Are Shortfall Systemic Risk Measures One Dimensional?

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Abstract

Shortfall systemic (multivariate) risk measures ρ defined through an N-dimensional multivariate utility function U and random allocations can be represented as classical (one dimensional) shortfall risk measures associated to an explicitly determined 1-dimensional function constructed from U. This finding allows for simplifying the study of several properties of ρ, such as dual representations, law invariance and stability.

Keywords: Systemic risk measures; Shortfall risk measures; Sup-convolution.

1 Introduction

We consider risky financial positions $(X^1, \ldots, X^N) := \mathbf{X}$ and assume, to simplify the exposition in this introduction, that $\mathbf{X} \in (L^\infty(\Omega, \mathcal{F}, P))^N := (L^\infty)^N$. We also let $\pi : (L^\infty)^N \to L^\infty$ be a pricing functional, $U : \mathbb{R}^N \to \mathbb{R}$ be a multivariate utility function and we set $\mathbb{U}(\mathbf{X}) := \mathbb{E}[U(X)]$ and $\mathcal{C} := \{\mathbf{Y} \in (L^\infty)^N \mid \pi(\mathbf{Y}) \in \mathbb{R}\}$. Then the functional $\rho_{\pi,\mathbb{U}}(\mathbf{X}) : (L^\infty)^N \to [-\infty, +\infty]$ defined by

$$\rho_{\pi,\mathbb{U}}(\mathbf{X}) := \inf \{\pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \, \mathbb{U}(\mathbf{X} + \mathbf{Y}) \geq 0\}, \quad \mathbf{X} \in (L^\infty)^N \tag{1}$$

represents a general capital requirement, as introduced in [9], as well as a shortfall systemic risk measure, as extensively analysed in [1], [2] and [3]. A related, but alternative approach, based on set-valued maps, is considered in [7]. Observe that the amount $\pi(\mathbf{Y})$ is enforced to be deterministic, even though the terminal-time allocations $\mathbf{Y} \in (L^\infty)^N$ are allowed to be scenario-dependent. The map $\pi(\mathbf{Y}) = \sum_{i=1}^N Y^i$ is a classical example for a pricing functional. Frequently used multivariate utility functions have the form $U(\mathbf{x}) = \sum_{i=1}^N U^i(x^i)$ for univariate utility functions $U^i : \mathbb{R} \to \mathbb{R}$, as in e.g. [3], where also a detailed discussion on scenario-dependent allocation can be found. A conditional version of (1) was treated in [5].

In this paper, we aim at establishing whether the functional $\rho_{\pi,\mathbb{U}}$ can be reduced to a classical *univariate* shortfall risk measure

$$\rho_{\mathbb{E}[g]}(X) := \inf \{\alpha \in \mathbb{R} \mid \mathbb{E}[g(X + \alpha)] \geq 0\}, \quad X \in L^\infty, \tag{2}$$

based on some function $g : \mathbb{R} \to \mathbb{R}$ that can be *explicitly recovered* from U and π. As a corollary of one of our main results, Theorem 2.9, we show that under suitable conditions on π and U

$$\rho_{\pi,\mathbb{U}}(\mathbf{X}) = \rho_{\mathbb{E}[g]}(\pi(\mathbf{X})) = \inf \{\alpha \in \mathbb{R} \mid \mathbb{E}[g(\pi(\mathbf{X})) + \alpha)] \geq 0\}, \quad \mathbf{X} \in (L^\infty)^N, \tag{3}$$

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for $g(x) = \square_\pi U(x) := \sup\{U(\mathbf{w}) \mid \mathbf{w} \in \mathbb{R}^N, \pi(\mathbf{w}) = x\}$, $x \in \mathbb{R}$. Moreover, we prove that $\rho_{\pi,\mathbb{U}}(\mathbf{X})$ admits a (unique) optimum $\mathbf{Y}_\mathbf{X}$, given by $\mathbf{Y}_\mathbf{X} = -\mathbf{X} + \Theta(\pi(\mathbf{X}) + \rho_{\pi,\mathbb{U}}(\mathbf{X})))$ for a continuous explicit function $\Theta : \mathbb{R} \to \mathbb{R}^N$ depending on U and π.

The fact that a particular class of shortfall systemic risk measures (based on the particular choices $U(\mathbf{x}) = \sum_{i=1}^N U^i(x^i)$ and $\pi(\mathbf{Y}) = \sum_{i=1}^N Y^i$) could be reduced to univariate ones, was already observed in [2] Proposition 5.3, [3] Proposition 3.1 (ii), as well as in [10] Theorem 3.16. One main difference from our work is that we provide very explicit representations (i.e. (3) and the more general case in Theorem 2.9 below) and produce explicit formulas for the function g and for the optimum.

The representation (3) is obtained as a particular instance of a more general result. Indeed, in (1) $\mathbb{U}$ needs not be an expected utility, but can rather be taken to be a general (multivariate) utility functional $\mathbb{U} : (L^\infty)^N \to \mathbb{R}$. In such a case, we prove that (3) takes the form

$$\rho_{\pi,\mathbb{U}}(\mathbf{X}) = \inf\{\alpha \in \mathbb{R} \mid \square_\pi \mathbb{U}(\pi(\mathbf{X}) + \alpha) \geq 0\}, \quad \mathbf{X} \in (L^\infty)^N, \tag{4}$$

where $\square_\pi \mathbb{U}(Z) := \sup\{\mathbb{U}(\mathbf{W}) : \mathbf{W} \in (L^\infty)^N, \pi(\mathbf{W}) = Z\}$, $Z \in L^\infty$, is the functional counterpart of $\square_\pi U$. Our results cover also cases when (i) we allow for unbounded positions $\mathbf{X}$ and (ii) π is $\mathbb{R}^M$-valued, for some $M \geq 1$. When $M \geq 2$, the analogue of (3), namely (13) or its functional counterpart (8) in the more general case, can be seeing as a dimensionality reduction property induced by random allocations. In this case, the RHS of (3) takes the form of a shortfall type systemic risk measure with deterministic allocations, i.e. in the form (5) below, as those treated e.g. in [1]. The case $M \geq 2$ also covers grouping examples, in which terminal-time exchanges are allowed only within certain subgropus of the whole system (see Section 2.1).

One first application of these findings is the law invariance of multivariate shortfall risk measures in the form (1) (see Section 3.1). Section 3.2 is devoted to establishing a Law of Large Numbers - type result in the style of [12] for systemic shortfall risk measures in the form (1). Our approach here is inspired by the one of [4].

2 Systemic risk measures can be reduced to univariate risk measures

We consider a vector subspace L of $L^0(\Omega, \mathcal{F}, P)$, with $\mathbb{R} \subseteq L$, and we induce on the Cartesian product L^N, $N \geq 1$, the order from the standard componentwise P-a.s. ordering from $(L^0(\Omega, \mathcal{F}, P))^N$. For $N \geq M \geq 1$, let $\pi = (\pi^1, \dots, \pi^M)^T : L^N \to L^M$ and set $\mathcal{C} := \{\mathbf{Y} \in L^N \mid \pi(\mathbf{Y}) \in \mathbb{R}^M\}$.

Given the functions $\mathbb{U} : L^N \to [-\infty, +\infty)$ and $G : L^M \to [-\infty, +\infty)$ we define $\rho_{\pi,\mathbb{U}} : L^N \to [-\infty, +\infty]$ and $\rho_G : L^M \to [-\infty, +\infty]$ by

$$\rho_{\pi,\mathbb{U}}(\mathbf{X}) := \inf\left\{\sum_{m=1}^M \pi^m(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbb{U}(\mathbf{X} + \mathbf{Y}) \geq 0\right\}, \quad \mathbf{X} \in L^N,$$

$$\rho_G(\mathbf{X}) := \inf\left\{\sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M, \, G(\mathbf{X} + \alpha) \geq 0\right\}, \quad \mathbf{X} \in L^M. \tag{5}$$

Let $\mathbf{e}^i$ be the i-th element of the canonical basis of $\mathbb{R}^N$. We say that $\mathbb{U} : L^N \to [-\infty, +\infty)$ is strictly increasing in some component, if for any $\mathbf{X} \in L^N$ there exists some $j \in \{1, \dots, N\}$ such that the function $x \to \mathbb{U}(\mathbf{X} + \mathbf{e}^j x)$, $x \in \mathbb{R}$, is strictly increasing.

Assumption 2.1 .
a) $\pi : L^N \to L^M$ *is linear and satisfies, for* $L_+ = L \cap L^0_+(\Omega, \mathcal{F}, P)$

$$\pi\left(L_+^N\right) = L_+^M \quad \textit{and} \quad \pi(\mathbb{R}_+^N) = \mathbb{R}_+^M. \tag{6}$$

b) $\mathbb{U} : L^N \to \mathbb{R}$ *is concave, increasing and strictly increasing in some component.*

Remark 2.2 *Observe that* (6) *implies that* π *is monotone and that*

$$\pi\left(L^N\right) = L^M \quad \text{and} \quad \pi(\mathbb{R}^N) = \mathbb{R}^M. \tag{7}$$

Indeed, $\mathbf{Z} = \mathbf{Z}^+ - \mathbf{Z}^- \in L^M$*, for* $\mathbf{Z}^\pm$ *the componentwise positive and negative parts, so that* $\mathbf{Z}^\pm = \pi(\mathbf{X}_\pm)$ *for* $\mathbf{X}_\pm \in L^N_+$ *(by* (6)*) and by linearity* $\mathbf{Z} = \pi(\mathbf{X}_+ - \mathbf{X}_-)$*. The same works with deterministic vectors in particular, yielding the second equality.*

In the following we adopt the conventions $\inf \emptyset = +\infty, \sup \emptyset = -\infty$.

Definition 2.3 *We call the function* $\square_\pi \mathbb{U} : L^M \to [-\infty, +\infty]$ *defined by*

$$\square_\pi \mathbb{U}(\mathbf{Z}) := \sup\{\mathbb{U}(\mathbf{W}) \mid \mathbf{W} \in L^N, \pi(\mathbf{W}) = \mathbf{Z}\}, \quad \mathbf{Z} \in L^M$$

the sup-convolution of $\mathbb{U}$ *under* π*.*

In Section 5 the analogous concept, defined for *convex* functions f, is there named *image function of* f *under* π, a terminology mutuated from [11]. Our choice is motivated by the following observation. Take $M = 1$, suppose that $\mathbb{U}^i : L \to [-\infty, \infty)$, $i = 1, \dots, N$, are N given univariate utility functions, consider the multivariate utility $\mathbb{U} : L^N \to [-\infty, \infty)$ defined by $\mathbb{U}(\mathbf{W}) = \sum_{i=1}^N \mathbb{U}^i(W^i)$ and the functional $\pi : L^N \to L$ given by the sum of the components, namely $\pi(\mathbf{W}) = \sum_{i=1}^N W^i$. Then by computing the sup-convolution $\mathbb{U}^1 \square \dots \square \mathbb{U}^N$ of the functions $\mathbb{U}^i$ we get

$$(\mathbb{U}^1 \square \dots \square \mathbb{U}^N)(Z) = \square_\pi \mathbb{U}(Z), \quad Z \in L.$$

Observe that in case $M = 1$, $\rho_{(\square_\pi \mathbb{U})}$ is a *classical (univariate) risk measure*. One first interesting finding is that *any* systemic risk measure in the form $\rho_{\pi,\mathbb{U}}(\mathbf{X})$ can be written as a univariate risk measure $\rho_{(\square_\pi \mathbb{U})}(\pi(\mathbf{X}))$ associated to the sup-convolution $\square_\pi \mathbb{U}$, namely to an explicitly determined univariate function. For $1 \leq M < N$ we analogously obtain a reduction in dimensionality, as explicitly described in the following proposition, whose proof is in Section 4.

Proposition 2.4 *Suppose that Assumption 2.1 holds true and that* $\square_\pi \mathbb{U}(\mathbf{Z}) < +\infty$ *for every* $\mathbf{Z} \in L^M$*. Then*

1. *The functional* $\square_\pi \mathbb{U}$ *is finite valued, concave and increasing on* L^M*.*
2. *If* $\mathbf{X} \in L^N$ *satisfies* $\sup\{\mathbb{U}(\mathbf{X} + \mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0$*, then*

$$\rho_{\pi,\mathbb{U}}(\mathbf{X}) = \rho_{(\square_\pi \mathbb{U})}(\pi(\mathbf{X})). \tag{8}$$

Remark 2.5 *The assumption in Item 2 of Proposition 2.4 is automatic if* $L = L^\infty$ *and* $\sup\{\mathbb{U}(\mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0$*, by monotonicity of* $\mathbb{U}$*.*

Remark 2.6 *Recall the general notion of a capital requirement* $\rho_{\pi,\mathcal{A}} : L^N \to [-\infty, +\infty]$ *(see [9]) and of a monetary risk measure* $\rho_{\mathcal{B}} : L \to [-\infty, +\infty]$ *(see [8]) :*

$$\rho_{\pi,\mathcal{A}}(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbf{X} + \mathbf{Y} \in \mathcal{A}\}, \quad \mathbf{X} \in L^N, \tag{9}$$

$$\rho_{\mathcal{B}}(Z) := \inf\{\alpha \in \mathbb{R} \mid Z + \alpha \in \mathcal{B}\}, \quad Z \in L, \tag{10}$$

for some acceptance sets $\mathcal{A} \subseteq L^N$ *and* $\mathcal{B} \subseteq L$*. If* π *is linear and for* $\mathcal{C} := \{\mathbf{Y} \in L^N \mid \pi(\mathbf{Y}) \in \mathbb{R}\}$ *we have for all* $\mathbf{X} \in L^N$

$$\begin{aligned}
\rho_{\pi,\mathcal{A}}(\mathbf{X}) &= \inf\{\alpha \in \mathbb{R} \mid \pi(\mathbf{Y}) = \alpha, \mathbf{X} + \mathbf{Y} \in \mathcal{A}\} \\
&= \inf\{\alpha \in \mathbb{R} \mid \exists \mathbf{W} \in \mathcal{A} \ s.t. \ \pi(\mathbf{W}) = \pi(\mathbf{X}) + \alpha\} \\
&= \inf\{\alpha \in \mathbb{R} \mid \pi(\mathbf{X}) + \alpha \in \pi(\mathcal{A})\} = \rho_{\pi(\mathcal{A})}(\pi(\mathbf{X})).
\end{aligned}$$

Hence, any capital requirement (or systemic multivariate risk measure) of dimension N *in the form* (9) *with* π *linear can be reduced to a classical univariate risk measure in the form* (10)*.*

In the remaining of this section we work in the following

Setting 2.7

1. *We select* $L = L^\infty$
2. *The linear functional* $\pi : (L^\infty)^N \to (L^\infty)^M$ *is assigned by*
$$\pi(\mathbf{X}) = A\mathbf{X},$$
where the (deterministic) matrix A *in* $\mathbb{R}^{M\times N}$ *satisfies* $A(\mathbb{R}^N_+) = \mathbb{R}^M_+$.
3. *The multivariate utility function* $U : \mathbb{R}^N \to \mathbb{R}$ *is nondecreasing (w.r.t. the componentwise order), differentiable, strictly concave throughout all* $\mathbb{R}^N$ *with* $\sup\{U(\mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0$.
4. *The functional* $\mathbb{U} : (L^\infty)^N \to \mathbb{R}$ *has the form*
$$\mathbb{U}(\mathbf{X}) := \mathbb{E}[U(\mathbf{X})],\ \ \mathbf{X} \in (L^\infty)^N.$$

We point out that $A(\mathbb{R}^N_+) = \mathbb{R}^M_+$ implies that A has full rank, $\text{rank}(A) = M$, and that π in Item 2 satisfies $\pi((L^\infty)^N_+) = (L^\infty)^M_+$. Moreover, the function $\mathbb{U}$ in Item 4 is also strictly increasing (in any component). Thus, in the Setting 2.7 the Assumption 2.1 holds true. The choices made in the Setting 2.7 lead to the classical shortfall systemic risk measure:

$$\rho_{\pi,\mathbb{U}}(\mathbf{X}) := \inf\left\{\sum_{m=1}^{M} \pi^m(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbb{E}\left[U(\mathbf{X}+\mathbf{Y})\right] \geq 0\right\}, \quad \mathbf{X} \in (L^\infty)^N, \tag{11}$$

which is a monotone increasing, convex, cash additive map. By definition, the sup-convolution $\square_\pi U : \mathbb{R}^M \to \mathbb{R}$ of U under π is assigned by:

$$\square_\pi U(\mathbf{y}) := \sup\{U(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^N, \pi(\mathbf{x}) = \mathbf{y}\}, \quad \mathbf{y} \in \mathbb{R}^M. \tag{12}$$

Assumption 2.8 *For some* $\mathbf{y} \in \mathbb{R}^M$, *the problem in* (12) *admits an optimum, namely there exists* $\mathbf{x} = \mathbf{x}(\mathbf{y}) \in \mathbb{R}^N$ *such that* $\pi(\mathbf{x}) = \mathbf{y}$ *and* $\square_\pi U(\mathbf{y}) = U(\mathbf{x})$.

We provide in Lemma A.1 in Appendix mild conditions which guarantee the validity of Assumption 2.8.

The main result of this note is described in the following Theorem. Shortfall systemic risk measures $\rho_{\pi,\mathbb{U}}(\mathbf{X})$ defined through a N-dimensional multivariate utility function U can be represented as a shortfall risk measure $\rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{X}))$ associated to the M-dimensional function $\square_\pi U$. Additionally, we provide the explicit formula for the optimum. The proof is deferred to Section 6.

Theorem 2.9 *Suppose that Assumption 2.8 is satisfied. Then*

1. $\square_\pi \mathbb{U}(\mathbf{Z}) = \mathbb{E}\left[(\square_\pi U)(\mathbf{Z})\right]$ *for every* $\mathbf{Z} \in (L^\infty)^M$.
2. *For every* $\mathbf{X} \in (L^\infty)^N$ *we have*
$$\rho_{\pi,\mathbb{U}}(\mathbf{X}) = \rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{X})) := \inf\left\{\sum_{m=1}^{M} \alpha^m \mid \alpha \in \mathbb{R}^M,\, \mathbb{E}\left[\square_\pi U(\pi(\mathbf{X})+\alpha)\right] \geq 0\right\}. \tag{13}$$
If additionally there exists an optimum $\widehat{\alpha} \in \mathbb{R}^M$ *attaining the infimum in RHS of* (13), *then also* $\rho_{\pi,\mathbb{U}}(\mathbf{X})$ *admits a (unique) optimum* $\mathbf{Y}_\mathbf{X}$, *given by*
$$\mathbf{Y}_\mathbf{X} = -\mathbf{X} - \nabla U^*\Big(\pi^T \cdot \nabla(\square_\pi U)(\pi(\mathbf{X})+\widehat{\alpha})\Big) = -\mathbf{X} + \Theta\Big(\pi(\mathbf{X})+\widehat{\alpha}\Big), \tag{14}$$
where U^* *is the concave conjugate of* U, π^T *is the transposed map of* π *and* $\Theta : \mathbb{R}^M \to \mathbb{R}^N$, $\Theta(\mathbf{y}) := -\nabla U^*\Big(\pi^T \cdot \nabla(\square_\pi U)(\mathbf{y})\Big)$, *is continuous.*

In case $M = 1$, $\rho_{\pi,\mathbb{U}}$ is finite valued, the optimum in the RHS of (13) *always exists, and it is given by $\widehat{\alpha} = \rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{X}))$.*

In the case $M = 1$, as an immediate byproduct of Theorem 2.9, the *dual representation* for the systemic risk measure $\rho_{\pi,\mathbb{U}}(= \rho_{(\mathbb{E}[\square_\pi U])})$ can be directly obtained from the well known classical dual representation of the univariate convex risk measure $\rho_{(\mathbb{E}[\square_\pi U])}$. Indeed, letting $\ell(x) = -\square_\pi U(-x), x \in \mathbb{R}$ the dual representation of $\rho_{(\mathbb{E}[\square_\pi U])}$ follows from [8] Theorem 4.115 with minimal penalty function $\alpha^{\min}$ in [8] Theorem 4.115 explicitly given, since $\ell^*(z) = -U^*(A^T z), z \in \mathbb{R}$.

2.1 Grouping case

In Lemma 2.10 below, whose simple proof is omitted, we show how the dimensionality reduction put in evidence in (13) covers also the grouping case in Example 5.2 of [3] and Definition 5.1 of [5]. More precisely, we show that with an appropriate choice of π we get:

$$\mathcal{C} := \{\mathbf{Y} \in L^N \mid \pi(\mathbf{Y}) \in \mathbb{R}^M\} = \left\{\mathbf{Y} \in (L^\infty)^N \mid \sum_{n \in I_m} Y^n \in \mathbb{R} \quad \forall\, m = 1, \ldots, M\right\}. \tag{15}$$

Lemma 2.10 *Let $I_1, \ldots, I_M \subseteq \{1, \ldots, N\}$ be a partition of $\{1, \ldots, N\}$, clearly with $M \leq N$. Define the matrix $A = (a_{mn})_{mn} \in \mathbb{R}^{M \times N}$ via*

$$a_{mn} = \begin{cases} 1 & \text{if } n \in I_m \\ 0 & \text{otherwise} \end{cases} \qquad n = 1, \ldots, N; m = 1, \ldots, M.$$

Furthermore, set $L = L^\infty$ and $\pi(\mathbf{X}) = A\mathbf{X}$ (as a matrix-vector product). Then A is full rank, the first Item in Assumption 2.1 and the second Item in Setting 2.7 are satisfied, and (15) *holds.*

3 Applications

3.1 Law invariance

We use the same notation of Section 2, we write P_X (resp. $P_{\mathbf{X}}$) for the law of a random variable X (resp. vector $\mathbf{X}$) on $\mathbb{R}$ (resp. $\mathbb{R}^N$) and $X \overset{P}{\sim} Y$ if the random variables (or vectors) X, Y have the same law under P.

Proposition 3.1 *Assume that $\pi(\mathbf{X}) \overset{P}{\sim} \pi(\mathbf{Y})$ with $\mathbf{X}, \mathbf{Y} \in L^N$. Then*

(1) If $\pi(\mathcal{A})$ is law invariant then $\rho_{\pi,\mathcal{A}}(\mathbf{X}) = \rho_{\pi,\mathcal{A}}(\mathbf{Y})$.

(2) If (8) *holds and if $\mathcal{B} = \{\mathbf{Z} \in L^M \mid \square_\pi \mathbb{U}(\mathbf{Z}) \geq 0\}$ is law invariant then $\rho_{\pi,\mathbb{U}}(\mathbf{X}) = \rho_{\pi,\mathbb{U}}(\mathbf{Y})$.*

Proof. Item (1) is an immediate consequence of $\rho_{\pi,\mathcal{A}}(\mathbf{X}) = \rho_{\pi(\mathcal{A})}(\pi(\mathbf{X}))$ proven in Remark 2.6. (2) From $\pi(\mathbf{X}) \overset{P}{\sim} \pi(\mathbf{Y})$ and the law invariance of $\mathcal{B}$ we get: $\square_\pi \mathbb{U}(\pi(\mathbf{X}) + \alpha) \geq 0$ iff $\square_\pi \mathbb{U}(\pi(\mathbf{Y}) + \alpha) \geq 0$. From $\rho_{\pi,\mathbb{U}}(\mathbf{X}) = \rho_{(\square_\pi \mathbb{U})}(\pi(\mathbf{X}))$, we deduce that $\rho_{\pi,\mathbb{U}}(\mathbf{X}) = \rho_{\pi,\mathbb{U}}(\mathbf{Y})$. ■

Remark 3.2 *Obviously, if π is law invariant and $\mathbf{X} \overset{P}{\sim} \mathbf{Y}$ then the assumption in the previous proposition holds, so that Proposition 3.1 gives in particular sufficient conditions for the law invariance of the systemic risk measures $\rho_{\pi,\mathcal{A}}$ and $\rho_{\pi,\mathbb{U}}$.*

Corollary 3.3 *Let $U : \mathbb{R}^N \to \mathbb{R}$ be nondecreasing, differentiable, strictly concave throughout all $\mathbb{R}^N$ and satisfying $\sup\{U(\mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0$ and Assumption 2.8. Then ρ defined by*

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N Y^i \mid \mathbf{Y} \in (L^\infty)^N, \sum_{i=1}^N Y^i \in \mathbb{R}, \mathbb{E}\left[U(\mathbf{X} + \mathbf{Y})\right] \geq 0 \right\}, \quad \mathbf{X} \in (L^\infty)^N, \tag{16}$$

is finite valued and law invariant.

Proof. Observe that $\rho = \rho_{\pi,\mathbb{U}}(\mathbf{X})$ for $\mathbb{U}(\cdot) := \mathbb{E}[U(\cdot)]$ and $\pi(\mathbf{Y}) := Y_1 + ... + Y_N$, the latter being law invariant. Thus the assumptions in the Setting 2.7 hold. By Proposition 5.7, $\square_\pi U$ is continuous on $\mathbb{R}$ and, by Theorem 2.9, Item 1, $\square_\pi\mathbb{U}(Z) = \mathbb{E}[\square_\pi U(Z)]$. Thus, $\mathcal{B} := \{Z \in L^\infty \mid \square_\pi\mathbb{U}(Z) \geq 0\} = \{Z \in L^\infty \mid \mathbb{E}[\square_\pi U(Z)] \geq 0\}$ is law invariant. The conclusion follows from Theorem 2.9 and Proposition 3.1 (2). ■

3.2 Stability

Let $(\Omega, \mathcal{F}, P)$ be an atomless standard probability space. If $\rho : (L^\infty)^N \to \mathbb{R}$ is a law invariant functional, then it is possible to think of ρ as defined on the class of probability measures on $\mathbb{R}^N$. Indeed, whenever $P_\mathbf{X} = P_\mathbf{Y}$ on $\mathbb{R}^N$ we have $\mathbf{X} \overset{P}{\sim} \mathbf{Y}$ and $\rho(X) = \rho(Y)$, and since the underlying space is non atomic every probability measure on $\mathbb{R}^N$ is realized as the law under P of some random vector $\mathbf{Z}$ defined on Ω by the Skorokhod Theorem. By a slight abuse of notation we write $\rho(Q)$ meaning $\rho(\mathbf{Z})$ for every $\mathbf{Z}$ having law Q on $\mathbb{R}^N$.

Corollary 3.4 *In Setup 2.7 with $M = 1$ and $A\mathbf{x} = \sum_{j=1}^N x^j$ for every $\mathbf{x} \in \mathbb{R}^N$, suppose that the assumptions of Corollary 3.3 are satisfied. Take probability measures $\{Q_n\}_n$ on $\mathbb{R}^N$, for $n = 1, \dots, +\infty$, such that for some positive radius $r > 0$ we have $Q_n(B_r) = 1$ for every $1 \leq n \leq +\infty$, B_r being the ball of radius r in $\mathbb{R}^N$. Suppose additionally that Q_n converges to Q_∞ in the weak sense for probability measures. Then ρ defined in (16) satisfies*

$$\lim_{n\to+\infty} \rho(Q_n) = \rho(Q_\infty).$$

Proof. By the Skorokhod Representation Theorem there exist N-dimensional random vectors $\mathbf{Z}_n, 1 \leq n \leq +\infty$ on $(\Omega, \mathcal{F}, P)$ such that Q_n is the law of $\mathbf{Z}_n$ under P and $\mathbf{Z}_n \to \mathbf{Z}_\infty$ P-a.s. In particular then $\mathbf{Z}_n \in (L^\infty)^N$, $\pi(\mathbf{Z}_n) \to \pi(\mathbf{Z}_\infty)$ and $\|\pi(\mathbf{Z}_n)\|_\infty \leq Nr\, P-$a.s. for every n. Then for $1 \leq n \leq +\infty$ we have $\rho(Q_n) = \rho(\mathbf{Z}_n) = \rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{Z}_n))$ by Theorem 2.9, where $\square_\pi U : \mathbb{R} \to \mathbb{R}$ is increasing and nonconstant by Proposition 5.7. We now show that $\rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{Z}_n)) \to_n \rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{Z}_\infty))$. By [8] Proposition 4.113 $\rho_{(\mathbb{E}[\square_\pi U])}$ is continuous from below. Then it has the Lebesgue property ([8] Corollary 4.35) and the desired convergence follows. ■

Take now $\mathbf{X} \in (L^\infty)^N$. Replacing the law $P_\mathbf{X}$ in $\rho(P_\mathbf{X})$ with the empirical measure $\widehat{P}_n$ based on an i.i.d. sample $(\mathbf{X}_1, \dots, \mathbf{X}_n)$, we obtain the empirical estimate/historical estimate $\rho(\widehat{P}_n)$ of $\rho(P_\mathbf{X}) = \rho(\mathbf{X})$. Under the assumptions of Corollary 3.4, we have in particular $\lim_n \rho(\widehat{P}_n) = \rho(P_\mathbf{X})$ $P-$a.s. by weak convergence (a.s.) of $\widehat{P}_n$ to $P_\mathbf{X}$. This can be exploited in conjunction with the explicit formula for the optima (14), since in the case $M = 1$ we know $\widehat{\alpha} = \rho(\mathbf{X})$, to guarantee a.s. convergence of the approximated optimal allocation functions $\mathbf{x} \mapsto -\mathbf{x} + \Theta(\pi(\mathbf{x}) + \rho(\widehat{P}_n))$.

4 Proof of Proposition 2.4

Proof of Item 1. The functional $\square_\pi\mathbb{U}$ is finite valued since, by (7), $\square_\pi\mathbb{U}(\mathbf{Z}) > -\infty$ for any $\mathbf{Z} \in L^M$.

Concavity. By (7), given any $\mathbf{Z}_1, \mathbf{Z}_2 \in L^M$, there exist $\mathbf{W}_1, \mathbf{W}_2 \in L^N$ such that $\pi(\mathbf{W}_i) = \mathbf{Z}_i$ for $i = 1, 2$. Hence, for any $\alpha \in [0, 1]$

$$\square_\pi\mathbb{U}(\alpha\mathbf{Z}_1 + (1-\alpha)\mathbf{Z}_2) \geq \mathbb{U}(\alpha\mathbf{W}_1 + (1-\alpha)\mathbf{W}_2) \geq \alpha\mathbb{U}(\mathbf{W}_1) + (1-\alpha)\mathbb{U}(\mathbf{W}_2),$$

where the former inequality is due to the definition of $\square_\pi\mathbb{U}$ and the linearity of π, the latter from concavity of $\mathbb{U}$. Concavity of $\square_\pi\mathbb{U}$ then follows by taking the supremum over all $\mathbf{W}_1, \mathbf{W}_2 \in L^N$ such that $\pi(\mathbf{W}_1) = \mathbf{Z}_1$ and $\pi(\mathbf{W}_2) = \mathbf{Z}_2$.

Monotonicity. Take $\mathbf{Z}_i \in L^M$ such that $\mathbf{Z}_1 \leq \mathbf{Z}_2$ and take by (7) $\mathbf{W}_1 \in L^N$ s.t. $\pi(\mathbf{W}_1) = \mathbf{Z}_1$. Now $\mathbf{Z}_2 - \mathbf{Z}_1 \in L^M_+$, so that $\mathbf{Z}_2 - \mathbf{Z}_1 = \pi(\mathbf{W})$ for some $\mathbf{W} \in (L_+)^N$ by (6). Hence, $\mathbf{W}_2 := \mathbf{W}_1 + \mathbf{W} \geq \mathbf{W}_1$ satisfies $\pi(\mathbf{W}_2) = \mathbf{Z}_2$ and $\mathbb{U}(\mathbf{W}_1) \leq \mathbb{U}(\mathbf{W}_2) \leq \square_\pi\mathbb{U}(\mathbf{Z}_2)$. Take now a supremum over $\mathbf{W}_1$ satisfying $\pi(\mathbf{W}_1) = \mathbf{Z}_1$ to get $\square_\pi \mathbb{U}(\mathbf{Z}_1) \leq \square_\pi \mathbb{U}(\mathbf{Z}_2)$.

Proof of Item 2. Observe first that under the additional assumption in Item 2, we have $\rho_{\pi,\mathbb{U}}(\mathbf{X}) < +\infty$. From the linearity of π and the definition of $\mathcal{C}$, we have for any $\mathbf{X} \in L^N$

$$\begin{aligned}
\rho_{\pi,\mathbb{U}}(\mathbf{X}) &= \inf\left\{\sum_{m=1}^{M} \pi^m(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbb{U}(\mathbf{X}+\mathbf{Y}) \geq 0\right\} \\
&= \inf\left\{\sum_{m=1}^{M} \pi^m(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbb{U}(\mathbf{X}+\mathbf{Y}) > 0\right\} \qquad (17) \\
&= \inf\left\{\sum_{m=1}^{M} \alpha^m \mid \alpha \in \mathbb{R}^M \text{ satisfies } \pi(\mathbf{Y}) = \alpha \text{ for some } \mathbf{Y} \in L^N, \mathbb{U}(\mathbf{X}+\mathbf{Y}) > 0\right\} \\
&= \inf\left\{\sum_{m=1}^{M} \alpha^m \mid \alpha \in \mathbb{R}^M, \exists \mathbf{W} \in L^N, \mathbb{U}(\mathbf{W}) > 0 \text{ s.t. } \pi(\mathbf{W}) = \pi(\mathbf{X}) + \alpha\right\} \\
&= \inf\left\{\sum_{m=1}^{M} \alpha^m \mid \alpha \in \mathbb{R}^M, \sup\left\{\mathbb{U}(\mathbf{W}) \mid \mathbf{W} \in L^N, \pi(\mathbf{W}) = \pi(\mathbf{X}) + \alpha\right\} > 0\right\} \\
&= \inf\left\{\sum_{m=1}^{M} \alpha^m \mid \alpha \in \mathbb{R}^M, \square_\pi \mathbb{U}(\pi(\mathbf{X}) + \alpha) > 0\right\} \\
&= \inf\left\{\sum_{m=1}^{M} \alpha^m \mid \alpha \in \mathbb{R}^M, \square_\pi \mathbb{U}(\pi(\mathbf{X}) + \alpha) \geq 0\right\} \qquad (18) \\
&= \rho_{(\square_\pi \mathbb{U})}(\pi(\mathbf{X}))
\end{aligned}$$

where only the equalities (17) and (18) are not evident. To prove the equality in (17), observe first that, obviously,

$$\rho_{\pi,\mathbb{U}}(\mathbf{X}) \leq \inf\left\{\sum_{m=1}^{M} \pi^m(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \mathbb{U}(\mathbf{X}+\mathbf{Y}) > 0\right\} := a$$

with $a \in \mathbb{R}$. Suppose by contradiction that $\rho_{\pi,\mathbb{U}}(\mathbf{X}) < a$ and let $0 < \varepsilon < \frac{a - \rho_{\pi,\mathbb{U}}(\mathbf{X})}{2}$. Then there exists $\mathbf{Y} \in \mathcal{C}$ such that $\mathbb{U}(\mathbf{X}+\mathbf{Y}) \geq 0$ and

$$\sum_{m=1}^{M} \pi^m(\mathbf{Y}) < \rho_{\pi,\mathbb{U}}(\mathbf{X}) + \varepsilon. \qquad (19)$$

Since $\mathbb{U}$ is strictly increasing on one component, say component i, take $\widehat{\mathbf{Y}} := \mathbf{Y} + \varepsilon \frac{\mathbf{e}^i}{\sum_{m=1}^{M} \pi^m(\mathbf{e}^i)+1}$, noticing that this is well defined as $\pi^m(\mathbf{e}^i) \geq 0$, for all m, by (6). Since π is linear and $\pi(\mathbb{R}^N) \subseteq \mathbb{R}^M$ (Remark 2.2) then $\sum_{m=1}^{M} \pi^m(\widehat{\mathbf{Y}}) = \left(\sum_{m=1}^{M} \pi^m(\mathbf{Y}) + \varepsilon \frac{\sum_{m=1}^{M} \pi^m(\mathbf{e}^i)}{\sum_{m=1}^{M} \pi^m(\mathbf{e}^i)+1}\right) \in \mathbb{R}$ and $\widehat{\mathbf{Y}} \in \mathcal{C}$. Moreover, $\mathbb{U}(\mathbf{X}+\widehat{\mathbf{Y}}) = \mathbb{U}(\mathbf{X}+\mathbf{Y}+\varepsilon \frac{\mathbf{e}^i}{\sum_{m=1}^{M} \pi^m(\mathbf{e}^i)+1}) > \mathbb{U}(\mathbf{X}+\mathbf{Y}) \geq 0$ so that $a \leq \sum_{m=1}^{M} \pi^m(\widehat{\mathbf{Y}})$. But this is a contradiction using (19): $a \leq \sum_{m=1}^{M} \pi^m(\widehat{\mathbf{Y}}) = \sum_{m=1}^{M} \pi^m(\mathbf{Y}) + \varepsilon \frac{\sum_{m=1}^{M} \pi^m(\mathbf{e}^i)}{\sum_{m=1}^{M} \pi^m(\mathbf{e}^i)+1} < \rho_{\pi,\mathbb{U}}(\mathbf{X}) + 2\varepsilon < a$. To prove the equality in (18), we set

$$g_{\mathbf{X}}(\alpha) := \square_\pi \mathbb{U}(\pi(\mathbf{X}) + \alpha), \quad \alpha \in \mathbb{R}^M$$

and show that

$$\inf\left\{\sum_{m=1}^{M} \alpha^m \mid \alpha \in \mathbb{R}^M, g_{\mathbf{X}}(\alpha) > 0\right\} = \inf\left\{\sum_{m=1}^{M} \alpha^m \mid \alpha \in \mathbb{R}^M, g_{\mathbf{X}}(\alpha) \geq 0\right\}.$$

The fact that LHS≥RHS is clear, and the equality is trivial if the set in RHS is empty. Then we assume this is not the case and prove LHS≤RHS. Take a minimizing sequence $(\alpha_n)_n$ such that $g_{\mathbf{X}}(\alpha_n) \geq 0$ for each n and $\sum_{m=1}^M \alpha_n^m \downarrow_n \inf\{\sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M,\, g_{\mathbf{X}}(\alpha) \geq 0\}$.

Case 1: $(\alpha_n)_n$ admits a subsequence $(\alpha_{n_k})_k$ with $g_{\mathbf{X}}(\alpha_{n_k}) > 0$ for each k. Clearly we get LHS$\leq \sum_{m=1}^M \alpha_{n_k}^m \downarrow_k$RHS, which is the desired remaining inequality.

Case 2: $g_{\mathbf{X}}(\alpha_n) = 0$ definitely in n. We assume the equality holds for each n w.l.o.g.. Define now the functions $h_n(\beta) := g_{\mathbf{X}}(\alpha_n + \beta\mathbf{1})$, $\beta \in \mathbb{R}$, and observe that Proposition 2.4 Item 1 ensures that, for each n, $h_n : \mathbb{R} \to \mathbb{R}$ is increasing and concave on $\mathbb{R}$ and thus continuous. Moreover,

$$g_{\mathbf{X}}(\alpha) \leq h_n\left(\max_m |\alpha_m^n| + \max_m |\alpha_m|\right), \quad \forall\, \alpha \in \mathbb{R}^M$$

guaranteeing that $\sup_{\beta\in\mathbb{R}} h_n(\beta) = \sup_{\alpha\in\mathbb{R}^M} g_{\mathbf{X}}(\alpha) := \widehat{g}$. By the linearity of π we then get

$$\begin{aligned}
\sup_{\beta\in\mathbb{R}} h_n(\beta) &= \sup_{\alpha\in\mathbb{R}^M} g_{\mathbf{X}}(\alpha) = \sup_{\alpha\in\mathbb{R}^M} \left\{\sup\{\mathbb{U}(\mathbf{W}) \mid \mathbf{W} \in L^N, \pi(\mathbf{W}) = \pi(\mathbf{X}) + \alpha\}\right\} \\
&= \sup_{\alpha\in\mathbb{R}^M} \{\sup\{\mathbb{U}(\mathbf{X}+\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \pi(\mathbf{Y}) = \alpha\}\} \\
&= \sup\{\mathbb{U}(\mathbf{X}+\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}\} \qquad (20) \\
&\geq \sup\{\mathbb{U}(\mathbf{X}+\mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0, \qquad (21)
\end{aligned}$$

where we used: in (20) the equality $\pi(\mathcal{C}) = \mathbb{R}^M$ (a consequence of (7)); in the first inequality in (21) the fact that $\pi(\mathbb{R}^N) \subseteq \mathbb{R}^M$, and the last strict inequality is guaranteed by hypothesis. Observe that since $g_{\mathbf{X}}(\alpha_n) = 0$, we also have $h_n(0) = 0 < \sup_{\beta\in\mathbb{R}} h_n(\beta) = \widehat{g}$. Let $\widehat{\beta}_n = \inf\{\beta \in \mathbb{R} \mid h_n(\beta) = \widehat{g}\} \leq +\infty$. From $h_n(0) < \widehat{g}$, the continuity and monotonicity of h_n, we have $\widehat{\beta}_n > 0$ for every n. Additionally, h_n as a univariate concave and increasing function is strictly increasing on $(-\infty, \widehat{\beta}_n)$. Thus, for some $0 < \varepsilon_n < \min(\frac{1}{n}, \widehat{\beta}_n)$ we have $0 = h_n(0) < h_n(\varepsilon_n) = g_{\mathbf{X}}(\alpha_n + \varepsilon_n\mathbf{1})$. Thus $\beta_n := \alpha_n + \varepsilon_n\mathbf{1}$ defines a minimizing sequence, with $g_{\mathbf{X}}(\beta_n) > 0$:

$$\sum_{m=1}^M \beta_n^m = \sum_{m=1}^M \alpha_n^m + M\varepsilon_n \downarrow_n \inf\{\sum_{m=1}^M \alpha^m \mid \alpha \in \mathbb{R}^M,\, g_{\mathbf{X}}(\alpha) \geq 0\}$$

and one can argue as in Case 1.

5 Image functions on $\mathbb{R}^N$

We now present some key properties of image functions. All definitions, as well as the notation, are mutuated from [11]. For convenience of the reader and to simplify the comparison with this reference, we thus opted to present the concepts and results for *convex* functions f and linear maps A, which replace the function $(-U)$ and the linear map π in the previous sections.

In this Section 5 we work again in Setting 2.7 without further mention. We set $f = -U : \mathbb{R}^N \to \mathbb{R}$ and denote by f^* the usual convex conjugate function of f,

$$f^*(\mathbf{z}) := \sup_{\mathbf{x}\in\mathbb{R}^N} \left(\sum_{j=1}^N x^j z^j - f(\mathbf{x})\right) \in (-\infty, +\infty], \quad \mathbf{z} \in \mathbb{R}^N.$$

Definition 5.1 *The image function of f under A is the function $\square^A f : \mathbb{R}^M \to [-\infty, +\infty]$*

$$\square^A f(\mathbf{y}) := \inf\left\{f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^N, A\mathbf{x} = \mathbf{y}\right\}, \ \mathbf{y} \in \mathbb{R}^M, \qquad (22)$$

with the usual convention $\inf\emptyset = +\infty$. We say that, for a given $\mathbf{y} \in \mathbb{R}^M$, the problem in (22) admits an optimum if there exists $\mathbf{x} = \mathbf{x}(\mathbf{y}) \in \mathbb{R}^N$ such that $A\mathbf{x} = \mathbf{y}$ and $\square^A f(\mathbf{y}) = f(\mathbf{x})$.

In [11] the image function $\Box^A f$ is denoted by Af. Observe that $\Box^A f = -\Box_\pi U$, as in (12), for $f = -U$ and $A = \pi$ and that $\Box^A f$ is a convex function (Th. 5.7 [11]) and, since f is real valued on the whole $\mathbb{R}^N$ and A has full range, $\Box^A f(\mathbf{y}) < +\infty$ for every $\mathbf{y} \in \mathbb{R}^M$. We stress that by strict convexity of f, the problem (22) admits at most one optimum.

Remark 5.2 *Under Assumption 2.8, we have that for some $\mathbf{y} \in \mathbb{R}^M$, the problem in* (22) *admits an optimum.*

We refer to [11] Chapter 26 for definitions of essentially smooth functions and Legendre type pairs. In our setting, f is essentially smooth on the whole $\mathbb{R}^N$ and thus $(f, \mathbb{R}^N)$ is of Legendre type. We also briefly recall the following key results:

Theorem 5.3 ([11] Theorem 26.5) *Let $h : \mathbb{R}^D \to (-\infty, +\infty]$ be a closed (i.e. proper lower semicontinuous w.r.t. the usual Euclidean topology) convex function. Set $C := \operatorname{int}\operatorname{dom}(h), C^* := \operatorname{int}\operatorname{dom}(h^*)$. Then (h, C) is a convex function of Legendre type iff so is (h^*, C^*). When these conditions hold, ∇h is one-to-one from the open convex set C onto C^*, continuous in both directions, and $(\nabla h)^{-1} = \nabla h^*$.*

Theorem 5.4 ([11] Theorem 16.3) *Suppose that there exists $\lambda \in \mathbb{R}^M$ such that $A^T\lambda \in \operatorname{ri}\operatorname{dom}(f^*)$. Then for every $\mathbf{y} \in \mathbb{R}^M$ there exists an optimum for* (22).

Proof. Since f is real valued and convex we have $f = (f^*)^*$ on $\mathbb{R}^N$. Moreover $(A^T)^T = A$ and thus we can rewrite

$$\inf\left\{f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^N, A\mathbf{x} = \mathbf{y}\right\} = \inf\left\{(f^*)^*(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^N, (A^T)^T\mathbf{x} = \mathbf{y}\right\}$$

Setting $g = f^*$, which is a convex function on $\mathbb{R}^N$, we recognize the setup of the last part in [11] Theorem 16.3 with $f, \lambda, \mathbf{x}, \mathbf{y}, A^T$ here in place of $g^*, \mathbf{x}, \mathbf{y}^*, \mathbf{x}^*, A$ in the reference respectively. By hypothesis we have $A^T\lambda \in \operatorname{ri}\operatorname{dom}(g)$ and so by [11] Theorem 16.3 the infimum in (22) is attained. ■

Proposition 5.5 *Take $\mathbf{y} \in \mathbb{R}^M$ such that $\Box^A f(\mathbf{y}) > -\infty$. Then $\Box^A f(\mathbf{y}) \in \mathbb{R}$ and the following are equivalent:*

(i) there exists the optimum $\mathbf{x} = \mathbf{x}(\mathbf{y}) \in \mathbb{R}^N$ for (22)

(ii) there exist $(\mathbf{x}, \lambda) = (\mathbf{x}(\mathbf{y}), \lambda(\mathbf{y})) \in \mathbb{R}^N \times \mathbb{R}^M$ solving

$$\begin{cases} \nabla f(\mathbf{x}) = A^T\lambda \\ A\mathbf{x} = \mathbf{y} \end{cases} \tag{23}$$

Proof. We already know that $\Box^A f(\mathbf{y}) < +\infty$ for every $\mathbf{y} \in \mathbb{R}^M$, thus $\Box^A f(\mathbf{y}) \in \mathbb{R}$. Observe that, once $\mathbf{y} \in \mathbb{R}^M$ is fixed, (22) is what is called in [11] Chapter 28 an ordinary convex program admitting a feasible solution (since f is real valued). Its set of constraint is given by $\mathbf{y} - A\mathbf{x} = 0$, and $C = \operatorname{ri}(C) = \mathbb{R}^N$ in the notation of [11]. By the Kuhn-Tucker Theorem ([11] Corollary 28.3.1, whose hypotheses are met since we are assuming $\Box^A f(\mathbf{y}) > -\infty$), (i) in the statement is equivalent to: there exists a pair $(\mathbf{x}, \lambda) \in \mathbb{R}^N \times \mathbb{R}^N$ satisfying conditions (a),(b),(c) in [11] Theorem 28.3 (with $\mathbf{x}$ in place of $\overline{\mathbf{x}}$ and λ in place of $\mathbf{u}^*$). By the discussion following the proof of [11] Theorem 28.3, since f and the functions enforcing the constraints are differentiable, condition (c) can be rewritten as $\nabla f(\mathbf{x}) - A^T\lambda = \mathbf{0}$. Condition (b) is $A\mathbf{x} = \mathbf{y}$, and conditions (a) can actually be ignored since we have no inequality constraints. This proves that (i), (ii) are equivalent. ■

Proposition 5.6 *The following are equivalent:*

(1) there exists the optimum for (22) *for **all** $\mathbf{y} \in \mathbb{R}^M$.*

(2) there exists the optimum for (22) *for **some** $\hat{\mathbf{y}} \in \mathbb{R}^M$.*

Proof. Clearly (1) implies (2). As to the converse, observe that since f is essentially smooth, by Theorem 5.3 $\nabla f(\mathbb{R}^N) = \operatorname{int}\operatorname{dom} f^* \subseteq \operatorname{ri}\operatorname{dom} f^*$. Then, by the equivalence established in Proposition 5.5, $A^T\lambda(\hat{\mathbf{y}}) = \nabla f(\mathbf{x}(\hat{\mathbf{y}})) \in \operatorname{int}\operatorname{dom} f^* \subseteq \operatorname{ri}\operatorname{dom} f^*$ and Theorem 5.4 yields the existence of the optimum for every $\mathbf{y} \in \mathbb{R}^M$. ■

Proposition 5.7 *Under Assumption 2.8, the map $\square^A f : \mathbb{R}^M \to \mathbb{R}$ is continuously differentiable and strictly convex on $\mathbb{R}^M$. Its conjugate is given by $(\square^A f)^*(\mathbf{z}) = f^*(A^T\mathbf{z})$, $\mathbf{z} \in \mathbb{R}^M$, which is continuously differentiable on the interior of its domain. The gradient $\nabla\square^A f(\cdot)$ is a homeomorphism between $\mathbb{R}^M$ and $\operatorname{int}\operatorname{dom}(\square^A f)^* = \operatorname{int}\{\mathbf{z} \in \mathbb{R}^M \mid A^T\mathbf{z} \in \operatorname{dom} f^*\} =: \mathcal{O} \subseteq \mathbb{R}^M$, and its (continuous) inverse is given by $\nabla(\square^A f)^*(\mathbf{z}), \mathbf{z} \in \mathcal{O}$. Finally, the unique optimum $\mathbf{x} = \mathbf{x}(\mathbf{y})$ of problem (22) is given by*

$$\mathbf{x} = \Theta(\mathbf{y}) = \nabla f^*\left(A^T \cdot \nabla(\square^A f)(\mathbf{y})\right) \tag{24}$$

where $\Theta : \mathbb{R}^M \to \mathbb{R}^N$ is continuous on $\mathbb{R}^M$.

Proof. By Remark 5.2 and Proposition 5.6 there exists an optimum of (22) for all $\mathbf{y} \in \mathbb{R}^M$. In particular $\square^A f(\mathbf{y}) \in \mathbb{R}\,\forall \mathbf{y} \in \mathbb{R}^M$. As argued in the proof of Proposition 5.6 there exists $\lambda \in \mathbb{R}^M$ s.t. $A^T\lambda \in \operatorname{int}\operatorname{dom}(f^*) \subseteq \operatorname{ri}\operatorname{dom}(f^*)$. Since f is essentially smooth on the whole $\mathbb{R}^N$, by [11] Corollary 26.3.3 $\square^A f$ is itself essentially smooth throughout the whole $\mathbb{R}^M$. Existence of optima yields by standard arguments the strict convexity of $\square^A f$, which is induced by the one of f, and $(\square^A f, \mathbb{R}^M)$ is then of Legendre type. By Theorem 5.3 applied to $h = \square^A f$, $\square^A f : \mathbb{R}^M \to \mathbb{R}$ is continuously differentiable on $\mathbb{R}^M$. Its conjugate $(\square^A f)^*$ is continuously differentiable on $\mathcal{O}$, the gradient $\nabla(\square^A f)(\cdot)$ is a homeomorphism between $\mathbb{R}^M$ and $\mathcal{O}$, and its (continuous) inverse is given by $\nabla(\square^A f)^*(\cdot)$. Now, fix $\mathbf{y} \in \mathbb{R}^M$ and take $(\mathbf{x}, \lambda) = (\mathbf{x}(\mathbf{y}), \lambda(\mathbf{y})) \in \mathbb{R}^N \times \mathbb{R}^M$ solving (23). In particular $A^T\lambda \in \operatorname{int}\operatorname{dom}(f^*)$, and $\mathbf{x} = (\nabla f)^{-1}(A^T\lambda) = \nabla f^*(A^T\lambda)$, by Theorem 5.3. Since $A\mathbf{x} = \mathbf{y}$, we get $\mathbf{y} = A\nabla f^*(A^T\lambda)$. The last step is to prove that $\lambda \in \operatorname{int}\operatorname{dom}(\square^A f)^* = \mathcal{O}$ and $A\nabla f^*(A^T\lambda) = \nabla(\square^A f)^*(\lambda)$, as this would give $\lambda = (\nabla(\square^A f)^*)^{-1}(\mathbf{y}) = \nabla(\square^A f)(\mathbf{y})$ by Theorem 5.3 so that $\mathbf{x} = \nabla f^*\left(A^T\lambda\right) = \nabla f^*\left(A^T \cdot \nabla(\square^A f)(\mathbf{y})\right)$.

We come to these verifications. First, observe that as argued before $A^T\lambda \in \operatorname{int}\operatorname{dom} f^*$, which is open. Then, λ belongs to the pre-image of $\operatorname{int}\operatorname{dom} f^*$ under A^T, which is open by continuity of A^T. Set now $\mathcal{O}' := (A^T)^{-1}(\operatorname{int}\operatorname{dom} f^*)$. Since $A^T(\mathcal{O}') = \operatorname{int}\operatorname{dom} f^* \subseteq \operatorname{dom} f^*$, then $\lambda \in \mathcal{O}' \subseteq \mathcal{O}$. To conclude we prove that

$$\nabla(\square^A f)^*(\mathbf{z}) = A\nabla f^*(A^T\mathbf{z}), \quad \mathbf{z} \in \mathcal{O}' \subseteq \mathcal{O}.$$

First, by [11] Theorem 16.3 $(\square^A f)^*(\mathbf{z}) = f^*(A^T\mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}^M$. The map $(\square^A f)^*$ on $\mathcal{O}'$ is then the composition of the map A^T, differentiable on $\mathcal{O}'$ and taking values in $\operatorname{int}\operatorname{dom} f^*$, and f^*, differentiable on the latter set. The formula is then the usual chain rule. Continuity of Θ follows observing that for every $\mathbf{y} \in \mathbb{R}^M$, $A^T\lambda(\mathbf{y}) \in \operatorname{int}\operatorname{dom} f^*$, where ∇f^* is continuous, and that $\lambda(\mathbf{y}) = \nabla(\square^A f)(\mathbf{y})$, the latter being continuous on $\mathbb{R}^M$. ■

6 Proof of Theorem 2.9

We work again in the Setting 2.7. By Remark 5.2 we may apply the results in Proposition 5.7. The proof is indeed based on the following two facts that are proven in Proposition 5.7, using there the notation $f := -U$, $A\mathbf{X} := \pi(\mathbf{X})$ and $\square^A f = -\square_\pi U$.

a) The function $\square_\pi U : \mathbb{R}^M \to \mathbb{R}$ is continuous on $\mathbb{R}^M$;

b) Fix any $\mathbf{z} \in \mathbb{R}^M$. There exist a unique optimum $\mathbf{x} = \Theta(\mathbf{z}) \in \mathbb{R}^N$ for the problem $\square_\pi U(\mathbf{z})$ with $\Theta : \mathbb{R}^M \to \mathbb{R}^N$ being the continuous function on $\mathbb{R}^M$ defined in (24). In particular we have: (b1) $\square_\pi U(\mathbf{z}) = U(\Theta(\mathbf{z}))$ and (b2) $\pi(\Theta(\mathbf{z})) = \mathbf{z}$.

Proof of Item 1. Fix $\mathbf{Z} \in (L^\infty)^M$. We now prove

$$\sup\left\{\mathbb{E}\left[U(\mathbf{Y})\right] \mid \mathbf{Y} \in (L^\infty)^N, \pi(\mathbf{Y}) = \mathbf{Z}\right\} =: \square_\pi \mathbb{U}(\mathbf{Z}) = \mathbb{E}\left[\square_\pi U(\mathbf{Z})\right]. \tag{25}$$

Fix $\mathbf{z} \in \mathbb{R}^M$ and observe that if $\mathbf{y} \in \mathbb{R}^N$ satisfies $\pi(\mathbf{y}) = \mathbf{z}$ then

$$U(\mathbf{y}) \le \sup \Big\{ U(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^N, \pi(\mathbf{x}) = \mathbf{z} \Big\} = \square_\pi U(\mathbf{z}) .$$

Now, we can plug in $\mathbf{Y} \in (L^\infty)^N$ s.t. $\pi(\mathbf{Y}) = \mathbf{Z}$, to get $U(\mathbf{Y}) \le \square_\pi U(\mathbf{Z})$. From (a) we know that $\square_\pi U$, as well as U, is a continuous function and so no measurability issues arise and both $U(\mathbf{Y})$ and $\square_\pi U(\mathbf{Z})$ are bounded random variables. We can then take expectations on both sides of the latter inequality and deduce $(\square_\pi \mathbb{U})(\mathbf{Z}) \le \mathbb{E}\left[(\square_\pi U)(\mathbf{Z})\right]$. We prove the converse inequality. Consider the continuous function Θ in (b) and set $\hat{\mathbf{Y}} := \Theta(\mathbf{Z})$. Then $\hat{\mathbf{Y}} \in (L^\infty)^N$ and by (b2) above, $\pi(\hat{\mathbf{Y}}) = \pi(\Theta(\mathbf{Z})) = \mathbf{Z}$. Thus, $\hat{\mathbf{Y}}$ satisfies the constraints in LHS of (25). Consequently,

$$\sup \Big\{ \mathbb{E}\left[U(\mathbf{Y})\right] \mid \mathbf{Y} \in (L^\infty)^N, \pi(\mathbf{Y}) = \mathbf{Z} \Big\} \ge \mathbb{E}\Big[U\Big(\hat{\mathbf{Y}}\Big)\Big] = \mathbb{E}\Big[U\Big(\Theta(\mathbf{Z})\Big)\Big] = \mathbb{E}\left[\square_\pi U(\mathbf{Z})\right],$$

by (b1), which concludes the proof of (25).

Proof of Item 2. Recall that in Setting 2.7 Assumption 2.1 holds true. Fix $\mathbf{Z} \in (L^\infty)^M$ and $\mathbf{X} \in (L^\infty)^N$. From (a) we deduce that $\mathbb{E}\left[\square_\pi U(\mathbf{Z})\right] < \infty$ and by (25), $\square_\pi \mathbb{U}(\mathbf{Z}) = \mathbb{E}\left[\square_\pi U(\mathbf{Z})\right] < \infty$. By the assumption $\sup\{U(\mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0$ we obtain $\sup\{\mathbb{U}(\mathbf{X} + \mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0$. Thus all the assumptions in Proposition 2.4 are satisfied and hence $\rho_{\pi,\mathbb{U}}(\mathbf{X}) = \rho_{(\square_\pi \mathbb{U})}(\pi(\mathbf{X})) = \rho_{(\mathbb{E}[\square_\pi U])}(\pi(\mathbf{X}))$, by (25). Recalling the definition in (5), we thus proved (13). Regarding the optimality of $\mathbf{Y}_\mathbf{X} := -\mathbf{X} + \Theta(\pi(\mathbf{X}) + \widehat{\alpha})$, observe that $\mathbb{U}(\mathbf{X} + \mathbf{Y}_\mathbf{X}) = \mathbb{U}(\Theta(\pi(\mathbf{X}) + \widehat{\alpha})) = \mathbb{E}\left[U(\Theta(\pi(\mathbf{X}) + \widehat{\alpha}))\right] = \mathbb{E}\left[\square_\pi U(\pi(\mathbf{X}) + \widehat{\alpha})\right] \ge 0$, where in the last equality we used (b1), and the inequality follows from the optimality of $\widehat{\alpha}$ in (13). Thus $\mathbf{Y}_\mathbf{X}$ satisfies the inequality constraint in (11) Moreover, using the linearity of π and (b2) we get

$$\pi(\mathbf{Y}_\mathbf{X}) = \pi\Big(-\mathbf{X} + \Theta(\pi(\mathbf{X}) + \widehat{\alpha})\Big) = -\pi(\mathbf{X}) + \pi\Big(\Theta(\pi(\mathbf{X}) + \widehat{\alpha})\Big) = -\pi(\mathbf{X}) + \pi(\mathbf{X}) + \widehat{\alpha} = \widehat{\alpha}$$

so that $\mathbf{Y}_\mathbf{X} \in \mathcal{C}$. Finally, $\sum_{m=1}^M \pi^m(\mathbf{Y}_\mathbf{X}) = \sum_{m=1}^M \widehat{\alpha}^m = \rho_{\pi,\mathbb{U}}(\mathbf{X})$, by optimality of $\widehat{\alpha}$. Thus $\mathbf{Y}_\mathbf{X}$ is the desired optimum, which is unique by the strict concavity of $\mathbb{U}$.

Conclusion, for the case $M = 1$. If $\rho_{\pi,\mathbb{U}}(\mathbf{X})$ is finite, then the optimality of $\widehat{\alpha} = \rho_{\pi,\mathbb{U}}(\mathbf{X})$ is directly checked by monotone convergence theorem, considering that $\square_\pi U$ is continuous on $\mathbb{R}$ and nondecreasing by Proposition 5.7. Thus, we only need to show that $\rho_{\pi,\mathbb{U}}(\mathbf{X}) \in \mathbb{R}$ for every $\mathbf{X} \in (L^\infty)^N$. Since we are in Setting 2.7, by Remark 2.5 we have $\sup\{\mathbb{U}(\mathbf{X} + \mathbf{y}) \mid \mathbf{y} \in \mathbb{R}^N\} > 0$ which yields $\rho_{\pi,\mathbb{U}}(\mathbf{X}) < +\infty$. Suppose now by contradiction that $\rho_{\pi,\mathbb{U}}(\mathbf{X}) = -\infty$ and take a minimizing sequence $\mathbf{Y}_n \in \mathcal{C}$ with $\pi(\mathbf{Y}_n) \downarrow_n -\infty$ and $\mathbb{E}\left[U(\mathbf{X} + \mathbf{Y}_n)\right] \ge 0$ for every n. By Proposition 5.5, since we are under Assumption 2.8 and Remark 5.2 applies, there exists a $\lambda \in \mathbb{R}$ such that $A^T\lambda \in \nabla f(\mathbb{R}^N)$. By Theorem 5.3 we have $A^T\lambda \in \operatorname{intdom}(f^*) \subseteq (-\infty, 0)^N$, the latter following from monotonicity of $f = -U$, which implies $\lambda < 0$: indeed all the components of A are nonnegative, as $A(\mathbb{R}^N_+) = \mathbb{R}_+$. Now by Fenchel inequality we have $-f^*(A^T\lambda) - f(\mathbf{x}) \le (-\lambda)A\mathbf{x}$. Substituting $\mathbf{x}$ with $\mathbf{X} + \mathbf{Y}_n$ and taking expectations yields a contradiction, as $f^*(A^T\lambda) \in \mathbb{R}$ and $\mathbb{E}\left[-f(\mathbf{X} + \mathbf{Y}_n)\right] \ge 0$ for each n, while RHS tends to $-\infty$, as $\pi(\mathbf{Y}_n) \downarrow_n -\infty$.

A Appendix

A function $\Phi : [0, +\infty)^N \to \mathbb{R}$ is called multivariate Orlicz function if it is null in 0, convex, continuous, increasing in the usual componentwise order and satisfies: there exist $A > 0$, B constants such that $\Phi(\mathbf{x}) \ge A \sum_{j=1}^N x^j - B$ for every $\mathbf{x} \in [0, +\infty)^N$. We refer to [1] and [6] for further details. Inspired by [6] Definition 3.4, we say that a function $U : \mathbb{R}^N \to \mathbb{R}$ is well controlled if there exist a multivariate Orlicz function $\widehat{\Phi} : R^N \to \mathbb{R}$ and a function $h : [0, +\infty) \to \mathbb{R}$ such that $U(\mathbf{x}) \le -\widehat{\Phi}((x)^-) + \varepsilon \sum_{j=1}^N \left|x^j\right| + h(\varepsilon)$ for every $\varepsilon > 0$.

Lemma A.1 *Suppose $U : \mathbb{R}^N \to \mathbb{R}$ is strictly concave, strictly increasing in the componentwise order and also well controlled. Suppose also that π satisfies Assumption 2.1 (a), and that $\sum_{m=1}^M \pi^m(\mathbf{x}) = \sum_{j=1}^N x^j$ for every $\mathbf{x} \in \mathbb{R}^N$. Then Assumption 2.8 is satisfied.*

Proof. Take $\mathbf{z} = \mathbf{0} \in \mathbb{R}^M$ and take a maximizing sequence $(\mathbf{x}_n)_n$ for $\Box_\pi U(\mathbf{0})$, w.l.o.g. assuming that $U(\mathbf{x}_n) \geq \Box_\pi U(\mathbf{0}) - 1$ for every n. By [6] Lemma 3.5.(iv) we have for some $a > 0, b \in \mathbb{R}$ that

$$\Box_\pi U(\mathbf{0}) - 1 \leq U(\mathbf{x}_n) \leq a \sum_{j=1}^N (x_n^j)^+ - 2a \sum_{j=1}^N (x_n^j)^- + b$$

$$= a \sum_{j=1}^N x_n^j - a \sum_{j=1}^N (x_n^j)^- + b = a \sum_{m=1}^M \pi^m(\mathbf{x}_n) - a \sum_{j=1}^N (x_n^j)^- + b = a \sum_{m=1}^M z^m + b - a \sum_{j=1}^N (x_n^j)^-$$

It follows that $\sum_{j=1}^N (x_n^j)^-$ needs to be bounded, and since also $\sum_{j=1}^N (x_n^j)^+ = \sum_{j=1}^N x_n^j + \sum_{j=1}^N (x_n^j)^- = \sum_{m=1}^M z^m + \sum_{j=1}^N (x_n^j)^-$ the same holds for $\sum_{j=1}^N (x_n^j)^+$. Thus $(\mathbf{x}_n)_n$ is bounded in $\mathbb{R}^N$. Passing to a subsequence converging to some $\mathbf{x}_\infty \in \mathbb{R}^N$ we get by continuity of π (which is linear on $\mathbb{R}^N$ and takes values in $\mathbb{R}^M$ by hypothesis) that $\pi(\mathbf{x}_\infty) = \mathbf{z}$, and since U is continuous on $\mathbb{R}^N$ (by [11] Theorem 10.4 applied to $f = -U$, since it is finite-valued on the whole $\mathbb{R}^N$ by assumption) we have $U(\mathbf{x}_\infty) = \lim_n U(\mathbf{x}_n) = \Box_\pi U(\mathbf{0})$. This proves the optimality of $\mathbf{x}_\infty$. ■

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