ERRATUM TO: REMARKS ON MEAN CURVATURE FLOW SOLITONS IN WARPED PRODUCTS

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Dedicated to Patrizia Pucci on her 65^{th} *birthday*

Abstract. We provide full details of the proof of Theorem 3.4 (Theorem C in the Introduction) in Colombo, Mari and Rigoli, *Remarks on mean curvature flow solitons in warped products.* Discrete Contin. Dyn. Syst. Ser. S 13 (2020), no. 7, 1957-1991.

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1. Introduction

The purpose of this note is to provide full details of the rather involved proof of the following result, [2, Theorem 3.4] (also, Theorem C in the Introduction). Indeed, the one appearing in [2] was incomplete, especially in the concluding topological part. At the same time, we simplify various arguments and reorganize the proof to make it more readable.

In what follows, $(\mathbb{P}^m, \langle\ ,\ \rangle_{\mathbb{P}})$ is a complete m-dimensional Riemannian manifold, $\overline{M} = I \times_h \mathbb{P}$ is the product of the interval $I \subset \mathbb{R}$ and of $\mathbb{P}$, endowed with the warped product metric

$$\bar{g} = \mathrm{d}t^2 + h(t)^2 \langle\ ,\ \rangle_{\mathbb{P}},$$

and $\pi_I : I \times_h \mathbb{P} \to I$ is the standard projection. With our chosen normalization, a soliton with respect to the vector field $X = h(t)\partial_t$, with soliton constant $c \in \mathbb{R}$, is an isometric immersion $\psi : M \to \overline{M}$ which solves

$$cX^\perp = m\mathbf{H},$$

where $\mathbf{H}$ is the normalized mean curvature vector and $\perp$ is the projection onto the normal bundle of M. Writing $\eta = \overline{\eta} \circ \psi$, where $\overline{\eta} \in C^\infty(\overline{M})$ is such that $\overline{\nabla}\overline{\eta} = X$, [2, Theorem 3.4] classifies complete, stable solitons in manifolds $\overline{M}$ with constant sectional curvature, whose umbilicity tensor Φ satisfies $|\Phi| \in L^2(M, e^{c\eta})$, where $L^2(M, e^{c\eta})$ is the space of functions v such that

$$\int_M v^2 e^{c\eta} \mathrm{d}x < \infty.$$

If $\overline{M}$ has constant sectional curvature $\bar{\kappa}$, we note that necessarily $\mathbb{P}$ has constant sectional curvature too, say κ, and by Gauss equations

$$-\frac{h''}{h} = \bar{\kappa} = \frac{\kappa}{h^2} - \left(\frac{h'}{h}\right)^2, \tag{1}$$

thus

$$\kappa + h''h - (h')^2 \equiv 0. \tag{2}$$

In this case, the stability (Jacobi) operator of M is given by

$$L = \Delta_{-c\eta} + (|\mathrm{II}|^2 + m\bar{\kappa} - ch').$$

We refer to [2] for further notation.

Theorem 1.1. *Let $\psi : M^m \to \overline{M}^{m+1} = I \times_h \mathbb{P}$ be a connected, complete, stable mean curvature flow soliton with respect to $X = h(t)\partial_t$ with soliton constant c. Assume that $\overline{M}$ is complete and has constant sectional curvature $\bar\kappa$, with*

$$ch'(\pi_I \circ \psi) \le m\bar\kappa \quad \text{on } M. \tag{3}$$

Let $\Phi = \mathrm{II} - \langle\ ,\ \rangle_M \otimes \mathbf{H}$ be the umbilicity tensor of ψ and suppose that

$$|\Phi| \in L^2(M, e^{c\eta}) \tag{4}$$

with η defined as above. Then one of the following cases occurs:

(i) *ψ is totally geodesic (and, if $c \neq 0$, $\psi(M)$ is invariant by the flow of X), or*

(ii) *$I = \mathbb{R}$, h is constant on $\mathbb{R}$, $\overline{M}$ is isometric to the product $\mathbb{R} \times F$ with F a complete flat manifold and M is also flat. By introducing the universal coverings $\pi_M : \mathbb{R}^m \to M$, $\pi_F : \mathbb{R}^m \to F$ and $\pi_{\overline{M}} = \mathrm{id}_{\mathbb{R}} \times \pi_F : \mathbb{R}^{m+1} \to \overline{M}$, the map ψ lifts to an immersion $\hat\psi : \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^m$ satisfying $\pi_{\overline{M}} \circ \hat\psi = \psi \circ \pi_M$, which up to an isometry of $\mathbb{R}^m$ and a translation along the $\mathbb{R}$ factor of $\mathbb{R}^{m+1}$ is given by*

$$\hat\psi : \mathbb{R}^m \to \mathbb{R}^{m+1}, \quad (x^1, x^2, \dots, x^m) \mapsto (\sigma_1(x^1), \sigma_2(x^1), x^2, \dots, x^m)$$

where $\gamma = (\sigma_1, \sigma_2) : \mathbb{R} \to \mathbb{R}^2$ is the grim reaper curve with image

$$\sigma(\mathbb{R}) = \left\{ (x, y) \in \mathbb{R}^2 : x = -\frac{1}{ch_0} \log(\cos(ch_0 y)), |y| < \frac{2}{\pi |c| h_0} \right\}$$

and h_0 is the constant value of h on $\mathbb{R}$. Furthermore, there exists a Riemannian submersion $\pi_\Omega : M \to \Omega$ onto a compact, flat manifold Ω with 1-dimensional, noncompact geodesic fibers of the type $\pi_M(\mathbb{R} \times \{(x^2, \dots, x^m)\})$, for constant $(x^2, \dots, x^m) \in \mathbb{R}^{m-1}$. Such fiber is mapped by ψ into the grim reaper curve $\pi_{\overline{M}}(\sigma(\mathbb{R}) \times \{(x^2, \dots, x^m)\})$.

Furthermore, any of the solitons in (ii) is stable, while a soliton in (i) is stable if and only if $L = \Delta_{-c\eta} + (m\bar\kappa - ch')$ is non-negative.

Remark 1. The completeness assumption on $\overline{M} = I \times_h \mathbb{P}$ forces $I = \mathbb{R}$. However, completeness can be weakened to the combination of the following two requirements:

(i) $\mathbb{P}$ is complete;

(ii) either $I = \mathbb{R}$ or h extends continuously to $\partial I \subset \mathbb{R}$ with value 0.

Proof. We split the proof into several steps.

Step 1: the function $u = |\Phi|^2$ either vanishes identically, or it is everywhere positive and satisfies

$$u\Delta_{-c\eta} u + 2(|\mathrm{II}|^2 + m\bar\kappa - ch'(\pi_I \circ \psi))u^2 = \frac{1}{2}|\nabla u|^2. \tag{5}$$

Proof of Step 1. Having fixed a local unit normal ν and set $H = \langle \mathbf{H}, \nu \rangle$ we clearly have

$$|\Phi|^2 = |\mathrm{II}|^2 - mH^2 \ge 0.$$

Furthermore,

$$|\nabla\Phi|^2 = |\nabla \mathrm{II}|^2 - m|\nabla H|^2. \tag{6}$$

We recall, see for instance equation (9.37) in [1], that in the present setting we have the validity of the Simons' type formula

$$\frac{1}{2}\Delta_{-c\eta}|\mathrm{II}|^2 = -(ch'(\pi_I \circ \psi) + |\mathrm{II}|^2)|\mathrm{II}|^2 + m\bar\kappa|\Phi|^2 + |\nabla \mathrm{II}|^2. \tag{7}$$

On the other hand, from [2, Formula (65)], valid in the present assumptions, we have

$$\frac{1}{2}\Delta_{-c\eta}H^2 = -(ch'(\pi_I\circ\psi) + |\mathrm{II}|^2)H^2 + |\nabla H|^2. \tag{8}$$

Putting together (7) and (8) and using (6) we obtain

$$\frac{1}{2}\Delta_{-c\eta}|\Phi|^2 + (ch'(\pi_I\circ\psi) + |\mathrm{II}|^2 - m\bar{\kappa})|\Phi|^2 - |\nabla\Phi|^2 = 0. \tag{9}$$

The function $u = |\Phi|^2$ therefore solves

$$u\Delta_{-c\eta}u + 2(|\mathrm{II}|^2 + m\bar{\kappa} - ch'(\pi_I\circ\psi))u^2 = 2|\nabla\Phi|^2u + 4(m\bar{\kappa} - ch'(\pi_I\circ\psi))u^2. \tag{10}$$

Now by (3) we deduce

$$m\bar{\kappa} - ch' \geq 0 \quad \text{on } M,$$

while from Kato's inequality $|\nabla|\Phi||^2 \leq |\nabla\Phi|^2$ we get

$$2u|\nabla\Phi|^2 \geq \frac{1}{2}|\nabla u|^2. \tag{11}$$

Substituting in the above we eventually have

$$u\Delta_{-c\eta}u + 2(|\mathrm{II}|^2 + m\bar{\kappa} - ch'(\pi_I\circ\psi))u^2 \geq \frac{1}{2}|\nabla u|^2. \tag{12}$$

The stability of the soliton implies the existence of $v > 0$ on M solving

$$\Delta_{-c\eta}v + (|\mathrm{II}|^2 + m\bar{\kappa} - ch')v = 0. \tag{13}$$

We now apply [2, Theorem 3.1] with the choices

$$a(x) = 2(|\mathrm{II}|^2 + m\bar{\kappa} - ch'(\pi_I\circ\psi))(x), \quad f = -c\eta, \quad \mu = \frac{1}{2}, \quad A = -\frac{1}{2}, \quad K = 0.$$

In case M is non-compact, by choosing the admissible $\beta = -\frac{1}{2}$ we see that (4) implies

$$\left(\int_{\partial B_r} ue^{c\eta}\right)^{-1} \notin L^1(+\infty),$$

that corresponds to [2, Formula (60)] for the choice $p = 2$. Applying [2, Theorem 3.1] we deduce that either $u \equiv 0$ (so $\psi : M \to \overline{M}$ is totally umbilical) or $u > 0$ and $u^{\frac{1}{2}}$ satisfies the equation

$$\Delta_{-c\eta}u^{1/2} + (|\mathrm{II}|^2 + m\bar{\kappa} - ch'(\pi_I\circ\psi))u^{1/2} = 0$$

or equivalently

$$u\Delta_{-c\eta}u + 2(|\mathrm{II}|^2 + m\bar{\kappa} - ch'(\pi_I\circ\psi))u^2 = \frac{1}{2}|\nabla u|^2. \tag{14}$$

This proves Step 1.

Step 2: if $u \equiv 0$, then ψ is totally geodesic and, if $c \neq 0$, X is tangent to $\psi(M)$. The stability of ψ is equivalent to the non-negativity of $L = \Delta_{-c\eta} + (m\bar{\kappa} - ch')$.

Proof of Step 2. $\overline{M}$ is a space of constant curvature, so the tensor field II is Codazzi. Fixing a local orthonormal coframe $\{\theta^i\}_i$ on M we write $\mathrm{II} = a_{ij}\theta^i\otimes\theta^j\otimes\nu$, $\nabla\mathrm{II} = a_{ijk}\theta^k\otimes\theta^i\otimes\theta^j\otimes\nu$, $dH = H_k\theta^k$. Since $u \equiv 0$, the umbilicity tensor $\Phi = \mathrm{II} - \langle\ ,\ \rangle\otimes\mathbf{H}$ vanishes, so we have $a_{ij} = H\delta_{ij}$ and $a_{ijk} = \delta_{ij}H_k$ by parallelism of the metric. For $1 \leq k \leq m$ and for any index $t \neq k$ we have $H_k = \delta_{tt}H_k = a_{ttk} = a_{tkt} = \delta_{tk}H_t = 0$, where $a_{ttk} = a_{tkt}$ holds true as II is Codazzi. It follows that $dH \equiv 0$, therefore H is constant and so is $|\mathrm{II}|^2 = mH^2$. Plugging this into (8) we get

$$H^2(ch'(\pi_I\circ\psi) + mH^2) \equiv 0 \quad \text{on } M.$$

In particular, either $H \equiv 0$, and ψ is totally geodesic, or $H \neq 0$, $ch'(\pi_I \circ \psi) \neq 0$, $ch'(\pi_I \circ \psi) = -mH^2$ on M. We now prove that the second case cannot occur.

Suppose, by contradiction, that $u \equiv 0$, $H \neq 0$ and that $\pi_I \circ \psi$ is constant: then $\psi(M)$ is a slice $\{t_0\} \times \mathbb{P}$ for some $t_0 \in I$ such that $ch'(t_0) = -mH^2 = -m\frac{h'(t_0)^2}{h(t_0)^2}$ and from (1) the stability operator of ψ can be written as

$$\begin{aligned}\Delta_{-c\eta} + (|\mathrm{II}|^2 + m\bar{\kappa} - ch'(t_0)) &= \Delta + m\left(H^2 + \frac{\kappa - h'(t_0)^2}{h(t_0)^2} + H^2\right) \\ &= \Delta + m\,\frac{\kappa + h'(t_0)^2}{h(t_0)^2},\end{aligned} \tag{15}$$

where Δ is the Laplace-Beltrami operator of $(\mathbb{P}, h(t_0)^2\langle\ ,\ \rangle_{\mathbb{P}})$. Condition (3) now reads as

$$ch'(t_0) = -m\,\frac{h'(t_0)^2}{h(t_0)^2} \leq m\bar{\kappa} = m\,\frac{\kappa}{h(t_0)^2} - m\,\frac{h'(t_0)^2}{h(t_0)^2},$$

that is $\kappa \geq 0$. Since $(\mathbb{P}, h(t_0)^2\langle\ ,\ \rangle_{\mathbb{P}})$ has constant sectional curvature $\frac{\kappa}{h(t_0)^2} \geq 0$, the bottom of the spectrum of $-\Delta$ is zero (cf. [5]). It follows that the stability operator (15) is non-negative if and only if $\kappa = 0$ and $h'(t_0) = 0$. But $h'(t_0) = 0$ implies that $H^2 = \frac{h'(t_0)^2}{h(t_0)^2} = 0$, contradicting the assumption that $H \neq 0$.

Suppose, by contradiction again, that $u \equiv 0$, $H \neq 0$ and that $\pi_I \circ \psi$ is not constant on M. Then h' is constant on the nondegenerate interval $(\pi_I \circ \psi)(M) \subseteq I$ and by (1) this forces $\bar{\kappa} = 0$. From $|\mathrm{II}|^2 = mH^2$ and $ch'(\pi_I \circ \psi) = -mH^2$, the stability operator becomes

$$L = \Delta_{-c\eta} + (|\mathrm{II}|^2 + m\bar{\kappa} - ch'(\pi_I \circ \psi)) = \Delta_{-c\eta} + 2mH^2.$$

The Gauss equation and the fact that ψ is totally umbilic imply that M has constant sectional curvature $H^2 > 0$, whence it is compact. Therefore, the first eigenvalue of L is $-2mH^2 < 0$, contradiction.

So far, we have proved that if $u \equiv 0$ then ψ must be totally geodesic. The stability operator reduces to $\Delta_{-c\eta} + (m\bar{\kappa} - ch')$. If $c \neq 0$, the fact that X is tangent to $\psi(M)$ follows by the soliton equation $cX^{\perp} = m\mathbf{H}$, which concludes the proof of Step 2.

Step 3: if $u > 0$ solves (14), then $I = \mathbb{R}$, $\overline{M}$ is flat, $X = h_0\partial_t$ for some constant $h_0 > 0$, and u is not constant on any open subset of M.

Proof of Step 3. From (10) and (14) we have

$$\frac{1}{2}|\nabla|\Phi|^2|^2 = 2u|\nabla\Phi|^2 + 4(m\bar{\kappa} - ch')u^2.$$

Since $m\bar{\kappa} - ch' \geq 0$ and $u > 0$, from the above and Kato's inequality (11) we deduce

$$m\bar{\kappa} \equiv ch'(\pi_I \circ \psi), \qquad |\nabla\Phi|^2 = |\nabla|\Phi||^2 \quad \text{on } M. \tag{16}$$

Note that u cannot be constant on an open set of M, since otherwise equation (14) would reduce to $0 = 2|\mathrm{II}|^2u^2$, which is absurd since $|\mathrm{II}|^2 \geq u > 0$.

We prove that $\overline{M}$ is flat. Indeed, if $c = 0$ then $\bar{\kappa} = 0$ by (16); if $c \neq 0$, then again by (16) and since $\psi(M)$ is not a slice (as slices are totally umbilical), we see that h' is constant on the nondegenerate interval $(\pi_I \circ \psi)(M) \subseteq I$, so $\bar{\kappa} = -h''/h \equiv 0$ by (1). Inserting this into (16) we see that $h' \equiv 0$, so h is constant and the completeness of $\overline{M}$ or Remark 1 imply that $I = \mathbb{R}$. Moreover, $\kappa = 0$ by (1) and we conclude that $\mathbb{P}$ is flat and $X = h_0\partial_t$ for some $h_0 > 0$. This concludes the proof of Step 3.

Step 4: if $u > 0$ solves (14), then M is isometric to a cylinder $\mathbb{R} \times \Sigma$ for some complete Σ, with metric and second fundamental form given by

$$\langle\ ,\ \rangle = \mathrm{d}s \otimes \mathrm{d}s + \langle\ ,\ \rangle_\Sigma, \qquad \mathrm{II} = \mu_1(s)\,\mathrm{d}s \otimes \mathrm{d}s \otimes \nu, \tag{17}$$

where ν is a unit normal vector to $M \to \overline{M}$. Moreover, $\mu_1 \neq 0$ on $\mathbb{R}$, and the soliton constant satisfies $c \neq 0$.

Proof of Step 4. By Step 3, we know that u cannot be constant on any open set of M, thus the set $\{\nabla u \neq 0\}$ is nonempty and dense in M. Fix a point $p \in M$ such that $\nabla u(p) \neq 0$. The tensor field II is Codazzi, as observed in Step 2, and its traceless part Φ attains the equality in Kato's inequality by (16). By applying [2, Lemma 3.3] with $A = \mathrm{II}$, we obtain the existence of a neighbourhood U of p that splits as a Riemannian product $(-\varepsilon, \varepsilon) \times \Sigma^{m-1}$ and such that the metric $\langle\ ,\ \rangle$ of M and the tensor field II can be written as

$$\langle\ ,\ \rangle = \mathrm{d}s \otimes \mathrm{d}s + \langle\ ,\ \rangle_\Sigma, \qquad \mathrm{II} = \Big(\mu_1(s)\,\mathrm{d}s \otimes \mathrm{d}s + \mu_2(\pi_\Sigma)\langle\ ,\ \rangle_\Sigma\Big) \otimes \nu, \tag{18}$$

for some smooth functions $\mu_1 : (-\varepsilon, \varepsilon) \to \mathbb{R}$, $\mu_2 : \Sigma \to \mathbb{R}$ satisfying

$$\mu_1(s) \neq \mu_2(x) \quad \text{for each } s \in (-\varepsilon, \varepsilon),\ x \in \Sigma. \tag{19}$$

Up to reparametrizing $(-\varepsilon, \varepsilon)$, we can write

$$p = (0, q) \qquad \text{for some } q \in \Sigma. \tag{20}$$

Now we prove that $\mu_1(s)\mu_2(\pi_\Sigma) \equiv 0$ on U. Let $\{\theta^i\}$ be a local orthonormal coframe on U as the one described in the last part of the proof of [2, Lemma 3.3]. In particular, we assume that $\theta^1 = ds$ and then we have $\theta^1_j \equiv 0$ on U for $1 \leq j \leq m$. Writing

$$\mathrm{II} = a_{ij}\theta^i \otimes \theta^j \otimes \nu,$$

we have

$$a_{11} = \mu_1, \qquad a_{ii} = \mu_2 \quad \text{for } 2 \leq i \leq m \qquad \text{and} \qquad a_{ij} = 0 \quad \text{for each } i \neq j.$$

On the other hand, since $\overline{M}$ is flat, Gauss' equations give

$$R_{ijkt} = a_{ik}a_{jt} - a_{it}a_{jk} \qquad \text{for } 1 \leq i, j, k, t \leq m \tag{21}$$

where R_{ijkt} are the components of the Riemann curvature tensor of M along $\{\theta^i\}$. Recalling that $\theta^1_j \equiv 0$ for $1 \leq j \leq m$, by Cartan structural equations

$$\frac{1}{2}R_{ijkt}\theta^k \wedge \theta^t = \mathrm{d}\theta^i_j + \theta^i_k \wedge \theta^k_j \qquad \text{for } 1 \leq i, j \leq m$$

we immediately see that

$$R_{1jkt} = 0 \qquad \text{for } 1 \leq j, k, t \leq m.$$

Putting together these facts, from (21) we obtain

$$\mu_1(s)\mu_2(\pi_\Sigma) = a_{11}a_{22} = a_{11}a_{22} - a_{12}a_{12} = R_{1212} = 0.$$

Note that μ_1 and μ_2 can never be both zero at the same point by (19). Since they depend on disjoint sets of variables, this implies that exactly one of them identically vanishes on its domain while the other one never attains the zero value. In the 2-dimensional case where $m = 2$ we can assume without loss of generality that Σ is an interval and then $\mu_2 \equiv 0$, up to renaming indices. We claim that μ_2 identically vanishes on Σ also in case $m \geq 3$. Suppose, by contradiction, that $\mu_2 \neq 0$. Then $\mu_1 \equiv 0$ on $(-\varepsilon, \varepsilon)$ while μ_2 has constant (by [2, Lemma 3.3]) nonzero value. By (18), ψ has constant mean curvature $H = \frac{m-1}{m}\mu_2$ in U. Putting this constant value of H into equation (8) we obtain $0 = |\mathrm{II}|^2 H^2$, contradiction. So, we have proved that $\mu_2 \equiv 0$ on Σ. Moreover, the mean curvature of ψ is given by $H = \frac{1}{m}\mu_1(s) \neq 0$ on U, so ψ is not a minimal hypersurface and therefore $c \neq 0$. Also, the rank of II is exactly 1 in a neighbourhood of p. Since $u > 0$, we observe that M does not possess totally geodesic points, hence the rank of II it at least 1 everywhere. Since the rank of II is lower-semicontinuous and it is 1 on the dense subset $\{\nabla u \neq 0\}$, we deduce that II has rank one everywhere. In particular, the distribution corresponding to the nullity of II is smooth, totally geodesic and integrable (cf. [3, Proposition 1.18]). Hence, the entire M splits as $\mathbb{R} \times \Sigma$ for some complete Σ, and the metric and second fundamental form write as claimed on the entire M.

Step 5: if $u > 0$ solves (14), then ψ lifts to a (possibly tilted) grim reaper immersion $\hat{\psi} : \mathbb{R}^m \to \mathbb{R}^{m+1}$.

Proof of Step 5. By Step 3 and by completeness of $\overline{M}$, $I \times_h \mathbb{P} = \mathbb{R} \times F$, with $(F, \langle \ , \ \rangle_F) = (\mathbb{P}, h_0^2 \langle \ , \ \rangle_{\mathbb{P}})$ a complete flat manifold, and ψ is a translating soliton with respect to the parallel vector field ∂_t with soliton constant ch_0, which is non-zero by Step 4. Without loss of generality, we can therefore assume $c > 0$.

Let $\pi : \mathbb{R}^m \to F$ be the universal Riemannian covering of F. Then

$$\pi_{\overline{M}} = \mathrm{id}_{\mathbb{R}} \times \pi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \times F = \overline{M}$$

is the universal Riemannian covering of $\overline{M}$. The deck transformation group of the covering π is a discrete subgroup Γ_F of the isometries of $\mathbb{R}^m$, and $F = \mathbb{R}^m / \Gamma_F$, while the deck transformation group of the covering $\pi_{\overline{M}}$ consists of the maps of the form $\mathrm{id}_{\mathbb{R}} \times T : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$, with $T \in \Gamma_F$. Consider the immersion $\tilde{\psi} = \psi \circ \pi_M : \mathbb{R}^m \to \overline{M}$. Then, $\tilde{\psi}$ uniquely lifts to an immersion

$$\hat{\psi} : \mathbb{R}^m \to \mathbb{R}^{m+1} \quad \text{such that} \quad \pi_{\overline{M}} \circ \hat{\psi} = \tilde{\psi}.$$

It is easy to see that $\hat{\psi}$ is again a translating mean curvature flow soliton with respect to the lift $\hat{\partial}_t \in \mathfrak{X}(\mathbb{R}^{m+1})$ of ∂_t with soliton constant ch_0. Furthermore, by Step 4 the universal covering of M splits as $\mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m-1}$, where $\mathbb{R}^{m-1}$ covers Σ and the metric and second fundamental form on $\mathbb{R}^m$ have the expression

$$\langle \ , \ \rangle = \mathrm{d}s \otimes \mathrm{d}s + \langle \ , \ \rangle_{\mathbb{R}^{m-1}}, \qquad \hat{\mathrm{II}} = \mu_1(s)\, \mathrm{d}s \otimes \mathrm{d}s \otimes \hat{\nu},$$

with $\hat{\nu}$ the local normal vector field along $\hat{\psi}$ given by the lift of ν. In particular, Σ is totally geodesic in $\mathbb{R}^m$.

From the expression of metric and second fundamental form of M, and from Gauss equations, we deduce that M is flat. Since the nullity of $\hat{\mathrm{II}}$ has dimension $m-1$, a classical theorem of Hartman [4] guarantees that $\hat{\psi}$ is a flat cylinder over a plane curve. More precisely, for each $q \in \mathbb{R}^{m-1}$ we define

$$\gamma_q : \mathbb{R} \times \mathbb{R}^m, \qquad s \mapsto (s, q).$$

Then, $\hat{\psi}(\gamma_q)$ is contained in the 2-plane $\Pi_q = (\hat{\psi}_* T_q \mathbb{R}^{m-1})^\perp \subseteq T_{\hat{\psi}(0,q)} \mathbb{R}^{m+1}$, and the planes Π_q are all parallel. We denote with Π the plane associated to $q = 0$. As observed in the proof of [2, Theorem 2.3], γ_0 is itself a translating soliton with soliton constant ch_0 in Π, with respect to the orthogonal projection of the vector field $\hat{\partial}_t$ onto Π. We let V denote such orthogonal projection and define

$$\alpha \in [0, \pi/2) \qquad \text{such that } ||V|| = \cos\alpha. \tag{22}$$

In fact, V is a nonzero vector, since otherwise $\hat{\psi}(\gamma_0)$ would be a straight line and ψ would be totally geodesic. Then, $\hat{\psi}(\gamma_0)$ is a translating soliton curve with soliton constant

$$k = ch_0 \cos\alpha > 0 \tag{23}$$

with respect to a parallel unit vector field in the Euclidean plane and therefore, under a suitable choice of cartesian coordinates (x^1, x^2) on Π such that $V = \cos\alpha\,\partial_1$, it can be reparametrized as the grim reaper curve

$$\begin{array}{rcl} \sigma & : & \left(-\frac{\pi}{2k}, \frac{\pi}{2k}\right) \;\; \to \;\; \mathbb{R}^2 \\ & & \tau \;\; \mapsto \;\; \sigma(\tau) = (\sigma_1(\tau), \sigma_2(\tau)) = \left(-\frac{1}{k}\log(\cos(k\tau)), \tau\right). \end{array} \tag{24}$$

Indeed, a translating soliton curve with respect to ∂_1 in $\mathbb{R}^2$ with soliton constant k can always be locally written as a graph $x_1 = f(x_2)$ with f satisfying

$$k = \sqrt{1+(f')^2}\left(\frac{f'}{\sqrt{1+(f')^2}}\right)' = \frac{f''}{1+(f')^2} = (\arctan(f'))'.$$

Hereafter, the point $\sigma(0)$ of the grim reaper curve parametrized as above will be referred to as the vertex of the grim reaper. So, $\hat{\psi}$ is a (possibly tilted) grim reaper cylinder, and up to an isometry of the second factor of the ambient space $\mathbb{R}^{m+1} = \mathbb{R}\times\mathbb{R}^m$ and a translation in the first factor (which does not affect the validity of (4)) we can assume

$$\begin{aligned} &\hat{\psi}(s, x_2, x_3, \ldots, x_m) \\ &\quad = (x_2\sin\alpha + \sigma_1(\tau(s))\cos\alpha, \sigma_2(\tau(s)), x_2\cos\alpha - \sigma_1(\tau(s))\sin\alpha, x_3, \ldots, x_m), \end{aligned} \tag{25}$$

where $\tau(s)$ is the change of parameter from arclength s to τ in (24). To introduce Step 6, let $\Omega_0 = \{0\}\times\mathbb{R}^{m-1} \subset \mathbb{R}^m$ be the "valley" of the grim reaper, that is mapped by $\hat{\psi}$ to the vertices of the grim reaper curves $\hat{\psi}(\gamma_q)$, $q \in \mathbb{R}^{m-1}$. Also, let $\Omega = \pi_M(\Omega_0)$.

Step 6: the grim reaper $\hat{\psi}$ is not tilted (that is, the angle α defined in (22) vanishes).

Proof of Step 6. Suppose, by contradiction, that $\alpha \neq 0$. Let $\hat{W}$ be the orthogonal projection of $\hat{\partial}_t$ onto the subspace $\hat{\psi}_* T\Omega_0$ of $T\mathbb{R}^{m+1}$, so $\hat{W} \neq 0$, and let $W \in T\Omega_0$ be the (never vanishing) induced vector field on Ω_0. Explicitely, we have

$$\begin{array}{rcl} \hat{W} & \equiv & (\sin^2\alpha, 0, \sin\alpha\cos\alpha, 0, \ldots, 0) \in \mathbb{R}^{m+1}, \\ W & \equiv & (0, \sin\alpha, 0, \ldots, 0) \in \mathbb{R}^m. \end{array}$$

As W is parallel on Ω_0, it allows to split Ω_0 as the product $\Sigma_0^{m-2}\times\mathbb{R}$, where the tangent vector to the $\mathbb{R}$ direction is W. Let $\ell_W \subset \Omega_0$ be a line in the universal covering $\mathbb{R}\times\Sigma_0\times\mathbb{R}$ of M of the form $\ell_W(t) = (0, z, t)$ for $t \in \mathbb{R}$ and fixed $z \in \Sigma_0$. Next, observe that $\hat{W}$ is not tangent to the "horizontal" factor $\mathbb{R}^m$ of $\mathbb{R}^{m+1} = \mathbb{R}\times\mathbb{R}^m$, so for any deck transformation $\hat{T} = \mathrm{id}_{\mathbb{R}}\times T \in \mathrm{deck}(\pi_{\overline{M}})$, condition $\hat{T}(\hat{\psi}(\mathbb{R}^m)) = \hat{\psi}(\mathbb{R}^m)$ forces the map T to act as the identity in the direction given by the projection of $\hat{W}$ on the horizontal $\mathbb{R}^m$. In particular, this implies the following properties:

$$\begin{array}{ll} (i) & \text{there exists no nontrivial deck transformation of } \pi_{\overline{M}} \text{ fixing } \hat{\psi}(\ell_W); \\ (ii) & (\pi_{\overline{M}})_*\hat{W} = \hat{W}; \\ (iii) & \pi_{\overline{M}} \text{ is injective on } \hat{\psi}(\ell_W). \end{array} \tag{26}$$

From (iii), $\psi\circ\pi_M = \pi_{\overline{M}}\circ\hat{\psi}$ is injective on ℓ_W, since $\hat{\psi}$ is injective. As a consequence, π_M is injective on ℓ_W and ψ is injective on $\pi_M(\ell_W)$. Note also that $\psi\circ\pi_M(\ell_W) = \pi_{\overline{M}}\circ\hat{\psi}(\ell_W)$ is a proper curve in $\overline{M}$, and thus the curve $\pi_M(\ell_W)$ is proper in M (and contained into Ω). Indeed, for each compact set $K \subset M$,

$$(\pi_M\circ\ell_W)^{-1}(K) \subset (\psi\circ\pi_M\circ\ell_W)^{-1}(\psi(K))$$

is therefore closed in a compact set, hence it is compact. Summarizing, $\pi_M(\ell_W)$ is a proper and injective immersion, hence a proper embedding. Next, observe that if two lines ℓ_W, ℓ'_W are different (hence, they do not intersect), then either $\pi_M(\ell_W) \cap \pi_M(\ell'_W) = \emptyset$ or $\pi_M(\ell_W) = \pi_M(\ell'_W)$. Indeed, if $\pi_M(\ell_W)$ and $\pi_M(\ell'_W)$ intersect but do not coincide, since they are geodesics in M they have to intersect transversely. However, the two curves $\hat\psi(\ell_W)$ and $\hat\psi(\ell'_W)$ are parallel straight lines on $\mathbb{R}^{m+1} = \mathbb{R} \times \mathbb{R}^m$ in the direction of $\hat W$, hence by (ii) in (26) their projections cannot be transverse.

Set $\Sigma = \pi_M(\Sigma_0) \subset \Omega$, and take a small, contractible and relatively compact open subset Σ' of Σ. By a compactness argument that uses the properness of $\pi_M(\ell_W)$, up to reducing Σ' we can guarantee that each $\pi_M(\ell_W)$ meets Σ' at most once. Summarizing the above properties, since $\pi_M(\ell_W)$ is a geodesic, the exponential map $\exp^\perp : T\Sigma'^\perp \to \Omega$ realizes a diffeomorphism between $T\Sigma'^\perp$ and the union $E \subset \Omega_0$ of all curves $\pi_M(\ell_W)$ passing through Σ'. Note that $T\Sigma'^\perp$ is diffeomorphic to $\Sigma' \times \mathbb{R}$ since Σ' is contractible. From $\Omega_0 = \Sigma_0 \times \mathbb{R}$ and the constructions of ℓ_W and Σ', we deduce that the pulled-back metric via $\exp^\perp$ on $T\Sigma'^\perp$ is the product metric, whence E is isometric to $\Sigma' \times \mathbb{R}$. In conclusion, M contains a subset M_0 of positive m-dimensional measure that splits as $\mathbb{R} \times E = \mathbb{R} \times \mathbb{R} \times \Sigma'$. Fix $U \subseteq \Sigma_0$ an open subset such that $\pi_{M|U} : U \to \Sigma'$ is a diffeomorphism. Then

$$\pi_{M|\mathbb{R}^2 \times U} : \mathbb{R} \times \mathbb{R} \times U \to M_0$$

is an isometry, and from

$$(27) \qquad \begin{aligned} \bar\eta(\pi_I) &= h_0(\pi_I - t_0) \quad \text{for some } t_0 \in \mathbb{R}, \\ |\Phi|^2 &= |\mathrm{II}|^2 - 2H^2 = (\mu_1)^2 - 2\left(\frac{\mu_1}{2}\right)^2 = \frac{(\mu_1)^2}{2} = \frac{1}{2}\left(\frac{k}{||\dot\sigma||}\right)^2, \\ ||\dot\sigma(\tau)||^2 &= 1 + \tan^2(k\tau) = \frac{1}{\cos^2(k\tau)} \end{aligned}$$

we can estimate

$$\begin{aligned} \int_M |\Phi|^2 e^{c\eta} \ge \int_{M_0} |\Phi|^2 e^{c\eta} &= |\Sigma'| \int_{\mathbb{R}^2} \frac{\mu_1(s)^2}{2} e^{ch_0(x_2 \sin\alpha + \sigma_1(\tau(s))\cos\alpha - t_0)} \,\mathrm{d}s\,\mathrm{d}x_2 \\ &= |\Sigma'| \frac{k\pi e^{-ch_0 t_0}}{2} \int_{\mathbb{R}} e^{(ch_0 \sin\alpha)x_2} \,\mathrm{d}x_2 = +\infty \end{aligned}$$

where $|\Sigma'|$ is the $(m-2)$-dimensional volume of Σ' and we have used (23) and

$$(28) \qquad \begin{aligned} \int_{\mathbb{R}} \mu_1(s)^2 e^{k\sigma_1(\tau(s))} &= k^2 \int_{\mathbb{R}} \cos^2(k\tau(s)) e^{-\log(\cos(k\tau(s)))} \,\mathrm{d}s = k^2 \int_{\mathbb{R}} \cos(k\tau(s)) \,\mathrm{d}s \\ &= k^2 \int_{\mathbb{R}} \frac{\mathrm{d}s}{\|\dot\sigma(\tau(s))\|} = k^2 \int_{\mathbb{R}} \tau'(s) \,\mathrm{d}s = k^2 \int_{-\pi/(2k)}^{\pi/(2k)} \mathrm{d}\tau = k\pi\,. \end{aligned}$$

So, $|\Phi| \notin L^2(M, e^{c\eta})$, contradicting (4).

Step 7: if $u > 0$ solves (14), then there exists a Riemannian submersion $\pi_\Omega : M \to \Omega$ with 1-dimensional, noncompact, totally geodesic fibers that are sent, via $\hat\psi$, to the grim reaper curves (24).

Proof of Step 7. Having shown that $\alpha = 0$, $\hat\psi$ writes as

$$(29) \qquad \hat\psi(s, x_2, x_3, \ldots, x_m) = (\sigma_1(\tau(s)), \sigma_2(\tau(s)), x_2, x_3, \ldots, x_m)\,.$$

We first describe the structure of translations $\hat T = \mathrm{id}_{\mathbb{R}} \times T \in \mathrm{deck}(\pi_{\overline{M}})$. Choose Cartesian coordinates $(t, y_1, y_2, \ldots, y_m) = (t, y_1, y')$ on $\mathbb{R}^{m+1}$, so by (29) the image of $\hat\psi$ can be written as

$t = \sigma_1(\tau(s))$, $y_1 = \sigma_2(\tau(s))$. To preserve $\hat{\psi}(\mathbb{R}^m)$, the component T of $\hat{T}$ shall satisfy $T(y_1, y') = (\pm y_1, T'(y_1, y'))$, since the image of lines obtained by fixing y_1 have bounded y_1 coordinate and since $y_1(\hat{\psi}(\mathbb{R}^m))$ is invariant by $\hat{T}$. As T is an isometry, T' only depends on y'. In particular, either $T(\hat{\psi}(\gamma_q)) = \hat{\psi}(\gamma_{q'})$ or $T(\hat{\psi}(\gamma_q)) \cap \hat{\psi}(\gamma_{q'}) = \emptyset$. Therefore, either $\pi_{\overline{M}}(\hat{\psi}(\gamma_q))$ and $\pi_{\overline{M}}(\hat{\psi}(\gamma_{q'}))$ coincide or they have empty intersection, in particular, they cannot be transverse.

We next study the geodesics $\pi_M(\gamma_q)$. First, again since deck transformations of $\pi_{\overline{M}}$ act as the identity on the first component, $\pi_{\overline{M}} \circ \psi(\gamma_q)$ is a proper curve in $\overline{M}$, thus, as in Step 6, $\pi_M(\gamma_q)$ is a proper geodesic in M. We claim that $\pi_M(\gamma_q)$ is injectively immersed, hence embedded because of its properness. If $\pi_M(s_1, q) = \pi_M(s_2, q)$ for some $s_1 \neq s_2$, then $\hat{\psi}(s_1, q)$ and $\hat{\psi}(s_2, q)$ project onto the same point in $\overline{M}$. Comparing the first component of the two points in $\mathbb{R}^{m+1}$, we deduce $s_1 = \pm s_2$, hence $s_1 = -s_2$. Let $\hat{T} \in \mathrm{deck}(\pi_{\overline{M}})$ be the deck transformation that satisfies $\hat{T}(\hat{\psi}(s_1, q)) = \hat{\psi}(-s_1, q)$, and consider the geodesic $\sigma \subset \mathbb{R}^{m+1}$ joining $\hat{\psi}(s_1, q)$ and $\hat{\psi}(-s_1, q)$. Then, $\hat{T}(\sigma) = \sigma$, and since $\hat{T}(\hat{\psi}(\mathbb{R}^m)) = \hat{\psi}(\mathbb{R}^m)$ the middle point of the segment joining $\hat{\psi}(s_1, q)$ and $\hat{\psi}(-s_1, q)$ shall necessarily be a fixed point of $\hat{T}$. Hence $\hat{T} = \mathrm{id}$, contradiction.

Having proved that $\pi_M(\gamma_q)$ is injectively immersed, we claim that for $q \neq q'$ either $\pi_M(\gamma_q) \cap \pi_M(\gamma_{q'}) = \emptyset$ or they coincide. We proceed by contradiction: since both the curves are geodesics, we assume that $\pi_M(\gamma_q)$ and $\pi_M(\gamma_{q'})$ are transverse somewhere. Then, also their images $\psi \circ \pi_M(\gamma_q) = \pi_{\overline{M}} \circ \hat{\psi}(\gamma_q)$ and $\psi \circ \pi_M(\gamma_{q'}) = \pi_{\overline{M}} \circ \hat{\psi}(\gamma_{q'})$ are transverse somewhere, which contradicts the observations at the beginning of this Step.

With the above preparation, let $x \in M$ and $(s, q), (s', q') \in \pi_M^{-1}(x)$. From the fact that $\pi_M(\gamma_q)$ and $\pi_M(\gamma_{q'})$ either coincide or they do not intersect, we also deduce that $\pi_M(0, q) = \pi_M(0, q')$. Hence, the map

$$\pi_\Omega : M \to \Omega, \qquad \pi_\Omega(x) = \pi_M\big((0, q)\big) \quad \text{for any chosen } (s, q) \in \pi_M^{-1}(x)$$

is well defined. Fix a contractible, small open subset $\Omega' \subset \Omega$, and let $\Omega_0' \subset \Omega_0$ be one of its diffeomorphic lifts by π_M. Proceeding as in the end of Step 6, we can prove that the union $E \subset M$ of lines $\pi_M(\gamma_q)$ passing through points $q \in \Omega'$ is isometric to $\mathbb{R} \times \Omega_0'$, the isometry being $(s, q) \to \pi_M(\gamma_q(s))$. It follows that π_Ω is a fibration and a Riemannian submersion.

Step 8: Ω is compact.

Proof of Step 8. A straightforward computation that uses (27) and (28) then shows

$$\begin{aligned} \textstyle\int_M |\Phi|^2 e^{c\eta} &= |\Omega| e^{-ch_0 t_0} \int_{\mathbb{R}} \frac{\mu_1(s)^2}{2} e^{k\sigma_1(\tau(s))} \mathrm{d}s \\ &= |\Omega| \frac{\pi k e^{-ch_0 t_0}}{2}, \end{aligned}$$

where $|\Omega|$ is the $(m-1)$-dimensional volume of Ω. So in this case $|\Phi| \in L^2(M, e^{c\eta})$ holds true if and only if the manifold Ω has finite volume. Being Ω flat, Ω must be compact.

Step 9: Each of the solitons ψ in Item (ii) is stable.

Proof of Step 9. It is known that the embedding $\hat{\psi} : \mathbb{R}^m \to \mathbb{R}^{m+1}$ is stable: to show this, denoting with $\hat{\nu}$ a global choice of the normal vector, it is enough to observe that the function $\hat{v} = \langle \hat{\nu}, \hat{\partial}_t \rangle$ has a sign, say it is positive up to suitably choosing $\hat{\nu}$, and satisfies $0 = \Delta_{-c\eta} \hat{v} + |\mathrm{II}|^2 \hat{v} = L\hat{v}$ on M by [2, Proposition 1]. Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be a deck transformation of π_M. For fixed $x \in M$, we compare $\hat{v}(\tilde{x})$ to $\hat{v}(T(\tilde{x}))$, where $\tilde{x} \in \pi_M^{-1}(x)$. Since every deck transformation of $\pi_{\overline{M}}$ act as the identity in the first factor of $\mathbb{R} \times \mathbb{R}^m$, the product of $\hat{\nu}$ with $\hat{\partial}_t$ is constant on the fiber $\pi_{\overline{M}}^{-1}(\psi(x))$.

Therefore, since $\hat{\psi}(\tilde{x}), \hat{\psi}(T(\tilde{x})) \in \pi_{\overline{M}}^{-1}(\psi(x))$, we deduce that $\hat{v}(\tilde{x}) = \hat{v}(T(\tilde{x}))$, hence $\hat{v}$ induces a smooth, positive function $v : M \to \mathbb{R}$ which solves $Lv = 0$, proving the stability of ψ. □

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