



# Quantization and soliton-like solutions for the $\Phi\Psi$ -model in an optic fiber

Francesco Belgiorno<sup>1,2,4</sup>, Sergio L. Cacciatori<sup>3,4</sup>, Simone Trevisan<sup>3,4</sup>, Adriano Viganò<sup>4,5,a</sup> 

<sup>1</sup> Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo 32, 20133 Milan, Italy

<sup>2</sup> INdAM-GNFM, Città Universitaria-P.le Aldo Moro 5, 00185 Roma, Italy

<sup>3</sup> Department of Science and High Technology, Università dell'Insubria, Via Valleggio 11, 22100 Como, Italy

<sup>4</sup> INFN sezione di Milano, via Celoria 16, 20133 Milan, Italy

<sup>5</sup> Dipartimento di Fisica, Università degli Studi di Milano, Via Celoria 16, 20133 Milan, Italy

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**Abstract** In the framework of a mesoscopical model for dielectric media we provide an analytical description for the electromagnetic field confined in a cylindrical cavity containing a finite dielectric sample. This system is apted to simulate the electromagnetic field in a optic fiber, in which two different regions, a vacuum region and a dielectric one, appear. A complete description for the scattering basis is introduced, together with field quantization and the two-point function. Furthermore, we also determine soliton-like solutions in the dielectric, propagating in the sample of nonlinear dielectric medium.

## 1 Introduction

Dielectric media in the framework of analogue gravity are an active subject of investigation, with particular reference to the Hawking effect in nonlinear dielectrics. See e.g. [1–13]. As to experiments with dielectric media and their debate one may refer to [4, 5, 7, 14–16] and also to the (uncontroversial) experiment in a optic fiber reported in [17]. In general, the problem is quite difficult, because of dispersive effects associated with condensed matter systems. Notwithstanding, a framework can be provided where, in the limit of weak dispersive effects, in a precise mathematical sense, one is able to find how the Hawking effect manifests itself when the system is affected by the presence of horizon(s) (mathematically, turning point(s)) [13]. Mostly, calculations are carried out for a dielectric medium filling all the space. Furthermore, in order to avoid technical difficulties arising mainly because of the gauge field nature of the electromagnetic field, which arise naturally in the Hopfield model (see [8, 19, 20]), we have introduced a simplified model, called the  $\Phi\Psi$ -model, where

the original fields of the Hopfield model are replaced by two scalar fields:  $\Phi$  in place of the electromagnetic field, and  $\Psi$  in place of the polarization [8]. An exact quantization for the fully relativistically covariant version of the model have been provided in [18]. We have also taken into account the case of dielectric medium filling only an half-space [21]. We have verified that, in the latter case, spectral boundary conditions are required, because of the peculiar role played by the polarization field.

Herein, we extend our analysis by taking into account a cylindrical geometry, where the dielectric field fills only a finite cylindrical region of length  $2L$  and radius  $R$ . The remaining region of radius  $R$  is filled by vacuum. This simplified setting can be still interesting because the vacuum regions could be also replaced by regions containing dielectrics with different refractive index.

In the first part of our analysis, we discuss in details the problem of the boundary conditions to be imposed on the fields, and a complete scattering basis for the problem is introduced through separation of variables allowed by our peculiar geometrical setting. We also provide the full propagator for the model at hand.

In the second part, we take into account a fully nonlinear dielectric, where the nonlinearity is simulated by introducing a term proportional to the fourth power of the polarization field  $\Psi$ . Our aim is to show that solitonic solutions exist, representing a dielectric perturbation travelling with constant velocity in the direction of the cylindrical fiber axis. We can show that, by taking into account 'homogeneous' solutions which do not vary in the radial direction and also in the azimuthal one, soliton-like solutions exist, with different characteristics depending on suitable parameters. A linearization around the solitonic solution is the natural set-up for studying the analog Hawking effect also in the present

<sup>a</sup> e-mail: [adriano.vigano@mi.infn.it](mailto:adriano.vigano@mi.infn.it) (corresponding author)

case. The present analysis is in preparation of the analysis of the Hawking effect in the geometrical setting described above.

### 2 The relativistic Kerr- $\Phi\Psi$ model in a cylindrical fibre

Let us consider the electromagnetic field in a cylindrical cavity, along the  $z$  direction, where an Hopfield dielectric is at rest in the lab, filling the region  $C_\chi = \{(t, x, y, z) | -L \leq z \leq L, x^2 + y^2 \leq R^2\}$ . In general, we consider inertial frames which are boosted in the  $z$  direction with respect to the Lab frame. See also Fig. 1.

If  $\underline{n}$  is the four-vector with covariant components  $\underline{n} = (0, 0, 0, 1)$  in the Lab frame, then, in an arbitrary inertial frame centered in  $\mathbf{0}$  (the origin of the Lab frame) the confining cylindrical region is  $C = \{\mathbf{x} \in M_0^{1,3} | -(\mathbf{x} - (\mathbf{x} \cdot \mathbf{v})\mathbf{v} + (\mathbf{x} \cdot \mathbf{n})\mathbf{n})^2 \leq R^2\}$ , where  $\underline{v} = (1, 0, 0, 0)$  in the Lab frame, and the dielectric region is  $C_\chi = \{\mathbf{x} \in C_\chi | -L \leq \mathbf{x} \cdot \mathbf{n} \leq L\}$ .

The  $\phi - \psi$  model is thus described by the action principle as follows:

$$S[\phi, \psi] = \frac{1}{2} \int_C \partial_\mu \phi \partial^\mu \phi d^4x + \int_{C_\chi} \left( \frac{1}{2} v^\mu \partial_\mu \psi v^\nu \partial_\nu \psi - \frac{\omega_0^2}{2} \psi^2 - g \phi v^\mu \partial_\mu \psi \right) d^4x. \tag{2.1}$$

This is because we require for  $\psi$  to vanish outside  $C_\chi$  by definition.

#### 2.1 Equations of motion

There are several interesting discussions regarding the deduction of the equations of motion. Nevertheless we will follow a simple deduction, by using local variations and then by choosing boundary conditions. With ‘‘local variations’’ we mean the following. Fix a point  $p$  internal to  $C_\chi$  (in the topological sense) or internal to  $C \setminus C_\chi$ . Thus, there is at least an open set  $U(p)$  such that is completely contained in  $C_\chi$  (or in  $C \setminus C_\chi$ ). A local variation is a variation of the fields with support in such a  $U(p)$ . Using local variations we get for the equations of motion

$$\square \phi = 0, \tag{2.2}$$

$$\psi = 0, \tag{2.3}$$

outside  $C_\chi$ , and

$$\square \phi + g v^\mu \partial_\mu \psi = 0, \tag{2.4}$$

$$(v^\mu \partial_\mu)^2 \psi + \omega_0^2 \psi - g v^\mu \partial_\mu \phi = 0, \tag{2.5}$$

inside  $C_\chi$ . Now, we are left with the choice of the boundary conditions in order to completely define the theory. The boundary consists in  $\partial C$ , which includes the conditions at infinity and  $C_\chi \cap \partial C$ , and  $\partial C_\chi$  that adds  $\partial C_\chi - C_\chi \cap \partial C =: \Sigma_{-L} \cup \Sigma_L$  in obvious notations. In order to choose such conditions, let us start by considering (global) variations in  $\psi$ . The support of  $\psi$  is compact, so we choose to work with variations which are  $C^\infty(C_\chi)$  with support in  $C_\chi$ . In particular, we do not require for them to be continuous on  $\Sigma_{-L}$  and  $\Sigma_L$ . Nevertheless, since  $\mathbf{v}$  is orthogonal to  $\mathbf{N}$ , where the latter is the suitably oriented normal field to  $\partial C$  and  $\partial C_\chi$ , there are no boundary terms in  $\delta S$  under variations of  $\psi$ , and then we are not required to choose any particular condition on  $\psi$  (apart from requiring that it must be at least  $C^2(C_\chi)$ ).

A little bit more involved is the (global) variation in  $\phi$ . In this case we have to tackle the variation

$$\begin{aligned} \delta \frac{1}{2} \int_C \partial_\mu \phi \partial^\mu \phi d^4x &= \int_C \partial_\mu \delta \phi \partial^\mu \phi d^4x \\ &= \int_C \partial_\mu (\delta \phi \partial^\mu \phi) d^4x - \int_C \delta \phi \partial_\mu \partial^\mu \phi d^4x. \end{aligned} \tag{2.6}$$

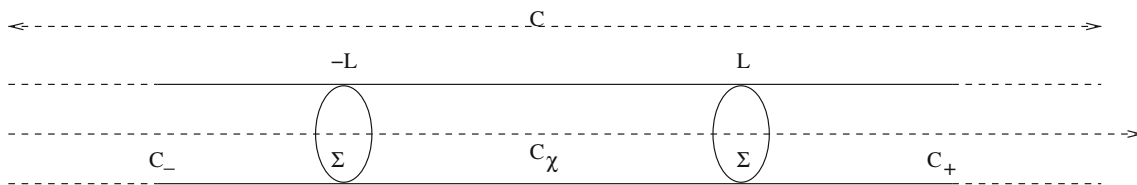
Requiring for  $\phi$  to be continuous in  $C$  implies that  $\delta \phi$  must be continuous. However, requiring also the continuity of  $\partial_\mu \phi$  looks too much restrictive in general. In order to manipulate the divergence in the last expression, let us notice that if we do not require for  $\partial_\mu \phi$  to be continuous on  $\Sigma_0$  and  $\Sigma_L$ , then we cannot apply the divergence theorem directly but we need to separate  $C$  into three regions as  $C = C_\chi \cup C_- \cup C_+$ , where  $C_- \cup C_+ := C - C_\chi$  (with obvious notation). This way

$$\begin{aligned} \int_C \partial_\mu (\delta \phi \partial^\mu \phi) d^4x &= \int_{C_-} \partial_\mu (\delta \phi \partial^\mu \phi) d^4x \\ &+ \int_{C_+} \partial_\mu (\delta \phi \partial^\mu \phi) d^4x + \int_{C_\chi} \partial_\mu (\delta \phi \partial^\mu \phi) d^4x, \end{aligned} \tag{2.7}$$

and  $\delta \phi \partial^\mu \phi$  is continuous and, indeed, smooth in each of the three regions. So we can apply the divergence theorem to each of the three regions. The result is that, if we assume that  $n^\mu \partial_\mu \phi$  is continuous in an open neighbourhood of  $\Sigma_A$ ,  $A = -L, L$ , then

$$\begin{aligned} \int_C \partial_\mu (\delta \phi \partial^\mu \phi) d^4x &= \int_{\partial C} N^\mu \partial_\mu \phi \delta \phi d^3\sigma. \end{aligned} \tag{2.8}$$

Since we do not mean to fix the value of  $\phi$  on the boundary, we can get rid of the boundary term by imposing the Neuman condition  $N^\mu \partial_\mu \phi|_{\partial C} = 0$ . More precisely this condition is clear on the cylindrical boundary but it should also include a condition at  $z \rightarrow \pm\infty$ . There, the above Neuman condition looks not suitable if we want to allow for sources or, say, fluxes. In this sense it seems that at infinity some other condition could be better, but it is not a case of interest here.



**Fig. 1** The geometry for the problem at hand is displayed. The region  $C_\chi$  contains the dielectric medium, the other two cylindrical regions  $C_\pm$  with the same radius  $R$  are void. The inner boundaries indicated

in the text as  $\Sigma_{\pm L}$  are for simplicity both indicated with  $\Sigma$ . Again, for simplicity, we have not indicated the other boundaries we take into consideration in the main text

On the boundaries  $\Sigma_A$  of the dielectric we are left with the condition of continuity of the normal derivative of  $\phi$ . Let us investigate a little bit more at this condition by looking at the Eqs. (2.2) and (2.4). They show that  $\square\phi$  is not continuous on  $\Sigma_A$ . Now, since  $\psi$  vanishes outside  $C_\chi$ , we can write both these equations as

$$\partial^\mu (\partial_\mu \phi + g v_\mu \psi) = 0. \tag{2.9}$$

Since  $\psi$  is discontinuous,  $\partial^\mu \psi$  is expected to produce  $\delta$ -function contributions supported on  $\Sigma_A$ . However,  $\mathbf{v} \cdot \mathbf{n} = 0$ , so that  $v^\mu \partial_\mu \psi$  is discontinuous but does not contains  $\delta$  contributions. Thus, the same happens for  $\square\phi$ . Now, since  $C$  is contractible we can add to  $\mathbf{n}$  and  $\mathbf{v}$  two other vectors on  $C$ ,  $\mathbf{e}_i$  ( $i = 1, 2$ ) in order to get a complete constant orthonormal frame. Thus, we can write

$$0 = \partial_\mu (v^\mu \partial_\nu \phi - n^\mu \partial_n \phi - e_1^\mu \partial_{\mathbf{e}_1} \phi - e_2^\mu \partial_{\mathbf{e}_2} \phi) = \partial_\nu \partial_\nu \phi - \partial_n \partial_n \phi - \partial_{\mathbf{e}_1} \partial_{\mathbf{e}_1} \phi - \partial_{\mathbf{e}_2} \partial_{\mathbf{e}_2} \phi. \tag{2.10}$$

Now,  $\partial_{\mathbf{e}_1} \phi$ ,  $\partial_{\mathbf{e}_2} \phi$  and  $\partial_\nu \phi$  are discontinuous, but no  $\delta$  contribution arises in further deriving, since  $\partial_{\mathbf{e}_1}$ ,  $\partial_{\mathbf{e}_2}$  and  $\partial_\nu$  derive in directions orthogonal to  $\mathbf{n}$  (and, so, tangent to the separating hypersurface). The remaining term  $\partial_n \phi$  is continuous and the further derivative  $\partial_n$  introduce at most new discontinuities. This shows that no further conditions are necessary for having consistent equations: all the condition we have to require inside  $C$  are the continuity of  $\phi$  and  $\partial_n \phi$  everywhere, with the last vanishing on  $\partial C$ .

### 2.2 General solution

Let us work in a frame with four-velocity

$$\underline{v} = \gamma(v)(1, 0, 0, v), \tag{2.11}$$

so that

$$\underline{n} = \gamma(v)(v, 0, 0, 1). \tag{2.12}$$

#### 2.2.1 Outside the dielectric

The dielectric region is defined by  $-vt - L/\gamma(v) \leq z \leq -vt + L/\gamma(v)$ .

If we choose cylindrical coordinates, outside the dielectric the equations of motion are simply  $\psi = 0$  and

$$\partial_t^2 \phi - \partial_z^2 \phi - \partial_\rho^2 \phi - \frac{1}{\rho} \partial_\rho \phi - \frac{1}{\rho^2} \partial_\theta^2 \phi = 0. \tag{2.13}$$

Separating the variables as

$$\phi(t, \rho, z, \theta) = K \phi_T(t) \phi_R(\rho) \phi_Z(z) \phi_\Theta(\theta), \tag{2.14}$$

with  $K$  a constant, we find that

$$\phi_T(t) = e^{-ik_0 t}, \quad \phi_Z(z) = e^{ik_z z}, \quad \phi_\Theta(\theta) = e^{im\theta}, \tag{2.15}$$

with  $m \in \mathbb{Z}$ , and

$$\phi_R'' + \frac{1}{\rho} \phi_R' + \left( k_\rho^2 - \frac{m^2}{\rho^2} \right) \phi = 0, \tag{2.16}$$

where  $k_\rho$  must satisfy  $k_0^2 - k_z^2 - k_\rho^2 = 0$ . The only solutions continuous in  $\rho = 0$  are  $\phi_R(\rho) = J_m(k_\rho \rho)$ , where  $J_m$  are the usual Bessel functions. The boundary condition on  $\partial C$  reduces to

$$J'_m(k_\rho R) = 0, \tag{2.17}$$

so that at the end we have

$$\phi_R(\rho) = J_{|m|,s}(\rho) = J_{|m|} \left( z_{ms} \frac{\rho}{R} \right), \quad |m|, s \in \mathbb{N}, \tag{2.18}$$

where  $z_{ms}$  is the  $s$ -th positive zero of  $J'_m$ . The corresponding dispersion relations are  $k_0^2 = k_z^2 + z_{ms}^2/R^2$ , which can be codified in

$$K = c_{m,s}(k_0, k_z) \delta \left( k_0^2 - k_z^2 - \frac{z_{ms}^2}{R^2} \right). \tag{2.19}$$

In conclusion, we can write the general solution outside the dielectric in the form

$$\phi = \sum_{s \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{dk_z}{4\pi k^0} \left( c_{m,s}(k_z) e^{-ik^0 t + ik_z z + im\theta} + c_{m,s}^*(k_z) e^{ik^0 t - ik_z z - im\theta} \right) J_{|m|} \left( z_{ms} \frac{\rho}{R} \right), \tag{2.20}$$

$$\psi = 0, \tag{2.21}$$

where

$$k^0(k_z, m, s) = \sqrt{k_z^2 + \frac{z_{ms}^2}{R^2}}. \tag{2.22}$$

### 2.2.2 Inside the dielectric

Inside the dielectric, the equations take the form ( $v^0 = \gamma(v)$  and  $v = v\gamma(v)$ )

$$\begin{aligned} \partial_t^2 \phi - \partial_z^2 \phi - \partial_\rho^2 \phi - \frac{1}{\rho} \partial_\rho \phi \\ - \frac{1}{\rho^2} \partial_\theta^2 \phi + gv^0 \partial_t \psi + gv \partial_z \psi = 0, \end{aligned} \tag{2.23}$$

$$(v^0 \partial_t + v \partial_z)^2 \psi + \omega_0^2 \psi - gv^0 \partial_t \phi - gv \partial_z \phi = 0. \tag{2.24}$$

By means of the separation ansatz

$$\phi(t, \rho, z, \theta) = \tilde{\phi} \phi_T(t) \phi_R(\rho) \phi_Z(z) \phi_\Theta(\theta), \tag{2.25}$$

$$\psi(t, \rho, z, \theta) = \tilde{\psi} \psi_T(t) \psi_R(\rho) \psi_Z(z) \psi_\Theta(\theta), \tag{2.26}$$

we find that

$$\begin{aligned} \psi_T(t) = \phi_T(t) = e^{-ik_0 t}, \quad \psi_Z(z) = \phi_Z(z) = e^{ik_z z}, \quad \psi_\Theta(\theta) \\ = \phi_\Theta(\theta) = e^{im\theta}, \end{aligned} \tag{2.27}$$

$$\begin{aligned} \psi_R(\rho) = \phi_R(\rho) = J_{|m|,s}(\rho) \\ = J_{|m|} \left( z_{ms} \frac{\rho}{R} \right), \quad s \in \mathbb{N}, \quad m \in \mathbb{Z}, \end{aligned} \tag{2.28}$$

and  $\tilde{\phi}, \tilde{\psi}$  must satisfy the algebraic system

$$\begin{pmatrix} -k_0^2 + k_z^2 + k_\rho^2 & ig\omega \\ -ig\omega & \omega_0^2 - \omega^2 \end{pmatrix} \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.29}$$

where we have put  $k_\rho = z_{ms}/R$  and

$$\omega = v^\mu k_\mu = k^0 v^0 - vk_z. \tag{2.30}$$

This has nontrivial solutions if the determinant of the matrix vanishes, which means

$$DR := k_0^2 - k_z^2 - k_\rho^2 + \frac{g^2 \omega^2}{\omega_0^2 - \omega^2} = 0. \tag{2.31}$$

Notice that this is an implicit equation in  $k_0 = k^0$ , since  $\omega$  is a function of  $k_0$ . With this condition, the solution of the algebraic system takes the form

$$\tilde{\phi} = b(k_0, k_z, m, s) \delta(DR), \tag{2.32}$$

$$\tilde{\psi} = \frac{k_0^2 - k_z^2 - k_\rho^2}{ig\omega} b(k_0, k_z, m, s) \delta(DR). \tag{2.33}$$

If we set

$$DR' := \partial_{k_0} DR = 2k^0 \left( 1 + \frac{g^2 \omega_0^2 v^0}{(\omega_0^2 - \omega^2)^2} \right), \tag{2.34}$$

then we can write the general solution inside the dielectric as

$$\begin{aligned} \phi &= \sum_{a=1}^2 \sum_{s \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi DR'_{(a)}} \\ &\quad \times \left( b_{(a)ms}(k_z) e^{-ik_{(a)}^0 t + ik_z z + im\theta} + c.c. \right) J_{|m|} \left( z_{ms} \frac{\rho}{R} \right), \\ \psi &= \sum_{a=1}^2 \sum_{s \in \mathbb{N}} \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi DR'_{(a)}} \\ &\quad \times \left( b_{(a)ms}(k_z) e^{-ik_{(a)}^0 t + ik_z z + im\theta} - c.c. \right) \frac{-ig\omega_{(a)}}{\omega_{(a)}^2 - \omega_0^2} \\ &\quad \times J_{|m|} \left( z_{ms} \frac{\rho}{R} \right), \end{aligned}$$

where  $(a)$  indicates the branch like in [18,20].

### 2.2.3 Gluing conditions

At this point we must impose the continuity of  $\phi$  and  $\partial_n \phi$ . From now on, we will work in the lab frame where  $\underline{v} \equiv (1, 0, 0, 0)$  and  $\omega = k^0$ . In this case, the normal direction is  $z$  and the gluing conditions are

$$\begin{aligned} \phi(-L^-) = \phi(-L^+), \quad \phi(L^-) = \phi(L^+), \\ \partial_z \phi(-L^-) = \partial_z \phi(-L^+), \\ \partial_z \phi(L^-) = \partial_z \phi(L^+), \end{aligned} \tag{2.35}$$

where we defined for short

$$\phi(z^\pm) := \lim_{\epsilon \rightarrow 0} \phi(t, \rho, z \pm |\epsilon|, \theta). \tag{2.36}$$

It is convenient to define a scattering basis, defined by replacing the Fourier modes with solutions of the form (omitting the angular and radial parts, at time  $t = 0$ )

$$\begin{aligned} \phi_k(z) &= \left( e^{ikz} + R_{ksm} e^{-ikz} \right) \chi_{(-\infty, -L)}(z) \\ &\quad + \left( M_{ksm} e^{iq_{s,m}(k)z} + N_{ksm} e^{-iq_{sm}(k)z} \right) \chi_{[-L, L]}(z) \\ &\quad + T_{ksm} e^{ikz} \chi_{(L, \infty)}(z), \end{aligned} \tag{2.37}$$

going from the left to the right, and the analogous left moving modes. Here  $R_{ksm}, T_{ksm}, M_{ksm}$ , and  $N_{ksm}$  are reflection and transmission coefficients. Moreover, we choose the measure to be  $dk_z/(2\pi)2k^0$  everywhere so that inside the dielectric  $b_{(a)ms}$  and  $c_{ms}$  reabsorb the normalization factor  $1 + \frac{g^2 \omega_0^2 v^0}{(\omega_0^2 - \omega^2)^2}$  from  $DR'$ .

We have defined the functions  $\chi_{(a,b)}(z) = 1$  for  $z \in (a, b)$ , 0 otherwise.

Writing the conditions (2.35) explicitly, we obtain the following algebraic system

$$\begin{cases} e^{-ikL} + R_{ksm}e^{ikL} = M_{ksm}e^{-iqL} + N_{ksm}e^{iqL} \\ ke^{-ikL} - R_{ksm}ke^{ikL} = M_{ksm}qe^{-iqL} - N_{ksm}qe^{iqL} \\ T_{ksm}e^{ikL} = M_{ksm}e^{iqL} + N_{ksm}e^{-iqL} \\ T_{ksm}ke^{ikL} = M_{ksm}qe^{iqL} - N_{ksm}qe^{-iqL} \end{cases} \quad (2.38)$$

that has solution

$$R_{ksm} = \frac{2i(k^2 - q_{sm}^2(k)) \sin(2q_{sm}(k)L)e^{2i(q_{sm}(k)-k)L}}{(k - q_{sm}(k))^2 e^{4iq_{sm}(k)L} - (k + q_{sm}(k))^2}, \quad (2.39)$$

$$M_{ksm} = -\frac{2k(k + q_{sm}(k))e^{i(q_{sm}(k)-k)L}}{(k - q_{sm}(k))^2 e^{4iq_{sm}(k)L} - (k + q_{sm}(k))^2}, \quad (2.40)$$

$$N_{ksm} = \frac{2k(k - q_{sm}(k))e^{i(3q_{sm}(k)-k)L}}{(k - q_{sm}(k))^2 e^{4iq_{sm}(k)L} - (k + q_{sm}(k))^2}, \quad (2.41)$$

$$T_{ksm} = -\frac{4kq_{sm}(k)e^{2iq_{sm}(k)L}e^{-2ikL}}{(k - q_{sm}(k))^2 e^{4iq_{sm}(k)L} - (k + q_{sm}(k))^2}. \quad (2.42)$$

### 2.3 The scattering basis

The positive energy scattering basis consists in the dielectric modes and the gap modes. In the lab frame, the dielectric modes are (we write the  $\phi$ -component only):

$$\begin{aligned} \phi_{D,ksm}^R(t, \rho, z, \theta) &= \kappa_{ksm}e^{-i\omega_{ksm}t}e^{im\theta}J_{|m|}\left(z_{ms}\frac{\rho}{R}\right) \\ &\times \left[ \left( e^{ikz} + R_{ksm}e^{-ikz} \right) \chi_{(-\infty, -L)}(z) \right. \\ &+ \left( M_{ksm}e^{iq_{s,m}(k)z} + N_{ksm}e^{-iq_{sm}(k)z} \right) \chi_{[-L, L]}(z) \\ &+ \left. T_{ksm}e^{ikz} \chi_{(L, \infty)}(z) \right], \end{aligned} \quad (2.43)$$

$$\phi_{D,ksm}^L(t, \rho, z, \theta) = \phi_{D,ksm}^R(t, \rho, -z, \theta), \quad (2.44)$$

where

$$\omega_{ksm} = \sqrt{k^2 + k_\rho^2}, \quad (2.45)$$

$$q_{sm}(k)^2 + k_\rho^2 = (k^2 + k_\rho^2) \frac{\omega_0^2 + g^2 - k^2 - k_\rho^2}{\omega_0^2 - k^2 - k_\rho^2}, \quad (2.46)$$

$$m \in \mathbb{Z}, \quad s \in \mathbb{N}, \quad k > 0, \quad (2.47)$$

and the coefficients are

$$R_{ksm} = \frac{2i(k^2 - q_{sm}^2(k)) \sin(2q_{sm}(k)L)e^{2i(q_{sm}(k)-k)L}}{(k - q_{sm}(k))^2 e^{4iq_{sm}(k)L} - (k + q_{sm}(k))^2}, \quad (2.48)$$

$$M_{ksm} = -\frac{2k(k + q_{sm}(k))e^{i(q_{sm}(k)-k)L}}{(k - q_{sm}(k))^2 e^{4iq_{sm}(k)L} - (k + q_{sm}(k))^2}, \quad (2.49)$$

$$N_{ksm} = \frac{2k(k - q_{sm}(k))e^{i(3q_{sm}(k)-k)L}}{(k - q_{sm}(k))^2 e^{4iq_{sm}(k)L} - (k + q_{sm}(k))^2}, \quad (2.50)$$

$$T_{ksm} = -\frac{4kq_{sm}(k)e^{2iq_{sm}(k)L}e^{-2ikL}}{(k - q_{sm}(k))^2 e^{4iq_{sm}(k)L} - (k + q_{sm}(k))^2}. \quad (2.51)$$

Notice that

$$|T_{ksm}|^2 + |R_{ksm}|^2 = 1, \quad (2.52)$$

so that there is no trapping in the dielectric. The dielectric modes are defined in the range for  $\omega_{k,s,m}$  not in the gap  $[\omega_0, \bar{\omega}]$ , and do not form a complete basis for all possible initial conditions.

In order to get a complete basis we have to add the gap modes

$$\begin{aligned} \phi_{G,ksm}^R(t, \rho, z, \theta) &= \tilde{\kappa}_{ksm}e^{-i\omega_{ksm}t}e^{im\theta}J_{|m|}\left(z_{ms}\frac{\rho}{R}\right) \sin((k + L)z) \chi_{(-\infty, -L]}(z), \\ & \quad (2.53) \end{aligned}$$

$$\begin{aligned} \phi_{G,ksm}^L(t, \rho, z, \theta) &= \tilde{\kappa}_{ksm}e^{-i\omega_{ksm}t}e^{im\theta}J_{|m|}\left(z_{ms}\frac{\rho}{R}\right) \sin((k - L)z) \chi_{[L, \infty)}(z), \\ & \quad (2.54) \end{aligned}$$

defined for

$$\omega_0^2 \leq k^2 + k_\rho^2 \leq \bar{\omega}^2. \quad (2.55)$$

These represent modes that are totally reflected by the dielectric.

The normalisation constants  $\kappa_{ksm}$  and  $\tilde{\kappa}_{ksm}$  can be computed by using the results in appendix A. If we choose

$$\kappa_{ksm} = \frac{\tilde{\kappa}_{ksm}}{2} = \frac{1}{\left(1 - \frac{m^2}{z_{ms}^2}\right)^{\frac{1}{2}} \sqrt{\pi} R J_{|m|}(z_{ms})}, \quad (2.56)$$

then the scattering solution are orthonormalised (with measure  $dk/2\pi(2k^0)$ ).

### 3 Quantization

Here we determine the scalar product and invert the expressions to compute the amplitudes fields  $c$ ,  $b$  in terms of the fields and their conjugate momenta, and impose the equal time canonical commutation relations (ETCCR).

We write the full field as a superposition of the component fields  $\phi_D^R$  etc, so:

$$\begin{aligned} \phi(t, z, \rho, \theta) &= \sum_{s,m} \int_{R_{m,s}} \frac{dk}{4\pi k^0} \left\{ a_{D,ksm}^R \phi_{D,ksm}^R(t, z, \rho, \theta) \right. \\ & \quad \left. + a_{D,ksm}^L \phi_{D,ksm}^L(t, z, \rho, \theta) \right\} \end{aligned}$$

$$\begin{aligned}
 &+d_{G,ksm}^R \phi_{G,ksm}^R(t, z, \rho, \theta) \\
 &+d_{G,ksm}^L \phi_{G,ksm}^L(t, z, \rho, \theta) + \text{h.c.} \Big\}, \quad (3.1)
 \end{aligned}$$

where  $R_{m,s}$  is the range of  $k$  satisfying the right spectral conditions for any given  $s, m$ . In the particular case when  $s = m = 0$  one has that the integration is on  $[0, \omega_0] \cup [\bar{\omega}, \infty)$  for  $D$  modes and  $[\omega_0, \bar{\omega}]$  for the  $G$  modes (see [18]).

Now, our purpose is to determine the commutation relations between the operators  $a$ 's and  $d$ 's: to do this, we note that the scalar product defined in appendix A can also be written as

$$(f|\tilde{f}) = \langle \Psi, \tilde{\Psi} \rangle := \frac{i}{2} \int d^3 \Psi \Omega \tilde{\Psi}, \quad (3.2)$$

where

$$\Psi = \begin{pmatrix} \phi \\ \psi \\ \pi_\phi \\ \pi_\psi \end{pmatrix} \quad (3.3)$$

and

$$\Omega = \begin{pmatrix} 0_{2 \times 2} & 1_{2 \times 2} \\ -1_{2 \times 2} & 0_{2 \times 2} \end{pmatrix}. \quad (3.4)$$

Given the orthogonality relations, we can write the coefficients as the scalar product

$$a_{D,ksm}^R = \langle \Psi_{D,ksm}^R, \Psi_{ksm} \rangle, \quad (3.5)$$

and similarly for the other coefficients.

With some algebra, we can evaluate the products  $a_{D,ksm}^R a_{D,k's'm'}^R$  and  $a_{D,k's'm'}^R a_{D,ksm}^R$ , and imposing the ETCCR  $[\phi(x), \pi_\phi(y)] = i\delta^{(3)}(x - y)$ , we get that the commutator is equal to the scalar product

$$[a_{D,ksm}^R, a_{D,k's'm'}^R] = \langle \Psi_{D,ksm}^R, \Psi_{D,k's'm'}^R \rangle. \quad (3.6)$$

Therefore we can easily evaluate the commutation relations, using the orthogonality relations calculated in appendix A. The relevant commutators are

$$[a_{D,ksm}^R, a_{D,k's'm'}^R] = 4\pi \omega_{ksm} \delta_{mm'} \delta_{ss'} \delta(k' - k), \quad (3.7)$$

$$[a_{D,ksm}^L, a_{D,k's'm'}^L] = 4\pi \omega_{ksm} \delta_{mm'} \delta_{ss'} \delta(k' - k), \quad (3.8)$$

$$[d_{G,ksm}^R, d_{G,k's'm'}^R] = 4\pi \omega_{ksm} \delta_{mm'} \delta_{ss'} \delta(k' - k), \quad (3.9)$$

$$[d_{G,ksm}^L, d_{G,k's'm'}^L] = 4\pi \omega_{ksm} \delta_{mm'} \delta_{ss'} \delta(k' - k), \quad (3.10)$$

and all other vanish.

### 4 The two-point function of the Kerr- $\Phi\Psi$ model

The full quantum theory, in absence of the Kerr nonlinear term, is fully defined by the free propagator. Let us compute the two-point function of the free theory

$$iG_{\psi\psi}^0(x, x') = \langle \psi(x)\psi(x') \rangle|_{\lambda=0}. \quad (4.1)$$

In general, its explicit expression will depend on where we choose the points  $x$  and  $x'$ . We will not consider the gap modes. In fact, the right and left gap modes can be included by noting that they are equivalent to the  $\phi^R$  and  $\phi^L$  modes respectively, with reflection coefficient  $R = 1$  and all other coefficients vanishing, so that, if necessary, the corresponding contributions can be deduced by taking the limit  $R \rightarrow 1$ .

We start by discussing the case  $x, x' \in C_\chi$ , which is the most important one for computing Feynman diagrams in perturbation theory, when the nonlinearity is included. For simplicity, we restrict to the case  $\rho = \rho', \theta = \theta'$  in the following computation, but at the end we will give the general result:

$$\begin{aligned}
 iG_{\psi\psi}^0(x, x') &= \theta(t - t') \sum_{s,m} \int_0^\infty \frac{dk}{4\pi \omega_{ksm}} \left[ \psi_{ksm}^R(x) \psi_{ksm}^R(x')^* \right. \\
 &\quad \left. + \psi_{ksm}^L(x) \psi_{ksm}^L(x')^* \right] + (x \leftrightarrow x') \\
 &= \frac{i}{2\pi} \int d\omega \sum_{s,m} \int_0^\infty \frac{dk}{4\pi \omega_{ksm}} \frac{e^{-i(\omega + \omega_{ksm})(t-t')}}{\omega + i\epsilon} \\
 &\quad \times \left[ \left( M_{ksm} e^{iq(k)z} + N_{ksm} e^{-iq(k)z} \right) \right. \\
 &\quad \times \left( M_{ksm}^* e^{-iq(k)z'} + N_{ksm}^* e^{iq(k)z'} \right) \\
 &\quad \left. + \left( M_{ksm} e^{-iq(k)z} + N_{ksm} e^{iq(k)z} \right) \right. \\
 &\quad \left. \times \left( M_{ksm}^* e^{iq(k)z'} + N_{ksm}^* e^{-iq(k)z'} \right) \right] \\
 &\quad \times \frac{g^2 \omega_{ksm}^2 \kappa_{sm}^2 |J_m(\rho)|^2}{(\omega_{ksm}^2 - \omega_0^2)^2} \\
 &\quad - \frac{i}{2\pi} \int d\omega \sum_{s,m} \int_0^\infty \frac{dk}{4\pi \omega_{ksm}} \frac{e^{i(\omega + \omega_{ksm})(t-t')}}{\omega + i\epsilon} \\
 &\quad \times \left[ \left( M_{ksm} e^{iq(k)z'} + N_{ksm} e^{-iq(k)z'} \right) \right. \\
 &\quad \times \left( M_{ksm}^* e^{-iq(k)z} + N_{ksm}^* e^{iq(k)z} \right) \\
 &\quad \left. + \left( M_{ksm} e^{-iq(k)z'} + N_{ksm} e^{iq(k)z'} \right) \right. \\
 &\quad \left. \times \left( M_{ksm}^* e^{iq(k)z} + N_{ksm}^* e^{-iq(k)z} \right) \right] \\
 &\quad \times \frac{g^2 \omega_{ksm}^2 \kappa_{sm}^2 |J_m(\rho)|^2}{(\omega_{ksm}^2 - \omega_0^2)^2} \\
 &= \frac{i}{2\pi} \int d\bar{\omega} \sum_{s,m} \int_0^\infty \frac{dk}{4\pi \omega_{ksm}} \\
 &\quad \times \left[ 2 \left( |M_{ksm}|^2 + |N_{ksm}|^2 \right) \cos(q(k)(z - z')) \right. \\
 &\quad \left. + 2\text{Re}(M_{ksm} N_{ksm}^*) \cos(q(k)(z + z')) \right] \\
 &\quad \times \frac{g^2 \omega_{ksm}^2 \kappa_{sm}^2 |J_m(\rho)|^2}{(\omega_{kms}^2 - \omega_0^2)^2} e^{-i\bar{\omega}(t-t')}
 \end{aligned}$$

$$\times \left( \frac{1}{\tilde{\omega} - \omega_{kms} + i\epsilon} - \frac{1}{\tilde{\omega} + \omega_{kms} - i\epsilon} \right). \tag{4.2}$$

In the second step we used the integral representation  $\theta(\tau) = \frac{i}{2\pi} \int d\omega \frac{e^{-i\omega\tau}}{\omega + i\epsilon}$  for the Heaviside  $\theta$ -function, and in the third step we have performed the change of variables  $\tilde{\omega} = \omega + \omega_{kms}$  in the first integral, and  $\tilde{\omega} = -\omega - \omega_{kms}$  in the second one. Notice that the two-point function (4.2) can be written as the sum of two functions

$$G_{\psi\psi}^0(x, x') = G_1(t - t', \rho, \theta, z - z') + G_2(t - t', \rho, \theta, z + z'). \tag{4.3}$$

This makes evident the breaking of translation invariance along the  $z$  axis. While  $G_1$  is translationally invariant,  $G_2$  can be interpreted as depending on the reflections at the boundaries of the dielectric region.

Let us focus first on the  $G_1$  part. By performing a change of variable  $k = k_a(q)$ , where the subscript  $a = \pm$  denotes the two branches of the dispersion relation, we find

$$G_1(t - t', \rho, \theta, z - z') = \sum_{s,m,a} \int \frac{d\omega}{2\pi} \int_0^\infty dq \frac{d\omega}{2\pi DR'_a(q)} e^{-i\tilde{\omega}(t-t')} \times \frac{g^2 \omega_{asm}^2(q) \kappa_{sm}^2 |J_m(\rho)|^2}{(\omega_{asm}^2(q) - \omega_0^2)^2} \times \frac{2\omega_{asm}(q)}{\omega^2 - \omega_{asm}^2(q) + i\epsilon} \times (|M_{asm}(q)|^2 + |N_{asm}(q)|^2) \times (e^{iq(z-z')} + e^{-iq(z-z')}), \tag{4.4}$$

where

$$\omega_{asm}(q) := \omega_{k_a(q)sm}, \tag{4.5}$$

$$M_{asm}(q) := M_{k_a(q)sm}, \tag{4.6}$$

$$DR'_a(q) = 2\omega_{asm}(q) \left( 1 + \frac{g^2 \omega_0^2}{(\omega_{asm}^2(q) - \omega_0^2)^2} \right). \tag{4.7}$$

Notice that  $\omega_{asm}(q) \equiv \omega_a(\sqrt{q^2 + k_\rho^2})$ , where  $\omega_a(\cdot)$  denotes the two solutions of the dispersion relation as in [18]. After defining  $k_a(-q) := -k_a(q)$ , we can rewrite the integral in  $q$  over the whole  $(-\infty, +\infty)$  range. Performing the sum over  $a$  explicitly and extending and noting that  $M_{asm}(-q) =$

$M_{asm}^*(q)$ , we obtain the final expression

$$G_1(t - t', \rho, \theta, z - z') = \sum_{s,m} \int \frac{d\omega}{2\pi} \frac{dq}{2\pi} \kappa_{sm}^2 |J_m(\rho)|^2 e^{-i\omega(t-t') + iq(z-z')} \times \frac{\omega^2 - q^2 - k_\rho^2}{(\omega^2 - q^2 - k_\rho^2)(\omega^2 - \omega_0^2) - g^2 \omega^2} \times \frac{1}{(\omega^2 - q^2 - k_\rho^2)(\omega_{+sm}^2(q) - \omega_{-sm}^2(q))} \times [(\omega^2 - \omega_{-sm}^2)(\omega_{+sm}^2 - q^2 - k_\rho^2) \times (|M_{+sm}(q)|^2 + |N_{+sm}(q)|^2) - (\omega^2 - \omega_{+sm}^2) \times (\omega_{-sm}^2 - q^2 - k_\rho^2)(|M_{-sm}(q)|^2 + |N_{-sm}(q)|^2)] =: \sum_{s,m} \int \frac{d\omega}{2\pi} \frac{dq}{2\pi} \kappa_{sm}^2 |J_m(\rho)|^2 e^{-i\omega(t-t') + iq(z-z')} \times D(\omega, q, s, m) \ell_1(\omega, q, s, m), \tag{4.8}$$

where  $D(\omega, q, s, m)$  is the factor in the first line, which equals the free propagator  $G_{\psi\psi}^0$  computed in [18] for the case of the infinite dielectric medium. Also we defined  $\ell_1(\omega, q, s, m)$ , which includes the corrections due to the reflections and transmissions appearing in the finite dielectric. Notice that in the case  $|M_q|^2 + |N_q|^2 = 1$  we have  $\ell_1(\omega, q) = 1$ . From the expression (4.8) we can read the Fourier transform of the  $G_1$  part of the propagator. It has the same poles as the ones in the infinite dielectric case, corresponding to the Sellmeier dispersion relation.

It can be of interest to understand the asymptotic behaviour of the factor  $\ell_1$  for large momenta. For the  $|M_{asm}(q)|^2 + |N_{asm}(q)|^2$  factor, we have

$$\frac{|M_{asm}(q)|^2 + |N_{asm}(q)|^2}{(k_a^2(q) + q^2)^2 \sin^2(2qL) + 4q^2 k_a^2(q) \cos^2(2qL)} = \frac{k_a^2(q) (k_a^2(q) + q^2)}{(k_a^2(q) + q^2)^2 \sin^2(2qL) + 4q^2 k_a^2(q) \cos^2(2qL)}. \tag{4.9}$$

In the case  $a = +$ , we have  $k_a(q) \sim q$  for  $q \gg \omega_0$ . Therefore, in the limit  $q \rightarrow \infty$  the whole factor (4.9) tends to  $1/2$ . In the case  $a = -$ , instead,  $\lim_{q \rightarrow \infty} k_a(q) = \omega_0$ , and Eq. (4.9) has a point dependent limit. Indeed, we can take a succession  $q_n$  such that  $\sin(2q_n L) = 0$ , in such a way that for  $n \rightarrow \infty$  equation (4.9) tends to  $1/4$ . For all other successions such that  $\sin(2q_n L) = C \neq 0$ , we have

$$|M_{-sm}(q)|^2 + |N_{-sm}(q)|^2 \sim \frac{\omega_0^2}{q^2 \sin^2 C} \rightarrow 0.$$

Therefore, we see that the Fourier transform of the propagator (4.8) is  $\sim q^{-2}$ , with the exception of small neighbourhoods of the points  $q_n = n\pi/(2L)$ , where sharp peaks of height  $1/4$  appear.

For the  $G_2$  part of the propagator, with similar manipulations, we obtain

$$\begin{aligned}
 &G_2(t-t', \rho, \theta, z+z') \\
 &= \sum_{s,m} \int \frac{d\omega}{2\pi} \frac{dq}{2\pi} \kappa_{sm}^2 |J_m(\rho)|^2 e^{-i\omega(t-t') + iq(z+z')} \\
 &\quad \times \frac{\omega^2 - q^2 - k_\rho^2}{(\omega^2 - q^2 - k_\rho^2)(\omega^2 - \omega_0^2) - g^2\omega^2} \\
 &\quad \times \frac{(\omega^2 - \omega_{-sm}^2)(\omega_{+sm}^2 - q^2 - k_\rho^2) \text{Re}(M_{+sm}(q)N_{+sm}(q)^*) - (\omega^2 - \omega_{+sm}^2)(\omega_{-sm}^2 - q^2 - k_\rho^2) \text{Re}(M_{-sm}(q)N_{-sm}(q)^*)}{(\omega^2 - q^2 - k_\rho^2)(\omega_{+sm}^2(q) - \omega_{-sm}^2(q))} \\
 &=: \sum_{s,m} \int \frac{d\omega}{2\pi} \frac{dq}{2\pi} \kappa_{ms}^2 |J_m(\rho)|^2 e^{-i\omega(t-t') + iq(z+z')} D(\omega, q, s, m) \ell_2(\omega, q, s, m)
 \end{aligned} \tag{4.10}$$

The factors  $M_{asm}(q)N_{asm}(q)^*$  have a very similar behaviour as the factors  $|M_{asm}(q)|^2 + |N_{asm}(q)|^2$  studied before, so also the function  $G_2$  is vanishing for large  $q$ .

The final expression of the propagator, in the general case  $\rho' \neq \rho, \theta' \neq \theta$ , is therefore:

$$\begin{aligned}
 &G_{\psi\psi}^0(t-t', \rho, \rho', \theta - \theta', z, z') \chi_{[-L,L]}(z) \chi_{[-L,L]}(z') \\
 &= \sum_{s,m} \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} e^{-i\omega(t-t')} J_{|m|,s}(\rho) J_{|m|,s}(\rho') e^{im(\theta-\theta')} \\
 &\quad \times \kappa_{sm}^2 D(\omega, q, s, m) \left[ \ell_1(\omega, q, s, m) e^{iq(z-z')} \right. \\
 &\quad \left. \times + \ell_2(\omega, q, s, m) e^{iq(z+z')} \right].
 \end{aligned} \tag{4.11}$$

where  $\tau = t-t', \Theta = \theta - \theta', \xi_{\mp} = z \mp z'$ . The matrices  $\mathcal{G}_a, a = 1, 2$ , result as the Fourier transform in  $\tau$  and  $\xi_{\mp}$  of

$$\begin{aligned}
 &\tilde{\mathcal{G}}_a(\omega, \rho, \rho', \Theta, k) \\
 &= \sum_{s,m} \kappa_{sm}^2 J_{|m|,s}(\rho) J_{|m|,s}(\rho') e^{im\Theta} \\
 &\quad \times \frac{1}{(\omega^2 - k^2 - k_\rho^2)(\omega^2 - \omega_0^2) - g^2\omega^2} \mathcal{M}_a,
 \end{aligned} \tag{4.13}$$

The matrix  $\mathcal{M}_1$  has the following components (here we omit the obvious dependences on  $m, s$  and  $q$ ):

$$\mathcal{M}_1^{\phi\phi} = \frac{(\omega^2 - \omega_-^2)(\omega_+^2 - \omega_0^2)(|M_+|^2 + |N_+|^2) - (\omega^2 - \omega_+^2)(\omega_-^2 - \omega_0^2)(|M_-|^2 + |N_-|^2)}{(\omega_+^2 - \omega_-^2)}, \tag{4.14}$$

$$\mathcal{M}_1^{\phi\psi} = \left(\mathcal{M}_1^{\psi\phi}\right)^* = -ig\omega \frac{(\omega^2 - \omega_-^2)(|M_+|^2 + |N_+|^2) - (\omega^2 - \omega_+^2)(|M_-|^2 + |N_-|^2)}{(\omega_+^2 - \omega_-^2)}, \tag{4.15}$$

$$\mathcal{M}_1^{\psi\psi} = \frac{(\omega^2 - \omega_-^2)(\omega_+^2 - q^2 - k_\rho^2)(|M_+|^2 + |N_+|^2) - (\omega^2 - \omega_+^2)(\omega_-^2 - q^2 - k_\rho^2)(|M_-|^2 + |N_-|^2)}{(\omega_+^2 - \omega_-^2)}. \tag{4.16}$$

The remaining components of the propagator,  $G_{\phi\phi}$  and  $G_{\psi\psi}$ , can be derived in a very similar way. We express the result as a matrix propagator

$$\begin{aligned}
 &\begin{pmatrix} G_{\phi\phi}^0 & G_{\phi\psi}^0 \\ G_{\psi\phi}^0 & G_{\psi\psi}^0 \end{pmatrix} \chi_{[-L,L]}(z) \chi_{[-L,L]}(z') \\
 &= \mathcal{G}_1(\tau, \rho, \rho', \Theta, \xi_-) + \mathcal{G}_2(\tau, \rho, \rho', \Theta, \xi_+),
 \end{aligned} \tag{4.12}$$

The matrix  $\mathcal{M}_2$  is obtained from  $\mathcal{M}_1$  by replacing the factors  $(|M_a|^2 + |N_a|^2)$  by  $\text{Re}(M_a N_a^*)$ .

The calculations for the case  $z < -L, z' > L$  and  $z, z' < -L$  are very similar, and actually simpler, to the previous case. The  $\phi$ - $\phi$  propagator in this case is the only non vanishing one, and it is given by

$$\begin{aligned}
 &G_{\phi\phi}^0(t-t', \rho, \theta, z, z') \chi_{[-\infty, -L]}(z) \chi_{[L, +\infty]}(z') \\
 &= \sum_{s,m} \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} e^{-i\omega(t-t')} \kappa_{sm}^2 J_{|m|,s}(\rho) J_{|m|,s}(\rho') e^{im(\theta-\theta')} \\
 &\quad \times \frac{1}{\omega^2 - \omega_{ksm}^2} T_{ksm}^* e^{ik(z-z')}.
 \end{aligned} \tag{4.17}$$



$$\begin{aligned}
 G_{\phi\phi}^0(t-t', \rho, \theta, z, z') & \chi_{[-\infty, -L]}(z) \chi_{[-\infty, -L]}(z') \\
 &= \sum_{s,m} \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} e^{-i\omega(t-t')} \kappa_{sm}^2 J_{|m|,s}(\rho) J_{|m|,s}(\rho') e^{im(\theta-\theta')} \\
 & \times \frac{1}{\omega^2 - \omega_{ksm}^2} \left[ e^{ik(z-z')} + R_{ksm}^* e^{ik(z+z')} \right].
 \end{aligned}
 \tag{4.18}$$

In the second case again we find a dependence on  $z + z'$  which is due to reflections; in the first case this does not happen due to the fact that  $R_{kms} T_{kms}^*$  is purely imaginary, so that  $R_{kms} T_{kms}^* + R_{kms}^* T_{kms} = 0$ . We also notice that for  $T = 1$  and  $R = 0$  we find the usual Feynman propagator for a scalar field.

The propagators in the cases  $z < -L, -L < z' < L$  and  $-L < z < L, z' > L$  have less trivial dependence on  $z$  and  $z'$ . The non vanishing components in these cases are

$$\begin{aligned}
 G_{\phi\phi}^0(t-t', \rho, \theta, z, z') & \chi_{[-\infty, -L]}(z) \chi_{[-L, L]}(z') \\
 &= \sum_{s,m} \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} e^{-i\omega(t-t')} \\
 & \times \frac{\kappa_{sm}^2 J_{|m|,s}(\rho) J_{|m|,s}(\rho') e^{im(\theta-\theta')}}{\omega^2 - \omega_{ksm}^2} \\
 & \times \left[ M_{ksm}^* e^{ikz-iq(k)z'} + N_{ksm}^* e^{ikz+iq(k)z'} \right],
 \end{aligned}
 \tag{4.19}$$

$$\begin{aligned}
 G_{\phi\psi}^0(t-t', \rho, \theta, z, z') & \chi_{[-\infty, -L]}(z) \chi_{[-L, L]}(z') \\
 &= \sum_{s,m} \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} e^{-i\omega(t-t')} \frac{\kappa_{sm}^2 J_{|m|,s}(\rho) J_{|m|,s}(\rho') e^{im(\theta-\theta')}}{\omega^2 - \omega_{ksm}^2} \\
 & \times \frac{-ig\omega}{\omega_{ksm}^2 - \omega_0^2} \left[ M_{ksm}^* e^{ikz-iq(k)z'} + N_{ksm}^* e^{ikz+iq(k)z'} \right],
 \end{aligned}
 \tag{4.20}$$

$$\begin{aligned}
 G_{\psi\phi}^0(t-t', \rho, \theta, z, z') & \chi_{[-\infty, -L]}(z) \chi_{[-L, L]}(z') \\
 &= \sum_{s,m} \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} e^{-i\omega(t-t')} \frac{\kappa_{sm}^2 J_{|m|,s}(\rho) J_{|m|,s}(\rho') e^{im(\theta-\theta')}}{\omega^2 - \omega_{ksm}^2} \\
 & \times \frac{ig\omega}{\omega_{ksm}^2 - \omega_0^2} \left[ M_{ksm}^* e^{ikz-iq(k)z'} + N_{ksm}^* e^{ikz+iq(k)z'} \right],
 \end{aligned}
 \tag{4.21}$$

where the factor in square brackets is the same for all three components. These expressions are obtained by making use of the following equalities, that are easily checked:

$$M_{kms} = R_{kms} N_{kms}^* + T_{kms} M_{kms}^*, \tag{4.22}$$

$$N_{kms} = R_{kms} M_{kms}^* + T_{kms} N_{kms}^*. \tag{4.23}$$

### 5 Solitonic solutions

In this section we introduce a nonlinearity in the model, with the aim of describing the perturbation of refractive index propagating in the nonlinear dielectric medium when a strong laser pulse is shot into the dielectric and the Kerr effect is

stimulated. The propagating perturbation breaks the homogeneity of the dielectric sample described in the previous section. Still, the solutions for the homogeneous case represent a good asymptotic scattering basis for the full nonlinear problem in the linearisation of the theory around the dielectric perturbation represented by the solitonic solutions we are going to describe.

With this aim, we add a fourth order term in the polarization field  $\psi$ , as in [21]. A fourth order term in nonlinear optical media appears also e.g. in [22], where a fourth order term in the displacement field can be introduced in the case of an optical fiber. We have discussed a fourth order term in the polarization field for the Hopfield model in [11], where we have shown that our solitonic solutions can be associated with the Kerr effect in a proper way. It is also to be remarked that our approach does not represent the standard way to approach the Kerr effect (see also [23,24]), and that our solitonic solutions are more constrained than the usual solutions of the nonlinear Schrödinger equation studied in [24]. See also the discussion in [25].

Our non-linear theory has the following action

$$\begin{aligned}
 S[\phi, \psi] &= \int_C \frac{1}{2} \partial_\mu \phi \partial^\mu \phi d^4x \\
 &+ \int_{C_x} \left[ \frac{1}{2} (v^\mu \partial_\mu \psi)^2 \right. \\
 &\left. - \frac{\omega_0^2}{2} \psi^2 - g\phi v^\mu \partial_\mu \psi - \frac{\lambda}{4!} \psi^4 \right] d^4x;
 \end{aligned}
 \tag{5.1}$$

the equations of motion are

$$\square\phi + g v^\mu \partial_\mu \psi = 0, \tag{5.2}$$

$$\begin{aligned}
 (v^\mu \partial_\mu)^2 \psi + \omega_0^2 \psi \\
 - g v^\mu \partial_\mu \phi + \frac{\lambda}{3!} \psi^3 = 0.
 \end{aligned}
 \tag{5.3}$$

In lab  $v^\mu = (1, 0, 0, 0)$ , so the equations become

$$\square\phi + g \dot{\psi} = 0, \tag{5.4}$$

$$\ddot{\psi} + \omega_0^2 \psi - g \dot{\phi} + \frac{\lambda}{3!} \psi^3 = 0. \tag{5.5}$$

We are not looking for the general solution of the system above, still we are interested in finding out analytical solutions. We attempt the ansatz

$$\phi(t, z, \rho, \theta) = f(z - Vt)Y(\rho, \theta), \tag{5.6}$$

$$\psi(t, z, \rho, \theta) = h(z - Vt)Y(\rho, \theta); \tag{5.7}$$

the radial and angular parts can be separated if  $Y = \text{const}$ <sup>1</sup> to obtain the equations

$$(1 - V^2)f'' + gVh' = 0, \tag{5.8}$$

$$V^2h'' + gVf' + \omega_0^2h + \frac{Y^2\lambda}{3!}h^3 = 0, \tag{5.9}$$

where the prime stands for the derivative with respect to the argument  $z - Vt$ .

Integrating the first equation and inserting in the second, after a new integration we obtain

$$\frac{Vh'}{i\sqrt{\frac{Y^2\lambda}{12}h^4 + (\omega_0^2 - g^2V^2\gamma^2)h^2 + 2g\kappa Vh - 2\chi}} = 1, \tag{5.10}$$

where  $\kappa$  and  $\chi$  are constants of integration, and  $\gamma = (1 - V^2)^{-\frac{1}{2}}$ . We will also use the notation

$$v := \gamma V. \tag{5.11}$$

It is worth to mention here that we accept solution having finite energy. It is immediate to see that the energy density inside the dielectric is

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{2}\dot{\psi}^2 \\ &+ \frac{1}{2}\omega_0^2\psi^2 + \frac{\lambda}{4!}\psi^4 = Y^2 \\ &(\chi + (gV\gamma^2h(z - Vt) - \kappa)^2 - (1 - V^2)\kappa). \end{aligned} \tag{5.12}$$

Thus, the energy is finite if  $h$  has no poles. Notice that, in particular,  $h$  has to be limited, which implies that the quartic radicand in (5.10) must have real roots. So, the constants must be constrained in order to ensure this condition.

### 5.1 $\kappa = \chi = 0$

In this case, the solution inside the dielectric is the solitonic solution obtained in [21]. With  $a = \sqrt{\frac{12}{\lambda Y^2}(g^2v^2 - \omega_0^2)}$  and  $b = \frac{1}{v}\sqrt{g^2v^2 - \omega_0^2}$ , we find (for  $-L \leq z \leq L$ )

$$h = \frac{a}{\cosh(b\gamma(z - Vt))}, \tag{5.13}$$

$$f = \frac{2agv}{b} \arctan\left[\tanh\left(\frac{b}{2}\gamma(z - Vt)\right)\right], \tag{5.14}$$

while the solution in vacuum is a superposition of a progressive and a regressive wave, whose form is determined by the continuity of  $\phi$  and  $\partial_z\phi$ , as discussed in Sect. 2.1:

$$\partial_z f =$$

<sup>1</sup> This would be equivalent to a calculation involving only  $s$ -waves in presence of spherical symmetry.

$$\begin{cases} \frac{1+V}{2} \frac{ag\gamma v}{\cosh[b\gamma(L(1-V)-V(z-t))]} + \frac{1-V}{2} \frac{ag\gamma v}{\cosh[b\gamma(L(1+V)+V(z+t))]}, & z \leq -L \\ \frac{1+V}{2} \frac{ag\gamma v}{\cosh[b\gamma(L(1-V)+V(z-t))]} + \frac{1-V}{2} \frac{ag\gamma v}{\cosh[b\gamma(L(1+V)-V(z+t))]}, & z \geq L \end{cases}. \tag{5.15}$$

Note that the solution exists only if  $v > \omega_0/g$ , that is

$$V^2 > \frac{\omega_0^2}{g^2 + \omega_0^2}. \tag{5.16}$$

### 5.2 $\kappa \neq 0, \chi \neq 0$

We look for a Möbius transformation

$$h = \frac{as + b}{cs + d}, \tag{5.17}$$

which maps (5.10) into the form

$$\frac{i}{V} = \frac{s'}{\sqrt{4s^3 - g_2s - g_3}}. \tag{5.18}$$

The assumption that the original quartic equation has at least one real root ensures that such a transformation exists with real coefficients  $a, b, c, d$  that can be chosen to satisfy  $ad - bc = \pm 1$ , see Appendix B. Equation (5.18) has general solution

$$\begin{aligned} s(x) &= \wp(g_2, g_3; i(x - x_0)/V) \\ &= -\wp(g_2, -g_3; (x - x_0)/V), \end{aligned} \tag{5.19}$$

where  $g_2, g_3$  are defined in Appendix B and  $\wp$  is the Weierstrass elliptic function defined by

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} \\ &+ \sum_{(n,m) \in \mathbb{Z}^2 - \{0,0\}} \left( \frac{1}{(z + n\omega_1 + m\omega_2)^2} - \frac{1}{(n\omega_1 + m\omega_2)^2} \right) \end{aligned} \tag{5.20}$$

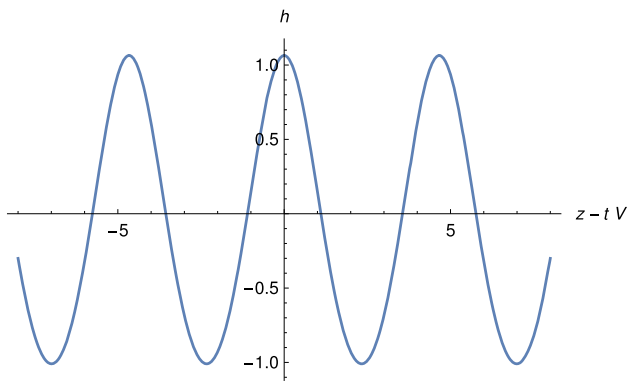
with  $\omega_1, \omega_2$  the two periods satisfying  $\tau := \omega_2/\omega_1 \notin \mathbb{R}$ , and  $x_0$  is an integration constant. Indeed, we can be more precise and notice that there are two distinct situations, when all three roots of  $4z^3 - g_2z - g_3$  are distinct. In our case,  $g_2$  and  $g_3$  are real and so we may have three real roots  $e_3 < e_2 < e_1$ , or one real root  $e_2$  and two complex roots  $e_1, e_3$ , with  $e_1 = \bar{e}_3$ . Let us shortly discuss the two cases.

#### 5.2.1 3 Real roots

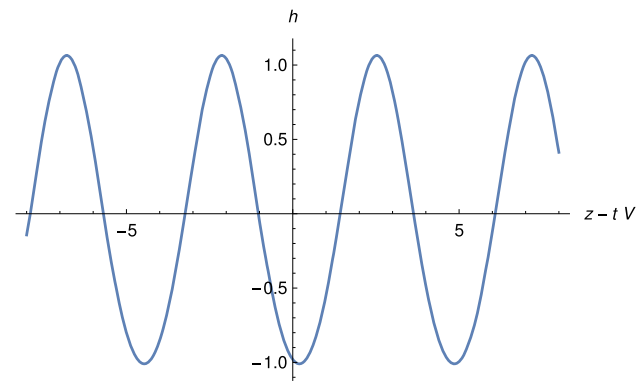
$e_3 < e_2 < e_1$ . The periods of  $\wp(g_2, -g_3, z)$  are

$$\omega_1 = 2 \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z + g_3}} \in \mathbb{R}_{>}, \tag{5.21}$$

$$\omega_2 = 2i \int_{-\infty}^{e_3} \frac{dz}{\sqrt{-4z^3 + g_2z - g_3}} \in i\mathbb{R}_{>}, \tag{5.22}$$



**Fig. 2** Plot of the solution (5.23) corresponding to the three real root case. The parameters have the following values:  $Y = \omega_0 = g = 1$ ,  $\lambda = V = 0.5$ ,  $\chi = 2$ ,  $\kappa = 3$ ,  $z_0 = 0$



**Fig. 3** Plot of the solution (5.25) corresponding to the three real root case. The parameters have the following values:  $Y = \omega_0 = g = 1$ ,  $\lambda = V = 0.5$ ,  $\chi = 2$ ,  $\kappa = 3$ ,  $z_0 = 0$

and we get two distinct solutions for  $h$ . The first one is

$$h(z - Vt) = \frac{a\wp(g_2, -g_3; t - (z - z_0)/V) - b}{c\wp(g_2, -g_3; t - (z - z_0)/V) - d}. \quad (5.23)$$

In this case  $\wp$  assumes all values in  $[e_1, \infty)$ . Since we are interested in solutions with  $h$  of class  $C^2$  everywhere inside the dielectric, we must discard solutions such that the denominator above vanishes somewhere. This happens only if the condition

$$ce_1 - d > 0 \quad (5.24)$$

is satisfied. Analysing this condition in general depends on several details. A partial analysis can be found in appendix B. This solutions represent trains of pulses having period  $\omega_1$ , and moving with constant velocity as we can see from Fig. 2.

The second solution is

$$h(z - Vt) = \frac{a\wp(g_2, -g_3; t - (z - z_0)/V + \frac{\omega_2}{2}) - b}{c\wp(g_2, -g_3; t - (z - z_0)/V + \frac{\omega_2}{2}) - d} \quad (5.25)$$

In this case  $\wp$  oscillates in the interval  $[e_3, e_2]$ , which is acceptable if the condition

$$\frac{d}{c} \notin [e_3, e_2] \quad (5.26)$$

is satisfied. Again, we get a train of pulses which is shifted along the horizontal axis, as shown in Fig. 3.

### 5.2.2 1 Real root

$e_2 \in \mathbb{R}$ . In this case we have two complex conjugate periods  $\omega_1 = \omega$  and  $\omega_2 = \bar{\omega}$  with

$$\omega = \int_{e_2}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z + g_3}} + i \int_{-\infty}^{e_2} \frac{dz}{\sqrt{-4z^3 + g_2z - g_3}}. \quad (5.27)$$

In this case there is only one kind of solutions, having the form (5.23) with condition (5.24). This represents a train of pulses having period  $\omega + \bar{\omega}$ , like for example in Fig. 4.

### 5.2.3 Elementary solutions

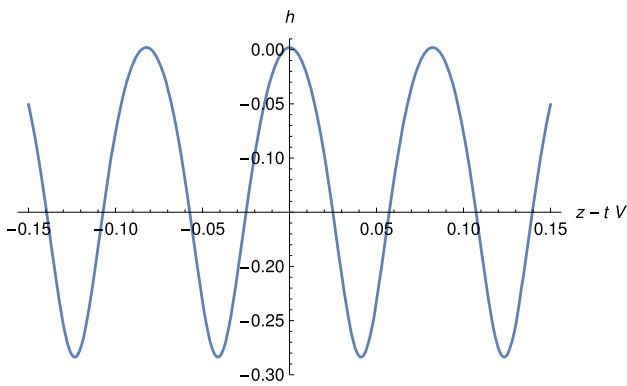
These solutions correspond to the cases of degenerate roots and could be directly deduced as particular cases of the Weierstrass cases. However, since classifying all possible Weierstrass configurations is quite cumbersome, as shown in Appendix B, it is easier to construct them directly. In order to obtain a solution expressible in an algebraic form, we set equal to zero the discriminant of the fourth degree polynomial at the denominator of (5.10), and resolve it for the constant  $\lambda$  (here  $\Omega := \omega_0^2 - g^2v^2$ ):

$$Y^2\lambda_{\pm} = -\frac{3}{64\chi^3} \left( 27g^4\kappa^4V^4 + 72g^2\kappa^2V^2\chi\Omega + 32\chi^2\Omega^2 \pm \sqrt{g^2\kappa^2V^2(9g^2\kappa^2V^2 + 16\chi\Omega^3)} \right); \quad (5.28)$$

$\lambda_{\pm}$  is real if and only if  $\Omega\chi \geq -9g^2\kappa^2V^2/16$ . Recall that at least one between  $\lambda_+$  and  $\lambda_-$  must be positive in order to be consistent with our physical assumptions.

On the other hand, with the aid of the general theory of quartic equations [27], we discover the nature of the roots of our polynomial, depending of how the parameters change. It holds (one must choose the positive one between  $\lambda_+$  and  $\lambda_-$ ):

- if  $\Omega < 0$  and  $-\frac{3\Omega^2}{2\lambda_{\pm}} < \chi < \frac{\Omega^2}{2\lambda_{\pm}}$ , then we have four real roots, of which two are double;
- if  $\Omega < 0$  and  $\chi < -\frac{3\Omega^2}{2\lambda_{\pm}}$ , there are a double real root and two complex conjugate roots;
- if  $\Omega > 0$  and  $\chi < 0$ , then there are again a double real root and two complex conjugate roots.



**Fig. 4** Plot of the solution (5.23) corresponding to the one real root case. The parameters have the following values:  $Y = 10^3$ ,  $\omega_0 = g = 1$ ,  $\lambda = V = 0.5$ ,  $\chi = 2$ ,  $\kappa = 10^3$ ,  $z_0 = 0$

This is what the theory tells us. We can say something more: from the equation  $\frac{\lambda}{12}h^4 + \Omega h^2 + 2g\kappa Vh - 2\chi = 0$ , we see that if  $\chi > 0$ , it can be chosen  $h$  such that the polynomial is zero. So, we conclude that for every  $\chi > 0$  the equation has two double real roots.

Note that not all of these conditions are compatible with our choice of  $\lambda_{\pm}$ . Consider, for example,  $\lambda_+$ : if we want to guarantee the positivity of  $\lambda_+$ , then the combination  $\Omega > 0$ ,  $\chi > 0$  is not acceptable, while the other combinations allow  $\lambda_+ > 0$ . Moreover, the reality condition  $\chi\Omega \geq -9g^2\kappa^2V^2/16$  provide further constraints. Independently by the value of the parameters, the general solution of (5.10) can be easily written: if  $\alpha$  is the double real root, and  $\beta, \delta$  are real (or complex), then, recalling that in our case  $\delta + \beta = -2\alpha$ ,

$$h = \left[ \frac{\beta - \delta}{2(\alpha - \beta)(\alpha - \delta)} \cos\left(\sqrt{\frac{\lambda(\alpha - \beta)(\alpha - \delta)}{12}} \frac{z - Vt}{V}\right) - \frac{2\alpha}{(\alpha - \beta)(\alpha - \delta)} \right]^{-1} + \alpha. \tag{5.29}$$

The term  $(\beta - \delta)$  shows that if  $\beta$  and  $\delta$  are not real, then the final solution is not real too and must be excluded. So, the parameters must be such that the roots are all real. In this case

$$(\alpha - \beta)(\alpha - \delta) = 4\alpha^2 - (\alpha + \beta)^2 = 4\alpha^2 - (\alpha + \delta)^2 \tag{5.30}$$

shows that we may have both signs for  $(\alpha - \beta)(\alpha - \delta)$ , so that we have

$$h = \left[ \frac{\beta - \delta}{2(\alpha - \beta)(\alpha - \delta)} \cos\left(\sqrt{\frac{\lambda(\alpha - \beta)(\alpha - \delta)}{12}} \frac{z - Vt}{V}\right) - \frac{2\alpha}{(\alpha - \beta)(\alpha - \delta)} \right]^{-1} + \alpha, \text{ if } (\alpha - \beta)(\alpha - \delta) > 0, \tag{5.31}$$

$$h = \left[ \frac{\beta - \delta}{2(\alpha - \beta)(\alpha - \delta)} \cosh\left(\sqrt{\frac{\lambda(\alpha - \beta)(\alpha - \delta)}{12}} \frac{z - Vt}{V}\right) - \frac{2\alpha}{(\beta - \alpha)(\alpha - \delta)} \right]^{-1} + \alpha, \text{ if } (\alpha - \beta)(\alpha - \delta) < 0, \tag{5.32}$$

The last case includes the simplest situation  $k = 0$ ,  $\chi = 0$  studied above. The first case, instead, includes the simple case studied below.

### 5.3 $\kappa \neq 0, \chi = 0$

As a particular subcase, we can study Eq. (5.10) with  $\chi = 0$ ; here the analysis is simplified, because we have to study a cubic equation. Requiring the discriminant of the cubic to be zero, we find

$$\lambda = -\frac{4\Omega^3}{9g^2\kappa^2V^2}, \tag{5.33}$$

so the positivity of  $\lambda$  requires  $\Omega < 0$ , that is  $v > \omega_0/g$  as usual.

With this value for  $\lambda$  there are three real roots, of which two are double. The double root is  $-3g\kappa V/\Omega$ , and the simple root is  $6g\kappa V/\Omega$ . The solution of (5.10) is

$$h = -\frac{3g\kappa V}{\omega_0^2 - g^2v^2} \left[ \frac{3}{\cos\left(\frac{3g\kappa\sqrt{\lambda}}{2(\omega_0^2 - g^2v^2)}(z - Vt)\right) - 2} + 1 \right]. \tag{5.34}$$

## 6 Conclusions

We have studied, in the simplified framework of the  $\phi\psi$ -model, the propagation of the electromagnetic field in a spatially finite sample of dielectric medium. This situation is physically relevant in the Analogue Gravity picture for the Hawking effect, as experiments involve necessarily finite samples of dielectrics. We have chosen to work in a cylindrical geometry, where the dielectric field fills only a finite cylindrical region of length  $2L$  and radius  $R$ . The remaining region of radius  $R$  is filled by vacuum. This may be considered as a model for a optic fiber, which are an active benchmark for experiments in Analogue Gravity [17].

Our present study concerns analytical properties of the solutions for the equations of motion of the involved fields. As a preliminary analysis, we have considered the boundary conditions to be imposed on the fields, together with a complete scattering basis and the quantization of the fields in the case of a still homogeneous dielectric sample. We have also described the propagator for the fields in the given setting.

Then we have introduced a nonlinearity in the model, as the dielectric media we are interested in must be associated

with the Kerr effect. Indeed, as pointed out firstly in [1], a possibility to obtain analogous black hole in dielectrics consists in generating strong laser pulses which propagate inside a nonlinear dielectric medium. The Kerr effect gives rise to a propagating perturbation of the refractive index which plays the role of the analogous black hole, and is indeed involved with a horizon. Our interest has been to find out solitonic solutions describing the aforementioned perturbation, i.e. the background solutions around which a linearization is performed, and the perturbations are quantized. We have shown that solitonic solutions exist, representing a dielectric perturbation travelling with constant velocity in the direction of the cylindrical fiber axis.

Further developments can involve different aspects, all with a noticeable physical interest. One may study perturbation theory for the model, looking for quantum effects induced by surface effects, e.g. transition radiation [26]. Also, one can study absorption in the model, associated with the fourth order perturbation. Our main research focus, which is represented by the analogous Hawking effect, requires the analysis of the linearization of the model around the solitonic solution and its quantization. One may also limit to consider, in the comoving frame of the dielectric perturbation, the dependence of the dielectric susceptibility and of the resonance frequency on space (induced by the Kerr effect), and analyze the Hawking effect with simpler background profiles. Future works will be devoted to the aforementioned goals.

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Funded by SCOAP<sup>3</sup>.

### Appendix A: Orthogonality relations

For

$$f = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad \tilde{f} = \begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \end{pmatrix}, \tag{A.1}$$

we define

$$j_{f, \tilde{f}}^\mu(\mathbf{x}) := i[\phi^*(\mathbf{x})\partial^\mu\tilde{\phi}(\mathbf{x}) - \tilde{\phi}(\mathbf{x})\partial^\mu\phi^*(\mathbf{x}) + v^\mu(\psi^*(\mathbf{x})\dot{\tilde{\psi}}(\mathbf{x}) - \tilde{\psi}(\mathbf{x})\dot{\psi}^*(\mathbf{x})) + gv^\mu(\psi^*(\mathbf{x})\tilde{\phi}(\mathbf{x}) - \phi^*(\mathbf{x})\tilde{\psi}(\mathbf{x}))]. \tag{A.2}$$

It is a conserved current

$$\partial_\mu j_{f, \tilde{f}}^\mu(\mathbf{x}) = 0, \tag{A.3}$$

and the scalar product is

$$(f|\tilde{f}) = \int_{C_t} j_{f, \tilde{f}}^0(\mathbf{x}), \tag{A.4}$$

where  $C_t$  is the slice of  $C$  obtained by fixing  $t$ .

The conservation law is particularly helpful for computing the scalar product among plane wave solutions or scattering wave solutions. These solutions have the form

$$\phi(\mathbf{x}) = e^{-i\omega_{ksm}t} \varphi_{ksm}(\mathbf{x}), \tag{A.5}$$

so if we take

$$f(\mathbf{x}) = e^{-i\omega_{ksm}t} \begin{pmatrix} \varphi_{ksm}(\mathbf{x}) \\ Q_{ksm}(\mathbf{x}) \end{pmatrix},$$

$$\tilde{f}(\mathbf{x}) = e^{-i\omega_{k's'm'}t} \begin{pmatrix} \varphi_{k's'm'}(\mathbf{x}) \\ Q_{k's'm'}(\mathbf{x}) \end{pmatrix}, \tag{A.6}$$

then

$$\partial_0 j_{f, \tilde{f}}^0(\mathbf{x}) = i(\omega_{ksm} - \omega_{k's'm'}) j_{f, \tilde{f}}^0(\mathbf{x}), \tag{A.7}$$

and integrating over the spatial slice and using the continuity equation we get

$$(f|\tilde{f}) = -\frac{1}{i(\omega_{ksm} - \omega_{k's'm'})} \int_{\partial C_t} \mathbf{j}_{f, \tilde{f}}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d^2\sigma(\mathbf{x}), \tag{A.8}$$

where we have used the continuity of  $\mathbf{j}_{f, \tilde{f}}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$ , as a consequence of the boundary conditions. In order to compute this integral let us restrict it on the compact cylinder

$$C_t^Z = \{(\rho, \theta, z) \in C_t | -Z \leq z \leq Z\}, \tag{A.9}$$

so that

$$(f|\tilde{f}) = -\frac{1}{i(\omega_{ksm} - \omega_{k's'm'})} \lim_{Z \rightarrow +\infty} \int_{\partial C_t^Z} \mathbf{j}_{f, \tilde{f}}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d^2\sigma(\mathbf{x}). \tag{A.10}$$

Finally, by taking into account the boundary condition for the fields we get

$$(f|\tilde{f}) = -\frac{1}{i(\omega_{ksm} - \omega_{k's'm'})} \lim_{Z \rightarrow +\infty} \left( \int_{D_Z} j_{f,\tilde{f}}^z(\mathbf{x}) \rho^2 d\rho d\theta - \int_{D_{-Z}} j_{f,\tilde{f}}^z(\mathbf{x}) \rho^2 d\rho d\theta \right), \tag{A.11}$$

where

$$D_{\pm Z} = \{(\rho, \theta, z) \in C_t | z = \pm Z\}. \tag{A.12}$$

Using

$$\int_0^{2\pi} e^{i(m'-m)\theta} d\theta = 2\pi \delta_{mm'}, \tag{A.13}$$

for two right dielectric scattering functions we get

$$\begin{aligned} \frac{(f|\tilde{f})}{\kappa_{ksm}^* \kappa_{ksm}} &= -\frac{e^{i(\omega_{ksm} - \omega_{k's'm'})t}}{i(\omega_{ksm} - \omega_{k's'm'})} 2\pi \delta_{mm'} \\ &\times \int_0^R J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) J_{|m|}\left(z_{ms'} \frac{\rho}{R}\right) \rho d\rho \\ &\times \lim_{Z \rightarrow +\infty} \left[ (k - k') R_{ksm}^* e^{-i(k+k')Z} \right. \\ &- (k - k') R_{k's'm} e^{i(k+k')Z} + (k + k') \\ &\times \left. \left( T_{ksm}^* T_{k's'm} + R_{ksm}^* R_{k's'm} \right) e^{-(k-k')Z} \right. \\ &\left. - e^{i(k-k')Z} \right]. \end{aligned} \tag{A.14}$$

We first show that

**Lemma 1** *It holds*

$$\begin{aligned} &\int_0^R J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) J_{|m|}\left(z_{ms'} \frac{\rho}{R}\right) \rho d\rho \\ &= \delta_{ss'} \frac{R^2}{2z_{ms}^2} (z_{ms}^2 - m^2) J_m^2(z_{ms}). \end{aligned} \tag{A.15}$$

*Proof* The Bessel equation can be written in the form

$$\begin{aligned} &\frac{d}{d\rho} \left( \rho \frac{d}{d\rho} J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) \right) \\ &+ \left( \rho \frac{z_{ms}^2}{R^2} - \frac{m^2}{\rho} \right) J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) = 0, \end{aligned} \tag{A.16}$$

from which we get

$$\begin{aligned} &\frac{d}{d\rho} \left[ \rho J_{|m|}\left(z_{ms'} \frac{\rho}{R}\right) \frac{d}{d\rho} J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) \right. \\ &- \rho J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) \frac{d}{d\rho} J_{|m|}\left(z_{ms'} \frac{\rho}{R}\right) \left. \right] \\ &+ \frac{z_{ms}^2 - z_{ms'}^2}{R^2} \rho J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) \\ &J_{|m|}\left(z_{ms'} \frac{\rho}{R}\right) = 0, \end{aligned} \tag{A.17}$$

that integrated from 0 to  $R$  in  $d\rho$  gives

$$\int_0^R J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) J_{|m|}\left(z_{ms'} \frac{\rho}{R}\right) \rho d\rho = 0, \tag{A.18}$$

if  $s \neq s'$ . Moreover, from (A.16) we get

$$\begin{aligned} 0 &= \frac{d}{d\rho} \left( \rho J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) \right) \right) \\ &+ \frac{z_{ms}^2}{R^2} 2\rho J_m^2\left(z_{ms} \frac{\rho}{R}\right) \\ &+ \left( \rho^2 \frac{z_{ms}^2}{R^2} - m^2 \right) 2J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) \frac{d}{d\rho} J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) \\ &= \frac{d}{d\rho} \left( \rho J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} J_{|m|}\left(z_{ms'} \frac{\rho}{R}\right) \right) \right) \\ &+ \frac{z_{ms}^2}{R^2} 2\rho J_m^2\left(z_{ms} \frac{\rho}{R}\right) \\ &- \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} J_{|m|}\left(z_{ms} \frac{\rho}{R}\right) \right)^2. \end{aligned} \tag{A.19}$$

After integration in  $d\rho$  from 0 to  $R$ , and using the definition of  $z_{ms}$ , we get

$$\begin{aligned} &2 \frac{z_{ms}^2}{R^2} \int_0^R J_{|m|}^2\left(z_{ms} \frac{\rho}{R}\right) \rho d\rho = -R^2 J_{|m|}^2(z_{ms}) \\ &\frac{d^2}{d\rho^2} J_{|m|}^2\left(z_{ms} \frac{\rho}{R}\right) \Big|_{\rho=R}. \end{aligned} \tag{A.20}$$

Using again the Bessel equation we finally get the assert.  $\square$

Thus

$$\begin{aligned} &\frac{(f|\tilde{f})}{\kappa_{ksm}^* \kappa_{ksm}} \\ &= -\frac{e^{i(\omega_{ksm} - \omega_{k's'm'})t}}{i(\omega_{ksm} - \omega_{k's'm'})} 2\pi \delta_{mm'} \delta_{ss'} \\ &\frac{R^2}{2z_{ms}^2} (z_{ms}^2 - m^2) J_m^2(z_{ms}) \\ &\lim_{Z \rightarrow +\infty} \left[ (k - k') R_{ksm}^* e^{-i(k+k')Z} \right. \\ &- (k - k') R_{k's'm} e^{i(k+k')Z} + (k + k') \\ &\left. \left( (T_{ksm}^* T_{k's'm} + R_{ksm}^* R_{k's'm}) e^{-(k-k')Z} - e^{i(k-k')Z} \right) \right]. \end{aligned} \tag{A.21}$$

In order to compute this limit, we rewrite it in the form

$$\begin{aligned} &\lim_{Z \rightarrow \infty} \frac{k - k'}{\omega_{ksm} - \omega_{k's'm}} \\ &\times \left[ (k + k') \frac{e^{-(k-k')Z} - e^{i(k-k')Z}}{k' - k} \right. \\ &+ \left( R_{ksm}^* e^{-i(k+k')Z} - R_{k's'm} e^{i(k+k')Z} \right) \\ &\left. + (k + k') \frac{T_{ksm}^* T_{k's'm} + R_{ksm}^* R_{k's'm} - 1}{k' - k} e^{-(k-k')Z} \right]. \end{aligned} \tag{A.22}$$

Since  $k$  and  $k'$  are positive, the second term in the square brackets vanishes in the limit because of the Riemann–Lebesgue theorem. The second term vanishes for the same reason unless  $k = k'$ . Since  $|T_{ksm}|^2 + |R_{ksm}|^2 = 1$ , it stays finite for  $k = k'$  (if we take the continuation by the limit  $k' \rightarrow k$ ). Thus, in the distributional sense, it vanishes. So, the surviving limit is

$$\lim_{Z \rightarrow \infty} \frac{k - k'}{\omega_{ksm} - \omega_{k'sm}} \left[ (k + k') \frac{2i \sin((k - k')Z)}{k' - k} \right] = 2\pi k \frac{dk}{d\omega} 2i \delta(k' - k), \tag{A.23}$$

and we finally get

$$(f|\tilde{f}) = |\kappa_{ksm}|^2 4\omega_{ksm} \pi^2 R^2 \times \left( 1 - \frac{m^2}{z_{ms}^2} \right) J_m^2(z_{ms}) \delta_{mm'} \delta_{ss'} \delta(k' - k). \tag{A.24}$$

The same result is true for two left dielectric wave functions.

The same procedure can be used to compute the scalar product between two right (or left) gap wave functions  $g, \tilde{g}$

$$(g|\tilde{g}) = |\tilde{\kappa}_{ksm}|^2 \omega_{ksm} \pi^2 R^2 \times \left( 1 - \frac{m^2}{z_{ms}^2} \right) J_m^2(z_{ms}) \delta_{mm'} \delta_{ss'} \delta(k' - k). \tag{A.25}$$

All other combinations vanish. It is worth to mention that for the particular case  $m = 0$  there is also the zero  $z_{0,0} = 0$ , for which

$$(f|\tilde{f}) = |\kappa_{k00}|^2 4k\pi^2 R^2 \delta(k' - k), \tag{A.26}$$

$$(g|\tilde{g}) = |\tilde{\kappa}_{k00}|^2 k\pi^2 R^2 \delta(k' - k). \tag{A.27}$$

**Appendix B: Study of Eq. (5.10)**

Let us write the quartic as

$$\frac{Y^2 \lambda}{12} h^4 + (\omega_0^2 - g^2 V^2 \gamma^2) h^2 + 2g\kappa V h - 2\chi = p(h), \tag{B.1}$$

with

$$p(x) = \alpha_0 x^4 + \alpha_2 x^2 + \alpha_3 x + \alpha_4 \equiv \alpha_0 (x - E_0)(x - E_1)(x - E_2)(x - E_3), \tag{B.2}$$

where  $E_j$  are the polynomial roots, satisfying  $E_0 + E_1 + E_2 + E_3 = 0$ . We are assuming that there is at least one

real root and so define  $E_0$  to be the largest real root. Let us consider the change of variables

$$h = \frac{as + b}{cs + d}, \tag{B.3}$$

with  $a, b, c, d$  all real. These parameters are defined up to a global real rescaling, which can be fixed so that  $ad - bc = \varepsilon$ , with  $\varepsilon = \pm 1$ . We get

$$h' = \frac{\varepsilon s'}{(cs + d)^2}, \tag{B.4}$$

so that

$$\frac{h'}{\sqrt{p(h)}} = \frac{\varepsilon s'}{q(s)}, \tag{B.5}$$

where

$$q(s) = a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4, \tag{B.6}$$

with

$$a_0 = c^4 p\left(\frac{a}{c}\right), \tag{B.7}$$

which we easily set to zero by imposing

$$a = cE_0. \tag{B.8}$$

With this position, the remaining coefficients are

$$a_1 = c^3 (b - E_0 d) p'(E_0), \tag{B.9}$$

$$a_2 = (b - E_0 d) c^2 d p'(E_0) - (b - E_0 d) \varepsilon \frac{c}{2} p''(E_0), \tag{B.10}$$

$$a_3 = 4cd^3 p\left(\frac{b}{d}\right) + \varepsilon d^2 p'\left(\frac{b}{d}\right), \tag{B.11}$$

$$a_4 = d^4 p\left(\frac{b}{d}\right). \tag{B.12}$$

We have to impose the condition  $a_2 = 0$  and  $a_1 = 4$ . Notice that  $b - E_0 d = -\varepsilon/c$ . Moreover, since we are assuming the roots are generic, therefore all distinct,  $p'(E_0) \neq 0$ . We finally get:

$$a = E_0 c, \quad b = \frac{c}{4} \left( p'(E_0) - \frac{1}{2} E_0 p''(E_0) \right), \\ c = \frac{2}{\sqrt{|p'(E_0)|}} \quad d = -\frac{p''(E_0)}{8} c. \tag{B.13}$$

and  $\varepsilon = -\text{sign}(p'(E_0))$ . With the assumption  $E_0$  real, these coefficients are all real and lead us to Eq. (5.18), with

$$g_2 = 4cd^3 p\left(\frac{b}{d}\right) + \varepsilon d^2 p'\left(\frac{b}{d}\right),$$

$$g_3 = d^4 p\left(\frac{b}{d}\right). \tag{B.14}$$

The conditions leading to this solution are essentially the ones guaranteeing the existence of at least one real solution of  $p(x) = 0$ .

The discriminant of our quartic equation is

$$\Delta = 256\alpha_0^3\alpha_4^3 - 128\alpha_0^2\alpha_2^2\alpha_4^2 + 144\alpha_0^2\alpha_2\alpha_3^2\alpha_4 - 27\alpha_0^2\alpha_3^4 + 16\alpha_0\alpha_2^4\alpha_4 - 4\alpha_0\alpha_2^3\alpha_3^2. \tag{B.15}$$

From [27, Theorem 7], we see that if  $\Delta < 0$  there are always 2 real roots and 2 complex conjugate roots. For  $\Delta = 0$  we boil down to the case of degenerate solutions, studied apart in the main text. Finally, if  $\Delta > 0$  the only case with real roots is when the conditions  $M \equiv \alpha_0\alpha_2 < 0$  and  $N \equiv 4\alpha_0\alpha_4 - \alpha_2^2 < 0$  are satisfied. If the case, then there are four real solutions.

More explicitly, since  $\alpha_0 = \frac{\lambda}{12}Y^2 > 0$ , we have to consider the sign of

$$\tilde{\Delta} \equiv \frac{\Delta}{16\alpha_0} = -128\alpha_0^2\chi^3 - 32\alpha_0\Omega^2\chi^2 - 72\alpha_0\Omega g^2\kappa^2V^2\chi - 27\alpha_0g^4\kappa^4V^4 - 2\Omega^4\chi - g^2\kappa^2V^2\Omega^3, \tag{B.16}$$

where

$$\Omega = \omega_0^2 - g^2V^2\gamma^2. \tag{B.17}$$

Moreover,

$$\frac{M}{\alpha_0} \equiv \alpha_2 = \Omega, \tag{B.18}$$

$$N \equiv 4\alpha_0\alpha_4 - \alpha_2^2 = -8\alpha_0\chi - \Omega^2. \tag{B.19}$$

Therefore, if  $\chi \geq 0$  and  $\Omega > 0$ ,  $\tilde{\Delta} < 0$  we have always two real roots and two complex conjugate roots. If  $\chi \geq 0$  and  $\omega < 0$  then  $\tilde{\Delta}$  may have any sign but  $M$  and  $N$  are both negative, so we always have two or four real roots. This was also evident from the fact that  $p(0) = -2\chi$  so, if  $\chi > 0$ , we always have at least one real root (two if  $\chi > 0$  or if  $\kappa \neq 0$  when  $\chi = 0$ . For  $\chi = \kappa = 0$ ,  $x = 0$  is a double root of  $p$  and the discriminant vanishes).

When  $\chi$  is negative things are little bit more complicate. In this case one has to study more carefully the sign of  $\tilde{\Delta}$ . When it is negative then we are done, while when it is positive then  $\Omega$  must be negative, providing the same condition  $v > \omega_0/g$  as for the case  $\chi = \kappa = 0$ . In this case, we have also to impose the condition  $N < 0$ , which gives  $g^2v^2 > \omega_0^2 + \sqrt{8\alpha_0|\chi|}$ , that is

$$V^2 > \frac{\omega_0^2 + |Y|\sqrt{\frac{2}{3}\lambda|\chi|}}{g^2 + \omega_0^2 + |Y|\sqrt{\frac{2}{3}\lambda|\chi|}}, \tag{B.20}$$

which generalizes condition (5.16).

So, one is left with the study of the general conditions for which  $\tilde{\Delta}$  have a specific sign when  $\chi$  is negative. We will not pursue this here, but we limit ourselves to the following considerations. We can look at  $\tilde{\Delta} > 0$  as a second order inequality in  $\alpha_0$ , recalling the physical constraint  $\alpha_0 > 0$ . Since  $\chi$  is negative, this is always true if the discriminant is negative, while if it is positive we are led to the condition  $\alpha_0 > \max(0, \alpha_+)$ ,  $\alpha_+$  being the higher root of the quadric. A complete classification of all possibilities is not difficult but quite cumbersome and out of the task of the present work.

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