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Original Paper

A Novel Chain of Lie Algebras and its Coalgebra Symmetry

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Abstract. We study a novel $n(n+1)/2$-dimensional non-semisimple Lie algebra $\mathfrak{g}_n$, a generalisation of both $\mathfrak{sl}_2(\mathbb{K})$ and the two-photon Lie algebra $\mathfrak{h}_6$. We investigate its properties, including its structure, representations, and its Casimir elements. In particular, we prove that there exists only one non-trivial Casimir polynomial of degree n given by the determinant of an $n \times n$ symmetric matrix. We then associate this Lie algebra to a hierarchy of Hamiltonian systems with integrability properties depending on n and describe their first integrals as sums of squares of linear combinations of the components of the angular momentum. In particular, we obtain that these systems are integrable for $n = 2$, quasi-integrable for $n = 3$, and of Poincaré–Lyapunov–Nekhoroshev type for $n \geq 4$.

Contents

1. Introduction

Lie algebras play a pivotal role in the study of integrable systems, serving as a fundamental tool for understanding and classifying these mathematical

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structures. Crucial to the advancement of this field was the development of quantum mechanics. Indeed, the Lie algebraic approach to integrable systems was initiated in Hendrik Casimir's Ph.D. thesis [24] on rigid body motion in quantum mechanics and later continued with the seminal paper by Fock [33] on the SO(4) symmetry of the hydrogen atom. A similar finding was made around of the same time by Demkov [27] and Fradkin [34] about the harmonic oscillator, whose underlying symmetry algebra was found to be related to the special unitary algebra, see also [32]. This research left an everlasting mark in the history of the topic as it is witnessed by the famous book of Herman Weyl on the subject [90]. On the other hand, the importance of Lie algebras in classical mechanics was established only later, mostly through the fundamental work of Olshanetsky and Perelomov. In their most famous paper [71], Olshanetsky and Perelomov established a connection between semisimple Lie algebras and Calogero–Moser like integrable Hamiltonian systems [19], see also the book [74].

The systematisation of these ideas led to the so-called algebraic approach to integrable systems. This field uses tools from Lie or Poisson algebra theory to construct new integrable systems and explicitly solve existing ones. The first problem, also known as the "search problem", is of fundamental importance in the theory of integrable systems and in mathematical physics in general, since integrable systems are ubiquitous in applications, as they arise as a limiting form of a broad families of nonlinear systems [20]. Given a "generic system", it is possible to obtain information on its behaviour using perturbation theory of an integrable system, see e.g. [3, Chap. 11].

The strength of most algebraic methods in integrable systems is that, in contrast with various existing direct methods, see e.g. [28,29,35,39,41,42,48–51,77–81,84,85,89], they can be employed to construct integrable systems in arbitrary dimensions. In contrast, direct methods are often limited by cumbersome computations as the number of degrees of freedom increases.

As in any top-down approach, the main limitation of algebraic methods in integrable systems is our knowledge of the properties of a given mathematical structure. For this reason, the techniques based on Lie algebras in most of the cases are built on (semi)simple Lie algebras, which are essentially completely understood over the complex field since the times of Killing [52] and Cartan [23]. When the real field is considered, for instance for applications in classical mechanics, the situation is more involved, since complex simple Lie algebras have multiple possible real forms [73]. Besides the results on simple Lie algebras, the classification of (all) real Lie algebras is nowadays limited up to dimension 6, with an history going back to Luigi Bianchi [17] and completed only about 70 years later by Turkowski [88] with many contributions given by Mubarakzyanov [63–65]. For more details, see the very complete introduction on the topic in [75, §2]. Classification in higher dimension becomes impractical as the number of Lie algebras grows exponentially (see the estimates in [54]), so usually one confines the study to Lie algebras with interesting properties, where of course the interest depends on the intended application.

This paper is devoted to the study of a novel $n(n+1)/2$-dimensional non-semisimple Lie algebra, which we call $\mathfrak{g}_n$ for $n \geq 2$. Up to isomorphism, the Lie algebra $\mathfrak{g}_n$ is a generalisation of another non-semisimple Lie algebra called the two-photon Lie algebra $\mathfrak{h}_6$ [91], which arises as $\mathfrak{g}_3$. Moreover, it is well known that the two-photon Lie algebra admits as Levi factor a copy of the three-dimensional special linear algebra $\mathfrak{sl}_2(\mathbb{K})$, which arises in our construction as $\mathfrak{g}_2$. The Lie algebra $\mathfrak{g}_n$ generalises this property, and in particular, we will show that for every fixed n it contains as Lie subalgebras all the Lie algebras $\mathfrak{g}_k$ with $2 \leq k \leq n-1$. The Lie algebra $\mathfrak{g}_n$ was first proposed in [43] in the context of the application of Lie algebras to *discrete time* integrable systems through the so-called coalgebra symmetry approach [6,11,14]. Roughly speaking, the coalgebra symmetry approach is an algebraic machinery that given a (classical) one degree of freedom Hamiltonian system associated with a Lie–Poisson algebra turns it into one with N degrees of freedom, and many first integrals are obtained from the Casimir elements of the underlying Lie–Poisson algebra. In some particular cases, these first integrals are enough to give both integrability and even superintegrability of the N degrees of freedom Hamiltonian system. In [43], it was proposed that the Lie algebra $\mathfrak{g}_n$ could be useful to explain integrability of both discrete and continuous time systems, making it a "good" candidate Lie algebra to be studied in arbitrary dimension, i.e. arbitrary n. However, the idea given in [43] requires the computation of the Casimir elements of the Lie algebra $\mathfrak{g}_n$ for generic n. This is a difficult problem to tackle because the algebra is non-semisimple, and for this kind of Lie algebras the standard ways of computing the Casimir elements rapidly become untreatable because of the complexity of the calculation involved.

In this paper, we address the problem of determining the Casimir elements of the Lie algebra $\mathfrak{g}_n$ by initially employing the vector field approach based on its coadjoint representation. We then rigorously prove that the obtained polynomial is indeed the Casimir invariant for any $n \geq 2$, within a framework grounded in the representation theory of Lie algebras [36]. We are able to show that there exists, up to a scalar multiple, only one non-trivial Casimir polynomial of degree n. This result generalises the well-known Casimir elements of $\mathfrak{sl}_2(\mathbb{K})$, a degree two polynomial [86, §16.5], and of $\mathfrak{h}_6$, a degree three polynomial [8], which are recovered as the $n = 2, 3$ cases, respectively. Then, in the spirit of [6,11,14] we build a hierarchy of Hamiltonian systems in N degrees of freedom associated with the Lie algebra $\mathfrak{g}_n$. In particular, we show that each of these systems possesses a total number of $2(N-n+1)$ first integrals divided into two families of $N-(n-2)$ pairwise commuting ones. In particular, this implies that the given systems are integrable for $n = 2$, quasi-integrable for $n = 3$, and of Poincaré–Lyapunov–Nekhoroshev type with rank $N-(n-2)$ [37,38,69] for $n \geq 4$.

The plan of the paper is the following: in Sect. 2, we discuss the background material we need throughout the paper. In particular, we focus on the basic notions on Lie algebras and their representations that we are going to use, and we give a short introduction to the coalgebra symmetry method applied to

Classical Mechanics. In Sect. 3, we introduce the Lie algebra $\mathfrak{g}_n$ and establish its main properties. In particular, we show that there is a chain of inclusions of Lie subalgebras, prove that $\mathfrak{g}_n$ can have at most one non-trivial Casimir element, and then construct a matrix representation in $\mathfrak{sl}_{2(n-1)}(\mathbb{K})$ and also its coadjoint representation in the space of vector fields. Section 4 contains the main result of the paper: the proof of the form of the Casimir element of $\mathfrak{g}_n$ for arbitrary n. We start by employing the method of vector fields to examine the low-dimensional examples $n = 2, 3, 4$. Carefully analysing those examples, we guess the general form of the Casimir polynomial for generic n as the determinant of a given $n \times n$ symmetric matrix with coefficients in $\mathfrak{g}_n$. Then, we use the matrix representation constructed in Sect. 3 to prove that such a determinant is indeed a Casimir polynomial, being well-behaved with respect to the action of the group $\mathrm{SL}_n(\mathbb{K})$, and recalling that such an action has the property of leaving invariant the determinant of a symmetric matrix. Then, in Sect. 5 we proceed to associate to $\mathfrak{g}_n$ a hierarchy of classical Hamiltonian systems using a one degree of freedom symplectic realisation. Moreover, using the information on the Casimir element of $\mathfrak{g}_n$, we are able to describe the first integrals of the obtained Hamiltonian systems. In particular, we provide a expression of the first integrals as sum of squares of “building blocks” that are n-indices objects arising as linear combinations of the angular momentum components $L_{i,j}$, plus constant terms coming from the $\mathfrak{g}_n$ central elements under the chosen symplectic realisation. We also provide a determinantal formula for these n-indices objects, which clarifies their rather intricate structure. Of course, the known cases $n = 2, 3$ are properly recovered. Finally, in Sect. 6 we discuss our results and suggest further directions of research in the field.

2. Background and Notations

In this section, we discuss the background tools we need to fully develop our arguments, and fix the notations we are going to use throughout the manuscript.

2.1. Notions from Representation Theory

Consider a finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$. In this subsection, we recall a few notions from representation theory, see for instance [36]. A *representation* ρ of the Lie algebra $\mathfrak{g}$ on a vector space V (finite or infinite dimensional) is a linear map of $\mathfrak{g}$ into the space of linear operators acting on V, denoted by $\mathscr{L}(V)$:

$$\begin{aligned} \rho\colon \mathfrak{g} &\longrightarrow \mathscr{L}(V) \\ x &\longmapsto \rho(x), \end{aligned} \tag{2.1}$$

such that:

$$\rho([x,y]) = \rho(x)\circ\rho(y) - \rho(y)\circ\rho(x). \tag{2.2}$$

Note that, within this definition, we implicitly used that $\mathscr{L}(V)$ is a Lie algebra with Lie bracket given by the difference of compositions: $[F,G]_{\mathscr{L}(V)} = F\circ G-$

$G \circ F$, and a representation preserves the Lie bracket: $\rho([x, y]) = [\rho(x), \rho(y)]$. If the map ρ is injective, the representation is called *faithful*.

When V is a finite-dimensional vector space over $\mathbb{K}$ we have that $\mathscr{L}(V) \cong \mathfrak{gl}(V)$, the space of linear operators on V. If $\dim V = n$, after choosing a basis, this is the space of $n \times n$ matrices and it is denoted by $\mathfrak{gl}_n(\mathbb{K})$. The commutator is $[X, Y] = XY - YX$, cf. (2.2). Representations $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}_n(\mathbb{K})$ are called *matrix representations*.

A basic example of a representation of a Lie algebra is the *adjoint representation* ρ_{ad}, which we will use later. It is defined as follows:

$$\begin{aligned} \rho_{\text{ad}}\colon \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ x &\longmapsto \rho_{\text{ad}}(x)(-) = \text{ad}_x(-) = [x, -]. \end{aligned} \tag{2.3}$$

The Lie bracket is preserved due to the Jacobi identity for a Lie algebra. Given a finite-dimensional representation $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, we can define representations on tensor products of V, see [53, §4.2.2]. In particular, we are interested in the representation $\rho^{\otimes 2}$ induced on $V^{\otimes 2}$, namely $\rho^{\otimes 2}\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V^{\otimes 2})$, where for every $x \in \mathfrak{g}$ we have:

$$\begin{aligned} \rho^{\otimes 2}(x)\colon V^{\otimes 2} &\longrightarrow V^{\otimes 2} \\ v \otimes w &\longmapsto (\rho(x)v) \otimes w \,+\, v \otimes (\rho(x)w)\ . \end{aligned} \tag{2.4}$$

We observe that that $\rho(x)$ acts as a derivation on $V^{\otimes 2}$ and this is also true for $V^{\otimes n}$. This action can be made more explicit by choosing a basis $e_1, \dots, e_n$ of V and identifying the tensor product with the vector space of $n \times n$ matrices in the following way:

$$\begin{aligned} V^{\otimes 2} &\xleftrightarrow{\ \cong\ } M_n(\mathbb{K}) \\ e_i \otimes e_j &\longleftrightarrow E_{i,j}\ , \end{aligned} \tag{2.5}$$

where $E_{i,j}$ is an $n \times n$ matrix whose only nonzero entry is the element (i, j):

$$E_{i,j} = (\delta_{ik}\delta_{jl})_{k,l=1}^{n}. \tag{2.6}$$

It is then an easy computation to verify that the representation $\rho^{\otimes 2}$ on $V^{\otimes 2}$ is (equivalent to) the representation, still denoted by $\rho^{\otimes 2}$, on $M_n(\mathbb{K})$:

$$\begin{aligned} \rho^{\otimes 2}\colon \mathfrak{g} &\longrightarrow M_n(\mathbb{K}) \\ x &\longmapsto \rho^{\otimes 2}(x)(M) = \rho(x)M + M\rho(x)^T. \end{aligned} \tag{2.7}$$

In the decomposition $V^{\otimes 2} = \operatorname{Sym}^2 V \oplus \Lambda^2 V$, the subspaces of symmetric and skew symmetric tensors are mapped into themselves by any $\rho^{\otimes 2}(x)$. So, it makes sense to consider the representations on the two invariant subsets. In particular, we denote the representation $\rho^{\otimes 2}$ restricted to $\operatorname{Sym}^2 V$ (resp. $\Lambda^2 V$) by $\rho^{(2)}$ (resp. $\rho^{[2]}$).

More generally, for each $n \geq 0$ we have the representations $\rho^{(n)}$ on $\operatorname{Sym}^n V$. This leads to consider the symmetric algebra SV of V, an infinite dimensional vector space

$$SV := \bigoplus_{n=0}^{\infty} \operatorname{Sym}^n V \ , \tag{2.8}$$

with the convention that $\operatorname{Sym}^0 V = \mathbb{K}$. An element $v \in V$ defines a linear map on the dual vector space $v\colon V^* \longrightarrow \mathbb{K}$ simply mapping $\ell \in V^*$ to $\ell(v)$. The vector space $\operatorname{Sym}^n V$ can then be identified with linear combinations of products $v_{i_1} v_{i_2} \dots v_{i_n}$ of functions on V^* that are homogeneous of degree n. This leads to the identification $SV = \operatorname{Pol}(V^*)$, the algebra of polynomial functions on V^*.

We are particularly interested in the case that $V = \mathfrak{g}$ and $\rho = \rho_{\mathrm{ad}}$, the adjoint representation on V, now extended, by derivations, to all of SV. That is, $\rho_{\mathrm{ad}}\colon \mathfrak{g} \longrightarrow \mathscr{L}(S\mathfrak{g})$, where for all $x \in \mathfrak{g}$ we have:

$$\begin{aligned} \rho_{\mathrm{ad}}(x)\colon S\mathfrak{g} &\longrightarrow S\mathfrak{g} \\ x_{i_1} x_{i_2} \dots x_{i_n} &\longmapsto [x, x_{i_1}] x_{i_2} \dots x_{i_n} + \dots + [x, x_{i_n}] x_{i_1} \dots x_{i_{n-1}} . \end{aligned} \tag{2.9}$$

Then, a *polynomial Casimir element* $C \in S\mathfrak{g}$ is by definition an element annihilated by the adjoint action:

$$\rho_{\mathrm{ad}}(x)(C) = 0 \qquad \forall x \in \mathfrak{g} \ . \tag{2.10}$$

Clearly, the elements of the centre of a Lie algebra $Z(\mathfrak{g})$ are Casimir elements. These elements are usually called *trivial Casimir elements*, since they belong to the Lie algebra $\mathfrak{g}$ itself.

More explicitly, if we choose a basis $\mathfrak{g} = \operatorname{span}_{\mathbb{K}} \{ x_1, \dots, x_d \}$, then we have the identification $S\mathfrak{g} \cong \mathbb{K}[x_1, \dots, x_d]$, the polynomial algebra in d variables. The Lie algebra structure is determined by the structure constants c_{ij}^k, i.e.:

$$[x_i, x_j] := \sum_{k=1}^{d} c_{ij}^k x_k \ . \tag{2.11}$$

The Lie bracket of $\mathfrak{g}$ induces on $S\mathfrak{g} = \operatorname{Pol}(\mathfrak{g}^*)$, and more in general on the algebra of functions $\mathscr{F}(\mathfrak{g}^*)$ of all smooth/analytic functions on $\mathfrak{g}^*$, the so-called Lie–Poisson bracket:

$$\{f, g\}_{S\mathfrak{g}} = \sum_{i,j,k=1}^{d} c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \tag{2.12}$$

see also [60, Chap. 7]. Notice that:

$$\rho_{\mathrm{ad}}(x_i)(x_j) = [x_i, x_j] = \{x_i, x_j\}_{S\mathfrak{g}}, \tag{2.13}$$

and hence, since $x \in \mathfrak{g}$ acts by derivations:

$$\rho_{\mathrm{ad}}(x)(f) = \{x, f\}_{S\mathfrak{g}}. \tag{2.14}$$

Consequently, polynomial Casimir elements C can be constructed as elements of the symmetric algebra by solving:

$$\{C, x\}_{S\mathfrak{g}} = 0, \quad \forall x \in \mathfrak{g}. \tag{2.15}$$

Since $\mathfrak{g}$ acts on $\mathscr{F}(\mathfrak{g}^*)$ by derivations, we actually have a map $\mathfrak{g} \longrightarrow \mathfrak{X}(\mathfrak{g}^*)$, the latter being the space of vector fields over $\mathfrak{g}^*$. In fact, the formula for $\rho_{\mathrm{ad}}(x_i)(x_j)$ implies that x_i acts as the vector field:

$$\widehat{x}_i := \sum_{j,k=1}^{d} c_{ij}^k x_k \frac{\partial}{\partial x_j}. \tag{2.16}$$

The vector fields are a Lie algebra with Lie bracket given by

$$[X,Y]_{LB}(f) := X(Y(f)) - Y(X(f)), \qquad X,Y \in \mathfrak{X}(\mathfrak{g}^*), \quad f \in \mathscr{F}(\mathfrak{g}^*). \tag{2.17}$$

Then $x \mapsto \widehat{x}$ is a Lie algebra representation since the vector fields $\widehat{x}_i$ satisfy the same commutation relations as x_i in the Lie algebra $\mathfrak{g}$, i.e. the following identity holds:

$$[\widehat{x}_i, \widehat{x}_j]_{LB} = \widehat{[x_i, x_j]_{\mathfrak{g}}}. \tag{2.18}$$

Thus, following for instance [86], we say that the vector fields $\{ \widehat{x}_1, \ldots, \widehat{x}_d \} \subset \mathfrak{X}(\mathfrak{g}^*)$ form a basis of the so-called coadjoint representation of $\mathfrak{g}$, acting by derivations on $\mathscr{F}(\mathfrak{g}^*)$.

Remark 2.1. Note that if $Z(\mathfrak{g}) \neq 0$, the adjoint representation is not faithful, as all the elements in the centre are mapped into the zero vector field of $\mathfrak{X}(\mathfrak{g}^*)$.

For the search of Casimirs, we then have the following result:

Proposition 2.2 ([86, §3.2]). *Given a d-dimensional Lie algebra $\mathfrak{g}$, then its Casimir elements can be found as the functionally independent solutions of the system of PDEs:*

$$\widehat{x}_i(f) = 0, \quad i = 1, \ldots, d, \tag{2.19}$$

where the $\widehat{x}_i$ are the basis of the coadjoint representation in the space of vector fields, see equation (2.16).

This result tells us that one does not need to check the polynomial Casimir condition (2.15) for every element, but just a suitably chosen basis is enough. For applications and further discussion of this method to find Casimir elements of Lie algebras, we refer to [86]. A short discussion of this approach in our case of interest, along with some motivations on why it tends to become ineffective as the dimension of the Lie algebra grows, is presented in Sect. 4.

We also recall the following important result on the number of Casimir elements admitted by a Lie algebra:

Theorem 2.3 (Beltrametti–Blasi [16]). *Let $\mathfrak{g}$ be a d-dimensional Lie algebra over $\mathbb{K}$ with structure constants c_{ij}^k. Then, the number of functionally independent Casimir invariants, ν, of $\mathfrak{g}$ is the following:*

$$\nu = \dim \mathfrak{g} - \operatorname{rank} A, \qquad A = \left(\sum_{k=1}^{d} c_{ij}^k x_k \right)_{i,j=1}^{d}, \tag{2.20}$$

where the x_i are commuting variables on the base field.

We conclude by recalling that for some other authors Casimir elements are considered in the universal enveloping algebra (UEA) $U\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ rather than in its symmetric algebra. For instance, in [53] a Casimir element is defined to be an element of $U\mathfrak{g}$ invariant with respect to the adjoint action of $\mathfrak{g}$, defined by extending the adjoint action of $\mathfrak{g}$ on $U\mathfrak{g}$. In fact, this definition is analogous to the one we gave, because there exists a map $\Sigma\colon S\mathfrak{g} \longrightarrow U\mathfrak{g}$, called the *symmetrisation map*, mapping polynomial Casimir elements in $S\mathfrak{g}$ into polynomial Casimir elements in $U\mathfrak{g}$ in the sense just described. For this reason, within this paper we will only consider Casimir elements in the symmetric algebra $S\mathfrak{g}$, see also [22, §2].

2.2. The Coalgebra Symmetry Approach for Hamiltonian Systems

The concept of coalgebra was introduced in quantum group theory during the 1980s [30], see also the book [25]. However, its application to (super)integrable systems did not emerge until the late 1990s through the pioneering work of Ballesteros, Ragnisco, and their collaborators in [11,14]. Over the years, this approach has been significantly expanded to encompass diverse scenarios, leading to a deeper understanding of various integrable and superintegrable systems. As evident from comprehensive reviews such as [6], the coalgebra symmetry framework has also been successfully applied in other areas, including the study of superintegrable systems on non-Euclidean spaces [4,7]. Additionally, it has been employed to investigate models with spin–orbital interactions [83], discrete quantum mechanical systems [56], and superintegrable systems related to the generalised Racah algebra $R(n)$ [26,57–59]. Furthermore, adaptations of this method have been made for use with comodule algebras in [10] and loop coproducts in [68]. Recent work has also explored its application to discrete time systems[31,43,44]. This subsection will provide a brief overview of the coalgebra symmetry construction, highlighting the main results and ideas in this theory.

The starting point of this approach is the following definition:

Definition 2.4. A *coalgebra* is a pair $(\mathscr{A}, \Delta)$ where $\mathscr{A}$ is a unital, associative algebra and $\Delta\colon \mathscr{A} \longrightarrow \mathscr{A}^{\otimes 2}$ is a *coassociative* map. That is, Δ is an algebra homomorphism from $\mathscr{A}$ to $\mathscr{A}^{\otimes 2}$:

$$\Delta(x \cdot y) = \Delta(x) \cdot \Delta(y), \qquad \forall x, y \in \mathscr{A}\,, \tag{2.21}$$

such that:

$$(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta\,, \tag{2.22}$$

that is, the following diagram commutes:

$$\begin{tikzcd} & \mathscr{A}^{\otimes 2} \arrow[dr, "\Delta \otimes \mathrm{Id}"] & \\ \mathscr{A} \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & \mathscr{A}^{\otimes 3} \\ & \mathscr{A}^{\otimes 2} \arrow[ur, "\mathrm{Id} \otimes \Delta"'] & \end{tikzcd} \tag{2.23}$$

The map Δ is called the *coproduct map*.

When there is no possible confusion on the coproduct map, it is customary to denote the coalgebra simply by $\mathscr{A}$. Our case of interest are *Poisson coalgebras*. That is, we will not assume $\mathscr{A}$ to be a generic unital associative algebra, but rather we will consider it to be a *Poisson algebra*, i.e. a commutative associative algebra admitting a bilinear derivation $\{,\}$, satisfying the Jacobi identity. The bilinear operation $\{,\}$ is called a *Poisson bracket*. To build a Poisson coalgebra, the coalgebra structure and the Poisson algebra structure have to be compatible in the sense of the following definition:

Definition 2.5. A *Poisson coalgebra* is a triple $(\mathscr{A},\Delta,\{\,,\,\})$ where $(\mathscr{A},\Delta)$ is a coalgebra and $(\mathscr{A},\{,\})$ is a Poisson algebra, such that the coproduct map is a *Poisson homomorphism* with respect to the induced Poisson bracket on $\mathscr{A}^{\otimes 2}$:

$$\{x\otimes y, w\otimes z\}_{\mathscr{A}^{\otimes 2}} = \{x,w\}_{\mathscr{A}}\otimes yz + xw\otimes\{y,z\}_{\mathscr{A}}\,,\quad x,y,z,w\in\mathscr{A}, \tag{2.24}$$

i.e. $\Delta(\{x,y\}_{\mathscr{A}}) = \{\Delta(x),\Delta(y)\}_{\mathscr{A}^{\otimes 2}}$.

The coproduct provides us a way to obtain relevant elements living in the twofold tensor product of a Poisson algebra. In particular, the coproduct is not surjective, but its image is a copy of the original Poisson algebra $\mathscr{A}$ in $\mathscr{A}^{\otimes 2}$: $\mathscr{A}\cong\Delta(\mathscr{A})\subset\mathscr{A}^{\otimes 2}$.

The construction of the coproduct can be extended in two different ways for all $m>2$ [6]. Indeed, by recursion one can define the *left and right* higher-order mth *coproducts* $\Delta_R^{(m)}\colon\mathscr{A}\longrightarrow\mathscr{A}^{\otimes m}$ and $\Delta_L^{(m)}\colon\mathscr{A}\longrightarrow\mathscr{A}^{\otimes m}$, acting as:

$$\Delta_L^{(m)} = \big(\overbrace{\mathrm{Id}\otimes\mathrm{Id}\otimes\cdots\otimes\mathrm{Id}}^{m-2}\otimes\Delta\big)\circ\Delta_L^{(m-1)}, \tag{2.25a}$$

$$\Delta_R^{(m)} = \big(\Delta\otimes\overbrace{\mathrm{Id}\otimes\mathrm{Id}\otimes\cdots\otimes\mathrm{Id}}^{m-2}\big)\circ\Delta_R^{(m-1)}. \tag{2.25b}$$

By induction on m, it can be proved that the mth coproducts $\Delta_L^{(m)}$ and $\Delta_R^{(m)}$ are also Poisson maps on $\mathscr{A}^{\otimes m}$ and define a copy of the original algebra: $\mathscr{A}\cong\Delta_i^{(m)}(\mathscr{A})\subset\mathscr{A}^{\otimes m}$, $i=L,R$.

In particular, if $\{\,x_1,\ldots,x_d\,\}$ is the generating set of $\mathscr{A}$, for a generic element $F\in\mathscr{A}$ we can write:

$$\Delta_L^{(m)}(F)(x_1,\ldots,x_d) = F(\Delta_L^{(m)}(x_1),\ldots,\Delta_L^{(m)}(x_d)), \tag{2.26a}$$

$$\Delta_R^{(m)}(F)(x_1,\ldots,x_d) = F(\Delta_R^{(m)}(x_1),\ldots,\Delta_R^{(m)}(x_d)). \tag{2.26b}$$

Now, let us fix an $N\in\mathbb{N}$. Then, following for example [66] we can consider a chain of tensor products $\mathscr{A}^{\otimes m}$ for all $m\leq N$ with left/right embeddings:

$$\begin{aligned}\iota_{m,L}\colon \mathscr{A}^{\otimes m} &\longrightarrow \mathscr{A}^{\otimes N}\\ x &\longmapsto \iota_{m,L}(x) = x\otimes\underbrace{1\otimes 1\otimes\cdots\otimes 1}_{N-m},\end{aligned} \tag{2.27}$$

and:

$$\begin{aligned}\iota_{m,R}\colon \mathscr{A}^{\otimes m} &\longrightarrow \mathscr{A}^{\otimes N}\\ x &\longmapsto \iota_{m,R}(x) = \underbrace{1\otimes 1\otimes\cdots\otimes 1}_{N-m}\otimes x.\end{aligned} \tag{2.28}$$

So, we can consider all the elements as lying in the final tensor product $\mathscr{A}^{\otimes N}$. In particular, we can consider all the Poisson brackets in $\mathscr{A}^{\otimes N}$ in the following way: let $x \in \mathscr{A}^{\otimes m}$ and $y \in \mathscr{A}^{\otimes m'}$, then:

$$\{x, y\}_{\mathscr{A}^{\otimes N}} = \{\iota_{m,L}(x), \iota_{m',L}(y)\}_{\mathscr{A}^{\otimes N}}, \tag{2.29}$$

and, similarly, if we consider the tensor product within the right embedding, or the mixed case.

Now, a crucial ingredient in the construction of Hamiltonian systems from coalgebras is the fact that Poisson algebras admit Casimir elements. This is a direct generalisation of what we discussed in the previous section about Casimir elements for Lie algebras. Clearly, given a Poisson algebra $(\mathscr{A}, \{.,.\})$, a Casimir element $C \in \mathscr{A}$ is an element such that $\{C, x\}_{\mathscr{A}} = 0$ for all $x \in \mathscr{A}$, to be compared with (2.15). We denote the space of Casimir elements of a Poisson algebra by:

$$\operatorname{Cas}(\mathscr{A}, \{.,.\}) = \{\, C \in \mathscr{A} \mid \{C, x\} = 0, \forall x \in \mathscr{A}\,\}. \tag{2.30}$$

This space is a subalgebra and also an ideal of $\mathscr{A}$ [60, Chap. 1]. If $\operatorname{Cas}(\mathscr{A}, \{.,.\})$ is *finitely generated*, we can take a (minimal) set of its generators $\{\, C_1, \dots, C_s \,\}$. We will say that such a set of generators is a set of *functionally independent Casimirs*. This name is justified from the fact that in the case of the symmetric algebra $S\mathfrak{g}$ of a d-dimensional Lie algebra $\mathfrak{g}$, through the isomorphism $S\mathfrak{g} \cong \mathbb{K}[x_1, \dots, x_d]$ this notion reduces to the standard analytic notion of functional independence.

Then, we can state the following fundamental result adapted to our notations:

Theorem 2.6 ([6,14]). *Assume we are given a finitely generated Poisson coalgebra* $(\mathscr{A}, \Delta, \{,\})$*, with* $\mathscr{A}$ *generated by the set* $\{x_1, \dots, x_d\}$*, whose Casimir subalgebra is finitely generated* $\operatorname{Cas}(\mathscr{A}, \{\,,\})= \{\, C_1, \dots, C_s \,\}$*. Then, the two sets of functions:*

$$\mathrm{L} = \left\{\, C_j^{(m)} := (\iota_{m,L}\circ \Delta_L^{(m)})(C_j)(x_1, \dots, x_d) \,\right\}_{m=1,\dots,N, j=1,\dots,s}, \tag{2.31a}$$

$$\mathrm{R} = \left\{\, C_{j,(m)} := (\iota_{m,R}\circ \Delta_R^{(m)})(C_j)(x_1, \dots, x_d) \,\right\}_{m=1,\dots,N, j=1,\dots,s}, \tag{2.31b}$$

are made of Poisson commuting functions on $\mathscr{A}^{\otimes N}$*. Furthermore, they Poisson commute with* $\Delta_L^{(N)}(x_i) = \Delta_R^{(N)}(x_i)$*,* $i = 1, \dots, d$ *and with all functions* $\Delta_L^{(N)}(H) = \Delta_R^{(N)}(H) \in \mathscr{A}^{\otimes N}$*, where* $H \in \mathscr{A}$*.*

This result was obtained in [14] in the case of the left coproduct. The extension to incorporate also the right coproducts originally introduced in [15] can be found, for example, in [6]. The functions $C_j^{(m)}$ and $C_{j,(m)}$ are usually called *left and right Casimir functions*. Note that despite the name these functions are not Casimir elements in the standard sense, and in general they do not commute with each other [57]. Moreover, because of the coassociativity property we have $C_j^{(N)} = C_{j,(N)}$, at a fixed $j = 1, \ldots s$.

Now, to discuss integrable systems, we need to put this algebraic construction with a Hamiltonian system. This is done through the so-called symplectic realisation of a Poisson algebra. A symplectic realisation is, in general, a Poisson algebra homomorphism from a given Poisson algebra $(\mathscr{A}, \{,\})$ to the algebra of smooth functions on an open subset U of a symplectic manifold $(\mathscr{M}, \Omega)$, with $\dim \mathscr{M} = 2\,M$. Since we are working locally, from Darboux theorem, we can consider on the open subset U the canonical coordinates $(\mathbf{q}_1, \mathbf{p}_1)$, and the symplectic form to be the canonical one ω. Under such hypotheses, a *M degrees of freedom canonical symplectic realisation* is the following Poisson homomorphism:

$$\begin{aligned} \mathrm{D}\colon (\mathscr{A}, \{\,,\,\}) &\longrightarrow (\mathscr{C}^\infty(U), \{\,,\,\}_\omega) \\ x_i &\longmapsto x_i(\mathbf{q}_1, \mathbf{p}_1). \end{aligned} \tag{2.32}$$

If the Poisson algebra is a Poisson coalgebra, from a M-degrees of freedom symplectic realisation we can find a NM-degrees of freedom symplectic realisation by taking the tensor product of realisations: $\mathrm{D}^{\otimes N}$. The factors of the tensor product are usually physically interpreted as "sites", where on each of them resides a "particle" with M-degrees of freedom. We denote the canonical coordinates in the NM-degree of freedom realisation by:

$$(\mathbf{q}, \mathbf{p}) = (\mathbf{q}_1, \mathbf{p}_1, \ldots, \mathbf{q}_N, \mathbf{p}_N), \tag{2.33a}$$

where:

$$\mathbf{q}_j = (q_{j,1}, \ldots, q_{j,M})^T, \quad \mathbf{p}_j = (p_{j,1}, \ldots, p_{j,M})^T, \qquad j = 1, \ldots, N. \tag{2.33b}$$

In general, by applying the tensor product realisation we can construct the first integrals, using the left and right Casimir introduced in Theorem 2.6. Moreover, using the last part of Theorem 2.6 we see that we can lift a one-particle, M-degrees of freedom Hamiltonian $H \in \mathscr{C}^\infty(U)$ to generate an N particle, NM-degrees of freedom Hamiltonian equipped with *many* first integrals, usually called *universal integrals* [4,57]. This is the content of the following result:

Theorem 2.7. *Assume we are given a finitely generated Poisson coalgebra* $(\mathscr{A}, \Delta, \{\,,\,\})$ *whose Casimir subalgebra is finitely generated* $\mathrm{Cas}(\mathscr{A}, \{\,,\,\}) = \{\, C_1, \ldots, C_s \,\}$*, and assume we are given a M-degrees of freedom canonical realisation $D\colon \mathscr{A} \longrightarrow \mathscr{C}^\infty(U)$. Then, given $H \in \mathscr{A}$ and the corresponding M-degrees of freedom Hamiltonian* $H(\mathbf{q}_1, \mathbf{p}_1) := \mathrm{D}(H) \in \mathscr{C}^\infty(U)$*, the NM-degrees of freedom Hamiltonian:*

$$H^{(N)}(\mathbf{q}, \mathbf{p}) := (\mathrm{D}^{\otimes N} \circ \Delta_L^{(N)})(H) \tag{2.34}$$

admits two sets of sN commuting first integrals given by:

$$L = \left\{ C_j^{(m)}(\mathbf{q},\mathbf{p}) := \mathrm{D}^{\otimes N}(C_j^{(m)}) \right\}_{m=1,\dots,N,j=1,\dots,s}, \tag{2.35a}$$

$$R = \left\{ C_{j,(m)}(\mathbf{q},\mathbf{p}) := \mathrm{D}^{\otimes N}(C_{j,(m)}) \right\}_{m=1,\dots,N,j=1,\dots,s}, \tag{2.35b}$$

where the functions $C_j^{(m)}$ and $C_{j,(m)}$ are defined in equation (2.31).

Let us now discuss the main case of interest, that is, the one of Lie–Poisson algebras $(\mathscr{F}(\mathfrak{g}^*), \{,\})$, or if only polynomials are involved the symmetric algebra $S\mathfrak{g} \cong \mathrm{Pol}(\mathfrak{g}^*) \subset \mathscr{F}(\mathfrak{g}^*)$ of a Lie algebra $\mathfrak{g}$. All Lie–Poisson algebras admit a common coproduct, called the *primitive coproduct* Δ [87], which can be defined on the (linear) generators of the symmetric algebra as:

$$\begin{aligned} \Delta\colon S\mathfrak{g} &\longrightarrow (S\mathfrak{g})^{\otimes 2} \\ x_i &\longmapsto \Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i, \end{aligned} \tag{2.36}$$

then extended to generic elements of $S\mathfrak{g}$ through the homomorphism property (2.21), and eventually to the whole algebra of smooth functions $\mathscr{F}(\mathfrak{g}^*)$. Many different coproducts are build upon deformation of the primitive coproduct, see e.g. [25] and papers [12,61]. The primitive coproduct allows great simplifications in the theory. For instance, it makes the construction of the tensor realisations from a basic one immediate. For the sake of convenience, we enclose this result in the following lemma:

Lemma 2.8. *Consider a Lie algebra $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{ e_1, \dots, e_d \}$, the induced primitive coproduct on $S\mathfrak{g} \cong \mathbb{R}[x_1, \dots, x_d]$ (2.36) and a M-degrees of freedom realisation $\mathrm{D}\colon (\mathscr{A}, \{,\}) \longrightarrow (\mathscr{C}^\infty(U), \{,\}_\omega)$ (2.32). Then, we have the following expressions for the NM-degrees of freedom canonical symplectic realisation of the generators:*

$$\mathrm{D}^{\otimes N}(\iota_{m,L} \circ \Delta_L^{(m)}(x_i)) = \sum_{j=1}^{m} x_i(\mathbf{q}_j, \mathbf{p}_j), \tag{2.37a}$$

$$\mathrm{D}^{\otimes N}(\iota_{m,R} \circ \Delta_R^{(m)}(x_i)) = \sum_{j=N-m+1}^{N} x_i(\mathbf{q}_j, \mathbf{p}_j), \tag{2.37b}$$

where $\mathbf{q}_j$ and $\mathbf{p}_j$ are given in Eq. (2.33b).

Let us summarise the construction and place it in the context of integrability of Hamiltonian systems. From Theorem 2.7, we have that a Hamiltonian system endowed with coalgebra symmetry is naturally equipped with two sets of first integrals whose number depends on the Casimir of the underlying Lie (or Poisson) algebra. However, for a "generic" Poisson algebra such a Hamiltonian system is not necessarily integrable, while for some specific ones it can be *even more* than integrable. To this end, we recall some basic definitions in the theory of integrable systems:

- a Hamiltonian system possessing N functionally independent first integrals in involution, Hamiltonian included, is called *Liouville integrable* [3];

- a Liouville integrable system possessing $N+k$, $1 \leq k \leq N-1$ functionally independent first integrals is called *superintegrable*, in particular if $k = N-1$ it is called *maximally superintegrable (MS)*, and if $k = N-2$ it is called *quasi-maximally superintegrable (QMS)* [62,82];
- a Hamiltonian system possessing l with $1 \leq l \leq N-1$ commuting first integrals is called *a system of Poincaré–Lyapunov–Nekhoroshev (PLN) type with rank* l, in particular if $l = N-1$ it is called *quasi-integrable* [37, 38,69].

Now, for a Hamiltonian system with coalgebra symmetry there are algebraic conditions to be fulfilled on the algebra and on its realisation to obtain automatically Liouville integrability, see [6]. Moreover, for some specific realisation, the first integrals computed through formulas (2.35) can be trivial up to a certain N, i.e. a constant. Without entering into the details, we simply note that the only aspect that may fail in terms of integrability concerns the functional independence of the first integrals produced via Theorem 2.7. However, for any given Casimir, the left and right integrals generated from it are, by construction, mutually functionally independent. We will show the details of this construction in the explicit example of the Lie algebra $\mathfrak{g}_n$ in Sect. 5.

3. The $\mathfrak{g}_n$ Chain of Lie Algebras

In this section, we rigorously introduce the $\mathfrak{g}_n$ Lie algebra and derive its first properties. In particular, we present a matrix representation of $\mathfrak{g}_n$ in the special linear algebra of a properly chosen dimension.

Throughout this section and the rest of the paper, we will not explicitly indicate the base field for the Lie algebras. In general, the reader can assume either the complex or the real numbers. Field-dependent results or statements will be specified when needed.

3.1. Definition and Main Properties

The Lie algebra $\mathfrak{g}_n$ is a Lie algebra of dimension $T_n = n(n+1)/2$ characterised by the following generators.[1]:

- the 3 generators $\{ h, x_-, x_+ \}$;
- the $2(n-2)$ generators $\{ y_{i,-}, y_{i,+} \}$ for $i = 1, \dots, n-2$, $(n \geq 3)$;
- the T_{n-2} central elements $z_{i,j}$ with $1 \leq i \leq j \leq n-2$, $(n \geq 3)$;

with commutation relations:

$$\begin{array}{lll} [x_+, x_-] = h, & [h, x_\pm] = \pm 2x_\pm, & [h, y_{i,\pm}] = \pm y_{i,\pm}, \\ {[x_-, y_{i,-}]} = [x_+, y_{i,+}] = 0, & [x_-, y_{i,+}] = y_{i,-}, & [x_+, y_{i,-}] = y_{i,+}, \\ {[y_{i,+}, y_{j,+}]} = [y_{i,-}, y_{j,-}] = 0 & [y_{i,+}, y_{j,-}] = z_{\min(i,j),\max(i,j)} & [z_{i,j}, -] = 0. \end{array} \tag{3.1}$$

As stated above, the centre of $\mathfrak{g}_n$ is

$$Z(\mathfrak{g}_n) = \operatorname{span}_{\mathbb{K}} \{ z_{i,j} \}_{1 \leq i \leq j \leq n-2} . \tag{3.2}$$

[1]We use the notation T_n because the dimensions of the Lie algebras $\mathfrak{g}_n$ are the *triangular numbers*.

Note that $\dim Z(\mathfrak{g}_n) = T_{n-2} = (n-1)(n-2)/2$.

In the following, we denote by $\overline{\mathfrak{g}}_n$ the quotient algebra obtained as the quotient of $\mathfrak{g}_n$ by the ideal $Z(\mathfrak{g}_n)$, i.e. $\overline{\mathfrak{g}}_n = \mathfrak{g}_n/Z(\mathfrak{g}_n)$, see, for instance, [36, §9.1].

By construction, for $n = 2$ we have $\mathfrak{g}_2 = \mathfrak{sl}_2(\mathbb{K})$. Indeed, in this case the generators $y_{i,\pm}$ and $z_{i,j}$ are not present. Next, for $n = 3$ we get a six-dimensional Lie algebra with generators $\{h, x_+, x_-, y_+ := y_{1,+}, y_- := y_{1,-}, z := z_{1,1}\}$. This Lie algebra is isomorphic to the two-photon Lie algebra $\mathfrak{h}_6$ [91], which is in turn isomorphic to the $(1+1)$-Schrödinger algebra [45,70]. For $n = 4$, we have a ten-dimensional Lie algebra isomorphic to the $\mathfrak{A}_{10}$ Lie algebra introduced in [43]. The peculiarity of these Lie algebras is the fact that they form a chain of inclusions:

$$\mathfrak{sl}_2(\mathbb{K}) \subset \mathfrak{h}_6 \subset \mathfrak{A}_{10}, \tag{3.3}$$

where each Lie algebra is "embedded" as a Lie subalgebra in the next one. This result can be made more general in the following lemma:

Lemma 3.1. *For a fixed $n \geq 2$, we have the following chain of inclusions*

$$\mathfrak{g}_2 \subset \mathfrak{g}_3 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n, \tag{3.4}$$

where any Lie algebra is a Lie subalgebra of the following one.

Proof. Let us note that, given the isomorphisms discussed above, we can restrict to prove this statement for $n \geq 3$. To be more precise, we observe that for all $2 < k \leq n$ the following decomposition *as vector spaces* holds true:

$$\mathfrak{g}_k = \mathfrak{g}_{k-1} \oplus I_k \supset \mathfrak{g}_{k-1}, \tag{3.5}$$

where

$$I_k = \mathrm{span}_{\mathbb{K}} \{\, y_{k-2,-}, y_{k-2,+} \,\} \oplus \mathrm{span}_{\mathbb{K}} \{\, z_{1,k-2}, \ldots, z_{k-2,k-2} \,\}. \tag{3.6}$$

Now, we see that $[\mathfrak{g}_k, I_k] \subset I_k$, that is, I_k is an ideal in $\mathfrak{g}_k$, and that $[\mathfrak{g}_{k-1}, \mathfrak{g}_{k-1}] \subset \mathfrak{g}_{k-1} \subset \mathfrak{g}_k$, that is, $\mathfrak{g}_{k-1}$ is a Lie subalgebra of $\mathfrak{g}_k$. Both these statements can be read from the following decomposition of the commutation table:

$$\begin{array}{c|c|cc|ccc}
 & \mathfrak{g}_{k-1} & y_{k-2,+} & y_{k-2,-} & z_{1,k-2} & \cdots & z_{k-2,k-2} \\
\hline
\mathfrak{g}_{k-1} & L_{k-1} & \multicolumn{2}{c|}{W_k} & \mathbf{0} & \cdots & \mathbf{0} \\
\hline
y_{k-2,+} & \multirow{2}{*}{$-W_k^T$} & 0 & -z_{k-2,k-2} & 0 & \cdots & 0 \\
y_{k-2,-} & & z_{k-2,k-2} & 0 & 0 & \cdots & 0 \\
\hline
z_{1,k-2} & \mathbf{0}^T & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
z_{k-2,k-2} & \mathbf{0}^T & 0 & 0 & 0 & \cdots & 0 \\
\hline
\end{array} \tag{3.7}$$

where L_{k-1} is the commutation table for $\mathfrak{g}_{k-1}$, and

$$W_k := \begin{pmatrix} -y_{k-2,-} & y_{k-2,+} \\ 0 & y_{k-2,-} \\ y_{k-2,+} & 0 \\ 0 & -z_{1,k-2} \\ z_{1,k-2} & 0 \\ 0 & 0 \\ 0 & -z_{2,k-2} \\ z_{2,k-2} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -z_{3,k-2} \\ z_{3,k-2} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -z_{k-3,k-2} \\ z_{k-3,k-2} & 0 \end{pmatrix}. \tag{3.8}$$

This ends the proof of the lemma. □

So, following the statement of the previous lemma, we say that the sequence of Lie algebras $\{ \mathfrak{g}_n \}_{n\geq 2}$ form a *chain of Lie algebras*. Next, we show the Levi decomposition [86, §2.2] of the Lie algebra $\mathfrak{g}_n$:

Lemma 3.2. *The Lie algebra $\mathfrak{g}_n$ has the following Levi decomposition:*

$$\mathfrak{g}_n = \mathfrak{sl}_2(\mathbb{K}) \oplus_S \mathfrak{r}_n, \tag{3.9}$$

where $\mathfrak{r}_n$ is the radical, generated by the elements $y_{i,\pm}$, $i = 1,\dots,n-2$ and $z_{i,j}$, $i \leq j = 1,\dots,n-2$, whose nonzero commutation relations are:

$$[y_{i,+}, y_{j,-}] = z_{\min(i,j),\max(i,j)}. \tag{3.10}$$

Proof. This statement follows trivially from the commutation relations of $\mathfrak{g}_n$ (3.1). □

Remark 3.3. The radical $\mathfrak{r}_n$ is a two-step nilpotent Lie algebra which can be interpreted as a generalised Heisenberg algebra. We remark that in the case $n = 3$, it is exactly the Heisenberg algebra $\mathfrak{h}(1)$, or in the notation of [86] the nilpotent algebra $\mathfrak{n}_{1,3}$.

Since our goal is to determine the Casimir elements of the Lie algebra $\mathfrak{g}_n$, we begin by establishing how many such invariants the algebra admits:

Proposition 3.4. *The Lie algebra $\mathfrak{g}_n$ admits*

$$\nu_n = T_{n-2} + 1, \tag{3.11}$$

Casimir elements of which T_{n-2} are the central elements $z_{i,j}$ and only one is non-trivial up to multiplication by scalars and by central elements.

Proof. The proof of this result relies on the Betrametti–Blasi formula in Theorem 2.3. Consider the commutative variables in $\mathbb{K}$:

$$h, x_-, x_+, y_{1,-}, y_{1,+}, \dots, y_{n-2,-}, y_{n-2,+}, z_{1,n-2}, \dots, z_{n-2,n-2}, \tag{3.12}$$

associated with the corresponding generators of $\mathfrak{g}_n$ and taken in this order. Then, using formula (3.7) we can write the matrix $A(n)$ in (2.20) in a recursive way as:

$$A(n) = \begin{pmatrix} A(n-1) & Y(n) & O \\ -Y^T(n) & Z(n) & O \\ O & O & O \end{pmatrix}, \tag{3.13}$$

where O denotes a zero matrix of the appropriate size, and we defined:

$$Y(n) := \begin{pmatrix} -y_{n-2,-} & y_{n-2,+} \\ 0 & y_{n-2,-} \\ y_{n-2,+} & 0 \\ 0 & -z_{1,n-2} \\ z_{1,n-2} & 0 \\ 0 & 0 \\ 0 & -z_{2,n-2} \\ z_{2,n-2} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -z_{3,n-2} \\ z_{3,n-3} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -z_{n-3,n-2} \\ z_{n-3,n-2} & 0 \end{pmatrix}, \qquad Z(n) := \begin{pmatrix} 0 & -z_{n-2,n-2} \\ z_{n-2,n-2} & 0 \end{pmatrix}, \tag{3.14}$$

and the initial datum:

$$A(2) := \begin{pmatrix} 0 & -2x_- & 2x_+ \\ 2x_- & 0 & -h \\ -2x_+ & h & 0 \end{pmatrix}. \tag{3.15}$$

We aim to prove that $\operatorname{rank} A(n) = 2(n-1)$. To do so, we will extract from $A(n)$ a $2(n-1) \times 2(n-1)$ submatrix which evaluated for a particular choice of the variables (3.12) will have non-vanishing determinant. From the continuity of the determinant, this is enough to prove the statement.

Firstly, let us remove all zero lines and rows in $A(n)$: these correspond to the coefficients associated with the central elements $z_{i,j}$. This gives us a matrix with $T_n - T_{n-2} = 2(n-1)+1$ columns and rows of the following shape:

$$\tilde{A}(n) = \begin{pmatrix} 0 & -2x_- & 2x_+ & -y_{1,-} & y_{1,+} & \cdots & -y_{n-2,-} & y_{n-2,+} \\ & 0 & -h & 0 & y_{1,-} & \cdots & 0 & y_{n-2,-} \\ & & 0 & y_{1,+} & 0 & \cdots & y_{n-2,+} & 0 \\ & & & 0 & -z_{1,1} & \cdots & 0 & -z_{1,n-2} \\ & & & & \ddots & & & \vdots \\ & & & & & & & -z_{n-2,n-2} \\ & & & & & & & 0 \end{pmatrix}. \tag{3.16}$$

where we omit the subdiagonal elements since the matrix is skew symmetric. Formula (3.16) is proved easily using the recursive formula for $A(n)$ (3.13). This matrix is a skew symmetric matrix of odd dimension, meaning that its determinant must be zero and its rank is *at most* $2(n-1)$.

Now, consider the submatrix of $\tilde{A}(n)$ obtained by deleting the third row and third column and evaluating on the following value of the variables (3.12):

$$y_{i,-} = y_{i,+} = 0, \qquad z_{i,j} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases} \tag{3.17}$$

that is, the $2(n-1) \times 2(n-1)$ matrix:

$$\tilde{A}_0(n) = \begin{pmatrix} 0 & -2x_- & & & & & \\ 2x_- & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix}. \tag{3.18}$$

The determinant of the matrix $\tilde{A}_0(n)$ is clearly nonzero, and as stated above the continuity of the determinant implies the following chain of equalities:

$$\operatorname{rank} A(n) = \operatorname{rank} \tilde{A}(n) = \operatorname{rank} \tilde{A}_0 = 2(n-1). \tag{3.19}$$

This last finding implies formula (3.11) using Beltrametti–Blasi's formula in Theorem 2.3. Moreover, since the number of central elements is exactly T_{n-2}, we conclude that these are part of the above number, and then only a non-trivial Casimir invariant exists (up to a scalar multiple). This ends the proof of the proposition. □

3.2. Matrix Representations of $\mathfrak{g}_n$

In this subsection, following the ideas of [91] for the $\mathfrak{h}_6$ Lie algebra, we construct a faithful representation of $\mathfrak{g}_n$ in $\mathfrak{sl}_{2n-2}(\mathbb{K})$. The faithful representation also implies that the Jacobi identity holds in $\mathfrak{g}_n$. This is the content of the following proposition:

Proposition 3.5. *The Lie algebra $\mathfrak{g}_n$ has the following faithful representation:*

$$\begin{aligned} \rho\colon \mathfrak{g}_n &\longrightarrow \mathfrak{sl}_{2(n-1)}(\mathbb{K}) \\ x &\longmapsto X. \end{aligned} \tag{3.20}$$

where

$$x = ah + bx_+ + cx_- + \sum_{i=1}^{n-2}(d_i y_{i,+} + e_i y_{i,-}) + \sum_{1\le i\le j}^{n-2} m_{ij} z_{i,j}, \tag{3.21}$$

and

$$X = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ d_1 & d_2 & \cdots & d_{n-2} & a & b & 0 & \cdots & 0 \\ e_1 & e_2 & \cdots & e_{n-2} & c & -a & 0 & \cdots & 0 \\ 2m_{11} & m_{12} & \cdots & m_{1,n-2} & -e_1 & d_1 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots & & & \vdots \\ m_{1,n-2} & m_{2,n-2} & \cdots & 2m_{n-2,n-2} & -e_{n-2} & d_{n-2} & 0 & \cdots & 0 \end{pmatrix}. \tag{3.22}$$

Proof. We start by noting that the subspace of $\mathfrak{sl}_{2(n-1)}(\mathbb{K})$ consisting of the matrices X of the form (3.22) can be described synthetically as:

$$X = \begin{pmatrix} 0 & 0 & 0 \\ P & A & 0 \\ M & Q & 0 \end{pmatrix}, \tag{3.23}$$

where A is the 2×2 block representing $\mathfrak{sl}_2(\mathbb{K})$:

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{3.24}$$

P is the $(n-2)\times 2$ block:

$$P = \begin{pmatrix} d_1 & \dots & d_{n-2} \\ e_1 & \dots & e_{n-2} \end{pmatrix}, \tag{3.25}$$

and Q is a $2\times(n-2)$ block related to P by the formula:

$$Q = P^T J, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.26}$$

and finally M is a symmetric matrix. We will prove that this subspace is a Lie subalgebra of $\mathfrak{sl}_{2(n-1)}(\mathbb{K})$ and that its commutation relations agree with those of $\mathfrak{g}_n$, thus proving the proposition.

We start by showing that the subspace is closed under the Lie bracket of $\mathfrak{sl}_{2(n-1)}(\mathbb{K})$. By definition, $[X, X'] = XX' - X'X$, where the matrix X' has submatrices denoted with primes. So, after some easy algebraic manipulations we obtain:

$$[X, X'] = \begin{pmatrix} 0 & 0 & 0 \\ AP' - A'P & AA' - A'A & 0 \\ QP' - Q'P & QA' - Q'A & 0 \end{pmatrix}. \tag{3.27}$$

Observing that if $A, A' \in \mathfrak{sl}_2(\mathbb{K})$ then $AA' - A'A \in \mathfrak{sl}_2(\mathbb{K})$, we are left to check that the submatrices in (3.27) satisfy the two conditions:

$$QA' - Q'A = (AP' - A'P)^T J, \tag{3.28a}$$

$$(QP' - Q'P)^T = QP' - Q'P. \tag{3.28b}$$

To prove (3.28b), it is enough to observe that J is a skew symmetric matrix, so that:

$$(QP' - Q'P)^T = (P')^T (PJ)^T - P^T (P'J)^T = -(P')^T JP + P^T JP' = -Q'P + QP'. \tag{3.29}$$

On the other hand, to prove (3.28a), we observe that $A \in \mathfrak{sl}_2(\mathbb{K})$ one has $JA = -A^T J$, so that:

$$QA' - Q'A = P^T JA' - (P')^T JA = -P^T (A')^T J + (P')^T A^T J = (AP' - A'P)^T J. \tag{3.30}$$

The commutation relations can be proved using the basic matrices $E_{i,j}$ introduced in Eq. (2.6) and their commutation relations, see e.g. [46, Section IV.6]:

$$[E_{i,j}, E_{k,l}] = \delta_{jk} E_{i,l} - \delta_{il} E_{k,j}. \tag{3.31}$$

Indeed, it is easy to see that:

$$\begin{gathered}
\rho(x_+) = E_{n-1,n}, \quad \rho(x_-) = E_{n,n-1}, \quad \rho(h) = E_{n-1,n-1} - E_{n,n}, \\
\rho(y_{i,+}) = E_{n-1,i} + E_{n+i,n}, \quad \rho(y_{i,-}) = E_{n,i} - E_{n+i,n-1}, \quad i = 1, \ldots, n-2, \\
\rho(z_{i,j}) = E_{i+n,j} + E_{j+n,i}, \quad 1 \leq i \leq j \leq n-2.
\end{gathered} \tag{3.32}$$

Note that the elements $\rho(z_{i,j})$ automatically commute with all the others since the indices of their base matrices never intersect with the ones of the others and Eq. (3.31) vanishes in such a case. As a title of example, let us compute the commutation relations of h with the other elements:

$$\begin{aligned}
[\rho(h), \rho(x_+)] &= [E_{n-1,n-1} - E_{n,n}, E_{n-1,n}] = [E_{n-1,n-1}, E_{n-1,n}] - [E_{n,n}, E_{n-1,n}] \\
&= \delta_{n-1,n-1} E_{n-1,n} - \delta_{n-1,n} E_{n-1,n-1} - (\delta_{n,n-1} E_{n,n} - \delta_{n,n} E_{n-1,n}) \\
&= E_{n-1,n} + E_{n-1,n} = 2E_{n-1,n} = 2\rho(x_+), && (3.33a) \\
[\rho(h), \rho(x_-)] &= [E_{n-1,n-1} - E_{n,n}, E_{n,n-1}] \\
&= -E_{n,n-1} - E_{n,n-1} = -2E_{n,n-1} = -2\rho(x_-), && (3.33b) \\
[\rho(h), \rho(y_{i,+})] &= [E_{n-1,n-1} - E_{n,n}, E_{n-1,i} + E_{n+i,n}] \\
&= [E_{n-1,n-1}, E_{n-1,i}] - [E_{n,n}, E_{n+i,n}] \\
&= \delta_{n-1,n-1} E_{n-1,i} - \delta_{n-1,i} E_{n-1,n-1} - (\delta_{n,n+i} E_{n,n} - \delta_{n,n} E_{n+i,n}) \\
&= E_{n-1,i} + E_{n+i,n} = \rho(y_{i,+}), && (3.33c) \\
[\rho(h), \rho(y_{i,-})] &= [E_{n-1,n-1} - E_{n,n}, E_{n,i} + E_{n+i,n-1}] \\
&= -[E_{n,n}, E_{n,i}] - [E_{n-1,n-1}, E_{n+i,n-1}] \\
&= -E_{n,i} + E_{n+i,n-1} = -\rho(y_{i,-}). && (3.33d)
\end{aligned}$$

Then, all the commutation relations of the elements with h coincide with those given in (3.1). As a last check, we prove the commutation relations of the elements $y_{i,+}$ with the elements $y_{j,-}$:

$$
\begin{aligned}
[\rho(y_{i,+}),\rho(y_{j,-})] &= [E_{n-1,i}+E_{n+i,n},E_{n,j}-E_{n+j,n-1}]\\
&= [E_{n+i,n},E_{n,j}]-[E_{n-1,i},E_{n+j,n-1}]\\
&= \delta_{n,n}E_{n+i,j}-\delta_{n+i,j}E_{n,n}-(\delta_{i,n+j}E_{n-1,n-1}-\delta_{n-1,n-1}E_{n+j,i})\\
&= E_{n+i,j}+E_{n+j,i}=\rho(z_{\min(i,j),\max(i,j)}).
\end{aligned}\tag{3.34}
$$

The last equality in Eq. (3.34) follows from the fact that formula (3.32) is defined for $1\le i\le j\le n-2$. So, also these commutation relations coincide with those given in (3.1). The remaining commutation relations can be proved in the same way, thus ending the proof of the proposition. $\square$

We conclude this subsection with the following corollary which gives a faithful representation $\overline{\rho}$ of the quotient algebra $\overline{\mathfrak{g}}_n$ with image in the $n\times n$ matrices with trace zero. We leave the proof to the reader, but we observe that the matrix $\overline{\rho}(\overline{x})$ is the upper left block in the matrix $\rho(x)$.

Corollary 3.6. *The quotient algebra $\overline{\mathfrak{g}}_n=\mathfrak{g}_n/Z(\mathfrak{g}_n)$ has the following representation in $\mathfrak{sl}_n(\mathbb{K})$:*

$$
\begin{aligned}
\overline{\rho}\colon \overline{\mathfrak{g}}_n &\longrightarrow \mathfrak{sl}_n(\mathbb{K})\\
\overline{x} &\longmapsto \overline{X},
\end{aligned}\tag{3.35}
$$

where $\overline{x}$ has representative

$$
x\;=\;ah+bx_++cx_-+\sum_{i=1}^{n-2}(d_iy_{i,+}+e_iy_{i,-}),\tag{3.36}
$$

and

$$
\overline{X}=\begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 0\\
\vdots & & & & & \vdots\\
0 & 0 & \dots & 0 & 0 & 0\\
d_1 & d_2 & \dots & d_{n-2} & a & b\\
e_1 & e_2 & \dots & e_{n-2} & c & -a
\end{pmatrix}.\tag{3.37}
$$

3.3. The Coadjoint Representation in the Space of Vector Fields

We conclude this section on the generalities of the Lie algebra $\mathfrak{g}_n$ by presenting the explicit form of its coadjoint representation in the space of vector fields on $\mathfrak{g}_n^*$. This is the content of the following proposition:

Proposition 3.7. *The coadjoint representation of the Lie algebra $\mathfrak{g}_n$ in the space of vector fields $\mathfrak{X}(\mathfrak{g}^*)$ is generated by the following vector fields:*

$$
\widehat{h}=-2x_-\frac{\partial}{\partial x_-}+2x_+\frac{\partial}{\partial x_+}+\sum_{j=1}^{n-2}\left[y_{j,+}\frac{\partial}{\partial y_{j,+}}-y_{j,-}\frac{\partial}{\partial y_{j,-}}\right],\tag{3.38a}
$$

$$\widehat{x}_{\mp} = \pm 2x_{\mp}\frac{\partial}{\partial h} \mp h\frac{\partial}{\partial x_{\pm}} + \sum_{j=1}^{n-2} y_{j,\mp}\frac{\partial}{\partial y_{j,\pm}}, \tag{3.38b}$$

$$\widehat{y}_{i,\mp} = \pm y_{i,\mp}\frac{\partial}{\partial h} - y_{i,\pm}\frac{\partial}{\partial x_{\pm}} \mp \sum_{j=1}^{n-2} z_{j,i}\frac{\partial}{\partial y_{j,\pm}}, \tag{3.38c}$$

$$\widehat{z}_{i,j} = 0. \tag{3.38d}$$

The proof of Proposition 3.7 is an immediate consequence of the general result presented in Eq. (2.16). For example, from the commutation relations (3.1), one has $\rho_{\mathrm{ad}}(x_-)(x) = [x_-, x] \neq 0$ for basis element x of $\mathfrak{g}_n$ only if $x = h, x_+, y_{i,+}$ and one has

$$[x_-, h] = 2x_-, \quad [x_-, x_+] = -h, \quad [x_-, y_{i,+}] = y_{i,-}\ . \tag{3.39}$$

Thus, the vector field $\widehat{x}_-$ is given by

$$\widehat{x}_- = 2x_-\frac{\partial}{\partial h} - h\frac{\partial}{\partial x_+} + \sum_{j=1}^{n-2} y_{j,-}\frac{\partial}{\partial y_{j,+}}, \tag{3.40}$$

since $\widehat{x}_-(x) = [x_-, x]$ for any $x \in \mathfrak{g}_n$.

We observe that, as expected, the coadjoint representation is not faithful because the central elements all map to the null vector field. Thus, the coadjoint representation is in fact a representation of $\bar{\mathfrak{g}}_n = \mathfrak{g}_n/Z(\mathfrak{g}_n)$.

4. Construction of the Casimir Invariants of $\mathfrak{g}_n$

In this section, we review the construction of the Casimir elements for the Lie algebras $\mathfrak{g}_2 \cong \mathfrak{sl}_2(\mathbb{K})$ and $\mathfrak{g}_3 \cong \mathfrak{h}_6$ with the standard method of vector fields, and we computationally extend the result to the case of $\mathfrak{g}_4$. From these low-dimensional cases, we are able to guess the general form of the Casimir polynomial. After a short digression on the properties of the $\mathfrak{sl}_n(\mathbb{K})$ Lie algebra and its invariants, we prove the guess in the main theorem of this section and of the whole paper.

4.1. Low-Dimensional Cases: $n = 2, 3, 4$

Let us start with the Lie algebra $\mathfrak{g}_2 \cong \mathfrak{sl}_2(\mathbb{K})$, and let us find its Casimir element using the coadjoint action on the space of vector fields. In such a case, we have to solve the following system of overdetermined PDEs for the function $F = F(h, x_-, x_+)$, obtained from (3.38) with $n = 2$:

$$\widehat{h}(F) = -2x_-\frac{\partial F}{\partial x_-} + 2x_+\frac{\partial F}{\partial x_+} = 0, \quad \widehat{x}_{\mp}(F) = \pm 2x_{\mp}\frac{\partial F}{\partial h} \mp h\frac{\partial F}{\partial x_{\pm}} = 0. \tag{4.1}$$

Solving these three equations using for instance the computer algebra system `Maple`, we get the following solution:

$$F = F\left(h^2 + 4x_+x_-\right), \tag{4.2}$$

where F is an arbitrary function of its argument, to which corresponds the only functionally independent polynomial Casimir element in $S\mathfrak{g}_2$:

$$C_2 = h^2 + 4x_+x_-. \tag{4.3}$$

This is of course a well-known result, see, for instance, [86, §16.5].

We observe that one can write the (functional independent) Casimir of $S\mathfrak{g}_2$ (4.3) as the determinant of a 2×2 symmetric matrix in the following way:

$$C_2 = -\det \begin{pmatrix} -2x_- & h \\ h & 2x_+ \end{pmatrix}. \tag{4.4}$$

Remark 4.1. As stated in Sect. 2 Casimir elements in $S\mathfrak{g}$ are usually found just by looking at homogeneous polynomial solutions of the system of PDEs (2.19). In this simple case, we choose to present the general solution of the system (4.1) since it is one of the few cases where using a computer algebra system it is possible to obtain such a general solution in a reasonable amount of time.

Let us now consider the next Lie algebra in the hierarchy, namely $\mathfrak{g}_3 \cong \mathfrak{h}_6$. To find the Casimir elements in $S\mathfrak{g}_3$, we use again the PDE system obtained from the coadjoint representation from (3.38) with $n = 3$:

$$\widehat{h}(F) = -2x_-\frac{\partial F}{\partial x_-} + 2x_+\frac{\partial F}{\partial x_+} + y_{1,+}\frac{\partial F}{\partial y_{1,+}} - y_{1,-}\frac{\partial F}{\partial y_{1,-}} = 0, \tag{4.5a}$$

$$\widehat{x}_{\mp}(F) = \pm 2x_{\mp}\frac{\partial F}{\partial h} \mp h\frac{\partial F}{\partial x_{\pm}} + y_{1,\mp}\frac{\partial F}{\partial y_{1,\pm}} = 0, \tag{4.5b}$$

$$\widehat{y}_{1,\pm}(F) = \pm y_{1,\mp}\frac{\partial F}{\partial h} - y_{1,\pm}\frac{\partial F}{\partial x_{\pm}} \mp z_{1,1}\frac{\partial F}{\partial y_{1,\pm}} = 0, \tag{4.5c}$$

$$\widehat{z}_{1,1}(F) = 0, \tag{4.5d}$$

for the unknown function $F = F(h, x_-, x_+, y_{1,-}, y_{1,+}, z_{1,1})$. Now, instead of a generic function F, since we work in the symmetric algebra $S\mathfrak{g}_n$ we can restrict our search to homogeneous Casimir polynomials P of degree l, namely:

$$P^{(l)} = \sum_{i_1+\ldots+i_6=l} c_{i_1,\ldots,i_6} h^{i_1} x_-^{i_2} x_+^{i_3} y_{1,-}^{i_4} y_{1,+}^{i_5} z_{1,1}^{i_6}. \tag{4.6}$$

Substituting the expression (4.6) into the system (4.5) for $l = 1, 2$, we obtain only trivial solutions involving the central element $z_{1,1}$, but we find, up to an unessential multiplicative constant, the following degree three Casimir element:

$$C_3 = 2x_+y_{1,-}^2 - 2x_-y_{1,+}^2 + 2hy_{1,-}y_{1,+} + z_{1,1}(h^2 + 4x_-x_+), \tag{4.7}$$

where $C_3 = P^{(3)}|_{\text{solution of (4.5)}}$. This Casimir element C_3 corresponds, mutatis mutandis, to the known Casimir element of the two-photon Lie algebra $\mathfrak{h}_6$, see, for instance, [5, 8, 18].

Even more interestingly, we observe that that it is possible to write this degree three Casimir of $S\mathfrak{g}_3$ (4.7) as the determinant of a 3×3 symmetric matrix with coefficients in $\mathfrak{g}_3 \subset S\mathfrak{g}_3 = \mathrm{Pol}(\mathfrak{g}_3^*)$:

$$C_3 = -\det \begin{pmatrix} z_{1,1} & -y_{1,-} & y_{1,+} \\ -y_{1,-} & -2x_- & h \\ y_{1,+} & h & 2x_+ \end{pmatrix}. \tag{4.8}$$

The idea of considering a determinant came from a study of this polynomial Casimir. As a homogeneous cubic in six variables, it defines a hypersurface in $\mathbb{P}^5$. One finds that it has a 2-dimensional singular locus, which is rather special. At this point, one remembers that the determinant of a 3×3 symmetric matrix defines a similar hypersurface, which is singular in the symmetric matrices of rank one; this gives a 2-dimensional singular locus. With some trial and error, one finds the expression for C_3 given above.

In the case $n = 4$, for $l = 1, 2, 3, 4$, we have to look for homogeneous polynomial solutions of the system of PDEs

$$\begin{aligned}
\widehat{h}(P^{(l)}) &= -2x_- \frac{\partial P^{(l)}}{\partial x_-} + 2x_+ \frac{\partial P^{(l)}}{\partial x_+} + \sum_{j=1}^{2} \left[y_{j,+} \frac{\partial P^{(l)}}{\partial y_{j,+}} - y_{j,-} \frac{\partial P^{(l)}}{\partial y_{j,-}} \right] = 0, \\
\widehat{x}_{\mp}(P^{(l)}) &= \pm 2x_{\mp} \frac{\partial P^{(l)}}{\partial h} \mp h \frac{\partial P^{(l)}}{\partial x_\pm} + \sum_{j=1}^{2} y_{j,\mp} \frac{\partial P^{(l)}}{\partial y_{j,\pm}} = 0, \\
\widehat{y}_{i,\pm}(P^{(l)})) &= \pm y_{i,\mp} \frac{\partial P^{(l)}}{\partial h} - y_{i,\pm} \frac{\partial P^{(l)}}{\partial x_\pm} \mp \sum_{j=1}^{2} z_{j,i} \frac{\partial P^{(l)}}{\partial y_{j,\pm}} = 0, \quad i = 1, 2 \\
\widehat{z}_{1,1}(P^{(l)}) &= \widehat{z}_{1,2}(P^{(l)}) = \widehat{z}_{2,2}(P^{(l)}) = 0.
\end{aligned} \tag{4.9}$$

With the same notation as above, also in this case for $l = 1, 2, 3$ one finds trivial solutions related to the centre of $\mathfrak{g}_4$ and finally a new non-trivial element for $l = 4$. This non-trivial element can be written down as the determinant of a symmetric matrix with coefficients in $\mathfrak{g}_4 \subset S\mathfrak{g}_4 = \mathrm{Pol}(\mathfrak{g}_4^*)$:

$$C_4 = -\det \begin{pmatrix} z_{1,1} & z_{1,2} & -y_{1,-} & y_{1,+} \\ z_{1,2} & z_{2,2} & -y_{2,-} & y_{2,+} \\ -y_{1,-} & -y_{2,-} & -2x_- & h \\ y_{1,+} & y_{2,+} & h & 2x_+ \end{pmatrix}. \tag{4.10}$$

So, from the result above one can guess that the form of the Casimir polynomial for general $n \geq 2$ is:

$$C_n = -\det(M_n), \tag{4.11}$$

where M_n is the following $n \times n$ symmetric matrix, with coefficients in $\mathfrak{g}_n$:

$$M_n := \begin{pmatrix} z_{1,1} & \cdots & z_{1,n-2} & -y_{1,-} & y_{1,+} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ z_{1,n-2} & \cdots & z_{n-2,n-2} & -y_{n-2,-} & y_{n-2,+} \\ -y_{1,-} & \cdots & -y_{n-2,-} & -2x_- & h \\ y_{1,+} & \cdots & y_{n-2,+} & h & 2x_+ \end{pmatrix}. \tag{4.12}$$

Experimentally, one can check that this formula provides the Casimir polynomials also for $n = 5, 6, 7, 8$. However, by simple observation it is not possible to establish that this will hold for all n. To do so requires a change of perspective.

4.2. The $\mathfrak{sl}_n(\mathbb{K})$ Case

In this subsection, we recall the crucial facts about determinants and $\mathrm{SL}_n(\mathbb{K})$ invariance, interpreted in terms of representation theory of $\mathfrak{sl}_n(\mathbb{K})$.

The determinant of a symmetric $n \times n$ matrix N is a homogeneous polynomial of degree n in the coefficients $(N)_{ij} = n_{ij}$, $1 \leq i \leq j \leq n$, of N. It is invariant under the map $N \mapsto ANA^T$ for $A \in \mathrm{SL}_n(\mathbb{K})$, that is $\det(N) = \det\big(ANA^T\big)$. The vector space of symmetric matrices can be identified with $\mathrm{Sym}^2(\mathbb{K}^n)$, and $N \mapsto ANA^T$ is the representation of the Lie group $\mathrm{SL}_n(\mathbb{K})$ (for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) on $\mathrm{Sym}^2(\mathbb{K}^n)$.

Let $A(t) \in \mathrm{SL}_n(\mathbb{K})$ be a one parameter subgroup with $A(0) = I$, $A'(0) = X \in \mathfrak{sl}_n(\mathbb{K})$ and $A(s+t) = A(s)A(t)$. Differentiating the matrix $A(t)NA(t)^T$ with respect to t and evaluating at $t = 0$ gives:

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t)NA(t)^T \bigg|_{t=0} = XN + NX^T , \tag{4.13}$$

which is indeed $\rho^{(2)}(X)(N)$ for the representation $\rho^{(2)}$ of $\mathfrak{sl}_n(\mathbb{K})$ on $\mathrm{Sym}^2(\mathbb{K}^n)$, see (2.7).

We now consider the coefficients n_{ij}, $1 \leq i \leq j \leq n$, of a symmetric $n \times n$ matrix N as independent polynomial variables. Then, the representation $\rho^{(2)}$ on the linear forms n_{ij} above induces, by derivations, a representation of $\mathfrak{sl}_n(\mathbb{K})$ on the polynomials in $\mathbb{K}[\ldots, n_{ij}, \ldots]$. Thus, as we have seen in Sect. 2, we get a representation of $\mathfrak{sl}_n(\mathbb{K})$ in the vector fields on $\mathbb{K}^{n(n+1)/2}$, defined as:

$$\begin{aligned} \mathfrak{sl}_n(\mathbb{K}) &\longrightarrow \mathfrak{X}(\mathbb{K}^{n(n+1)/2}) \\ X &\longmapsto \widehat{X} := \sum_{1 \leq i \leq j \leq n} \ell_{ij}(X) \frac{\partial}{\partial n_{ij}} , \end{aligned} \tag{4.14}$$

where

$$\ell_{ij}(X) := (XN + NX^T)_{ij} . \tag{4.15}$$

This representation has the property that for $G \in \mathbb{K}[\ldots, n_{ij}, \ldots]$ one has

$$\widehat{X}(G) = \frac{\mathrm{d}}{\mathrm{d}t} G(A(t)NA(t)^T) \bigg|_{t=0} = \sum_{1 \leq i \leq j \leq n} \frac{\partial G}{\partial n_{ij}} (XN + NX^T)_{ij} , \tag{4.16}$$

where $A(t) = \exp(tX) \in \mathrm{SL}_n(\mathbb{K})$ is the one parameter subgroup generated by $X \in \mathfrak{sl}_n(\mathbb{K})$.

The function $\det(N) \in \mathbb{K}[\ldots, n_{ij}, \ldots]$ is a polynomial, of degree n, in the n_{ij}. The function $\det(N)$, being invariant under the action of $\mathrm{SL}_n(\mathbb{K})$ on N so $\det\big(A(t)NA(t)^T\big)$ is constant in t, is annihilated by the vector fields $\widehat{X}$:

$$\widehat{X}(\det(N)) = 0, \qquad \forall\, X \in \mathfrak{sl}_n(\mathbb{K}) . \tag{4.17}$$

Example 4.2. We consider the case $n = 2$. The Lie algebra $\mathfrak{sl}_n(\mathbb{K})$ consists of the matrices with trace zero, so we have:

$$N = \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix}, \qquad \mathfrak{sl}_2(\mathbb{K}) = \left\{ X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : \, a, b, c \in \mathbb{K} \right\} . \tag{4.18}$$

Since

$$XN + NX^T = \begin{pmatrix} 2an_{11} + 2bn_{12} & cn_{11} + bn_{22} \\ cn_{11} + bn_{22} & 2cn_{12} - 2an_{22} \end{pmatrix} \tag{4.19}$$

we find for $X \in \mathfrak{sl}_2(\mathbb{K})$ the vector field in $\mathfrak{X}(\mathbb{K}^3)$:

$$\widehat{X} = 2(an_{11} + bn_{12})\frac{\partial}{\partial n_{11}} + (cn_{11} + bn_{22})\frac{\partial}{\partial n_{12}} + 2(cn_{12} - an_{22})\frac{\partial}{\partial n_{22}}, \tag{4.20}$$

or more explicitly considering the standard basis of $\mathfrak{sl}_2(\mathbb{K})$ we have three vector fields:

$$\widehat{X}_h = 2n_{11}\frac{\partial}{\partial n_{11}} - 2n_{22}\frac{\partial}{\partial n_{22}} \tag{4.21a}$$

$$\widehat{X}_{x_+} = 2n_{12}\frac{\partial}{\partial n_{11}} + n_{22}\frac{\partial}{\partial n_{12}}, \tag{4.21b}$$

$$\widehat{X}_{x_-} = n_{11}\frac{\partial}{\partial n_{12}} + 2n_{12}\frac{\partial}{\partial n_{22}}, \tag{4.21c}$$

such that $\widehat{X} = a\widehat{X}_h + b\widehat{X}_{x_+} + c\widehat{X}_{x_-}$. As guaranteed by the way we defined these vector fields, we then have

$$\widehat{X}_x(\det N) = \widehat{X}_x(n_{11}n_{22} - n_{12}^2) = 0, \quad x = h, x_+, x_-, \tag{4.22}$$

and this is of course also easy to verify. This is clearly equivalent, up to a change of basis, to the vector field approach we presented in Sect. 4.

4.3. The $\mathfrak{g}_n$ Case

Now we prove the following statement:

Theorem 4.3. *The functionally independent Casimir invariants of the Lie algebra $\mathfrak{g}_n$ are the linear central elements $z_{i,j}$ and the nth degree polynomial C_n given in equation 4.11.*

To prove Theorem 4.3, we need to show that the determinant of the matrix M_n (4.12) is annihilated by the vector field $\widehat{x}$ on $\mathfrak{g}_n^*$ for all $x \in \mathfrak{g}_n$. Since M_n is a symmetric matrix, we want to apply the ideas discussed in the previous subsection.

Notice that $\mathfrak{g}_n \subset S\mathfrak{g}_n = \mathbb{K}[h, x_-, x_+, \ldots, z_{n-2,n-2}]$ and that each of the $n(n+1)/2$ variables appears as a coefficient (up to constants) of M_n, so that M_n is basically the matrix N above, if we make the appropriate identification of the variables.

We will show in Lemma 4.4 that for each $x \in \mathfrak{g}_n$ there is a matrix $X_x \in \mathfrak{sl}_n(\mathbb{K})$ such that $\rho_{\text{ad}}(x)(M_n) = X_x M_n + M_n X_x$. Here by $\rho_{\text{ad}}(x)(M_n)$ we mean the matrix obtained by applying $\rho_{\text{ad}}(x)$ to all coefficients of M_n. From this, we will easily deduce that $\hat{x}(\det(M_n)) = 0$.

Since the $z_{i,j}$ are in the centre of $\mathfrak{g}_n$, which is the kernel of adjoint representation, the vector fields $\widehat{z}_{i,j}$ are zero. Thus, we only need to consider the quotient Lie algebra $\overline{\mathfrak{g}}_n$. This quotient algebra has a faithful representation in

$\mathfrak{sl}_n(\mathbb{K})$, see Corollary 3.6. We will find it convenient to consider the composition:

$$\begin{tikzcd} \mathfrak{g}_n \arrow[r] \arrow[dr] & \overline{\mathfrak{g}}_n \arrow[d] \\ & \mathfrak{sl}_n(\mathbb{K}) \end{tikzcd} \tag{4.23}$$

again denoted by $\overline{\rho}$, and whose action is:

$$\begin{aligned} \overline{\rho}\colon \mathfrak{g}_n &\longrightarrow \mathfrak{sl}_n(\mathbb{K}) \\ x &\longmapsto \overline{\rho}(x) \;=\; \begin{pmatrix} 0 & 0 \\ P & A \end{pmatrix} \;, \end{aligned} \tag{4.24}$$

where:

$$x \;=\; ah + bx_+ + cx_- + \sum_{j=1}^{n-2} (d_i y_{i,+} + e_i y_{i,-}), \tag{4.25}$$

with P and A defined by Eqs. (3.24) and (3.25), respectively. We now show that $X_x = -\overline{\rho}(x)$.

Lemma 4.4. *For $x \in \mathfrak{g}_n$, we have:*

$$\rho_{\mathrm{ad}}(x)(M_n) \;=\; -\big(\overline{\rho}(x)M_n \;+\; M_n\overline{\rho}(x)^T\big) \;. \tag{4.26}$$

Proof. For convenience, we write M_n in block form

$$M_n = \begin{pmatrix} S & U \\ U^T & R \end{pmatrix}, \tag{4.27}$$

where the (sub)matrices S, U, and R are defined as follows:

$$S = \begin{pmatrix} z_{1,1} & \cdots & z_{1,n-2} \\ \vdots & \ddots & \vdots \\ z_{1,n-2} & \cdots & z_{n-2,n-2} \end{pmatrix}, \quad U = \begin{pmatrix} -y_{1,-} & y_{1,+} \\ \vdots & \vdots \\ -y_{n-2,-} & y_{n-2,+} \end{pmatrix}, \quad R = \begin{pmatrix} -2x_- & h \\ h & 2x_+ \end{pmatrix}. \tag{4.28}$$

Note that the matrix S is symmetric. Since $\rho_{\mathrm{ad}}(x)(z_{i,j}) = [x, z_{i,j}] = 0$, the upper left block of $\rho_{\mathrm{ad}}(X)(M_n)$ is zero. For the upper right block of $\rho_{\mathrm{ad}}(X)(M_n)$, we use that

$$\rho_{\mathrm{ad}}(ah + bx_+ + cx_-)(-y_{i,-}) \;=\; ay_{i,-} - by_{i,+}, \tag{4.29a}$$

$$\rho_{\mathrm{ad}}(ah + bx_+ + cx_-)(y_{i,+}) \;=\; ay_{i,+} + cy_{i,-} \;, \tag{4.29b}$$

leading to the term $-UA^T$ in the upper right block and its transpose in the lower right block. Similarly:

$$\rho_{\mathrm{ad}}\left(\sum_{j=1}^{n-2}(d_j y_{+,j} + e_j y_{j,-})\right)(-y_{i,-}) \;=\; -\sum_{j=1}^{n-2} d_j z_{i,j}, \tag{4.30a}$$

$$\rho_{\mathrm{ad}}\left(\sum_{j=1}^{n-2}(d_j y_{+,j} + e_j y_{j,-})\right)(y_{i,+}) \;=\; -\sum_{j=1}^{n-2} e_j z_{i,j}, \tag{4.30b}$$

hence $\rho_{\mathrm{ad}}(\sum_j (d_j y_{+,i} + e_j y_{j,-}))$ maps U to $-SP^T$. For the lower right block, we compute:

$$\begin{aligned}\rho_{\mathrm{ad}}(ah + bx_+ + cx_-)(R) &= \begin{pmatrix} 4ax_- - 2bh & -2bx_+ + 2cx_- \\ -2bx_+ + 2cx_- & 2ch - 4ax_+ \end{pmatrix} \\ &= -\left(AR + RA^T\right) .\end{aligned} \tag{4.31}$$

Similarly:

$$\begin{aligned}\rho_{\mathrm{ad}}(\sum (d_j y_{+,j} + e_j y_{j,-})(R) &= \begin{pmatrix} 2\sum_j d_j y_{j,-} & \sum_j (-d_j y_{j,+} + e_j y_{j,-}) \\ \sum_j (-d_j y_{j,+} + e_j y_{j,-}) & -2\sum_j d_j y_{j,+} \end{pmatrix} \\ &= -\left(PU + U^T P^T\right) .\end{aligned} \tag{4.32}$$

Putting everything together, we showed that:

$$\rho_{\mathrm{ad}}(x)(M_n) = \begin{pmatrix} 0 & -(UA^T + SP^T) \\ -(AU^T + PS) & -(AR + RA^T + PU + U^T P^T) \end{pmatrix}. \tag{4.33}$$

On the other hand, we can compute the action $\overline{\rho}(x)$ directly and we obtain the desired equality:

$$\begin{aligned}\overline{\rho}(x) M_n + M_n \overline{\rho}(x)^T &= \begin{pmatrix} 0 & 0 \\ P & A \end{pmatrix} \begin{pmatrix} S & U \\ U^T & R \end{pmatrix} + \begin{pmatrix} S & U \\ U^T & R \end{pmatrix} \begin{pmatrix} 0 & P^T \\ 0 & A^T \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ PS + AU^T & PU + AR \end{pmatrix} + \begin{pmatrix} 0 & SP^T + UA^T \\ 0 & U^T P^T + RA^T \end{pmatrix} \\ &= \begin{pmatrix} 0 & SP^T + UA^T \\ PS + AU^T & PU + U^T P^T + AR + RA^T \end{pmatrix}. \\ &= -\rho_{\mathrm{ad}}(x)(M_n) .\end{aligned} \tag{4.34}$$

The ends the proof of the lemma. □

We are in place to complete the proof of Theorem 4.3.

Proof of Theorem 4.3. For $x \in \mathfrak{g}_n$, we define $X_x := -\overline{\rho}(x) \in \mathfrak{sl}_n(\mathbb{K})$. Then the vector field $\widehat{X}_x$ has the property that $\widehat{X}_x(y) = \rho_{\mathrm{ad}}(x)(y)$ for all $y \in \mathfrak{g}_n \subset S\mathfrak{g}_n$ and thus $\widehat{X}_x = \rho_{\mathrm{ad}}(x)$ for all $x \in \mathfrak{g}_n$. For $x \in \mathfrak{g}_n$, let $A(t) = \exp(tX_x) \in \mathrm{SL}_n(\mathbb{K})$, then:

$$\rho_{\mathrm{ad}}(x)(\det M_n) = \frac{\mathrm{d}}{\mathrm{d}t} \det\left(A(t) N A(t)^T\right)\Big|_{t=0} = 0, \tag{4.35}$$

where the first equality follows from Lemma 4.4 and the second from the invariance of the determinant. Thus, $C_n := -\det(M_n)$ is a polynomial Casimir for $\mathfrak{g}_n$. □

In the next section, as a physical application in classical mechanics within the framework of coalgebra symmetry, we derive a one degree of freedom canonical symplectic realisation of the Lie–Poisson coalgebra $(\mathscr{F}(\mathfrak{g}_n^*), \{,\}, \Delta)$ and we use the information on its polynomial invariant to construct Hamiltonian systems in arbitrary dimensions with a given number of first integrals obtained by the coalgebra symmetry through the Casimir of degree n reported in equation 4.11.

5. Canonical Symplectic Realisations and Hamiltonian Systems Associated with the Lie–Poisson Coalgebra Constructed from $\mathfrak{g}_n$

In this section, we present a hierarchy of Hamiltonian models arising from the Lie–Poisson algebra $S\mathfrak{g}_n = \mathrm{Pol}(\mathfrak{g}_n^*) \subset \mathscr{F}(\mathfrak{g}_n^*)$ using the coalgebra symmetry approach as detailed in Sect. 2. To do so, we will consider as a base case the simplest symplectic manifold possible, namely $\mathbb{R}^2$ with local coordinates (q, p), the canonical symplectic form $\omega = \mathrm{d}q \wedge \mathrm{d}p$, and Poisson bracket:

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}, \quad f, g \in \mathscr{C}^\infty(\mathbb{R}^2). \tag{5.1}$$

Our final result will be that to the Lie–Poisson algebra constructed from the Lie algebra $\mathfrak{g}_n$ corresponds a hierarchy of Hamiltonian systems in N degrees of freedom (obtained from a canonical one degree of freedom realisation) possessing a grand total of $2N-(2n-2)$ first integrals, of which $N-(n-2)$ are in involution. These Hamiltonian systems will be, in general, non-integrable. The first two cases, namely $n = 2, 3$, are well known in the literature, having being studied in many different papers [8,9,11,13,14]. Now, we will detail this construction, starting from the basic cases $n = 2, 3, 4$ and confront our results with the literature.

5.1. The Case $n = 2$: $\mathfrak{g}_2$ Coalgebra Symmetry

The case $n = 2$ corresponds to the Lie–Poisson coalgebra associated with $\mathfrak{sl}_2(\mathbb{R})$. The coalgebra symmetry approach is well known in the literature, see e.g. [6,9]. We use this example to illustrate in detail the construction outlined in Sect. 2.2. The reader who already knows the method can safely jump to Sect. 5.4.

It is possible to show that the following map:

$$\begin{aligned} \mathrm{D}_2 \colon S\mathfrak{g}_2 &\longrightarrow \mathscr{C}^\infty(\mathbb{R}^2) \\ (h, x_-, x_+) &\longmapsto (qp, -q^2/2, p^2/2), \end{aligned} \tag{5.2}$$

gives rise to a one degree of freedom realisation of the linear elements of the symmetric algebra $S\mathfrak{g}_2$. Indeed, by direct computation we have that:

$$\{\mathrm{D}_2(x_+), \mathrm{D}_2(x_-)\} = \mathrm{D}_2(h), \quad \{\mathrm{D}_2(h), \mathrm{D}_2(x_\pm)\} = \pm 2\,\mathrm{D}_2(x_\pm), \tag{5.3}$$

where we used the canonical Poisson bracket (5.1). When we consider the canonical realisation the Casimir assumes the value:

$$C_2(q, p) = \mathrm{D}_2(C_2) = \mathrm{D}_2(h)^2 + 4\,\mathrm{D}_2(x_-)\,\mathrm{D}_2(x_+) \equiv 0. \tag{5.4}$$

At this level, we can take as one degree of freedom Hamiltonian any polynomial (and then extend to any smooth/analytic function) of the generators of the Lie–Poisson algebra $S\mathfrak{g}_2$, i.e.:

$$H_2 = H_2(\mathrm{D}_2(h), \mathrm{D}_2(x_-), \mathrm{D}_2(x_+)) = H_2(qp, -q^2/2, p^2/2). \tag{5.5}$$

To give a concrete example, the harmonic oscillator arises with the choice:

$$H_{\mathrm{ho}} = \mathrm{D}_2(x_+) - \omega^2\, \mathrm{D}_2(x_-) \tag{5.6}$$

in (5.5). Now, we can use the primitive coproduct Δ to obtain the generators of $S\mathfrak{g}_2$ within $(S\mathfrak{g}_2)^{\otimes 2}$, namely:

$$\Delta(h) = h \otimes 1 + 1 \otimes h, \quad \Delta(x_\pm) = x_\pm \otimes 1 + 1 \otimes x_\pm. \tag{5.7}$$

Applying this to the Hamiltonian (5.5), we obtain its two degrees of freedom version:

$$\begin{aligned} H_2^{(2)} &:= \Delta(H_2) = H_2(\Delta(h), \Delta(x_-), \Delta(x_+)) \\ &= H_2(h \otimes 1 + 1 \otimes h, x_- \otimes 1 + 1 \otimes x_-, x_+ \otimes 1 + 1 \otimes x_+). \end{aligned} \tag{5.8}$$

In the same way, we obtain the Casimir element:

$$\begin{aligned} C_2^{(2)} &:= \Delta(C_2) = \Delta(h)^2 + 4\Delta(x_-)\Delta(x_+) \\ &= C_2 \otimes 1 + 1 \otimes C_2 + 2\big(h \otimes h + 2(x_- \otimes x_+ + x_+ \otimes x_-)\big). \end{aligned} \tag{5.9}$$

From this last formula, we see how the coproduct produces non-trivial elements by "intertwining" elements coming from different sites.

Now, we come to the realisation. By direct computation, we see that applying the realisation $\mathrm{D}_2 \colon S\mathfrak{g}_2 \longrightarrow \mathscr{C}^\infty(\mathbb{R}^2)$ to the generators (5.7) we obtain:

$$\mathrm{D}^{\otimes 2}(\Delta(h)) = q_1p_1 + q_2p_2, \quad \mathrm{D}^{\otimes 2}(\Delta(x_-)) = -\frac{q_1^2 + q_2^2}{2}, \quad \mathrm{D}^{\otimes 2}(\Delta(x_+)) = \frac{p_1^2 + p_2^2}{2}. \tag{5.10}$$

That is, we obtain the following canonical two degrees of freedom realisation:

$$\begin{aligned} (\mathrm{D}_2)^{\otimes 2} \colon S\mathfrak{g}_2^{\otimes 2} &\longrightarrow \mathscr{C}^\infty(\mathbb{R}^4) \\ (\Delta(h), \Delta(x_-), \Delta(x_+)) &\longmapsto (q_1p_1 + q_2p_2, -(q_1^2 + q_2^2)/2, (p_1^2 + p_2^2)/2). \end{aligned} \tag{5.11}$$

Clearly, this is the same as an application of Lemma 2.8. So, in the realisation we have the two degrees of freedom Hamiltonian:

$$H_2^{(2)}(q_1, p_1, q_2, p_2) = H_2(q_1p_1 + q_2p_2, -(q_1^2 + q_2^2)/2, (p_1^2 + p_2^2)/2), \tag{5.12}$$

and the Casimir function becomes:

$$C_2^{(2)}(q_1, p_1, q_2, p_2) = -L_{12}^2, \quad L_{12} = q_1p_2 - q_2p_1. \tag{5.13}$$

The minus sign is of course irrelevant. Note that taking into account the first integral $C_2^{(2)}$ the Hamiltonian system generated by $H_2^{(2)}$ is automatically Liouville integrable, which for $N = 2$ coincides with the notion of quasi-maximal superintegrability, being $N = 2N - 2$ for $N = 2$. For example, for $H_2 = H_{\mathrm{ho}}$ we obtain:

$$H_{\mathrm{ho}}^{(2)}(q_1, p_1, q_2, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\omega^2}{2}(q_1^2 + q_2^2), \tag{5.14}$$

i.e. the two-dimensional isotropic harmonic oscillator. Such a system admits additional first integrals making it maximally superintegrable. As far as we know today, the additional integrals are not of coalgebraic origin, see for instance [76] for a discussion of this problem in the quantum case.

Again by direct computation, or by application of Lemma 2.8 we produce the following canonical N degrees of freedom realisation with Darboux coordinates $(\mathbf{q},\mathbf{p}) = (q_1,\dots,q_N,p_1,\dots,p_N)$

$$\begin{aligned} (\mathrm{D}_2)^{\otimes N}\colon S\mathfrak{g}_2^{\otimes N} &\longrightarrow \mathscr{C}^\infty(\mathbb{R}^{2N}) \\ (\Delta^{(N)}(h),\Delta^{(N)}(x_-),\Delta^{(N)}(x_+)) &\longmapsto (\mathbf{q}\cdot\mathbf{p}, -\mathbf{q}^2/2, \mathbf{p}^2/2), \end{aligned} \tag{5.15}$$

where we used the vector notation. In the same way, we obtain the following abstract Hamiltonian:

$$H_2^{(N)} := \Delta(H_2) = H_2(\Delta^{(N)}(h),\Delta^{(N)}(x_-),\Delta^{(N)}(x_+))\,, \tag{5.16}$$

which in the realisation (5.15) becomes the function:

$$H_2^{(N)}(\mathbf{q},\mathbf{p}) = H_2(\mathbf{q}\cdot\mathbf{p}, -\mathbf{q}^2/2, \mathbf{p}^2/2). \tag{5.17}$$

Applying Theorem 2.7, we have that the above Hamiltonian is endowed with the following left and right Casimir functions:

$$C_2^{(m)}(\mathbf{q},\mathbf{p}) = -\sum_{1\le i<j}^{m} L_{ij}^2, \quad C_{2,(m)}(\mathbf{q},\mathbf{p}) = -\sum_{N-m+1\le i<j}^{N} L_{ij}^2, \quad m=2,\dots,N, \tag{5.18}$$

where $L_{ij} = q_i p_j - q_j p_i$ are the components of the angular momentum tensor. Let us remark that the subscripts appearing in the left and right Casimirs refer to the $n=2$ case of $\mathfrak{g}_n$. Again, the minus sign is irrelevant. Still following the statement of Theorem 2.7 the $C_2^{(m)}$ and the $C_{2,(m)}$ form two sets of commuting first integrals. We also recall that, from the general theory, $C_2^{(N)} = C_{2,(N)}$.

Adding the Hamiltonian $H_2^{(N)}$ to the two sets of integrals $L_2 := \{C_2^{(2)},\dots, C_2^{(N)}\}$ and $R_2 := \{\, C_{2,(2)},\dots,C_{2,(N)}\,\}$ we obtain two sets of N functionally independent commuting first integrals:

$$\widetilde{L}_2 := \{\, H_2^{(N)}, C_2^{(2)},\dots,C_2^{(N)}\,\}, \quad \widetilde{R}_2 := \{\, H_2^{(N)}, C_{2,(2)},\dots,C_{2,(N)}\,\}, \tag{5.19}$$

thus proving that the obtained Hamiltonian system is always Liouville integrable, see also [6]. This kind of property is called *multi-integrability.* In total, it is easy to check, as all first integrals except $H_2^{(N)}$ and $C_2^{(N)} = C_{2,(N)}$ are local, that we have a set of $[(N-1)+(N-1)-1]+1 = 2N-2$ functionally independent first integrals:

$$I_2 := \{\, H_2^{(N)}, C_2^{(2)},\dots,C_2^{(N)}, C_{2,(2)},\dots,C_{2,(N-1)}\,\}\,. \tag{5.20}$$

So, the Hamiltonian system is quasi-maximally superintegrable (QMS), being just one constant missing for maximal superintegrability. In some particular cases, like the harmonic oscillator discussed above, there are additional functionally independent constants leading to maximal superintegrability, see, for instance, [55] and references therein.

5.2. The Case $n = 3$: $\mathfrak{g}_3$ Coalgebra Symmetry

The case $n = 3$ is associated with the Lie algebra $\mathfrak{h}_6$ [91]. The coalgebra symmetry approach to this case has been developed in [6,8,18].

It is possible to show by direct computation that the map $\mathrm{D}_3 \colon S\mathfrak{g}_3 \longrightarrow \mathscr{C}^\infty(\mathbb{R}^2)$ whose action on the generating set of $S\mathfrak{g}_3$, $\{\, h, x_-, x_+, y_{1,-}, y_{1,+}, z_{1,1} \,\}$, is given by:

$$h \mapsto qp\,, \quad x_- \mapsto -q^2/2, \quad x_+ \mapsto p^2/2\,, \quad y_{1,-} \mapsto -\alpha q\,, \quad y_{1,+} \mapsto \alpha p\,, \quad z_{1,1} \mapsto \alpha^2, \tag{5.21}$$

where $\alpha \in \mathbb{R}$ provides a one degree of freedom canonical symplectic realisation of $S\mathfrak{g}_3$. So, as the Hamiltonian we can take any smooth function of the generators in the realisation (5.21):

$$\begin{aligned} H_3 &= H_3(\mathrm{D}_3(h), \mathrm{D}_3(x_-), \mathrm{D}_3(x_+), \mathrm{D}_3(y_{1,-}), \mathrm{D}_3(y_{1,+}), \mathrm{D}_3(z_{1,1})) \\ &= H_3(qp, -q^2/2, p^2/2, -\alpha q, \alpha p, \alpha^2). \end{aligned} \tag{5.22}$$

Then, through a straightforward application of Lemma 2.8 we obtain a N degrees of freedom canonical symplectic realisation $\mathrm{D}_3^{\otimes N} \colon S\mathfrak{g}_3^{\otimes N} \longrightarrow \mathscr{C}^\infty(\mathbb{R}^{2N})$ with Darboux coordinates $(\mathbf{q}, \mathbf{p}) = (q_1, \dots, q_N, p_1, \dots, p_N)$ whose action on the generating set of $S\mathfrak{g}_3$, $\{\, h, x_-, x_+, y_{1,-}, y_{1,+}, z_{1,1} \,\}$, is given by:

$$h \mapsto \mathbf{q}\cdot\mathbf{p}, \quad x_- \mapsto -\mathbf{q}^2/2, \quad x_+ \mapsto \mathbf{p}^2/2, \quad y_{1,-} \mapsto -\boldsymbol{\alpha}\cdot\mathbf{q}, \quad y_{1,+} \mapsto \boldsymbol{\alpha}\cdot\mathbf{p}, \quad z_{1,1} \mapsto \boldsymbol{\alpha}^2, \tag{5.23}$$

where we used the vector notation and $\boldsymbol{\alpha} \in \mathbb{R}^N$ is a vector of arbitrary constants. So, from the one degree of freedom Hamiltonian we obtain the following N degrees of freedom one:

$$\begin{aligned} H_3^{(N)} &= \mathrm{D}_3^{\otimes N}\left[H_3(\Delta^{(N)}(h), \Delta^{(N)}(x_-), \Delta^{(N)}(x_+), \Delta^{(N)}(y_{1,-}), \Delta^{(N)}(y_{1,+}), \Delta^{(N)}(z_{1,1}))\right] \\ &= H_3\left(\mathbf{q}\cdot\mathbf{p}, -\mathbf{q}^2/2, \mathbf{p}^2/2, -\boldsymbol{\alpha}\cdot\mathbf{q}, \boldsymbol{\alpha}\cdot\mathbf{p}, \boldsymbol{\alpha}^2\right). \end{aligned} \tag{5.24}$$

Then, applying again Theorem 2.7, we obtain the left and right Casimir functions arising from the degree three Casimir polynomial (4.7):

$$C_3^{(m)}(\mathbf{q}, \mathbf{p}) = -\sum_{1 \le i < j < k}^{m} L_{ijk}^2, \quad m = 3, \dots, N, \tag{5.25a}$$

$$C_{3,(m)}(\mathbf{q}, \mathbf{p}) = -\sum_{N-m+1 \le i < j < k}^{N} L_{ijk}^2, \quad m = 3, \dots, N, \tag{5.25b}$$

where

$$L_{ijk} := \alpha_i L_{jk} - \alpha_j L_{ik} + \alpha_k L_{ij}. \tag{5.26}$$

We observe that the index m in this case starts from 3, since it easy to show that $C_3^{(m)}$ and $C_{3,(m)}$ vanish identically for $m = 1, 2$. In summary, we have that the two sets $L_3 := \{\, C_3^{(m)} \,\}_{m=3}^{N}$ and the $R_3 := \{\, C_{3,(m)} \,\}_{m=3}^{N}$ are made of commuting first integrals. Again, from the general theory we have that $C_3^{(N)} = C_{3,(N)}$.

So, for $N \ge 3$ adding the Hamiltonian $H_3^{(N)}$ to the sets L_3 and R_3 we obtain two sets composed by $N - 1$ functionally independent commuting first integrals:

$$\widetilde{L}_3 := \{\, H_3^{(N)}, C_3^{(3)}, \ldots, C_3^{(N)} \,\}, \quad \widetilde{R}_3 := \{\, H_3^{(N)}, C_{3,(3)}, \ldots, C_{3,(N)} \,\}, \tag{5.27}$$

thus proving that the obtained Hamiltonian system is always quasi-integrable. In total, it is easy to check, as all first integrals except $H_3^{(N)}$ and $C_3^{(N)} = C_{3,(N)}$ are local, we have a set of $[(N-2)+(N-2)-1]+1 = 2N-4$ functionally independent first integrals:

$$I_3 := \{\, H_3^{(N)}, C_3^{(3)}, \ldots, C_3^{(N)}, C_{3,(3)}, \ldots, C_{3,(N-1)} \,\}. \tag{5.28}$$

So, despite the Hamiltonian system is not integrable, it possesses a wealth of first integrals. In some particular cases such that this additional first integral exists, the Hamiltonian system immediately becomes superintegrable with $2N-3$ first integral. These particular cases have been studied in the literature with various methods, including the so-called subalgebra integrability approach, see, for instance, [13,18]. In particular, we observe that the $\mathfrak{g}_3$ symmetry algebra also allows the possibility of anisotropic extensions of known models, see, for instance, [18, §5.1.1].

5.3. The Case $n = 4$: $\mathfrak{g}_4$ Coalgebra Symmetry

The case $n = 4$ is associated with the ten-dimensional Lie algebra $\mathfrak{g}_4$. This algebra was first introduced (in a different basis) in [43], where it was observed that it could be used to interpret a N degrees of freedom generalisation of the autonomous discrete Painlevé equation, see [40,47]. However, we observe that in [43] the details of the coalgebra construction for such a Lie algebra were not presented. In this section, we fill this gap presenting an outline of such a construction.

It is possible to show by direct computation that the map $\mathrm{D}_4 \colon S\mathfrak{g}_4 \longrightarrow \mathscr{C}^\infty(\mathbb{R}^2)$ whose action on the generating set of $S\mathfrak{g}_4$, $\{h, x_-, x_+, y_{1,-}, y_{1,+}, z_{1,1}, y_{2,-}, y_{2,+}, z_{1,2}, z_{2,2}\}$, is given by:

$$\begin{gathered} h \mapsto qp, \quad x_- \mapsto -q^2/2,\ x_+ \mapsto p^2/2,\ y_{1,-} \mapsto -\alpha q,\ y_{1,+} \mapsto \alpha p, \\ z_{1,1} \mapsto \alpha^2,\ y_{2,-} \mapsto -\beta q,\ y_{2,+} \mapsto \beta p, \quad z_{1,2} \mapsto \alpha\beta, \quad z_{2,2} \mapsto \beta^2, \end{gathered} \tag{5.29}$$

where $\alpha, \beta \in \mathbb{R}$ provides a one degree of freedom canonical symplectic realisation of $S\mathfrak{g}_4$. So, as a Hamiltonian we can take any smooth function of the generators in the realisation (5.29):

$$\begin{aligned} H_4 &= \mathrm{D}_4\big[H_4(h, x_-, x_+, y_{1,-}, y_{1,+}, z_{1,1}, y_{2,-}, y_{2,+}, z_{1,2}, z_{2,2})\big] \\ &= H_4\big(qp, -q^2/2, p^2/2, -\alpha q, \alpha p, \alpha^2, -\beta q, \beta p, \alpha\beta, \beta^2\big). \end{aligned} \tag{5.30}$$

Then, with a straightforward application of Lemma 2.8 we obtain a N degrees of freedom canonical symplectic realisation $(\mathrm{D}_4)^{\otimes N} \colon S\mathfrak{g}_4^{\otimes N} \longrightarrow \mathscr{C}^\infty(\mathbb{R}^{2N})$ with Darboux coordinates $(\mathbf{q}, \mathbf{p}) = (q_1, \ldots, q_N, p_1, \ldots, p_N)$ whose action on the generating set of $S\mathfrak{g}_4$, $\{\, h, x_-, x_+, y_{1,-}, y_{1,+}, z_{1,1}, y_{2,-}, y_{2,+}, z_{1,2}, z_{2,2} \,\}$, is given by:

$$\begin{gathered} h \mapsto \mathbf{q}\cdot\mathbf{p},\ x_- \mapsto -\mathbf{q}^2/2, \quad x_+ \mapsto \mathbf{p}^2/2,\ y_{1,-} \mapsto -\boldsymbol{\alpha}\cdot\mathbf{q}, y_{1,+} \mapsto \boldsymbol{\alpha}\cdot\mathbf{p}, \\ z_{1,1} \mapsto \boldsymbol{\alpha}^2, y_{2,-} \mapsto -\boldsymbol{\beta}\cdot\mathbf{q}, y_{2,+} \mapsto \boldsymbol{\beta}\cdot\mathbf{p}, \quad z_{1,2} \mapsto \boldsymbol{\alpha}\cdot\boldsymbol{\beta}, \quad z_{2,2} \mapsto \boldsymbol{\beta}^2, \end{gathered} \tag{5.31}$$

where we used the vector notation and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^N$ are vectors of arbitrary constants. So, from the one degree of freedom Hamiltonian we obtain the following N degrees of freedom one:

$$\begin{aligned} H_4^{(N)} &= \mathrm{D}_4^{\otimes N}\left[\Delta^{(N)}(H_4)(h, x_-, x_+, y_{1,-}, y_{1,+}, z_{1,1}, y_{2,-}, y_{2,+}, z_{1,2}, z_{2,2})\right] \\ &= H_4\big(\mathbf{q}\cdot\mathbf{p}, -\mathbf{q}^2/2, \mathbf{p}^2/2, -\boldsymbol{\alpha}\cdot\mathbf{q}, \boldsymbol{\alpha}\cdot\mathbf{p}, \boldsymbol{\alpha}^2, -\boldsymbol{\beta}\cdot\mathbf{q}, \boldsymbol{\beta}\cdot\mathbf{p}, \boldsymbol{\alpha}\cdot\boldsymbol{\beta}, \boldsymbol{\beta}^2\big). \end{aligned} \tag{5.32}$$

Then, applying again Theorem 2.7, we obtain the left and right Casimir functions arising from the degree four Casimir polynomial (4.10):

$$C_4^{(m)}(\mathbf{q}, \mathbf{p}) = -\sum_{1\le i<j<k<l}^{m} L_{ijkl}^2, \quad m = 4, \ldots, N, \tag{5.33a}$$

$$C_{4,(m)}(\mathbf{q}, \mathbf{p}) = -\sum_{N-m+1\le i<j<k<l}^{N} L_{ijkl}^2, \quad m = 4, \ldots, N, \tag{5.33b}$$

where

$$\begin{aligned} L_{ijkl} &= (\alpha_k\beta_l - \alpha_l\beta_k)\, L_{ij} + (\alpha_l\beta_j - \alpha_j\beta_l)\, L_{ik} + (\alpha_j\beta_k - \alpha_k\beta_j)\, L_{il} \\ &\quad + (\alpha_i\beta_l - \alpha_l\beta_i)\, L_{jk} + (\alpha_k\beta_i - \alpha_i\beta_k)\, L_{jl} + (\alpha_i\beta_j - \alpha_j\beta_i)\, L_{kl}. \end{aligned} \tag{5.34}$$

We observe again that the index m in this case starts from 4, since it is easy to show that $C_4^{(m)}$ and $C_{4,(m)}$ vanish identically for $m = 1, 2, 3$. In summary, we have that the two sets $L_4 := \{\, C_4^{(m)} \,\}_{m=4}^{N}$ and the $R_4 := \{\, C_{4,(m)} \,\}_{m=4}^{N}$ are made of commuting first integrals. Again, from the general theory we have that $C_4^{(N)} = C_{4,(N)}$.

So, for $N \ge 4$ adding the Hamiltonian $H_4^{(N)}$ to the sets L_4 and R_4 we obtain two sets composed by $N-2$ functionally independent commuting first integrals:

$$\widetilde{L}_4 := \{\, H_4^{(N)}, C_4^{(4)}, \ldots, C_4^{(N)} \,\}, \quad \widetilde{R}_4 := \{\, H_4^{(N)}, C_{4,(4)}, \ldots, C_{4,(N)} \,\}, \tag{5.35}$$

thus proving that the obtained Hamiltonian system is of PLN type with rank $N-2$. In total, it is easy to check, as all first integrals except $H^{(N)}$ and $C_4^{(N)} = C_{4,(N)}$ are local, we have a set of $[(N-3)+(N-3)-1]+1 = 2N-6$ functionally independent first integrals:

$$I_4 := \{\, H_4^{(N)}, C_4^{(4)}, \ldots, C_4^{(N)}, C_{4,(4)}, \ldots, C_{4,(N-1)} \,\}. \tag{5.36}$$

So, despite that the Hamiltonian system is not integrable, it possesses several first integrals. In some particular cases such that two additional first integral exist, the Hamiltonian system immediately becomes superintegrable with $2N-4$ first integrals. We defer the study of such cases to future research.

5.4. The Case n Generic: $\mathfrak{g}_n$ Coalgebra Symmetry

It is now possible to obtain the expression for generic n. We start with the following result characterising a one degree of freedom symplectic realisation of $\mathfrak{g}_n$ and the associated N degrees of freedom one:

Proposition 5.1. *The Lie–Poisson algebra $S\mathfrak{g}_n$ has a one degree of freedom canonical symplectic realisation $\mathrm{D}_n\colon S\mathfrak{g}_n \longrightarrow \mathscr{C}^\infty(\mathbb{R}^2)$ mapping the T_n generators of $\mathfrak{g}_n$ as:*

$$\begin{gathered} h \mapsto qp, \qquad x_- \mapsto -q^2/2, \quad x_+ \mapsto p^2/2, \\ y_{i,-} \mapsto -\alpha^{(i)} q,\ y_{i,+} \mapsto \alpha^{(i)} p,\ z_{i,j} \mapsto \alpha^{(i)}\alpha^{(j)}, \end{gathered} \tag{5.37}$$

where $\alpha^{(i)} \in \mathbb{R}$, $i = 1, \ldots, n-2$, are $n-2$ real constants. Moreover, this realisation gives rise to the following N degrees of freedom canonical symplectic realisation $\mathrm{D}_n^{\otimes N}\colon S\mathfrak{g}_n^{\otimes N} \longrightarrow \mathscr{C}^\infty(\mathbb{R}^{2N})$ mapping the generators of $\mathfrak{g}_n$ as:

$$\begin{gathered} h \mapsto \mathbf{q}\cdot\mathbf{p}, \qquad x_- \mapsto -\mathbf{q}^2/2, \qquad x_+ \mapsto \mathbf{p}^2/2, \\ y_{i,-} \mapsto -\boldsymbol{\alpha}^{(i)}\cdot\mathbf{q},\ y_{i,+} \mapsto \boldsymbol{\alpha}^{(i)}\cdot\mathbf{p},\ z_{i,j} \mapsto \boldsymbol{\alpha}^{(i)}\cdot\boldsymbol{\alpha}^{(j)}, \end{gathered} \tag{5.38}$$

where $\boldsymbol{\alpha}^{(i)} \in \mathbb{R}^N$, $i = 1, \ldots, n-2$, are $n-2$ real vectors.

Remark 5.2. In Proposition 5.1, we denoted with $\alpha^{(i)}$ the real constants appearing in the realisation. For example, comparing with the previous case $n = 4$, in this notation we should identify $\alpha^{(1)} \equiv \alpha$ and $\alpha^{(2)} \equiv \beta$.

Proof. The proof is from a direct computation. For the first part of the proposition, we verify that $\mathrm{D}_n(\{y_{i,+}, y_{j,-}\}) = \{\mathrm{D}_n(y_{i,+}), \mathrm{D}_n(y_{j,-})\}$, while the other relations are proven similarly:

$$\begin{aligned} \{\mathrm{D}_n(y_{i,+}), \mathrm{D}_n(y_{j,-})\} &= \left\{\alpha^{(i)}p, -\alpha^{(j)}q\right\} = -\alpha^{(i)}\alpha^{(j)}\{p, q\} \\ &= \alpha^{(i)}\alpha^{(j)} = \mathrm{D}_n(z_{i,j}) = \mathrm{D}_n(\{y_{i,+}, y_{j,-}\}). \end{aligned} \tag{5.39}$$

In turn, the second part of the statement follows from a direct application of Lemma 2.8. This concludes the proof of the proposition. ☐

At this point, we are in place to define the corresponding N degrees of freedom Hamiltonian system. From formula (5.37), we have that as Hamiltonian we can take any smooth function of the T_n elements of the generating set of $S\mathfrak{g}_n$:

$$\begin{aligned} H_n &= \mathrm{D}_n\big[H_n\big(h, x_-, x_+, \{\,y_{i,-}\,\}_i, \{\,y_{i,+}\,\}_i, \{\,z_{i,j}\,\}_{i\le j}\big)\big] \\ &= H_n\big(qp, -q^2/2, p^2/2, \{\,-\alpha^{(i)}q\,\}_i, \{\,\alpha^{(i)}p\,\}_i, \{\,\alpha^{(i)}\alpha^{(j)}\,\}_{i\le j}\big), \end{aligned} \tag{5.40}$$

where for the sake of brevity we consider understood that $i = 1, \ldots, n-2$ and $1 \le i \le j \le n-2$. The N degrees of freedom extension of the Hamiltonian is obtained through the application of $\mathrm{D}_n^{\otimes N}$ realisation from equation (5.38) and reads as:

$$H^{(N)} = H\big(\mathbf{q}\cdot\mathbf{p}, -\mathbf{q}^2/2, \mathbf{p}^2/2, \{\,-\boldsymbol{\alpha}^{(i)}\cdot\mathbf{q}\,\}_i, \{\,\boldsymbol{\alpha}^{(i)}\cdot\mathbf{p}\,\}_i, \{\,\boldsymbol{\alpha}^{(i)}\cdot\boldsymbol{\alpha}^{(j)}\,\}_{i\le j}\big). \tag{5.41}$$

Now, let us discuss the first integral we can build for the Hamiltonian system defined by equation (5.41) using Theorem 2.7. Along the line of the previous results, we can generate first integrals only from the non-trivial degree n Casimir polynomial C_n. Then, upon careful direct inspection, the left and right Casimir integrals arising from the above-mentioned polynomial can be expressed as:

$$C_n^{(m)}(\mathbf{q},\mathbf{p}) = -\sum_{1\leq i_1<\cdots<i_n}^{m} L_{i_1\ldots i_n}^2\,, \quad m=n,\ldots,N \tag{5.42a}$$

$$C_{n,(m)}(\mathbf{q},\mathbf{p}) = -\sum_{N-m+1\leq i_1<\cdots<i_n}^{N} L_{i_1\ldots i_n}^2, \quad m=n,\ldots,N \tag{5.42b}$$

where

$$L_{i_1,\ldots,i_n} := \det\begin{pmatrix} \alpha_{i_1}^{(1)} & \ldots & \alpha_{i_n}^{(1)} \\ \vdots & \ddots & \vdots \\ \alpha_{i_1}^{(n-2)} & \ldots & \alpha_{i_n}^{(n-2)} \\ q_{i_1} & \ldots & q_{i_n} \\ p_{i_1} & \ldots & p_{i_n} \end{pmatrix}. \tag{5.43}$$

From the expression (5.43), we see immediately that the $C_n^{(m)} = C_{n,(m)} \equiv 0$ for $m < n$. Indeed, in such cases the building blocks $L_{i_1,\ldots,i_n}$ are identically zero because one should fill the defining matrices with at least two equal indices, causing its determinant to vanish. In summary, we have that the two sets $L_n := \{\, C_n^{(m)} \,\}_{m=n}^N$ and $R_n := \{\, C_{n,(m)} \,\}_{m=n}^N$ consist of commuting first integrals. Again, from the general theory, we have that $C_n^{(N)} = C_{n,(N)}$ due to coassociativity.

Remark 5.3. We remark that formula (5.43) reduces to the component of the angular momentum L_{i_1,i_2} in the case $n = 2$ because of the absence of the constants $\alpha_{i_l}^{(j)}$.

A different yet instructive expression of the building blocks $L_{i_1,\ldots,i_n}$ can be obtained by expanding the determinant in (5.43) with respect to the last two rows and then resumming. This yields the following alternative expression:

$$L_{i_1,\ldots,i_n} = \sum_{1\leq a<b\leq n} (-1)^{a+b-1} \det(A_{a,b}) L_{i_a,i_b}, \tag{5.44}$$

where

$$A_{a,b} = \begin{pmatrix} \alpha_{i_1}^{(1)} & \ldots & \widehat{\alpha}_{i_a}^{(1)} & \ldots & \widehat{\alpha}_{i_b}^{(1)} & \ldots & \alpha_{i_n}^{(1)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_{i_1}^{(n-2)} & \ldots & \widehat{\alpha}_{i_a}^{(n-2)} & \ldots & \widehat{\alpha}_{i_b}^{(n-2)} & \ldots & \alpha_{i_n}^{(n-2)} \end{pmatrix}, \tag{5.45}$$

where the hat means that the column is missing. This last form is quite informative, since it shows that the building blocks $L_{i_1,\ldots,i_n}$ are linear in the components of the angular momentum L_{i_a,i_b} multiplied by some degree $n-2$ polynomials in the real constants $\alpha_{i_l}^{(j)}$. Moreover, formula (5.44) reduces immediately to formulas (5.26, 5.34), holding for the cases $n = 3, 4$, respectively.

So, for $N \geq n$ adding the Hamiltonian $H_n^{(N)}$ to the sets L_n and R_n we obtain two sets composed by $N-(n-2)$ functionally independent commuting first integrals:

$$\widetilde{L}_n := \{ H_n^{(N)}, C_n^{(n)}, \ldots, C_n^{(N)} \}, \quad \widetilde{R}_n := \{ H_n^{(N)}, C_{n,(n)}, \ldots, C_{n,(N)} \}, \tag{5.46}$$

thus proving that the obtained Hamiltonian system is of PLN type with rank $N-(n-2)$. In total, it is easy to check, as all first integrals except $H_n^{(N)}$ and $C_n^{(N)} = C_{n,(N)}$ are local, we have a set composed by $[(N-n+1)+(N-n+1)-1]+1 = 2(N-n)+2$ functionally independent first integrals:

$$I_n := \{ H_n^{(N)}, C_n^{(n)}, \ldots, C_n^{(N)}, C_{n,(n)}, \ldots, C_{n,(N-1)} \}. \tag{5.47}$$

So, despite the Hamiltonian system is not integrable for $n > 2$, it admits a large number of first integrals. In some particular cases such that $n-2$ additional first integrals exist, the Hamiltonian system immediately becomes superintegrable with $2(N-n)+2$ first integrals. To summarise, the hierarchy of Hamiltonians defined by the chain of Lie–Poisson algebras coming from $\mathfrak{g}_n$ turns out to be composed by an infinite family of integrable systems ($n = 2$), then quasi-integrable systems ($n = 3$), and PLN type systems with rank $N-(n-2)$ for $n > 3$.

6. Conclusions and Outlook

In this paper, we introduced and studied the novel T_n-dimensional Lie algebra $\mathfrak{g}_n$. This Lie algebra was suggested by two of us in a previous work [43] when dealing with discrete integrable systems. The main feature of this Lie algebra is being non-semisimple, but also being a generalisation of both $\mathfrak{sl}_2(\mathbb{K})$ and the two-photon Lie algebra $\mathfrak{h}_6$. This algebra also has an unusually large centre $Z(\mathfrak{g}_n)$ of dimension T_{n-2}.

Throughout this paper, we proved the main properties of the $\mathfrak{g}_n$ Lie algebra. In particular, in Proposition 3.4 we showed that it can admit at most one non-trivial Casimir element while in Proposition 3.5 we gave a matrix representation of this algebra. Our main result, stated in Theorem 4.3, showed that such a non-trivial Casimir element exists and it is a polynomial of degree n, for $n \geq 2$. Our proof was based on some elements of representation theory, after guessing the general form of the Casimir invariant as a determinant of a $n \times n$ matrix through the standard method of vector fields. To be more specific, the key point is that we can obtain a representation of $\mathfrak{sl}_n(\mathbb{K})$ in the space of vector fields over $\mathbb{K}^{n(n+1)/2}$ such that the determinant of a symmetric matrix is annihilated by its vector fields. This property encapsulates the well-known property that the determinant of a matrix is a polynomial of degree n and it is invariant with respect to the action of the special linear group $\mathrm{SL}_n(\mathbb{K})$. This allowed us to conclude that the determinant of the matrix M_n is the Casimir polynomial we were looking for.

Finally, in the last section we used the coalgebra symmetry approach to associate to the Lie algebra $\mathfrak{g}_n$ a hierarchy of N degrees of freedom Hamiltonian systems with Hamiltonian $H_n^{(N)}$ (5.41). This was achieved by constructing an appropriate one degree of freedom canonical symplectic realisation and then using the known properties of the primitive coproduct Δ, see Sect. 5.4. The construction of the non-trivial Casimir polynomial was necessary to ensure the existence of two sets of $N-(n-2)$ commuting first integrals, see equation (5.42). These first integrals are expressed as sums of squares of n-indices objects arising as linear combinations of the components of the angular momentum (see (5.44)). We observed how the integrability properties of the obtained Hamiltonian depend crucially on n, with $n=2$ being integrable, $n=3$ quasi-integrable, and for $n \geq 4$ a Hamiltonian system of Poincaré–Lyapunov–Nekhoroshev type.

The study of the Lie algebra $\mathfrak{g}_n$ we carried out in this paper also paves the way to many other directions and investigations in the algebraic theory of integrable systems. We outline some of them now, highlighting why we believe they are important.

For instance, a natural question is the geometric interpretation of the coalgebra symmetry with respect to the Lie algebra $\mathfrak{g}_n$. Indeed, for $n=2$ the coalgebra symmetry with respect to $\mathfrak{g}_2 \cong \mathfrak{sl}_2(\mathbb{R})$, in the chosen realisation, can be understood as radial symmetry, i.e. invariance with respect to the Lie group of transformations $\mathrm{SO}(N)$. In a similar way, for $n=3$, the coalgebra symmetry with respect to $\mathfrak{g}_3 \cong \mathfrak{h}_6$ can be understood as symmetry around a fixed axis, i.e. invariance with respect to the Lie group of transformations $\mathrm{SO}(N-1) \subset \mathrm{SO}(N)$. For generic n, we don't know yet a similar interpretation. A related question is if there is any relevant associated symmetry algebra structure for the universal left and right invariants $C_n^{(m)}$ and $C_{n,(m)}$ like there is in the case of $\mathfrak{g}_2 \cong \mathfrak{sl}_2(\mathbb{R})$, see [57]. Specifically, it remains unclear whether these symmetries close in a Lie algebra, or in a polynomial Poisson algebra.

Another area requiring further exploration is the generalisation of the $\mathfrak{g}_n$ construction to other semi-direct sums with different semisimple Lie algebras, coming from Cartan's classification, and the study of the associated Casimir elements. Since $\mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{sp}_2$, the most natural generalisation would be to substitute the $\mathfrak{sl}_2$ Levi factor with a generic orthosymplectic Lie algebra $\mathfrak{sp}_{2k}$. Let us note that the Casimir elements of two Lie algebras having the orthosymplectic Lie algebra $\mathfrak{sp}_{2k}$ as Levi factor, namely the perfect Lie algebra $w\mathfrak{sp}_k$ and the inhomogenous Lie algebra $I\mathfrak{sp}_{2k}$, have been found in [21] as coefficients of the characteristic polynomials of an associated matrix. Considering that a determinant is the known term of a characteristic polynomial, it is plausible that our results can generalise to the orthosymplectic case in similar way as in [21].

In Sect. 5, we remarked how the coalgebra symmetry property with respect to the Lie algebra $\mathfrak{g}_n$ in the realisation $\mathrm{D}_n \colon S\mathfrak{g}_n \longrightarrow \mathscr{C}^\infty(\mathbb{R}^2)$ is not able to automatically yield integrability as soon as $n \geq 3$. To this end, we believe that a careful application of the subalgebra integrability method [18] can yield many interesting (super)integrable Hamiltonian systems. This method allows to build Hamiltonian with additional first integrals using the fact that corresponding to some subalgebras one can obtain additional nonlinear Casimir invariants. Our strategy to find such integrable case will be to classify all the optimal subalgebras of $\mathfrak{g}_n$, i.e. all the different subalgebras conjugated up to adjoint action of $G_n = \exp(\mathfrak{g}_n)$ on $\mathfrak{g}_n$, see [72, Definition 3.11]. We observe that recently the slightly stricter notion of optimal *families* of subalgebras has been introduced [1,2]. This notion is particularly convenient for computational purposes, allowing to isolate subalgebras with desired properties algorithmically, giving then a strong starting point to tackle the general problem.

A final relevant open problem is a classification of the various symplectic realisations of the Lie algebra $\mathfrak{g}_n$. Indeed, in this paper we confined ourselves to one degree of freedom canonical symplectic realisations. However, it is for instance known [67] that the Lie algebra $\mathfrak{g}_3 \cong \mathfrak{h}_6$ admits a non-trivial two degrees of freedom canonical symplectic realisation $\mathrm{D}_{3,2}\colon \mathfrak{g}_3 \longrightarrow \mathscr{C}^\infty(U)$, where $U = (\mathbb{R}^2_{(q_1,q_2)} \setminus \{\, \mu_2 q_1 = \mu_1 q_2 \,\}) \times \mathbb{R}^2_{(p_1,p_2)}$ and its action on the generating set, $\{\, h, x_-, x_+, y_{1,-}, y_{1,+}, z_{1,1} \,\}$, is given by:

$$
\begin{aligned}
&h \mapsto q_1 p_1 + q_2 p_2, \qquad x_- \mapsto -\tfrac{q_1^2+q_2^2}{2}, \qquad x_+ \mapsto \tfrac{p_1^2+p_2^2}{2} + \tfrac{\kappa}{(\mu_2 q_1 - \mu_1 q_2)^2},\\
&y_{1,-} \mapsto -\mu_1 q_1 - \mu_2 q_2,\ y_{1,+} \mapsto \mu_1 p_1 + \mu_2 p_2,\ z_{1,1} \mapsto \mu_1^2 + \mu_2^2,
\end{aligned}
\tag{6.1}
$$

μ_1, μ_2, κ being real parameters. It is therefore natural to address the classification of such symplectic realisations and to investigate their potential use in the construction of non-trivial integrable systems, including possible applications of the loop coproduct method [66–68]. We observe that such a method, which is an extension of the coalgebra symmetry approach that we explained in Sect. 2.2, is a promising approach for the construction of integrable systems, but its application was limited because of the lack of suitable Lie algebras where the construction may apply. We believe that the newly introduced Lie algebra $\mathfrak{g}_n$ can be used to fill this gap and produce novel interesting result in such a theory.

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Data availability All data that support the findings of this study are included within the article (and any supplementary files).

Declarations

Conflict of interest The authors have no conflict of interest to declare that is relevant to the content of this article.

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