, \bar{Z}\right])[\left[\varphi, \bar{Y}\right], \mathcal{U}].$$

Note that $[\left[\varphi, \bar{Y}\right], \mathcal{U}] = [\left[\varphi, \bar{Y}\right], \emptyset]$: indeed taking $\mathcal{U} \supseteq \mathcal{U}, \emptyset$ we have in $\underline{P}(B)$

$$\left[\varphi, \bar{Y}\right] \wedge \underline{P}(!) \bigwedge_{i=1}^{n} \left[\psi_{j_i}^{B_i}, \bar{X}_{\star}^{(B_i, j_i)}\right] = \left[\varphi, \bar{Y}\right].$$

Moreover, since $\left[\varphi, \bar{Y}\right] \in \underline{P}(B)$ in particular $\left[\varphi, \bar{Y}\right] = \left[\varphi_j^B, \bar{X}^{(B,j)}\right]$ for some $j \in \Lambda$, with

$$\varphi_j^B \in P(\prod_{a=1}^{n^{(B,j)}} X_a^{(B,j)} \times B).$$

First of all compute $\underset{\to}{\exists}{}_{\mathbf{t}}^{B}[\left[\varphi, \bar{Y}\right], \mathcal{U}] = \underset{\to}{\exists}{}_{\mathbf{t}}^{B}[\left[\varphi_j^B, \bar{X}^{(B,j)}\right], \emptyset] = [\underline{\exists}{}_{\mathbf{t}}^{B} \left[\varphi_j^B, \bar{X}^{(B,j)}\right], \emptyset]$. Then define $\left[c, (B, d_j^B)\right] : \mathbf{t} \dashrightarrow B$ as the equivalence class of the identity

$$c = \mathrm{id}_B \colon B \to B,$$

and compute

$$\underset{\to}{P}(\left[c, (B, d_j^B)\right])[\left[\varphi, \bar{Y}\right], \mathcal{U}] = \underset{\to}{P}(\left[c, (B, d_j^B)\right])[\left[\varphi_j^B, \bar{X}^{(B,j)}\right], \emptyset] = [\left[\varphi_j^B, \bar{X}_{\star}^{(B,j)}\right], \emptyset]$$

Then, in $\underset{\to}{P}(\mathbf{t})$ we have

$$\underset{\to}{\exists}{}_{\mathbf{t}}^{B}[\left[\varphi, \bar{Y}\right], \mathcal{U}] \leq \underset{\to}{P}(\left[c, (B, d_j^B)\right])[\left[\varphi, \bar{Y}\right], \mathcal{U}]$$

if and only if

$$[[\top, \emptyset], \emptyset] \leq \underset{\to}{\exists}{}_{\mathbf{t}}^{B}[\left[\varphi, \bar{Y}\right], \mathcal{U}] \longrightarrow \underset{\to}{P}(\left[c, (B, d_j^B)\right])[\left[\varphi, \bar{Y}\right], \mathcal{U}],$$

i.e.

$$[[\top, \emptyset], \emptyset] \leq [\underline{\exists}{}_{\mathbf{t}}^{B} \left[\varphi_j^B, \bar{X}^{(B,j)}\right], \emptyset] \longrightarrow [\left[\varphi_j^B, \bar{X}_{\star}^{(B,j)}\right], \emptyset];$$

but then compute the implication in $\underline{P}(\mathbf{t})$ as seen in (3.1) to get

$$[[\top, \emptyset], \emptyset] \leq [\left[\psi_j^B, \bar{X}_{\star}^{(B,j)}\right], \emptyset]$$

which holds since $[\left[\psi_j^B, \bar{X}_{\star}^{(B,j)}\right], \emptyset]$ is the top element of $\underset{\to}{P}(\mathbf{t})$ by definition: take $\{(B, j)\} \supseteq \emptyset$ and observe that in $\underline{P}(\mathbf{t})$:

$$[\top, \emptyset] \wedge \left[\psi_j^B, \bar{X}_{\star}^{(B,j)}\right] = \left[\psi_j^B, \bar{X}_{\star}^{(B,j)}\right] \wedge \left[\psi_j^B, \bar{X}_{\star}^{(B,j)}\right].$$

This concludes the proof that $\underset{\to}{P}$ is rich. $\square$

3.5 Consistency of $\underrightarrow{P}$

Definition 3.5.1. A doctrine $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ is *consistent* if there exists a pair $a, b \in R(\mathbf{t})$ such that $a \nleq b$. Moreover, R is *two-valued* if it is consistent and there exists a pair $a, b \in R(\mathbf{t})$ such that $a \nleq b$ and for all $c \in R(\mathbf{t})$ one has $a \leq c$ or $b \leq c$.

> For the whole chapter, $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is a fixed bounded implicational existential consistent doctrine, with a small base category, unless otherwise specified.

Our goal is to show that the new doctrine $\underrightarrow{P}$ is consistent: we must be careful not to collapse fibers of $\underrightarrow{P}$ to the trivial poset.

Lemma 3.5.2. If $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ is a doctrine with both $\top$ and $\bot$. Then the following are equivalent:

(i) $R(\mathbf{t}) \neq \{\star\}$;

(ii) $\top_{\mathbf{t}} \nleq \bot_{\mathbf{t}}$;

(iii) R is consistent;

(iv) R is two-valued.

Proof. ((i) $\implies$ (ii)) If $\top_{\mathbf{t}} \leq \bot_{\mathbf{t}}$, then for all $a \in R(\mathbf{t})$ we have $\bot_{\mathbf{t}} \leq a \leq \top_{\mathbf{t}} \leq \bot_{\mathbf{t}}$, hence for all a we have $a = \bot_{\mathbf{t}}$, hence $R(\mathbf{t})$ is a singleton.
((ii) $\implies$ (i)) Trivial.
((iv) $\implies$ (iii)) By definition.
((iii) $\implies$ (ii)) If $\top_{\mathbf{t}} \leq \bot_{\mathbf{t}}$, then for all $a, b \in R(\mathbf{t})$ we have $a \leq \top_{\mathbf{t}} \leq \bot_{\mathbf{t}} \leq b$, hence R cannot be consistent.
((ii) $\implies$ (iv)) Take $a = \top_{\mathbf{t}}$ and $b = \bot_{\mathbf{t}}$ and observe that for all $c \in R(\mathbf{t})$ we have $b = \bot_{\mathbf{t}} \leq c$. ☐

Remark 3.5.3. Let R be an existential doctrine with bottom element. If R is consistent and rich, then each of its fiber is non-trivial—i.e. it is not a singleton. Indeed, suppose $R(D) = \{\bot_D = \top_D\}$ for some object D in the base category. Then there exists a $d\colon \mathbf{t} \to D$ such that $\exists^D_{\mathbf{t}} \top_D = R(d)\top_D = \exists^D_{\mathbf{t}} \bot_D = R(d)\bot_D$, in particular $\top_{\mathbf{t}} = \bot_{\mathbf{t}}$, which is absurd since R is consistent.

We want to find the conditions making $\underrightarrow{P}$ a consistent doctrine as well. Using the lemma above, we want $[[\top, \emptyset], \emptyset] \nleq [[\bot, \emptyset], \emptyset]$ in $\underrightarrow{P}(\mathbf{t})$.
However, $[[\top, \emptyset], \emptyset] \leq [[\bot, \emptyset], \emptyset]$ if and only if there exists $\mathcal{U} = \{(B_1, j_1), \ldots, (B_n, j_n)\} \in I$ such that

$$\bigwedge_{i=1}^{q} \left[\psi^{B_i}_{j_i}, \bar{X}^{(B_i,j_i)}_{\star}\right] \leq [\bot, \emptyset] \text{ in } \underline{P}(\mathbf{t}). \tag{3.2}$$

We want to prove this to be a contradiction by induction on q. If $q = 0$, we get $[\top, \emptyset] \leq [\bot, \emptyset]$, i.e. there exists $\bar{Y} = \big((Y_1, y_1), \ldots, (Y_m, y_m)\big) \in J$ such that in $P(\prod_{b=1}^{m} Y_b)$

$$P(!_{\Pi Y_b})(\top) \leq P(!_{\Pi Y_b})(\bot)$$

i.e. $\top \leq \bot$ in $P(\prod_{b=1}^{m} Y_b)$. It follows from this that a stronger requirement on P is needed: not only $P(\mathbf{t})$ must not be a singleton, but also each $P(A)$ must not be a singleton, for every $A \in \mathrm{ob}\mathbb{C}$. Otherwise, $\underline{P}(\mathbf{t})$ is trivial, hence also $\underset{\rightarrow}{P}(\mathbf{t})$ is trivial. So, from now on we suppose that P has bottom element and has each $P(A)$ non-trivial.

For the whole chapter, $P \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is a fixed bounded implicational existential doctrine, with non-trivial fibers, and with a small base category, unless otherwise specified.

With this additional assumption, we get a contradiction in the case $q = 0$. Suppose now (3.2) to be a contradiction for q; we will take the rest of the section to understanding when also $q + 1$ gives a contradiction. Suppose

$$\bigwedge_{i=1}^{q+1} \left[\psi_{j_i}^{B_i}, \bar{X}_{\star}^{(B_i, j_i)}\right] \leq [\bot, \emptyset] \text{ in } \underline{P}(\mathbf{t}),$$

$$\text{i.e. } \bigwedge_{i=1}^{q} \left[\psi_{j_i}^{B_i}, \bar{X}_{\star}^{(B_i, j_i)}\right] \wedge \left[\psi_{j_{q+1}}^{B_{q+1}}, \bar{X}_{\star}^{(B_{q+1}, j_{q+1})}\right] \leq [\bot, \emptyset] \text{ in } \underline{P}(\mathbf{t}).$$

For the sake of simplicity we write ψ instead of $\psi_{j_{q+1}}^{B_{q+1}}$. Moreover, up to a permutation of the indices $i = 1, \ldots, q + 1$, we can suppose that $d_{j_{q+1}}^{B_{q+1}} \geq d_{j_i}^{B_i}$ for $i = 1, \ldots, q$.
Compute $\bigwedge_{i=1}^{q} \left[\psi_{j_i}^{B_i}, \bar{X}_{\star}^{(B_i, j_i)}\right]$ as the class of some θ paired with a list $\bar{T}$ of J with entries in

$$\mathcal{F} \coloneqq \bigcup_{i=1}^{q} \left\{ \left(X_a^{(B_i, j_i)}, x_a^{(B_i, j_i)}\right) \right\}_{a=1}^{n^{(B_i, j_i)}} \cup \bigcup_{i=1}^{q} \left\{ \left(B_i, d_{j_i}^{B_i}\right) \right\}.$$

Then call

$$\mathcal{G} \coloneqq \left\{ \left(X_a^{(B_{q+1}, j_{q+1})}, x_a^{(B_{q+1}, j_{q+1})}\right) \right\}_{a=1}^{n^{(B_{q+1}, j_{q+1})}};$$

and rename the pairs:

$$\begin{aligned} \mathcal{F} \cap \mathcal{G} &= \{(Z_b, z_b)\}_{b=1}^{\overline{b}}, \\ \mathcal{F} \smallsetminus (\mathcal{F} \cap \mathcal{G}) &= \{(W_c, w_c)\}_{c=1}^{\overline{c}}, \\ \mathcal{G} \smallsetminus (\mathcal{F} \cap \mathcal{G}) &= \{(V_e, v_e)\}_{e=1}^{\overline{e}}. \end{aligned}$$

Observe that $(B_{q+1}, d_{j_{q+1}}^{B_{q+1}}) \notin \mathcal{G} \cup \mathcal{F}$: it does not belong to $\mathcal{G}$ by definition of $d_{j_{q+1}}^{B_{q+1}}$, it is different

from all the pairs $(B_i, d^{B_i}_{j_i})$ for $i = 1, \dots, q$ since we are taking the conjunction of $q + 1$ formulae by assumption, and it is different from all the pairs $(X^{(B_i,j_i)}_a, x^{(B_i,j_i)}_a)$ for $i = 1, \dots, q$ and $a = 1, \dots, n^{(B_i,j_i)}$ since $d^{B_{q+1}}_{j_{q+1}} \geq d^{B_i}_{j_i} > x^{(B_i,j_i)}_a$ for $i = 1, \dots, q$ and $a = 1, \dots, n^{(B_i,j_i)}$.
From now on, we write (B, d) instead of $(B_{q+1}, d^{B_{q+1}}_{j_{q+1}})$ in order to lighten the notation. We compute $\left[\theta, \bar{T}\right] \wedge \left[\psi, \bar{X}^{(B,j_{q+1})}_\star\right]$ as the equivalence class of an element in

$$P(\overbrace{\prod W_c \times \underbrace{\prod Z_b}^{\mathcal{F}} \times \prod V_e}_{\mathcal{G}} \times B)$$

paired with the list

$$\bar{S} = \big(\dots, (W_c, w_c), \dots, (Z_b, z_b), \dots, (V_e, v_e), \dots, (B, d)\big).$$

We can assume $\theta \in P(\Pi W_c \times \Pi Z_b)$ and

$$[\psi', (\dots, (Z_b, z_b), \dots, (V_e, v_e), \dots, (B, d))] = \left[\psi, \bar{X}^{(B,j_{q+1})}_\star\right]$$

where $\psi' \in P(\Pi Z_b \times \Pi V_e \times B)$ is a reindexing along a suitable permutation of ψ. We can do so recalling that

$$\mathcal{G} = \left\{\left(X^{(B_{q+1},j_{q+1})}_a, x^{(B_{q+1},j_{q+1})}_a\right)\right\}_{a=1}^{n^{(B_{q+1},j_{q+1})}} = \{(Z_b, z_b)\}_{b=1}^{\bar{b}} \cup \{(V_e, v_e)\}_{e=1}^{\bar{e}}.$$

Then

$$\left[\theta, \bar{T}\right] \wedge \left[\psi, \bar{X}^{(B,j_{q+1})}_\star\right] = \left[P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)\theta \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_4\rangle)\psi', \bar{S}\right] \in \underline{P}(\mathbf{t}).$$

Then $\left[\theta, \bar{T}\right] \wedge \left[\psi, \bar{X}^{(B,j_{q+1})}_\star\right] \leq [\bot, \emptyset]$ if and only if there exists a set $\{(Y_h, y_h)\}_{h=1}^{\bar{h}}$, disjoint from $\mathcal{F} \cup \mathcal{G} \cup \{(B, d)\}$ such that in $P(\Pi W_c \times \Pi Z_b \times \Pi V_e \times B \times \Pi Y_h)$ one has

$$P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)\theta \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_4\rangle)\psi' \leq \bot$$

if and only if in in $P(\Pi W_c \times \Pi Z_b \times \Pi V_e \times \Pi Y_h \times B)$ one has

$$P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)\theta \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_5\rangle)\psi' \leq \bot = P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_4\rangle)\bot$$

if and only if, using $\exists^B_{\Pi W \times \Pi Z \times \Pi V \times \Pi Y} \dashv P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_4\rangle)$, in $P(\Pi W_c \times \Pi Z_b \times \Pi V_e \times \Pi Y_h)$ one has

$$\exists^B_{\Pi W \times \Pi Z \times \Pi V \times \Pi Y} \left(P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)\theta \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_5\rangle)\psi'\right) \leq \bot;$$

then use Frobenius reciprocity, and note that $P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle) = P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_4\rangle)P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)$ as the composition of the projections from $\Pi W_c \times \Pi Z_b \times \Pi V_e \times \Pi Y_h \times B$ to $\Pi W_c \times \Pi Z_b \times \Pi V_e \times \Pi Y_h$

to $\Pi W_c \times \Pi Z_b$ in order to get

$$\exists^B_{\Pi W \times \Pi Z \times \Pi V \times \Pi Y} P(\langle \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_5 \rangle)\psi' \wedge P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)\theta \leq \bot.$$

Claim 3.5.4. $\top \leq \exists^B_{\Pi W \times \Pi Z \times \Pi V \times \Pi Y} P(\langle \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_5 \rangle)\psi'$.

If this is the case, then we get $P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)\theta \leq \bot$, hence we have

$$[P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)\theta, (\dots, (W_c, w_c), \dots, (Z_b, z_b), \dots, (V_e, v_e), \dots, (Y_h, y_h), \dots)] =$$

$$= \left[\theta, \bar{T}\right] = \bigwedge_{i=1}^{q} \left[\psi^{B_i}_{j_i}, \bar{X}^{(B_i, j_i)}_{\star}\right] \leq [\bot, \emptyset]$$

which is (3.2), a contradiction for our inductive hypothesis.
Now recall the definition of

$$\psi = \psi^{B_{q+1}}_{j_{q+1}} = P(\mathrm{pr}_1)\exists^B_{\Pi X_a^{(B_{q+1}, j_{q+1})}} \varphi^{B_{q+1}}_{j_{q+1}} \longrightarrow \varphi^{B_{q+1}}_{j_{q+1}}.$$

Using the same permutation that defines ψ' and naturality of the existential quantifier, the claim above becomes equivalent to

$$\top \leq \exists^B_{\Pi W \times \Pi X \times \Pi Y} P(\langle \mathrm{pr}_2, \mathrm{pr}_4 \rangle)\psi.$$

We have

$$\begin{aligned} \exists^B_{\Pi W \times \Pi X \times \Pi Y} P(\langle \mathrm{pr}_2, \mathrm{pr}_4 \rangle)\psi &= \exists^B_{\Pi W \times \Pi X \times \Pi Y} P(\langle \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_4 \rangle) P(\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\psi \\ &= P(\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle)\exists^B_{\Pi X \times \Pi Y} P(\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\psi, \end{aligned}$$

so it is sufficient to prove $\top \leq \exists^B_{\Pi X \times \Pi Y} P(\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\psi$. Substituting ψ with its definition, omitting superscripts and subscripts of $\varphi^{B_{q+1}}_{j_{q+1}}$ and $X_a^{(B_{q+1}, j_{q+1})}$ we want to prove

Claim 3.5.5. $\top \leq \exists^B_{\Pi X \times \Pi Y}\Big(P(\mathrm{pr}_1)\exists^B_{\Pi X}\varphi \longrightarrow P(\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\varphi\Big)$ in $P(\Pi X \times \Pi Y)$.

For the whole chapter, $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is a fixed Boolean existential doctrine, with non-trivial fibers, and with a small base category, unless otherwise specified.

The doctrine P is Boolean, so we can suppose that

$$\top = \left(\exists^B_{\Pi X \times \Pi Y} P(\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\varphi\right) \vee \left(\neg \exists^B_{\Pi X \times \Pi Y} P(\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\varphi\right).$$

Then, use naturality of $\exists^B_{(-)}$ to write $P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)\exists^B_{\Pi X \times \Pi Y} P(\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle)\varphi$ instead of $P(\mathrm{pr}_1)\exists^B_{\Pi X}\varphi$.

Hence now it is sufficient to prove

$$\begin{aligned}&\exists^B_{\Pi X\times\Pi Y}P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi\\&\qquad\leq \exists^B_{\Pi X\times\Pi Y}\big(P(\langle \mathrm{pr}_1,\mathrm{pr}_2\rangle)\exists^B_{\Pi X\times\Pi Y}P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi\to P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi\big)\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}&\neg\exists^B_{\Pi X\times\Pi Y}P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi\\&\qquad\leq \exists^B_{\Pi X\times\Pi Y}\big(P(\langle \mathrm{pr}_1,\mathrm{pr}_2\rangle)\exists^B_{\Pi X\times\Pi Y}P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi\to P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi\big),\end{aligned}\tag{3.4}$$

so that the Claim 3.5.5 follows by taking the join of (3.3) and (3.4).
To prove (3.3) it is sufficient to see that

$$P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi\leq P(\langle \mathrm{pr}_1,\mathrm{pr}_2\rangle)\exists^B_{\Pi X\times\Pi Y}P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi\to P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi\tag{3.5}$$

if and only if

$$P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi\wedge P(\langle \mathrm{pr}_1,\mathrm{pr}_2\rangle)\exists^B_{\Pi X\times\Pi Y}P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi\leq P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi,$$

which is trivially verified; then get (3.3) by applying $\exists^B_{\Pi X\times\Pi Y}$ on both sides of (3.5).
Now write φ' instead of $P(\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)\varphi$, and we prove (3.4) by showing first

$$\neg\exists^B_{\Pi X\times\Pi Y}\varphi'\leq\exists^B_{\Pi X\times\Pi Y}P(\langle \mathrm{pr}_1,\mathrm{pr}_2\rangle)\neg\exists^B_{\Pi X\times\Pi Y}\varphi'\tag{3.6}$$

and then

$$\exists^B_{\Pi X\times\Pi Y}P(\langle \mathrm{pr}_1,\mathrm{pr}_2\rangle)\neg\exists^B_{\Pi X\times\Pi Y}\varphi'\leq\exists^B_{\Pi X\times\Pi Y}\big(P(\langle \mathrm{pr}_1,\mathrm{pr}_2\rangle)\exists^B_{\Pi X\times\Pi Y}\varphi'\to\varphi'\big).\tag{3.7}$$

The proof of (3.7) is quite immediate: observe that in general in a Boolean algebra we have $\neg\alpha\leq\alpha\to\beta$—if and only if $\bot=\neg\alpha\wedge\alpha\leq\beta$—, hence take $\alpha=P(\langle \mathrm{pr}_1,\mathrm{pr}_2\rangle)\exists^B_{\Pi X\times\Pi Y}\varphi'$, $\beta=\varphi'$ and apply $\exists^B_{\Pi X\times\Pi Y}$ to get (3.7).
To conclude, we show that given $\gamma\in P(\Pi X\times\Pi Y)$ we have $\gamma\leq\exists^B_{\Pi X\times\Pi Y}P(\langle \mathrm{pr}_1,\mathrm{pr}_2\rangle)\gamma$, so that we get (3.6) by taking $\gamma=\neg\exists^B_{\Pi X\times\Pi Y}\varphi'$. To do so, we need to look at the set $\{(Y_h,y_h)\}_{h=1}^{\overline{h}}$ defined above. We can suppose that one the Y_h's is actually the object B—in which case the associated ordinal y_h is different from d. If this is not the case, we add the element (B,k) to $\{(Y_h,y_h)\}_{h=1}^{\overline{h}}$ for some ordinal $k\in\Lambda$ that does not appear in any second entry of (B,λ) belonging to $\mathcal{F}\cup\mathcal{G}\cup\{(B,d)\}$—note that such new pair does not belong to $\{(Y_h,y_h)\}_{h=1}^{\overline{h}}$: if it did, we did not have to add it to such set. So, up to a permutation of indices and up to a change of $\overline{h}$ with $\overline{h}+1$, we can suppose that in the set $\{(Y_h,y_h)\}_{h=1}^{\overline{h}}$ we have $Y_{\overline{h}}=B$.

So now we look at the adjunction:

$$\begin{tikzcd}
P(\prod_{a=1}^{\bar{b}+\bar{e}} X_a \times \prod_{h=1}^{\bar{h}} Y_h) \arrow[r, shift left, "P(\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle)"] \arrow[r, phantom, "\top"] \arrow[d, equal] & P(\prod_{a=1}^{\bar{b}+\bar{e}} X_a \times \prod_{h=1}^{\bar{h}} Y_h \times B) \arrow[l, shift left, "\exists^B_{\Pi X \times \Pi Y}"] \arrow[d, equal] \\
P(\prod_{a=1}^{\bar{b}+\bar{e}} X_a \times \prod_{h=1}^{\bar{h}-1} Y_h \times B) \arrow[r, shift left, "P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \rangle)"] \arrow[r, phantom, "\perp"] & P(\prod_{a=1}^{\bar{b}+\bar{e}} X_a \times \prod_{h=1}^{\bar{h}-1} Y_h \times B \times B) \arrow[l, shift left, "\exists^B_{\Pi X \times \Pi Y}"] \arrow[l, bend left, "P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_3 \rangle)"]
\end{tikzcd}$$

so if we look at our claim in the lower part of the diagram we want that given $\gamma \in P(\Pi X \times \Pi Y)$, then $\gamma \leq \exists^B_{\Pi X \times \Pi Y} P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \rangle)\gamma$. Now, consider the unit of the adjunction at the level $P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \rangle)\gamma$, hence

$$P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \rangle)\gamma \leq P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \rangle)\exists^B_{\Pi X \times \Pi Y} P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \rangle)\gamma;$$

now, apply $P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_3 \rangle)$, so we get exactly $\gamma \leq \exists^B_{\Pi X \times \Pi Y} P(\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3 \rangle)\gamma$ as claimed. In particular we proved the following:

Proposition 3.5.6. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a Boolean existential doctrine such that each fiber is non-trivial, and the base category $\mathbb{C}$ is small, then the doctrine $\underset{\rightarrow}{P}$ is consistent.

Actually, we will later slightly weaken the assumption that P is Boolean, and prove the consistency of $\underset{\rightarrow}{P}$ anyway.

3.6 Weak universal property of $\underset{\rightarrow}{P}$

For the whole chapter, $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is a fixed implicational existential doctrine with a small base category, unless otherwise specified.

Theorem 3.6.1. Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be an implicational existential doctrine with a small base category. The 1-arrow $(F, \mathfrak{f})\colon P \to \underset{\rightarrow}{P}$ is implicational existential and it is such that $\underset{\rightarrow}{P}$ is rich, and it is weakly universal with respect to this property, i.e. for any implicational existential morphism $(H, \mathfrak{h})\colon P \to R$ where $R\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ is an implicational rich doctrine, there exists an implicational existential 1-arrow $(G, \mathfrak{g})\colon \underset{\rightarrow}{P} \to R$ such that $(G, \mathfrak{g})(F, \mathfrak{f}) = (H, \mathfrak{h})$.
Moreover, if P, R and $(H, \mathfrak{h})$ are respectively bounded, universal, elementary, then such $(G, \mathfrak{g})$ is respectively bounded, universal, elementary.

$$\begin{tikzcd}
P \arrow[rr, "(H, \mathfrak{h})"] \arrow[dr, "(F, \mathfrak{f})"'] & & R \\
& \underset{\rightarrow}{P} \arrow[ur, dashed, "(G, \mathfrak{g})"'] &
\end{tikzcd}$$

Proof. Recall the colimit diagrams:

$$\begin{tikzcd}
 & P_{\Pi X_a} \arrow[dr, "(F'_{\bar X}, \mathfrak{f}'_{\bar X})"] \arrow[dd, "(F_{\bar X\bar Y}, \mathfrak{f}_{\bar X\bar Y})"] & \\
P \arrow[ur, "(F_{\bar X}, \mathfrak{f}_{\bar X})"] \arrow[dr, "(F_{\bar Y}, \mathfrak{f}_{\bar Y})"'] \arrow[rr, bend left, "(\underline{F}, \underline{\mathfrak{f}})"] & & \underline{P} \\
 & P_{\Pi Y_b} \arrow[ur, "(F'_{\bar Y}, \mathfrak{f}'_{\bar Y})"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \underline{P}^{\mathcal{U}} \arrow[dr, "(\mathrm{id}, \mathfrak{q}_{\mathcal{U}})"] \arrow[dd, "(\mathrm{id}, \mathfrak{f}_{\mathcal{U}\mathcal{V}})"] & \\
\underline{P} \arrow[ur, "(\mathrm{id}, \mathfrak{f}_{\mathcal{U}})"] \arrow[dr, "(\mathrm{id}, \mathfrak{f}_{\mathcal{V}})"'] \arrow[rr, bend left, "(\mathrm{id}, \underrightarrow{\mathfrak{f}})"] & & \underrightarrow{P} \\
 & \underline{P}^{\mathcal{V}} \arrow[ur, "(\mathrm{id}, \mathfrak{q}_{\mathcal{V}})"'] &
\end{tikzcd}$$

First of all, fix a well-ordering of $\mathrm{ob}\mathbb{C}$, and consider the lexicographic order on $\mathrm{ob}\mathbb{C} \times \Lambda$. We need to define a constant in $\mathbb{D}$ for all new constant of $\underline{\mathbb{C}}$.
Recall that for any given object B we have $\underline{P}(B) = \left\{ \left[\varphi_j^B, \bar{X}^{(B,j)} \right] \right\}_{j \in \Lambda}$ where

$$\varphi_j^B \in P(\prod_{a=1}^{n^{(B,j)}} X_a^{(B,j)} \times B).$$

We start from $(B_0, 0)$: consider $\varphi_0^{B_0} \in P(\prod_{a=1}^{n^{(B_0,0)}} X_a^{(B_0,0)} \times B_0)$ which is used to define the last entry of the list $\bar{X}_\star^{(B_0,0)}$ in Section 3.2. Take $\mathfrak{h}_{\Pi X_a \times B_0} \varphi_0^{B_0} \in R(\prod HX_a^{(B_0,0)} \times HB_0)$, hence there exists a constant in $\mathbb{D}$—which is actually a list of constants

$$c^{(B_0,0)} = \langle c_{(X_1^{(B_0,0)}, x_1^{(B_0,0)})}, \dots, c_{(X_{n^{(B_0,0)}}^{(B_0,0)}, x_{n^{(B_0,0)}}^{(B_0,0)})}, c_{(B_0, d_0^{B_0})} \rangle \colon \mathbf{t} \to \prod_{a=1}^{n^{(B_0,0)}} HX_a^{(B_0,0)} \times HB_0$$

such that

$$\exists_{\mathbf{t}}^{\Pi HX_a \times HB_0} \mathfrak{h}_{\Pi X_a \times B_0} \varphi_0^{B_0} \leq R(c^{(B_0,0)}) \mathfrak{h}_{\Pi X_a \times B_0} \varphi_0^{B_0}$$

by using the richness property of R. This defines an assignment $(Y, \lambda) \mapsto [c_{(Y,\lambda)} \colon \mathbf{t} \to HY]$ for some pair (Y, λ): our goal is to extend this to every pair of such kind. Consider now $(B, j) > (B_0, 0)$—i.e. $B > B_0$ in $\mathrm{ob}\mathbb{C}$, or $B = B_0$ and $j > 0$—, and take $\varphi_j^B \in P(\prod_{a=1}^{n^{(B,j)}} X_a^{(B,j)} \times B)$. Take all the pairs $(X_b^{(B,j)}, x_b^{(B,j)})$ that have already appeared as subscripts in the components of some $c^{(A,i)}$ for some $(A, i) < (B, j)$. Their indexes form a subset $K^{(B,j)} \subseteq \{1, \dots, n^{(B,j)}\}$.

Evaluate the element $\mathfrak{h}_{\Pi X_a \times B}\varphi_j^B \in R(\prod_{a=1}^{n^{(B,j)}} HX_a^{(B,j)} \times HB)$ in the corresponding constants:

$$\begin{gathered} R(\langle \mathrm{pr}_1, \dots, c_{(X_b^{(B,j)}, x_b^{(B,j)})}, \dots, \mathrm{pr}_{n^{(B,j)}}, \mathrm{pr}_{n^{(B,j)}+1}\rangle)\mathfrak{h}_{\Pi X_a \times B}\varphi_j^B \\ \Cap \\ R(HX_1^{(B,j)} \times \dots \times \widehat{HX_b^{(B,j)}} \times \dots HX_{n^{(B,j)}}^{(B,j)} \times HB) \end{gathered} \tag{3.8}$$

where each $\widehat{HX_b^{(B,j)}}$ for $b \in K^{(B,j)}$ is the terminal object $\mathbf{t}$. Let

$$\widehat{\prod} HX_a^{(B,j)} = \prod_{a \notin K^{(B,j)}} HX_a^{(B,j)},$$

and observe that there exists a canonical isomorphism

$$\omega_{(B,j)} \colon \widehat{\prod} HX_a^{(B,j)} \times HB \longrightarrow HX_1^{(B,j)} \times \dots \times \widehat{HX_b^{(B,j)}} \times \dots HX_{n^{(B,j)}}^{(B,j)} \times HB.$$

So now there exists a list of constants

$$c^{(B,j)} = \langle \dots, c_{(X_a^{(B,j)}, x_a^{(B,j)})}, \dots, c_{(B, d_j^B)}\rangle \colon \mathbf{t} \to \widehat{\prod} HX_a^{(B,j)} \times HB$$

such that

$$\begin{aligned} &\exists_{\mathbf{t}}^{\widehat{\Pi} HX_a \times HB} R(\omega_{(B,j)}) R(\langle \mathrm{pr}_1, \dots, c_{(X_b^{(B,j)}, x_b^{(B,j)})}, \dots, \mathrm{pr}_{n^{(B,j)}+1}\rangle)\mathfrak{h}_{\Pi X_a \times B}\varphi_j^B \\ &\leq R(c^{(B,j)}) R(\omega_{(B,j)}) R(\langle \mathrm{pr}_1, \dots, c_{(X_b^{(B,j)}, x_b^{(B,j)})}, \dots, \mathrm{pr}_{n^{(B,j)}+1}\rangle)\mathfrak{h}_{\Pi X_a \times B}\varphi_j^B \end{aligned} \tag{3.9}$$

by using again the richness property of R. Note that the reindexing over projections and constants is the same as above (3.8).

In this way, we are able to define $c_{(Y,\lambda)} : \mathbf{t} \to HY$ for all $Y \in \mathrm{ob}\mathbb{C}$ and $\lambda \in \Lambda$. Indeed, $d_i^B \geq i$ for all $i \in \Lambda$—see Section 3.2; then consider (Y, λ), so that we can surely find $c_{(Y, d_\lambda^Y)}$. But then, if $d_\lambda^Y = \lambda$, we defined $c_{(Y,\lambda)}$; otherwise $d_\lambda^Y > \lambda$, hence by choosing $c_{(Y, d_\lambda^Y)}$ we must have already fixed $c_{(Y,\lambda)}$. Once completed the assignments given by all pairs $(B, j) \in \mathrm{ob}\mathbb{C} \times \Lambda$, extend then the assignment $(Y, \lambda) \mapsto c_{(Y,\lambda)}$ to all the remaining pairs by choosing any constant $c_{(Y,\lambda)} \colon \mathbf{t} \to HY$. To do so, recall that since R is rich, for any object D in $\mathbb{D}$ there exists a map $\mathbf{t} \to D$.

Now, in order to find a 1-arrow $\underrightarrow{P} \to R$, we need to fix for all $\mathcal{U} \in I$, a 1-arrow $(S_{\mathcal{U}}, \mathfrak{s}_{\mathcal{U}}) \colon \underline{P}^{\mathcal{U}} \to R$ such that $(S_{\mathcal{U}}, \mathfrak{s}_{\mathcal{U}}) = (S_{\mathcal{V}}, \mathfrak{s}_{\mathcal{V}}) \circ (\mathrm{id}, \mathfrak{f}_{\mathcal{U}\mathcal{V}})$. In particular all functors must coincide $S_{\mathcal{U}} = S_{\mathcal{V}}$ for all $\mathcal{U}, \mathcal{V} \in I$, we will call it $G : \underline{\mathbb{C}} \to \mathbb{D}$; hence $\mathfrak{s}_{\mathcal{U}} = \mathfrak{s}_{\mathcal{V}}\mathfrak{f}_{\mathcal{U}\mathcal{V}}$ whenever $\mathcal{U} \subseteq \mathcal{V}$ in I.

However, since $\underline{P}^{\mathcal{U}}$—where $\mathcal{U} = \{(B_1, j_1), \dots, (B_n, j_n)\}$—is a Kleisli construction starting from $\underline{P}$ with respect to the pair $(\mathbf{t}, \bigwedge_{i=1}^n \left[\psi_{j_i}^{B_i}, \bar{X}_\star^{(B_i, j_i)}\right])$, we can equivalently define $(G, \mathfrak{p}_{\mathcal{U}}) \colon \underline{P} \to R$ such that $\bigwedge_{i=1}^n \left[\psi_{j_i}^{B_i}, \bar{X}_\star^{(B_i, j_i)}\right] \mapsto \top \in R(\mathbf{t})$ through $(\mathfrak{p}_{\mathcal{U}})_{\mathbf{t}}$; this allows us to get $\mathfrak{s}_{\mathcal{U}}$ such that $\mathfrak{p}_{\mathcal{U}} = \mathfrak{s}_{\mathcal{U}}\mathfrak{f}_{\mathcal{U}}$. Equivalently, each $\left[\psi_{j_i}^{B_i}, \bar{X}_\star^{(B_i, j_i)}\right]$ must be sent to the top element.

Moreover, for $\emptyset \in I$ we need to have also $\mathfrak{s}\colon \underline{P} \stackrel{\cdot}{\to} RG$. Since $\mathfrak{s} = \mathfrak{s}_{\mathcal{U}}\mathfrak{f}_{\mathcal{U}}$ for all $\mathcal{U} \in I$, we need—and it is also sufficient—to define $\mathfrak{s}$ such that it maps each $\left[\psi_j^B, \bar{X}_\star^{(B,j)}\right]$ to the top element of $R(\mathbf{t})$. Now use the fact that $\underline{P}$ is a colimit as well, so that in order to get $(G, \mathfrak{s})\colon \underline{P} \to R$ we need for all $\bar{X} = \big((X_1, x_1), \dots, (X_n, x_n)\big) \in J$ a 1-arrow $(H_{\bar{X}}, \mathfrak{h}_{\bar{X}})\colon P_{\Pi X_a} \to R$ such that $(H_{\bar{X}}, \mathfrak{h}_{\bar{X}}) = (H_{\bar{Y}}, \mathfrak{h}_{\bar{Y}}) \circ (F_{\bar{X}\bar{Y}}, \mathfrak{f}_{\bar{X}\bar{Y}})$ for any $\bar{X} \leq \bar{Y}$ in J. Since $P_{\Pi X_a}$ is the Kleisli construction, starting from P with respect to the pair $(\prod_{a=1}^n X_a, \top)$, we want a 1-arrow $P \to R$ and a choice of a constant $\mathbf{t} \to \prod_{a=1}^n X_a$. Of course, take the arrow $(H, \mathfrak{h})\colon P \to R$ and the constant $\langle c_{(X_1,x_1)}, \dots, c_{(X_n,x_n)}\rangle$. We dedicate the rest of the section to the check that the induced $\mathfrak{s}$ maps each $\left[\psi_j^B, \bar{X}_\star^{(B,j)}\right]$ to the top element. Consider the $\underline{\mathbb{C}}$-arrow

$$\left[\mathrm{id}, \bar{X}_\star^{(B,j)}\right] \colon \mathbf{t} \dashrightarrow \prod_{a=1}^{n^{(B,j)}} X_a^{(B,j)} \times B,$$

equivalence class of the identity arrow in $\mathbb{C}$

$$\mathrm{id}\colon \prod_{a=1}^{n^{(B,j)}} X_a^{(B,j)} \times B \to \prod_{a=1}^{n^{(B,j)}} X_a^{(B,j)} \times B.$$

The reindexing in $\underline{P}$ along this map is the evaluation in the corresponding new constants. Compute now $\mathfrak{s}_{\mathbf{t}}\left[\psi_j^B, \bar{X}_\star^{(B,j)}\right]$, using the naturality of $\mathfrak{s}$ and the commutativity of the triangle $(H, \mathfrak{h}) = (G, \mathfrak{s}) \circ (\underline{F}, \underline{\mathfrak{f}})$:

$$\begin{aligned}\mathfrak{s}_{\mathbf{t}}\left[\psi_j^B, \bar{X}_\star^{(B,j)}\right] &= \mathfrak{s}_{\mathbf{t}}\underline{P}\left(\left[\mathrm{id}, \bar{X}_\star^{(B,j)}\right]\right)\left[\psi_j^B, \emptyset\right] = RG\left(\left[\mathrm{id}, \bar{X}_\star^{(B,j)}\right]\right)\mathfrak{s}_{\Pi X_a \times B}\left[\psi_j^B, \emptyset\right]\\ &= R(\langle c_{(X_1^{(B,j)}, x_1^{(B,j)})}, \dots, c_{(X_{n^{(B,j)}}^{(B,j)}, x_{n^{(B,j)}}^{(B,j)})}, c_{(B, d_j^B)}\rangle)\mathfrak{h}_{\Pi X_a \times B}\psi_j^B.\end{aligned}$$

For simplicity, write $\overline{c} = \langle c_{(X_1^{(B,j)}, x_1^{(B,j)})}, \dots, c_{(X_{n^{(B,j)}}^{(B,j)}, x_{n^{(B,j)}}^{(B,j)})}, c_{(B, d_j^B)}\rangle$ for the list of $\mathbb{D}$-constants above, and recall that

$$\psi_j^B = P(\langle \mathrm{pr}_1, \dots, \mathrm{pr}_{n^{(B,j)}}\rangle)\exists^B_{\Pi X_a^{(B,j)}}\varphi_j^B \longrightarrow \varphi_j^B.$$

So $\top \leq \mathfrak{s}_{\mathbf{t}}\left[\psi_j^B, \bar{X}_\star^{(B,j)}\right]$ if and only if

$$R(\overline{c})\mathfrak{h}_{\Pi X_a \times B}P(\langle \mathrm{pr}_1, \dots, \mathrm{pr}_{n^{(B,j)}}\rangle)\exists^B_{\Pi X_a}\varphi_j^B \leq R(\overline{c})\mathfrak{h}_{\Pi X_a \times B}\varphi_j^B;$$

using naturality of $\mathfrak{h}$ and the fact that H preserves products, and then the fact that $\mathfrak{h}$ preserves the existential quantifier, we get

$$\begin{aligned}R(\overline{c})\mathfrak{h}_{\Pi X_a\times B}P(\langle \mathrm{pr}_1,\dots,\mathrm{pr}_{n^{(B,j)}}\rangle)\exists^B_{\Pi X_a}\varphi^B_j \\ = R(\overline{c})R(\langle \mathrm{pr}_1,\dots,\mathrm{pr}_{n^{(B,j)}}\rangle)\mathfrak{h}_{\Pi X_a}\exists^B_{\Pi X_a}\varphi^B_j \\ = R(\overline{c})R(\langle \mathrm{pr}_1,\dots,\mathrm{pr}_{n^{(B,j)}}\rangle)\exists^{HB}_{\Pi HX_a}\mathfrak{h}_{\Pi X_a\times B}\varphi^B_j,\end{aligned}$$

so we need to prove

$$R(\langle c_{(X_1^{(B,j)},x_1^{(B,j)})},\dots,c_{(X^{(B,j)}_{n^{(B,j)}},x^{(B,j)}_{n^{(B,j)}})}\rangle)\exists^{HB}_{\Pi HX_a}\mathfrak{h}_{\Pi X_a\times B}\varphi^B_j\le R(\overline{c})\mathfrak{h}_{\Pi X_a\times B}\varphi^B_j.$$

Observe that the right-hand side of this inequality we have exactly same element of the right-hand side of (3.9).

$$\begin{tikzcd}
\mathbf{t} \arrow[d, "c^{(B,j)}=\langle\dots,c_{(X_a^{(B,j)},x_a^{(B,j)})},\dots,c_{(B,d_j^B)}\rangle"] \arrow[ddd, bend right=60, "\overline{c}"'] \\
\widehat{\prod} HX_a^{(B,j)}\times HB \arrow[d, "\omega_{(B,j)}", "\wr"'] \\
HX_1^{(B,j)}\times\cdots\times\widehat{HX_b^{(B,j)}}\times\dots HX_{n^{(B,j)}}^{(B,j)}\times HB \arrow[d, "\langle \mathrm{pr}_1,\dots,c_{(X_b^{(B,j)},x_b^{(B,j)})},\dots,\mathrm{pr}_{n^{(B,j)}+1}\rangle"] \\
HX_1^{(B,j)}\times\cdots\times HX_b^{(B,j)}\times\dots HX_{n^{(B,j)}}^{(B,j)}\times HB
\end{tikzcd}$$

So it is enough to prove

$$\begin{aligned}R(\langle c_{(X_1^{(B,j)},x_1^{(B,j)})},\dots,c_{(X^{(B,j)}_{n^{(B,j)}},x^{(B,j)}_{n^{(B,j)}})}\rangle)\exists^{HB}_{\Pi HX_a}\mathfrak{h}_{\Pi X_a\times B}\varphi^B_j \\ \le \exists_{\mathbf{t}}^{\widehat{\Pi}HX_a\times HB}R(\omega_{(B,j)})R(\langle \mathrm{pr}_1,\dots,c_{(X_b^{(B,j)},x_b^{(B,j)})},\dots,\mathrm{pr}_{n^{(B,j)}}\rangle\times \mathrm{id}_{HB})\mathfrak{h}_{\Pi X_a\times B}\varphi^B_j.\end{aligned}$$

Write τ for the list $\langle \mathrm{pr}_1,\dots,c_{(X_b^{(B,j)},x_b^{(B,j)})},\dots,\mathrm{pr}_{n^{(B,j)}}\rangle$, so that

$$\langle \mathrm{pr}_1,\dots,c_{(X_b^{(B,j)},x_b^{(B,j)})},\dots,\mathrm{pr}_{n^{(B,j)}+1}\rangle=\tau\times\mathrm{id}_{HB},$$

write σ for every component except for the last one for the map $c^{(B,j)}$, so that $c^{(B,j)}=\langle\sigma,c_{(B,d_j^B)}\rangle$, and write ω' for the canonical isomorphism

$$\widehat{\prod} HX_a^{(B,j)}\longrightarrow HX_1^{(B,j)}\times\cdots\times\widehat{HX_b^{(B,j)}}\times\dots HX_{n^{(B,j)}}^{(B,j)},$$

so that $\omega_{(B,j)}=\omega'\times\mathrm{id}_{HB}$. In particular $\overline{c}=(\tau\times\mathrm{id}_{HB})(\omega'\times\mathrm{id}_{HB})c^{(B,j)}$, so we can compute the list $\langle c_{(X_1^{(B,j)},x_1^{(B,j)})},\dots,c_{(X^{(B,j)}_{n^{(B,j)}},x^{(B,j)}_{n^{(B,j)}})}\rangle$:

$$
\begin{aligned}
\langle c_{(X_1^{(B,j)},x_1^{(B,j)})},\dots,c_{(X_{n(B,j)}^{(B,j)},x_{n(B,j)}^{(B,j)})}\rangle &= \langle \mathrm{pr}_1,\dots,\mathrm{pr}_{n^{(B,j)}}\rangle\overline{c}\\
&= \langle \mathrm{pr}_1,\dots,\mathrm{pr}_{n^{(B,j)}}\rangle(\tau\times \mathrm{id}_{HB})(\omega'\times\mathrm{id}_{HB})c^{(B,j)}\\
&= \langle \mathrm{pr}_1,\dots,\mathrm{pr}_{n^{(B,j)}}\rangle(\tau\omega'\times \mathrm{id}_{HB})c^{(B,j)} = \tau\omega'\sigma.
\end{aligned}
$$

So now we have:

$$
\begin{aligned}
R(\langle c_{(X_1^{(B,j)},x_1^{(B,j)})},\dots,c_{(X_{n(B,j)}^{(B,j)},x_{n(B,j)}^{(B,j)})}\rangle)\exists^{HB}_{\Pi HX_a} &= R(\sigma)R(\tau\omega')\exists^{HB}_{\Pi HX_a}\\
&= R(\sigma)\exists^{HB}_{\widehat{\Pi} HX_a}R(\tau\omega'\times \mathrm{id}_B);
\end{aligned}
$$

hence, we are left to prove that $R(\sigma)\exists^{HB}_{\widehat{\Pi}HX_a}\leq \exists_{\mathbf{t}}^{\widehat{\Pi}HX_a\times HB}$.
Now, since $\exists_{\mathbf{t}}^{\widehat{\Pi}HX_a\times HB}=\exists_{\mathbf{t}}^{\widehat{\Pi}HX_a}\exists^{HB}_{\widehat{\Pi}HX_a}$, we should prove $R(\sigma)\leq\exists_{\mathbf{t}}^{\widehat{\Pi}HX_a}$, but this holds since $\sigma\colon\mathbf{t}\to\widehat{\Pi}HX_a$ and we can apply $R(\sigma)$ to the unit $\mathrm{id}_{R(\widehat{\Pi}HX_a)}\leq R(!_{\widehat{\Pi}HX_a})\exists_{\mathbf{t}}^{\widehat{\Pi}HX_a}$.
Since we defined $(G,\mathfrak{g})$ through directed colimits and Kleisli constructions, implicational and existential structure are preserved by $(G,\mathfrak{g})$; moreover, if R has as additional structure any between bottom element, universal quantifier, elementary structure, preserved by $(H,\mathfrak{h})$, then also $(G,\mathfrak{g})$ does. ☐

3.6.1 2-arrows and weak universal property

We extend the result to 2-arrows.

Proposition 3.6.2. Let $P\colon\mathbb{C}^{\mathrm{op}}\to\mathbf{Pos}$ be an implicational existential doctrine with a small base category. Consider the 1-arrow $(F,\mathfrak{f})\colon P\to\underset{\to}{P}$, and let $(H,\mathfrak{h})\colon P\to R$ be an implicational existential morphism where $R\colon\mathbb{D}^{\mathrm{op}}\to\mathbf{Pos}$ is an implicational rich doctrine and let $(G,\mathfrak{g})\colon\underset{\to}{P}\to R$ be an implicational existential 1-arrow such that $(G,\mathfrak{g})(F,\mathfrak{f})=(H,\mathfrak{h})$. Then precomposition with $(F,\mathfrak{f})$ induces an equivalence between the coslice categories

$$
-\circ(F,\mathfrak{f})\colon(G,\mathfrak{g})\downarrow\mathbf{Dct}_{\wedge,\top,\to,\exists}(\underset{\to}{P},R)\longrightarrow(H,\mathfrak{h})\downarrow\mathbf{Dct}_{\wedge,\top,\to,\exists}(P,R).
$$

Proof. Take any two objects $\gamma\colon(G,\mathfrak{g})\to(M,\mathfrak{m}),\mu\colon(G,\mathfrak{g})\to(N,\mathfrak{n})\in(G,\mathfrak{g})\downarrow\mathbf{Dct}_{\wedge,\top,\to,\exists}(\underset{\to}{P},R)$, for some $(M,\mathfrak{m}),(N,\mathfrak{n})\colon\underset{\to}{P}\to R$; then take an arrow $\delta\colon\gamma\to\mu$. Since the functor F acts as the identity on objects, precomposition with F applied to the natural transformations γ,μ and δ is the identity:

$$(G,\mathfrak{g}) \downarrow \mathbf{Dct}_{\wedge,\top,\to,\exists}(\underrightarrow{P}, R) \xrightarrow{-\circ(F,\mathfrak{f})} (H,\mathfrak{h}) \downarrow \mathbf{Dct}_{\wedge,\top,\to,\exists}(P, R)$$

$$\begin{tikzcd}
& (G,\mathfrak{g}) \arrow[dl, "\gamma"'] \arrow[dr, "\mu"] & & & (H,\mathfrak{h}) \arrow[dl, "\gamma"'] \arrow[dr, "\mu"] & \\
(M,\mathfrak{m}) \arrow[rr, "\delta"'] & & (N,\mathfrak{n}) & (M,\mathfrak{m})(F,\mathfrak{f}) \arrow[rr, "\delta"'] & & (N,\mathfrak{n})(F,\mathfrak{f})
\end{tikzcd}$$

In particular, faithfulness of the precomposition functor follows trivially. We show that the functor is essentially surjective.

Take a 2-arrow $\gamma\colon (H,\mathfrak{h}) \to (K,\mathfrak{k})$ where $(K,\mathfrak{k})\colon P \to R$ is an implicational existential morphism. We want to find a morphism $(M,\mathfrak{m})\colon \underrightarrow{P} \to R$ and a 2-arrow $(G,\mathfrak{g}) \to (M,\mathfrak{m})$, where $(M,\mathfrak{m})$ makes the triangle with $(K,\mathfrak{k})$ commute.

$$\begin{tikzcd}
P \arrow[rr, bend left=30, "{(H,\mathfrak{h})}", ""{name=U, below}] \arrow[rr, bend left=10, "{(K,\mathfrak{k})}"', ""{name=D}] \arrow[dr, bend right, "{(F,\mathfrak{f})}"'] & & R \\
& \underrightarrow{P} \arrow[ur, bend right, "{(G,\mathfrak{g})}", ""{name=G, below}] \arrow[ur, dotted, bend right=50, "{(M,\mathfrak{m})}"', ""{name=M}] &
\arrow[Rightarrow, from=U, to=D, "\gamma"] \arrow[Rightarrow, dotted, from=G, to=M]
\end{tikzcd}$$

Recall that $(G,\mathfrak{g})$ is uniquely determined by $(H,\mathfrak{h})$ and a choice of $c_{(X,x)}\colon \mathbf{t} \to HX$ for each $(X,x) \in J$. Moreover, having a 2-arrow γ means that we have a natural transformation $\gamma\colon H \xrightarrow{\cdot} K$ such that $\mathfrak{h}_X \leq R(\gamma_X)\mathfrak{k}_X$ for all $X \in \mathrm{ob}\mathbb{C}$. To define $(M,\mathfrak{m})$, we look for a constant $d_{(X,x)}\colon \mathbf{t} \to KX$ for any $(X,x) \in J$ such that the corresponding induced map $\underline{P} \to R$ maps each $\left[\psi_j^B, \bar{X}_\star^{(B,j)}\right] \in \underline{P}(\mathbf{t})$ in the top element of $R(\mathbf{t})$. Define $d_{(X,x)} \coloneqq \gamma_X \cdot c_{(X,x)}$, and then we check that in $R(\mathbf{t})$

$$\top \leq R(\langle d_{(X_1^{(B,j)},x_1^{(B,j)})}, \ldots, d_{(X_{n(B,j)}^{(B,j)},x_{n(B,j)}^{(B,j)})}, d_{(B,d_j^B)}\rangle)\mathfrak{k}_{\Pi X_a \times B}\psi_j^B.$$

By using naturality of γ and the fact that both H and K preserve products, we get the following commutative triangle

$$\begin{tikzcd}
\mathbf{t} \arrow[rr, "{\langle \ldots, d_{(X_i^{(B,j)},x_i^{(B,j)})}, \ldots, d_{(B,d_j^B)}\rangle}"] \arrow[dr, "{\langle \ldots, c_{(X_i^{(B,j)},x_i^{(B,j)})}, \ldots, c_{(B,d_j^B)}\rangle}"'] & & \prod KX_a \times KB \\
& \prod HX_a \times HB \arrow[ur, "\gamma_{\Pi X_a \times B}"'] &
\end{tikzcd}$$

Then, using the definition of $c_{(X,x)}$'s and the fact that γ is a 2-arrow we have:

$$\begin{aligned}\top \leq R(\langle c_{(X_1^{(B,j)},x_1^{(B,j)})},\dots,c_{(X^{(B,j)}_{n(B,j)},x^{(B,j)}_{n(B,j)})},c_{(B,d^B_j)}\rangle)\mathfrak{h}_{\Pi X_a\times B}\psi^B_j\\ \leq R(\langle c_{(X_1^{(B,j)},x_1^{(B,j)})},\dots,c_{(X^{(B,j)}_{n(B,j)},x^{(B,j)}_{n(B,j)})},c_{(B,d^B_j)}\rangle)R(\gamma_{\Pi X_a\times B})\mathfrak{k}_{\Pi X_a\times B}\psi^B_j\end{aligned}$$

as claimed, so we defined a morphism $(M,\mathfrak{m})$ such that $(M,\mathfrak{m})(F,\mathfrak{f})=(K,\mathfrak{k})$.

To conclude essential surjectivity, we show that γ is actually a 2-arrow also between $(G,\mathfrak{g})$ and $(M,\mathfrak{m})$. Take any $\underline{\mathbb{C}}$-arrow $\big[f,\bar{X}\big]: A \dashrightarrow B$, where $f\colon\prod_{a=1}^n X_a\times A\to B$ is a $\mathbb{C}$-arrow and $\bar{X}=\big((X_1,x_1),\dots,(X_n,x_n)\big)$ is a list in J. Naturality means that the following square commutes:

$$\begin{tikzcd} GA \arrow[r, "\gamma_A"'] \arrow[d, "{G[f,\bar{X}]}"'] & MA \arrow[d, "{M[f,\bar{X}]}"] \\ GB \arrow[r, "\gamma_B"] & MB \end{tikzcd}$$

Observe that the $\mathbb{D}$-arrow $\gamma_A\colon HA\to KA$ is indeed an arrow from GA to MA, because the functors G and M act like H and K on objects respectively. Use now the definition of $G\,\big[f,\bar{X}\big]$ and $M\,\big[f,\bar{X}\big]$, so that we need to prove the commutativity of the outer rectangle:

$$\begin{tikzcd} HA \arrow[r, "\gamma_A"] \arrow[d, "{\langle\overline{c}\cdot!,\mathrm{id}_{HA}\rangle}"'] & KA \arrow[d, "{\langle\overline{d}\cdot!,\mathrm{id}_{KA}\rangle}"] \\ \prod HX_a\times HA \arrow[r, dotted, "\gamma_{\Pi X_a\times A}"] \arrow[d, "H(f)"'] & \prod KX_a\times KA \arrow[d, "K(f)"] \\ HB \arrow[r, "\gamma_B"] & KB \end{tikzcd}$$

where $\overline{c}=\langle c_{(X_1,x_1)},\dots,c_{(X_n,x_n)}\rangle$ and symilarly $\overline{d}=\langle d_{(X_1,x_1)},\dots,d_{(X_n,x_n)}\rangle$. The rectangle can be easliy divided in two commutative squares: the lower one is clearly commutative by naturality of γ, while the upper one is commutative too since $\gamma_{\Pi X_a\times A}=\Pi\gamma_{X_a}\times\gamma_A$ and $\langle\overline{d}\cdot!,\mathrm{id}_{KA}\rangle=(\Pi\gamma_{X_a}\times\mathrm{id}_{KA})\langle\overline{c}\cdot!,\mathrm{id}_{KA}\rangle$. So we get $\gamma\colon G\dot{\to}M$, as claimed.

To conclude, we show that it is indeed a 2-arrow: take any $[\big[\alpha,\bar{X}\big],\mathcal{U}]\in\underset{\to}{P}(A)$ for some element $\alpha\in P(\prod_{a=1}^n X_a\times A)$ and $\bar{X}=\big((X_1,x_1),\dots,(X_n,x_n)\big)\in J$, we prove that in $R(GA)$

$$\mathfrak{g}_A[\big[\alpha,\bar{X}\big],\mathcal{U}]\leq R(\gamma_A)\mathfrak{m}_A[\big[\alpha,\bar{X}\big],\mathcal{U}].$$

Using the same notation we used above for $\overline{c}$ and $\overline{d}$, we compute:

$$\begin{aligned}\mathfrak{g}_A[\big[\alpha,\bar{X}\big]\,\mathcal{U}]=\mathfrak{g}_A[\big[\alpha,\bar{X}\big]\,\emptyset]=R(\langle\overline{c}\cdot!,\mathrm{id}_{HA}\rangle)\mathfrak{h}_{\Pi X_a\times A}\alpha\leq R(\langle\overline{c}\cdot!,\mathrm{id}_{HA}\rangle)R(\gamma_{\Pi X_a\times A})\mathfrak{k}_{\Pi X_a\times A}\alpha\\ =R(\gamma_A)R(\langle\overline{d}\cdot!,\mathrm{id}_{KA}\rangle)\mathfrak{k}_{\Pi X_a\times A}\alpha=R(\gamma_A)\mathfrak{m}_A[\big[\alpha,\bar{X}\big],\mathcal{U}]\end{aligned}$$

as claimed.

At last, we check that the functor $-\circ(F,\mathfrak{f})$ is a full functor between the coslice categories. Suppose to have $\gamma\colon(G,\mathfrak{g})\to(M,\mathfrak{m}),\mu\colon(G,\mathfrak{g})\to(N,\mathfrak{n})\in(G,\mathfrak{g})\downarrow\mathbf{Dct}_{\wedge,\top,\to,\exists}(\underset{\rightarrow}{P},R)$, for some $(M,\mathfrak{m}),(N,\mathfrak{n})\colon\underset{\rightarrow}{P}\to R$. Moreover, let $\delta\colon(M,\mathfrak{m})(F,\mathfrak{f})\to(N,\mathfrak{n})(F,\mathfrak{f})$ be a 2-arrow making the triangle on the right commute.

$$\begin{tikzcd}
 & (G,\mathfrak{g}) \arrow[dl, "\gamma"'] \arrow[dr, "\mu"] & \\
(M,\mathfrak{m}) \arrow[rr, dashed, "\delta"'] & & (N,\mathfrak{n})
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (H,\mathfrak{h}) \arrow[dl, "\gamma"'] \arrow[dr, "\mu"] & \\
(M,\mathfrak{m})(F,\mathfrak{f}) \arrow[rr, "\delta"'] & & (N,\mathfrak{n})(F,\mathfrak{f})
\end{tikzcd}$$

We prove that δ is also a 2-arrow between $(M,\mathfrak{m})$ and $(N,\mathfrak{n})$. Similarly to what we did before, define for any $(X,x)\in J$ the $\mathbb{D}$-arrows $d_{(X,x)}\colon=M\,[\mathrm{id}_X,(X,x)]:\mathbf{t}\to MX$ and $e_{(X,x)}\colon=N\,[\mathrm{id}_X,(X,x)]:\mathbf{t}\to NX$. Apply naturality of $\gamma\colon G\to M$ and $\mu\colon G\to N$ to the arrow $[\mathrm{id}_X,(X,x)]$ to obtain respectively $\gamma_X c_{(X,x)}=d_{(X,x)}$ and $\mu_X c_{(X,x)}=e_{(X,x)}$. However, since $\delta_X\cdot\gamma_X=\mu_X$, we get

$$\delta_X d_{(X,x)}=e_{(X,x)}. \tag{3.10}$$

Now fix a $\underline{\mathbb{C}}$-arrow $\left[f,\bar{X}\right]:A\dashrightarrow B$, where $f\colon\prod_{a=1}^n X_a\times A\to B$ is a $\mathbb{C}$-arrow and $\bar{X}=\big((X_1,x_1),\ldots,(X_n,x_n)\big)$ is a list in J; moreover write $\overline{d}=\langle d_{(X_1,x_1)},\ldots,d_{(X_n,x_n)}\rangle$ and symilarly $\overline{e}=\langle e_{(X_1,x_1)},\ldots,e_{(X_n,x_n)}\rangle$. Naturality of $\delta\colon M\to N$ means that the following square commutes:

$$\begin{tikzcd}
MA \arrow[r, "\delta_A"] \arrow[d, "\langle\overline{d}\cdot!,\mathrm{id}_{MA}\rangle"'] & NA \arrow[d, "\langle\overline{e}\cdot!,\mathrm{id}_{NA}\rangle"] \\
\prod MX_a\times MA \arrow[r, dotted, "\delta_{\prod X_a\times A}"] \arrow[d, "MF(f)"'] & \prod NX_a\times NA \arrow[d, "NF(f)"] \\
MB \arrow[r, "\delta_B"] & NB
\end{tikzcd}$$

Commutativity of the lower square follows from naturality of $\delta\colon MF\to NF$, while the upper square commutes if and only if $\delta_{X_i}d_{(X_i,x_i)}=e_{(X_i,x_i)}$, but this follows from (3.10). This concludes the proof. □

3.7 Consistency of $\underset{\rightarrow}{P}$, weaker assumptions

Recall that, given a doctrine $P\colon\mathbb{C}^{\mathrm{op}}\to\mathbf{Pos}$ which is Boolean, existential, with non-trivial fibers, and with a small base category, the construction $\underset{\rightarrow}{P}$ is consistent and rich. As hinted at the end of Section 3.5, weaken the assumption as follows.

Proposition 3.7.1. Let $P\colon\mathbb{C}^{\mathrm{op}}\to\mathbf{Pos}$ be a bounded existential implicational doctrine such that each fiber is non-trivial, and the base category $\mathbb{C}$ is small, then the doctrine $\underset{\rightarrow}{P}$ is consistent.

Proof. We start from P, and we build the boolean completion $P_{\neg\neg}\colon\mathbb{C}^{\mathrm{op}}\to\mathbf{Pos}$—see Section 1.4.

We have the following commutative diagram:

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & P_{\neg\neg} \arrow[d] \\
\underrightarrow{P} \arrow[r] & \underrightarrow{P_{\neg\neg}}
\end{tikzcd}$$

The map $P \to \underrightarrow{P}$ is $(F_P, \mathfrak{f}_P)$ defined in Remark 3.3.1, the map $P \to P_{\neg\neg}$ is $(\mathrm{id}, \neg\neg)$ given by the completion, the map $P_{\neg\neg} \to \underrightarrow{P_{\neg\neg}}$ is $(F_{P_{\neg\neg}}, \mathfrak{f}_{P_{\neg\neg}})$ again defined in Remark 3.3.1 corresponding to the construction applied to the doctrine $P_{\neg\neg}$. Then, use the weak universal property of $P \to \underrightarrow{P}$—see Theorem 3.6.1: the doctrine $\underrightarrow{P_{\neg\neg}}$ is implicational, rich, and the composition of the upper morphism with the one on the right preserves the bounded implicational existential structure because both arrows do; so there exists a map $\underrightarrow{P} \to \underrightarrow{P_{\neg\neg}}$ closing the diagram and endowed with the structure just mentioned. Note that all $P_{\neg\neg}(X)$ are non-trivial, since top and bottom element are computed in $P(X)$, in which these are distinct element by assumption. In particular, since $P_{\neg\neg}$ is also Boolean, it follows from Proposition 3.5.6 that $\underrightarrow{P_{\neg\neg}}$ is consistent. But then, since there exists a map $\underrightarrow{P} \to \underrightarrow{P_{\neg\neg}}$ preserving, among others, $\top$ and $\bot$, if $\underrightarrow{P_{\neg\neg}}$ is consistent, $\underrightarrow{P}$ must be consistent too. ☐

3.8 A model of a rich doctrine

Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a bounded consistent existential implicational rich doctrine. Let $\nabla \subseteq P(\mathbf{t})$ be an ultrafilter and $P/\nabla\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ the quotient doctrine. Such ultrafilter exists since $\top \neq \bot$ in $P(\mathbf{t})$, and we can take an extension of the proper filter $\{\top\}$. The doctrine P/∇ is again bounded existential implicational, and all of these structures are preserved by the quotient morphism $(\mathrm{id}_{\mathbb{C}}, \mathfrak{q})\colon P \to P/\nabla$. See Section 1.5.1 for more details.

3.8.1 Definition of a model

We now build a model of P/∇ in the doctrine $\mathscr{P}_*\colon \mathrm{Set}_*^{\mathrm{op}} \to \mathbf{Pos}$, meaning a doctrine homomorphism $(\Gamma, \mathfrak{g})\colon P/\nabla \to \mathscr{P}_*$. Also, this model preserves the bounded existential implicational structure. Define $\Gamma \coloneqq \mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, -)\colon \mathbb{C} \to \mathrm{Set}_*$. It is well defined since P is rich, and this clearly preserves the products. Then, define for a given $X \in \mathrm{ob}\mathbb{C}$, $\mathfrak{g}_X\colon P/\nabla(X) \to \mathscr{P}_*(\mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, X))$:

$$\begin{aligned}
\mathfrak{g}_X[\varphi] &= \{c\colon \mathbf{t} \to X \mid [\top] \leq P/\nabla(c)[\varphi]\} = \\
&= \{c\colon \mathbf{t} \to X \mid [\top] \leq [P(c)\varphi]\} = \\
&= \{c\colon \mathbf{t} \to X \mid P(c)\varphi \in \nabla\}.
\end{aligned}$$

Proposition 3.8.1. Let P be a bounded consistent implicational existential rich doctrine, let $\nabla \subseteq P(\mathbf{t})$ be an ultrafilter, and let P/∇ be the quotient doctrine. Then the pair $(\Gamma, \mathfrak{g})$, where

$\Gamma = \mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, -)$ and $\mathfrak{g}_X[\varphi] = \{c\colon \mathbf{t} \to X \mid P(c)\varphi \in \nabla\}$ for any object X and any $[\varphi] \in P/\nabla(X)$ is a bounded existential implicational morphism.

Proof. $\mathfrak{g}_X$ **is monotone:** Suppose $[\varphi] \leq [\psi]$ in $P/\nabla(X)$, i.e. there exists $\theta \in \nabla$ such that $P(!_X)\theta \leq \varphi \to \psi$; we show that $\mathfrak{g}_X[\varphi] \subseteq \mathfrak{g}_X[\psi]$. Let $c\colon \mathbf{t} \to X$ be an arrow in $\mathbb{C}$ such that $P(c)\varphi \in \nabla$. Apply $P(c)$ to the inequality above and get $\theta \leq P(c)(\varphi \to \psi)$; so $P(c)(\varphi \to \psi) \in \nabla$. Then, $P(c)\varphi \wedge P(c)(\varphi \to \psi) \leq P(c)\psi \in \nabla$, i.e. $c \in \mathfrak{g}_X[\psi]$.

$\mathfrak{g}_X$ **is a natural transformation:** Take $f\colon X \to Y$ an arrow in $\mathbb{C}$. We want to show that the following diagram commutes:

$$\begin{tikzcd}
Y & P/\nabla(Y) \arrow[r, "\mathfrak{g}_Y"] \arrow[d, "P/\nabla(f)"] & \mathscr{P}(\mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, Y)) \arrow[d, "(f\circ -)^{-1}"] \\
X \arrow[u, "f"] & P/\nabla(X) \arrow[r, "\mathfrak{g}_X"] & \mathscr{P}(\mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, X))
\end{tikzcd}$$

Consider $c\colon \mathbf{t} \to X$; $c \in \mathfrak{g}_X P/\nabla(f)[\varphi]$ if and only if $P(c)P(f)\varphi \in \nabla$. On the other hand, $c \in (f \circ -)^{-1}\mathfrak{g}_Y[\varphi]$ if and only if $fc \in \mathfrak{g}_Y[\varphi]$ if and only if $P(fc)\varphi \in \nabla$.

In particular, $(\mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, -), \mathfrak{g})$ is a morphism of doctrines. We now prove that all the other properties are preserved.

$\mathfrak{g}_X$ **preserves top and bottom elements:** Compute

$$\mathfrak{g}_X[\top_X] = \{c\colon \mathbf{t} \to X \mid P(c)\top_X \in \nabla\} = \mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, X),$$

since $P(c)\top_X = \top_{\mathbf{t}} \in \nabla$ for any c.

$$\mathfrak{g}_X[\bot_X] = \{c\colon \mathbf{t} \to X \mid P(c)\bot_X \in \nabla\} = \emptyset,$$

since $P(c)\bot_X = \bot_{\mathbf{t}} \notin \nabla$ for any c. Moreover,

$\mathfrak{g}_X$ **preserves meets:** Compute

$$\begin{aligned}
\mathfrak{g}_X([\varphi] \wedge [\psi]) = \mathfrak{g}_X([\varphi \wedge \psi]) &= \{c\colon \mathbf{t} \to X \mid P(c)\varphi \wedge P(c)\psi \in \nabla\} \\
&= \{c\colon \mathbf{t} \to X \mid P(c)\varphi \in \nabla \text{ and } P(c)\psi \in \nabla\} = \mathfrak{g}_X[\varphi] \cap \mathfrak{g}_X[\psi].
\end{aligned}$$

$\mathfrak{g}_X$ **preserves implication:** Compute

$$\mathfrak{g}_X([\varphi] \to [\psi]) = \mathfrak{g}_X([\varphi \to \psi]) = \{c\colon \mathbf{t} \to X \mid P(c)\varphi \to P(c)\psi \in \nabla\};$$
$$\mathfrak{g}_X[\varphi] \Rightarrow \mathfrak{g}_X[\psi] = \{c\colon \mathbf{t} \to X \mid P(c)\psi \in \nabla\} \cup \{c\colon \mathbf{t} \to X \mid P(c)\varphi \notin \nabla\}.$$

First of all, suppose $c\colon \mathbf{t} \to X$ be such that $P(c)\varphi \to P(c)\psi \in \nabla$; then consider $P(c)\varphi$. If $P(c)\varphi \in \nabla$, we get $P(c)\varphi \wedge (P(c)\varphi \to P(c)\psi) \leq P(c)\psi \in \nabla$; otherwise, $P(c)\varphi \notin \nabla$. In both cases $c \in \mathfrak{g}_X[\varphi] \Rightarrow \mathfrak{g}_X[\psi]$. For the converse, take at first c such that $P(c)\psi \in \nabla$. Since

$P(c)\psi \leq P(c)\varphi \to P(c)\psi$, we get $P(c)\varphi \to P(c)\psi \in \nabla$. Then, take c such that $P(c)\varphi \notin \nabla$; since ∇ is an ultrafilter, $P(c)\varphi \to \bot \in \nabla$. But then, $P(c)\varphi \to \bot \leq P(c)\varphi \to P(c)\psi$ since $P(c)\varphi \to (-)$ is monotone; so $P(c)\varphi \to P(c)\psi \in \nabla$.

$\mathfrak{g}_X$ preserves existential quantifier: Recall that, given a function between two sets $h\colon A \to B$, the left adjoint to the preimage $h^{-1}\colon \mathscr{P}(B) \to \mathscr{P}(A)$ acts on any subset of A as the image $\exists_h = h\colon \mathscr{P}(A) \to \mathscr{P}(B)$.

So now we show $\exists_{\mathrm{pr}_1 \circ -}\mathfrak{g}_{X\times Y}[\varphi] = \mathfrak{g}_X\exists_X^Y[\varphi]$ for any pair X, Y of objects in $\mathbb{C}$. First of all, observe that the inclusion $(\subseteq)$ holds if and only if $\mathfrak{g}_{X\times Y}[\varphi] \subseteq (\mathrm{pr}_1 \circ -)^{-1}\mathfrak{g}_X[\exists_X^Y\varphi]$ but

$$(\mathrm{pr}_1 \circ -)^{-1}\mathfrak{g}_X[\exists_X^Y\varphi] = \mathscr{P}(\mathrm{pr}_1 \circ -)\mathfrak{g}_X[\exists_X^Y\varphi] = \mathfrak{g}_{X\times Y}P/\nabla(\mathrm{pr}_1)[\exists_X^Y\varphi]$$

and $[\varphi] \leq P/\nabla(\mathrm{pr}_1)[\exists_X^Y\varphi]$. Concerning the converse, observe that

$$\begin{aligned}\exists_{\mathrm{pr}_1 \circ -}\mathfrak{g}_{X\times Y}[\varphi] &= \{c\colon \mathbf{t} \to X \mid \text{there exists } d\colon \mathbf{t} \to Y \text{ such that } \langle c, d\rangle \in \mathfrak{g}_{X\times Y}[\varphi]\} \\ &= \{c\colon \mathbf{t} \to X \mid \text{there exists } d\colon \mathbf{t} \to Y \text{ such that } P(\langle c, d\rangle)\varphi \in \nabla\}.\end{aligned}$$

Then take $c\colon \mathbf{t} \to X$ such that $P(c)\exists_X^Y\varphi = \exists_{\mathbf{t}}^Y P(\langle c!, \mathrm{id}_Y\rangle)\varphi \in \nabla$. Since P is rich, we can take $d\colon \mathbf{t} \to Y$ such that

$$\exists_{\mathbf{t}}^Y P(\langle c!, \mathrm{id}_Y\rangle)\varphi = P(d)P(\langle c!, \mathrm{id}_Y\rangle)\varphi = P(\langle c, d\rangle)\varphi,$$

so that $c \in \exists_{\mathrm{pr}_1 \circ -}\mathfrak{g}_{X\times Y}[\varphi]$. $\square$

Example 3.8.2. A counterexample to universality. We prove that in general, if we add the universal quantifier to our structure, it is not necessarily preserved by the model. We will consider a slight change of the domain in the realizability doctrine, defined in [HJP80]: $R\colon \mathrm{Set}_*^{\mathrm{op}} \to \mathbf{Pos}$ takes value from the opposite category of non-empty sets. For each non-empty set I, define the following preorder in $\mathscr{P}(\mathbb{N})^I = \{p\colon I \to \mathscr{P}(\mathbb{N})\}$: we say that $p \leq q$ if there exists a partial recursive function $\varphi\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that for all $i \in I$ the restriction $\varphi_{|p(i)}\colon p(i) \to q(i)$ is a total function; reflexivity is witnessed by the identity $\mathrm{id}_{\mathbb{N}}$, while transitivity by the composition of the two partial functions. Then, define $R(I)$ to be the poset reflection of this preorder. The reindexing along a function $\alpha\colon J \to I$ is given by precomposition $- \circ \alpha\colon R(I) \to R(J)$; note that if $p \leq q$ in $\mathscr{P}(\mathbb{N})^I$ is witnessed by $\varphi\colon \mathbb{N} \dashrightarrow \mathbb{N}$, also $p\alpha \leq q\alpha$ in $\mathscr{P}(\mathbb{N})^J$ is again witnessed by φ.

R is primary: First of all observe that in each $R(I)$, the constant function $T_I\colon I \to \mathscr{P}(\mathbb{N})$ sending each $i \mapsto \mathbb{N}$ is the top element: take any other $p\colon I \to \mathscr{P}(\mathbb{N})$ and consider $\mathrm{id}_{\mathbb{N}}$, so that the inclusion $\mathrm{id}_{\mathbb{N}|p(i)}\colon p(i) \to \mathbb{N}$ is a total function for every $i \in I$, giving $p \leq T_I$. Moreover, for any $\alpha\colon J \to I$, precomposition $T_I\alpha = T_J$ is again the constant function to the element $\mathbb{N}$, so the top element is preserved by reindexing. Then, for any $p, q\colon I \to \mathscr{P}(\mathbb{N})$, define for each $i \in I$, $(p \wedge q)(i) \coloneqq \{\langle a, b\rangle \in \mathbb{N} \mid a \in p(i), b \in q(i)\}$; here $\langle -, -\rangle\colon \mathbb{N} \times \mathbb{N} \overset{\sim}{\leftrightarrow} \mathbb{N}\colon \langle \pi_1, \pi_2\rangle$ are Cantor's pairing and unpairing functions. The inequalities $p \wedge q \leq p$ and $p \wedge q \leq q$ are witnessed by the—total—functions $\pi_1\colon \mathbb{N} \to \mathbb{N}$ and $\pi_2\colon \mathbb{N} \to \mathbb{N}$ respectively. Suppose now $r \leq p$ and $r \leq q$, with given recursive functions φ and ψ; then define $\langle \varphi, \psi\rangle\colon \mathbb{N} \dashrightarrow \mathbb{N}$ whose domain is the intersection of the domains of φ and ψ, sending $n \in \mathrm{dom}\varphi \cap \mathrm{dom}\psi$ to $\langle \varphi(n), \psi(n)\rangle$, so that $\langle \varphi, \psi\rangle$ is partial recursive and witnesses $r \leq p \wedge q$. As before, take $\alpha\colon J \to I$: for any $j \in J$ we have $(p \wedge q)(\alpha(j)) = \{\langle a, b\rangle \in \mathbb{N} \mid a \in p\alpha(j), b \in q\alpha(j)\} = (p\alpha \wedge q\alpha)(j)$, so the meet is preserved by reindexings, hence R is a primary doctrine.

R has bottom elements: In each $R(I)$, the constant function $B_I\colon I \to \mathscr{P}(\mathbb{N})$ sending each $i \mapsto \emptyset$ is the bottom element: take any other $p\colon I \to \mathscr{P}(\mathbb{N})$ and consider $\mathrm{id}_{\mathbb{N}}$, so that the inclusion $\mathrm{id}_{\mathbb{N}|\emptyset}\colon \emptyset \to p(i)$ is a total function for every $i \in I$, giving $B_I \leq p$. Moreover, for any $\alpha\colon J \to I$, precomposition $B_I\alpha = B_J$ is again the constant function to the element $\emptyset$, so the bottom element is preserved by reindexing.

R is implicational: For any $p, q\colon I \to \mathscr{P}(\mathbb{N})$, define for each $i \in I$, $(p \to q)(i)$ as the set $\{e \in \mathbb{N} \mid e$ encodes a partial recursive function $\theta\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that θ maps $p(i)$ in $q(i)\}$. To prove that this is indeed the implication in $R(I)$, take $r \in R(I)$ and suppose $r \wedge p \leq q$, if and only if there exists $\varphi\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that for every $i \in I$, $\varphi_{|(r\wedge p)(i)}\colon (r \wedge p)(i) \to q(i)$ is a total function. For a given $n \in \mathbb{N}$, we can consider the partial function $\varphi(\langle n, -\rangle)\colon \mathbb{N} \dashrightarrow \mathbb{N}$, $m \mapsto \varphi(\langle n, m\rangle)$ when it exists; define $\psi\colon \mathbb{N} \to \mathbb{N}$ the—total—function that maps n to the natural number that encodes $\varphi(\langle n, -\rangle)$. For each $i \in I$, the restriction $\psi_{|r(i)}$ is defined over all $r(i)$, and its image is in $(p \to q)(i)$, proving $r \leq p \to q$: indeed, take $n \in r(i)$, then $\psi(n) \in (p \to q)(i)$ if and only if $\varphi(\langle n, -\rangle)$ maps $p(i)$ to $q(i)$, but if we take any $m \in p(i)$, then $\langle n, m\rangle \in (r \wedge p)(i)$, so that $\varphi(\langle n, m\rangle) \in q(i)$. Now, to prove the converse, suppose $r \leq p \to q$, if and only if there exists $\psi\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that for every $i \in I$, $\psi_{|r(i)}\colon r(i) \to (p \to q)(i)$ is a total function. For any $k \in \mathbb{N}$, recall that $k = \langle n, m\rangle$ where $n = \pi_1(k)$ and $m = \pi_2(k)$; if $\psi(n)$ exist, call $\theta_n\colon \mathbb{N} \dashrightarrow \mathbb{N}$ the partial function encoded by the natural number $\psi(n)$. Define $\varphi\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that $\langle n, m\rangle \mapsto \theta_n(m)$ whenever both $\psi(n)$ and $\theta_n(m)$ are defined. For each $i \in I$, the restriction $\varphi_{|(r\wedge p)(i)}$ is defined over all $(r \wedge p)(i)$, and its image is in $q(i)$, proving $r \wedge p \leq q$: indeed, take $k = \langle n, m\rangle \in (r \wedge p)(i)$, hence $n \in r(i)$ and $m \in p(i)$; then $\psi(n)$ is defined and belongs to $(p \to q)(i)$, hence encodes a partial recursive function θ_n that maps $p(i)$ to $q(i)$. Since $m \in p(i)$, we have $\varphi(k) = \theta_n(m) \in q(i)$, as claimed.

Take then $\alpha\colon J \to I$: for any $j \in J$ we have on the one hand $(R(\alpha)(p \to q))(j) = (p \to q)(\alpha(j)) = \{e \in \mathbb{N} \mid e$ encodes a partial recursive function $\theta\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that θ maps $p(\alpha(j))$ in $q(\alpha(j))\}$, and on the other hand

$$\begin{aligned}
&(R(\alpha)(p) \to R(\alpha)(q))(j)\\
&\qquad = \{d \in \mathbb{N} \mid d \text{ encodes a partial recursive function } \tau\colon \mathbb{N} \dashrightarrow \mathbb{N}\\
&\qquad\qquad \text{such that } \tau \text{ maps } R(\alpha)(p)(j) \text{ in } R(\alpha)(q)(j)\},
\end{aligned}$$

so the implication is preserved by reindexings, hence R is an implicational doctrine.

R is existential: For each pair of non-empty sets I, J, consider $\mathrm{pr}_1\colon I \times J \to I$ and define $\exists^J_I\colon R(I \times J) \to R(I)$ that maps a function $q\colon I \times J \to \mathscr{P}(\mathbb{N})$ to $\exists^J_I q\colon I \to \mathscr{P}(\mathbb{N})$, $(\exists^J_I q)(i) = \bigcup_{j\in J} q(i,j)$. This is the left adjoint to $R(\mathrm{pr}_1)$: $\exists^J_I q \leq p$ if and only if there exists $\varphi\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that for all $i \in I$, $\varphi_{|(\exists^J_I q)(i)}\colon \bigcup_{j\in J} q(i,j) \to p(i)$ is a total function, if and only if there exists $\varphi\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that for all $i \in I$ and $j \in J$, $\varphi_{|q(i,j)}\colon q(i,j) \to p(i)$ is a total function, if and only if $q \leq R(\mathrm{pr}_1)p$.

To show naturality in I, take a function $\alpha\colon K \to I$: for any $q \in R(I \times J)$ and $k \in K$,

$$R(\alpha)(\exists^J_I q)(k) = (\exists^J_I q)(\alpha(k)) = \bigcup_{j\in J} q(\alpha(k), j)$$

and also

$$(\exists^J_K R(\alpha \times \mathrm{id}_J)q)(k) = \bigcup_{j\in J} (R(\alpha \times \mathrm{id}_J)q)(k,j) = \bigcup_{j\in J} q(\alpha(k), j)$$

so that $R(\alpha)(\exists^J_I q) = \exists^J_K R(\alpha \times \mathrm{id}_J)q$, hence naturality holds.

To show Frobenius reciprocity, for any $q \in R(I \times J)$, $p \in R(I)$, and $i \in I$

$$\exists^J_I (q \wedge R(\mathrm{pr}_1)p)(i) = \bigcup_{j\in J} ((q \wedge R(\mathrm{pr}_1)p)(i,j)) = \bigcup_{j\in J} \{\langle a,b\rangle \in \mathbb{N} \mid a \in q(i,j), b \in p(i)\}$$

and also

$$(\exists^J_I q \wedge p)(i) = \{\langle a,b\rangle \in \mathbb{N} \mid a \in \bigcup_{j\in J} q(i,j), b \in p(i)\}$$

so that $\exists^J_I (q \wedge R(\mathrm{pr}_1)p) = \exists^J_I q \wedge p$, hence Frobenius reciprocity holds.

R is consistent: Take $R(\{\star\}) = \mathscr{P}(\mathbb{N})$; $T_{\{\star\}} \not\leq B_{\{\star\}}$ since for any partial recursive function $\varphi\colon \mathbb{N} \dashrightarrow \mathbb{N}$ is it not the case that $\varphi_{|\mathbb{N}}\colon \mathbb{N} \to \emptyset$ can be defined.

R is rich: Take any $q \in R(J)$ for a non-empty set J, we then look for a function $\bar{c}\colon \{\star\} \to J$, hence an element $c = \bar{c}(\star) \in J$, such that $\exists^J_{\{\star\}} q \leq R(\bar{c})q$, i.e. such that there exists a partial recursive function $\varphi\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that $\varphi_{|\bigcup_{j\in J} q(j)}\colon \bigcup_{j\in J} q(j) \to q(c)$ is a total function. Here is the point where the usual realizability doctrine defined over Set does not satisfy the needed assumption, and we need to remove the empty set from the base category. If $\bigcup_{j\in J} q(j) = \emptyset$, choose any $c \in J$ and $\varphi = \mathrm{id}_{\mathbb{N}}$, so that $\mathrm{id}_{\mathbb{N}|\emptyset}\colon \emptyset \to q(c)$ is a total function, as claimed. On the other hand, if $\bigcup_{j\in J} q(j) \neq \emptyset$, there exist $n \in \mathbb{N}$ and $c \in J$ such that $n \in q(c)$; choose $\varphi\colon \mathbb{N} \to \mathbb{N}$ to be the constant function to n, so that the restriction $\varphi_{|\bigcup_{j\in J} q(j)}\colon \bigcup_{j\in J} q(j) \to q(c)$ is a total function, again as wanted.

R is universal: For each pair of non-empty sets I, J, consider $\mathrm{pr}_1\colon I \times J \to I$ and define $\forall^J_I\colon R(I \times J) \to R(I)$ that maps a function $q\colon I \times J \to \mathscr{P}(\mathbb{N})$ to $\forall^J_I q\colon I \to \mathscr{P}(\mathbb{N})$, $(\forall^J_I q)(i) = \bigcap_{j\in J} q(i,j)$. This is the right adjoint to $R(\mathrm{pr}_1)$: $p \leq \forall^J_I q$ if and only if there exists $\varphi\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that for all $i \in I$, $\varphi_{|p(i)}\colon p(i) \to \bigcap_{j\in J} q(i,j)$ is a total function, if and only if there exists $\varphi\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that for all $i \in I$ and $j \in J$, $\varphi_{|p(i)}\colon p(i) \to q(i,j)$ is a total function, if and only if $q \leq R(\mathrm{pr}_1)p$. To show naturality in I, take a function $\alpha\colon K \to I$: for any $q \in R(I \times J)$ and $k \in K$,

$$R(\alpha)(\forall_I^J q)(k) = (\forall_I^J q)(\alpha(k)) = \bigcap_{j \in J} q(\alpha(k), j)$$

and also

$$(\forall_K^J R(\alpha \times \mathrm{id}_J)q)(k) = \bigcap_{j \in J} (R(\alpha \times \mathrm{id}_J)q)(k, j) = \bigcap_{j \in J} q(\alpha(k), j)$$

so that $R(\alpha)(\forall_I^J q) = \forall_K^J R(\alpha \times \mathrm{id}_J)q$, hence naturality holds.

Universal quantifier not preserved—expanding the cofinite sets: Our next goal is to find an ultrafilter $\nabla \subseteq R(\{\star\}) = \mathscr{P}(\mathbb{N})$ such that the morphism we built above $(\Gamma, \mathfrak{g})\colon R/\nabla \to \mathscr{P}_*$ does not preserve the universal quantifier: in particular we will find a non-empty set J and a $q \in R(J)$ such that $\forall_{!_J} \mathfrak{g}_J[q] \not\subseteq \mathfrak{g}_{\{\star\}} \underline{\forall}_{\{\star\}}^J [q]$. Recall that, given a function between two sets $h\colon A \to B$, the right adjoint to the preimage $h^{-1}\colon \mathscr{P}(B) \to \mathscr{P}(A)$ sends a subset S of A to the set $\forall_h S \coloneqq \{b \in B \mid h^{-1}(b) \subseteq S\}$. In our case, we have $\forall_{!_J} \mathfrak{g}_J[q] \neq \emptyset$ if and only if $J \subseteq \mathfrak{g}_J[q] = \{j \in J \mid q(j) \in \nabla\}$. Then, observe that $\mathfrak{g}_{\{\star\}}[\forall_{\{\star\}}^J q] \neq \emptyset$ if and only if $\forall_{\{\star\}}^J q \in \nabla$.

$$\begin{tikzcd}
R(J) \arrow[r, "\mathfrak{q}_J"] \arrow[d, "\forall_{\{\star\}}^J"'] & R/\nabla(J) \arrow[r, "\mathfrak{g}_J"] \arrow[d, "\underline{\forall}_{\{\star\}}^J"'] & \mathscr{P}(J) \arrow[d, "\forall_{!_J}"] \\
R(\{\star\}) \arrow[r, "\mathfrak{q}_{\{\star\}}"'] & R/\nabla(\{\star\}) \arrow[r, "\mathfrak{g}_{\{\star\}}"'] & \mathscr{P}(\{\star\})
\end{tikzcd}$$

Suppose $\nabla \subseteq \mathscr{P}(\mathbb{N})$ is an ultrafilter that contains all cofinite sets of $\mathbb{N}$; then take $J \coloneqq \mathbb{N}$ and $q\colon \mathbb{N} \to \mathscr{P}(\mathbb{N})$ such that $q(n) \coloneqq \mathbb{N} \setminus \{n\}$. We show that for all $j \in J$, $q(j) \in \nabla$, but $\forall_{\{\star\}}^J q \notin \nabla$, so that $\forall_{!_J} \mathfrak{g}_J[q] \not\subseteq \mathfrak{g}_{\{\star\}} \underline{\forall}_{\{\star\}}^J [q]$. Since $q(j)$ is clearly cofinite for every j, each $q(j) \in \nabla$; then compute $\forall_{\{\star\}}^J q = \bigcap_{j \in J} q(j) = \bigcap_{n \in \mathbb{N}} \mathbb{N} \setminus \{n\} = \emptyset \notin \nabla$. To conclude our proof, we need to show the existence of an ultrafilter over $\mathscr{P}(\mathbb{N})$ that contains every cofinite set. It is enough to prove that the filter generated by cofinite sets is a proper filter—i.e. does not contain the bottom element. Take the filter $F = \langle \mathscr{C} \rangle$ where $\mathscr{C}$ is the set of all cofinite set of $\mathbb{N}$ and suppose that it contains the bottom element. Recall from above that the bottom is $\emptyset$ and the meet of two subsets A, B of $\mathbb{N}$ is computed as $A \wedge B = \{\langle a, b \rangle \in \mathbb{N} \mid a \in A, b \in B\}$. Note that if A and B are cofinite, $A \wedge B$ is not in general cofinite, hence $\mathscr{C}$ is not a filter, as it is instead by taking the intersection as meet. However, suppose that $A \wedge B \leq \emptyset$ for a given pair $A, B \subseteq \mathbb{N}$, i.e. there exists a partial recursive function $\varphi\colon \mathbb{N} \dashrightarrow \mathbb{N}$ such that $\varphi_{|A \wedge B}\colon A \wedge B \to \emptyset$ is total, hence $A \wedge B = \emptyset$. In particular, it follows that at least one between A and B must be the empty set: if both $A \neq \emptyset$ and $B \neq \emptyset$, we can take $a \in A$ and $b \in B$, so that $\langle a, b \rangle \in A \wedge B \neq \emptyset$. Having noticed this, if it were the case that $\emptyset \in F$, there would exist $A_1, \ldots, A_n \in \mathscr{C}$ such that $((A_1 \wedge A_2) \wedge \cdots \wedge A_{n-1}) \wedge A_n \leq \bot$, so that one between $((A_1 \wedge A_2) \wedge \cdots \wedge A_{n-1})$ and A_n would be the empty set; since $A_n \in \mathscr{C}$, we must have $((A_1 \wedge A_2) \wedge \cdots \wedge A_{n-1}) = \emptyset$; by induction we get to a contradiction, so $\emptyset \notin F$, hence F is a proper filter.

Remark 3.8.3. Suppose that the starting doctrine in Proposition 3.8.1 $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is also boolean, meaning that we have the additional condition that $\neg\neg$ is the identity on each $P(X)$. Then, in particular, also P/∇ is a boolean algebra, since the quotient preserves both implication and bottom element. Under this assumption, we obtain that the model $(\Gamma, \mathfrak{g})$ is boolean. In particular, since the morphism is existential and boolean, it is also universal.

3.8.2 Definition of a model, elementary case

A little more work must be done in general if the starting doctrine is also elementary—in addition to the bounded implicational existential rich structure—and we want the model to preserve elementary structure. So this time we define a morphism $(\Omega, \mathfrak{h})\colon P/\nabla \to \mathscr{P}_*$ preserving the bounded elementary existential implicational structure. Define for each object X the following equivalence relation $\sim^X_\nabla$ on $\mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, X)$: given $c, d\colon \mathbf{t} \to X$, se say that $c \sim^X_\nabla d$ if and only if $P(\langle c, d\rangle)\AE^X_{\mathbf{t}}(\top_X) \in \nabla$.

- Reflexivity: $P(\langle c, c\rangle)\AE^X_{\mathbf{t}}(\top_X) = P(c)P(\Delta_X)\AE^X_{\mathbf{t}}(\top_X) \geq P(c)\top_X = \top_{\mathbf{t}} \in \nabla$, so $c \sim^X_\nabla c$;
- symmetry: suppose $P(\langle c, d\rangle)\AE^X_{\mathbf{t}}(\top_X) \in \nabla$, then

  $$P(\langle d, c\rangle)\AE^X_{\mathbf{t}}(\top_X) = P(\langle d, c\rangle)P(\langle \mathrm{pr}_2, \mathrm{pr}_1\rangle)\AE^X_{\mathbf{t}}(\top_X) = P(\langle c, d\rangle)\AE^X_{\mathbf{t}}(\top_X) \in \nabla,$$

  this follows from the fact that $\AE^X_{\mathbf{t}} = P(\langle \mathrm{pr}_2, \mathrm{pr}_1\rangle)\AE^X_{\mathbf{t}}$;
- transitivity: suppose $c \sim^X_\nabla d$ and $d \sim^X_\nabla a$, then apply $P(\langle c, d, a\rangle)$ to transitivity for equality $P(\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle)\AE^X_{\mathbf{t}}(\top_X) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle)\AE^X_{\mathbf{t}}(\top_X) \leq P(\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle)\AE^X_{\mathbf{t}}(\top_X)$ to get $P(\langle c, a\rangle)\AE^X_{\mathbf{t}}(\top_X) \in \nabla$, hence $c \sim^X_\nabla a$.

Given $f\colon X \to Y$, post-composition $f \circ -\colon \mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, X) \to \mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, Y)$ is well defined on the quotients: take $c \sim^X_\nabla d$ for some $c, d\colon \mathbf{t} \to X$, i.e. $P(\langle c, d\rangle)\AE^X_{\mathbf{t}}(\top_X) \in \nabla$, we show that $fc \sim^Y_\nabla fd$. From $\top_Y \leq P(\Delta_Y)\AE^Y_{\mathbf{t}}(\top_Y)$ apply $P(f)$ to get

$$\top_X \leq P(f)P(\Delta_Y)\AE^Y_{\mathbf{t}}(\top_Y) = P(\Delta_X)P(f \times f)\AE^Y_{\mathbf{t}}(\top_Y)$$

so that $\AE^X_{\mathbf{t}}(\top_X) \leq P(f \times f)\AE^Y_{\mathbf{t}}(\top_Y)$ and $P(\langle c, d\rangle)\AE^X_{\mathbf{t}}(\top_X) \leq P(\langle c, d\rangle)P(f \times f)\AE^Y_{\mathbf{t}}(\top_Y) = P(\langle fc, fd\rangle)\AE^Y_{\mathbf{t}}(\top_Y) \in \nabla$, as claimed. Hence, we can define the functor

$$\Omega \coloneqq \mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, -)/{\sim^{(-)}_\nabla}\colon \mathbb{C} \to \mathrm{Set}_*.$$

This preserves the products: take $a, c\colon \mathbf{t} \to X$ and $b, d\colon \mathbf{t} \to Y$, we have $\langle a, b\rangle \sim^{X\times Y}_\nabla \langle c, d\rangle$ if and only if $P(\langle a, b, c, d\rangle)\AE^{X\times Y}_{\mathbf{t}}(\top_{X\times Y}) \in \nabla$. Applying $P(\langle a, b, c, d\rangle)$ to the property $\AE^{X\times Y}_{\mathbf{t}}(\top_{X\times Y}) = P(\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle)\AE^X_{\mathbf{t}}(\top_X) \wedge P(\langle \mathrm{pr}_2, \mathrm{pr}_4\rangle)\AE^Y_{\mathbf{t}}(\top_Y)$, we get

$$P(\langle a, b, c, d\rangle)\AE^{X\times Y}_{\mathbf{t}}(\top_{X\times Y}) = P(\langle a, c\rangle)\AE^X_{\mathbf{t}}(\top_X) \wedge P(\langle b, d\rangle)\AE^Y_{\mathbf{t}}(\top_Y),$$

so that $P(\langle a,b,c,d\rangle)\AE_{\mathbf{t}}^{X\times Y}(\top_{X\times Y})\in\nabla$ if and only if both

$$P(\langle a,c\rangle)\AE_{\mathbf{t}}^{X}(\top_X)\in\nabla \text{ and } P(\langle b,d\rangle)\AE_{\mathbf{t}}^{Y}(\top_Y)\in\nabla,$$

if and only if $a\sim_{\nabla}^{X} c$ and $b\sim_{\nabla}^{Y} d$; so we proved that

$$\mathrm{Hom}_{\mathbb{C}}(\mathbf{t},X\times Y)/{\sim_{\nabla}^{X\times Y}}=\mathrm{Hom}_{\mathbb{C}}(\mathbf{t},X)/{\sim_{\nabla}^{X}}\times\mathrm{Hom}_{\mathbb{C}}(\mathbf{t},Y)/{\sim_{\nabla}^{Y}}.$$

Then, define for a given $X\in\mathrm{ob}\mathbb{C}$, $\mathfrak{h}_X\colon P/\nabla(X)\to\mathscr{P}_*(\mathrm{Hom}_{\mathbb{C}}(\mathbf{t},X)/{\sim_{\nabla}^{X}})$:

$$\mathfrak{h}_X[\varphi]=\{[c\colon\mathbf{t}\to X]\mid P(c)\varphi\in\nabla\}.$$

This is well defined, since whenever $c\sim_{\nabla}^{X} d$ and $[c]\in\mathfrak{h}_X[\varphi]$ we can apply $P(\langle c,d\rangle)$ to the property $\AE_{\mathbf{t}}^{X}(\top_X)\wedge P(\mathrm{pr}_1)\varphi\le P(\mathrm{pr}_2)\varphi$ to get $P(c)\varphi\to P(d)\varphi\in\nabla$, and hence $P(d)\varphi\in\nabla$.

Proposition 3.8.4. Let P be a bounded consistent implicational elementary existential rich doctrine, let $\nabla\subseteq P(\mathbf{t})$ be an ultrafilter, and let P/∇ be the quotient doctrine. Then the pair $(\Omega,\mathfrak{h})$, where $\Omega:=\mathrm{Hom}_{\mathbb{C}}(\mathbf{t},-)/{\sim_{\nabla}^{(-)}}$ and $\mathfrak{h}_X[\varphi]=\{[c\colon\mathbf{t}\to X]\mid P(c)\varphi\in\nabla\}$ for any object X and any $[\varphi]\in P/\nabla(X)$ is a bounded elementary existential implicational morphism.

Proof. All proofs from Proposition 3.8.1 can be rearranged in this scenario to prove that $(\Omega,\mathfrak{h})$ is a morphism of doctrines, preserving bounded implicational existential structure. The last thing left to prove is that $(\Omega,\mathfrak{h})$ preserves elementary structure: for a given $[\varphi]\in P/\nabla(X\times Y)$, the inclusion $=_X^Y\ \mathfrak{h}_{X\times Y}[\varphi]\subseteq\mathfrak{h}_{X\times Y\times Y}\underline{\AE}_X^Y[\varphi]$ follows from adjointness; for the converse, take $([a],[c],[d])\in\mathfrak{h}_{X\times Y\times Y}\underline{\AE}_X^Y$, i.e. $P(\langle a,c,d\rangle)\AE_X^Y\varphi\in\nabla$. By naturality of $\AE_{-}^{Y}$, we know that $\AE_{\mathbf{t}}^{Y}(\top_Y)=P(\langle a\cdot!_{Y\times Y},\mathrm{pr}_1,\mathrm{pr}_2\rangle)\AE_X^Y(\top_{X\times Y})$, hence

$$\begin{aligned}P(\langle c,d\rangle)\AE_{\mathbf{t}}^{Y}(\top_Y)&=P(\langle c,d\rangle)P(\langle a\cdot!_{Y\times Y},\mathrm{pr}_1,\mathrm{pr}_2\rangle)\AE_X^Y(\top_{X\times Y})\\&=P(\langle a,c,d\rangle)\AE_X^Y(\top_{X\times Y})\ge P(\langle a,c,d\rangle)\AE_X^Y\varphi\in\nabla,\end{aligned}$$

so that $c\sim_{\nabla}^{Y} d$, i.e. $[c]=[d]$, hence $([a],[c],[d])\in=_X^Y\ \mathfrak{h}_{X\times Y}[\varphi]$. □

We now have all the ingredients to generalize Henkin's Theorem.

Theorem 3.8.5. Let P be a bounded existential implicational doctrine, with non-trivial fibers and with a small base category. Then there exists a bounded existential implicational model of P in the doctrine of subsets $\mathscr{P}_*\colon\mathrm{Set}_*^{\mathrm{op}}\to\mathbf{Pos}$.

Proof. Do the construction in Remark 3.3.1 to get a morphism $(F,\mathfrak{f})\colon P\to\underset{\rightarrow}{P}$ that preserves bounded implicational existential structure; moreover by Proposition 3.7.1 the doctrine $\underset{\rightarrow}{P}$ is consistent. So $\underset{\rightarrow}{P}$ is an existential, implicational doctrine with bottom element, consistent and rich, then we can chose an ultrafilter $\nabla \subseteq \underset{\to}{P}(\mathbf{t})$ and take the quotient over it, and then the model $(\Gamma, \mathfrak{g})$ of such quotient. The composition

$$P \xrightarrow{(F,\mathfrak{f})} \underset{\to}{P} \xrightarrow{(\mathrm{id},\mathfrak{q})} \underset{\to}{P}/\nabla \xrightarrow{(\Gamma,\mathfrak{g})} \mathscr{P}_*$$

is a model of P, preserving all said structure. ☐

Theorem 3.8.6. Let P be a bounded elementary existential implicational doctrine, such that each of its fiber non-trivial and with a small base category. Then there exists a bounded elementary existential implicational model of P in the doctrine of subsets $\mathscr{P}_*\colon \mathrm{Set}_*^{\mathrm{op}} \to \mathbf{Pos}$.

Proof. Do as above but take $(\Omega, \mathfrak{h})$ instead of $(\Gamma, \mathfrak{g})$. ☐

Chapter 4

Left adjoint to precomposition in elementary doctrines

In Chapter 2 we extensively explored the process of adding a constant and an axiom to a doctrine P in a universal way, using the Kleisli object for some suitable comonads on P. Moreover we observed that applying these two constructions to the doctrine of well-formed formulae for some language and theory we obtain new doctrines that are isomorphic to the doctrine of well-formed formulae for the extended language—or respectively the extended theory. Of course, in algebra adding structure or axioms is a widely used technique: classical results say that for a given category of algebraic structure—e.g. monoids—, adding some structure or axioms—e.g. groups, commutative monoids—defines a forgetful functor from the new category to the original one, with a left adjoint. The category **ED** of elementary doctrines provides a natural framework for studying algebraic theories, with each theory $\mathbb{T}$ for a particular algebraic language Σ described by some doctrine of formulae $\mathtt{HF}^{\Sigma}_{\mathbb{T}}$; the models of such theories are morphisms in **ED** from the doctrine of formulae to the doctrine of subsets $\mathscr{P}_{*}\colon \mathrm{Set}_{*}^{\mathrm{op}} \to \mathbf{Pos}$, and the process of adding structure and axiom to a theory can be described by another morphism in **ED** between two doctrines of formulae $\mathtt{HF}^{\Sigma}_{\mathbb{T}} \to \mathtt{HF}^{\Sigma'}_{\mathbb{T}'}$. In particular the forgetful functor can be translated in **ED** as the precomposition with this last morphism. In this chapter we extend this classical result in **ED** by considering the subobject doctrine from a Grothendieck topos instead of the doctrine of subsets, and precomposition with any morphism $(F, \mathfrak{f})\colon P \to R$ instead of the forgetful functor.

4.1 The definition of the functor

Fix in the category **ED** of elementary doctrines a morphism $(F, \mathfrak{f})$ between two doctrines:

$$\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}"] \arrow[dr, "P"'] \arrow[rr, phantom, bend right=20, "\mathfrak{f}\ \dot{\to}"] & & \mathbb{D}^{\mathrm{op}} \arrow[dl, "R"] \\
& \mathbf{Pos} &
\end{tikzcd}$$

where $F\colon \mathbb{C} \to \mathbb{D}$ is a product preserving functor, $\mathfrak{f}\colon P \dot{\to} RF^{\mathrm{op}}$ is a natural transformation that preserves meets, top element and the elementary structure. Moreover, suppose that $\mathbb{C}$ is small. Consider a Grothendieck topos $\mathbb{E}$, and the associated subobjects doctrine $\mathrm{Sub}\colon \mathbb{E}^{\mathrm{op}} \to \mathbf{Pos}$, which is elementary—indeed, it is enough to ask for a finitely complete base category, see Example 1.1.10. Trivially we can precompose any morphism $(K, \mathfrak{k})\colon R \to \mathrm{Sub}$ in **ED** with $(F, \mathfrak{f})$ to obtain a morphism $(K, \mathfrak{k})(F, \mathfrak{f})\colon P \to \mathrm{Sub}$; this gives a functor

$$- \circ (F, \mathfrak{f})\colon \mathbf{ED}(R, \mathrm{Sub}) \to \mathbf{ED}(P, \mathrm{Sub}).$$

We look for a left adjoint for this precomposition.

$$\begin{tikzcd}
\mathbb{C}^{\mathrm{op}} \arrow[rr, "H^{\mathrm{op}}"] \arrow[dr, "F^{\mathrm{op}}"] \arrow[ddr, bend right=40, "P"'] & & \mathbb{E}^{\mathrm{op}} \arrow[ddl, bend left=40, "\mathrm{Sub}"] \\
& \mathbb{D}^{\mathrm{op}} \arrow[ur, dashed] \arrow[d, "R"] & \\
& \mathbf{Pos} &
\end{tikzcd}$$

with $\mathfrak{h}$ from P to Sub, $\mathfrak{f}$ from P to R, and a dashed arrow from R to Sub drawn as natural transformations between the legs.

The whole chapter is devoted to the proof of the following:

Theorem 4.1.1. Let $(F, \mathfrak{f})\colon P \to R$ be a morphism in **ED**, and suppose the base category of P to be small. Moreover let $\mathbb{E}$ be a Grothendieck topos and $\mathrm{Sub}\colon \mathbb{E}^{\mathrm{op}} \to \mathbf{Pos}$ be the subobject doctrine. Then, the functor induced by precomposition

$$- \circ (F, \mathfrak{f})\colon \mathbf{ED}(R, \mathrm{Sub}) \to \mathbf{ED}(P, \mathrm{Sub})$$

has a left adjoint.

We start from a morphism $(H, \mathfrak{h})\colon P \to \mathrm{Sub}$, our first goal is to find a functor $\mathbb{D} \to \mathbb{E}$. An easy choice is to take the left Kan extension of H along F, whose existence is granted by the fact that $\mathbb{E}$ is a Grothendieck topos—since it is cocomplete, see Chapter X in [Mac71]. Recall that the left Kan extension comes with a natural transformation $\mu\colon H \dot{\to} (\mathrm{Lan}_F\, H)F$ such that for any other functor $K\colon \mathbb{D} \to \mathbb{E}$ and any other natural transformation $\theta\colon H \dot{\to} KF$ there exists a unique $\widehat{\theta}\colon \mathrm{Lan}_F\, H \dot{\to} K$ making the obvious diagrams commute:

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "H"] \arrow[dr, "F"'] & {} \arrow[d, phantom, "\Downarrow\mu"] & \mathbb{E} \\
& \mathbb{D} \arrow[ur, "\mathrm{Lan}_F\, H"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \arrow[rr, "H"] \arrow[dr, "F"'] & {} \arrow[d, phantom, "\Downarrow\theta"] & \mathbb{E} \\
& \mathbb{D} \arrow[ur, "K"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \arrow[rr, "H"] \arrow[dr, "F"'] & {} \arrow[d, phantom, "\Downarrow\theta"] & \mathbb{E} \\
& \mathbb{D} \arrow[ur, bend left=15, "\mathrm{Lan}_F\, H"] \arrow[ur, bend right=25, "K"'] \arrow[ur, phantom, "\Downarrow\widehat{\theta}"] &
\end{tikzcd}$$

Before we continue, we need $\mathrm{Lan}_F\, H$ to be product preserving. Recall from [Mac71] that for any object $D \in \mathbb{D}$, we have $(\mathrm{Lan}_F\, H)(D) = \operatorname{colim}\big((F \downarrow D) \xrightarrow{\mathrm{pr}} \mathbb{C} \xrightarrow{H} \mathbb{E}\big)$.

Proposition 4.1.2. Let $\mathbb{C}, \mathbb{D}, \mathbb{E}$ be categories with finite products such that $\mathbb{C}$ is small and $\mathbb{E}$ is cocomplete and cartesian closed, and let $F\colon \mathbb{C} \to \mathbb{D}$ and $H\colon \mathbb{C} \to \mathbb{E}$ be finite product preserving functors. Then $\mathrm{Lan}_F\, H\colon \mathbb{D} \to \mathbb{E}$ preserves finite products.

Proof. Take two objects $D, D' \in \mathbb{D}$ and consider the two projections $\mathrm{pr}_1\colon D \times D' \to D$ and $\mathrm{pr}_2\colon D \times D' \to D'$. Since $\mathrm{Lan}_F\, H$ is a functor we have an arrow in $\mathbb{E}$:

$$\psi := \langle \underset{F}{\mathrm{Lan}}\, H(\mathrm{pr}_1), \underset{F}{\mathrm{Lan}}\, H(\mathrm{pr}_2)\rangle\colon \underset{F}{\mathrm{Lan}}\, H(D \times D') \to \underset{F}{\mathrm{Lan}}\, H(D) \times \underset{F}{\mathrm{Lan}}\, H(D').$$

The arrow $\mathrm{Lan}_F\, H(\mathrm{pr}_1)$ is the unique one such that

$$(\underset{F}{\mathrm{Lan}}\, H(\mathrm{pr}_1))\iota_{(\widetilde{C}, \widetilde{c}: F\widetilde{C} \to D \times D')} = \iota_{(\widetilde{C}, \mathrm{pr}_1 \widetilde{c}: \widetilde{C} \to D)}\colon H\widetilde{C} \to \underset{F}{\mathrm{Lan}}\, HD$$

and similarly $(\mathrm{Lan}_F\, H(\mathrm{pr}_2))\iota_{(\widetilde{C}, \widetilde{c}: F\widetilde{C} \to D \times D')} = \iota_{(\widetilde{C}, \mathrm{pr}_2 \widetilde{c}: \widetilde{C} \to D')}$. We want to prove that ψ has an inverse. Since $\mathbb{E}$ is cartesian closed, the product functor with a fixed object is a left adjoint, hence it preserves colimits, so we have:

$$\underset{F}{\mathrm{Lan}}\, HD \times \underset{F}{\mathrm{Lan}}\, HD' \cong \left(\operatorname*{colim}_{(C, c: FC \to D)} HC\right) \times \left(\operatorname*{colim}_{(C', c': FC' \to D')} HC'\right) \cong$$

$$\cong \operatorname*{colim}_{(C, c: FC \to D)} \operatorname*{colim}_{(C', c': FC' \to D')} (HC \times HC')$$

The arrow from the double colimit to the product above is the unique $\omega = \langle \omega_1, \omega_2 \rangle$ such that $\omega_1 \iota_{(C,c),(C',c')} = \iota_{(C,c)}\mathrm{pr}_1$ and $\omega_2 \iota_{(C,c),(C',c')} = \iota_{(C',c')}\mathrm{pr}_2$. So now we look for the inverse φ of $\omega^{-1}\psi$, defining an arrow

$$\varphi\colon \operatorname*{colim}_{(C,c)} \operatorname*{colim}_{(C',c')} (HC \times HC') \to \operatorname*{colim}_{(\widetilde{C}, \widetilde{c})} H\widetilde{C}.$$

Build the following cocone: for any $(C, c), (C', c')$ we take the arrow

$$\iota_{(C \times C', c \times c')}\colon HC \times HC' \to \operatorname*{colim}_{(\widetilde{C}, \widetilde{c})} H\widetilde{C}.$$

Observe that here we use that both F and H preserve binary products. Now take another pair $(\bar{C}, \bar{c}), (\bar{C}', \bar{c}')$ and two arrows $f\colon C \to \bar{C}$ and $f'\colon C' \to \bar{C}'$ such that the following triangles commute:

$$\begin{tikzcd} FC \arrow[r, "Ff"] \arrow[rd, "c"'] & F\bar{C} \arrow[d, "\bar{c}"] \\ & D \end{tikzcd} \qquad \begin{tikzcd} FC' \arrow[r, "Ff'"] \arrow[rd, "c'"'] & F\bar{C}' \arrow[d, "\bar{c}'"] \\ & D' \end{tikzcd}$$

we need $\iota_{(C \times C', c \times c')} = (Hf \times Hf')\iota_{(\bar{C} \times \bar{C}', \bar{c} \times \bar{c}')}$, but this holds by definition of inclusions in

$\operatorname{colim}_{(\widetilde{C},\widetilde{c})} H\widetilde{C}$.

$$\begin{tikzcd}
HC \times HC' \arrow[rr, "Hf\times Hf'"] \arrow[dr, "\iota_{(C,c),(C',c')}"] \arrow[ddr, bend right, "\iota_{(C\times C',c\times c')}"'] & & H\bar{C} \times H\bar{C}' \arrow[dl, "\iota_{(\bar{C},\bar{c}),(\bar{C}',\bar{c}')}"'] \arrow[ddl, bend left, "\iota_{(\bar{C}\times\bar{C}',\bar{c}\times\bar{c}')}"] \\
& \operatorname{colim}\operatorname{colim}(HC \times HC') \arrow[d, "\varphi"'] & \\
& \operatorname{colim} H\widetilde{C} &
\end{tikzcd}$$

Instead of proving that ψ is an isomorphism we prove that $\omega^{-1}\psi$ is; so look at the following arrows:

$$\begin{tikzcd}
\operatorname{Lan}_F H(D\times D') \arrow[r, "\psi"] & \operatorname{Lan}_F HD \times \operatorname{Lan}_F HD' \\
& \operatorname{colim}_{(C,c)} \operatorname{colim}_{(C',c')}(HC \times HC') \arrow[ul, bend left=10, "\varphi"] \arrow[u, "\omega"', "\wr"]
\end{tikzcd}$$

On the one side we want to show $(\omega^{-1}\psi)\varphi = \mathrm{id}_{\operatorname{colim}\operatorname{colim}(HC\times HC')}$, and then that $\varphi(\omega^{-1}\psi) = \mathrm{id}_{\operatorname{Lan}_F H(D\times D')}$. The first equality holds if and only if $\psi\varphi = \omega$ if and only if $\mathrm{pr}_1\psi\varphi = \mathrm{pr}_1\omega$ and $\mathrm{pr}_2\psi\varphi = \mathrm{pr}_2\omega$ if and only if $\mathrm{pr}_1\psi\varphi\iota_{(C,c),(C',c')} = \mathrm{pr}_1\omega\iota_{(C,c),(C',c')}$ and $\mathrm{pr}_2\psi\varphi\iota_{(C,c),(C',c')} = \mathrm{pr}_2\omega\iota_{(C,c),(C',c')}$ for every $\iota_{(C,c),(C',c')}$. However,

$$\begin{aligned}\mathrm{pr}_1\psi\varphi\iota_{(C,c),(C',c')} &= (\operatorname*{Lan}_F H\mathrm{pr}_1)\iota_{(C\times C',c\times c')} = \iota_{(C\times C',\mathrm{pr}_1(c\times c'))}\\ &= \iota_{(C\times C',c\mathrm{pr}_1)} = \iota_{(C,c)}\mathrm{pr}_1 = \mathrm{pr}_1\omega\iota_{(C,c),(C',c')};\end{aligned}$$

similarly $\mathrm{pr}_2\psi\varphi\iota_{(C,c),(C',c')} = \mathrm{pr}_2\omega\iota_{(C,c),(C',c')}$.

$$\begin{tikzcd}
C \times C' \arrow[d, "\mathrm{pr}_1"] \\
C
\end{tikzcd}
\qquad
\begin{tikzcd}
FC \times FC' \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, "c\mathrm{pr}_1"] & \\
FC \arrow[r, "c"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
HC \times HC' \arrow[r, "\mathrm{pr}_1"] \arrow[dr, "\iota_{(C\times C',c\mathrm{pr}_1)}"'] & HC \arrow[d, "\iota_{(C,c)}"] \\
& \operatorname{colim}_{(C,c)} HC
\end{tikzcd}$$

Concerning the second equality, it holds if and only if for every $\iota_{(\widetilde{C},\widetilde{c})}$ we have $\varphi\omega^{-1}\psi\iota_{(\widetilde{C},\widetilde{c})} = \iota_{(\widetilde{C},\widetilde{c})}$; however

$$\begin{aligned}\varphi\omega^{-1}\psi\iota_{(\widetilde{C},\widetilde{c})} &= \varphi\omega^{-1}\langle\iota_{(\widetilde{C},\mathrm{pr}_1\widetilde{c})},\iota_{(\widetilde{C},\mathrm{pr}_2\widetilde{c})}\rangle = \varphi\omega^{-1}\omega\iota_{(\widetilde{C},\mathrm{pr}_1\widetilde{c}),(\widetilde{C},\mathrm{pr}_2\widetilde{c})}\Delta_{H\widetilde{C}}\\ &= \iota_{(\widetilde{C}\times\widetilde{C},\mathrm{pr}_1\widetilde{c}\times\mathrm{pr}_2\widetilde{c})}\Delta_{H\widetilde{C}} = \iota_{(\widetilde{C},\widetilde{c})},\end{aligned}$$

as claimed.

$$\begin{tikzcd}
\widetilde{C} \arrow[d, "\Delta"] \\
\widetilde{C} \times \widetilde{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
F\widetilde{C} \arrow[d, "\Delta"'] \arrow[dr, "\widetilde{c}"] & \\
F\widetilde{C} \times F\widetilde{C} \arrow[r, "\mathrm{pr}_1\widetilde{c}\times\mathrm{pr}_2\widetilde{c}"'] & D \times D'
\end{tikzcd}
\qquad
\begin{tikzcd}
H\widetilde{C} \arrow[r, "\Delta"] \arrow[dr, "\iota_{(\widetilde{C},\widetilde{c})}"'] & H\widetilde{C} \times H\widetilde{C} \arrow[d, "\iota_{(\widetilde{C}\times\widetilde{C},\mathrm{pr}_1\widetilde{c}\times\mathrm{pr}_2\widetilde{c})}"] \\
& \operatorname{colim}_{(C,c)} HC
\end{tikzcd}$$

To see the equality $\langle\iota_{(\widetilde{C},\mathrm{pr}_1\widetilde{c})},\iota_{(\widetilde{C},\mathrm{pr}_2\widetilde{c})}\rangle = \omega\iota_{(\widetilde{C},\mathrm{pr}_1\widetilde{c}),(\widetilde{C},\mathrm{pr}_2\widetilde{c})}\Delta_{H\widetilde{C}}$ above observe that $\iota_{(\widetilde{C},\mathrm{pr}_1\widetilde{c})} = \iota_{(\widetilde{C},\mathrm{pr}_1\widetilde{c})}\mathrm{pr}_1\Delta_{H\widetilde{C}} = \omega_1\iota_{(\widetilde{C},\mathrm{pr}_1\widetilde{c}),(\widetilde{C},\mathrm{pr}_2\widetilde{c})}\Delta_{H\widetilde{C}}$, and similarly also the second projections coincide. To conclude the proof we need to show that the terminal object is preserved by $\mathrm{Lan}_F\, H$. Recall that $\mathrm{Lan}_F\, H\mathbf{t}_{\mathbb{D}} = \operatorname{colim}_{(C,!_{FC}:FC\to\mathbf{t}_{\mathbb{D}})} HC$; moreover for any object $C \in \mathbb{C}$ we have that $\iota_{(\mathbf{t}_{\mathbb{C}},\mathrm{id})}!_{HC} = \iota_{(C,!_{FC})}$:

$$\begin{tikzcd}
C \arrow[d, "!_C"] \\
\mathbf{t}_{\mathbb{C}}
\end{tikzcd}
\qquad
\begin{tikzcd}
FC \arrow[d, "!_{FC}"'] \arrow[dr, "!_{FC}"] & \\
\mathbf{t}_{\mathbb{D}} \arrow[r, "\mathrm{id}"'] & \mathbf{t}_{\mathbb{D}}
\end{tikzcd}
\qquad
\begin{tikzcd}
HC \arrow[r, "!_{HC}"] \arrow[dr, "\iota_{(C,!_{FC})}"'] & \mathbf{t}_{\mathbb{E}} \arrow[d, "\iota_{(\mathbf{t}_{\mathbb{C}},\mathrm{id})}"] \\
& \operatorname{colim}_{(C,c)} HC
\end{tikzcd}$$

In particular $!_{\mathrm{Lan}_F\, H\mathbf{t}_{\mathbb{D}}}\iota_{(\mathbf{t}_{\mathbb{C}},\mathrm{id})} = \mathrm{id}_{\mathbf{t}_{\mathbb{E}}}$; then $\iota_{(\mathbf{t}_{\mathbb{C}},\mathrm{id})}!_{\mathrm{Lan}_F\, H\mathbf{t}_{\mathbb{D}}} = \mathrm{id}_{\mathrm{Lan}_F\, H\mathbf{t}_{\mathbb{D}}}$ if and only if for any $(C,!_{FC})$ we have $\iota_{(\mathbf{t}_{\mathbb{C}},\mathrm{id})}!_{\mathrm{Lan}_F\, H\mathbf{t}_{\mathbb{D}}}\iota_{(C,!_{FC})} = \iota_{(C,!_{FC})}$ but

$$\iota_{(\mathbf{t}_{\mathbb{C}},\mathrm{id})}!_{\mathrm{Lan}_F\, H\mathbf{t}_{\mathbb{D}}}\iota_{(C,!_{FC})} = \iota_{(\mathbf{t}_{\mathbb{C}},\mathrm{id})}!_{HC} = \iota_{(C,!_{FC})},$$

as claimed; this concludes the proof. □

If $\mathbb{E}$ is a Grothendieck topos, the hypothesis of the proposition above are satisfied, so $\mathrm{Lan}_F\, H$ preserves finite products.

Define now a natural transformation $\mathfrak{l}\colon R \dot{\to} \mathrm{Sub}(\mathrm{Lan}_F\, H)^{\mathrm{op}}$. For any object $D \in \mathbb{D}$, and any $\gamma \in R(D)$, write

$$\mathfrak{l}_D(\gamma) = \bigwedge_{(K,\mathfrak{k}),\theta} \widehat{\theta}^*_D(\mathfrak{k}_D(\gamma))$$

where $(K,\mathfrak{k})\colon R \to \mathrm{Sub}$ is an arrow in **ED** and $\theta\colon (H,\mathfrak{h}) \to (K,\mathfrak{k})(F,\mathfrak{f})$ is a 2-arrow, i.e. $\mathfrak{h}_A(\alpha) \leq \theta^*_A(\mathfrak{k}_{FA}(\mathfrak{f}_A(\alpha)))$ for all $A \in \mathbb{C}$ and $\alpha \in P(A)$. Observe that $\mathfrak{k}_D(\gamma)$ is a subobject of KD, $\widehat{\theta}$ is defined by the universal property of the left Kan extension, and $\widehat{\theta}^*_D(\mathfrak{k}_D(\gamma))$ is the pullback of $\mathfrak{k}_D(\gamma)$ along $\widehat{\theta}_D\colon (\mathrm{Lan}_F\, H)(D) \to KD$, hence it is a subobject of $(\mathrm{Lan}_F\, H)(D)$. Since $\mathbb{E}$ is a complete category, the infimum of $\{\widehat{\theta}^*_D(\mathfrak{k}_D(\gamma))\}_{(K,\mathfrak{k}),\theta}$ exists, and we call it $\mathfrak{l}_D(\gamma)$.

Lemma 4.1.3. The following properties hold:

1. $\mathfrak{l}\colon R \dot{\to} \mathrm{Sub}(\mathrm{Lan}_F\, H)^{\mathrm{op}}$ is a natural transformation;
2. $\mathfrak{l}\colon R \dot{\to} \mathrm{Sub}(\mathrm{Lan}_F\, H)^{\mathrm{op}}$ preserves finite meets;
3. $\mathfrak{l}_{D\times D}(\delta_D) \in \mathrm{Sub}\big((\mathrm{Lan}_F\, H)(D) \times (\mathrm{Lan}_F\, H)(D)\big)$ is an equivalence relation for any object $D \in \mathbb{D}$.

Proof. 1. Take an arrow $g\colon D' \to D$ in $\mathbb{D}$, we prove that $\big((\mathrm{Lan}_F\, H)(g)\big)^*\mathfrak{l}_D(\gamma) = \mathfrak{l}_{D'}R(g)(\gamma)$ for any $\gamma \in RD$:

$$\begin{aligned}\big((\mathrm{Lan}_F\, H)(g)\big)^*\mathfrak{l}_D(\gamma) &= \big((\mathrm{Lan}_F\, H)(g)\big)^*\Big(\textstyle\bigwedge_{(K,\mathfrak{k}),\theta}\widehat{\theta}^*_D(\mathfrak{k}_D(\gamma))\Big)\\ &= \bigwedge_{(K,\mathfrak{k}),\theta}\big((\mathrm{Lan}_F\, H)(g)\big)^*\widehat{\theta}^*_D(\mathfrak{k}_D(\gamma)) = \textstyle\bigwedge_{(K,\mathfrak{k}),\theta}\widehat{\theta}^*_{D'}\big(K(g)\big)^*(\mathfrak{k}_D(\gamma))\\ &= \bigwedge_{(K,\mathfrak{k}),\theta}\widehat{\theta}^*_{D'}(\mathfrak{k}_{D'}R(g)(\gamma)) = \mathfrak{l}_{D'}(R(g)\gamma).\end{aligned}$$

Note that the second equality follows from the fact that pullback functors between subobjects categories preserve arbitrary limits—since in a regular categories they have a left adjoint—; the other equalities follow from naturality of $\widehat{\theta}$ and $\mathfrak{k}$.

2. The top element $\top_D \in RD$ for any object $D \in \mathbb{D}$ is preserved by $\mathfrak{l}_D$ since $\mathfrak{k}_D(\top_D)$ is $\mathrm{id}_{KD}\colon KD \to KD$ the top element in $\mathrm{Sub}(KD)$ by assumption, and its pullback along any $\widehat{\theta}_D$ is the identity of $(\mathrm{Lan}_F\, H)(D)$. Similarly, $\mathfrak{l}_D$ preserves binary meets since any $\mathfrak{k}_D$ and any pullback functor do.
3. Compute

$$\mathfrak{l}_{D\times D}(\delta_D) = \bigwedge_{(K,\mathfrak{k}),\theta}\widehat{\theta}^*_{D\times D}(\mathfrak{k}_{D\times D}(\delta_D)) = \bigwedge_{(K,\mathfrak{k}),\theta}\big(\widehat{\theta}_D\times\widehat{\theta}_D\big)^*(\Delta_{KD}).$$

Note that each $\big(\widehat{\theta}_D\times\widehat{\theta}_D\big)^*(\Delta_{KD})$ is an equivalence relation on $(\mathrm{Lan}_F\, H)(D)$, since it is the kernel pair of the map $\widehat{\theta}_D$. By Lemma 4.1.4, we know that the infimum of equivalence relations is itself an equivalence relation. Hence, we can conclude that $\mathfrak{l}_{D\times D}(\delta_D)$ is indeed an equivalence relation. ☐

Lemma 4.1.4. Let $\mathbb{E}$ be a complete category, X be an object of $\mathbb{E}$, and let

$$\{r^i = \langle r^i_1, r^i_2\rangle\colon R_i \rightarrowtail X\times X\}_{i\in I} \subseteq \mathrm{Sub}(X\times X)$$

be a family of equivalence relations on X. Then the infimum $r = \bigwedge_{i\in I} r^i$ is again an equivalence relation on X.

Proof.

$$\begin{tikzcd} X \arrow[d, "\delta_i"'] \arrow[rd, "\Delta_X"] & \\ R_i \arrow[r, "r^i"'] & X\times X \end{tikzcd} \qquad \begin{tikzcd} R_i \arrow[d, "\sigma_i"'] \arrow[rd, "{\langle r^i_2, r^i_1\rangle}"] & \\ R_i \arrow[r, "r^i"'] & X\times X \end{tikzcd} \qquad \begin{tikzcd} R_i\times_X R_i \arrow[d, "\tau_i"'] \arrow[rd, "{\langle p^i_1, p^i_3\rangle}"] & \\ R_i \arrow[r, "r^i"'] & X\times X \end{tikzcd}$$

For every $i \in I$, call $\delta_i\colon X \to R_i$ the arrow such that $r^i\delta_i = \Delta_X$—reflexivity—, $\sigma_i\colon Ri \to R_i$ the arrow such that $r^i\sigma_i = \langle r^i_2, r^i_1\rangle$—symmetry—and $\tau_i\colon R_i\times_X R_i \to R_i$ the arrow such that

$r^i\tau_i = \langle p_1^i, p_3^i\rangle$—transitivity—, where $R_i \times_X R_i, p_1^i$ and p_3^i are defined as follows:

$$\begin{tikzcd}
 & & R_i \times_X R_i \arrow[dl, "\alpha^i"'] \arrow[dr, "\beta^i"] \arrow[ddll, bend right, "p_1^i"'] \arrow[ddrr, bend left, "p_3^i"] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & R_i \arrow[dl, "r_1^i"'] \arrow[dr, "r_2^i"] & & R_i \arrow[dl, "r_1^i"'] \arrow[dr, "r_2^i"] & \\
X & & X & & X
\end{tikzcd}$$

First of all, observe that r is a generalized pullback: it is the datum of an arrow $r: R \to X \times X$ and a family of arrows $\{m_i: R \to R_i\}_{i\in I}$ such that $r^i m_i = r$ for all $i \in I$, with the property that for any arrow $s = \langle s_1, s_2\rangle: S \to X \times X$ and family $n_i: S \to R_i$ such that $r^i n_i = s$, then there exists a unique $n: S \to R$ such that $rn = s$.

- r *is a monomorphism.* Take $f, g: A \to R$ such that $rf = rg$; so for any $i \in I$ we have $r^i m_i f = r^i m_i f$, but each r^i is a mono so $m_i f = m_i g$. Now $\{m_i f\}_{i\in I} = \{m_i g\}_{i\in I}$ is a family of arrow such that $r^i m_i f = rf = rg$, so there exist a unique n such that $rn = rf = rg$, hence $f = g$.
- r *is the infimum of* $\{r^i\}_{i\in I}$ *in* $\mathrm{Sub}(X \times X)$. Using the existence of m_i we have $r \leq r^i$. Then, suppose s above to be a mono, so that $s \leq r^i$ for all $i \in I$ and it follows that $s \leq r$.
- r *is reflexive.* From the pair $(\Delta_X: X \to X \times X, \{\delta_i\}_{i\in I})$ such that $r^i\delta_i = \Delta_X$, define $\delta: X \to R$ such that $r\delta = \Delta_X$.
- r *is symmetric.* From the pair $(\langle r_2, r_1\rangle: R \to X\times X, \{\sigma_i m_i\}_{i\in I})$ such that $r^i\sigma_i m_i = \langle r_2, r_1\rangle$, define $\sigma: R \to R$ such that $r\sigma = \langle r_2, r_1\rangle$.
- r *is transitive.* Call α, β the projection maps in the pullback of r_1 along r_2, $p_1 = r_1\alpha$ and $p_3 = r_2\beta$. Define $t_i: R \times_X R \to R_i \times_X R_i$ the unique maps such that $\alpha^i t_i = m_i\alpha$ and $\beta^i t_i = m_i\beta$. From the pair $(\langle p_1, p_3\rangle: R\times_X R \to X\times X, \{\tau_i t_i\}_{i\in I})$ such that $r^i\tau_i t_i = \langle p_1, p_3\rangle$, define $\tau: R \times_X R \to R$ such that $r\tau = \langle p_1, p_3\rangle$. ☐

Recall from Section 4 of [MR12] that given any elementary doctrine $P: \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ one can build the category $\mathcal{R}_P$ of equivalence relations of P where objects are pairs (A, ρ), with $\rho \in P(A \times A)$ an equivalence relation on A, and arrows $f: (A, \rho) \to (B, \sigma)$ are arrows $f: A \to B$ such that $\rho \leq_{A\times A} P(f \times f)(\sigma)$. Composition and identities are computed in $\mathbb{C}$. Then, $(P)_q: \mathcal{R}_P^{\mathrm{op}} \to \mathbf{Pos}$ will be given by $(P)_q(A, \rho) = \mathscr{D}es_\rho = \{\alpha \in P(A) \mid P(\mathrm{pr}_1)(\alpha) \wedge \rho \leq P(\mathrm{pr}_2)(\alpha)\}$, and $(P)_q(f) = P(f)$. Since $\mathfrak{l}_{D\times D}(\delta_D)$ is an equivalence relation on $(\mathrm{Lan}_F H)(D)$, we can define a functor $\mathscr{L} = \langle \mathrm{Lan}_F H(-), \mathfrak{l}_{-\times -}(\delta_-)\rangle: \mathbb{D} \to \mathcal{R}_{\mathrm{Sub}}$, where $\mathcal{R}_{\mathrm{Sub}}$ is the category of equivalence relations of $\mathrm{Sub}: \mathbb{E}^{\mathrm{op}} \to \mathbf{Pos}$. Given an arrow $g: D' \to D$ in $\mathbb{D}$, we define

$$\mathscr{L}(g) = (\operatorname*{Lan}_F H)(g): \big((\operatorname*{Lan}_F H)(D'), \mathfrak{l}_{D'\times D'}(\delta_{D'})\big) \to \big((\operatorname*{Lan}_F H)(D), \mathfrak{l}_{D\times D}(\delta_D)\big).$$

$\mathscr{L}$ **is well defined on arrows:** In order to prove this is well defined we need

$$\mathfrak{l}_{D'\times D'}(\delta_{D'}) \leq \big((\operatorname*{Lan}_F H)(g) \times (\operatorname*{Lan}_F H)(g)\big)^*(\mathfrak{l}_{D\times D}(\delta_D))$$

in $\mathrm{Sub}((\mathrm{Lan}_F H)(D \times D))$, so we need a map $\mathrm{dom}\, \mathfrak{l}_{D'\times D'}(\delta'_D) \to \mathrm{dom}\, \mathfrak{l}_{D\times D}(\delta_D)$ making the external diagram commute.

$$\begin{tikzcd}
\mathrm{dom}\, \mathfrak{l}_{D'\times D'}(\delta_{D'}) \arrow[dr, tail, dashed, "\leq"] \arrow[drr, bend left] \arrow[ddr, tail, bend right, "\mathfrak{l}_{D'\times D'}(\delta_{D'})"'] & & \\
& \mathrm{dom}((\mathrm{Lan}_F H)(g \times g))^*(\mathfrak{l}_{D\times D}(\delta_D)) \arrow[r] \arrow[d, tail, "((\mathrm{Lan}_F H)(g\times g))^*(\mathfrak{l}_{D\times D}(\delta_D))"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{dom}\, \mathfrak{l}_{D\times D}(\delta_D) \arrow[d, tail, "\mathfrak{l}_{D\times D}(\delta_D)"] \\
& (\mathrm{Lan}_F H)(D' \times D') \arrow[r, "(\mathrm{Lan}_F H)(g\times g)"] & (\mathrm{Lan}_F H)(D \times D)
\end{tikzcd}$$

So now consider for each $(K, \mathfrak{k})\colon R \to \mathrm{Sub}$ and $\theta\colon (H, \mathfrak{h}) \to (K, \mathfrak{k})(F, \mathfrak{f})$.

$$\begin{tikzcd}
& KD' \arrow[dr, "Kg"] & \\
\mathrm{dom}\, \widehat{\theta}^*_{D'\times D'}(\Delta_{KD'}) \arrow[ur, "\widehat{\theta}_{D\times D}{}_{|\mathrm{dom}\, \widehat{\theta}^*_{D\times D}(\Delta_{KD})}"] \arrow[r, dashed] \arrow[d, tail, "\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'})"'] & \mathrm{dom}\, \widehat{\theta}^*_{D\times D}(\Delta_{KD}) \arrow[r, "\widehat{\theta}_{D\times D}{}_{|\mathrm{dom}\, \widehat{\theta}^*_{D\times D}(\Delta_{KD})}"] \arrow[d, tail, "\widehat{\theta}^*_{D\times D}(\Delta_{KD})"] \arrow[dr, phantom, "\lrcorner", very near start] & KD \arrow[d, tail, "\Delta_{KD}"] \\
(\mathrm{Lan}_F H)(D' \times D') \arrow[r, "(\mathrm{Lan}_F H)(g\times g)"'] \arrow[rr, bend right, "K(g\times g)\widehat{\theta}_{D'\times D'}"'] & (\mathrm{Lan}_F H)(D \times D) \arrow[r, "\widehat{\theta}_{D\times D}"'] & KD \times KD
\end{tikzcd}$$

By definition of $\mathfrak{l}$ as infimum we can find the wanted arrow $\mathrm{dom}\, \mathfrak{l}_{D'\times D'}(\delta_{D'}) \to \mathrm{dom}\, \mathfrak{l}_{D\times D}(\delta_D)$.

$\mathscr{L}$ **preserves products:** Moreover, $\mathscr{L}$ preserves products. To see this, compute $\mathscr{L}(D \times D')$ and $\mathscr{L}(D) \times \mathscr{L}(D')$: the first projections $\mathrm{Lan}_F H(D \times D') = \mathrm{Lan}_F H(D) \times \mathrm{Lan}_F H(D')$ coincide, since $\mathrm{Lan}_F H$ preserves products; so we need to show that also the equivalence relations $\mathfrak{l}_{D\times D'\times D\times D'}(\delta_{D\times D'})$ and $\mathfrak{l}_D(\delta_D) \boxtimes \mathfrak{l}_{D'}(\delta_{D'})$ are the same subobject of $(\mathrm{Lan}_F H)(D\times D'\times D\times D')$. First of all, we have that:

$$\mathfrak{l}_{D\times D'\times D\times D'}(\delta_{D\times D'}) = \bigwedge_{(K,\mathfrak{k}),\theta} \widehat{\theta}^*_{D\times D'\times D\times D'}(\Delta_{KD\times KD'});$$

on the other hand we have:

$$\begin{aligned}
\mathfrak{l}_D(\delta_D) \boxtimes \mathfrak{l}_{D'}(\delta_{D'}) &= \langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^* \left(\bigwedge_{(K,\mathfrak{k}),\theta} \widehat{\theta}^*_{D\times D}(\Delta_{KD}) \right) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^* \left(\bigwedge_{(K,\mathfrak{k}),\theta} \widehat{\theta}^*_{D'\times D'}(\Delta_{KD'}) \right) \\
&= \bigwedge_{(K,\mathfrak{k}),\theta} \Big(\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^* \widehat{\theta}^*_{D\times D}(\Delta_{KD}) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^* \widehat{\theta}^*_{D'\times D'}(\Delta_{KD'}) \Big);
\end{aligned}$$

we prove with some diagram computation that for each $(K, \mathfrak{k}), \theta$ the arguments in the meets are the same.

$$\begin{tikzcd}
\operatorname{dom}\widehat{\theta}^*_{D\times D'\times D\times D'}(\Delta_{KD\times KD'}) \arrow[r, "\widehat{\theta}_{D\times D'\times D\times D'}{}_{|\operatorname{dom}\widehat{\theta}^*_{D\times D'\times D\times D'}(\Delta_{KD\times KD'})}"] \arrow[d, tail, "\widehat{\theta}^*_{D\times D'\times D\times D'}(\Delta_{KD\times KD'})"'] \arrow[dr, phantom, "\lrcorner", very near start] & KD\times KD' \arrow[d, tail, "\Delta_{KD\times KD'}"] \\
(\operatorname{Lan}_F H)(D\times D'\times D\times D') \arrow[r, "\widehat{\theta}_{D\times D'\times D\times D'}"'] & KD\times KD'\times KD\times KD'
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{dom}\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*\widehat{\theta}^*_{D\times D}(\Delta_{KD}) \arrow[rr, "(\widehat{\theta}_{D\times D}\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle)_{|\operatorname{dom}\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle^*\widehat{\theta}^*_{D\times D}(\Delta_{KD})}"] \arrow[d, tail, "\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle^*\widehat{\theta}^*_{D\times D}(\Delta_{KD})"'] \arrow[dr, phantom, "\lrcorner", very near start] & & KD \arrow[d, tail, "\Delta_{KD}"] \\
(\operatorname{Lan}_F H)(D\times D'\times D\times D') \arrow[r, "\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle"] \arrow[dr, "\widehat{\theta}_{D\times D'\times D\times D'}"'] & (\operatorname{Lan}_F H)(D\times D) \arrow[r, "\widehat{\theta}_{D\times D}"] & KD\times KD \\
& KD\times KD'\times KD\times KD' \arrow[ur, "\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{dom}\langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'}) \arrow[rr, "(\widehat{\theta}_{D'\times D'}\langle \mathrm{pr}_2,\mathrm{pr}_4\rangle)_{|\operatorname{dom}\langle \mathrm{pr}_2,\mathrm{pr}_4\rangle^*\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'})}"] \arrow[d, tail, "\langle \mathrm{pr}_2,\mathrm{pr}_4\rangle^*\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'})"'] \arrow[dr, phantom, "\lrcorner", very near start] & & KD' \arrow[d, tail, "\Delta_{KD'}"] \\
(\operatorname{Lan}_F H)(D\times D'\times D\times D') \arrow[r, "\langle \mathrm{pr}_2,\mathrm{pr}_4\rangle"] \arrow[dr, "\widehat{\theta}_{D\times D'\times D\times D'}"'] & (\operatorname{Lan}_F H)(D'\times D') \arrow[r, "\widehat{\theta}_{D'\times D'}"] & KD'\times KD' \\
& KD\times KD'\times KD\times KD' \arrow[ur, "\langle \mathrm{pr}_2,\mathrm{pr}_4\rangle"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{dom}(\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*\widehat{\theta}^*_{D\times D}(\Delta_{KD}) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'})) \arrow[r, tail, bend left, "\omega_1"] \arrow[d, tail, "\omega_2"'] \arrow[dr, tail, "\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle^*\widehat{\theta}^*_{D\times D}(\Delta_{KD})\wedge\langle \mathrm{pr}_2,\mathrm{pr}_4\rangle^*\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'})" description] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{dom}\langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'}) \arrow[d, tail, "\langle \mathrm{pr}_2,\mathrm{pr}_4\rangle^*\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'})"] \\
\operatorname{dom}\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*\widehat{\theta}^*_{D\times D}(\Delta_{KD}) \arrow[r, tail, "\langle \mathrm{pr}_1,\mathrm{pr}_3\rangle^*\widehat{\theta}^*_{D\times D}(\Delta_{KD})"'] & (\operatorname{Lan}_F H)(D\times D'\times D\times D')
\end{tikzcd}$$

Now, for each $(K, \mathfrak{k}), \theta$:

$$\widehat{\theta}^*_{D\times D'\times D\times D'}(\Delta_{KD\times KD'}) \leq \langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*\widehat{\theta}^*_{D\times D}(\Delta_{KD}) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'})$$

if and only if both

$$\widehat{\theta}^*_{D\times D'\times D\times D'}(\Delta_{KD\times KD'}) \leq \langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*\widehat{\theta}^*_{D\times D}(\Delta_{KD}) \text{ and}$$

$$\widehat{\theta}^*_{D\times D'\times D\times D'}(\Delta_{KD\times KD'}) \leq \langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'});$$

to show the first inequality, take the pair $\widehat{\theta}^*_{D\times D'\times D\times D'}(\Delta_{KD\times KD'})$ and the first projection of

$$\widehat{\theta}_{D\times D'\times D\times D'}{}_{|\operatorname{dom}\widehat{\theta}^*_{D\times D'\times D\times D'}(\Delta_{KD\times KD'})},$$

then use the universal property of the second pullback above. Similarly, prove the second one by taking the pair $\widehat{\theta}^*_{D\times D'\times D\times D'}(\Delta_{KD\times KD'})$ and the second projection of

$$\widehat{\theta}_{D\times D'\times D\times D'_{|\mathrm{dom}\,\widehat{\theta}^*_{D\times D'\times D\times D'}(\Delta_{KD\times KD'})}},$$

then use the universal property of the third pullback above. For the converse, take the pair $\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*\widehat{\theta}^*_{D\times D}(\Delta_{KD}) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'})$ and the arrow that has as a first component

$$(\widehat{\theta}_{D\times D}\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle)_{|\mathrm{dom}\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*\widehat{\theta}^*_{D\times D}(\Delta_{KD})}\omega_2$$

and as second component

$$(\widehat{\theta}_{D'\times D'}\langle \mathrm{pr}_2, \mathrm{pr}_4\rangle)_{|\mathrm{dom}\langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*\widehat{\theta}^*_{D'\times D'}(\Delta_{KD'})}\omega_1,$$

then use the universal property of the first pullback.

$(\mathscr{L}, \mathfrak{l})$ **is well defined:** We are now in the situation:

$$\begin{tikzcd}
\mathbb{D}^{\mathrm{op}} \arrow[rr, "\mathscr{L}^{\mathrm{op}}"] \arrow[dr, "R"', ""{name=A}] & & \mathcal{R}^{\mathrm{op}}_{\mathrm{Sub}} \arrow[dl, "(\mathrm{Sub})_q", ""'{name=B}] \\
& \mathbf{Pos} &
\arrow[from=A, to=B, bend left, "\mathfrak{l}", "\cdot"']
\end{tikzcd}$$

We prove that $\mathfrak{l}\colon R \stackrel{\cdot}{\rightarrow} (\mathrm{Sub})_q\mathscr{L}^{\mathrm{op}}$ is well defined by showing that for each $\gamma \in RD$, we have $\mathfrak{l}_D(\gamma) \in \mathscr{D}es_{\mathfrak{l}_{D\times D}(\delta_D)}$, i.e. $\mathrm{pr}_1^*\mathfrak{l}_D(\gamma) \wedge \mathfrak{l}_{D\times D}(\delta_D) \leq \mathrm{pr}_2^*\mathfrak{l}_D(\gamma)$ in $\mathrm{Sub}(\mathrm{Lan}_F\, H)(D\times D)$. Indeed

$$\begin{aligned}
\mathrm{pr}_1^*\left(\bigwedge_{(K,\mathfrak{k}),\theta} \widehat{\theta}^*_D(\mathfrak{k}_D(\gamma))\right) \wedge \bigwedge_{(K,\mathfrak{k}),\theta} (\widehat{\theta}_D \times \widehat{\theta}_D)^*(\Delta_{KD}) \\
= \bigwedge_{(K,\mathfrak{k}),\theta} \left(\mathrm{pr}_1^*\widehat{\theta}^*_D(\mathfrak{k}_D(\gamma)) \wedge (\widehat{\theta}_D \times \widehat{\theta}_D)^*(\Delta_{KD})\right) \\
= \bigwedge_{(K,\mathfrak{k}),\theta} (\widehat{\theta}_D \times \widehat{\theta}_D)^*\left(\mathrm{pr}_1^*(\mathfrak{k}_D(\gamma)) \wedge \Delta_{KD}\right) \\
\leq \bigwedge_{(K,\mathfrak{k}),\theta} (\widehat{\theta}_D \times \widehat{\theta}_D)^*\left(\mathrm{pr}_2^*(\mathfrak{k}_D(\gamma))\right) = \mathrm{pr}_2^*\mathfrak{l}_D(\gamma).
\end{aligned}$$

So we proved that we have a 1-arrow between the doctrines $(\mathscr{L}, \mathfrak{l})\colon R \to (\mathrm{Sub})_q$.

$(\mathscr{L}, \mathfrak{l})$ **is in ED:** By construction $\mathfrak{l}$ is a natural transformation and preserves finite meets. Note that in general in the doctrine $(P)_q\colon \mathcal{R}_P^{\mathrm{op}} \to \mathbf{Pos}$ of descent objects from the quotient completion of a doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$, the equality in $(P)_q\big((A,\rho)\times(A,\rho)\big) = \mathscr{D}es_{\rho\boxtimes\rho}$ is ρ itself. Indeed:

- $\rho \in \mathscr{D}es_{\rho\boxtimes\rho}$, using transitivity and symmetry of ρ;
- $\top_{(A,\rho)} \leq (P)_q(\Delta_{(A,\rho)})(\rho)$, since $\top_A \leq P(\Delta_A)(\delta_A) \leq P(\Delta_A)(\rho)$ using reflexivity of ρ;
- $(P)_q(A,\rho) = \mathscr{D}es_\rho$, since descent objects with respect to P and $(P)_q$ are the same;

- $(P)_q(\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle)(\rho) \wedge (P)_q(\langle \mathrm{pr}_2, \mathrm{pr}_4\rangle)(\sigma) = \rho \boxtimes \sigma$ by definition of product $(X, \rho) \times (Y, \sigma)$ in $\mathcal{R}_P$.

So applying this to the case $(\mathrm{Sub})_q$, we obtain that

$$\mathfrak{l}_{D\times D}\colon R(D \times D) \to (\mathrm{Sub})_q\big(((\operatorname*{Lan}_F H)(D), \mathfrak{l}_{D\times D}(\delta_D)) \times ((\operatorname*{Lan}_F H)(D), \mathfrak{l}_{D\times D}(\delta_D))\big)$$

or, computing the codomain,

$$\mathfrak{l}_{D\times D}\colon R(D \times D) \to \mathscr{D}es_{\mathfrak{l}_{D\times D}(\delta_D)\boxtimes\mathfrak{l}_{D\times D}(\delta_D)}$$

so $\mathfrak{l}$ preserves the fibered equality and $(\mathscr{L}, \mathfrak{l})$ is in **ED**.

From $(\mathrm{Sub})_q$ **to** Sub**:** Recall that our goal is to define for each $(H, \mathfrak{h})\colon P \to \mathrm{Sub}$ in **ED** a suitable 1-arrow from R to Sub. So we look for an arrow $(Q, \mathfrak{q})\colon (\mathrm{Sub})_q \to \mathrm{Sub}$ in order to define the wanted map by the composition $(Q, \mathfrak{q})(\mathscr{L}, \mathfrak{l})$.

$$\begin{tikzcd}
\mathcal{R}^{\mathrm{op}}_{\mathrm{Sub}} \arrow[rr, "Q^{\mathrm{op}}"] \arrow[dr, "(\mathrm{Sub})_q"'] \arrow[rr, phantom, bend right=20, "\xrightarrow{\mathfrak{q}}"] & & \mathbb{E}^{\mathrm{op}} \arrow[dl, "\mathrm{Sub}"] \\
& \mathbf{Pos} &
\end{tikzcd}$$

To to this, we want to use the universal property of $(\mathrm{Sub})_q$: from Theorem 4.5 of [MR12] there is an essential equivalence of categories

$$- \circ (J, j)\colon \mathbf{QED}((\mathrm{Sub})_q, Z) \to \mathbf{ED}(\mathrm{Sub}, Z)$$

for every Z in **QED**, where **QED** is the 2-full 2-subcategory of **ED** whose objects are elementary doctrines $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ in which every P-equivalence relation has a P-quotient that is a descent morphism; the 1-morphism are those arrows $(G, \mathfrak{g})\colon P \to Z$ in **ED** such that G preserves quotients—meaning, if $q\colon A \to C$ is a quotient of a P-equivalence relation ρ on A, then Gq is a quotient of the Z-equivalence relation $\mathfrak{g}_{A\times A}(\rho)$ on GA. So we prove that Sub is in **QED**, and define $(Q, \mathfrak{q})$ as the essentially unique 1-morphism such that $(Q, \mathfrak{q})(J, j) = \mathrm{id}_{\mathrm{Sub}}$. We show that every equivalence relation $\langle s_1, s_2\rangle\colon S \rightarrowtail X \times X$ in $\mathbb{E}$ has a quotient that is a descent morphism. Since Grothendieck topos are cocomplete, the quotient exists, and it is the coequalizer $q\colon X \to X/S$ of s_1 and s_2. We then need q to be such that $q^*\colon \mathrm{Sub}(X/S) \to \mathrm{Sub}(X)$ is full. Take $y\colon Y \rightarrowtail X/S$ and $y'\colon Y' \rightarrowtail X/S$ such that $q^*(y) \leq q^*(y')$.

$$\begin{tikzcd}
\operatorname{dom} q^*y \arrow[r, tail, "\ell"] \arrow[dr, tail, "q^*y"'] \arrow[rr, bend left, "q_y"] & \operatorname{dom} q^*y' \arrow[d, tail, "q^*y'"] \arrow[rr, bend left, "q_{y'}"] & Y \arrow[r, tail, dashed] \arrow[dr, tail, "y"'] & Y' \arrow[d, tail, "y'"'] \\
& X \arrow[rr, "q"'] & & X/S
\end{tikzcd}$$

We claim that $y \leq y'$. Since $\mathbb{E}$ is regular, we know that q_y is a regular epimorphism—as a pullback of the regular epimorphism q. In particular, q_y is the coequaliser of its kernel pair, call it $[T, t_1, t_2]$. Similarly $q_{y'}$ is the coequaliser of its kernel pair $[T', t_1', t_2']$.

$$\begin{tikzcd}
T \arrow[r, shift left, "t_1"] \arrow[r, shift right, "t_2"'] \arrow[d, dashed, "t"] & \operatorname{dom} q^*y \arrow[r, "q_y"] \arrow[d, tail, "\ell"'] \arrow[dd, bend left, tail, "q^*y"] & Y \arrow[d, dashed, "\ell'"] \\
T' \arrow[r, shift left, "t_1'"] \arrow[r, shift right, "t_2'"'] & \operatorname{dom} q^*y' \arrow[r, "q_{y'}"'] \arrow[d, tail, "q^*y'"'] & Y' \\
 & C &
\end{tikzcd}$$

By definition of kernel pair of $q_{y'}$, we can define $t\colon T \to T'$ making the two squares on the left commute if and only if $q_{y'}\ell t_1 = q_{y'}\ell t_2$, if and only if $y' q_{y'}\ell t_1 = y' q_{y'}\ell t_2$—since y' is a monomorphism—, i.e. $qq^*(y')\ell t_1 = qq^*(y')\ell t_2$, i.e. $qq^*(y)t_1 = qq^*(y)t_2$, i.e. $yq_y t_1 = yq_y t_2$, if and only if $q_y t_1 = q_y t_2$, but this is true since q_y is the coequaliser of t_1, t_2. So now we define a map $\ell'\colon Y \to Y'$ making the square on the right commute: it is equivalent to ask that $q_{y'}\ell t_1 = q_{y'}\ell t_2$, if and only if $q_{y'}t_1' t = q_{y'}t_2' t$, which is true. At last, we check that $y'\ell' = y$, if and only if $y'\ell' q_y = yq_y$—since q_y is an epimorphism—, if and only if $y' q_{y'}\ell = qq^*(y)$, if and only if $qq^*(y')\ell = qq^*(y)$. This concludes the proof.

Claim 4.1.5. The assignment defined above, sending $(H, \mathfrak{h})$ to $(Q, \mathfrak{q})(\mathscr{L}, \mathfrak{l})$ extends to a left adjoint to $- \circ (F, \mathfrak{f})$

We look for the universal arrow

$$\eta_{(H,\mathfrak{h})}\colon (H, \mathfrak{h}) \to (Q, \mathfrak{q})(\mathscr{L}, \mathfrak{l})(F, \mathfrak{f}).$$

In particular, we need a natural transformation $\eta_{(H,\mathfrak{h})}\colon H \dot{\to} Q\mathscr{L}F$, i.e. for any object $A \in \mathbb{C}$

$$\big(\eta_{(H,\mathfrak{h})}\big)_A\colon HA \xrightarrow{\mu_A} (\underset{F}{\mathrm{Lan}}\, H)(FA) \xrightarrow{\rho_{FA}} Q\big((\underset{F}{\mathrm{Lan}}\, H)(FA), \mathfrak{l}_{FA \times FA}(\delta_{FA})\big)$$

To define $\rho\colon \mathrm{Lan}_F H \dot{\to} Q\mathscr{L}$, note that for any $(X, s = \langle s_1, s_2\rangle\colon S \rightarrowtail X \times X) \in \mathcal{R}_{\mathrm{Sub}}$, there is an arrow $\mathrm{id}_X\colon (X, \Delta_X) \to (X, s)$ in $\mathcal{R}_{\mathrm{Sub}}$—since s is an equivalence relation. Apply Q to obtain $Q(\mathrm{id}_X)\colon X \to Q(X, s)$ in $\mathbb{E}$. So we can define $\rho_D = Q(\mathrm{id}_{(\mathrm{Lan}_F H)(D)})$; this is clearly a natural transformation, since the naturality square

$$\begin{tikzcd}
D' \arrow[d, "g"'] & (\mathrm{Lan}_F H)(D') \arrow[r, "Q(\mathrm{id}_{(\mathrm{Lan}_F H)(D')})"'] \arrow[d, "(\mathrm{Lan}_F H)(g)"'] & Q((\mathrm{Lan}_F H)(D'), \mathfrak{l}_{D' \times D'}(\delta_{D'})) \arrow[d, "Q(\mathrm{Lan}_F H)(g)"'] \\
D & (\mathrm{Lan}_F H)(D) \arrow[r, "Q(\mathrm{id}_{(\mathrm{Lan}_F H)(D)})"] & Q((\mathrm{Lan}_F H)(D), \mathfrak{l}_{D \times D}(\delta_D))
\end{tikzcd}$$

is the image through Q of

$$\begin{tikzcd}
((\operatorname{Lan}_F H)(D'), \Delta_{(\operatorname{Lan}_F H)(D')}) \arrow[r, "\mathrm{id}_{(\operatorname{Lan}_F H)(D')}"'] \arrow[d, "(\operatorname{Lan}_F H)(g)"'] & ((\operatorname{Lan}_F H)(D'), \mathfrak{l}_{D' \times D'}(\delta_{D'})) \arrow[d, "(\operatorname{Lan}_F H)(g)"'] \\
((\operatorname{Lan}_F H)(D), \Delta_{(\operatorname{Lan}_F H)(D)}) \arrow[r, "\mathrm{id}_{(\operatorname{Lan}_F H)(D)}"] & ((\operatorname{Lan}_F H)(D), \mathfrak{l}_{D \times D}(\delta_D))
\end{tikzcd}$$

So $\eta_{(H,\mathfrak{h})}$ is defined, and it is a natural transformation, since it is the composition of natural transformations.

$\eta_{(H,\mathfrak{h})}$ **is a 2-arrow:** We need to show that

$$\mathfrak{h}_A(\alpha) \leq \left(\eta_{(H,\mathfrak{h})}\right)^*_A(\mathfrak{q}_{\mathscr{L}FA}\mathfrak{l}_{FA}\mathfrak{f}_A(\alpha)) = \mu^*_A \rho^*_{FA}(\mathfrak{q}_{\mathscr{L}FA}\mathfrak{l}_{FA}\mathfrak{f}_A(\alpha)).$$

Observe that from naturality of $\mathfrak{q}$, we have for any $(X, s) \in \mathcal{R}_{\mathrm{Sub}}$ the following commutative diagram:

$$\begin{tikzcd}
(X, \Delta_X) \arrow[d, "\mathrm{id}_X"] & (\mathrm{Sub})_q(X, \Delta_X) \arrow[r, "\mathfrak{q}_{(X,\Delta_X)}"] & \operatorname{Sub} Q(X, \Delta_X) \\
(X, s) & (\mathrm{Sub})_q(X, s) \arrow[u, "\mathrm{id}_X^*"] \arrow[r, "\mathfrak{q}_{(X,s)}"'] & \operatorname{Sub} Q(X, s) \arrow[u, "Q(\mathrm{id}_X)^*"']
\end{tikzcd}$$

Note that id_X^* is just the inclusion $\mathscr{D}es_s \subseteq \operatorname{Sub} X$, and $\mathfrak{q}_{(X,\Delta_X)} = \mathfrak{q}_{JX}$ is such that $\mathfrak{q}_{JX} j_X = \mathrm{id}_{\operatorname{Sub} X}$, hence $\mathfrak{q}_{(X,\Delta_X)} = \mathrm{id}_{\operatorname{Sub} X}$, so to conclude we have that $Q(\mathrm{id}_X)^* \mathfrak{q}_{(X,s)}$ is the inclusion $\mathscr{D}es_s \subseteq \operatorname{Sub} X$. Apply this when

$$(X, s) = ((\operatorname*{Lan}_F H)(FA), \mathfrak{l}_{FA \times FA}(\delta_{FA})) = \mathscr{L}(FA)$$

to get that $\rho^*_{FA}\mathfrak{q}_{\mathscr{L}FA}$ acts as the identity, so our claim becomes

$$\mathfrak{h}_A(\alpha) \leq \mu^*_A(\mathfrak{l}_{FA}\mathfrak{f}_A(\alpha)).$$

But now

$$\mu^*_A(\mathfrak{l}_{FA}\mathfrak{f}_A(\alpha)) = \mu^*_A\Big(\bigwedge_{(K,\mathfrak{k}),\theta} \widehat{\theta}^*_{FA}(\mathfrak{k}_{FA}\mathfrak{f}_A(\alpha))\Big) = \bigwedge_{(K,\mathfrak{k}),\theta} \mu^*_A \widehat{\theta}^*_{FA}(\mathfrak{k}_{FA}\mathfrak{f}_A(\alpha)) = \bigwedge_{(K,\mathfrak{k}),\theta} \theta^*_A(\mathfrak{k}_{FA}\mathfrak{f}_A(\alpha)),$$

but every θ is a 2-arrow, so $\mathfrak{h}_A(\alpha) \leq \theta^*_A(\mathfrak{k}_{FA}\mathfrak{f}_A(\alpha))$, hence $\eta_{(H,\mathfrak{h})}$ is indeed a 2-arrow.

$\eta_{(H,\mathfrak{h})}$ **has the universal property:** We now prove that for every arrow $(K, \mathfrak{k})\colon R \to \operatorname{Sub}$ in **ED** and every 2-arrow $\theta\colon (H, \mathfrak{h}) \to (K, \mathfrak{k})(F, \mathfrak{f})$ there exists a unique 2-arrow $\overline{\theta}\colon (Q, \mathfrak{q})(\mathscr{L}, \mathfrak{l}) \to (K, \mathfrak{k})$, making the following diagram commute:

$$\begin{tikzcd}
(H,\mathfrak{h}) \arrow[rr, "\eta_{(H,\mathfrak{h})}"] \arrow[dr, "\theta"'] & & (Q,\mathfrak{q})(\mathscr{L},\mathfrak{l})(F,\mathfrak{f}) \arrow[dl, "\overline{\theta}\circ(F,\mathfrak{f})"] \\
& (K,\mathfrak{k})(F,\mathfrak{f}) &
\end{tikzcd}$$

For any object $D \in \mathbb{D}$, observe that the image of $\widehat{\theta}_D\colon \operatorname{Lan}_F HD \to KD$ in $\mathcal{R}_{\mathrm{Sub}}$ through J factors as follows:

$$\begin{tikzcd}
(\operatorname{Lan}_F HD, \Delta_{\operatorname{Lan}_F HD}) \arrow[r, "\mathrm{id}_{\operatorname{Lan}_F HD}"] \arrow[rr, bend right, "\widehat{\theta}_D"'] & (\operatorname{Lan}_F HD, \mathfrak{l}_{D\times D(\delta_D)}) \arrow[r, "\widehat{\theta}_D"] & (KD, \Delta_{KD})
\end{tikzcd}$$

The first map trivially exists; the second one exists if and only if there exists a dotted arrow in the diagram below:

$$\begin{tikzcd}
\operatorname{dom}\mathfrak{l}_{D\times D}(\delta_D) \arrow[drr, bend left, "\mathfrak{l}_{D\times D}(\delta_D)"] \arrow[ddr, dashed] & & \\
& \operatorname{dom}\widehat{\theta}^*_{D\times D}(\Delta_{KD}) \arrow[r, "\widehat{\theta}^*_{D\times D}(\Delta_{KD})"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Lan}_F HD \times \operatorname{Lan}_F HD \arrow[d, "\widehat{\theta}_D\times\widehat{\theta}_D"] \\
& KD \arrow[r, "\Delta_{KD}"'] & KD \times KD
\end{tikzcd}$$

but it exists by definition of $\mathfrak{l}_{D\times D}(\delta_D)$ as the infimum of all subobject of the form $\widehat{\theta}^*_{D\times D}(\Delta_{KD})$. Apply Q to the factorization of $\widehat{\theta}_D$ above to obtain in $\mathbb{E}$:

$$\begin{tikzcd}
\operatorname{Lan}_F HD \arrow[r, "\rho_D"] \arrow[rr, bend right, "\widehat{\theta}_D"'] & Q(\operatorname{Lan}_F HD, \mathfrak{l}_{D\times D(\delta_D)}) \arrow[r, "Q(\widehat{\theta}_D)"] & KD
\end{tikzcd}$$

Define $\overline{\theta}_D = Q(\widehat{\theta}_D)\colon Q\mathscr{L}D \to KD$. For any given $g\colon D' \to D$ in $\mathbb{E}$, we have that the image through Q of the square on the right in the diagram below gives naturality of $\overline{\theta}$:

$$\begin{tikzcd}
(\operatorname{Lan}_F HD', \Delta_{\operatorname{Lan}_F HD'}) \arrow[r, "\mathrm{id}_{\operatorname{Lan}_F HD'}"'] \arrow[d, "\operatorname{Lan}_F H(g)"'] \arrow[rr, bend left, "\widehat{\theta}_{D'}"] & (\operatorname{Lan}_F HD', \mathfrak{l}_{D'\times D'(\delta_{D'})}) \arrow[r, "\widehat{\theta}_{D'}"'] \arrow[d, "\operatorname{Lan}_F H(g)"] & (KD', \Delta_{KD'}) \arrow[d, "K(g)"] \\
(\operatorname{Lan}_F HD, \Delta_{\operatorname{Lan}_F HD}) \arrow[r, "\mathrm{id}_{\operatorname{Lan}_F HD}"] \arrow[rr, bend right, "\widehat{\theta}_D"'] & (\operatorname{Lan}_F HD, \mathfrak{l}_{D\times D(\delta_D)}) \arrow[r, "\widehat{\theta}_D"] & (KD, \Delta_{KD})
\end{tikzcd}$$

Now, to prove that $\overline{\theta}$ is a 2-arrow we show that for any object $D \in \mathbb{D}$ and any $\gamma \in RD$

$$\mathfrak{q}_{\mathscr{L}D}\mathfrak{l}_D(\gamma) \le \overline{\theta}^*_D(\mathfrak{k}_D(\gamma)).$$

Since $\mathfrak{l}_D(\gamma) \le \widehat{\theta}^*_D(\mathfrak{k}_D(\gamma))$, it is enough to prove that $\mathfrak{q}_{\mathscr{L}D}\widehat{\theta}^*_D(\mathfrak{k}_D(\gamma)) \le Q(\widehat{\theta}_D)^*(\mathfrak{k}_D(\gamma))$. If ρ^*_D is full, the last inequality is equivalent to $\rho^*_D\mathfrak{q}_{\mathscr{L}D}\widehat{\theta}^*_D(\mathfrak{k}_D(\gamma)) \le \rho^*_D Q(\widehat{\theta}_D)^*(\mathfrak{k}_D(\gamma))$, i.e. $\widehat{\theta}^*_D(\mathfrak{k}_D(\gamma)) \le Q(\mathrm{id}_{\mathrm{Lan}_F HD})^* Q(\widehat{\theta}_D)^*(\mathfrak{k}_D(\gamma)) = \widehat{\theta}^*_D\mathfrak{k}_D(\gamma)$. So we check the following:

Claim 4.1.6. The arrow ρ_D is a regular epimorphism.

Proof. For any $(X, s = \langle s_1, s_2\rangle \colon S \rightarrowtail X \times X) \in \mathcal{R}_{\mathrm{Sub}}$, consider $\mathrm{id}_X \colon (X, \Delta_X) \to (X, s)$ in $\mathcal{R}_{\mathrm{Sub}}$ and $Q(\mathrm{id}_X) \colon X \to Q(X, s)$ in $\mathbb{E}$. We prove that $Q(\mathrm{id}_X)$ is a regular epimorphism.
Note that given any $(X, r) \in \mathcal{R}_{\mathrm{Sub}}$, an $\mathcal{R}_{\mathrm{Sub}}$-equivalence relation on (X, r) is an element s in $(\mathrm{Sub})_q((X, r) \times (X, r)) = \mathscr{D}es_{r \boxtimes r} \subseteq \mathrm{Sub}(X \times X)$ that is an equivalence relation on X such that $r \le s$. It follows that $\mathrm{id}_X \colon (X, r) \to (X, s)$ is an arrow in $\mathcal{R}_{\mathrm{Sub}}$. Moreover, it is an $\mathcal{R}_{\mathrm{Sub}}$-quotient of s: it is an arrow such that $s \le (\mathrm{id} \times \mathrm{id})^*(s) = s$, and for every morphism $g \colon (X, r) \to (Y, u)$—i.e. $g \colon X \to Y$ such that $r \le (g \times g)^*(u)$—such that $s \le (g \times g)^*(u)$, we find a unique morphism $h \colon (X, r) \to (Y, u)$ such that $g = h\,\mathrm{id}$, indeed $h = g$, and it is an arrow in $\mathcal{R}_{\mathrm{Sub}}$ since $s \le (g \times g)^*(u)$. If we take $r = \Delta_X$, we have that $\mathrm{id}_X \colon (X, \Delta_X) \to (X, s)$ is an $\mathcal{R}_{\mathrm{Sub}}$-quotient of s, hence $Q(\mathrm{id}_X) \colon X \to Q(X, s)$ is a Sub-quotient of a Sub-equivalence relation, hence it is a regular epimorphism. □

Now we check commutativity of the triangle for the universal property:

$$\overline{\theta}_{FA}\big(\eta_{(H,\mathfrak{h})}\big)_A = \overline{\theta}_{FA}\rho_{FA}\mu_A = \widehat{\theta}_{FA}\mu_A = \theta_A.$$

To conclude, we show that $\overline{\theta}$ is unique. Suppose we have a 2-arrow $\lambda \colon (Q, \mathfrak{q})(\mathscr{L}, \mathfrak{l}) \to (K, \mathfrak{k})$ making the triangle commute. In particular, for any object $A \in \mathbb{C}$ we have $\lambda_{FA}\rho_{FA}\mu_A = \theta_A$ and for any $D \in \mathbb{D}, \gamma \in RD$ we have $\mathfrak{q}_{\mathscr{L}D}\mathfrak{l}_D(\gamma) \le \lambda^*_D(\mathfrak{k}_D(\gamma))$. Consider the natural transformation $\lambda \circ \rho \colon \mathrm{Lan}_F H \xrightarrow{\cdot} K$. It is such that $(\lambda \circ \rho) \circ \mu = \theta$, so by the universal property of μ we have $\lambda \circ \rho = \widehat{\theta}$, but then we have $\widehat{\theta}_D = \lambda_D\rho_D = \overline{\theta}_D\rho_D$, so that $\lambda_D = \overline{\theta}_D$.
This concludes the proof of Claim 4.1.5, hence of Theorem 4.1.1.

4.2 Examples

Before we dive into some examples, we prove a general result for some first-order theories. This generalizes Example 2.5.a of [MR12].
We refer to [Car17] for the definitions about first-order calculus. Note that here, in contrast to what we defined in Example 1.1.2 about the doctrine of well-formed formulae, which had Boolean elementary existential structure, we just consider the fragment of Horn logic. Moreover, a theory in this context is not a set of closed formulae, but is instead a set of Horn sequents over Σ. We

write in this case $\mathtt{HF}^{\Sigma}_{\mathbb{T}} \colon \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\Sigma} \to \mathbf{Pos}$ for the elementary doctrine of Horn formulae: the base category is the same defined in Example 1.1.2, each list of variable is sent to the poset reflection of Horn formulae—defined inductively as the smallest set containing relations, equalities, true constant and conjunctions of formulae—ordered by provable consequence in $\mathbb{T}$; reindexing are again defined as substitutions.

Proposition 4.2.1. Let Σ be a first-order language and $\mathbb{T}$ a first-order theory in the language Σ such that its axioms are Horn sequent. Then there exists an equivalence of categories:

$$\mathbf{ED}(\mathtt{HF}^{\Sigma}_{\mathbb{T}}, \mathrm{Sub}) \cong \underset{\mathbb{T}}{\overset{\Sigma}{\mathrm{Mod}}}$$

where $\mathtt{HF}^{\Sigma}_{\mathbb{T}} \colon \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\Sigma} \to \mathbf{Pos}$ is the elementary doctrine of Horn formulae in the language Σ of the theory $\mathbb{T}$, $\mathrm{Sub} \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is the elementary doctrine of subobject for a given category $\mathbb{C}$ with finite limits, and $\mathrm{Mod}^{\Sigma}_{\mathbb{T}}$ is the category whose objects are models of the theory $\mathbb{T}$ in the category $\mathbb{C}$, and whose arrows are Σ-homomorphism.

Proof. Since there is no confusion, we write $\mathtt{HF}$ instead of $\mathtt{HF}^{\Sigma}_{\mathbb{T}}$ and Mod instead of $\mathrm{Mod}^{\Sigma}_{\mathbb{T}}$. For any given $\theta \colon (H, \mathfrak{h}) \to (H', \mathfrak{h}')$ in $\mathbf{ED}(\mathtt{HF}, \mathrm{Sub})$, define $\theta_{(x)} \colon H(x) \to H'(x)$. Observe that $H(x)$ is indeed a model of the theory $\mathbb{T}$: each n-ary function symbol f in the language defines an arrow $f(x_1, \dots, x_n) \colon (x_1, \dots, x_n) \to (x)$ in $\mathbb{C}\mathrm{tx}$, hence its image through H—that preserves products—defines a map $f^H \colon \big(H(x)\big)^n \to H(x)$, which is the interpretation of f in $H(x)$; each n-ary predicate symbol R defines $R^H = \mathfrak{h}_{(x_1,\dots,x_n)}(R(x_1, \dots, x_n)) \colon \mathrm{dom}(R^H) \rightarrowtail \big(H(x)\big)^n$. From now on, we write $\vec{x}$ instead of the list $(x_1, \dots, x_n)$. Satisfiability of axioms follows by the fact that $\mathfrak{h}_{\vec{x}}(\alpha(\vec{x}))$ is the interpretation of α in $H(x)$ for each $\alpha(\vec{x}) \in \mathtt{HF}(\vec{x})$, and $\mathfrak{h}$ is monotone, so if we have an axiom $\alpha(\vec{x}) \vdash \beta(\vec{x})$ in $\mathbb{T}$ we have $\alpha^H \leq \beta^H$, i.e. $H(x)$ satisfies $\alpha \vdash \beta$. To check that $\mathfrak{h}_{\vec{x}}(\alpha(\vec{x})) = \alpha^H$ we work recursively on the complexity of α:

- $\alpha = \top$: $\alpha^H = \mathrm{id}_{\left(H(x)\right)^n} = \mathfrak{h}_{\vec{x}}(\top)$ trivially holds;
- $\alpha = R(\vec{x})$: $\alpha^H = R^H = \mathfrak{h}_{\vec{x}}(R)$ by the definition given above;
- $\alpha = \alpha_1 \wedge \alpha_2$: $\alpha^H = \alpha_1^H \wedge \alpha_2^H = \mathfrak{h}_{\vec{x}}(\alpha_1) \wedge \mathfrak{h}_{\vec{x}}(\alpha_2) = \mathfrak{h}_{\vec{x}}(\alpha_1 \wedge \alpha_2)$ since $\mathfrak{h}_{\vec{x}}$ preserves meets;
- $\alpha = \big(t_1(\vec{x}) = t_2(\vec{x})\big)$: $\alpha^H = \mathrm{Eq}(t_1^H, t_2^H)$, the equalizer of the interpretations t_1^H, t_2^H of the terms t_1, t_2 in $H(x)$.

It is left to prove then that if $\alpha = \big(t_1(\vec{x}) = t_2(\vec{x})\big)$, we have $\alpha^H = \mathfrak{h}_{\vec{x}}\big(t_1(\vec{x}) = t_2(\vec{x})\big)$. Naturality of $\mathfrak{h}$ with respect to the arrow $(t_1(\vec{x}), t_2(\vec{x})) \colon \vec{x} \to (y_1, y_2)$ in $\mathbb{C}\mathrm{tx}$, applied to the formula $\big(y_1 = y_2\big) \in \mathtt{HF}(y_1, y_2)$ gives:

$$\begin{tikzcd}
\mathtt{HF}(y_1, y_2) \arrow[r, "\mathfrak{h}_{(y_1,y_2)}"] \arrow[d, "\vec{t}(\vec{x})/\vec{y}"'] & \mathrm{Sub}((H(x))^2) \arrow[d, "\langle t_1^H, t_2^H \rangle^*"] \\
\mathtt{HF}(\vec{x}) \arrow[r, "\mathfrak{h}_{\vec{x}}"'] & \mathrm{Sub}((H(x))^n)
\end{tikzcd}$$

in order to get:

$$\mathfrak{h}_{\vec{x}}\big(t_1(\vec{x}) = t_2(\vec{x})\big) = \langle t_1^H, t_2^H\rangle^*(\Delta_{H(x)}) = \mathrm{Eq}(t_1^H, t_2^H),$$

as claimed. This proves that the association $\mathbf{ED}(\mathtt{HF}, \mathrm{Sub}) \to \mathrm{Mod}$ is well defined on objects. Concerning arrows, first of all observe that since H, H' preserve products and θ is a natural transformation, $\theta_{\vec{x}} = \theta_{(x)} \times \cdots \times \theta_{(x)}$—$n$ times. So the naturality diagram of θ with respect to an arrow defined by an n-ary function symbol $f(\vec{x})\colon \vec{x} \to (x)$ gives the fact that $\theta_{(x)}$ preserves the interpretation of the function symbol f:

$$\theta_{(x)} f^H = f^{H'} \theta_{\vec{x}} = f^{H'}(\theta_{(x)} \times \cdots \times \theta_{(x)});$$

moreover, for any n-ary predicate symbol R, since θ is a 2-arrow, we have

$$R^H = \mathfrak{h}_{\vec{x}}(R) \leq \theta_{\vec{x}}^*\big(\mathfrak{h}'_{\vec{x}}(R)\big) = \theta_{\vec{x}}^* R^{H'},$$

so that $\theta_{(x)}$ is indeed a homomorphism in Mod. Now that the functor is well defined, we prove that it is full, faithful and essentially surjective. Faithfulness is trivial since, as seen above, each component of θ is uniquely determined by its component on the context with one variable.
Take now $g\colon H(x) \to H'(x)$ an homomorphism in Mod, define $\theta_{\vec{x}}^g = g \times \cdots \times g$—$|\vec{x}|$ times, where $|\vec{x}|$ is the length of the list. This defines a natural transformation $\theta\colon H \to H'$: naturality with respect to projections follows by definition, moreover for any function symbol the naturality square commutes since g preserves interpretations, and then recursively since any other arrow is composition of projections and terms—defined by composition of function symbols—, naturality holds for any arrow in $\mathbb{C}\mathrm{tx}$.

$$\begin{tikzcd}
(\vec{x}) \arrow[d, "t(\vec{x})"] & H(x)^{|\vec{x}|} \arrow[r, "g\times\cdots\times g"] \arrow[d, "t^H"'] & H'(x)^{|\vec{x}|} \arrow[d, "t^{H'}"] \\
(x) & H(x) \arrow[r, "g"'] & H'(x)
\end{tikzcd}$$

We show that θ^g is an arrow in $\mathbf{ED}(\mathtt{HF}, \mathrm{Sub})$, i.e. that for any $\alpha(\vec{x}) \in \mathtt{HF}(\vec{x})$ we have $\mathfrak{h}_{\vec{x}}\big(\alpha(\vec{x})\big) \leq \theta_{\vec{x}}^{g\,*}\big(\mathfrak{h}'_{\vec{x}}\big(\alpha(\vec{x})\big)\big)$. Recursively on the complexity of α we observe that if $\alpha = \top$ or $\alpha = \beta \wedge \gamma$, the inequality holds since $\mathfrak{h}, \mathfrak{h}'$ and $\theta_{\vec{x}}^{g\,*}$ preserve the top element and meets; if α is an equality of terms $\alpha(\vec{x}) = \big(t_1(\vec{x}) = t_2(\vec{x})\big)$, we show $\mathrm{Eq}(t_1^H, t_2^H) \leq \theta_{\vec{x}}^{g\,*}(\mathrm{Eq}(t_1^{H'}, t_2^{H'}))$, but this holds looking at the diagram below:

$$\begin{tikzcd}
\operatorname{dom}(\alpha^H) \arrow[d, dashed] \arrow[dd, bend right] \arrow[dr, tail, "\alpha^H"] & & \\
\bullet \arrow[r, tail, "\theta^{g\,*}_{\vec{x}}(\alpha^{H'})"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \big(H(x)\big)^n \arrow[r, shift left, "t_1^H"] \arrow[r, shift right, "t_2^H"'] \arrow[d, "\theta^g_{\vec{x}}"'] & H(x) \arrow[d, "g"] \\
\operatorname{dom}(\alpha^{H'}) \arrow[r, tail, "\alpha^{H'}"'] & \big(H'(x)\big)^n \arrow[r, shift left, "t_1^{H'}"] \arrow[r, shift right, "t_2^{H'}"'] & H'(x)
\end{tikzcd}$$

the arrow $\operatorname{dom}(\alpha^H) \to \operatorname{dom}(\alpha^{H'})$ exists and makes the outer left square commute if and only if $t_1^{H'}\theta^g_{\vec{x}}\alpha^H = t_2^{H'}\theta^g_{\vec{x}}\alpha^H$, but this is true since $t_i^{H'}\theta^g_{\vec{x}} = gt_i^H$ for $i = 1, 2$. So the dashed arrow above exists by the universal property of the pullback, hence $\alpha^H \leq \theta^{g\,*}_{\vec{x}}(\alpha^{H'})$, as claimed. Finally, if $\alpha = R$ for some predicate symbol R, we have to check $R^H \leq \theta^{g\,*}_{\vec{x}}(R^{H'})$, but this holds by definition of Σ-homomorphism. So $\theta^g\colon (H, \mathfrak{h}) \to (H', \mathfrak{h}')$ is well defined, and its image is g, so the functor is full. To conclude, take M a model of $\mathbb{T}$, and write $f^M\colon M^n \to M$ for the interpretation in M of any n-ary function symbol f in the language and $R^M\colon \operatorname{dom}(R^M) \rightarrowtail M^n$ for the interpretation of any n-ary predicate symbol R. We define a functor $H^M\colon \mathbb{C}\mathrm{tx} \to \mathbb{C}$ that maps $\vec{x} \mapsto M^{|\vec{x}|}$, projections in projections, $f(\vec{x}) \mapsto f^M$, and this trivially extends to lists of terms, defining a product preserving functor. Now define $\mathfrak{h}^M\colon \mathtt{HF} \overset{\cdot}{\to} \operatorname{Sub} H^{\mathrm{op}}$:

$$\mathfrak{h}^M_{\vec{x}}(\alpha(\vec{x})) = \alpha^M,$$

the interpretation of α in M. It is well defined because M is a model. By definition of interpretation, $\mathfrak{h}^M$ preserves top element, meet and fibered equality. To prove that it is a natural transformation, take a list of terms $\vec{t}(\vec{x}) = (t_1(\vec{x}), \dots, t_{|\vec{y}|}(\vec{x}))\colon \vec{x} \to \vec{y}$ and we prove that the following diagram is commutative:

$$\begin{tikzcd}
\mathtt{HF}(\vec{y}) \arrow[r, "\mathfrak{h}^M_{\vec{y}}"] \arrow[d, "\vec{t}(\vec{x})/\vec{y}"'] & \operatorname{Sub}(M^{|\vec{y}|}) \arrow[d, "\langle t_1^M, \dots, t_{|\vec{y}|}^M \rangle^*"] \\
\mathtt{HF}(\vec{x}) \arrow[r, "\mathfrak{h}^M_{\vec{x}}"'] & \operatorname{Sub}(M^{|\vec{x}|})
\end{tikzcd}$$

but this is true by definition of interpretation. Clearly $(H^M, \mathfrak{h}^M) \mapsto M$ so the functor is essentially surjective and defines the equivalence of categories. ☐

Example 4.2.2. Some algebraic examples. We prove, using the equivalence of Proposition 4.2.1 in some specific theories, that many adjunction results in algebra can be obtained as a particular case of the adjunction shown in Theorem 4.1.1. Suppose we have an algebraic language Σ, and extend the language with some new function symbols to obtain a new algebraic language Σ'. Then suppose to extend the theory $\mathbb{T}$—which is a theory also in the language Σ'—with some new axioms of the form $\top \vdash \big(t(\vec{x}) = s(\vec{x})\big)$, where t and s are terms in the language Σ'. Note that we could have $\Sigma = \Sigma'$, so we can just extend the theory, or $\mathbb{T} = \mathbb{T}'$, so we just extend the language. This extension can be translated in a morphism $(E, \mathfrak{e})\colon \mathtt{HF}^{\Sigma}_{\mathbb{T}} \to \mathtt{HF}^{\Sigma'}_{\mathbb{T}'}$ in **ED**

$$\begin{tikzcd}
\mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\Sigma} \arrow[rr, "E^{\mathrm{op}}"] \arrow[dr, "\mathtt{HF}^{\Sigma}_{\mathbb{T}}"'] & {} \arrow[d, phantom, "\overset{\mathfrak{e}}{\overset{\cdot}{\longrightarrow}}"] & \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\Sigma'} \arrow[dl, "\mathtt{HF}^{\Sigma'}_{\mathbb{T}'}"] \\
& \mathbf{Pos} &
\end{tikzcd}$$

The functor E is the inclusion of terms written in the language Σ in the terms of the language Σ'; each component $\mathfrak{e}_{\vec{x}}$ of the natural transformation $\mathfrak{e}$ is the composition of the inclusion of $\mathtt{HF}^{\Sigma}_{\mathbb{T}}(\vec{x})$ in the poset $\mathtt{HF}^{\Sigma'}_{\mathbb{T}}(\vec{x})$ of Horn formulae in the extended language with respect to the same theory, with the quotient from $\mathtt{HF}^{\Sigma'}_{\mathbb{T}}(\vec{x})$ into $\mathtt{HF}^{\Sigma'}_{\mathbb{T}'}(\vec{x})$, that sends the equivalence class of a formula—with respect to reciprocal provability in the theory $\mathbb{T}$—to the equivalence class of the same formula, with respect to reciprocal provability in the theory $\mathbb{T}'$. In any such extension, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Mod}^{\Sigma'}_{\mathbb{T}'} \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mathrm{Mod}^{\Sigma}_{\mathbb{T}} \arrow[d, phantom, sloped, "\cong"] \\
\mathbf{ED}(\mathtt{HF}^{\Sigma'}_{\mathbb{T}'}, \mathscr{P}_{*}) \arrow[r, "-\circ(E,\mathfrak{e})"] & \mathbf{ED}(\mathtt{HF}^{\Sigma}_{\mathbb{T}}, \mathscr{P}_{*})
\end{tikzcd}$$

where $\mathscr{P}_{*}\colon \mathrm{Set}^{\mathrm{op}}_{*} \to \mathbf{Pos}$ is the elementary doctrine of subsets, and the arrow between the categories of models is the functor that forgets both the added structure from Σ' that is not in Σ and the axioms in $\mathbb{T}'$ that are not in $\mathbb{T}$. So the left adjoint to the precomposition with $(E, \mathfrak{e})$ described in the first section generalizes all such adjunctions in algebra. Some examples include the adjunction between: sets and pointed sets, groups and abelian groups, monoids and semigroups, non-unitary rings and unitary rings, and so on.

Example 4.2.3. Extension and restriction of scalars. In a similar way, let R, S be two commutative unitary rings, and let $a\colon R \to S$ be a ring homomorphism. One can obtain the category R**Mod** of modules over the ring R as the category of models in the language $\Sigma = \{0, +, -\} \cup \{r\cdot\}_{r\in R}$—where 0 is a constant, $+$ is a binary function symbol, and $-$ and each $r\cdot$ are unary function symbols—of the algebraic theory $\mathbb{T}$ with axioms making $\{0, +, -\}$ group operations, and $r\cdot$ the scalar multiplication with $r \in R$. As seen above, we can define the equivalence $\mathbf{ED}(\mathtt{HF}^{\Sigma}_{\mathbb{T}}, \mathscr{P}_{*}) \cong R\mathbf{Mod}$; similarly define the equivalence $\mathbf{ED}(\mathtt{HF}^{\Sigma'}_{\mathbb{T}'}, \mathscr{P}_{*}) \cong S\mathbf{Mod}$. Here Σ' and $\mathbb{T}'$ are not extension of Σ and $\mathbb{T}$. However we can define a functor $E\colon \mathbb{C}\mathrm{tx}_{\Sigma} \to \mathbb{C}\mathrm{tx}_{\mathcal{L}'}$ that maps $0\colon () \to (x), +\colon (x_1, x_2) \to (x), -\colon (x) \to (x)$ in themselves, and each $r\cdot\colon (x) \to (x)$ in $a(r)\cdot\colon (x) \to (x)$; moreover define $\mathfrak{e}_{\vec{x}}\colon \mathtt{HF}^{\Sigma}_{\mathbb{T}}(\vec{x}) \to \mathtt{HF}^{\Sigma'}_{\mathbb{T}'}(\vec{x})$, such that $\alpha(\vec{x}) \mapsto \alpha[a(r)/r](\vec{x})$, meaning that each formula is sent essentially in itself, but each occurrence of r in the terms that appear in α is substituted by $a(r)$, for every $r \in R$. This function preserves trivially top element, meets and fibered equality, and defines a natural transformation. The precomposition $- \circ (E, \mathfrak{e})$ recovers the adjunction between R**Mod** and S**Mod** given by extension and restriction of scalars.

Example 4.2.4. A multisorted example. Consider the two-sorted language Σ and the theory $\mathbb{T}$ that describes sets with an action of a monoid over it. The proof of Proposition 4.2.1 was done in the single sorted case, but holds also in the multisorted setting. Then $\mathbf{ED}(\mathtt{HF}^{\Sigma}_{\mathbb{T}}, \mathscr{R}_{*}) \cong \mathrm{Mon\,Set}$; then extend the language and the theory to describe sets with an action of a group over it, so $\mathbf{ED}(\mathtt{HF}^{\Sigma'}_{\mathbb{T}'}, \mathscr{R}_{*}) \cong \mathrm{Grp\,Set}$. We can again recover the left adjoint to the forgetful functor: for a given (M, X), where M is a monoid acting on a set X, let $\mathcal{F}(M)$ be the free group generated by M. Define the equivalence relation $\sim$ on the product $\mathcal{F}(M) \times X$ generated by $(mn, x) \sim (m, n \cdot x)$ for any $m \in \mathcal{F}(M)$ and $n \in M$; the action of $\mathcal{F}(M)$ on $\mathcal{F}(M) \times X/\sim$ maps $(m, [(m', x)])$ into $[(mm', x)]$ for any $m, m' \in \mathcal{F}(M)$ and $x \in X$. The universal arrow is given by $(\eta_M, \iota_X)\colon (M, X) \longrightarrow (\mathcal{F}(M), \mathcal{F}(M) \times X/\sim)$ where $\eta_M\colon M \to \mathcal{F}(M)$ is the inclusion of the monoid in the free group generated by it, and $\iota_X\colon X \to \mathcal{F}(M) \times X/\sim$ maps $x \in X$ to $[(e, x)]$, where e is the identity of M.

Example 4.2.5. Some quasi-algebraic examples. Suppose we have an algebraic language Σ and a quasi-algebraic theory $\mathbb{T}$, meaning that axioms can be quasi-identities—i.e. formulae of the form $\big(t_1(\vec{x}) = s_1(\vec{x})\big) \wedge \cdots \wedge \big(t_k(\vec{x}) = s_k(\vec{x})\big) \vdash \big(t(\vec{x}) = s(\vec{x})\big)$. In this case we can recover, for example, the left adjoint to the forgetful functor between torsion-free R**Mod** and R**Mod**, between cancellative semigropus and groups, between pseudocomplemented distributive lattices and boolean algebras.

Example 4.2.6. Some non-algebraic example. Let Σ be a first-order language with a binary relation R and $\mathbb{T}$ a theory such that the only axiom in $\mathbb{T}$ are reflexivity, transitivity, symmetry or antisymmetry.

We can easily recover some adjunction by adding axioms—of the kind defined above—to the theory, in the same way we did for the algebraic case: for example we can find the left adjoint to the forgetful functor from the category of sets with an equivalence relation to the category of sets with a reflexive and symmetric relation, or from the category of sets with a preorder to the category of sets with an order, and so on.

A little more work must be done to recover the adjunction between the category of posets and inf-semilattices. Recall that any inf-semilattice is a poset defining that an element is smaller than another one if their meet is the first element, so there is a forgetful functor from inf-semilattices to posets. This forgetful functor arises again from a precomposition between the doctrines of Horn formulae: take the language Σ with a binary predicate symbol, and a theory $\mathbb{T}$ with axioms of reflexivity, transitivity and antisymmetry; then take the algebraic language Σ' with a constant symbol $\top$ and a binary function symbol $\sqcap$, and the algebraic theory $\mathbb{T}'$ that defines inf-semilattices. The functor $E\colon \mathbb{C}\mathrm{tx}_{\Sigma} \to \mathbb{C}\mathrm{tx}_{\Sigma'}$ maps projections in projections and is extended to lists of projections; $\mathfrak{e}_{\vec{x}}\colon \mathtt{HF}^{\Sigma}_{\mathbb{T}}(\vec{x}) \to \mathtt{HF}^{\Sigma'}_{\mathbb{T}'}(\vec{x})$ is defined recursively: the top element, equalities of variables and conjunctions are sent to themselves, while the formula $R(x_i, x_j)$ is sent to the formula $\big(\sqcap(x_i, x_j) = x_i\big)$.

Example 4.2.7. An examples in Sub. Consider $\mathrm{Sub}\colon \mathbb{E}^{\mathrm{op}} \to \mathbf{Pos}$, where $\mathbb{E}$ is a Grothendieck topos. Consider the empty language and the empty theory, then

$$\mathbf{ED}(\mathtt{HF}, \mathrm{Sub}) \cong \mathbb{E}$$

Extend the language with one constant symbol, so we have an equivalence of category

$$\mathbf{ED}(\mathtt{HF}^{\{c\}}, \mathrm{Sub}) \cong \mathbb{E}_{\bullet}$$

where $\{c\}$ is the language with one constant symbol, and $\mathbb{E}_{\bullet}$ is the category of pointed object, meaning that its objects are pairs $(A, a\colon \mathbf{t} \to A)$ where A is an abject of $\mathbb{E}$, and arrows are morphism of $\mathbb{E}$ preserving the point.
So now we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{E}_{\bullet} \arrow[r, "\mathcal{U}"] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{E} \arrow[d, phantom, "\cong" rotate=90] \\
\mathbf{ED}(\mathtt{HF}^{\{c\}}, \mathrm{Sub}) \arrow[r, "-\circ(E,\mathfrak{e})"] & \mathbf{ED}(\mathtt{HF}, \mathrm{Sub})
\end{tikzcd}$$

where the upper arrow is the forgetful functor the leaves out the point, and $(E, \mathfrak{e})\colon \mathtt{HF} \to \mathtt{HF}^{\{c\}}$ is the usual arrow that arises from the extension of the empty language to the language with one constant symbol. The left adjoint to $\mathcal{U}$ generalizes in a Grothendieck topos the classical adjoint that adds a new element to a set.

Example 4.2.8. Adding an axiom. Consider an elementary doctrine P, take an element $\varphi \in P(\mathbf{t})$, and do the construction $(\mathrm{id}, P(!)\varphi \wedge -)\colon P \to P_{\varphi}$ as in Corollary 2.4.5. Then take $(H, \mathfrak{h})\colon P \to \mathrm{Sub}$ and suppose that $\mathfrak{h}_{\mathbf{t}}(\varphi) = \top$. We observe that applying the left adjoint functor to $(H, \mathfrak{h})$ we obtain exactly the unique $(H, \mathfrak{h}')\colon P_{\varphi} \to \mathrm{Sub}$ defined by the universal property of $(\mathrm{id}, P(!)\varphi \wedge -)\colon P \to P_{\varphi}$. Indeed, since the left Kan extension of H along the identity is H itself, it is enough to check that $\mathfrak{l}_A(\alpha) = \mathfrak{h}'(\alpha)$ for all $\alpha \in P_{\varphi}(A)$, so that $\mathscr{L}(A) = (HA, \mathfrak{l}_{A\times A}(\delta_A)) = (HA, \Delta_{HA})$, and $Q\mathscr{L}(A) = HA$. Consider the 1-arrow $(H, \mathfrak{h}')\colon P_{\varphi} \to \mathrm{Sub}$ and the identity 2-arrow $(H, \mathfrak{h}) \to (H, \mathfrak{h}')(\mathrm{id}, P(!)\varphi \wedge -)$ along all the 1-arrows $(K, \mathfrak{k})\colon P_{\varphi} \to \mathrm{Sub}$ and the 2-arrows $\theta\colon (H, \mathfrak{h}) \to (K, \mathfrak{k})(\mathrm{id}, P(!)\varphi \wedge -)$. By definition of $\mathfrak{l}_A$ we obtain

$$\mathfrak{l}_A(\alpha) = \bigwedge_{(K,\mathfrak{k}),\theta} \widehat{\theta}^*_A(\mathfrak{k}_A(\alpha)) \leq \mathfrak{h}'_A(\alpha).$$

Conversely, compute

$$\mathfrak{h}'_A(\alpha) = \mathfrak{h}'_A(P(!_A)\varphi \wedge \alpha) = \mathfrak{h}_A(\alpha) \leq \widehat{\theta}^*_A(\mathfrak{k}_A(P(!_A)\varphi \wedge \alpha)) = \widehat{\theta}^*_A(\mathfrak{k}_A(\alpha))$$

hence $\mathfrak{h}'_A(\alpha) \leq \mathfrak{l}_A(\alpha)$.

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