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A Fede,
per tutto quanto

Contents

Introduction

Black holes represent one of the most interesting litmus tests for the study of a complete quantum theory of gravity, since they can provide a bridge between two of the major paradigms of the last century in theoretical physics: general relativity and quantum field theory.

Black holes (BHs) appear as classical solutions in general relativity (GR). The first black hole metric was found by Schwarzschild in 1916 as a solution to the Einstein equations and depends on one single parameter, the mass. Many other black hole solutions have been constructed over the years, with the possible addition of rotations, charges or cosmological constant.
An important breakthrough took place in the early seventies, when formal analogies were discovered between the laws describing the dynamics of black holes and thermodynamics. Among the others, it was proposed by Bekenstein [1] and Hawking [2] that the entropy of a black hole is proportional to the area A_H of its event horizon:

$$S_{BH} = \frac{A_H}{4G_N}\,,$$

in natural units, where G_N represents the Newton constant. It thus states that the entropy of a black hole scales as the area of its event horizon, not as its volume.

As is well known, from a statistical mechanical point of view, entropy can establish a connection between the microscopic behavior of the elementary constituents (particles) of a system and its macroscopic thermodynamic properties, since it counts the number of possible microstates associated with a given macrostate. More in detail, according to Boltzmann's definition, the entropy of a system is proportional to the logarithm of the number of the microscopic configurations of the particles of the system, known as microstates, that lead to the observed macroscopic properties: in we denote by n the number of microstates, in natural units the entropy reads

$$S = \log n\,.$$

General relativity cannot give an interpretation of this relation, as classically there is only one stationary solution associated with the macroscopic parameter of the configuration, i.e. the black hole mass. To overcome this incompatibility a quantum theory of gravity is needed, to unify classical GR and quantum mechanics. String theory represents a promising candidate to achieve this aim.

String theory was first proposed in the late sixties to describe the strong nuclear force, but it seemed to fall out of flavor few years later with the introduction of quantum chromodynamics (QCD), that offered a more accurate understanding of strong nuclear interactions. Nevertheless, due to its development it made great progress in the last decades and nowadays it represents one of the most valuable candidates for a comprehensive unifying theory of quantum gravity.
It is based on the idea that the elementary components of the universe are one-dimensional objects, called strings, that can vibrate with different frequencies. Each observed particle corresponds to a different oscillation mode of the vibrating strings. To incorporate fermionic matter, the original bosonic string has been extended to include supersymmetry, originating five different superstring theories. The inclusion of supersymmetry has also solved one of the greatest issues of bosonic strings, namely the presence of tachyons in their spectrum, which are particles with negative squared mass. Additionally, in order to be consistent, superstring theory must be formulated in ten spacetime dimensions.
String theory as a quantum theory can be naturally studied perturbatively and the expansion can be organized in powers of the string coupling g_s. Nevertheless, starting from the eighties non-perturbative aspects of the theory have been widely studied, leading, among the others, to the discovery of a network of dualities connecting the five superstring theories. Furthermore, it was shown that all of them can be obtained, in the weakly-coupled limit, from a single eleven-dimensional theory, known as M-theory [3].
Besides the strings, superstring theory contains extended non-perturbative objects called Dp-branes, on which fundamental open strings end. They extend in p spatial directions and we can think of p-dimensional quantum field theories as existing on their worldvolume, arising as the massless modes of open strings attached to them.

In the low-energy[1] and weakly-coupled limit, M-theory and superstring theories can be described in terms of classical eleven- and ten-dimensional theories known as supergravities. Supergravity theories in lower dimensions can be obtained from the higher-dimensional ones by means of a mechanism called dimensional reduction. One of the best-known example of this procedure is the Kaluza–Klein reduction, which allows to construct a theory in one dimension less by compactifying one of the original coordinates on a circle of radius $r \to 0$. If the reduced theory still captures the physics of the mother theory, the truncation is said to be consistent.

Within the context of string theory, black hole solutions can arise as bound states of non-perturbative objects wrapped on compact spaces, e.g. a stack of N D-branes. The classical black hole behavior appears when N is large. On the other hand, for finite N, supergravity in general can no longer represent the solution and the whole string theory must be employed to give a reliable quantum description of the brane system.

A precursor work in which the entropy of a black hole was obtained from a microstate counting is represented by the paper of Strominger and Vafa, [4], in which they managed to reproduce the Bekenstein–Hawking area law for a class of five-dimensional asymptotically flat extremal black holes in string theory by enumerating the degeneracy of the associated bound states. This constituted a first important step towards a deeper identification between the laws of black hole dynamics and of thermodynamic, and a meaningful internal consistency check of string theory.

[1] Roughly speaking, in this limit strings become point-like particles.

In a parallel direction, inspired also by black hole area law, a significant breakthrough in theoretical physics was represented by the formulation of the holographic principle. Proposed in the nineties by 't Hooft and Susskind [5,6], in its original version it suggested that to combine gravity and quantum mechanics, our three-dimensional world has to be an image (a hologram) of data encoded in a two-dimensional surface. Generalizing this principle, it should be possible to extract all the information enclosed in a volume of space (often denoted as "bulk") in quantum gravity from the boundary of the region.

The prime realization of holography is the celebrated AdS/CFT correspondence, conjectured by Maldacena [7], which establishes a duality between string theory on $\mathrm{AdS}_5 \times S^5$ and the maximally supersymmetric Yang–Mills (SYM) theory in four-dimensions. It is also known as gauge/gravity duality, since in its more general formulation connects d-dimensional conformal field theories and gravitational theories in $d+1$ dimensions with AdS vacua. This proposal represents one of the most remarkable results of the last three decades in theoretical physics, giving new impetus to string theory and opening the way for new research areas.
Many of the most interesting features of the AdS/CFT correspondence come from the fact that it is a weak/strong duality, and thus connects strongly-coupled field theories to weakly-coupled gravitational theories and vice versa. Thus, on the one hand it can provide a non-perturbative formulation of string theory, while on the other hand it allows to investigate strongly-coupled quantum field theories. The applications of the correspondence span a wide range of research fields, from black hole physics to condensed matter.

The first part of this introduction is partially based on [8] and [9]. All the topics that we mentioned until now are exposed in more detail in the first part of this thesis.

In the present work we are mostly interested in the use of the AdS/CFT correspondence in the context of black hole thermodynamics. From a holographic point of view, the microstates of a black hole correspond to a set of states in the dual field theory with the same charges of the black hole. Thus, the counting of these states allows to reproduce the entropy of the dual black hole. Many successful results have been obtained in this direction over the years. However, the generalization to the asymptotically AdS_5 black holes in [10] has been problematic and no results have been found until very recently. The reason is that the field theory quantity that is a natural candidate to encode the information about the dual BH entropy, the so-called superconformal index (SCI), seemed not to account for the expected scaling of the degrees of freedom of the gravitational system. The superconformal index corresponds to the supersymmetric partition function on $S^3 \times S^1$ (coinciding with the black hole conformal horizon) and has been constructed as a generalization of the Witten index. The mismatch between the scaling of the black hole entropy and the one obtained from the SCI was due to the large amount of cancellations between bosonic and fermionic states, a consequence of the presence of the operator $(-1)^F$ in the index, where F is the fermionic number. The problem was overcome in [11,12] using two different approaches with one common denominator: they consider complex fugacities, such that their imaginary parts can obstruct the cancellations induced by the operator $(-1)^F$. These two approaches are the Cardy-like limit, in which the fugacities associated with the rotations are considered to be very small, and the Bethe Ansatz approach, consisting of an evaluation of the index in terms of a set of solutions to the so-called Bethe Ansatz Equations (BAEs). More precisely, the field theory quantity that can be extracted from the superconformal index and allows to reproduce the black hole entropy is known as entropy function. The entropy function was first defined for the $\mathcal{N} = 4$ SYM case in [13], where it was observed that its Legendre transform, evaluated at its critical point, coincides with the expected BH entropy. These black holes are rotating, electrically charged, supersymmetric and asymptotically AdS_5, first found in [10, 14].

In the last years this research field has been developed in many directions.

Recent frontiers

As we mentioned, in the Bethe Ansatz method the index is written as a sum over solutions to the Bethe Ansatz Equations (BAEs). It may be natural to ask whether this sum can be identified with the sum over the dual gravitational solutions, even though a priori there in no reason for a matching between the individual terms in the sum. In was recently found [15] that indeed such matching is present, at least for the index of $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group, and it holds both for the leading order-N^2 contributions and for the non-perturbative order-$1/N$ corrections. The first ones are dual to Euclidean black holes, while the second solutions are related to wrapped D-branes.
The Cardy-like limit of the index is computed using a saddle point approach, rewriting the index as a sum over saddles. A mapping for these saddle, analogous to the one for the Bethe Ansatz approach, is in general still missing, but the investigation of the relation between the two approaches in the computation of the index is an active research field.
The leading term in the Cardy-like limit of the index is also the one with the largest power in N and it gives rise to the entropy of the dual black hole. When considering subleading contributions to the leading saddle, a logarithmic correction appears, due to the degeneracy of saddles. For $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group, this term is proportional to $\log N$ and matches the one obtained from the dual supergravity one-loop computation, which represents an important check of the quantum origin of the theory, as pointed out by Sen [16]. One may ask how in different field theories the logarithmic term is modified and how this affects its holographic interpretation.
These open questions motivate the second part of this thesis, in which the Cardy-like limit of the index is studied for different field theories, and both the saddles and the subleading contributions are analyzed.

The extremization procedure that allows to compute the BH entropy from the superconformal index has been generalized to various different theories. Given a superconformal field theory (SCFT), one could in principle compute the index and extract the entropy of the black of the dual gravitational theory. Nevertheless, finding black hole solutions in supergravity is not a simple task. Thus, it often happens that the entropy computed from the field theory side is associated with a putative black hole in the dual theory, yet to be found as a supergravity solution.
On the other hand, extremization problems in superconformal field theories are related by the AdS/CFT correspondence to extremization problems in supergravity that resemble the original attractor mechanism of [17]. For example, they both provide useful tools to compute the entropy of certain classes of black holes. However, while extremizing the large-N limit of the superconformal index it is possible to recover the entropy of a class of rotating five-dimensional

black holes, the dual attractor mechanism is still unknown. An alternative construction has been developed in the last few years in [13] to circumvent this lack, based on dimensional reduction. The idea is to dimensionally reduce the five-dimensional supergravity solution to four dimensions along the Hopf fiber of the S^3 horizon of the 5d black hole. The black hole in four dimensions has the same entropy of the original five-dimensional one, which can be then computed using the four-dimensional attractor mechanism of [18]. This procedure will be the core of the third part of the present work.

Further evolutions related to extremization problems can be found in the framework of compactifications of SCFTs.
One can define new superconformal field theories compactifying higher-dimensional theories on curved spaces. To preserve some supersymmetry in many cases it is necessary to turn on suitable background fluxes for the global symmetries. Such mechanism, commonly referred to as topological twist, has been vastly studied in many stringy and holographic setups. The prototypical example was discussed in [19] in terms of branes wrapped on Riemann surfaces. From the gravitational side, the mechanism is usually denoted as flow across dimensions. Then in [20] such flows were generalized and related to the c-extremization principle of [21], where c is the central charge of the conformal field theory. The c-extremization principle is in turn related to a gravitational attractor mechanism.
Recently it has been observed that one can extend the notion of topological twist on manifolds with orbifold singularities [22]. When the considered orbifold is a spindle, namely topologically a two-sphere with deficit angles at the poles, the supersymmetry of the higher-dimensional theory is preserved in a new way. The peculiarity of some of such constructions is that even though it might not be possible to find the full analytic solution to the flow across dimensions, the central charge of the theory compactified on the spindle can be obtained by solving the supersymmetry equations only at the poles of the spindle. This allows to match with the result obtained from a field theory computation performed via c-extremization, finding another application of the AdS/CFT correspondence. We will focus on such mechanisms in the last part of the thesis, studying two specific examples obtained compactifying two different $\mathcal{N} = 2$ five-dimensional supergravity models on the spindle. The first model is the AdS_5 consistent truncation of the conifold and contains two vector multiplets and two hypermultiplets. The second one is given by the family of AdS_5 consistent truncations associated with M5-branes wrapped on holomorphic curves in a Calabi–Yau threefold, containing two vector multiplets and one hypermultiplet.

Outline

This thesis is organized as follows.
We begin with a review part. In chapter 1 we present a short introduction to supersymmetry and string theory. We start from supersymmetry, reviewing its algebra and representations and moving to some examples of supersymmetric theories. We then introduce the main concepts of string theory, from the bosonic string to the five different superstrings, concluding with D-branes and a sketch of M-theory and supergravity theories. Chapter 2 is dedicated to the AdS/CFT correspondence. Conformal field theories and AdS spacetime are introduced. The AdS/CFT duality is presented, with a particular focus on the original conjecture of Maldacena and a summary of the main quantities matched by the holographic dictionary between the two

theories: the "boundary" superconformal field theories and the "bulk" gravitational theories on AdS.

In the second part we analyze the boundary side of the correspondence. Chapter 3 collects some notions on quantum field theories, useful in the remainder of the work. We discuss toric quiver gauge theories, that represent the main class of superconformal field theories of our interest, and we make some comments on 4d super QCD, Seiberg duality and 3d Chern–Simons theories. In chapter 4 we present the main character of this part, namely the superconformal index. We begin with a review of the Witten index and then we outline the derivation of the superconformal index, concluding with a discussion on the computation of black hole entropy from the index. Chapter 5 is more technical and is focused on the Cardy-like limit of the superconformal index. We first calculate the Cardy-like limit of the index of $\mathcal{N} = 4$ SYM with real groups, identifying the saddle point solutions and computing the logarithmic corrections, and of the $\mathcal{N} = 1^*$ LS fixed point, for which we also compute the entropy of the dual black hole using an extremization procedure. We then propose a formula for the Cardy-like limit of generic $\mathcal{N} = 1$ theories with ABCD gauge algebra, including finite-order corrections, and we provide a series of example to validate it.

In the third part we move to the bulk. In chapter 6 we recall the main aspects of $\mathcal{N} = 2$ matter-coupled supergravity in four and five dimensions. We presents the multiplets of the theories and their moduli spaces and we outline the isometries and the possible gaugings, concluding with the Lagrangians of the two theories. Chapter 7 is devoted to the computation of the entropy of the Kerr–Newman black hole dual to the LS fixed point, using the attractor mechanism after dimensional reducing the theory.

In the final part we study compactifications on curved spaces. In chapter 8 we report the original cases studied by Maldacena and Nuñez of branes wrapped on Riemann surfaces. We then introduce the main features of compactifications on spindles, of which we give two explicit examples in the last two chapters. In chapter 9 we discuss the compactification on the spindle of the $\mathcal{N} = 2$ AdS_5 consistent truncation of the conifold, in presence of a Betti vector multiplet. We compute the central charge and we compare our result with the one obtained from the analysis of the dual field theory, finding exact agreement. Finally, in chapter 10 we perform a similar analysis on a spindle compactification of families of AdS_5 consistent truncations corresponding to M5-branes wrapped on complex curves in Calabi–Yau three-folds. These models are holographically dual to $\mathcal{N} = 1$ SCFTs obtained by gluing of T_N blocks. Again we obtain the central charges both from the gravity and from the field theory side, finding perfect matching.

The appendices contain technical details on various topics addressed in the main body. In appendix A we compute the partition functions for Chern–Simons theories on three-spheres with real groups, used in the computation of the Cardy-like limit of the index. In appendix B we perform an explicit Kaluza–Klein reduction from a five-dimensional supergravity theory to a four dimensions, needed in chapter 7. In appendix C we present some details about the quaternionic geometry of the supergravity model that constitutes the starting point of the analysis in chapter 9, while in appendix D we derive the BPS equations for the same theory.

Part I

String theory and AdS/CFT correspondence

Chapter 1
A short introduction to supersymmetry and string theory

As is well known, string theory is one of the most promising and fascinating fields in modern theoretical physics. It was originally formulated in the late sixties as an attempt to describe strong nuclear interactions. Even though quantum chromodynamics (QCD), developed in the seventies, managed to present a more successful theory describing the strong nuclear force, the investigation on string theory was not abandoned and its evolution over the years led to a progressive resolution of different technical issues. On top of that, thanks to the identification of a massless spin-two particle in the string spectrum which could mediate the gravitational interaction, string theory has turned out to be a promising candidate for a quantum theory unifying gravity and the other fundamental forces of nature.

The constituent idea of string theory is that the fundamental building blocks of the universe are not point-like particles but one-dimensional extended objects called *strings*. These strings can vibrate at different frequencies, and specific oscillation modes (i.e. quantized excited states) correspond to specific particles.
In order to take fermions into account, string theory needs to include *supersymmetry*, which is a spacetime symmetry relating bosons and fermions. Supersymmetry also solves some inconsistency problems of bosonic string theory, and in the eighties the so-called "first superstring revolution" took place, leading to the formulation of five finite and totally self-consistent superstring theories in ten spacetime dimensions.
The fact that there are five different yet consistent superstring theories has been puzzling for a while, until an underlying web of dualities between the various theories was discovered. In the same years, great progress in understanding non-perturbative aspects of string theory was achieved, leading to the introduction of *D-branes*. Dp-branes are objects extended in p-spatial dimensions on which fundamental open strings end. The culminating point of the "second superstring revolution", that took place in the nineties, was the formulation of *M-theory*, a new type of eleven-dimensional quantum theory emerging as the UV-completion of the ten-dimensional superstring theories, providing thus a (supposed) unifying non-perturbative theory. The low-energy approximation of M-theory is a classical gravitational theory called eleven-dimensional *supergravity*. Analogously, ten-dimensional supergravities provide a low-energy description of superstring theories.

This chapter is organized as follows. We begin in section 1.1 with a review of the fundaments of supersymmetry, focusing on supersymmetry algebra, its representations and supersymmetric gauge theories. In section 1.2 we summarize the basics of string theory, starting from the bosonic string and proceeding to its supersymmetric extension and quantization. In subsection 1.2.3 an introduction on D-branes is presented, followed by some comments on M-theory in subsection 1.2.4. Finally, in section 1.3 we briefly introduce supergravity theories.

1.1 Supersymmetry

Motivated by the discussion above, in this section we proceed to introduce supersymmetry (SUSY). Supersymmetry is a spacetime symmetry that maps particles and fields of integer spin (bosons) into particles and fields of half-integer spin (fermions) and vice versa.

SUSY can be thought as a way to enlarge the group of spacetime symmetries in a non-trivial way to a bigger symmetry group, including both the Poincaré group and the group on the internal symmetries of the theory, but such that the two subgroups do not commute. Originally this idea seemed in contrast with the no-go theorem formulated in 1967 by Coleman and Mandula, in which they stated that in a quantum field theory, under certain reasonable and physical assumptions, the only possible continuous symmetries are spacetime and internal symmetries, that can only combine in a trivial way. In other words, the most general symmetry group is given by

$$ISO(3,1) \times G, \tag{1.1}$$

where $ISO(3,1)$ is the Poincaré group in four dimensions and G is a semi-simple Lie group times abelian factors, representing the group of the internal symmetries of the theory. The generators of these two groups commute.
However, this theorem can be circumvented by relaxing one or more of its assumptions. One of such assumptions is that all the generators of the algebra are bosonic, satisfying commutation relations. Hence, if one allows for fermionic generators, whose symmetry algebra involves anticommutators, it can be proved that the set of allowed symmetries can be enlarged. This makes the Poincaré group becoming superPoincaré, as we will show in the next paragraphs.

In the remainder of this section we will present the supersymmetry algebra and its representations and we will give a short introduction on supersymmetric theories. The discussion in based on the reviews [23, 24], while our choice of conventions is similar to [25].

1.1.1 Supersymmetry algebra

We now move to the construction of the supersymmetry algebra.
In the following we use the mostly plus signature, i.e. $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$, where $\mu, \nu = 0, \ldots, d-1$ are spacetime indices. Moreover, in order to be more explicit, we work in $d = 4$ spacetime dimensions.

Lorentz and Poincaré groups

The Lorentz group has six generators, three associated with spatial rotations J_i and three associated with boosts $K_i, i = 1, 2, 3$, obeying the following commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k \,. \tag{1.2}$$

It is useful to introduce the linear combinations of the above generators

$$J_i^{\pm} = \frac{1}{2}(J_i \pm iK_i)\,, \tag{1.3}$$

in terms of which the Lorentz algebra splits into the following two $SU(2)$ algebras

$$[J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk}J_k^{\pm}, \qquad [J_i^{\pm}, J_i^{\mp}] = 0\,. \tag{1.4}$$

This implies that the representation of the Lorentz group can be organized into a couple of $SU(2)$ representations; the importance of this isomorphism will be clear later.[1]

The Poincaré group is given by a semi-direct product of the Lorentz group and the group of spacetime translations, generated by P_μ, where μ is a spacetime index. In addition to the algebra (1.2), the Poincaré algebra includes:

$$\begin{aligned} &[P_\mu, P_\nu] = 0, \qquad [J_i, P_j] = i\epsilon_{ijk}P_k, \qquad [J_i, P_0] = 0, \\ &[K_i, P_j] = -iP_0, \qquad [K_i, P_0] = -iP_j\,, \end{aligned} \tag{1.5}$$

where we have separated the time component P_0 from the space components P_i of the translation generators.

One can also introduce a notation for the Lorentz generators in terms of an antisymmetric tensor $M_{\mu\nu} = -M_{\nu\mu}$ defined as

$$M_{0i} = K_i\,, \qquad M_{ij} = \epsilon_{ijk}J_k\,, \tag{1.6}$$

in terms of which the full Poincaré algebra becomes

$$\begin{aligned} &[P_\mu, P_\nu] = 0, \\ &[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma}\,, \\ &[M_{\mu\nu}, P_\rho] = -i\eta_{\mu\rho}P_\nu + i\eta_{\rho\nu}P_\mu. \end{aligned} \tag{1.7}$$

Spinors

We define two-component spinors as the objects transforming in the fundamental representations of $SL(2,\mathbb{C})$. A spinor is thus a two complex component object

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{1.8}$$

[1] More precisely, the universal cover of the Lorentz group is a complexified version of $SU(2) \times SU(2)$, denoted as $SL(2,\mathbb{C})$.

where ψ_1 and ψ_2 are complex Grassmann numbers, that transforms under a complex 2×2 matrix $\mathcal{M} \in SL(2, \mathbb{C})$ as

$$\psi_\alpha \to \psi'_\alpha = \mathcal{M}_\alpha^{\ \beta} \psi_\beta, \qquad \alpha, \beta = 1, 2. \tag{1.9}$$

Notice that a representation of $SL(2, \mathbb{C})$ and its complex conjugate are not equivalent, and therefore $\mathcal{M}$ and $\mathcal{M}^*$ give inequivalent representations. We denote as dotted spinor $\overline{\psi}$ a two-component object transforming as

$$\overline{\psi}_{\dot{\alpha}} \to \overline{\psi}'_{\dot{\alpha}} = \mathcal{M}^{*\ \dot{\beta}}_{\dot{\alpha}} \overline{\psi}_{\dot{\beta}}, \qquad \dot{\alpha}, \dot{\beta} = 1, 2, \tag{1.10}$$

that we can identify with $(\psi_\alpha)^*$.
The two spinors can be labeled in terms of $SU(2)$ representations as

$$\psi^\alpha \equiv \left(\frac{1}{2}, 0\right), \qquad \overline{\psi}_{\dot{\alpha}} \equiv \left(0, \frac{1}{2}\right). \tag{1.11}$$

Both representations are irreducible.

We now introduce some notations and conventions, starting from the invariant tensors of $SU(2)$,

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.12}$$

used to raise and lower spinorial indices as follows

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \psi^\beta \epsilon_{\beta\alpha}, \quad \overline{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \overline{\psi}_{\dot{\beta}}, \quad \overline{\psi}_{\dot{\alpha}} = \overline{\psi}^{\dot{\beta}} \epsilon_{\dot{\beta}\dot{\alpha}}. \tag{1.13}$$

Finally, recalling that Grassmann variables anticommute, e.g. $\psi_1\chi_2 = -\chi_2\psi_1$ as well as $\psi_1\overline{\chi}_{\dot{2}} = -\overline{\chi}_{\dot{2}}\psi_1$, the scalar products for spinors are defined as[2]

$$\begin{aligned} \psi\chi &\equiv \psi^\alpha \chi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\alpha\beta}\chi_\alpha\psi_\beta = \epsilon^{\beta\alpha}\chi_\alpha\psi_\beta = \chi^\beta\psi_\beta = \chi\psi, \\ \overline{\psi}\overline{\chi} &\equiv \overline{\psi}^{\dot{\alpha}}\overline{\chi}_{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\overline{\psi}_{\dot{\beta}}\overline{\chi}_{\dot{\alpha}} = \ldots = \overline{\chi}^{\dot{\beta}}\overline{\psi}_{\dot{\beta}} = \overline{\chi}\overline{\psi}, \end{aligned} \tag{1.14}$$

and under Hermitian conjugation holds

$$(\psi\chi)^* = (\psi^\alpha\chi_\alpha)^* = \psi^{\alpha *}\chi^*_\alpha = \overline{\psi}^{\dot{\alpha}}\overline{\chi}_{\dot{\alpha}} = \overline{\psi}\overline{\chi}. \tag{1.15}$$

We now introduce the four 2×2 matrices σ^μ, where σ^0 is the identity matrix and $\sigma^i, i = 1, 2, 3$, are the three Pauli matrices. These σ^μ matrices have a dotted and an undotted index and they can be written as

$$(\sigma^\mu)^{\alpha\dot{\beta}} = (\mathbb{I}, \sigma^i)^{\alpha\dot{\beta}}, \qquad (\overline{\sigma}^\mu)^{\dot{\beta}\alpha} = (-\mathbb{I}, \sigma^i)^{\dot{\beta}\alpha}. \tag{1.16}$$

We close this section by introducing Dirac spinors.
Let us start considering a set of 4×4 matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}\,, \tag{1.17}$$

[2] We use the NW-SE rule to contract spinorial indices.

known as Dirac matrices, which are the generators of the Clifford algebra. In the Weyl representation they read

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.18}$$

where we defined also the chiral matrix γ_5. A four-component Dirac spinor is composed by a two-component undotted and a two-component dotted spinor as

$$\psi = \begin{pmatrix} \psi^\alpha \\ \overline{\chi}_{\dot\alpha} \end{pmatrix}, \tag{1.19}$$

transforming in the reducible representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of the Lorentz group. We can observe that

$$\begin{pmatrix} \psi^\alpha \\ 0 \end{pmatrix} = \frac{1}{2}(1+\gamma_5)\psi, \qquad \begin{pmatrix} 0 \\ \overline{\chi}_{\dot\alpha} \end{pmatrix} = \frac{1}{2}(1-\gamma_5)\psi \tag{1.20}$$

and thus they are Weyl (chiral Dirac) spinors, with chirality ± 1, respectively.

A fundamental building block when working with spinors is the Majorana spinor, defined by the relation $\psi = \psi^{\mathcal{C}}$. Here $\psi^{\mathcal{C}} = \mathcal{C}i(\gamma^0)^T\psi^*$ is the charge conjugate spinor, where the charge conjugation matrix $\mathcal{C}$ satisfies

$$\mathcal{C}\gamma^\mu\mathcal{C}^{-1} = -(\gamma^\mu)^T \tag{1.21}$$

and in the Weyl representation reads

$$\mathcal{C} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}. \tag{1.22}$$

Furthermore, it is real, $\mathcal{C}^T = -\mathcal{C} = \mathcal{C}^{-1}$, $\mathcal{C}^2 = -\mathbb{I}$ and $\mathcal{C}\gamma_5\mathcal{C}^{-1} = \gamma_5^T$.
Introducing the notion of conjugate Dirac spinor $\overline{\psi}_D \equiv \psi^\dagger(i\gamma^0)$ and Majorana conjugate $\overline{\psi}_M \equiv \psi^T\mathcal{C}$, the Majorana condition can be expressed as $\overline{\psi}_D = \overline{\psi}_M$. From this, it can be shown that a Majorana spinor is a Dirac spinor with $\chi = \psi$, i.e. it has the form $\begin{pmatrix} \psi^\alpha \\ \overline{\psi}_{\dot\alpha} \end{pmatrix}$.
Finally, the Lorentz generators take the form

$$\Sigma^{\mu\nu} \equiv \frac{1}{2}\gamma^{\mu\nu}, \qquad \gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \tag{1.23}$$

that can be rewritten as

$$\gamma^{\mu\nu} = \frac{1}{2}\begin{pmatrix} \sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu & 0 \\ 0 & \overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu \end{pmatrix} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix}, \tag{1.24}$$

where we have introduced

$$\sigma^{\mu\nu} = \frac{1}{2}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu), \qquad \overline{\sigma}^{\mu\nu} = \frac{1}{2}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu), \tag{1.25}$$

or in a more explicit notation

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha^{\ \beta} &= \frac{1}{2}\big(\sigma^\mu_{\alpha\dot\gamma}(\bar\sigma^\nu)^{\dot\gamma\beta} - (\mu \leftrightarrow \nu)\big), \\ (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} &= \frac{1}{2}\big((\bar\sigma^\mu)^{\dot\alpha\gamma}\sigma^\nu_{\gamma\dot\beta} - (\mu \leftrightarrow \nu)\big). \end{aligned} \tag{1.26}$$

The supersymmetry algebra

As we mentioned before, our aim is to construct a group that includes:

- The Poincaré group, with generators $P_\mu, M_{\mu\nu}$.
- A semi-simple Lie group G with generators B_l that are scalars under the Lorentz group, representing the internal symmetries of the theory.

The structure of this group is $ISO(3,1) \times G$ and its full symmetry algebra is given by

$$\begin{aligned}
[P_\mu, P_\nu] &= 0\,, \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma}\,, \\
[M_{\mu\nu}, P_\rho] &= -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu\,, \\
[B_l, B_m] &= i f_{lm}{}^n B_n\,, \\
[P_\mu, B_l] &= 0\,, \\
[M_{\mu\nu}, B_l] &= 0
\end{aligned} \tag{1.27}$$

where $f_{lm}{}^n$ are the structure constants of the Lie algebra $\mathfrak{g}$ of G.
To evade the Coleman–Mandula theorem, it is possible to generalize the notion of Lie algebra by introducing a *graded Lie algebra*. A graded Lie algebra L of grade n is a vector space obtained as a direct sum of other vector spaces L_i,

$$L = \oplus_{i=0}^{n} L_i\,, \tag{1.28}$$

such that the product

$$[\ ,\ \} : L \times L \to L \tag{1.29}$$

enjoys the following properties:

$$\begin{aligned}
&[L_i, L_j\} \in L_{i+j} \bmod n+1\,, \\
&[L_i, L_j\} = -(-1)^{ij}[L_j, L_i\}\,, \\
&[L_i, [L_j, L_k\}\}(-1)^{ik} + [L_j, [L_k, L_i\}\}(-1)^{ij} + [L_k, [L_i, L_j\}\}(-1)^{jk} = 0\,,
\end{aligned} \tag{1.30}$$

where from the first property emerges that L_0 is a Lie algebra while all the other L_i are not.
The supersymmetry algebra is a graded algebra of grade one,

$$L = L_0 \oplus L_1 \tag{1.31}$$

where $L_0 \equiv \mathcal{P} \oplus \mathfrak{g}$, where $\mathcal{P}$ is the Poincaré algebra, while $L_1 = (Q^I_\alpha, \overline{Q}^I_{\dot\alpha})$, with $I = 1, \ldots, \mathcal{N}$, where Q^I_α and $\overline{Q}^I_{\dot\alpha}$ are a set of $2\mathcal{N}$ anticommuting fermionic generators that transform in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group, respectively.
Since the generators of L_1 transform non-trivially under the Lorentz group, supersymmetry is not an internal symmetry but rather it is an extension of Poincaré spacetime symmetries. Furthermore, supersymmetry generators transform bosons into fermions and vice versa, and thus, physically, supersymmetry mixes matter and radiation.

The supersymmetry algebra contains the commutators in (1.27), plus the following relations:

$$\begin{aligned}
[P_\mu, Q^I_\alpha] &= 0\,, \\
[P_\mu, \overline{Q}^I_{\dot\alpha}] &= 0\,, \\
[M_{\mu\nu}, Q^I_\alpha] &= i(\sigma_{\mu\nu})_\alpha^{\ \beta} Q^I_\beta\,, \\
[M_{\mu\nu}, \overline{Q}^{I\dot\alpha}] &= i(\overline{\sigma}_{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \overline{Q}^{I\dot\beta}\,, \\
\{Q^I_\alpha, \overline{Q}^J_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ}\,, \\
\{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta} Z^{IJ}\,, \qquad Z^{IJ} = -Z^{JI}\,, \\
\{\overline{Q}^I_{\dot\alpha}, \overline{Q}^J_{\dot\beta}\} &= \epsilon_{\dot\alpha\dot\beta} (Z^{IJ})^*\,.
\end{aligned} \tag{1.32}$$

As mentioned above, Q^I_α and $\overline{Q}^I_{\dot\alpha}$ transform as spinors under the Lorentz group and moreover they commute with the translations. The index $I = 1, \ldots, \mathcal{N}$ labels the spinorial generators. From the fifth line we can see that the commutator of two supersymmetry transformations is a translation. Furthermore, one can notice that Q^I_1 and $(Q^I_2)^\dagger$ raise the z-component of the spin (helicity) by half a unit, while Q^I_2 and $(Q^I_1)^\dagger$ lower it by half a unit.

In the last two relations, the $Z^{IJ} = -Z^{JI}$ are central charges, meaning that they commute with all the generators of the algebra and within themselves. Nevertheless, they are not numbers but quantum operators, with values that may vary from state to state.
In the simplest case the algebra has $\mathcal{N} = 1$, corresponding to only two supersymmetry generators and no possibility of central charges. It is called minimal (or unextended) supersymmetry algebra.
On the other hand, for $\mathcal{N} > 1$, we have extended supersymmetry. In the case $\mathcal{N} = 2$ there is one central charge, $Z \equiv Z^{12}$. There is no limit on $\mathcal{N}$ coming from the alegbra. However, as $\mathcal{N}$ increases, the theory contains particles of increasing spin. In order to have consistent quantum field theories we require

- $\mathcal{N} \leq 4$ for theories without gravity, corresponding to spin ≤ 1.
- $\mathcal{N} \leq 8$ for theories with gravity, corresponding to spin ≤ 2.

Thus, for four-dimensional theories, $\mathcal{N} = 8$ is an upper bound.

Let us conclude with a comment on the commutators between supersymmetry generators and internal symmetry generators. In general, the first ones carry a representation of the internal symmetry group G and thus one expects

$$\begin{aligned}
[Q^I_\alpha, B_l] &= (b_l)^I_{\ J} Q^J_\alpha\,, \\
[\overline{Q}_{I\dot\alpha}, B_l] &= -\overline{Q}_{J\dot\alpha} (b_l)^J_{\ I}\,,
\end{aligned} \tag{1.33}$$

where the second commutator is obtained from the first under hermitian conjugation and the b_l are hermitian. Thus, from these relations[3] we can see that the largest possible internal symmetry

[3] Recall that hermitian generators correspond to unitary groups.

group that can act non-trivially on the supersymmetry generators is $U(\mathcal{N})$ and it is called the R-symmetry group. It can be proven that in presence of non-vanishing central charges the R-symmetry group reduces to $USp(\mathcal{N})$.

1.1.2 Representations of the supersymmetry algebra

Let us now move to the representations of the supersymmetry group.
The Poincaré algebra is a subalgebra of the full supersymmetry algebra and thus any representation of the suspersymmetry algebra gives a representation of the Poincaré algebra, although in general a reducible one. We begin focusing on the second one.

Representations of the Poincaré algebra

The Poincaré algebra has two Casimir operators[4]

$$P^2 = P_\mu P^\mu \qquad \text{and} \qquad W^2 = W_\mu W^\mu, \tag{1.34}$$

where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$ is the Pauli–Lubanski vector. The irreducible representations of the Poincaré group are what one usually calls particles.

- If we consider a **massive particle** with mass m, we can move to the rest frame where $P^\mu = (m, 0, 0, 0)$ and the two Casimir operators reduce to $P^2 = -m^2$ and $W^2 = m^2 j(j+1)$, where j is the spin of the particle. Massive particles are thus distinguished by their mass and their spin.
- If we consider a **massless particle**, we can allign the momentum along the z direction, having $P^\mu = (E, 0, 0, E)$, with $P^2 = W^2 = 0$ and $W^\mu = M_{12} P^\mu$. Thus, for a massless particle, the two Casimir operators are proportional with proportionality constant $M_{12} = \pm j$, which is the helicity. For these representations the spin j is fixed and the states are distinguished by their energy E and by the sign of the helicity.

Representations of the supersymmetry algebra

An irreducible representation of the supersymmetry algebra is called superparticle and corresponds to a collection of particles, related to each other by the action of the supersymmetry generators Q^I_α and $\overline{Q}^J_{\dot\beta}$ and thus having spins that differs by units of one half. These states form a supermultiplet, as a superparticle is often called.
Furthermore, as one can see from spin-statistics theorem, the Q and $\overline{Q}$ change bosons into fermions and vice versa.

In the following we list a few useful generic properties shared by any supermultiplet:

1. All particles belonging to an irreducible supersymmetry representation have the same mass but different spin. This is due to the fact that P^2 is still a Casimir operator for the supersymmetry algebra, while W^2 is not.

[4] A Casimir operator is constructed from the generators of the algebra and commutes with all the generators. In each irreducible representation, the Casimir operators are proportional to the identity matrix, and the representation is labeled by the proportionality constant.

2. In a supersymmetric theory the energy of any state is always greater than zero. The proof relies on the positivity of the Hilbert space.

3. A supermultiplet always contains an equal number of bosonic and fermionic degrees of freedom (i.e. physical states).

Massless supermultiplets

In order to construct massless supermultiplets, we first assume that all the central charges Z^{IJ} vanish (we will see below that this is the only relevant case, for massless representations, following from the positivity of the Hilbert space). In this case, all Q^I_α and $\overline{Q}^J_{\dot\beta}$ anticommute among themselves. We choose the frame in which $P^\mu = (E,0,0,E)$ so that $\sigma^\mu P_\mu = \begin{pmatrix} 0 & 0 \\ 0 & -2E \end{pmatrix}$ and thus

$$\{Q^I_\alpha, \overline{Q}^J_{\dot\beta}\} = \begin{pmatrix} 0 & 0 \\ 0 & -4E \end{pmatrix}^{\gamma\dot\delta} \epsilon_{\gamma\alpha}\epsilon_{\dot\delta\dot\beta}\delta^{IJ} \;\Rightarrow\; \{Q^I_2, \overline{Q}^J_{\dot 2}\} = 0 \quad \forall I, J. \tag{1.35}$$

On a positive definite Hilbert space this requires $Q^I_2 = \overline{Q}^I_{\dot 2} = 0,\ \forall I$, because

$$0 = \langle\Phi|\{Q^I_2, \overline{Q}^I_{\dot 2}\}|\Phi\rangle = ||Q^I_2|\Phi\rangle||^2 + ||\overline{Q}^I_{\dot 2}|\Phi\rangle||^2. \tag{1.36}$$

We are thus left with only Q^I_1 and $\overline{Q}^J_{\dot 1}$, i.e. half of the initial fermionic generators. We can now define the following creation and annihilation operators

$$a^\dagger_I \equiv \frac{1}{\sqrt{4E}}\overline{Q}^I_{\dot 1}, \qquad a_I \equiv \frac{1}{\sqrt{4E}}Q^I_1 \tag{1.37}$$

such that

$$\{a_I, a^\dagger_J\} = \delta^{IJ}, \qquad \{a_I, a_I\} = \{a^\dagger_I, a^\dagger_J\} = 0. \tag{1.38}$$

To construct a representation, one then chooses a vacuum state, which is a state annihilated by all the a_I. This state carries an irreducible representation of the Poincaré group and thus it is characterized by $m = 0$ and by some helicity λ_0. We denote it by $|\lambda_0\rangle$ and it holds

$$a_I|\lambda_0\rangle = 0. \tag{1.39}$$

From the commutators of Q^I_1 and $\overline{Q}^J_{\dot 1}$ with the helicity operator $J_3 \equiv M_{12}$ (in our frame), one can see that Q^I_1 lowers the helicity by one half, while $\overline{Q}^J_{\dot 1}$ raises it by one half. The full supermultiplet is then constructed by acting on the vacuum state with the creation operators as follows:

$$\begin{aligned}
&|\lambda_0\rangle\,, \\
a^\dagger_I|\lambda_0\rangle &= \left|\lambda_0 + \tfrac{1}{2}\right\rangle_I\,, \\
a^\dagger_I a^\dagger_J|\lambda_0\rangle &= |\lambda_0 + 1\rangle_{IJ}\,, \\
&\vdots \\
a^\dagger_1 a^\dagger_2 \ldots a^\dagger_{\mathcal{N}}|\lambda_0\rangle &= \left|\lambda_0 + \tfrac{\mathcal{N}}{2}\right\rangle\,.
\end{aligned} \tag{1.40}$$

Due to the antisymmetry in the indices $I, J, \ldots$, there are $\binom{\mathcal{N}}{k}$ states with helicity $\lambda = \lambda_0 + \frac{k}{2}, k = 0, 1, \ldots, \mathcal{N}$. The state with highest helicity in the supermultiplet has helicity $\lambda = \lambda_0 + \frac{\mathcal{N}}{2}$. Summing the coefficients we can see that the are $2^{\mathcal{N}}$ states in a superparticle, half of them with integer helicity (bosons), half of them with half-integer helicity (fermions).

In general, the helicities are not distributed symmetrically around zero. Since CPT transformations flip the sign of helicity, a supermultiplet is not CPT invariant. Thus, in order to have a CPT-invariant theory, one has to double each multiplet by adding its CPT conjugate, that has opposite helicity and opposite quantum numbers. The only case in which this is not needed, i.e. when the supermultiplet is self conjugate, is when $\lambda_0 = -\frac{\mathcal{N}}{4}$.

Here we list the physical interesting massless $\mathcal{N} = 1$ supermultiplets.

- The **matter multiplet** (or **chiral multiplet**):

$$\lambda_0 = 0 \quad : \quad \left(0, \frac{1}{2}\right) \underset{\text{CPT}}{\oplus} \left(-\frac{1}{2}, 0\right) , \tag{1.41}$$

  corresponding to a Weyl fermion and a complex scalar.

- The **gauge multiplet** (or **vector multiplet**):

$$\lambda_0 = \frac{1}{2} \quad : \quad \left(\frac{1}{2}, 1\right) \underset{\text{CPT}}{\oplus} \left(-1, -\frac{1}{2}\right) , \tag{1.42}$$

  corresponding to a gauge boson (massless vector) and a Weyl fermion, both transforming in the adjoint representation of the gauge group.

- The **gravitino multiplet**:

$$\lambda_0 = 1 \quad : \quad \left(1, \frac{3}{2}\right) \underset{\text{CPT}}{\oplus} \left(-\frac{3}{2}, -1\right) , \tag{1.43}$$

  corresponding to a gravitino (spin-3/2 fermion) and a gauge boson.

- The **graviton multiplet**:

$$\lambda_0 = \frac{3}{2} \quad : \quad \left(\frac{3}{2}, 2\right) \underset{\text{CPT}}{\oplus} \left(-2, -\frac{3}{2}\right) , \tag{1.44}$$

  corresponding to the graviton (spin-2 boson) and the gravitino. The gravitino is the supersymmetric partner of the graviton.

If we want to construct interacting local field theories we have to stop here. Furthermore, the spin-3/2 particle is associated with local supersymmetry and hence with gravity. Thus, in a theory with $\mathcal{N} = 1$ supersymmetry, the physical gravitino must be the one belonging to the graviton multiplet, and the gravitino multiplet cannot appear in minimal supersymmetry.

If we now consider the massless $\mathcal{N} = 2$ supermultiplet, restricting again ourselves to the cases in which the helicity does not exceed two, we have:

- The **gauge multiplet** (or **vector multiplet**):

$$\lambda_0 = 0 \quad : \quad \left(0, \frac{1}{2}, \frac{1}{2}, 1\right) \underset{\text{CPT}}{\oplus} \left(-1, -\frac{1}{2}, -\frac{1}{2}, 0\right) , \tag{1.45}$$

  corresponding to a vector, two Weyl fermions and a complex scalar, all transforming in the adjoint representation of the gauge group. In $\mathcal{N} = 1$ language, this is a vector and a matter multiplet. It is worth noting that, even if in principle this representation is CPT self-conjugate, we need to double the degrees of freedom for technical reasons.

- The **hypermultiplet**:

$$\lambda_0 = -\frac{1}{2} \quad : \quad \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right) \underset{\text{CPT}}{\oplus} \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right) , \tag{1.46}$$

  corresponding to two Weyl fermions and two complex scalars. In terms of $\mathcal{N} = 1$ representation, it contains two chiral multiplets with opposite chirality.

- The **gravitino multiplet**:

$$\lambda_0 = -\frac{3}{2} \quad : \quad \left(-\frac{3}{2}, -1, -1, -\frac{1}{2}\right) \underset{\text{CPT}}{\oplus} \left(\frac{1}{2}, 1, 1, \frac{3}{2}\right) , \tag{1.47}$$

  corresponding to a gravitino, two vectors and a Weyl fermion.

- The **graviton multiplet**:

$$\lambda_0 = -2 \quad : \quad \left(-2, -\frac{3}{2}, -\frac{3}{2}, -1\right) \underset{\text{CPT}}{\oplus} \left(1, \frac{3}{2}, \frac{3}{2}, 2\right) , \tag{1.48}$$

  corresponding to a graviton, two gravitini and the graviphoton (vector).

Finally, in the $\mathcal{N} = 4$ case, if we want to avoid gravity we have only one possible multiplet, which is always CPT self-conjugate:

- The **gauge multiplet** (or **vector multiplet**):

$$\lambda_0 = -1 \quad : \quad \left(-1, 4 \times -\frac{1}{2}, 6 \times 0, 4 \times \frac{1}{2}, 1\right) , \tag{1.49}$$

  corresponding to a vector, four Weyl fermions and three complex scalars. In $\mathcal{N} = 1$ language, it contains one vector multiplet and three matter multiplets, all transforming in the adjoint representation of the gauge group.

Notice that in this case it is not possible to have matter in the usual sense, since there are no fermions transforming in the fundamental representation.

Finally, theories with $\mathcal{N} > 4$ supersymmetry are all supergravity theories. In $\mathcal{N} = 8$ supergravity there is only one allowed representation with helicity not greater than two. Hence, $\mathcal{N} = 8$ is an upper bound on the number of supersymmetry generators on four spacetime dimensions, if one wants an interacting local field theory.
Using a dimension-independent language, this notion can be refrased by stating that the maximum number of supercharges of non-gravitational theories is 16, and for theories with gravity is 32.

Massive supermultiplets

In the remainder of this work we will not deal with massive representations of the supersymmetry algebra, and thus here we treat them very briefly. We refer the interested reader to [23,24,26] for a more detailed review about the topic.

If we consider a state with mass m, in its rest frame we have $P^\mu = (m, 0, 0, 0)$, from which we can see that the anticommutator between Q^I_α and $\overline{Q}^J_\beta$ does not trivialize and all the supersymmetric generators survive, giving rise to a set of $2\mathcal{N}$ creation and $2\mathcal{N}$ annihilation operators. This implies that, in general, massive representations are longer than massless ones. Another relevant difference is that the vacuum state is now defined by mass m and spin j, where $j(j+1)$ are the eigenvalues of J^2, and the vacuum itself has degeneracy.
Performing an appropriate $U(\mathcal{N})$ rotation, the matrix Z^{IJ} of central charges can be written in the block-diagonal form

$$Z^{IJ} = \begin{pmatrix} 0 & Z_1 & 0 & 0 & \dots \\ -Z_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & Z_2 & \dots \\ 0 & 0 & -Z_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.50}$$

with $Z_r \geq 0,\ r = 1, \dots, \frac{\mathcal{N}}{2}$.[5] If we define the creation operators a^r_α, b^r_α as

$$\begin{aligned} a^1_\alpha &= \frac{1}{\sqrt{2}}\left(Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) , \\ b^1_\alpha &= \frac{1}{\sqrt{2}}\left(Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger\right) , \\ a^2_\alpha &= \frac{1}{\sqrt{2}}\left(Q^3_\alpha + \epsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right) , \\ b^2_\alpha &= \frac{1}{\sqrt{2}}\left(Q^3_\alpha - \epsilon_{\alpha\beta}(Q^4_\beta)^\dagger\right) , \\ &\vdots \\ a^{\mathcal{N}/2}_\alpha &= \frac{1}{\sqrt{2}}\left(Q^{\mathcal{N}-1} + \epsilon_{\alpha\beta}(Q^{\mathcal{N}}_\beta)^\dagger\right) , \\ b^{\mathcal{N}/2}_\alpha &= \frac{1}{\sqrt{2}}\left(Q^{\mathcal{N}-1} - \epsilon_{\alpha\beta}(Q^{\mathcal{N}}_\beta)^\dagger\right) , \end{aligned} \tag{1.51}$$

then they satisfy the following algebra of harmonic oscillators with their hermitian conjugates:

$$\begin{aligned} &\{a^r_\alpha, (a^s_\beta)^\dagger\} = (2m + Z_r)\delta_{rs}\delta_{\alpha\beta} , \\ &\{b^r_\alpha, (b^s_\beta)^\dagger\} = (2m - Z_r)\delta_{rs}\delta_{\alpha\beta} , \\ &\{a^r_\alpha, (b^s_\beta)^\dagger\} = \{a^r_\alpha, a^s_\beta\} = \ldots = 0 . \end{aligned} \tag{1.52}$$

[5] Here we assume for simplicity that $\mathcal{N}$ is even. Otherwise the matrix has an extra zero eigenvalue, which can be treated trivially.

Positivity of the Hilbert space imposes the constraint

$$2m \geq |Z_r|\,, \quad r = 1, \dots, \frac{\mathcal{N}}{2}\,. \tag{1.53}$$

It immediately follows that in the massless case there cannot be central charges.
On the other hand, depending on whether this bound is saturated or not for (some of) the central charge eigenvalues, theories with extended supersymmetry admit massive multiplets with different lengths:

- **long multiplets**: when $2m > |Z_r|\;\forall r$. In this case, acting on the vacuum state λ_0 annihilated by all operators a^r_α, b^r_α and acting on it with the creation operators $(a^r_\alpha)^\dagger, (b^r_\alpha)^\dagger$ one creates $2^{2\mathcal{N}}$ states,
$$2^{2\mathcal{N}} = \left(2^{\mathcal{N}-1}\right)_B + \left(2^{\mathcal{N}-1}\right)_F\,, \tag{1.54}$$
where B indicates the bosonic states and F the fermionic ones.

- **short multiplets**: when $2m = |Z_r|$ for $k < \frac{\mathcal{N}}{2}$ of the Z_r. In this case there are only $2\mathcal{N} - 2k$ oscillators, and the multiplets contain $2^{2(\mathcal{N}-k)}$ states,
$$2^{2(\mathcal{N}-k)} = \left(2^{2(\mathcal{N}-k)-1}\right)_B + \left(2^{2(\mathcal{N}-k)-1}\right)_F\,. \tag{1.55}$$

- **ultra-short multiplets**: when $2m = |Z_r|\;\forall r$. In this case we get the shortest multiplets, whose dimension is identical to that of massless ones, containing only $2^{\mathcal{N}}$ states,
$$2^{\mathcal{N}} = \left(2^{\mathcal{N}-1}\right)_B + \left(2^{\mathcal{N}-1}\right)_F\,. \tag{1.56}$$

Due to the connection to Bogomolny-Prasad-Sommerfield (BPS) magnetic monopoles, the short multiplets are also called *BPS multiplets*, and the inequality in (1.53) is the *BPS bound*, which, as we will see, is a very important bound for supergravity solutions. Since quantum (and even non-perturbative) corrections cannot change tha size of a multiplet, the relation (1.53) for BPS states is an exact result.

It can be shown that supersymmetric solutions are BPS solutions.
More in general, in a supersymmetry or supergravity theory, the term "BPS" indicates a solution that is invariant under a subalgebra of the supersymmetry algebra of the action, containing at least one fermionic generator. This solution can be obtained by solving first-order differential equations that come from the supersymmetry variations of the fermions of the theory.
On the other hand, for each preserved supersymmetry there is a Killing spinor and for this reason a supergravity solution that admits Killing spinors is usually called a BPS solution, and the first-order differential equations coming from the fermionic transformation rules of the theory are called BPS equations.

1.1.3 Supersymmetric field theories

Let us now give an overview of supersymmetric theories.
As we mentioned before, since bosonic states are mapped by supersymmetry generators into fermionic states and vice versa, a supersymmetric theory must contain an equal number of

bosonic and fermionic degrees of freedom.
In order to be supersymmetric, a theory must contain at least a spin-1/2 fermion

$$\chi_A = \begin{pmatrix} \chi^\alpha \\ \overline{\chi}_{\dot\alpha} \end{pmatrix} \tag{1.57}$$

χ_A is a Dirac spinor in the $(0, 1/2) \oplus (1/2, 0)$ representation, which is reducible. Imposing the Majorana condition

$$\chi_A = \mathcal{C}_{AB}\overline{\chi}^B, \tag{1.58}$$

where $\mathcal{C}_{AB}$ is the charge conjugation matrix, the representation of the spinor becomes irreducible. The spinor is left with two independent complex components, corresponding to four off-shell real degrees of freedom. If χ satisfies the equations of motion, the degrees of freedom reduce to two, and thus we have two possible ways to construct a supersymmetric field theory, by adding two bosonic degrees of freedom:

- The free Wess–Zumino model, containing χ_A + 2 real scalars.
  The fields of the theory and their equation of motions are given by

$$\begin{cases} \chi_A \text{ Majorana} & \gamma^\mu\partial_\mu\chi = 0, \\ A \text{ scalar} & \Box A = 0, \\ B \text{ pseudoscalar} & \Box B = 0 \end{cases} \tag{1.59}$$

  and the Lagrangian reads

$$\mathscr{L} = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\overline{\chi}\partial\!\!\!/\chi. \tag{1.60}$$

- The supersymmetric electrodynamics, containing χ_A + vector A_μ.
  The fields of the theory and their equation of motions are given by

$$\begin{cases} \chi_A \text{ Majorana} & \gamma^\mu\partial_\mu\chi = 0, \\ F^{\mu\nu} \text{ vector} & \partial_\mu F^{\mu\nu} = 0 \end{cases} \tag{1.61}$$

  and the Lagrangian reads

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\overline{\chi}\partial\!\!\!/\chi. \tag{1.62}$$

It may be now natural to ask whether it is possible to have an off-shell supersymmetric theory. If we consider the Wess–Zumino theory, in the case when χ does not satisfy the equations of motion we need two extra scalar fields, say F and G, to balance the counting of degrees of freedom. To not alter the dynamics of the theory, these fields should not propagate, i.e. they must have the following equations of motion:

$$F = 0, \qquad G = 0. \tag{1.63}$$

Thus, the Lagrangian of the theory becomes

$$\mathscr{L}_{\text{WS}} \to \mathscr{L}_{\text{WS}} + \frac{1}{2}F^2 + \frac{1}{2}G^2. \tag{1.64}$$

F and G are called auxiliary fields.
Similarly, to have a supersymmetric electrodynamics off-shell, we modify the Lagrangian

$$\mathscr{L}_{\text{ed}} \to \mathscr{L}_{\text{ed}} + \frac{1}{2}D^2, \tag{1.65}$$

where D is a pseudoscalar auxiliary field.

Superspace formalism

We have seen two examples of simple supersymmetric theories. However, in general, to check whether a given action in invariant under supersymmetry is rather challenging, due to the fact that supersymmetry is not manifest in the usual formulation. In fact, Minkowski spacetime is the natural background in which ordinary field theories are defined, where it is easy to construct actions that respect Poincaré symmetry. Analogously, supersymmetric field theories are naturally defined on an extension of Minkowski space, called superspace, that besides ordinary coordinates x^μ contains anticommuting spinorial coordinates ϑ_α and $\overline{\vartheta}_{\dot{\alpha}}$.
In analogy with ordinary functions defined on ordinary space, one can then define functions of the superspace coordinates $(x^\mu, \vartheta_\alpha, \overline{\vartheta}_{\dot{\alpha}})$, known as superfields. Using this formalism, the integral in superspace of any arbitrary superfield is a supersymmetric invariant quantity, i.e.

$$\int d^4x d^2\vartheta d^2\overline{\vartheta}\, Y(x, \vartheta, \overline{\vartheta}) \tag{1.66}$$

is supersymmetric invariant if Y is a superfield.

We will not deepen the superspace formalism, and its detail are beyond the scope of this review. We just mention that the fields contained in the supersymmetry multiplets are also contained in the superfields, as can be seen by expanding them in the spinorial variables.

1.2 Strings and branes

In this section we present a short introduction to string theory. We start by recalling the bases of classical bosonic string and later proceed to include supersymmetry in the theory. After summarizing the quantization procedure, we review the classification of superstring theories. We conclude the section with a description of D-branes and some notes on M-theory.
In writing this section we have mainly followed the books [9,27–29] and the lecture notes [30–33]. The structure is partially based on [34,35]. We also refer to [36] for a pedagogical introduction to the topic.

1.2.1 The classical string

In the following we introduce the main aspects of classical string theory, both for the bosonic string and for the superstring.

The bosonic string

Let us consider a D-dimensional Minkowski space $\mathbb{R}^{D-1,1}$ with signature

$$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, \dots, +1), \tag{1.67}$$

where $\mu, \nu = 0, \dots, D-1$ label the coordinates of the spacetime.
It is common to think of a string as an "extension" of a point particle, in order to study its motion. A string sweeps out a worldsheet, which is $(1+1)$-dimensional surface embedded in Minkowski spacetime (also referred to as the target space), analogous to the worldline swept out by a particle. The worldsheet is parametrized by a timelike coordinate τ and a spacelike coordinate σ, packaged together as $\sigma^\alpha = (\tau, \sigma)$, $\alpha = 0, 1$, and the embedding is realized by the coordinates $X^\mu(\sigma^\alpha)$.
The dynamics of the string can be described in terms of an action, first formulated by Nambu and Goto. The Nambu–Goto action is proportional to the area of the string wordlsheet and takes the form

$$S_{\text{NG}} = -T \int d^2\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}, \tag{1.68}$$

where

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}, \qquad X^{\mu\prime} = \frac{\partial X^\mu}{\partial \sigma} \tag{1.69}$$

and the scalar products on flat spacetime are defined as $A \cdot B = \eta_{\mu\nu} A^\mu B^\nu$. The proportionality constant T represents the tension of the string, i.e. the mass per unit length, and is related to the universal Regge slope α' as

$$T = \frac{1}{2\pi\alpha'}. \tag{1.70}$$

The Regge slope traditionally sets the fundamental dimensions of the theory, the string length ℓ_s and the string mass m_s, as:

$$\ell_s = \sqrt{\alpha'}, \qquad m_s = \frac{1}{\sqrt{\alpha'}}. \tag{1.71}$$

The classical string motion extremizes the worldsheet area, as the orbits of classical particles are geodesics that extremize (minimize) the length of the worldline.
The presence of the square root in (1.68) makes its quantization complicated. Hence it is more convenient to work with an equivalent formulation of the action, called Polyakov action, that gives rise to the same equations of motion and reads

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}\, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{1.72}$$

Here $h_{\alpha\beta} = h_{\alpha\beta}(\tau, \sigma)$ denotes the metric on the worldsheet and $h \equiv \det(h_{\alpha\beta})$.
The equations of motion for X^μ and $h_{\alpha\beta}$ are given by

$$\partial_\alpha(\sqrt{-h}\, h^{\alpha\beta} \partial_\beta X^\mu) = 0, \tag{1.73}$$

$$T_{\alpha\beta} \equiv -\frac{2}{T} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{\alpha\beta}} = 0. \tag{1.74}$$

The Polyakov action for the bosonic string in Minkowski spacetime enjoys the following symmetries:

- Poincaré invariance: a global symmetry of the ambient spacetime under which the fields transform as

$$\begin{aligned} \delta X^\mu &= a^\mu_{\ \nu} X^\mu + b^\mu, \\ \delta h^{\alpha\beta} &= 0, \end{aligned} \tag{1.75}$$

where a^μ_ν (with $a_{\mu\nu} = -a_{\nu\mu}$) and b^μ describe infinitesimal Lorentz transformations and spacetime translations, respectively.

- Reparameterization invariance: a gauge symmetry on the worldsheet under which the fields transform as

$$\begin{aligned} X^\mu(\sigma) &\to \tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma), \\ h_{\alpha\beta}(\sigma) &\to \tilde{h}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta} h_{\gamma\delta}(\sigma). \end{aligned} \tag{1.76}$$

These local symmetries are also called diffeomorphisms. At the infinitesimal level, making the coordinate change $\sigma^\alpha \to \tilde{\sigma}^\alpha = \sigma^\alpha - \xi^\alpha(\sigma)$ for small ξ, the transformations of the fields become

$$\begin{aligned} \delta X^\mu &= \xi^\alpha \partial_\alpha X^\mu, \\ \delta h^{\alpha\beta} &= \xi^\gamma \partial_\gamma h^{\alpha\beta} - \partial_\gamma \xi^\alpha h^{\gamma\beta} - \partial_\gamma \xi^\beta h^{\alpha\gamma}, \\ \delta(\sqrt{h}) &= \partial_\alpha(\xi^\alpha \sqrt{h}). \end{aligned} \tag{1.77}$$

- Weyl invariance: a gauge symmetry of the string under which the fields transform as

$$\begin{aligned} X^\mu(\sigma) &\to X^\mu(\sigma), \\ h_{\alpha\beta}(\sigma) &\to e^{2\phi(\sigma)} h_{\alpha\beta}(\sigma). \end{aligned} \tag{1.78}$$

As a consequence of this local symmetry the energy-momentum tensor is traceless.

The three local symmetries (two reparameterizations and one Weyl scaling) of the theory can be used to choose a gauge in which the equation of motion in (1.73) simplifies. We use the two reparameterizations to make the wordlsheet metric locally conformally flat:

$$h_{\alpha\beta} = e^{2\phi(\sigma)} \eta_{\alpha\beta}. \tag{1.79}$$

This choice is known as conformal gauge. Finally, we use the Weyl scaling to set $\phi = 0$, such that

$$h_{\alpha\beta} = \eta_{\alpha\beta}. \tag{1.80}$$

We are thus left with a flat worldsheet with Minkowski metric, in which the string action simplifies to

$$S = \frac{T}{2}\int d^2\sigma(\dot{X}^2 - X'^2), \tag{1.81}$$

and the equation of motion (1.73) becomes

$$\Box X^\mu \equiv \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu = 0 \tag{1.82}$$

that is simply the two-dimensional wave equation.

To have a well-defined variational problem, one needs to specify the boundary conditions. First of all, a string can be either closed or open. We choose the spacelike coordinate to have the range $\sigma \in [0, \pi]$.
The stationary points of the action are those obtained requiring the invariance under the general variation

$$X^\mu \to X^\mu + \delta X^\mu. \tag{1.83}$$

The variation of the action under (1.83) contains a volume term proportional to (1.82) plus a surface term

$$-T \int d\tau \left[X'_\mu \delta X^\mu\right]_{\sigma=0}^{\sigma=\pi}. \tag{1.84}$$

The possible boundary conditions under which this last term vanishes are the following:

- Closed strings with periodic boundary conditions

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \pi). \tag{1.85}$$

- Open strings with Neumann (N) boundary conditions

$$\partial_\sigma X^\mu\big|_{\sigma=0,\pi} = 0, \tag{1.86}$$

  i.e. the component of the momentum normal to the boundary of the worldsheet vanishes. If this choice is made for all spacetime coordinates, D-dimensional Poincaré invariance is preserved.

- Open strings with Dirichlet (D) boundary conditions

$$\delta X^\mu\big|_{\sigma=0,\pi} = 0 \quad \Rightarrow \quad X^\mu\big|_{\sigma=0,\pi} = c^\mu_{0,\pi} \tag{1.87}$$

  i.e. the positions of the two endpoints of the string are fixed.
  If we impose these conditions on the coordinates $\mu = 1, \ldots, D - p - 1$, we have to impose Neumann boundary conditions to the other $p + 1$ coordinates. This choice restricts the endpoints of the open strings to move on $(p + 1)$-dimensional hypersurfaces, breaking Poincaré invariance. For this reason Dirichlet boundary conditions were not considered for many years, until it was shown that in certain cirumstances they are unavoidable, and the hypersurfaces are now interpreted as the worldvolumes of dynamical p-dimensional objects, called Dp-branes, on which we will focus in section 1.2.3.

The solution to the equations of motion can be found by using worldsheet light-cone coordinates

$$\sigma^\pm \equiv \tau \pm \sigma, \tag{1.88}$$

in terms of which the general solution to the wave equation (1.82) splits as

$$X^\mu(\sigma, \tau) = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma), \tag{1.89}$$

i.e. a sum of right-movers X_R^μ and left-movers X_L^μ. Furthermore, to find an explicit solution one has also to require the reality of $X^\mu(\tau,\sigma)$ and to impose the constraint

$$(\partial_- X_R)^2 = (\partial_+ X_L)^2 = 0. \tag{1.90}$$

For closed strings, the more general solution of the wave equation that satisfies the periodic boundary conditions can be expanded in Fourier modes as

$$\begin{aligned} X_L^\mu(\sigma^+) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+}, \\ X_R^\mu(\sigma^-) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}, \end{aligned} \tag{1.91}$$

where x^μ and p^μ are the position and momentum of the center of mass of the string, while the exponential terms represent the string excitation modes.

For open strings, the boundary conditions in light-cone coordinates read

$$\partial_+ X_L^\mu = \partial_- X_R^\mu \quad \text{(N)} \qquad \text{or} \qquad \partial_+ X_L^\mu = -\partial_- X_R^\mu \quad \text{(D)}. \tag{1.92}$$

The general solution obtained by imposing these boundary conditions is given by a Fourier expansion similar to (1.91). The main difference is that there is one single set of oscillators α_n^μ.

The superstring

Despite having many interesting features, the bosonic string theory described in the above section has a few issues. The two major ones concern the presence of tachyons and the absence of fermions.
While open-string tachyons may have a physical interpretation in terms of D-branes, closed-string ones do not. Tachyons imply an instability of the vacuum and thus they are not acceptable in a physical theory.
On the other hand, fermions play a crucial role in the description of nature and hence they must be incorporated in string theory. The inclusion of fermions requires supersymmetry and the obtained string theories are called superstring theories. There are two main possible constructions to develop a string theory that includes supersymmetry, which are equivalent, at least for ten-dimensional Minkowski spacetime.

- The Ramond-Neveu-Schwarz (RNS) formalism, where supersymmetry is introduced at the level of the worldsheet.

- The Green-Schwarz (GS) formalism, where supersymmetry is introduced at the level of the target space.

Here we will only expand on the first formalism, in which one includes two-dimensional Majorana spinors $\psi^\mu(\tau,\sigma)$, which are fermionic partners of the bosonic fields $X^\mu(\tau,\sigma)$. More in detail, we incorporate a set of D Majorana fermions ψ^μ, that are two-component spinors on the worldsheet and transform in the vector representation of the Lorentz group $SO(D-1,1)$. Thus, the on-shell action in the conformal gauge for the superstring in flat background is given by

$$S = -\frac{T}{2}\int d^2\sigma \left(\partial_\alpha X_\mu \partial^\alpha X^\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right), \tag{1.93}$$

where $\rho^\alpha, \alpha = 0, 1,$ are two-dimensional Dirac matrices obeying the algebra

$$\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}. \tag{1.94}$$

The action in (1.93) is invariant under the global supersymmetry transformations

$$\begin{aligned} \delta X^\mu &= \varepsilon\bar{\psi}^\mu, \\ \delta\psi^\mu &= -i\rho^\alpha\partial_\alpha X^\mu \varepsilon, \end{aligned} \tag{1.95}$$

where ε is a constant infinitesimal Majorana spinor. The equations of motion for X^μ and ψ^μ read

$$\Box X^\mu = 0, \qquad \rho^\alpha\partial_\alpha\psi^\mu = 0. \tag{1.96}$$

The bosonic solution of the equations of motion in (1.96) is again given by (1.89), and similarly the general solution to the fermionic equation can be splitted in worldsheet light-cone coordinates as

$$\psi^\mu(\tau,\sigma) = \psi^\mu_+(\tau+\sigma) + \psi^\mu_-(\tau-\sigma). \tag{1.97}$$

The boundary conditions and mode expansions for the bosonic fields are the same as in bosonic string theory.
As concerns the fermionic fields, in the case of open strings we have to impose

$$\psi^\mu_+ = \pm\psi^\mu_- \tag{1.98}$$

at each end of the string, in order to make the surface terms in the variation of the action vanish. The overall relative sign in the above equation is a matter of convention and thus one can set

$$\psi^\mu_+(\tau,0) = \psi^\mu_-(\tau,0) \tag{1.99}$$

without loss of generality. On the other hand, depending on the relative sign at the other end of the string, we are left with two possible cases:

- Ramond (R) boundary conditions:

$$\psi^\mu_+(\tau,\pi) = \psi^\mu_-(\tau,\pi), \tag{1.100}$$

giving rise to spacetime fermions. The mode expansion of the Dirac equation in this sector gives

$$\begin{aligned} \psi^\mu_-(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d^\mu_n e^{-in(\tau-\sigma)}, \\ \psi^\mu_+(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d^\mu_n e^{-in(\tau+\sigma)}. \end{aligned} \tag{1.101}$$

- Neveu–Schwarz (NS) boundary conditions:

$$\psi_+^\mu(\tau,\pi) = -\psi_-^\mu(\tau,\pi), \tag{1.102}$$

giving rise to spacetime bosons. The mode expansion of the Dirac equation in this sector gives

$$\begin{aligned}\psi_-^\mu(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau-\sigma)},\\ \psi_+^\mu(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir(\tau+\sigma)}.\end{aligned} \tag{1.103}$$

For closed strings, the surface terms vanish when the boundary conditions are periodic or antiperiodic for each component of ψ separately, i.e.

$$\psi_\pm(\tau,\sigma) = \pm\psi_\pm(\tau,\sigma+\pi), \tag{1.104}$$

giving rise to left- and right-moving modes. It is possible to impose the periodicity (R) or antiperiodicity (NS) of the right-movers

$$\begin{aligned}&\psi_-^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} \qquad (1.105)\\ \text{or}\qquad &\psi_-^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-2ir(\tau-\sigma)},\end{aligned}$$

and of the left-movers

$$\begin{aligned}&\psi_+^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)} \qquad (1.106)\\ \text{or}\qquad &\psi_+^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)}\end{aligned}$$

separately. Depending to the different pairing of left-moving and right-moving modes, there are four distinct closed-string sectors: states in the R-R and NS-NS sectors are spacetime bosons, while states in the R-NS and NS-R sectors are spacetime fermions.

1.2.2 The quantum string

Let us move to the quantization of the theory. In this section we will only outline the main steps, starting from the bosonic string, and we will conclude reporting the classification of superstring theories.

There are three main procedures that have been developed to quantize the bosonic string. All the formulations are equivalent and lead to the same results and here we will only sketch the first one.

- (Old) covariant quantization, based on a description in terms of the embedding coordinates X^μ only, with restrictions on the physical states coming from the constraints of the Virasoro algebra.
- Light-cone gauge quantization.
- BRST quantization, involving the introduction of Faddeev-Popov ghosts and manifestly covariant.

The canonical quantization of string theory follows the usual procedure: the Fourier modes α_n^μ and $\tilde{\alpha}_n^\mu$ are promoted to creation and annihilation operators on a Hilbert space, analogous to raising and lowering operators of quantum-mechanical harmonic oscillators, that allow to define a vacuum state, $|0\rangle$, destroyed by the annihilation operator. The Poisson brackets between operators are promoted to commutators and by acting on the vacuum state with the creation operators one can originate all the states of the theory. In other words, the oscillation modes of the string correspond to particles in a quantum field theory.

From the analysis of the first mass levels of the open string we can see that at the ground state there is a tachyon, which is a state with imaginary mass. The presence of such a state points out a problem, which we will tackle in a while: when interactions are included, the theory may not have a stable vacuum. The first excitation level gives rise to a massless vector boson, in the vector representation of $SO(24)$, while the second one gives a massive spin-2 state, transforming in the symmetric traceless second-rank representation of $SO(25)$.
For what concerns the closed strings, as we commented above, their modes can be constructed as tensor products of left- and right-movers, each of them having the same structure of an open-string mode. At the first two mass levels we have the following physical states:

- The ground state is a tachyon.
- The states at the first level transform in the $\mathbf{24} \otimes \mathbf{24}$ representation of $SO(24)$, that decompose into the following irreducible representations:

  $$\text{traceless symmetric} \oplus \text{antisymmetric} \oplus \text{singlet.}$$

  The three representations correspond to a massless symmetric $(0, 2)$-tensor (the graviton $g_{\mu\nu}$), a massless two-form (the Kalb–Ramond field $B_{\mu\nu}$) and a massless real scalar (the dilaton Φ), respectively.

The physical importance of the Kalb–Ramond field lies in the fact that a string is a source for it, as a charged particle is a source for an electromagnetic vector potential A_μ. In other words, the string carries an electric charge with respect to $B_{\mu\nu}$.

Unfortunately, during the quantization procedure, some issues occur: in fact, besides tachyons, also ghosts,[6] which are states with negative Hilbert norm, appear in the spectrum, making the theory unstable. Both ghosts and tachyons are unphysical and we need to remove them from

[6]These ghosts should not be confused with the Faddeev-Popov ghosts.

the theory. It can be shown that it is possible to have a spectrum free of ghosts only for certain values of the spacetime dimension D. It turns out that for bosonic string theory when $D = 26$ all negative-norm states decouple and thus all physical states have positive norm. Nevertheless, as we mentioned before, tachyons remain in the spectrum. In order to get rid of them, we need to introduce the superstring.

The canonical quantization of the superstring follows procedures analogous to the ones of the bosonic string and one finds out that the critical dimension to obtain a ghost-free theory is $D = 10$. Nevertheless, there are still some open problems. In the NS sector the ground state is a tachyon and there is no fermion in the spectrum with its same mass. Thus the spectrum is not spacetime supersymmetric.
It has been found that RNS string theory can become a consistent theory by means of the so-called GSO projection, which truncates the spectrum such that the tachyon is eliminated, leading to a supersymmetric theory in ten-dimensional spacetime. During this procedure the worldsheet fermion number F is introduced, which allows to separate the R and NS sectors into R$_\pm$ and NS$_\pm$ subsectors with respect to the operator $e^{i\pi F}$. Tachyons are contained only in the NS$_-$ subsector, which is projected out from the theory. The ground state in the R spectrum is a massless spinor while the ground state in the NS sector is a massless vector, both of them belonging to the proper representation of $SO(8)$.

If we consider open superstrings, their boundary conditions only allow a single set of bosonic and fermionic modes, and restrict the possible supersymmetry to $\mathcal{N} = 1$. There are no tachyons and the massless sector is the ground state of the spectrum, corresponding to 16-dimensional multiplet given by $\mathbf{8_v} \oplus \mathbf{8_s}$.[7]
On the other hand, the states of closed superstrings can be constructed using two copies of the open-string ones, as for the bosonic string, combining a left-moving sector (NS or R$_\pm$) with a right-moving sector (NS or R$_\pm$).

In the following we give an outline of the classification of superstring theories.

- **Type II superstring theories**, which are maximally supersymmetric theories in ten dimensions, meaning that they preserve 32 real supercharges and thus have $\mathcal{N} = 2$ supersymmetry. They contain only closed string sectors and according on how the chiralities of the R sectors are combined we have two different inequivalent theories: Type IIA for opposite chiralities and Type IIB for equal ones. The different sectors are organized in irreducible representations of $SO(8)$ and thus one has

$$\begin{aligned} \text{Type IIA}: &\qquad (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}) \,, \\ \text{Type IIB}: &\qquad (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) \,. \end{aligned} \tag{1.107}$$

  Expanding this products, we can see how the matter content of the different sectors emerge:

  - NS-NS sector:

$$\mathbf{8_v} \otimes \mathbf{8_s} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} = \Phi \oplus B_{\mu\nu} \oplus g_{\mu\nu}, \tag{1.108}$$

[7] **v**: vectorial, **s**: spinorial, **c**: spinorial conjugate **t**: tensorial.

corresponding to the dilaton, the Kalb–Ramond and the graviton. It is the same field content of the bosonic string and it is common to both Type IIA and Type IIB.

– NS-R and R-NS sectors:

$$\begin{aligned} \mathbf{8_v} \otimes \mathbf{8_c} &= \mathbf{8_s} \oplus \mathbf{56_c}\,, \\ \mathbf{8_v} \otimes \mathbf{8_s} &= \mathbf{8_c} \oplus \mathbf{56_s}\,, \end{aligned} \tag{1.109}$$

containing the spin-1/2 dilatino λ and the spin-3/2 gravitino Ψ_μ. In Type IIA (NS-R$_\pm$ $\oplus$ R$_\mp$-NS) the two gravitini have opposite chiralities, while in Type IIB (NS-R$_\pm$ $\oplus$ R$_\pm$-NS) they have the same one.

– R-R sector:

$$\begin{aligned} \text{Type IIA}: &\qquad \mathbf{8_s} \otimes \mathbf{8_c} = \mathbf{8_v} \oplus \mathbf{56_t}\,, \\ \text{Type IIB}: &\qquad \mathbf{8_s} \otimes \mathbf{8_s} = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+\,, \end{aligned} \tag{1.110}$$

formed by bosons obtained by the tensor product of two spinors. Type IIA (R$_\pm$-R$_\mp$) contains a 1-form $C^{(1)}_\mu$ and a 3-form $C^{(3)}_{\mu\nu\rho}$, whilst Type IIB (R$_\pm$-R$_\pm$) contains a 0-form $C^{(0)}$, a 2-form $C^{(2)}_{\mu\nu}$ and a 4-form $C^{(4)}_{\mu\nu\rho\sigma}$ with a self-dual field strength.

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- **Type I superstring theory**, which is the only string theories containing open strings. For consistency the theory must include a closed-superstring sector with the same spacetime supersymmetry, which can be constructed by modding out Type II theories with respect to the $\mathbb{Z}_2$ parity symmetry of the worldsheet coordinates. The ground state of the spectrum corresponds to a vector multiplet of $D = 10\;\mathcal{N} = 1$ spacetime supersymmetry, containing the graviton, the dilaton and a R-R two-form, together with a gravitino and a dilatino.

- **Heterotic superstring theories**, obtained by combining left-movers of the closed bosonic string with right-movers of the closed superstring. The 16 extra spacetime dimensions, that arise from the reduction from $D = 26$ to $D = 10$ of the bosonic string, must be compactified and they give rise to internal gauge symmetries, with gauge group $SO(32)$ or $E_8 \times E_8$. These theories have $\mathcal{N} = 1$ spacetime supersymmetry and their massless sector contains the graviton, the dilaton, the Kalb–Ramond two-form, together with a gravitino and a dilatino. Additionally, there are vector fields with their related spin-1/2 fermionic partners (called gaugini) that gauge the internal symmetry group.

It can be shown that the physical constraints on the R-R sector are equivalent to the Maxwell equation of motion and to the Bianchi identity for an antisymmetric tensor field, i.e. we can write

$$F^{(n)}_{\mu_1\ldots\mu_n} = \partial_{[\mu_1} C^{(n-1)}_{\mu_2\ldots\mu_n]}\,, \tag{1.111}$$

where the n-forms $F^{(n)}_{\mu_1\ldots\mu_n}$ are called Ramond–Ramond fields, while $C^{(n)}_{\mu_1\ldots\mu_n}$ are called Ramond–Ramond potentials. The R-R fields are also related by the isomorphism

$$F^{(n)}_{\mu_1\ldots\mu_n} \sim \epsilon_{\mu_1\ldots\mu_n}{}^{\nu_1\ldots\nu_{10-n}} F^{(10-n)}_{\nu_1\ldots\nu_{10-n}}\,, \tag{1.112}$$

which corresponds to an electric-magnetic duality that exchanges equations of motion and Bianchi identities and relates the fields $C^{(n)}$ and $C^{(8-n)}$. Furthermore one can see that, in contrast with what happens in the NS-NS sector, only the R-R fields couple to strings, i.e. only the field strengths and not the potentials. Hence, string states cannot carry any charge with respect to the R-R potentials. It turns out that the objects coupling to them are the Dp-branes, which we will introduce in the next section.

1.2.3 Branes

As anticipated in the introduction to this chapter, D-branes are non-perturbative objects contained in superstring theory that can be defined as the loci where open strings end and that emerge from Dirichlet boundary conditions.

If we step back to consider an open string with Neumann boundary conditions for $(p+1)$ coordinates and Dirichlet for the others, at both the endpoints of the string we have

$$\begin{aligned} \partial_\sigma X^a\big|_{\sigma=0,\pi} &= 0 \qquad \text{for} \quad a = 0, \dots, p\,, \\ X^I\big|_{\sigma=0,\pi} &= c^I_{0,\pi} \qquad \text{for} \quad I = p+1, \dots, D-1\,. \end{aligned} \tag{1.113}$$

This choice forces the ends of the string to move on a $(p+1)$-dimensional hypersurface, breaking the Lorentz invariance of the background to

$$SO(D-1,1) \to SO(p,1) \times SO(D-p-1). \tag{1.114}$$

This hypersurface is called Dp-brane. Dp-branes extend along p dimensional spatial directions and thus they sweep a $(p+1)$-dimensional worldvolume. They can be thought as other dynamical objects contained in string theory.

To better understand the physics of D-branes, it can be useful to observe the ground state of open strings ending on them: one finds that the massless modes of open strings are associated with the fluctuations of the D-branes. In fact, if we look at the first excited states of open strings with Dirichlet boundary conditions we have:[8]

- *Oscillations longitudinal to the brane*, generated by $(p-1)$ different creation operators that transform under the vector representation of the $SO(p,1)$ Lorentz group of the brane. They can be interpreted as arising from a massless $(p+1)$-dimensional gauge field that lives on the brane.
- *Oscillations transverse to the brane*, generated by $(9-p)$ different creation operators that transform as scalars under the $SO(p,1)$ Lorentz group. They can be interpreted as arising from $(9-p)$ scalar fields that live on the brane and describe the fluctuations of the brane in the transverse directions. However, althought these fields are scalars under the Lorentz group of the brane, they transform as a vector under the $SO(9-p)$ group transverse to the brane, which appears as a global symmetry on the brane worldvolume.

[8] In the remainder of this section we stick to ten-dimensionsal spacetime.

As we commented above, Dp-branes couple to $C^{(p+1)}$ R-R potentials. This defines the Ramond–Ramond charge μ_p of the Dp-brane with respect to the "gauge field" $C^{(p+1)}$. More in detail, from the isomorphism in (1.112), we can see that Dp-branes couple electrically to $C^{(p+1)}$ and magnetically to $C^{(7-p)}$ and thus, starting from their R-R spectrum, it is possible to derive the D-brane content of Type II superstring theories:

- Type IIA: the branes exist for all even values of p,

$$\mathrm{D}0, \quad \mathrm{D}2, \quad \mathrm{D}4, \quad \mathrm{D}6, \quad \mathrm{D}8. \tag{1.115}$$

  The case $p = 0$ is a "D-particle", while the D8-brane couples to a R-R potential with a field strength $F^{(10)}$ that admits no propagating states. The D0-brane and the D6-brane, as well as the D2-brane and the D4-brane, are electromagnetic duals of each other.

- Type IIB: the branes exist for all odd values of p,

$$\mathrm{D}(-1), \quad \mathrm{D}1, \quad \mathrm{D}3, \quad \mathrm{D}5, \quad \mathrm{D}7, \quad \mathrm{D}9. \tag{1.116}$$

  The case $p = -1$ describes an object localized in time, called "D-instanton", while the D1-brane is a "D-string". The D9-branes are spacetime filling branes, with no coupling to any R-R field and leading to Neumann boundary conditions in every dimension. The D3-brane is self-dual, the D-instanton and the D7-brane, as well as the D-string and the D5-brane, are electromagnetic duals of each other.

The dynamics of a D-brane, as well as of the fields defined on it, are closely related to the modes of the open strings attached to it. When the energy of the brane is small compared to the energy of the open strings, the brane dynamics is completely determined by the open-string massless modes. If we consider a Dp-brane on a background generated by the massless NS-NS modes of the closed string sector $g_{\mu\nu},\ B_{\mu\nu}$ and Φ, its effective action is given by

$$\begin{aligned} S_{\mathrm{D}p} = &- T_{\mathrm{D}p} \int d^{p+1}\sigma e^{-\Phi} \sqrt{-\det\left(g_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)} \\ &+ \mu_p \int e^{B+2\pi\alpha' F} \wedge \sum_k C^{(k)}, \end{aligned} \tag{1.117}$$

where $T_{\mathrm{D}p}$ is the brane tension and $\sigma = (X^0, \ldots, X^{p+1})$ denote the coordinates on the worldvolume. The tensor fields $g_{\alpha\beta}$ and $B_{\alpha\beta}$ are the pullback of the background metric and of the Kalb–Ramond field on the Dp-brane worldvolume, respectively, e.g. $g_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$. $F_{\alpha\beta}$ is the field strength of the worldvolume abelian gauge field A_α living on the brane. The first term is the Dirac–Born–Infeld action, which describes the interaction between the brane and the background. The second term is the Wess–Zumino action, that represents the interaction of the brane with the R-R potentials.
Finally, the tension of a Dp-brane is given by

$$T_{\mathrm{D}p} = \frac{1}{g_s (2\pi)^p (\alpha')^{(p+1)/2}}, \tag{1.118}$$

from which we can see that when $g_s \to \infty$ or $\alpha' \to \infty$ the branes are light. On the countrary, in the weakly-coupled regime $g_s \to 0$, the branes become heavy: their dynamics decouples from the background and thus the interactions between the open and the closed strings vanish.

We conclude this section by pointing out a different characterization of the D-branes. To this end, recall that Type II superstring theory, in absence of D-branes, has $\mathcal{N} = 2$ spacetime supersymmetry. However, the open string boundary conditions are invariant under only one of these supersymmetries, i.e. the $\mathcal{N} = 2$ supersymmetry of the Type II vacuum breaks to $\mathcal{N} = 1$ supersymmetry when D-branes are included. A D-brane can be thus described as a state that preserves half of the original spacetime supersymmetry, also known as a BPS/2 state. The reason why they are called BPS ("Bogomolny-Prasad-Sommerfeld") is that they saturate the BPS bound, which, as we mentioned, characterizes massive supersymmetric configurations and establishes a relation between their mass and their central charges. In this case, the BPS bound relates the mass of the brane (and thus its tension) to its R-R charge. Thus, the fact that the D-branes are charged objects is consistent with the fact that they are BPS states.

1.2.4 M-theory

Despite the many improvements in the understanding of string theory, by the end of the eighties there were still some open problem. Among them, the existence the end of the eighties, an open problem related to string theory was the existence of five finite and totally self-consistent superstring theories: Type I, Type IIA/B and Heterotic with gauge group $E_8 \times E_8$ or $SO(32)$. A hint towards the resolution of this issue came from the discovery of the first dualities between different string theories, culminating in the mid-nineties with the work of Witten [3], in which he took further the formulation of the network of string dualities among the five different ten-dimensional theories. Furthermore, he showed that the strong coupling limit of Type IIA supergravity, which is the low-energy limit of Type IIA superstring theory, is eleven-dimensional supergravity, the only consistent supergravity theory in eleven dimensions. This led to the idea of *M-theory*, a non-perturbative eleven-dimensional theory whose low-energy effective field theory is eleven-dimensional supergravity and that reduces to the superstring theories by Kaluza–Klein compactifications.

To give a more concrete idea, let us suppose to perform a Kaluza–Klein reduction along the $(D+1)$-th coordinate of a $(D+1)$-dimensional theory, compactifying it on a circle of radius R. This leads to the appearance of an infinite tower of massive modes, besides the massless ones, with mass

$$m_n = \frac{n}{R}, \tag{1.119}$$

where $n \in \mathbb{Z}$ labels the n-th excited state. Thus, if we take the limit $R \to 0$ we are left only with the massless modes: the compactified dimension goes to zero and we obtain a D-dimensional theory.
We can think of a backward procedure in the context of string theory. If we consider Type IIA string theory, the spectrum includes D0-branes, whose mass is given by

$$M_{\text{D0}} = T_{\text{D0}} = \frac{1}{g_s \ell_s}. \tag{1.120}$$

The mass of a stack of n D0-branes is $n\, M_{\text{D0}}$ and, by comparison with (1.119), we can interpret it as the n-th excited state of a KK tower coming from the reduction of an eleven-dimensional

theory compactified on an S^1 of radius $R_{10} = g_s \ell_s$. This eleven-dimensional theory is indeed M-theory. Thus, Type IIA string theory appears as the perturbative regime of M-theory, i.e. in the limit $g_s \to 0$, in which $R_{10} \to 0$. On the countrary, in the strong-coupling regime $g_s \to \infty$ the circular eleventh dimension is decompactified and M-theory appears. It is worth noting that D0-branes are non-perturbative excitations, since their tension diverges as $g_s \to 0$. Therefore this analysis provides a test of duality between M-theory and Type IIA string theory that goes beyond the perturbative regime.
All the perturbative objects of Type IIA can be included in the M-theory formulation, as D0-branes. On the other hand, the BPS branes of M-theory are the M2-branes and the M5-branes, which describe the embedding of non-perturbative objects of Type IIA string theory.

As we have already pointed out, M-theory lacks a Lagrangian formulation: if we think of the duality with Type IIA string theory, we can write the Lagrangian only at weak string coupling, where we are in the perturbative regime and we know what the theory is.
Nevertheless, many constructions in M-theory still lead to interesting results, even in the context of the AdS/CFT correspondence. For what concerns us, in the last part of this work, we will deal with four-dimensional superconformal field theories that arise as low-energy effective descriptions of configurations of M5-branes wrapped on two-dimensional curves, either negative curved Riemann surfaces [19] or three-punctured spheres [37]. Both these SCFTs are dual to AdS_5 warped compactifications of eleven-dimensional supergravity. We will expand on these theories later.

1.3 Supergravity

In this last section we present a brief introduction to supergravity, focusing on its relation with string theory and on how lower-dimensional supergravities can be obtained from higher-dimensional ones through a dimensional reduction.
The content of this section is partially based on [8,28,38]. We refer the interested reader to [39] for a short review on this topic and to [40] for an introductory book.

As we mentioned in the introduction to this chapter, supergravity in ten and eleven dimensions emerges as the low-energy limit of weakly-coupled string theory and M-theory, respectively.
The parameters involved in this limit are the following:

- The string coupling g_s, a dimensionless constant proportional to the ratio ℓ_P/ℓ_s, where ℓ_P is the Planck length, that controls the hierarchy of scales in string theory. For example, $g_s \ll 1$ means $\ell_P \ll \ell_s$ and hence that the excitations of the string are much less massive than the Planck scale: quantum effects are negligible and we can treat string theory as a classical theory.
  From a different point of view, g_s behaves as a coupling constant in a quantum field theory. One can describe a string perturbation theory in terms of the mathematical genus of the string worldsheet and every loop in the string diagram introduces a factor of $(g_s)^2$. Thus, the weak string coupling limit $g_s \to 0$ suppresses the loops and the quantum effects.

- The Regge slope α', related to the dimensions of the string by $\alpha' = \ell_s^2 = \frac{1}{m_s^2}$. Thus, the low-energy limit is the $\alpha' \to 0$ limit, in which the string length can be ignored and a theory of particle is recovered.

The various effective supergravities theories come from a double perturbative expansion, both in $\alpha' \to 0$ and $g_s \to 0$, of the string theories listed in the previous section.
More in detail, as we can see from (1.118), in these limits the tension of a D-brane becomes large and thus the brane becomes rigid: the gauge theory on the brane decouples from the gravity theory of the background, i.e. open and closed string modes decouple. Moreover, at low energy we have $\alpha' \sim \frac{1}{m_s^2} \to 0$ and thus the massive string modes become too heavy to be observed, and only the massless states forming the ground states are relevant. On the one hand, the massless modes of the open string spectrum define a supersymmetric quantum field theory on the worldvolume of the brane. On the other hand, the background is described by classical massless fields coming from the ground states of closed strings, which define a supergravity theory.

1.3.1 Lower-dimensional supergravities

We conclude by giving a flavor of how higher-dimensional supergravities can be dimensional reduced. We address to [35] for an accurate review on the topic and we refer also to [33] for a thorough discussion

Lower-dimensional supergravity theories can be extracted from the ten- and eleven-dimensional ones using various mechanisms based on *compactifications* and *dimensional reductions*. The idea is to wrap some directions of the higher-dimensional background on a compact manifold. From this procedure a new parameter arises, associated with the size of the compact manifold. In the limit in which this parameter goes to zero, a lower-dimensional description of the theory appears. This may have interesting consequences on the propagation of the strings, at the level of their spectrum. Since their modes are now defined on a curved manifold with some compact directions, in certain cases it might happen that the degrees of freedom associated with the compact directions decouple, leaving a finite set of modes defined only on the lower-dimensional background. When it happens, the dimensional reduction defines a *consistent truncation*, meaning that the higher-dimensional physics is completely captured by a finite number of fields that define a lower-dimensional supergravity theory whose solutions can consistently uplift to higher dimensions.

We will not go into the details of the mechanism of dimensional reduction. We just mention that, from a geometrical point a view, one can start by rewriting the supergravity background manifold as

$$M_D = M_d \times X_{D-d} \tag{1.121}$$

where X_{D-d} is a compact $(D-d)$-dimensional interal space. The D-dimensional background is interpreted as an X_{D-d}-fibration over the lower-dimensional space M_d.
There are different ways to compactify a theory: the two main methods are the above-mentioned Kaluza–Klein reduction [41, 42][9] and the twisted reductions [43]. Furthermore, there are many geometries that can be chosen for the internal manifold X_{D-d}, as torii or Calabi–Yau manifolds. For what concerns the remainder of this paper, relevant examples of truncations are the ones

[9]We will give a concrete example of Kaluza–Klein reduction in chapter 7.

on spheres S^{D-d} or on Sasaki–Einstein manifolds SE_{D-d}, which play an important role in the AdS/CFT correspondence.

In the fourth part of this work we will deal with a few five-dimensional supergravity models obtained as consistent truncation from ten- or eleven-dimensional supergravity.

Chapter 2

The AdS/CFT correspondence

The AdS/CFT establishes a duality between gravitational theories in $d+1$ dimensions with AdS vacua and non-gravitational d-dimensional conformal field theories. It represents one of the most fertile research grounds of the last two decades, having implications that extend to different fields, from black hole physics to condensed matter.

The first hint toward a formulation of the AdS/CFT correspondence can be found in the black hole area law formulated by Bekenstein and Hawking in the seventies [1,2], which states that the entropy of a black hole S_{BH} scales as the area A_H of its event horizon, not as its volume. More in detail:

$$S_{BH} = \frac{A_H}{4\ell_P^2} = \frac{A_H c^3}{4G_N \hbar}, \tag{2.1}$$

where $\ell_P = G_N \hbar / c^3$ stands for the Planck length and G_N is the Newton's gravity constant. The holographic principle was formulated in the nineties by 't Hooft and Susskind [5,6] as a generalization of the area law and it implies that in quantum gravity all the information encoded in a volume of space can be described in terms of the degrees of freedom of the boundary only. In '96 Strominger and Vafa [4] reproduced the Bekenstein–Hawking entropy with D-branes methods, in the limit of large charges, showing that a collection of D-branes can explain the microscopic origin of the black hole entropy. Developing this idea, Maldacena formulated the founding example of the AdS/CFT correspondence [7], which conjectures a duality between the maximally supersymmetric Yang–Mills theory in four dimensions and Type IIB string theory on $\mathrm{AdS}_5 \times S^5$, as we will present in more detail in the last section of this chapter.

The correspondence has been soon showed to hold between supergravity theories on AdS_d and suitable superconformal field theories living on their $(d-1)$-dimensional boundary. It has been greatly developed over the years and generalized to less and even non-supersymmetric cases. In this work we will mostly deal with dualities between four-dimensional field theories and gravitational theories in AdS_5, as the original one, but with less supersymmetry. Specifically, in the majority of cases we will consider conformal field theories with $\mathcal{N} = 1$ supersymmetry dual to supergravity theories on $\mathrm{AdS}_5 \times X_5$, where X_5 is a Sasaki–Einstein manifold, as will be more clear in a few chapters.

Many relevant and useful applications of the correspondence rely on the fact that it is a strong/weak duality. Therefore, on the one hand it allows to use a classical gravitational theory to investigate quantum effects in a strongly-coupled theory. On the other hand, it is an ideal setting to study quantum properties of black holes from a fully quantum microstate counting in the dual conformal field theory.

In this chapter we give a short review of the AdS/CFT correspondence, along the lines of the review [44]. In section 2.1 and 2.2 we summarize the most salient aspects of (super) conformal field theories and AdS spacetime, respectively. In section 2.3 we present the AdS/CFT duality: we start by focusing on the original Maldacena's conjecture, giving a flavor of its derivation from a brane construction, and we conclude with some remarks on the holographic dictionary. Besides the review already cited, this chapter in mainly based on [45,46]. We will report other helpful references throughout the chapter.

2.1 Conformal field theories

To approach conformal field theories (CFTs), one usually starts by employing the symmetries of the theory. Thus, in this section we present dimensional conformal field theories starting from the conformal group. We then introduce its supersymmetric extension and finally we highlight some relevant aspects of quantum conformal field theories.
We mainly follow [44,47].

2.1.1 Conformal group

We begin by introducing the conformal group in D dimensions.
Given a D-dimensional spacetime, transformations thereof that locally preserve the angle between any two lines are called conformal transformations. In more mathematical terms, a conformal transformation is a change of coordinates that rescales the line element as

$$x^\mu \to x'^\mu \qquad ds^2 \to ds'^2 = \Omega(x)^2 ds^2, \tag{2.2}$$

where $\Omega(x)$ is an arbitrary function of the coordinates. At the infinitesimal level we can expand it as

$$x'^\mu = x^\mu + v^\mu(x), \qquad \Omega(x) = 1 + \frac{\omega(x)}{2}, \tag{2.3}$$

from which we can read the condition

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \omega(x)\eta_{\mu\nu}. \tag{2.4}$$

The function $\omega(x)$ can be fixed by tracing the equation above with $\eta^{\mu\nu}$, that gives $D\,\omega = 2\partial_\mu v^\mu$. Substituting this expression in the previous formula, we find the condition on the transformation (2.3) to be conformal:

$$\partial_\mu v_\nu + \partial_\nu v_\mu - \frac{2}{D}(\partial_\rho v^\rho)\eta_{\mu\nu} = 0. \tag{2.5}$$

In two dimensions this equation admits infinite solutions, and thus the conformal group is infinite-dimensional. For $D \neq 2$, the general solution is given by

$$
\begin{array}{lccc}
 & \textbf{form} & \textbf{generators} & \textbf{type} \\
\delta x^\mu = & a^\mu & P_\mu & \text{translations} \\
 & \omega^\mu{}_\nu x^\nu & M_{\mu\nu} & \text{Lorentz transf.} \\
 & \lambda x^\mu & \mathrm{D} & \text{dilatation} \\
 & b^\mu x^2 - 2x^\mu (b \cdot x) & K_\mu & \text{special conf. transf.}
\end{array}
\tag{2.6}
$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. The finite form of the special conformal transformation is

$$
x^\mu \to \frac{x^\mu + c^\mu x^2}{1 + 2c \cdot x + (c \cdot x)^2} . \tag{2.7}
$$

Altogether there are $(D+1)(D+2)/2$ generators. It can be shown that the group is isomorphic to $SO(D,2)$. Furthermore, there is an extra discrete symmetry,

$$
x^\mu \to \frac{x^\mu}{x^2} \qquad ds^2 \to \frac{ds^2}{x^4} \tag{2.8}
$$

that acts as a conformal transformation. Adding this last transformation we recover the full conformal group $O(D,2)$.

Conformal algebra

One can check that the generators P, M, D, K close the following algebra:

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho} M_{\nu\sigma} - i\eta_{\nu\sigma} M_{\mu\rho} + i\eta_{\mu\sigma} M_{\nu\rho} + i\eta_{\nu\rho} M_{\mu\sigma} \, , \\
[M_{\mu\nu}, P_\rho] &= -i\eta_{\mu\rho} P_\nu + i\eta_{\rho\nu} P_\mu \, , \\
[M_{\mu\nu}, K_\rho] &= i\eta_{\mu\rho} K_\nu - i\eta_{\nu\rho} K_\mu \, , \\
[M_{\mu\nu}, \mathrm{D}] &= 0 \, , \\
[\mathrm{D}, P_\mu] &= i P_\mu \\
[\mathrm{D}, K_\mu] &= -i K_\mu \, , \\
[K_\mu, P_\nu] &= -2i M_{\mu\nu} - 2i\eta_{\mu\nu} \mathrm{D} \, .
\end{aligned}
\tag{2.9}
$$

All the generators can be assembled as

$$
J_{MN} = \begin{pmatrix} M_{\mu\nu} & \frac{K_\mu - P_\mu}{2} & -\frac{K_\mu + P_\mu}{2} \\ -\frac{K_\mu - P_\mu}{2} & 0 & \mathrm{D} \\ \frac{K_\mu + P_\mu}{2} & -\mathrm{D} & 0 \end{pmatrix} , \qquad M, N = 1, \ldots, D+2 \, . \tag{2.10}
$$

Notice that J_{MN} is antisymmetric and represents a rotation in a $(D+2)$-dimensional space, with signature $\eta_{MN} = \mathrm{diag}(-1, 1, \ldots, 1, -1)$, thus representing the algebra of the above-mentioned $SO(D,2)$ group. We can see that D is a scalar, while P_μ and K_μ are vectors, acting as raising and lowering operators on D, respectively.

2.1.2 The energy-momentum tensor

According to Noether's theorem, for every continuous symmetry in a field theory there is a conserved current J_μ, i.e. $\partial^\mu J_\mu = 0$. We are studying theories with a conformal symmetry (2.3) and thus we have a conserved current that can be written as

$$J_\mu = T_{\mu\nu}\delta x^\nu, \tag{2.11}$$

where $T_{\mu\nu}$ is the energy-momentum tensor.
Conservations of the currents corresponding to translations and Lorentz transformations are realized if the energy-momentum tensor is conserved ($\partial^\mu T_{\mu\nu} = 0$) and symmetric, respectively, while the current associated with dilatations is conserved if the tensor is traceless.

Let us make a short digression. Classical massless fields are conformally invariant under Weyl rescalings of the metric tensor $g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)$. This invariance breaks when the conformal field theory is coupled to an external metric or when the theory is defined on a curved space, giving rise to the so-called Weyl anomaly, which in four dimensions gives the following vacuum expectation value to the trace of the energy-momentum tensor:

$$\langle T_\mu{}^\mu \rangle = \frac{c}{16\pi^2} W^2_{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} \tilde{R}^2_{\mu\nu\rho\sigma}\,, \tag{2.12}$$

where the Weyl tensor and Euler densities are

$$\begin{aligned} W^2_{\mu\nu\rho\sigma} &= R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2\,, \\ \tilde{R}^2_{\mu\nu\rho\sigma} &= \left(\frac{1}{2}\epsilon_{\mu\nu}{}^{\tau\lambda} R_{\tau\lambda\rho\sigma}\right)^2 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2\,. \end{aligned} \tag{2.13}$$

The coefficients a and c in (2.12) are called *conformal anomalies* or *central charges* and characterize a four-dimensional CFT. Remarkably, they can be also expressed in terms of the superconformal R-symmetry (that we will introduce in the next subsection) as

$$a = \frac{3}{32}(3\,\mathrm{Tr}\,R^2 - \mathrm{Tr}\,R), \qquad c = \frac{1}{32}(9\,\mathrm{Tr}\,R^3 - 5\,\mathrm{Tr}\,R) \tag{2.14}$$

2.1.3 Superconformal group

Supersymmetry enhances the conformal group to a supergroup, which is obtained from $O(D,2)$ by adding the supercharges $Q^I, I = 1, \ldots, \mathcal{N}$, the conformal supercharges S^I and the generators of the R-symmetry that rotates them. The conformal supercharges are needed in order to close the superconformal algebra.
In four dimensions the superconformal group is $SU(2,2|\mathcal{N})$, describing a theory with $\mathcal{N}$ supersymmetries obtained by including $\mathcal{N}$ supercharges Q^I_α, $\mathcal{N}$ superconformal charges S^α_I and the generators of a $U(\mathcal{N})$ global symmetry R^I_J rotating them.

2.1.4 Conformal quantum field theories

As we have already mentioned, particles can be identified by the Casimir operators of the Poincaré group. However, when a theory is conformal invariant, the mass operator $P_\mu P^\mu$ does not commute anymore with other generators, such for example D, and thus it is no longer a Casimir operator. Energy and mass may be rescaled by a conformal transformation: dilatations can make the energy of a given state vary from zero to infinity and thus we need to introduce a new manner of labelling states. A good candidate is given by the dilatation operator itself. If we consider its action on a field $\phi(x)$

$$[\mathrm{D}, \phi(x)] = i(\Delta + x_\mu \partial^\mu)\phi(x), \tag{2.15}$$

it identifies fields of conformal dimension Δ. When we will deal with gauge theories, the physical objects will be described by gauge invariant operators with given conformal dimension. Fields (or operators) annihilated by the lowering operator K_μ are called *primary fields* (or operators), while the ones obtained from primaries by applying the raising operator P_μ are called *descendants*.

In general, the quantization of a CFT breaks the conformal invariance due to the introduction of a renormalization scale. In a theory with gauge fields, fermions and scalars, all the dimensionless couplings g run with the energy scale, and thus the dimension Δ of a field is given by its classical dimension d corrected by the anomalous dimension γ:

$$\Delta = d + \gamma(g), \qquad \gamma = \frac{1}{2}\mu \frac{d}{d\mu} \ln Z\,. \tag{2.16}$$

Nevertheless, conformally invariant quantum field theories can be obtained both as fixed points of the Renormalization Group and as finite theories.

In order to have a conformally invariant quantum theory, we are interested in unitary representations of the conformal group in which the generators P, J, D, K are implemented as hermitian operators.
Unitarity of the theory imposes bounds on the dimensions of the primary fields. For example, in four-dimensional theories, the dimension of a scalar field has to be greater than one, $\Delta \geq 1$, where the bound is saturated, $\Delta = 1$, if the operator obeys free field equations. Furthermore, in supersymmetric theories, the bounds relate the dimension of the fields to their spin and R-symmetry. In four-dimensional $\mathcal{N} = 1$ supersymmetric theories, for example, the scalar bound $\Delta \geq \frac{3}{2}R$ relates the dimension to the R-charge, and it is saturated by chiral operators.

2.2 AdS spacetime

Moving to the gravitational side of the correspondence, we now introduce AdS spacetime, which is the maximally symmetric solution of Einstein equations with negative cosmological constant ($\Lambda < 0$).

Let us consider $\mathbb{R}^{d-1,2}$ with coordinates X^A, $A = 0, \ldots, d$, and metric $\eta_{AB} = \mathrm{diag}(-1, 1, \ldots, 1, -1)$. Anti-de Sitter spacetime AdS$_d$ in d dimensions is defined as the hypersurface

$$\eta_{AB} X^A X^B = -l^2, \tag{2.17}$$

where l is the radius of AdS. The isometry group is $SO(d-1,2)$.[1] For example, for $d = 2$, the hypersurface is given by

$$-(X^0)^2 + (X^1)^2 - (X^2)^2 = -l^2. \tag{2.18}$$

From the Einstein–Hilbert action with cosmological constant Λ

$$S = \frac{1}{16\pi G_d} \int d^d x \sqrt{-g}(R - 2\Lambda), \tag{2.19}$$

where G_d is the d-dimensional Newton's constant, R is the scalar curvature and g is the determinant of the metric, we can derive the following Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R - 2\Lambda) = 0, \tag{2.20}$$

from which

$$R = \frac{2d}{d-2}\Lambda, \tag{2.21}$$

telling us that the solution is an Einstein space.
Notice that the cosmological constant is related to AdS radius by

$$\Lambda = -\frac{(d-1)(d-2)}{2l^2}. \tag{2.22}$$

Among the solutions of (2.20), the only one which is maximally symmetric, namely

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right), \tag{2.23}$$

is given by the anti-de Sitter space.

2.2.1 Coordinates on AdS

We now review the most common sets of coordinates used to parametrize AdS$_d$ metric.

Global coordinates

AdS$_d$ can be parametrized in global coordinates using the set of coordinates (τ, ρ, θ_i), with $i = 1, \ldots, d-2$, as

$$\begin{cases} X^0 = l \cosh\rho \cos\tau, \\ X^d = l \cosh\rho \sin\tau, \\ X^i = l \sinh\rho \, y_i, \qquad i = 1, \ldots, d-1, \quad \sum_{i=1}^{d-1} y_i = 1, \end{cases} \tag{2.24}$$

where θ_i are the coordinates on an n-sphere parametrized by y_i. In these coordinates the line element reads

$$ds^2 = l^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2_{d-2}), \tag{2.25}$$

where $d\Omega_n^2$ is the line element of the n-sphere with unit radius.
For $\rho \in \mathbb{R}^+$ and $\tau \in [0, 2\pi]$, these coordinates cover the hyperboloid (2.17) exactly once. Due to the identification $\tau \sim \tau + 2\pi$ we have closed timelike curves; in order to avoid it, one usually takes the universal cover where $\tau \in \mathbb{R}$.

[1] It would be $O(d-1,2)$, but we consider the part connected to the identity.

Poincaré coordinates

Another set of coordinates can be written in terms of the parameters (t, z, x^i), with $i = 1, \dots, d-2$, as

$$\begin{cases} X^0 = \frac{z^2+l^2+\vec{x}^2-t^2}{2z}, \\ X^i = \frac{l}{z}x^i, \\ X^{d-1} = -\frac{z^2-l^2+\vec{x}^2-t^2}{2z}, \\ X^d = \frac{lt}{z}, \end{cases} \tag{2.26}$$

with $z \in \mathbb{R}^+$. The metric reads

$$ds^2 = \frac{l^2}{z^2}(-dt^2 + dz^2 + d\vec{x}^2), \tag{2.27}$$

or alternatively, changing the coordinate $z = l^2/r$,

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}d\vec{x}^2, \tag{2.28}$$

or

$$ds^2 = l^2\left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2)\right) \tag{2.29}$$

for $z = 1/u$.
This metric has slices isomorphic to $(d-1)$-dimensional Minkowski spacetime, foliated over u that runs from 0 to ∞. The plane $u = \infty$ is a conformal boundary for AdS_d,[2] while $u = 0$ is a Killing horizon, since the Killing vector ∂_t has zero norm at $u = 0$. Notice that, although they are convenient and commonly used, Poincaré coordinates cover only half of the hyperboloid: in $u = 0$ there is not a singularity and the metric can be extended after the horizon, for example using global coordinates. Global anti-de Sitter spacetime contains an infinite number of copies of the Poincaré patch.

2.2.2 Euclidean AdS

We now consider the Euclidean continuation of the AdS_d metric, that has useful applications in AdS/CFT correspondence.
This can be obtained by performing a Wick rotation on X^0 in embedding coordinates, or equivalently by sending $\tau \to -i\tau_E$ or $t \to -it_E$ in each set of coordinates. The resulting metric is thus

$$\begin{aligned} ds^2 &= l^2(\cosh^2\rho d\tau_E^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-2}) \\ &= l^2\left(\frac{du^2}{u^2} + u^2(dt_E^2 + d\vec{x}^2)\right). \end{aligned} \tag{2.30}$$

We can notice that the boundary plane $(u = \infty)$ $\mathbb{R}^{d-2,1}$ of the Minkowskian spacetime is replaced by $\mathbb{R}^{d-1}$, whereas the $u = 0$ plane shrinks to a point.
Furthermore, by adding the point $u = 0$ to the boundary $\mathbb{R}^{d-1}$, it is possible to compactify the

[2] The boundary is conformal because it can be described by different metrics related by conformal transformations.

flat $(d-1)$-dimensional space to S^{d-1}. Euclidean AdS$_d$ is thus diffeomorphic to a d-dimensional ball in $\mathbb{R}^d$ with metric

$$ds^2 = \sum_{i=1}^{d} \frac{dy_i^2}{(l^2 - |y|^2)^2}, \qquad \sum_{i=1}^{d} \leq l^2. \tag{2.31}$$

2.3 The AdS/CFT correspondence

As we mentioned, the AdS/CFT correspondence was first conjectured as a duality between Type IIB string theory with AdS$_5 \times S^5$ background and $\mathcal{N}=4$ Super Yang–Mills in four dimensions, which is the maximally supersymmetric four-dimensional field theory. More in detail, concrete results were obtained in the limit in which string theory is well approximated by classical ten-dimensional Type IIB supergravity. This duality has been later broadened and now it includes a general correspondence between gravitational theories in $(d+1)$ spacetime dimensions and quantum field theories without gravity in d dimensions.
In contrast with the field theory that lives on its d-dimensional "boundary", the $(d+1)$-gravitational theory is often denoted as the theory "in the bulk".

In this section we describe the AdS/CFT duality, starting from the original formulation and giving an intuition of its derivation from a brane engineering. We then make a digression on four-dimensional $\mathcal{N}=4$ Super Yang–Mills with $SU(N)$ gauge group and we conclude presenting some interesting examples of the holographic matching between corresponding quantities in the two dual theories. We follow again [44–46].

2.3.1 Maldacena conjecture

Here we present in more detail the duality conjectured by Maldacena, establishing a correspondence between the bulk and the boundary theory.
On the bulk side of the correspondence, there is a ten-dimensional Type IIB string theory on the product space AdS$_5 \times S^5$. The Type IIB five-form flux through S^5 in an integer N and AdS$_5$ has the same radius as S^5, L, given by the relation $L^4 = 4\pi g_s N \alpha'^2$.
On the boundary side, there is a four-dimensional Super Yang–Mills (SYM) theory with maximal $\mathcal{N}=4$ supersymmetry and $SU(N)$ gauge group. The Yang–Mills coupling g_{YM} is such that $g_{\text{YM}}^2 = g_s$ in the conformal phase.
The AdS/CFT correspondence conjectures that these two theories are equivalent to one another, including operators, states, correlation functions and dynamics.

In its strong form, the conjecture must hold for all values of N and of $g_s \sim g_{\text{YM}}^2$. Nevertheless, due to the complexity to quantize string theory on general curved manifolds, it is convenient to analyze limits in which the correspondence becomes more tractable.
One important example it the 't Hooft limit, which consists in keeping the 't Hooft coupling $\lambda \equiv g_{\text{YM}}^2 N \sim g_s N$ fixed as $N \to \infty$, on the field theory side. On the AdS side this can be achieved by re-expressing the string coupling in terms of the 't Hooft coupling as $g_s = \lambda/N$, and thus it corresponds to weak coupling string perturbation theory. It establishes a correspondence between classical string theory and the large-N limit of gauge theories: using classical string

theory on $\mathrm{AdS}_5 \times S^5$ one should be able to build a classical Lagrangian formulation of the large-N dynamics of $\mathcal{N}=4$ SYM theory. Finding an explicit realization of such a correspondence is, however, a challenging problem.
A further limit, $\lambda \to \infty$, allows to reduce classical string theory to classical Type IIB supergravity on $\mathrm{AdS}_5 \times S^5$, and thus to map strongly-coupled dynamics of Super Yang–Mills theory, in the large-N limit, into classical low-energy dynamics in supergravity.

We now present the brane engineering of the two theories, showing how they emerge from two different points of view of the same system.

Open strings living on D3-branes

Consider N parallel D3-branes very close to each other, in Type IIB string theory in ten-dimensional Minkowski spacetime: the D3-branes are extended along the (x^0, x^1, x^2, x^3) direction. On this background, string theory contains two kinds of perturbative excitations, closed strings and open strings. The first ones represent the excitations of empty space, while the latter describe excitations of the D-branes, where open strings are attached to.
If we restrict to energies lower than the string scale, $E \ll 1/\ell_s \sim 1/\sqrt{\alpha'}$, only the massless string states can be excited. The closed and open string massless states give a ten-dimensional gravity supermultiplet and an $\mathcal{N}=4$ vector supermultiplet in $(3+1)$ dimensions, respectively. The low-energy effective Lagrangian for closed string states is Type IIB supergravity, while the one for open string states is $\mathcal{N}=4$ $SU(N)$ Super Yang–Mills theory.[3]
The complete effective action of the massless modes can be written as

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}, \tag{2.32}$$

where

- S_{bulk} is the action of ten-dimensional supergravity, plus some higher derivative corrections.
- S_{brane} is the brane action, defined on the $(3+1)$-dimensional brane worldvolume. It contains the $\mathcal{N}=4$ SYM Lagrangian plus some higher derivative corrections.
- S_{int} is the action that describes the interactions between the bulk modes and the brane modes.

In order to study the low-energy limit of this action, we take $\ell_s \to 0\,(\alpha' \to 0)$ while keeping the energy and all the dimensionless parameters fixed, including g_s and N. One can show that both interaction terms and higher derivative corrections are proportional to positive powers of $\kappa \sim g_s \alpha'^2$, and thus they all vanish in the low energy limit. Therefore the interaction Lagrangian vanishes and we are left with two decoupled systems: the pure $\mathcal{N}=4$ $SU(N)$ gauge theory in $(3+1)$ dimensions and a free supergravity theory in the bulk.

[3] In section 2.3.2 we give a more detailed overview of non-abelian theories from D-branes and Super Yang–Mills theory.

D3-brane solution of Type IIB supergravity

We can also analyze the same system from a different perspective. In fact, D-branes act also as a source for various supergravity fields.

We focus on the low-energy effective action of Type IIB supergravity in ten dimensions, in which we only need to consider a subset of the bosonic fields of the theory consisting of the metric g_{MN}, the five-form field strength F_{MNPQR} and the dilaton ϕ, while the other fields consistently decouple. The action is given by

$$S_{\text{IIB}} = \frac{1}{2k_{10}^2} \int d^{10}x\sqrt{-g} \left[\frac{e^{-2\phi}}{g_s^2} \left(R_{10} + 4(\partial\phi)^2 \right) - \frac{1}{4}|F_5|^2 \right], \tag{2.33}$$

where the five-form is self-dual, $F_5 = \tilde{F}_5$ and $k_{10}^2 = 64\pi^7\alpha'^4$. We use $x^\mu, \mu = 0,1,2,3$, as Cartesian coordinates on Minkowski space and we parametrize the flat Euclidean "transverse space" with a radial coordinate r and five angular coordinates of a five-sphere. We are interested in the solution describing a set of N coincident D3-branes located at $r = 0$, for which we can write the line element as

$$ds_{10}^2 = \frac{1}{\sqrt{f(r)}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{h(r)} \left(dr^2 + r^2 d\Omega_5^2 \right), \tag{2.34}$$

where $d\Omega_5^2$ is the $SO(6)$ invariant metric on the unit S^5 and

$$f(r) = h(r) \equiv H(r) = 1 + \frac{L^4}{r^4}, \qquad L^4 = 4\pi\alpha'^2 g_s N, \tag{2.35}$$

$$e^\phi = g_s, \qquad A = \frac{1}{H(r)}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{2.36}$$

Notice that g_{tt} is non-constant and thus the energy E measured by an observer at infinity is related to the energy E_p of an object measured by an observer at a constant position r by the relation

$$E = H(r)^{-1/4}E_p. \tag{2.37}$$

We now consider two different regions of the above solution, in the low-energy limit. On the one hand, when $r \to \infty$ we have $H(r) \approx 1$ and $E \approx E_p$: the solution describes an asymptotically flat spacetime and thus there is a free supergravity theory in the bulk.
On the other hand, when $r \to 0$, the spacetime has a horizon and the region $r \approx 0$ is a "throat". Near the horizon $H(r) \approx \frac{L^4}{r^4}$ and the metric becomes

$$ds_{10}^2 \approx \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2 dr^2}{r^2} + L^2 d\Omega_5^2 \approx \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu \right) + L^2 d\Omega_5^2, \tag{2.38}$$

where in the second line we have introduced a new radial coordinate $z = L^2/r$, with respect to which the boundary is at $z = 0$ and the horizon at $z = \infty$. Thus, in the near-horizon limit the ten-dimensional spacetime is a product space $M_5 \times S^5$, where M_5 is the Poincaré patch of AdS$_5$ with radius L, which is the same radius of the five-sphere S^5.
Besides the symmetry $SO(4,2) \times SO(6)$, which is the isometry group of AdS$_5 \times S^5$, there are also 32 conserved supercharges, that can be found from a Killing spinor analysis of the fermionic supersymmetry variation of Type IIB supergravity.

Near-horizon geometry from brane construction

In conclusion, we have seen that both from the point of view of a field theory of open strings living on branes and from the point of view of branes as sources of supergravity fields, in the low-energy limit we obtain two decoupled systems. In both descriptions, one of the decoupled theory is supergravity in flat Minkowski spacetime, and thus the second theories that appear in both cases are expected to be equivalent. This leads to Maldacena conjecture in its weak form, i.e. the duality between $\mathcal{N} = 4$ Super Yang–Mills with $SU(N)$ gauge group in $(3 + 1)$-dimensions and Type IIB supergravity on $\mathrm{AdS}_5 \times S^5$.

Strong/weak duality

Let us focus more carefully on the regimes of validity of the approximations we are using.
From the field theory side, one can show from a loop diagrams expansion that the perturbative analysis of the Yang–Mills theory is reliable as long as

$$g_{\mathrm{YM}}^2 N \sim g_s N \sim \frac{L^4}{\ell_s^4} \ll 1. \tag{2.39}$$

From the AdS side, we can trust the classical gravity description when the radius of curvature L of both AdS_5 and S^5 is large compared to the string length,

$$\frac{L^4}{\ell^4} \sim g_s N \sim g_{\mathrm{YM}}^2 N \gg 1. \tag{2.40}$$

We can immediately notice that these two regimes are incompatible, and this is the reason why AdS/CFT correspondence is called a duality.
The two theories are thus conjectured to be equivalent, but when one side is weakly-coupled the other one is strongly-coupled, making the correspondence both useful and hard to prove.

2.3.2 $\mathcal{N} = 4$ Super Yang–Mills theory and D3-branes

In this section we give a short review on how non-abelian gauge theories can be engineered from a D-brane construction and we present the main features of four-dimensional $\mathcal{N} = 4$ Super Yang–Mills theory.

Non-abelian gauge theories from D-branes

In string theory, vector multiplets arise from the quantization of open strings that end on D-branes. If an open string has both end points attached to the same brane it can have arbitrarily short length and must thus be massless, giving rise to a massless vector multiplet: in fact this excitation mode induces a massless $U(1)$ gauge theory on the worldbrane. If we consider a configuration of N parallel separated D-branes with open strings, in which the end points of each string are attached to the same brane, these strings can again have arbitrarily small length and must be massless. These excitation modes induce a massless $U(1)^N$ gauge theory.
On the other hand, if in the latter configuration we have a string that connects different branes, the length of such a string is bounded from below by the separation distance between the branes and thus its mass cannot be arbitrarily small. One can show that there are $N^2 - N$ such possible

strings and they give rise to massive vector multiplets. Nevertheless, when the N D-branes coincide, all string states become massless and we obtain N^2 vector fields with the $U(1)^N$ gauge symmetry enhanced to a full $U(N)$ gauge symmetry. Notice that the $U(1) = U(N)/SU(N)$ factor corresponds to the position of the mass center of the branes and can be thus ignored when considering dynamics on the branes, leaving only a $SU(N)$ gauge symmetry.

We now focus on D3-branes. If we consider an open string ending on a single D3-brane, its excitation mode induces a massless $U(1)$ gauge theory on an effectively four-dimensional flat spacetime. The brane breaks half of the total number of supersymmetries and thus this gauge theory has $\mathcal{N} = 4$ Poincaré supersymmetry and in the low-energy approximation is a free theory. If we instead consider N parallel coincident D3-branes, in the low-energy limit they support a four-dimensional $\mathcal{N} = 4$ Super Yang–Mills theory with $SU(N)$ gauge group.

$\mathcal{N} = 4$ Super Yang–Mills

$\mathcal{N} = 4$ SYM is the theory that has the maximal amount of global supersymmetry in four dimensions. It contains only a vector multiplet,

$$V_{\mathcal{N}=4} = \left(A_\mu, \lambda^a_\alpha, \Phi^i\right), \tag{2.41}$$

where A_μ is a gauge field, Φ^i, $i = 1, \dots, 6$, are real scalar fields parametrizing the six directions transverse to the branes and λ^a_α, $\alpha = 1, 2$, $a = 1, \dots, 4$, are left Weyl spinors.
A D3-brane has an $SO(3, 1) \times SO(6)$ global symmetry. The first factor is the Lorentz group, the symmetry group of the D3-brane worldvolume, while the second can be interpreted as the $SU(4)_R \cong SO(6)_R$ R-symmetry of the theory, that rotates the four supercharges Q^a_α and under which A_μ is a singlet, λ^a_α are in the fundamental representation of $SU(4)$ and Φ^i are in the fundamental of $SO(6)$ (or equivalently they are in a rank 2 antisymmetric representation of $SU(4)$).
The Lagrangian of $\mathcal{N} = 4$ Super Yang–Mills is given by

$$\begin{aligned} \mathscr{L} = \mathrm{Tr}\bigg\{ &-\frac{1}{2g^2_{\mathrm{YM}}} F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu \Phi^i D^\mu \Phi^i \\ &+ \sum_{a,b,i} g_{\mathrm{YM}} \Big(C^{ab}_i \lambda_a [\Phi^i, \lambda_b] + \bar{C}_{iab}\bar{\lambda}^a[\Phi^i, \bar{\lambda}^b] \Big) + \frac{g^2_{\mathrm{YM}}}{2} \sum_{i,j} [\Phi^i, \Phi^j]^2 \bigg\}, \end{aligned} \tag{2.42}$$

where g_{YM} is the gauge coupling, θ_I is the instanton angle, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the non-abelian field strength, $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is the Hodge dual of F and $D_\mu = \partial_\mu + i[A_\mu, \cdot\,]$, while the constants C^{ab}_i and $\bar{C}_{iab}$ are related to the Dirac matrices for the R-symmetry group. Classically, $\mathscr{L}$ is scale invariant. As we mentioned above, scale invariance and Poincaré invariance combine and form the conformal group $SO(4, 2) \cong SU(2, 2)$, that in presence of $\mathcal{N} = 4$ supersymmetry enhances to the superconformal group $SU(2, 2|4)$.

Upon perturbative quantization, $\mathcal{N} = 4$ Super Yang–Mills theory is believed to be UV finite and thus its renormalization group β-function is identically vanishing. Therefore the theory is exactly scale invariant at the quantum level, with quantum mechanical symmetry $SU(2, 2|4)$.

Finally, there is an additional discrete global symmetry of the theory, at the quantum level, stated by the Montonen–Olive or S-duality conjecture. In fact, combining the real coupling and the real instanton angle into a single complex coupling

$$\tau \equiv \frac{\theta_I}{2\pi} + \frac{4\pi i}{g_{\mathrm{YM}}^2}, \tag{2.43}$$

the quantum theory is invariant under $\tau \to -1/\tau$, besides being invariant under $\tau \to \tau + 1$. The combination of these two symmetries gives the S-duality group $SL(2, \mathbb{Z})$, generated by

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \text{with} \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}. \tag{2.44}$$

Notice that when θ_I vanishes, the S-duality transformation reduces to $g_{\mathrm{YM}} \to 1/g_{\mathrm{YM}}$, exchanging strong and weak coupling.

2.3.3 Holographic dictionary

In this final section we summarize the main correspondences between quantities in the two theories related by the AdS/CFT duality.
In the majority of cases we restrict ourselves to the founding example, for simplicity.

Relations between parameters

We begin by recalling the relations between the various parameters of the two theories.
The number of colors N of the $SU(N)$ gauge group on the field theory side appears on the string theory side as the flux of the five-form field strength through the five-sphere,

$$\int_{S^5} F_5 = N. \tag{2.45}$$

From the physics of D-branes arises the relation between the Yang–Mills coupling and the string coupling, given by

$$\tau \equiv \frac{4\pi i}{g_{\mathrm{YM}}^2} + \frac{\theta_I}{2\pi} = \frac{i}{g_s} + \frac{\chi}{2\pi}, \tag{2.46}$$

where χ is the Ramond–Ramond scalar, and thus $g_{\mathrm{YM}}^2 = 4\pi g_s$. Furthermore, as we saw from supergravity D3-brane solution,

$$\frac{L^4}{\alpha'^2} = 4\pi g_s \qquad \Rightarrow \qquad \lambda \equiv g_{\mathrm{YM}}^2 = \frac{L^4}{\alpha'^2}. \tag{2.47}$$

Symmetries

For what concerns the symmetries of the two theories, the isometry group of AdS$_5$ is $SO(4, 2)$, which is the symmetry of the conformal group in four dimensions. The isometry group of the five-sphere is the compact group $SO(6)$, whose algebra is isomorphic to $SU(4)$, that is also the global internal symmetry of $\mathcal{N} = 4$ Super Yang–Mills. Furthermore, the latter theory is also invariant under four Poincaré supercharges and four conformal supercharges, which combine with the other symmetries into the superalgebra $SU(2, 2|4)$. The near-horizon limit of the superstring background has the same superalgebra, and thus the symmetry of the two theories completely match. We summarize this matching in the following table:

$\mathcal{N}=4$ **Super Yang–Mills**	**Type IIB string theory** on AdS$_5 \times S^5$
Conformal symmetry $SO(4,2)$ (global)	AdS$_5$ isometries $SO(4,2)$ (local)
Global internal $SO(6)$	S^5 isometries: $SO(6)$ (local)
Global supersymmetry	Same local supersymmetry
$\Downarrow$	$\Downarrow$
Global symmetry	**Local symmetry**

Notice that the $SU(N)$ gauge symmetry of SYM theory has no counterpart in the duality, because it describes a redundancy.

Field $\leftrightarrow$ Operator correspondence

There is a correspondence between the Hilbert spaces in the boundary and in the bulk: both spaces are organized in terms of the representations of $SO(4,2)$. Local gauge-invariant operators in the boundary correspond to bulk fields.

For example, one may deform a quantum field theory by a marginal operator $\mathcal{O}$ that changes the value of the coupling constant. As we have already noted, this corresponds to changing the value of the string coupling in the dual theory, which is related to the expectation value of the dilaton ϕ, set by the boundary condition for the dilaton at infinity. Thus, changing the field theory coupling constant modifies the boundary value of the dilaton.
More explicitly, let us deform the gauge theory as

$$S_{\text{QFT}} \to S_{\text{QFT}} + \int d^4x\, \phi_0(\vec{x})\mathcal{O}(\vec{x}) \,, \tag{2.48}$$

where $\phi_0(\vec{x})$ represents the source for the operator $\mathcal{O}(\vec{x})$. This changes the boundary condition of the dilaton at the AdS boundary $z=0$ (using the coordinate system in (2.38)) to $\phi(\vec{x},z)|_{z=0} = \phi_0(\vec{x})$ and leads to a fundamental statement of AdS/CFT correspondence:

$$\boxed{\langle e^{\int d^4x\, \phi_0(\vec{x})\mathcal{O}(\vec{x})}\rangle_{\text{CFT}} = \mathcal{Z}_{\text{string}}\Big[\phi(\vec{x},z)|_{z=0} = \phi_0(\vec{x})\Big]} \,. \tag{2.49}$$

On the left hand side we have the generating function of the correlators in the conformal field theory. On the right hand side we have the string theory partition function, with the boundary condition for the field ϕ on the AdS$_5$ boundary given by $\phi_0(\vec{x})$.

The importance of formula (2.49) resides in the fact that is valid in general, for any field ϕ. It establishes a one-to-one correspondence between each field propagating in AdS space and an operator in the dual field theory.

Mass $\leftrightarrow$ Conformal dimension relation

An application of the above field/operator correspondence gives a relation between the mass of the field ϕ on the AdS side and the conformal dimension Δ of the corresponding operator in the dual CFT.
Let us consider the wave equation for a field of mass m in AdS_{d+1} spacetime, which has the expansion

$$\phi(\vec{x}, z) = A(x) z^{d-\Delta} + B(x) z^{\Delta} + \dots \qquad \text{as } z \to 0\,, \tag{2.50}$$

where

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + L^2 m^2}. \tag{2.51}$$

We can note that the first term is dominant close to AdS boundary and thus, if we want to get a consistent version of formula (2.49) in presence of a massive field, we have to change its right hand side to

$$\phi(\vec{x}, \epsilon) = \epsilon^{d-\Delta} \phi_0(\vec{x}) \tag{2.52}$$

and then eventually take the limit $\epsilon \to 0$. Being ϕ dimensionless, ϕ_0 must have dimensions of $[\text{ length }]^{\Delta - d}$, and thus its associated operator $\mathcal{O}$ has dimension Δ, as can be read from the left hand side of (2.49).
Similar relations between AdS fields and field theory operators exist also for fermions and tensors on AdS space, and they can be obtained by comparing the partition functions of the two theories. Here we report some results in four dimensions, showing the relation between masses in AdS_5 and conformal dimensions of the dual operators for fields of arbitrary spin:

$$\begin{aligned}
&\text{scalar } \phi: && m^2 = L^2 \Delta(\Delta - 4)\,, \\
&\text{vector } A_\mu: && m^2 = L^2 (\Delta - 1)(\Delta - 3)\,, \\
&\text{symmetric } g_{\mu\nu}: && m^2 = L^2 \Delta(\Delta - 4)\,, \\
&\text{antisymmetric } B_{\mu\nu}: && m^2 = L^2 (\Delta - 2)^2\,, \\
&\text{spin-1/2 } \psi: && m = L(\Delta - 2)\,, \\
&\text{spin-3/2 } \psi_\mu: && m = L(\Delta - 2)\,.
\end{aligned} \tag{2.53}$$

Part II

On the boundary

Chapter 3

Some useful remarks on quantum field theories

We have seen that the AdS/CFT correspondence allows to extract gravitational quantities, such for example the entropy of certain anti-de Sitter black holes, from a field theory defined on the boundary of the gravity theory. In the original duality, between Type IIB string theory in $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ Super Yang–Mills with $SU(N)$ gauge group, one can reproduce the entropy of a class of supersymmetric rotating electrically charged AdS_5 black holes starting from the superconformal index of $\mathcal{N} = 4$ SYM on $S^1 \times S^3$ with proper chemical potentials.
In the last two decades this operation has been generalized to different dimensions and extended to less supersymmetric theories. Specifically, most of the results of the present work have been obtained in the framework of the holographic duality between four-dimensional superconformal field theories and gravitational theories in AdS_5 with less supersymmetry than the pioneering case. In fact, on the gravity side, one can break part of the supersymmetry deforming the string theory background, giving rise to supergravity theories that correspond to SCFTs which preserve fewer supersymmetry than $\mathcal{N} = 4$. We will typically deal with superconformal field theories with $\mathcal{N} = 1$.

This chapter is intended to introduce the main traits of the $\mathcal{N} = 1$ superconformal field theories that we will consider in the rest of this work.
In section 3.1 we introduce quiver gauge theories, which constitute the major class of SCFTs we will deal with. We present the geometry of their dual gravitational counterparts, we review the construction of quiver diagrams and brane tilings and we conclude with an explicit example. In section 3.2 we give a short introduction of super quantum chromodynamics, which provides another important example of $\mathcal{N} = 1$ SCFT, and we summarize the so-called Seiberg duality. Finally, in section 3.3 we discuss Chern–Simons theories. The latter can be formulated only in odd spacetime dimensions; we will focus on three-dimensional Chern–Simons theories and on their partition functions, since we will encounter them in section 5.2, arising from a four-dimensional computation.

3.1 Quiver gauge theories

In this section we introduce a class of four-dimensional superconformal field theories, called *quiver gauge theories*, that arise on the worldvolume of a stack of D3-branes located at the tip of a toric Calabi–Yau cone and that have gained considerable importance in the context of the AdS/CFT correspondence. The peculiarity of these gauge theories comes from the fact that, under certain conditions, they are completely specified by a two-dimensional graph and by the combinatorics of the dimer models on it. The following discussion in mainly based on [48].

As we mentioned before, the AdS/CFT correspondence can be extended to theories different from the presented one in the original Maldacena's conjecture, for example by replacing $\mathcal{N} = 4$ Super Yang–Mills with a less supersymmetric field theory. More specifically, in this work we will be mostly interested in superconformal field theories preserving $\mathcal{N} = 1$ supersymmetry. From the gravity side, this can be achieved following different approaches. One of them consists in breaking supersymmetry by modifying the topology of the string theory background $\mathrm{AdS}_5 \times S^5$, for example by substituting S^5 with a different five-dimensional manifold X_5. If we want the dual field theory to be a SCFT with $\mathcal{N} = 1$ supersymmetry, X_5 has to be a *Sasaki–Einstein manifold*.

A Sasaki–Einstein manifold (see e.g. [49]) is a Riemannian manifold (S, g) that is both Sasakian and Einstein:

- A Riemannian manifold (S, g) is Sasakian if and only if its metric cone $C(S)$ is Kähler, where
$$C(S) = \mathbb{R}_{>0} \times S \quad \text{with metric} \quad ds^2 = dr^2 + r^2 ds^2_{X_5}\,. \tag{3.1}$$
(S, g) has odd dimension $2n - 1$, where n is the complex dimension of the Kähler cone.

- A metric g is Einstein if its Ricci tensor satisfies $\mathrm{Ric}_g = \lambda g$ for some constant λ.

Furthermore, it can be shown that a Sasakian metric g is Einstein with $\mathrm{Ric}_g = 2(n - 1)$ if and only if the cone metric $\bar{g}$ is Ricci-flat, $\mathrm{Ric}_{\bar{g}} = 0$, i.e. is a Calabi–Yau three-fold metric.
For example, the odd dimensional sphere S^{2n-1} equipped with its standard Einstein metric is a Sasaki–Einstein manifold CY_3.

To have a dual generic $\mathcal{N} = 1$ superconformal field theory, the Sasaki–Einstein manifold only needs to admit a single $U(1)$ isometry, dual to the R-symmetry. However, we are interested in toric Sasaki–Einstein manifolds, that are a particular class of Sasaki–Einstein manifolds for which both X_5 and its cone have $U(1)^3$ isometries. The dual four-dimensional SCFTs are *toric* quiver gauge theories that admit a $U(1)_R \times U(1)^2_F$ global symmetry, which is an abelian subgroup of the R-symmetry group times the flavor symmetry group.[1] Furthermore, there may be other $U(1)$ global "baryonic" symmetries in the conformal field theory, that are gauged in the string theory on AdS: they come from the reduction of the RR four-form on three-cycles of X_5, producing a $U(1)$ gauge field in AdS.
Familiar examples of this class of gauge theories are abelian orbifolds S^5/Γ, with $\Gamma \simeq \mathbb{Z}_n$ or $\mathbb{Z}_n \times Z_m$, and the conifold and its orbifolds.

Moving back to a brane description, given a particular toric Calabi–Yau three-fold, the gauge theory that lives on the worldvolume of a stack of D3-branes located at the tip of the Calabi–Yau cone is indeed of quiver type and it flows to a non-trivial superconformal fixed point in the IR. In other words, this is equivalent to consider the near-horizon region of the D3-branes, in which the product space $\mathbb{R}^{3,1} \times CY_3$ is replaced by $\mathrm{AdS}_5 \times S^5$.

[1] Global symmetries may also be non-abelian, but here we only focus on their Cartan subgroups.

This construction can be thought as an extension of the case of a stack of D3-branes placed at a non-singular point of a Calabi–Yau three-fold, that sees locally a smooth $\mathbb{C}^3$ geometry and this gives rise to $\mathcal{N}=4$ SYM theory in the infrared.

As we mentioned, great interest in toric quiver gauge theories comes also from the fact that they can be completely defined without knowing the metric of the Sasaki–Einstein manifold by using a set of combinatorial models [50–53], as we will summarize in the following paragraphs.

3.1.1 Quiver diagrams

Quiver gauge theories contain both gauge groups and matter that transform in two-index tensor representation and they can be depicted using directed graphs, called quivers, made by vertices and arrows. Here we will focus on theories with $\mathcal{N}=1$ supersymmetry.
Suppose to have a theory with k $SU(N_a), a=1,\ldots,k$, gauge groups.[2] Each vertex of the graph represents an $SU(N_a)$ gauge group plus a vector multiplet in the adjoint representation of $SU(N_a)$. An arrow from the a-node to the b-node of the graph corresponds to an $\mathcal{N}=1$ chiral multiplet in the fundamental representation of $SU(N_a)\times SU(N_b)$, while an arrow from an $SU(N_a)$ node to itself represents a chiral multiplet transforming in the adjoint of $SU(N_a)$.

Useful constraints on the charges of the quiver theory can be read from the exact NSVZ beta function for its gauge couplings g_a, which is given by

$$\beta(g_a)=\frac{N}{1-\frac{g_a^2 N}{8\pi^2}}\left(3-\frac{1}{2}\sum_{i\in a}(1-\gamma_i)\right), \tag{3.2}$$

where γ_i is the anomalous dimension of the field X_i and i runs over all chiral multiplets that transform under the gauge group G_a. Since the anomalous dimension of a field X_i is related to its conformal dimension $\Delta(X_i)$ and to its R-charge $R(X_i)$ by

$$\Delta(X_i)=1+\frac{1}{2}\gamma_i=\frac{3}{2}R(X_i)\,, \tag{3.3}$$

the requirement of conformal invariance, i.e. that the β-functions all vanish, implies

$$\sum_{i\in a}(1-\gamma_i)=6 \quad\Leftrightarrow\quad \sum_{i\in a}(1-R(X_i))=2\,. \tag{3.4}$$

Finally, the remaining piece of information needed to completely specify the gauge theory is the *superpotential* W, not encoded in the quiver graph. The superpotential must be a function of gauge invariant operators, that correspond to closed loops on the graph. Furthermore, in order to preserve superconformal invariance, it must transform with R-charge 2 under the $U(1)_R$ symmetry, and it must be invariant under the flavor symmetries of the theory.
In general these conditions are not enough to write the superpotential. Nevertheless, when the gauge theory admit a toric $U(1)^3$ global symmetry, the requirement of invariance under

[2] The original gauge groups are multiple $U(N)$. However, one of the $U(1)$ factors always decouples in the IR, while the others can be addressed as $U(1)$ baryonic symmetries. In fact, in every toric gauge theory with N_g gauge groups there are N_g-1 baryonic symmetries. Therefore we can consider the gauge groups as $SU(N)$.

the symmetries of the Calabi-Yau geometry, which act as global symmetries on the D3-branes, completely fixes W. This is indeed the case of our analysis.
This additional constraint corresponds to the fact that each field appears linearly in the superpotential, and precisely in only two terms with opposite sign. This is the so-called toric condition.

All the information related to the superpotential can be then added to the quiver diagram by defining another graph, called *planar quiver*.

Planar quivers and brane tilings

To engineer a planar quiver we use the terms of the superpotential as plaquettes constituting boundaries of polygons. Thus, when a field appears in two terms of the superpotential, the plaquettes are glued together along the corresponding edge. The sign of the superpotential terms determines the orientation of the plaquettes. Since in toric quiver theories each field appears linearly in the superpotential and in exactly two terms, the plaquette tiling obtained from these theories is a polygonal tiling of an orientable Riemann surface without boundary, which is the planar quiver.
Using the constraints imposed by superconformality and by the requirement that the superpotential has R-charge 2, one can show that this tiling has genus 1 and is topologically a two-torus.

Starting from a planar quiver, we can construct its dual graph, called *brane tiling*.[3] In the brane tiling each face of the dual planar quiver is replaced by a vertex and the edges separating two faces are replaced by dual edges, delimiting the new faces dual to the previous vertices. Also this dual graph lives on a two-torus.
The brane tiling is a bipartite graph: each vertex is black (positive) or white (negative), depending on the orientation of the plaquettes, and each node is only connected to nodes with opposite sign. Finally, the polygonal faces now represent the $SU(N)_a$ gauge groups, while the arrows represent the chiral multiplets and the vertices represent the superpotential interactions.

3.1.2 An example: the conifold $T^{1,1}$

In order to give an explicit example, we report here the quiver graph, planar quiver and brane tiling for the conifold theory, that will also be handled in chapter 9.

With "conifold", also denoted as Klebanov–Witten theory [54], we refer to the gauge theory describing the low-energy dynamics of a stack of N D3-branes placed at the singular point of the conifold geometry. The properly named conifold is a Calabi–Yau three-fold [55] defined by the quadratic relation in $\mathbb{C}^4$

$$\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 = 1\,, \tag{3.5}$$

which has an isolated singularity at $\omega_1 = \omega_2 = \omega_3 = \omega_4$.
The conifold can be thought as a real cone over the compact five-real dimensional manifold called $T^{1,1}$. The manifold $T^{1,1} = SU(2) \times SU(2)/U(1)$ admits a Sasaki–Einstein structure and has the topology of $S^2 \times S^3$. It can be seen as a $U(1)$ fibration over the regular Kähler–Einstein

[3] The name "brane tiling" comes from the fact that the physics that can be extracted from this graph arises from D-branes in string theory.

manifold $\mathbb{CP}^1 \times \mathbb{CP}^1$. Remarkably, $T^{1,1}$ is toric.

We are thus studying the $\mathcal{N} = 1$ superconformal field theory dual to Type IIB theory compactified on $\mathrm{AdS}_5 \times T^{1,1}$, which is the infrared limit of the theory of N coincident D3-branes located at the conifold singularity. The quiver diagram of this field theory is depicted in figure 3.1. As represented in the graph, the conifold field theory has the gauge group $SU(N) \times SU(N)$.

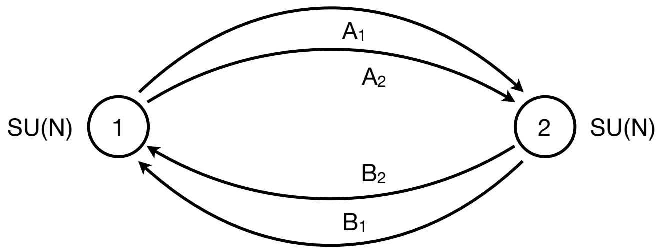
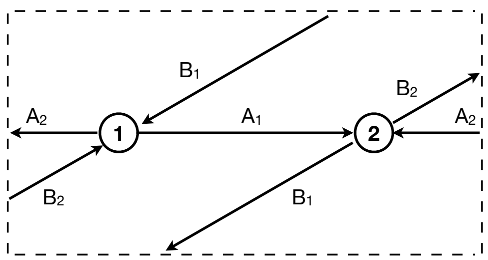


Figure 3.1: The quiver diagram of the conifold field theory.

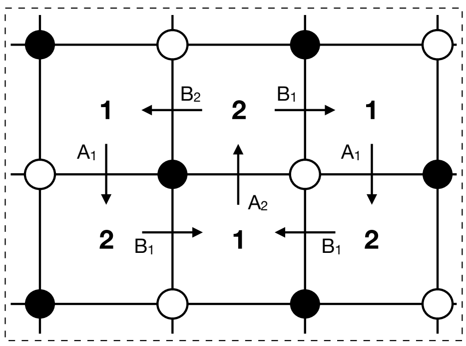
The chiral multiplets transform in the bifundamental representation of this gauge group: two superfields, A_1 and A_2, transform in the $(N, \bar{N})$, while the other two ones, B_1 and B_2, transform in the $(\bar{N}, N)$.
The superpotential W must preserve the $SU(2) \times SU(2) \times U(1)_R$ symmetry of the theory and is given by

$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1. \tag{3.6}$$

The planar quiver and the brane tiling of the conifold are represented in figure 3.2. The two



(a) Planar quiver



(b) Brane tiling

Figure 3.2: The planar quiver and the brane tiling for the conifold quiver.

$SU(2)$-factors of the global $SU(2) \times SU(2)$ flavor symmetry, which are associated with the isometries of $\mathbb{CP}^1 \times \mathbb{CP}^1$, act on the fields A_1, A_2 and B_1, B_2, respectively, that transform as doublets. In particular, the two $U(1)$ flavor symmetries are the Cartans of these two $SU(2)$s. The $U(1)_R$-symmetry, instead, comes from the fact that $T^{1,1}$ is a $U(1)$-fibration over $\mathbb{CP}^1 \times \mathbb{CP}^1$. The exact R-charges of this theory are all equal to $1/2$, since the $U(1)_R$ does not mix with the non-abelian flavor symmetries. Finally, the last global symmetry of the theory is a baryonic $U(1)_B$ symmetry, associated with the non-trivial three-cycle of the geometry, due to the presence of the S^3 cycle in the topology of $T^{1,1}$.[4]

The charges of the various fields under the symmetries of the theory are listed in the following table:

	$U(1)_{F_1}$	$U(1)_{F_2}$	$U(1)_B$	$U(1)_R$
A_1	1	0	1	$1/2$
A_2	-1	0	1	$1/2$
B_1	0	1	-1	$1/2$
B_2	0	-1	-1	$1/2$

. (3.7)

3.2 SQCD and Seiberg duality

We now present a very short summary of super quantum chromodynamics (SQCD) and Seiberg duality, since we will refer to these notions in chapter 5. We address to [23,24,56] for a thorough review on these topics.

Super quantum chromodynamics

SQCD is the supersymmetric version of quantum chromodynamics. It is a supersymmetric gauge theory with non-abelian gauge group $SU(N)$,[5] F flavors and no superpotential, $W = 0$. The quarks Q and $\widetilde{Q}$ are chiral superfields represented by $F \times N$ complex matrices. There are two independent flavor symmetries, $SU(F)_L$ and $SU(F)_R$, associated with Q and $\widetilde{Q}$ respectively. It is useful to split the matter indices as (i, a) to make the different symmetries manifest: $i = 1, \ldots, F$ is an index in the (anti)fundamental representation of the flavor group F, while $a = 1, \ldots, N$ is a index in the (anti)fundamental representation of the gauge group $SU(N)$. The theory has an $SU(F)_L \times SU(F)_R \times U(1)_B \times U(1)_R$ global symmetry. The charges of the chiral supermultiplets under the symmetries of the theory are summarized in the following table:

	$SU(N)$	$SU(F)_L$	$SU(F)_R$	$U(1)_B$	$U(1)_R$
Q^i_a	N	F	$\mathbf{1}$	1	$\frac{F-N}{F}$
$\widetilde{Q}^b_j$	$\bar{N}$	$\mathbf{1}$	$\bar{F}$	-1	$\frac{F-N}{F}$

. (3.8)

The convention for gauge indices is that lower indices are for objects transforming in the fundamental representation while upper indices for the anti-fundamental. The convention for flavor indices is the opposite.
The $SU(F)_L \times SU(F)_R$ global symmetry is analogous to the $SU(3)_L \times SU(3)_R$ chiral symmetry of the non-supersymmetric QCD with three flavors, while the $U(1)_B$ is analogous to the baryon number. With respect to QCD there is an extra $U(1)_R$, due to the presence of the gaugino in the

[4] As we mentioned above, we can see the $U(1)$ baryonic charge as coming from the IR decoupling of one of the $U(N)$ gauge group, which in fact reduces to $SU(N)$.

[5] It is also possible to formulate SQCD with a different non-abelian gauge group. Here we consider $SU(N)$ for definiteness.

supersymmetric theory.

If $F < N$, the only gauge invariant single trace operators that can be constructed are the mesons, given by

$$M^i{}_j = Q^i_a \widetilde{Q}^a_j \,, \tag{3.9}$$

where the contraction is taken over the N gauge indices. On the other hand, if $F \geq N$, besides the mesons there are other gauge invariant single trace operators, the baryons, made out of N fields Q and N fields $\tilde{Q}$ with fully anti-symmetrized indices. For example, in the case $F = N$ there are two baryons, defined as

$$\begin{aligned} B &= \epsilon^{a_1 a_2 \dots a_N} Q^1_{a_1} Q^2_{a_2} \dots Q^N_{a_N} \,, \\ \widetilde{B} &= \epsilon_{a_1 a_2 \dots a_N} \widetilde{Q}^{a_1}_1 \widetilde{Q}^{a_2}_2 \dots \widetilde{Q}^{a_N}_N \,. \end{aligned} \tag{3.10}$$

Conformal window and Seiberg duality

An interesting trait of SQCD is that when the number of flavors is in the range $\frac{3}{2}N < F < 3N$ the theory flows to an interacting fixed point in the infrared. This means that even if the theory is UV-free and thus the gauge coupling g increases towards the IR, at low energies g reaches a constant value g_*. The above range is called *conformal window*. Outside this range, for $F < \frac{3}{2}N$ the theory is in a different phase, while for $F \geq 3N$ SQCD is no longer asymptotically free.

Seiberg duality, proposed by Seiberg [57] in mid nineties, is an IR equivalence establishing a sort of electromagnetic duality. It states that the IR physics of SQCD, for $F > N + 1$, can be equivalently described by means of another supersymmetric gauge theory, called the magnetic dual theory. It is worth stressing that the two theories are not identical, but they flow to the same IR fixed point under an RG-flow. For this reason it is called an IR duality.
More precisely, the electric theory is given by SQCD with $SU(N)$ gauge group and $F > N + 1$, while the magnetic theory is a SQCD-like theory with $SU(\tilde{N})$, $\tilde{N} = F - N$, gauge group and F flavor, with an extra chiral superfield Φ and a non-zero superpotential W. The quark-like fields q and $\tilde{q}$ of the magnetic theory transform in the fundamental and anti-fundamental representation of $SU(\tilde{N})$, respectively. The baryons of the electric theory have a dual description in terms of these fields as

$$B_{i_1 i_2 \dots i_{\tilde{N}}} \sim \epsilon_{a_1 a_2 \dots a_{\tilde{N}}} q^{a_1}_{i_1} q^{a_2}_{i_2} \dots q^{a_{\tilde{N}}}_{i_{\tilde{N}}} \tag{3.11}$$

and similarly $\widetilde{B}$. The superfield Φ is a gauge singlet and transforms in the fundamental representation of $SU(F)_L$ and in the anti-fundamental of $SU(F)_R$. It interacts with q and $\tilde{q}$ through the cubic superpotential

$$W = h q_i \Phi^i_j \tilde{q}^j \,. \tag{3.12}$$

3.3 Chern–Simons theories

We conclude this chapter introducing Chern–Simons (CS) theories, which are gauge theories that can be formulated in any odd-dimensional spacetime.

To be more concrete, we restrict to three dimensions. The Lagrangian for a 3d Chern–Simons theory is given by

$$\mathscr{L}_{\text{CS}} = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}\,\text{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right), \tag{3.13}$$

where k is the Chern–Simons coupling constant (also called level of the CS term) and the gauge field A_μ takes values in some semi-simple Lie algebra $\mathfrak{g}$. If $\mathfrak{g} = u(1)$, i.e. if the Chern–Simons theory is abelian, the cubic term in the Lagrangian vanishes.

Under the non-abelian gauge transformation

$$A_\mu \to g^{-1}A_\mu g + g^{-1}\partial_\mu g\,, \tag{3.14}$$

where $g \in G$ and G is the gauge group, the Lagrangian transforms as

$$\mathscr{L}_{\text{CS}} \to \mathscr{L}_{\text{CS}} - \frac{k}{4\pi}\epsilon^{\mu\nu\rho}\partial_\mu\,\text{Tr}\left[(\partial_\nu g)g^{-1}A_\rho\right] - \frac{k}{12\pi}\,\text{Tr}\left[(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\rho g)\right], \tag{3.15}$$

where the first term is a boundary term that can be integrated out, while the second term is related to the winding density number

$$\omega(g) \equiv \frac{\epsilon^{\mu\nu\rho}}{24\pi^2}\,\text{Tr}\left[(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\rho g)\right]. \tag{3.16}$$

The integral of $\omega(g)$ is an integer number n. We can thus rewrite (3.15) as

$$S_{\text{CS}} \to S_{\text{CS}} - 2\pi k n \tag{3.17}$$

and we immediately see that the action is no longer gauge invariant. Nevertheless, if we consider the term appearing in the partition function, we have

$$e^{iS_{\text{CS}}} \to e^{iS_{\text{CS}}}e^{-i2\pi k n} = e^{iS_{\text{CS}}} \tag{3.18}$$

if require k to be integer.
Therefore, Chern–Simons action is classically not gauge invariant, but it can be made gauge invariant at the quantum level for an integer-valued k.

3.3.1 Chern–Simons partition functions

We now focus on the partition functions of three-dimensional pure Chern–Simons theories, that we will employ in a few chapters.
Chern–Simons partition function can be written in terms of matrix integrals [58], for example by using localization techniques [59] (see also [60] for a review on localization).

The partition function of a Chern–Simons theory is given by the path integral

$$Z = \int \mathcal{D}A e^{iS}. \tag{3.19}$$

Localization enables to compute the path integral of certain supersymmetric theories defined on curved spaces exactly. In fact, using supersymmetry it is possible to prove that the path

integral receives contributions only from the so-called localization locus, which is the locus of fixed points of supersymmetry.
More in detail, localization allows to lower the dimensionality of integrals: for example, using localization formulae, one can reduce the path integral of a quantum field theory on D-dimensional fields to a path integral on lower d-dimensional fields. Remarkably, if the localization locus consists of configurations of constant field only, and thus $d = 0$, we are left with the path integral of a zero-dimensional field theory, which is a finite-dimensional integral that can often be evaluated exactly. This is the case of Chern–Simons theories.

Let us start from the supersymmetric Chern–Simons term in flat Euclidean space

$$S = \int d^3x \, \mathrm{Tr} \left(\frac{k}{4\pi} \epsilon^{\mu\nu\rho} \Big(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \Big) - \lambda^\dagger \lambda + 2D\sigma \right), \tag{3.20}$$

where we have included also the auxiliary fermion λ and the auxiliary scalars σ and D.
To perform the localization, we work on a compact manifold rather than in flat space, in order to have a well-defined partition function. Being the action conformal, it can be moved to the unit three-sphere S^3 by simply multiplying the Lagrangian for an overall measure factor $\sqrt{g}$. Therefore, the partition function on S^3 for a supersymmetric Chern–Simons theory with gauge group G localizes to the following matrix integral, presented in [59],

$$Z = \frac{1}{|\mathcal{W}|} \int da \exp\left(-\pi i k \, \mathrm{Tr}(a^2)\right) \det{}_{Ad} 2\sinh(\pi a) \,, \tag{3.21}$$

where $|\mathcal{W}|$ is the order of the Weyl group of G, a runs over the Cartan of the Lie algebra of G and we have used the notation

$$\det{}_{Ad} f(a) \equiv \prod_\alpha f(\alpha(a)), \tag{3.22}$$

where Ad is the adjoint representation of G and the product runs over the roots α of the algebra.

Moving to a concrete example, the three-sphere partition function of supersymmetric $SU(N)$ Chern–Simons theory with level k is given by

$$Z^{\mathrm{CS}}_{SU(N)_k} = \frac{1}{N!} \int_{-\infty}^{+\infty} \prod_{\mu=1}^{N-1} d\lambda_\mu \cdot \exp\left[-\pi i k \sum_{j=1}^{N} \lambda_j^2 + \sum_{j\neq k} \log\big(2\sinh\pi(\lambda_j - \lambda_k)\big) \right] \tag{3.23}$$

with the constraint $\sum_{j=1}^N \lambda_j = 0$, where the $N!$ factor is due to the S_N Weyl group of $SU(N)$ and we have taken the Cartan as the set of diagonal matrices, $a = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$.[6]
The partition functions for supersymmetric Chern–Simons theories with real gauge groups, that we will use in section 5.2, have similar expressions. We report them in appendix A.

[6]Notice that in our formula the sum runs from 1 to $N-1$, due to $SU(N)$ extra constraint.

Chapter 4

The superconformal index

The four-dimensional superconformal index (SCI), originally defined in [61,62], is a generalization of the Witten index obtained by radially quantizing a superconformal field theory (SCFT). It counts a set of protected short multiplets that do not recombine into long ones. The index can equivalently be obtained by localization on $S^3 \times S^1$, except for an overall factor. The index is an excellent tool for the study of four-dimensional SCFTs, because it is a topological invariant, fully quantum, and protected quantity. For instance, it has been used to check dualities, propose new ones, study (super)symmetry enhancements, and analyze the conformal manifold. We refer e.g. to [63–65] for recent accounts on the subject.
The original motivation behind the introduction of the SCI of $\mathcal{N} = 4$ $SU(N)$ Super Yang–Mills was counting the 1/16-BPS states that should reproduce the entropy of the dual charged and rotating black hole in $\mathrm{AdS}_5 \times S^5$. Even if this expectation was not realized at the beginning, a refining in the computation of the index has finally allowed to recover the expected BH entropy, leading to a renewed interest in the field.

The structure of this chapter is the following. In section 4.1 we present a short review of the Witten index. Section 4.2 is devoted to the computation of the four-dimensional superconformal index. We mostly focus on its derivation as a trace and we conclude by mentioning its connection with the supersymmetric partition function on $S^3 \times S^1$. In section 4.3 we introduce the integral form of the SCI for a gauge theory containing flavor symmetry, while in section 4.4 we write a couple of explicit examples. Finally, in section 4.5 we discuss how the black hole entropy can be derived starting from the superconformal index. We present the class of supersymmetric black holes we will mostly deal with in this work and we summarize the extremization procedure needed to compute their entropy from a field theory approach.

4.1 The Witten index: a review

The Witten index was introduced by Edward Witten in [66] as an instrument to investigate whether the supersymmetry of a theory is spontaneously broken or not. It is defined as

$$\mathcal{I}_W = \mathrm{Tr}(-1)^F, \tag{4.1}$$

where the trace is over the Hilbert space of the theory and F represents the fermionic number. For the computation of the index it is convenient to consider supersymmetric field theories formulated in a finite spatial volume. On the one hand, this choice makes the spectrum of the

Hamiltonian discrete, allowing for a well-defined counting. On the other hand, if we furthermore impose periodic boundary conditions, it is a sufficient set-up for the study of supersymmetry breaking. In fact, as it is well known, supersymmetry is not broken if the energy of the vacuum is zero, in contrast with the internal symmetries, whose spontaneous symmetry breaking depends on the behavior of the theory in the infinite volume limit. However, since translations are part of the supersymmetry algebra, we have to adopt periodic boundary conditions (the same ones both for bosons and for fermions) in order to preserve supersymmetry. This request is equivalent to take the spatial manifold to be a three-dimensional torus, $\mathbb{T}^3$.

Given a supersymmetric theory, the energy E is greater than or equal to the magnitude of the momentum $|\mathbf{P}|$ for any state. Since we are interested in zero-energy states, we stick ourselves to the analysis of the $|\mathbf{P}| = 0$ subspace of the total Hilbert space, in which the supersymmetry algebra simplifies. We assume that $\mathcal{Q}_i,\ i = 1, \ldots, K$ (with $K = 4$ for $d = 4$, $\mathcal{N} = 1$) is a basis of normalized charges for this subspace. For our purpose, it is sufficient to work with any of the $\mathcal{Q}_i$, which we denote as $\mathcal{Q}$. Let $\mathcal{Q}^\dagger$ be its Hermitian conjugate. The algebra is

$$\begin{aligned} &\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 2\delta, \\ &\{\mathcal{Q}, \mathcal{Q}\} = \{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\} = 0. \end{aligned} \tag{4.2}$$

Supersymmetry maps bosonic states into fermionic states and vice versa. A bosonic state $|b\rangle$ satisfies $e^{2\pi i J_z}|b\rangle = |b\rangle$, while a fermionic state $|f\rangle$ satisfies $e^{2\pi i J_z}|f\rangle = -|f\rangle$, where J_z is the third component of the angular momentum in our basis.
Thus, the index is the trace of the operator

$$(-1)^F = e^{2\pi i J_z} \tag{4.3}$$

that distinguishes bosons from fermions.
Notice that states of non-zero energy are paired in two-dimensional supermultiplets by the action of $\mathcal{Q}$, while any bosonic or fermionic state of zero energy forms a trivial one-dimensional supermultiplet. In fact, if $|b\rangle$ is a normalized bosonic state of non-zero energy E, we can define a fermionic state $|f\rangle = (1/\sqrt{E})\mathcal{Q}|b\rangle$. The action of $\mathcal{Q}$ on the states is thus

$$\mathcal{Q}|b\rangle = \sqrt{E}|f\rangle, \qquad \mathcal{Q}|f\rangle = \sqrt{E}|b\rangle, \tag{4.4}$$

where the second equality satisfies $\mathcal{Q}^2 = \delta$.
Conversely, with $\mathcal{Q}^2 = \delta$ and $\mathcal{Q}$ Hermitian, each state annihilated by δ is also annihilated by $\mathcal{Q}$, i.e. any state of zero energy satisfies $\mathcal{Q}|b\rangle = 0$ or $\mathcal{Q}|f\rangle = 0$. Therefore there may be a number $n_B^{E=0}$ of zero-energy bosonic states and a (a priori different) number $n_F^{E=0}$ of zero-energy fermionic states. The crucial fact is that as we vary the parameters of the theory (the volume, the mass and the coupling constants), the states of non zero-energy change their energy, but they do it in Bose-Fermi pairs. Therefore, the difference

$$n_B^{E=0} - n_F^{E=0} \tag{4.5}$$

remains constant. The quantity (4.5) has two useful properties. First, being independent of all parameters, there is generally a convenient limit in which it can be computed. Second, if it is non-zero it means that supersymmetry is not spontaneously broken, because there are some

states with zero energy.
Notice that, on the contrary, if the difference in (4.5) is zero, we cannot tell whether both the number of states are zero, and thus supersymmetry is broken, or whether they are equal but non-zero, and thus supersymmetry is preserved.

The quantity (4.5) can be obtained as the trace of the operator in (4.3). In fact, states of non-zero energy do not contribute to the trace, since every bosonic state contributes with $+1$, while every fermionic state contributes with -1, and so they cancel. Thus (4.1) can be evaluated considering the zero-energy states only and can be written as

$$\mathcal{I}_W = \mathrm{Tr}(-1)^F = n_B^{E=0} - n_F^{E=0}. \tag{4.6}$$

Finally, one can regularize the index by considering

$$\mathcal{I}_W = \mathrm{Tr}(-1)^F e^{-\beta\delta} \tag{4.7}$$

for arbitrary positive β. In fact, this quantity is actually independent of β, since the states of non-zero energy, and thus $\delta \neq 0$, do not contribute to the index. This regularized index reduces to the original one in the limit $\beta \to 0$.

The superconformal index is defined as the Witten index of a four-dimensional superconformal field theory computed on S^3 as spatial manifold instead of $\mathbb{T}^3$, in radial quantization, refined by chemical potentials to take into account the relevant quantum numbers of the theory.

4.2 Computing the superconformal index

In the following we give a short overview of the four-dimensional superconformal index, mainly based on [64,65,67].
The index can equivalently be obtained using two different approaches. For theories that admit a Lagrangian description, it can be computed by enumerating, with signs, local gauge invariant operators made from elementary fields of the theory. This technique corresponds to perform a weighted trace over the states of the theory, quantized on $S^3 \times \mathbb{R}_t$. For more general theories, the four-dimensional counting problem can be reformulated in terms of a matrix integral, that can be equivalently constructed applying supersymmetric localization on the $S^3 \times S^1$ partition function. As we will see, these two definitions differ by an overall contribution, dubbed supersymmetric Casimir energy in [68,69].

4.2.1 Index as a trace

The four-dimensional superconformal index is defined for SCFTs with arbitrary amount of supersymmetry on $S^3 \times \mathbb{R}_t$ and it is schematically given by [61,62]

$$\mathcal{I} = \mathrm{Tr}\big[(-1)^F e^{-\beta\,\delta} e^{-\mu_i q_i}\big], \tag{4.8}$$

where the trace is taken over the Hilbert space of the theory on S^3. We denote by $\mathcal{Q}$ one of the Poincaré supercharges and by $\mathcal{S} = \mathcal{Q}^\dagger$ its superconformal partner. Again, $\delta \equiv \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\}$, while q_i are charges that commute with $\mathcal{Q}$ and μ_i the associated chemical potentials. As we noted before, the index is independent of β, since only zero-energy states contribute. Nevertheless, there is an infinite number of states with $\delta = 0$, even for a single short irreducible representation of the superconformal algebra. The chemical potentials μ_i are hence introduced both as regulators to avoid the divergence and to refine the counting.

We will now focus on $\mathcal{N} = 1$ supersymmetry. Recalling that the isometry group of S^3 is $SU(2)_1 \times SU(2)_2$, the supercharges are $\{\mathcal{Q}_\alpha, \mathcal{S}^\alpha \equiv \mathcal{Q}^{\alpha\dagger}, \widetilde{\mathcal{Q}}_{\dot\alpha}, \widetilde{\mathcal{S}}^{\dot\alpha} \equiv \widetilde{\mathcal{Q}}^{\dot\alpha\dagger}\}$, where $\alpha = \pm$ and $\dot\alpha = \dot\pm$ are $SU(2)_1$ and $SU(2)_2$ indices, respectively. The relevant anticommutators are given by

$$\begin{aligned} \{\mathcal{Q}_\alpha, \mathcal{Q}^{\beta\dagger}\} &= \Delta + 2M_\alpha^{\ \beta} + \frac{3}{2}R, \\ \{\widetilde{\mathcal{Q}}_{\dot\alpha}, \widetilde{\mathcal{Q}}^{\dot\beta\dagger}\} &= \Delta + 2\widetilde{M}_{\dot\alpha}^{\ \dot\beta} - \frac{3}{2}R, \end{aligned} \tag{4.9}$$

where Δ is the conformal dimension, $M_\alpha^{\ \beta}$ and $\widetilde{M}_{\dot\alpha}^{\ \dot\beta}$ are the $SU(2)_1$ and $SU(2)_2$ generators and R is the generator of the $U(1)_R$ R-symmetry. In these conventions $\mathcal{Q}_\alpha$ and $\widetilde{\mathcal{Q}}_{\dot\alpha}$ have respectively $R = -1$ and $R = +1$.
If we choose the supercharge $\mathcal{Q} \equiv \widetilde{\mathcal{Q}}_{\dot -}$, we have $\delta = \Delta - 2j_2 - \frac{3}{2}R$, where $j_{1,2}$ are the Cartan generators of $SU(2)_{1,2}$. We then choose the charges q_i that refine the index to be $\frac{1}{3}(\Delta + j_2) \pm j_1$.[1]
Therefore, the index (4.8) can be rewritten as

$$\mathcal{I}(p,q) = \mathrm{Tr}\left[(-1)^F p^{\frac{1}{3}(\Delta+j_2)+j_1} q^{\frac{1}{3}(\Delta+j_2)-j_1}\right] = \mathrm{Tr}\left[(-1)^F p^{j_1+j_2+\frac{1}{2}R} q^{j_2-j_1+\frac{1}{2}R}\right], \tag{4.10}$$

where p, q are the fugacities associated with the $SU(2)_{1,2}$ generators, such that $|p|, |q| < 1$. In the second equality we have taken advantage of the fact that only states with $\delta = 0$ contribute to the index, and thus $\Delta = 2j_2 + \frac{3}{2}R$.

Since we are dealing with conformal theories, we can use the state/operator map to interpret the states as local operators. Thus, to compute the index, one has to list all the possible operators that can be built from modes of the fields, projecting out gauge non-invariant ones. The different modes of the fields are called "letters" and are used to construct the operators as "words". Moreover, to include the gauge and flavor quantum numbers, we introduce characters: we denote by $\mathcal{R}$ the representation of the (gauge group) $\times$ (flavor group) under which the chiral multiplets transform and by $\chi_{\mathcal{R}}(U,V)$, $\chi_{\bar{\mathcal{R}}}(U,V)$ the characters of $\mathcal{R}$ and of its conjugate representation, with gauge group matrix U and flavor group matrix V.
The letters and their contributions to the index have been tabulated in literature (see e.g. Table 1 and 2 of [65]). From there, one can easily compute the single-letter index for a chiral multiplet Φ

$$i_\Phi(p,q,U,V) = \frac{(pq)^{\frac{1}{2}R}\chi_{\mathcal{R}}(U,V) - (pq)^{\frac{2-R}{2}}\chi_{\bar{\mathcal{R}}}(U,V)}{(1-p)(1-q)} \tag{4.11}$$

and the one for a vector multiplet v

$$i_v(p,q,U) = \frac{2pq - p - q}{(1-p)(1-q)}\chi_{\mathrm{adj}}(U), \tag{4.12}$$

[1] In principle one can choose an arbitrary basis of charges in this space as q_i. Our choice is in agreement with the existing literature.

where $\chi_{\text{adj}}(U)$ is the character of the adjoint representation of the gauge group.

The key point that makes the superconformal index easy to compute is the fact that it is invariant under variations of the gauge coupling. Therefore, the index of strongly-coupled theories can be computed by setting the gauge coupling to zero and, as we mentioned before, enumerating only the words that can be constructed using letters with $\delta = 0$. The partition function over such words is obtained by plethystic exponentiation of the single-letter index, that takes account of all the supersymmetric local operators that can be constructed in four dimensions. The plethystic exponential

$$\mathrm{PE}[i_k(p,q,U,V)] \equiv \exp\left\{\sum_{m=1}^{\infty}\frac{1}{m}i_k\left(p^m,q^m,V^m\right)\chi_{\mathcal{R}_k}\left(U^m,V^m\right)\right\}, \tag{4.13}$$

is used to implement the combinatorics of symmetrization of the single letters, where i_k represents the single-letter index of the k-th multiplet.
Since we are dealing with gauge theories, we only want to construct operators that are invariant under gauge transformations. To construct the index we thus enumerate and project onto gauge singlets, i.e. operators that transform under the trivial representation of the gauge group. This can be achieved integrating over the Haar measure of the gauge group. Schematically, the index takes the form

$$\mathcal{I}(p,q,V) = \int [dU]\prod_k \mathrm{PE}[i_k(p,q,U,V)]\,. \tag{4.14}$$

This functional integral can then be reduced to an integral over the maximal torus by gauge fixing the integral over the gauge group. This gives rise to an extra factor, that is the Vandermonde determinant.
The plethystic exponential of a single-letter index that gives the multi-letter contribution to the index of a chiral multiplet can be written as a product of elliptic Gamma functions. For example, for a chiral superfield in the representation $\mathcal{R}$ of the gauge group G with R-charge R, it holds

$$\mathrm{PE}[i_r(p,q,G)] \equiv \prod_{\rho}\Gamma_e\big((pq)^{\frac{R}{2}}z^{\rho};p,q\big), \quad \text{with} \quad \Gamma_e(z;p,q) \equiv \prod_{k,m=0}^{\infty}\frac{1-p^{k+1}q^{m+1}/z}{1-p^kq^mz}, \tag{4.15}$$

where ρ runs over the weight vectors of the representation $\mathcal{R}$ of the gauge group, z indicates the fugacities for the gauge symmetries and we are using the notation $z^{\rho} = \prod_{i=1}^{\mathrm{rk}_G} z_i^{\rho^i}$.
Analogously, combining the multi-letter contribution of a vector multiplet in the adjoint representation of the gauge group G with the Haar measure of G, one obtains the building block

$$\frac{\kappa^{\mathrm{rk}_G}}{|\mathrm{Weyl}(G)|}\oint_{\mathbb{T}^{\mathrm{rk}_G}}\prod_{i=1}^{\mathrm{rk}_G}\frac{dz}{2\pi i z_i}\prod_{\alpha}\frac{1}{\Gamma(z^{\alpha};p,q)}\cdots. \tag{4.16}$$

where the integral is taken over the maximal torus of G, α parametrizes the simple roots of the gauge algebra and $|\mathrm{Weyl}(G)|$ is the cardinality of the Weyl group. The integration contour is the product of rk_G unit circles $|z_i| = 1,\ i = 1,\ldots,\mathrm{rk}_G$, and $\kappa \equiv (p;p)_\infty(q;q)_\infty$, where

$$(a;b)_\infty \equiv \prod_{k=0}^{\infty}(1-ab^k) \tag{4.17}$$

are q-Pochhammer symbols.

4.2.2 Index as a partition function

An alternative way to introduce the superconformal index is as the supersymmetric partition function on $S^3 \times S^1_\tau$ [70,71], where τ is the radius of S^1. Evaluating this partition function with localization techniques, one arrives to a result that is almost equivalent to the one obtained computing the index as a counting problem, with a difference due to the vacuum normalization.

In particular, in the large radius limit $\tau \to \infty$, the index computed using the trace formula receives contributions only from the vacua. Therefore, if one assumes that there is a unique vacuum that preserves certain global symmetries, the index in this limit is 1, due to the normalization of the vacuum.
On the other hand, in the same limit, the index calculated as a partition function receives a contribution from the Casimir energy of the theory,

$$\lim_{\tau\to\infty} \mathcal{Z}_{S^3\times S^1_\tau} \sim e^{-\tau E_{\text{Casimir}}}. \tag{4.18}$$

Thus, the two formulations of the index differ only by a multiplicative factor related to the Casimir energy.

4.3 Adding flavor symmetry

We will now present the integral form of the index for a gauge theory with flavor symmetry.

First of all, if we add flavor symmetries to (4.10), we can write the index as

$$\mathcal{I} = \text{Tr}(-1)^F p^{J_1+\frac{R}{2}} q^{J_2+\frac{R}{2}} \prod_{b=1}^{\text{rk}_F} v_b^{q_b}, \tag{4.19}$$

where $J_{1,2}$ are the angular momenta on the three-sphere written in a different basis ($J_1 = j_1 + j_2,\ J_2 = j_2 - j_1$), q_b are the conserved charges commuting with the supercharges, where the index b runs over the Cartan subgroup of the flavor symmetry group F, $b = 1, \dots, \text{rk}_F$, and v_b are the fugacities associated with q_b.

Combining (4.15) and (4.16), with the addition of flavor symmetry, we can write the superconformal index for a generic $\mathcal{N} = 1$ theory with gauge group G, flavor symmetry group F and $U(1)_R$ R-symmetry. We consider a theory with matter content consisting of n_χ chiral multiplets Φ_I with weights ρ_I in representations $\mathcal{R}_I$ of G and flavor weights ω_I in some representations $\mathcal{F}_I$ of F, and having superconformal R-charges R_I. Furthermore, we turn on flavor fugacities v_α, with $\alpha = 1, \dots, \text{rk}_F$, that parametrize the maximal torus of F. The index for such a theory is given by:

$$\mathcal{I} = \frac{(p;p)_\infty^{\text{rk}_G} (q;q)_\infty^{\text{rk}_G}}{|\text{Weyl}(G)|} \oint_{\mathbb{T}^{\text{rk}_G}} \prod_{i=1}^{\text{rk}_G} \frac{dz_i}{2\pi i z_i} \frac{\prod_{I=1}^{n_\chi} \prod_{\rho_I} \Gamma_e\big((pq)^{\frac{R_I}{2}} z^{\rho_I} v^{\omega_I}\big)}{\prod_\alpha \Gamma_e\big(z^\alpha\big)}. \tag{4.20}$$

4.4 A few examples

To give some explicit examples, in this section we report the superconformal indices for two of the theories we have introduced so far: $\mathcal{N} = 4$ Super Yang–Mills theory with $SU(N)$ gauge group and the conifold.

The superconformal index of $\mathcal{N} = 4$ Super Yang–Mills

We start with the index of $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group, following the conventions of [64]. In $\mathcal{N} = 1$ notation, the theory contains one vector multiplet and three adjoint chiral fields, each with R-charge 2/3, rotated by an $SU(3)_t$ global symmetry. The superconformal index reads:

$$\mathcal{I}_{\text{SYM}} = \frac{\kappa^{N-1}}{N!} \oint \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j} \prod_{j\neq k} \frac{\Gamma_e((pq)^{\frac{1}{3}} t_1 z_j/z_k)\Gamma_e((pq)^{\frac{1}{3}} t_2 z_j/z_k)\Gamma_e((pq)^{\frac{1}{3}} \frac{1}{t_1 t_2} z_j/z_k)}{\Gamma_e(z_j/z_k)}. \tag{4.21}$$

The superconformal index of the conifold

To give another example, here we present the superconformal index for the conifold theory, introduced in subsection 3.1.2:

$$\begin{aligned} \mathcal{I}_{T^{1,1}} = &\prod_{k=1}^{2} \left[\frac{\kappa^{N-1}}{N!} \oint \prod_{i=1}^{N-1} \frac{dz_i^{(k)}}{2\pi i z_i^{(k)}} \prod_{i\neq j} \frac{1}{\Gamma_e\left(z_i^{(k)}/z_j^{(k)}\right)} \right] \\ &\times \prod_{i,j=1}^{N} \Gamma_e\left((pq)^{\frac{1}{4}} t_1 b \frac{z_i^{(1)}}{z_j^{(2)}} \right) \cdot \Gamma_e\left((pq)^{\frac{1}{4}} \frac{b}{t_1} \frac{z_i^{(1)}}{z_j^{(2)}} \right) \cdot \Gamma_e\left((pq)^{\frac{1}{4}} \frac{t_2}{b} \frac{z_i^{(2)}}{z_j^{(1)}} \right) \cdot \Gamma_e\left((pq)^{\frac{1}{4}} \frac{1}{t_2 b} \frac{z_i^{(2)}}{z_j^{(1)}} \right). \end{aligned} \tag{4.22}$$

4.5 Black hole entropy from the superconformal index

As we mentioned in the introduction, great interest in the superconformal index comes from the fact that it can be used as a powerful tool to extract the entropy of the (putative) black hole of the dual gravitational theory.
In this section we present an outline of this procedure, mostly based on [63], starting by the characterization of the class of black holes on which our analysis is focused.

The key ingredient of our study are supersymmetric black holes asymptotic to AdS vacua that can be embedded in string theory or M-theory, with a known field theory dual. Many of them can be embedded in maximally supersymmetric backgrounds, such as $\text{AdS}_5 \times S^5$ in Type IIB string theory, or $\text{AdS}_4 \times S^7$ and $\text{AdS}_7 \times S^4$ in M-theory.
Supersymmetric black holes have zero temperature and are extremal. Moreover, they satisfy a BPS condition that imposes a relation between their mass and the other conserved charges. The relevant black holes from a holographic point of view, in dimensions $d \geq 4$, can be divided in two classes depending on whether magnetic charges are present or not. Magnetically charged black holes arise as solutions in which supersymmetry is preserved by a topological twist. It is important to notice that supersymmetric non-magnetic black holes must be rotating, in order to avoid singularities.

4.5.1 Five-dimensional Kerr–Newman black holes

In this work we will mainly focus on supersymmetric electrically charged rotating black holes, known as Kerr–Newman black holes, asymptotic to an AdS_5 vacuum. The original examples were the ones embedded in Type IIB supergravity with $\mathrm{AdS}_5 \times S^5$ background, originally found by Gutowski and Reall [10] and later generalized in [14,72–74]. They are characterized by two angular momenta, corresponding to the Cartan isometries of AdS_5

$$(j_1, j_2): \qquad U(1)^2 \subset SO(4) \subset SO(4,2)\,, \tag{4.23}$$

and three electric charges coming from the Cartan isometries of S^5

$$(q_1, q_2, q_3): \qquad U(1)^3 \subset SO(6)\,, \tag{4.24}$$

that parametrize the rotation of the five-sphere, representing the internal space. Furthermore, there is an extra constraint between the conserved charges,

$$f(j_1, j_2, q_1, q_2, q_3) = 0, \tag{4.25}$$

coming from supersymmetry, that reduces the number of independent parameters. Since these black holes preserves two real supercharges out of the thirty-two of Type IIB supergravity, they are called 1/16-BPS. They are five-dimensional black holes, asymptotically AdS_5, with $S^3 \times \mathbb{R}_t$ conformal boundary.

We have seen that Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ is dual to $\mathcal{N} = 4$ Super Yang–Mills in four dimensions. Hence there must be an ensemble of states of $\mathcal{N} = 4$ SYM on $S^3 \times \mathbb{R}_t$ that are holographically dual to the black holes above. These states must preserve the same supersymmetries and have the same angular momenta and electric charges of the black holes. We expect that from the counting of these states one should be able to reproduce the entropy of the black holes.
We know that the entropy of this class of black holes can be written as [75]

$$S(q_a, j_i) = 2\pi \sqrt{q_1 q_2 + q_1 q_3 + q_2 q_3 - \frac{\pi}{4 G_N^{(5)} g^3}(j_1 + j_2)}\,. \tag{4.26}$$

The holographic dictionary relates the gravitational quantities to the number of colors N of the dual four-dimensional field theory through

$$N^2 = \frac{\pi}{2 G_N^{(5)} g^3}, \tag{4.27}$$

from which we can see that black holes with charges and angular momenta that scale as $\mathcal{O}(N^2)$ have an entropy of order $\mathcal{O}(N^2)$. It is worth pointing out that this entropy can be obtained as the Legendre transform of the following quantity, as noted in [13]:

$$S_E = \pi i N^2 \frac{\Delta_1 \Delta_2 \Delta_3}{\omega_1 \omega_2} \tag{4.28}$$

with the constraint among the chemical potentials dual to q_a and j_i

$$\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = \pm 1\,, \tag{4.29}$$

arising from regularity conditions on the Killing spinors. Thus, the entropy takes the form

$$S(q_a, j_i) = \pi i N^2 \frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2} + 2\pi i \left(\sum_{a=1}^{3} \Delta_a q_a + \sum_{i=1}^{2} \omega_i j_i\right)\Bigg|_{\bar{\Delta}_a, \bar{\omega}_i}\,, \tag{4.30}$$

which is extremized for complex values of the chemical potentials. Nevertheless, the on-shell value of (4.30) becomes real if we impose the constraint among the charges given in (4.25), required by supersymmetry. The quantity S_E in (4.28) is known as *entropy function* and can be obtained from the partition function of the dual field theory, as we will show in a few paragraphs.

We conclude this subsection by observing that, besides black holes that can be embedded in maximally supersymmetric backgrounds, similar supersymmetric black holes in more general Type IIB or M-theory are expected to exist, even though few examples are known in literature. For example, $\mathcal{N} = 1$ four-dimensional quiver gauge theories that we have introduced in the previous chapter should be dual to Type IIB rotating black holes in $\mathrm{AdS}_5 \times X_5$, with X_5 a five-dimensional Sasaki–Einstein manifold, charged under the isometries of X_5.

4.5.2 Computation of the entropy

Field theory BPS states can be enumerated by the grand canonical partition function

$$\mathcal{Z}(\Delta_a, \omega_i) = \mathrm{Tr}\Big|_{\mathcal{Q}=0} e^{i(\Delta_a Q_a + \omega_i J_i)} = \sum_{q_a, j_i} c(q_a, j_i) e^{i(\Delta_a q_a + \omega_i j_i)}\,, \tag{4.31}$$

where Q_a and J_i are the charge operators associated with the global symmetries and angular momenta of the field theory and Δ_a and ω_i are the chemical potential conjugated to them. The trace is taken over the Hilbert states on the three-sphere[2] that preserve the same supersymmetries of the black holes; supersymmetric states are in fact annihilated by the supercharges $\mathcal{Q}$. The coefficient $c(q_a, j_i)$ corresponds to the number of supersymmetric states with electric charge q_a and angular momentum j_i.
This partition function should also count the BPS states in the dual gravitational theory. Not all of these states are black holes microstates. Nevertheless, for large charges the dominant contribution to the density of supersymmetric states is given by macroscopic black holes, and thus we can write the black hole entropy as

$$S(q_a, j_i) = \log c(q_a, j_i)\,. \tag{4.32}$$

More in detail, in terms of the field theory data, the entropy of AdS_5 black holes scales as $\mathcal{O}(N^2)$, where N represents the number of colors of the gauge group of the dual field theory.

[2]This procedure can be also applied to the other supersymmetric AdS_d black holes that we have introduced, by counting the corresponding states in the dual field theory on $\mathbb{R} \times M_{d-2}$, that represents the conformal boundary of AdS_d. In this language, the trace has to be taken over the states on M_{d-2}.

Looking at (4.32), in principle, the entropy can be extracted as a Fourier coefficient from the grand canonical partition function, i.e.

$$e^{S(q_a,j_i)} = c(q_a, j_i) = \int \frac{d\Delta_a}{2\pi}\frac{d\omega_i}{2\pi}\mathcal{Z}(\Delta_a,\omega_i)e^{-i(\Delta_a q_a+\omega_i j_i)} \tag{4.33}$$

using a proper integration contour. In the limit of large charges, this integral can be computed using a saddle point approximation

$$S(q_a, j_i) = \log \mathcal{Z}(\Delta_a,\omega_i) - i(\Delta_a q_a + \omega_i j_i)\Big|_{\bar{\Delta}_a,\overline{\omega}_i}, \tag{4.34}$$

evaluated at the saddles $\bar{\Delta}_a$ and $\overline{\omega}_i$, obtained from the extremization of the functional

$$\mathcal{F}(\Delta_a,\omega_i) = \log \mathcal{Z}(\Delta_a,\omega_i) - i(\Delta_a q_a + \omega_i j_i)\,, \tag{4.35}$$

with respect to Δ_a and ω_i. In other words, we can see that the black hole entropy is given by the Legendre transform of the logarithm of the partition function of the dual field theory.

The problem with this procedure lies in the knowledge of the grand canonical partition function itself, which is in general too involved to compute. Nevertheless, as we mentioned above, it has been shown that this issue can be overcome by means of the superconformal index.
The four-dimensional superconformal index is the perfect candidate to reproduce the entropy of the class of black holes that we are considering since, as we saw, it can be identified with the supersymmetric partition function on $S^3 \times S^1$. For example, the superconformal index of $\mathcal{N} = 4$ Super Yang–Mills with $SU(N)$ gauge group counts the 1/16-BPS states dual to the microstates of the charged and rotating black hole in $\mathrm{AdS}_5 \times S^5$. However, this expectation has been puzzling for more than a decade, because the large-N index was found to be an order-one quantity, instead of order-N^2, as expected from the holographic dictionary. This is due to the large cancellation between fermionic and bosonic states counted by the $(-1)^F$ operator in the index. A solution to this problem was obtained only recently by noticing that allowing for complex fugacities an obstruction to such cancellation appears and the dual black hole entropy can indeed be extracted from the index. Two main approaches to compute the entropy from the index have been developed, implemented in the two following limits:

- The Cardy-like limit [11,76–79], in which the fugacities associated with rotations are taken to be very small, corresponding to $\omega_i \ll 1$ at fixed complex values of Δ_a, and thus to large black holes with charges scaling as

  $$\omega_1 \sim \omega_2 \sim \omega, \qquad q_a \sim \frac{1}{\omega^2}, \qquad j_i \sim \frac{1}{\omega^3}\,. \tag{4.36}$$

  A crucial point is that the chemical potentials are complex; their imaginary parts introduce phases that obstruct the cancellations between fermionic and bosonic states, allowing to reproduce the expected black hole entropy. We will focus on this limit in the next chapter.

- Large-N and equal angular momenta [12,80], in which the index is written in terms of a set of solutions to the so-called Bethe Ansatz Equations (BAE).

Many generalizations of these results have since then appeared [81–90], showing that the four-dimensional superconformal index can indeed be used to enumerate the microstates of the dual AdS_5 black hole. More in detail, the logarithm of the leading term of the index, in the large-N limit, gives exactly the entropy function associated with the dual black hole.

In this work we will only focus on the four-dimensional superconformal index. Nevertheless, similar indices have also been studied in different dimensions. For example, to reproduce the entropy of supersymmetric magnetically charged black holes in AdS_4, which are dual to topologically twisted superconformal field theories in three dimensions, the three-dimensional *topologically twisted index* was introduced [91–95].

Chapter 5

Subleading corrections to the Cardy-like limit of the superconformal index

In the Cardy-like limit, the chemical potentials associated with the angular momenta are taken to be very small. It can be thought as a generalization of the Cardy-limit [96], proposed for two-dimensional conformal field theories, in which the temperature in sent to infinity, corresponding to $\beta \to 0$.
As me mentioned at the end of the previous chapter, the Cardy-like limit of the superconformal index has played a crucial role in the understanding of how to extract the black hole entropy starting from the superconformal index. Indeed, in [11] the authors managed to reproduce the expected large-N behavior of the index, using an approach that relies on an appropriate analytic continuation of the fugacities to complex values, such as to obstruct the cancellations induced by the operator $(-1)^F$.

Many other examples and generalizations have since been worked out. It was observed that taking a Cardy-like limit of the index one could reproduce the entropy function expected from supergravity in various classes of models with a known holographic dual description. Furthermore, it was showed that this limit is controlled by universal combinations of the coefficients of the Weyl and Euler densities, i.e. by $\mathrm{Tr}\, R$ and $\mathrm{Tr}\, R^3$, calculated in terms of an opportunely defined set of charges, that generalize the R-charges (of the matter fields) to the curved background. These results have been extended in [76–79, 82–90, 97–109].
An interesting recent direction regards the calculation of subleading effects that correct the index. Such corrections have been studied in large detail in [106] for $\mathcal{N} = 4$ $SU(N_c)$ SYM and for the generalization to $\mathcal{N} = 1$ gauge theories representing a stack of D3-branes probing a toric Calabi–Yau three-fold singularity. The calculation has been carried out both in the Cardy-like limit, using a saddle point approximation to the matrix integral, and in the BAE approach, finding agreement between the two descriptions at large N_c. It has been observed that the leading saddle contributing to the index for an $\mathcal{N} = 1$ $SU(N_c)$ theory is corrected by a $\log N_c$ term (see [106, Eq. (3.53)]), an appealing result that should be recovered in a supergravity calculation. The presence of a $\log N_c$ correction is related to the $\mathbb{Z}_{N_c}$ center symmetry of $SU(N_c)$, as discussed in [101].
An analogous calculation in $USp(2N_c)/SO(2N_c + 1)$ and $SO(2N_c)$ gauge theories should then yield a $\log 2$ and $\log 4$ correction respectively.[1] In fact these are the dimensions of the centers of the universal covering groups $USp(2N_c)$ and $Spin(N_c)$ (2 or 4 for the latter, for odd and even

[1] See [110, 111] for similar results in 3d, where the center symmetry determines the logarithmic correction.

N_c respectively.) In all models considered in this paper we only have matter fields in the adjoint representation of the gauge group, and these do not break the center symmetry. (Moreover only the gauge algebra is captured by the SCI.[2]) In fact, as we will comment later, for models with other matter representations charged under the center the logarithmic correction corresponds to the order of the character lattice of the gauge algebra modulo the action of the Weyl symmetry. An analogous result has been discussed in [112] in terms of a spontaneously broken one-form symmetry.

Motivated by this expectation in the first part of this chapter we study the logarithmic corrections to the leading saddle contribution to the SCI of 4d $\mathcal{N} = 4$ SYM with symplectic and orthogonal gauge group. We find the expected $\log 2$ and $\log 4$ corrections to the (logarithm of the) SCI. As already noted in [106], we find that expanding the index in the Cardy-like limit one recovers a matrix integral that coincides with the three-sphere partition function of a 3d pure Chern–Simons (CS) theory. In the cases at hand the CS theories have gauge group $USp(2N_c)_{\pm(N_c+1)}$, $SO(2N_c+1)_{\pm(2N_c-1)}$, and $SO(2N_c)_{\pm 2(N_c-1)}$ (the subscript representing the CS level) and this integral can be evaluated exactly. The sign choice is related to a constraint (first discussed in [13, 113]) satisfied by the chemical potentials appearing in the SCI. Furthermore, in the $USp(2N_c)$ case we analyze in more detail the solutions of the saddle point equations, finding other subleading saddles. We analyze the Cardy-like limit for these solutions as well. All the models that we will study are examples of 4d *non-toric* gauge theories. Another interesting non-toric theory that we focus on is the Leigh–Strassler (LS) $\mathcal{N} = 1^*$ $SU(N_c)$ fixed point [114], for which we extract the contribution of the leading saddle to the index in the Cardy-like limit. We find that the entropy function, yielding the entropy of the holographic dual BH after a Legendre transform, is consistent with the result expected from the literature [11–13, 76–79, 82, 85, 98, 105], i.e. is formally obtained from the 4d central charge a. Furthermore, we extract the $\log N_c$ correction, consistently with the one obtained for the parent $\mathcal{N} = 4$ $SU(N_c)$ SYM. This part of the chapter is based on [115].

In the second part of this chapter we further extend our results and the ones of [106] to generic $\mathcal{N} = 1$ gauge theories with ABCD gauge algebra, focusing again on the case where the two fugacities associated with rotations are identified. Once again we find that the index in the Cardy-like limit is controlled by the traces $\mathrm{Tr}\, R^3$ and $\mathrm{Tr}\, R$, weighted by two factors that are universal in terms of the fugacity associated with the rotation parameter. Furthermore we find that there is a logarithmic correction related to the charges of the matter fields under the center of the gauge symmetry. (We will elaborate further on this point in section 5.4.) The main result is formula (5.80). This result is valid both for (non-toric) theories with $\mathrm{Tr}\, R = \mathcal{O}(1)$, i.e. for models that allow a weakly-coupled gravitational dual description, and for models with $\mathrm{Tr}\, R = \mathcal{O}(N_c^2)$, with N_c the rank of the gauge algebra. The second part of the chapter is based on [116].

This chapter is structured as follows. In section 5.1 we give a lighting review of the Cardy-like limit of the SCI for $\mathcal{N} = 1$ gauge theories. In section 5.2 we calculate the Cardy-like limit of the $\mathcal{N} = 4$ SCI for all classical gauge groups except $SU(N_c)$. In subsection 5.2.1 we focus on the $USp(2N_c)$ case, computing dominant contribution and subleading correction for the leading

[2] Keeping this in mind, in the rest of the paper we will be referring to the SCI of $SO(N_c)$ instead of that of $Spin(N_c)$.

(and other subleading) saddle(s). In subsection 5.2.2 we focus on the $SO(2N_c + 1)$ odd case, while in subsection 5.2.2 on the $SO(2N_c)$ even case. In section 5.3 we compute the Cardy-like limit of the SCI of the $\mathcal{N} = 1^*$ $SU(N_c)$ LS fixed point. Appendix A contains technical details on the calculation of three-dimensional pure Chern–Simons partition functions. In section 5.4 we propose and give a formal argument supporting our main formula (5.80) for the Cardy-like limit of generic $\mathcal{N} = 1$ theories with ABCD gauge algebra, including finite-order corrections, generalizing preexisting results. In later sections we test and validate this formula in a series of examples: holographic $\mathcal{N} = 1$ SCFTs (section 5.5), $\mathcal{N} = 1$ SCFTs without a weakly-coupled gravity dual (section 5.6), $\mathcal{N} = 2$ SCFTs (section 5.7).

5.1 Expanding on the Cardy-like limit

Following the strategy of [106], we start by rewriting the integral formula (4.20) in terms of modified elliptic Gamma functions $\tilde{\Gamma}$. This is done by expressing the holonomies and various fugacities as

$$p = e^{2\pi i\sigma}, \quad q = e^{2\pi i\tau}, \quad v_b = e^{2\pi i\xi_b}, \quad z_i = e^{2\pi i u_i}, \tag{5.1}$$

with $u_i \in (0, 1]$ and $0 \sim 1$. The R-symmetry chemical potential is given by the relation

$$v_R = \frac{1}{2}(\tau + \sigma)\ . \tag{5.2}$$

The modified elliptic Gamma functions are then

$$\tilde{\Gamma}(u; \tau, \sigma) = \tilde{\Gamma}(u) \equiv \Gamma_e(e^{2\pi i u}; e^{2\pi i\tau}, e^{2\pi i\sigma})\ , \tag{5.3}$$

such that the index (4.20) becomes

$$\mathcal{I}(\tau, \sigma, \Delta) = \frac{(p;p)_\infty^{\mathrm{rk}_G}(q;q)_\infty^{\mathrm{rk}_G}}{|\mathrm{Weyl}(G)|} \int_{\mathcal{C}} \prod_{i=1}^{\mathrm{rk}_G} du_i \frac{\prod_{I=1}^{n_\chi} \prod_{\rho_I} \tilde{\Gamma}(\rho_I(\vec{u}) + \Delta_I)}{\prod_\alpha \tilde{\Gamma}(\alpha(\vec{u}))} \tag{5.4}$$

where $\mathcal{C} = \bigcup_{i=1}^{\mathrm{rk}_G}(0, 1]$ and

$$\Delta_I \equiv \omega_I(\vec{\xi}) + R_I v_R. \tag{5.5}$$

There is one chemical potential Δ_I for each field in the theory, and they must satisfy the relations imposed by global symmetries, i.e. each superpotential term is uncharged under the flavor symmetry and it has R-charge two.

Next we restrict to the case $\tau = \sigma$ and expand the index in the Cardy-like limit $|\tau| \to 0$ at fixed $\arg \tau \in (0, \pi)$. In order to evaluate the index in this limit it is convenient to rewrite it as a matrix model by introducing the effective action S_{eff} through

$$\mathcal{I}_{\text{sc}}(\tau, \Delta) \equiv \frac{1}{|\mathrm{Weyl}(G)|} \int \prod_{i=1}^{\mathrm{rk}_G} du_i\, e^{S_{\text{eff}}(\vec{u};\tau,\Delta)}\ . \tag{5.6}$$

For a model with n_G gauge groups G_a and a set of n_χ matter fields Φ_I, the effective action takes the form

$$\begin{aligned} S_{\text{eff}}(\vec{u};\tau,\Delta) = &\sum_{I=1}^{n_\chi}\sum_{\rho_I}\log\tilde{\Gamma}\big(\rho_I(\vec{u})+\Delta_I\big) + \sum_{a=1}^{n_G}\sum_{\alpha_a}\log\theta_0\big(\alpha_a(\vec{u});\tau\big) \\ &+\sum_{a=1}^{n_G} 2\,\mathrm{rk}_{G_a}\log(q;q)_\infty\,. \end{aligned} \tag{5.7}$$

Observe that $\sum_{\rho_I}\rho_I(\vec{u})$ is a formal expression that repackages the sum over the weights of the representation $\mathcal{R}_I$. More explicitly let us consider a function f and a field Φ_I in the representation $\mathcal{R}_I$ of the gauge group: expressing the weights as $w_j(\vec{u})$, where $j=1,\dots,\dim\mathcal{R}_I$, we will write

$$f(\rho_I(\vec{u})) \equiv \sum_{j=1}^{\dim\mathcal{R}_I} f(w_j(\vec{u}))\,. \tag{5.8}$$

The notation $\sum_{\alpha_a}\alpha_a(\vec{u})$ then refers to the (sum of the) roots of the gauge group G_a, i.e. the weights of the adjoint representation. Moreover in (5.7) we introduced the elliptic theta function

$$\theta_0(u;\tau) \equiv \prod_{k=0}^{\infty}(1-e^{2\pi i(u+k\tau)})(1-e^{2\pi i(-u+(k+1)\tau)})\,, \tag{5.9}$$

which satisfies $\log\theta_0(u;\tau) = -\log\tilde{\Gamma}(u)$.[3]
Let us now define the τ-modded value of a complex $\mathbb{C}\ni u\equiv\tilde{u}+\tau\check{u}$ (with $\tilde{u},\check{u}\in\mathbb{R}$):

$$\{u\}_\tau \equiv u - \lfloor \mathrm{Re}(u) - \cot(\arg\tau)\,\mathrm{Im}(u)\rfloor\,, \tag{5.10}$$

where $\lfloor\cdot\rfloor$ is the floor function (of a real number). It satisfies

$$\{u\}_\tau = \{\tilde{u}\}_\tau + \tau\check{u}\,,\quad \{-u\}_\tau = \begin{cases} 1-\{u\}_\tau & \tilde{u}\notin\mathbb{Z} \\ -\{u\}_\tau & \tilde{u}\in\mathbb{Z}\end{cases}\,, \tag{5.11}$$

and for a real number $\tilde{u}$ it reduces to the usual modded value $\{\tilde{u}\}\equiv\tilde{u}-\lfloor\tilde{u}\rfloor$. At small $|\tau|$ and fixed $\arg\tau\in(0,\pi)$ we have the following asymptotic formulae (see e.g. [106, App. A]):

$$\log\,(q;q)_\infty = -\frac{i\pi}{12}\left(\tau+\frac{1}{\tau}\right) - \frac{1}{2}\log(-i\tau) + \mathcal{O}\left(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}}\right)\;; \tag{5.12}$$

$$\begin{aligned} \log\theta_0(u;\tau) = &\frac{\pi i}{\tau}\{u\}_\tau(1-\{u\}_\tau) + \pi i\{u\}_\tau - \frac{\pi i}{6\tau}(1+3\tau+\tau^2) \\ &+\log\left(\left(1-e^{-\frac{2\pi i}{\tau}(1-\{u\}_\tau)}\right)\left(1-e^{-\frac{2\pi i}{\tau}(\{u\}_\tau)}\right)\right) \\ &+\mathcal{O}\left(e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}}\right)\;; \end{aligned} \tag{5.13}$$

$$\log\tilde{\Gamma}(u) = 2\pi i Q(\{u\}_\tau;\tau) + \mathcal{O}\left(|\tau|^{-1}e^{-\frac{2\pi\sin(\arg\tau)}{|\tau|}\min(\{\tilde{u}\},1-\{\tilde{u}\})}\right)\,, \tag{5.14}$$

[3] To prove this identity, one can follow the steps explained below [106, Eq. (3.2)].

provided $\tilde{u} \nrightarrow \mathbb{Z}$. We will also need the quantity

$$Q(u;\tau) \equiv -\frac{B_3(u)}{6\tau^2} + \frac{B_2(u)}{2\tau} - \frac{5}{12}B_1(u) + \frac{\tau}{12}\,, \tag{5.15}$$

defined in terms of the Bernoulli polynomials[4]

$$B_3(u) = u^3 - \frac{3}{2}u^2 + \frac{1}{2}u\,, \quad B_2(u) = u^2 - u + \frac{1}{6}\,, \quad B_1(u) = u - \frac{1}{2}\,. \tag{5.17}$$

Using the above asymptotics we can expand the effective action (5.7) in τ for small $|\tau|$, and compute its saddle point equations *at leading order*:

$$0 = \frac{\partial S_{\text{eff}}(\vec{u};\tau,\Delta)}{\partial u_{i_a}} = -\frac{i\pi}{\tau^2}\sum_{I=1}^{n_\chi}\sum_{\rho_I}\frac{\partial \rho_I(\vec{u})}{\partial u_{i_a}}B_2(\{\rho_I(\vec{u})+\Delta_I\}_\tau)\,, \tag{5.18}$$

where u_{i_a} represents the i-th holonomy in the a-th gauge group, with $i_a = 1,\ldots,\text{rk}_{G_a}$ and $a = 1,\ldots,n_G$. This is a set of $\sum_{a=1}^{n_G}\text{rk}_{G_a}$ equations. We then look for solutions $\vec{u}$, namely the saddle points of the matrix model, which contain a constant part and a linear term in τ, i.e. we make an ansatz for the solutions of the form

$$\vec{u} = \left\{u_{i_a} = u_{*i_a} + \bar{u}_{i_a} \equiv u_{*i_a} + v_{i_a}\tau \;\middle|\; v_{i_a} \sim \mathcal{O}(|\tau|^0)\right\}\,. \tag{5.19}$$

We do this to capture the terms at finite order in τ in the expansion. In fact, when we plug this ansatz back into (5.7), we obtain leading and subleading contributions in τ, logarithmic corrections as well as finite terms.

5.2 $\mathcal{N}=4$ Super Yang–Mills with real gauge groups

Motivated by this expectation presented above, in this section we study the logarithmic corrections to the leading saddle contribution to the SCI of 4d $\mathcal{N}=4$ SYM with symplectic and orthogonal gauge group. We find the expected $\log 2$ and $\log 4$ corrections to the (logarithm of the) SCI.[5] As already noted in [106], we find that expanding the index in the Cardy-like limit one recovers a matrix integral that coincides with the three-sphere partition function of a 3d pure Chern–Simons (CS) theory. In the cases at hand the CS theories have gauge group $USp(2N_c)_{\pm(N_c+1)}$, $SO(2N_c+1)_{\pm(2N_c-1)}$, and $SO(2N_c)_{\pm 2(N_c-1)}$ (the subscript representing the CS level) and this integral can be evaluated exactly. The sign choice is related to a constraint (first discussed in [13, 113]) satisfied by the chemical potentials appearing in the SCI.
Furthermore in the $USp(2N_c)$ case we analyze in more detail the solutions of the saddle point equations finding other subleading saddles. We analyze the Cardy-like limit for these solutions

[4] In general, an explicit formula for the Bernoulli polynomials is given by

$$B_m(x) = \sum_{n=0}^{m}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+k)^m. \tag{5.16}$$

[5] The leading saddle is the one that dominates the index in the regime of charges that reproduces the entropy of the holographic dual black hole, corresponding to the M-wing in the terminology of [77]. See also [117] for an explicit matching of the saddles in the W-wing of the S-dual SO/USp $\mathcal{N}=4$ theories.

as well. All the models studied in this section are examples of 4d *non-toric* gauge theories. Another interesting non-toric theory that we focus on is the Leigh–Strassler (LS) $\mathcal{N}=1^*$ $SU(N_c)$ fixed point [114], for which we extract the contribution of the leading saddle to the index in the Cardy-like limit. We find that the entropy function, yielding the entropy of the holographic dual BH after a Legendre transform, is consistent with the result expected from the literature [11–13, 76–79, 82, 85, 98, 105], i.e. is formally obtained from the 4d central charge a. Furthermore, we extract the $\log N_c$ correction, consistently with the one obtained for the parent $\mathcal{N}=4$ $SU(N_c)$ SYM.

The SCI expressed in terms of modified elliptic Gamma functions in these cases reads:

- $G = USp(2N_c)$:

$$\mathcal{I}^{USp(2N_c)} = \frac{(p;p)_\infty^{N_c}(q;q)_\infty^{N_c}}{2^{N_c}N_c!}\prod_{a=1}^{3}\tilde{\Gamma}^{N_c}(\Delta_a)$$
$$\times\int\prod_{i=1}^{N_c}du_i\frac{\prod_{a=1}^{3}\prod_{i<j}^{N_c}\tilde{\Gamma}(\pm u_i\pm u_j+\Delta_a)}{\prod_{i<j}^{N_c}\tilde{\Gamma}(\pm u_i\pm u_j)}\cdot\frac{\prod_{a=1}^{3}\prod_{i=1}^{N_c}\tilde{\Gamma}(\pm 2u_i+\Delta_a)}{\prod_{i=1}^{N_c}\tilde{\Gamma}(\pm 2u_i)}\,, \tag{5.20}$$

where we used the shorthand $f(a\pm b)\equiv f(a+b)f(a-b)$ (and likewise for $f(\pm a\pm b)$).

- $G = SO(2N_c+1)$:

$$\mathcal{I}^{SO(2N_c+1)} = \frac{(p;p)_\infty^{N_c}(q;q)_\infty^{N_c}}{2^{N_c}N_c!}\prod_{a=1}^{3}\tilde{\Gamma}^{N_c}(\Delta_a)$$
$$\times\int\prod_{i=1}^{N_c}du_i\frac{\prod_{a=1}^{3}\prod_{i<j}^{N_c}\tilde{\Gamma}(\pm u_i\pm u_j+\Delta_a)}{\prod_{i<j}^{N_c}\tilde{\Gamma}(\pm u_i\pm u_j)}\frac{\prod_{a=1}^{3}\prod_{i=1}^{N_c}\tilde{\Gamma}(\pm u_i+\Delta_a)}{\prod_{i=1}^{N_c}\tilde{\Gamma}(\pm u_i)}\,. \tag{5.21}$$

- $G = SO(2N_c)$:

$$\mathcal{I}^{SO(2N_c)} = \frac{(p;p)_\infty^{N_c}(q;q)_\infty^{N_c}}{2^{N_c-1}N_c!}\prod_{a=1}^{3}\tilde{\Gamma}^{N_c}(\Delta_a)$$
$$\times\int\prod_{i=1}^{N_c}du_i\frac{\prod_{a=1}^{3}\prod_{i<j}^{N_c}\tilde{\Gamma}(\pm u_i\pm u_j+\Delta_a)}{\prod_{i<j}^{N_c}\tilde{\Gamma}(\pm u_i\pm u_j)}\,. \tag{5.22}$$

Using (5.7), we can write the effective action for each case as:

- $G = USp(2N_c)$:

$$S_{\text{eff}}^{USp(2N_c)} = \sum_{i\neq j}\left(\left(\sum_{a=1}^{3}\log\tilde{\Gamma}(u_{ij}^{(\pm)}+\Delta_a)\right)+\log\theta_0(u_{ij}^{(\pm)};\tau)\right)$$
$$+\sum_{i=1}^{N_c}\left(\left(\sum_{a=1}^{3}\log\tilde{\Gamma}(\pm 2u_i+\Delta_a)\right)+\log\theta_0(\pm 2u_i;\tau)\right)$$
$$+N_c\sum_{a=1}^{3}\log\tilde{\Gamma}(\Delta_a)+2N_c\log(q;q)_\infty\,. \tag{5.23}$$

• $G = SO(2N_c+1)$:

$$
\begin{aligned}
S_{\text{eff}}^{SO(2N_c+1)} = & \sum_{i\neq j}\left(\Big(\sum_{a=1}^{3}\log\tilde{\Gamma}(u_{ij}^{(\pm)}+\Delta_a)\Big)+\log\theta_0(u_{ij}^{(\pm)};\tau)\right)\\
&+\sum_{i=1}^{N_c}\left(\Big(\sum_{a=1}^{3}\log\tilde{\Gamma}(\pm u_i+\Delta_a)\Big)+\log\theta_0(\pm u_i;\tau)\right)\\
&+N_c\sum_{a=1}^{3}\log\tilde{\Gamma}(\Delta_a)+2N_c\log(q;q)_\infty\,.
\end{aligned} \tag{5.24}
$$

• $G = SO(2N_c)$:

$$
\begin{aligned}
S_{\text{eff}}^{SO(2N_c)} = & \sum_{i\neq j}\left(\Big(\sum_{a=1}^{3}\log\tilde{\Gamma}(u_{ij}^{(\pm)}+\Delta_a)\Big)+\log\theta_0(u_{ij}^{(\pm)};\tau)\right)\\
&+N_c\sum_{a=1}^{3}\log\tilde{\Gamma}(\Delta_a)+2N_c\log(q;q)_\infty\,.
\end{aligned} \tag{5.25}
$$

In the above expressions we have defined the shorthands

$$
\tilde{\Gamma}(u_{ij}^{(\pm)}+\Delta_a)\equiv\tilde{\Gamma}(u_i+u_j+\Delta_a)\tilde{\Gamma}(u_i-u_j+\Delta_a)\,, \tag{5.26}
$$

$$
\theta_0(u_{ij}^{(\pm)};\tau)\equiv\theta_0(u_{ij}^{(+)};\tau)\,\theta_0(u_{ij}^{(-)};\tau)\,. \tag{5.27}
$$

5.2.1 Symplectic gauge group

Let us start our analysis with the $USp(2N_c)$ case. The effective action in this case is (5.23). In this subsection we study the solutions to the saddle point equations $\frac{\partial}{\partial u_i}S_{\text{eff}}=0$ for the $USp(2N_c)$ case. These equations read:[6]

$$
\begin{aligned}
\sum_{a=1}^{3}\sum_{j=1}^{N_c}\Bigg(&B_2(\{u_{ij}^{(\pm)}+\Delta_a\}_\tau)-B_2(\{-u_{ij}^{(\pm)}+\Delta_a\}_\tau)+\\
&+B_2(\{2u_i+\Delta_a\}_\tau)-B_2(\{-2u_i+\Delta_a\}_\tau\Bigg)=0\,,
\end{aligned} \tag{5.28}
$$

for $i=1,\ldots,N_c$. We have found three sets of solutions.[7]

[6]Given (5.26), we have:

$$
\log\tilde{\Gamma}(u_{ij}^{(\pm)})=\log\tilde{\Gamma}(u_{ij}^{(+)})+\log\tilde{\Gamma}(u_{ij}^{(-)})\sim Q(u_{ij}^{(+)};\tau)+Q(u_{ij}^{(-)};\tau)\sim B_3(u_{ij}^{(+)})+B_3(u_{ij}^{(-)})+\ldots\,.
$$

Then in the following equation by $B_2(\{u_{ij}^{(\pm)}+\Delta_a\}_\tau)$ we mean $B_2(\{u_i+u_j+\Delta_a\}_\tau)+B_2(\{u_i-u_j+\Delta_a\}_\tau)$, and so on.

[7]Observe that we are not claiming that these are the only solutions; other isolated or continuous (sets of) solutions are possible for nongeneric values of Δ_a, compatibly with the constraint $\sum_a\Delta_a=2$. At any rate we will not investigate such sporadic possibilities.

i) The first set consists of L holonomies at $u=0$ and the remaining $K \equiv N_c - L$ at $u = \frac{1}{2}$. When studying the τ-expansion of the index for these solutions we will distinguish two cases. The first one consists of considering either all the holonomies at 0 or at $\frac{1}{2}$. We will see that they give the dominating contribution to the superconformal index, capturing the entropy function of the dual rotating black hole under the holographic correspondence. The other saddles correspond to subleading effects in this regime and their contributions are paired, i.e. the contribution of the saddle given by L holonomies at 0 and K holonomies at $\frac{1}{2}$ is equivalent to the contribution of K holonomies at 0 and L holonomies at $\frac{1}{2}$. In the case of N_c even there is also a single solution with $L = K$.

ii) The second set of solutions corresponds to placing L holonomies at $u = \frac{1}{4}$ and the remaining $K = N_c - L$ at $u = \frac{3}{4}$. By a symmetry argument we can actually send $u_i \to -u_i$, and this is equivalent to considering all the holonomies at $u = \frac{1}{4}$.

iii) The last possibility consists of considering P holonomies at $u=0$, P holonomies at $u = \frac{1}{2}$, and the remaining $Q \equiv N_c - 2P$ at $u = \frac{1}{4}$. Observe that if $Q = 0$ (which is possible only for even N_c) this case is equivalent to the first with $L = K$.

In the following we expand the effective action S_{eff} around these saddles.

Leading saddle: N_c **coincident holonomies at** $u_i = 0$ **or** $u_i = \frac{1}{2}$

The ansatz for the saddle point in this case is

$$\vec{u} = \left\{ u_j^{(m)} = \frac{m}{2} + \overline{u}_j \equiv \frac{m}{2} + v_j \tau \right\} \quad \text{with} \quad m = 0, 1 \, , \tag{5.29}$$

i.e. we have two possible sets of saddle point holonomies, consistently with the fact that the center of $USp(2N_c)$ is $\mathbb{Z}_2$. Expanding around the saddle point, the effective action becomes

$$\begin{aligned} S_{\text{eff}}|_{\vec{u}=\{0\}_{N_c} \text{ or } \{\frac{1}{2}\}_{N_c}} &= -\frac{(2i\pi\eta(N_c+1)) \sum_{i=1}^{N_c} \overline{u}_i^2}{\tau^2} \\ &+ \sum_{j \neq k} \log\left(2\sin\left(\frac{\pi \overline{u}_{jk}{}^{(\pm)}}{\tau}\right)\right) + 2\sum_{i=1}^{N_c} \log\left(2\sin\left(\frac{2\pi\overline{u}_i}{\tau}\right)\right) \\ &+ \frac{i\pi(6-5\eta)\left(2N_c^2+N_c\right)}{12} - i\pi N_c^2 \\ &- \frac{i\pi N_c(2N_c+1)}{\tau^2} \prod_{a=1}^{3} \left(\Delta_a - \frac{\eta+1}{2}\right) - N_c \log(\tau) \, . \end{aligned} \tag{5.30}$$

Making the change of variables $-i\sigma_j \equiv v_j \tau$, the SCI becomes

$$\mathcal{I}^{USp(2N_c)} = 2\tau^{N_c} e^{-i\pi \frac{N_c(2N_c+1)}{2}} \mathcal{I}_0^{USp(2N_c)} Z_{S^3}^{USp(2N_c)_{-\eta(N_c+1)}} \, , \tag{5.31}$$

where the last contribution corresponds to the three-sphere partition function of a 3d $USp(2N_c)$ pure Chern–Simons theory at level $-\eta(N_c+1)$.[8] We also defined

$$\mathcal{I}_0^{USp(2N_c)} \equiv \exp\left[-\frac{i\pi N_c(2N_c+1)}{\tau^2} \prod_{a=1}^{3} \left(\Delta_a - \frac{\eta+1}{2} \right) + \right. \\ \left. + \frac{1}{12} i\pi(6-5\eta)\left(2N_c^2+N_c\right) - i\pi N_c^2 - N_c \log(\tau) \right] . \quad (5.32)$$

We can evaluate $Z_{S^3}^{USp(2N_c)_{-\eta(N_c+1)}}$ exactly, as we show in formula (A.8). Adding the latter to (5.32) we obtain

$$e^{\frac{i\pi N_c(2N_c+1)}{2}} , \quad (5.33)$$

that cancels an analogous contribution in (5.31). All in all we are left with

$$\mathcal{I}^{USp(2N_c)} = 2\exp\left[-\frac{i\pi N_c(2N_c+1)}{\tau^2} \prod_{a=1}^{3} \left(\{\Delta_a\}_\tau - \frac{1+\eta}{2} \right) \right. \\ \left. + \mathcal{O}\left(e^{-\frac{1}{|\tau|}}\right) + \dots \right] \quad (5.34)$$

where the ellipsis represents the contributions from other saddles we ignored. In the following we will evaluate the contributions of these saddles, i.e. cases *ii)* and *iii)* described at the beginning of this section.
We see the appearance of the expected $\log 2$ correction to $\log \mathcal{I}^{USp(2N_c)}$, which is due to the degeneracies of the saddles (5.29) counted by m.

L holonomies at $u_i = 0$ and $L - N_c$ at $u_i = \frac{1}{2}$

The next saddle point that we discuss corresponds to an ansatz with L holonomies at $u = 0$ and $K \equiv N_c - L$ holonomies at $u = \frac{1}{2}$. Expanding around this ansatz we have

$$\vec{u} = \begin{cases} \overline{v}_i \equiv v_i\tau, & i = 1,\dots,L \\ \overline{w}_r + \frac{1}{2} \equiv w_r\tau + \frac{1}{2} , & r = 1,\dots,K \end{cases} . \quad (5.35)$$

The effective action in the limit $|\tau| \to 0$ can be rearranged as

$$S_{\text{eff}}|_{\vec{u}=\{\{0\}_L,\{\frac{1}{2}\}_K\}} = -\frac{2i\pi}{\tau^2}\left(\eta_1(L-K+1)+\eta_2 K\right)\sum_{i=1}^{L}\overline{v}_i^2 \quad (5.36) \\ -\frac{2i\pi}{\tau^2}\left(\eta_1(K-L+1)+\eta_2 L\right)\sum_{r=1}^{K}\overline{w}_r^2 + \sum_{i<j}\log\left(2\sin\left(\pm\frac{\pi\overline{v}_{ij}^{(\pm)}}{\tau}\right)\right) \\ + \sum_{r<s}\log\left(2\sin\left(\pm\frac{\pi\overline{w}_{rs}^{(\pm)}}{\tau}\right)\right) + 2\sum_{i=1}^{L}\log\left(2\sin\left(\frac{2\pi\overline{v}_i}{\tau}\right)\right)$$

[8] Where, as usual, the CS contribution to the partition function for the $USp(2N_c)$ case has an extra factor of 2 w.r.t. $SU(N_c)$ due to the normalization of the generators [118].

$$
\begin{aligned}
&+2\sum_{r=1}^{K}\log\left(2\sin\left(\frac{2\pi\overline{w}_r}{\tau}\right)\right)-\frac{i\pi LK}{\tau^2}\prod_{a=1}^{3}\left(\{2\Delta_a\}-\frac{1+\eta_2}{2}\right)\\
&-\frac{i\pi(2(L-K)^2+N_c)}{\tau^2}\prod_{a=1}^{3}\left(\{\Delta_a\}-\frac{1+\eta_1}{2}\right)\\
&+i\pi\left(\frac{i\pi(6-5\eta_1)(2(K-L)^2+N_c)}{12}+\frac{(12-5\eta_2)\,KL}{3}-N_c^2\right)-N_c\log\tau,
\end{aligned}
$$

where we used the relations

$$
\sum_{a=1}^{3}\{\Delta_a\}_\tau=2\tau+\frac{3+\xi_0}{2}\,,\qquad \sum_{a=1}^{3}\left\{\frac{1}{2}+\Delta_a\right\}_\tau=2\tau+\frac{3+\xi_1}{2} \tag{5.37}
$$

and $\xi_0=\pm1$, $\xi_1=\pm1$. We then defined $\eta_1=\xi_0$, while for η_2 we used the relation

$$
\sum_{a=1}^{3}\{\Delta_a\}_\tau+\left\{\frac{1}{2}+\Delta_a\right\}_\tau=\sum_{a=1}^{3}\left(\{2\Delta_a\}_\tau+\frac{1}{2}\right) \tag{5.38}
$$

such that

$$
\sum_{a=1}^{3}\{2\Delta_a\}_\tau=4\tau+\frac{3+\xi_1+\xi_0}{2}\equiv4\tau+\frac{3+\eta_2}{2}\,, \tag{5.39}
$$

providing a definition for η_2.

Again, changing variables as $-i\sigma_j\equiv v_j\tau$ and $-i\rho_r\equiv w_r\tau$, there appears a contribution in the index from the three-sphere partition function of a $USp(2L)\times USp(2K)$ pure Chern–Simons theory. The two symplectic groups have CS levels $k_{USp(2L)}=-\eta_1(L-K+1)-\eta_2K$ and $k_{USp(2K)}=-\eta_1(K-L+1)-\eta_2L$. These Chern–Simons integrals can be evaluated using the results presented in appendix A, but the result is not particularly illuminating and we do not report it here.

N_c coincident holonomies at $u_i=\frac{1}{4}$

The ansatz for the saddle point in this case is

$$
\vec{u}=\left\{u_j=\frac{1}{4}+\overline{u}_j=\frac{1}{4}+v_j\tau\right\}\,. \tag{5.40}
$$

Plugging this into the effective action and expanding for $|\tau|\to0$, the leading contribution becomes

$$
\begin{aligned}
S_{\text{eff}}|_{\vec{u}=\{\frac{1}{4}\}_{N_c}}=&-\frac{i\pi}{\tau^2}\left[\left(\xi_0N_c+\xi_1(N_c+2)\right)\sum_{i=1}^{N_c}\overline{u}_i^2-(\xi_0-\xi_1)\left(\sum_{i=1}^{N_c}\overline{u}_i\right)^2\right]\\
&+\sum_{i<j}\log\left(2\sin\left(\pm\frac{\pi\overline{u}_{ij}}{\tau}\right)\right)+\frac{i\pi N_c}{\tau^2}\prod_{a=1}^{3}\left(\{\Delta_a\}-\frac{1+\eta_1}{2}\right)
\end{aligned}
\tag{5.41}
$$

$$- \frac{i\pi(N_c^2+N_c)}{4\tau^2} \prod_{a=1}^{3} \left(\{2\Delta_a\} - \frac{1+\eta_2}{2} \right) + \frac{5i\pi N_c}{12} (\eta_1 - (N_c+1)\eta_2) + \frac{i\pi N_c}{2} - N_c \log\tau \, ,$$

where ξ_0 and ξ_1 are defined as in 5.2.1.

Once again, upon changing variables as $-i\sigma_j \equiv v_j \tau$ we see the emergence of the contribution of the three-sphere partition function of a $U(N_c)$ vector multiplet. There is also a Chern–Simons term, where the $SU(N_c)$ and $U(1)$ factors give different contributions. Indeed, using the results of [119, App. A], we can read off the CS terms from (5.41). While the $SU(N_c)$ factor has level $k_{SU(N_c)} = -N_c\eta_2 + 2(\eta_1 - \eta_2)$, the $U(1)$ term has Chern–Simons level $k_{U(1)} = 2(\eta_1 - \eta_2)(N_c+1)$. Also in this case the evaluation of the CS integrals does not lead to an illuminating expression and we do not report it here.

P holonomies at $u_i = 0$, P at $u_i = \frac{1}{2}$, and $N_c - 2P$ at $u_i = \frac{1}{4}$

The last case that we discuss corresponds to the ansatz with P holonomies at $u = 0$, P holonomies at $u = \frac{1}{2}$, and the remaining $Q \equiv N_c - 2P$ at $u = \frac{1}{4}$:

$$\vec{u} = \begin{cases} \overline{v}_i \equiv v_i \tau \, , & i = 1, \ldots, P \\ \overline{w}_r + \frac{1}{2} \equiv w_r \tau + \frac{1}{2} \, , & r = 1, \ldots, P \\ \overline{z}_m + \frac{1}{4} \equiv z_m \tau + \frac{1}{4} \, , & m = 1, \ldots, Q \end{cases} . \tag{5.42}$$

Expanding around this ansatz, the effective action in the limit $|\tau| \to 0$ can be rearranged as

$$\begin{aligned} & S_{\text{eff}}|_{\vec{u}=\{\{0\}_P, \{\frac{1}{2}\}_P, \{\frac{1}{4}\}_Q\}} = \\ & - \frac{i\pi}{\tau^2} \Bigg[(2(P+1)\xi_0 + 2P\xi_2 + Q(\xi_1+\xi_3)) \left(\sum_{i=1}^{P} \overline{v}_i^2 + \sum_{r=1}^{P} \overline{w}_i^2 \right) \\ & + (Q(\xi_0+\xi_2) + 2\xi_2 + 2P(\xi_1+\xi_3)) \sum_{m=1}^{Q} \overline{z}_m^2 - (\xi_0 - \xi_2) \left(\sum_{m=1}^{Q} \overline{z}_m \right)^2 \Bigg] \\ & + \sum_{i<j} \log\left(2\sin\left(\pm \frac{\pi \overline{v}_{ij}^{(\pm)}}{\tau} \right) \right) + \sum_{r<s} \log\left(2\sin\left(\pm \frac{\pi \overline{w}_{rs}^{(\pm)}}{\tau} \right) \right) \\ & + 2\sum_{i=1}^{P} \log\left(2\sin\left(\frac{2\pi \overline{v}_i}{\tau} \right) \right) + 2\sum_{r=1}^{P} \log\left(2\sin\left(\frac{2\pi \overline{w}_r}{\tau} \right) \right) \\ & + \sum_{m<n} \log\left(2\sin\left(\pm \frac{\pi \overline{z}_{mn}}{\tau} \right) \right) - \frac{i\pi(2P-Q)}{\tau^2} \prod_{a=1}^{3} \left(\{\Delta_a\} - \frac{1+\eta_1}{2} \right) \\ & - \frac{i\pi((2P-Q)^2+Q)}{4\tau^2} \prod_{a=1}^{3} \left(\{2\Delta_a\} - \frac{1+\eta_2}{2} \right) \\ & - \frac{i\pi PQ}{4\tau^2} \prod_{a=1}^{3} \left(\{4\Delta_a\} - \frac{1+\eta_4}{2} \right) - i\pi \left(Q^2 + 4P^2 \right) \end{aligned}$$

$$+\frac{1}{12}i\pi\,(6-5\eta_1)\,(2P-Q)+\frac{1}{12}i\pi\,(12-5\eta_2)\,\big((2P-Q)^2+Q\big)$$
$$+\frac{1}{3}i\pi\,(12-5\eta_4)\,QP-N_c\log\tau\,, \tag{5.43}$$

where $\xi_{0,1,2,3}=\pm1$ are defined by the relations

$$\sum_{a=1}^{3}\left\{\frac{J}{4}+\Delta_a\right\}_\tau=2\tau+\frac{3+\xi_J}{2}\,,\quad J=0,\ldots,3\,. \tag{5.44}$$

Furthermore we called $\eta_1\equiv\xi_0$, $\eta_2\equiv\xi_0+\xi_2$, and $\eta_4\equiv\xi_0+\xi_1+\xi_2+\xi_3$.

Changing variables as $-i\sigma_j\equiv v_j\tau$, $-i\rho_r\equiv w_r\tau$ and $-i\lambda_m\equiv z_m\tau$ we recognize in the expansion of the index a contribution from the three-sphere partition function of a $USp(2P)\times USp(2P)\times U(Q)$ pure CS theory. The two symplectic groups have the same Chern–Simons level $k_{USp(2P)}=-\frac{1}{2}(2(P+1)\xi_0+2P\xi_2+Q(\xi_1+\xi_3))$, while the $SU(Q)$ and the $U(1)$ subgroups of $U(Q)$ have different CS levels, $-(Q(\xi_0+\xi_2)+2\xi_2+2P(\xi_1+\xi_3)$ and $-2(\xi_2+Q\xi_2+P(-\xi_0+\xi_1+\xi_2+\xi_3))$ respectively.
Again the evaluation of the Chern–Simons integrals does not lead to an illuminating expression and we do not report it here.

5.2.2 Orthogonal gauge group

In order to study the orthogonal cases for generic rank we first discuss the SCI of $SO(N_c)$ with $N_c=3,\ldots,6$. In fact for these values of N_c the index can be extracted by leveraging the accidental isomorphisms of some classical Lie algebras.

• $SO(3)$: In this case, denoting y the holonomy of $USp(2)$ and u the holonomy of $SO(3)$ we can make the change of variables $u=2y$ and show, by direct inspection, that $\mathcal{I}^{SO(3)}=\mathcal{I}^{USp(2)}$.

• $SO(4)$: In this case, denoting $y_{1,2}$ the holonomies of $SU(2)\times SU(2)$ and $u_{1,2}$ the holonomies of $SO(4)$ we can make the change of variables

$$u_1=y_1+y_2\,,\quad u_2=y_1-y_2\,, \tag{5.45}$$

and show that $\mathcal{I}^{SO(4)}=\mathcal{I}^{SU(2)}\mathcal{I}^{SU(2)}$, where the right hand side corresponds to the index of two decoupled $\mathcal{N}=4$ $SU(2)$ models.

• $SO(5)$: In this case, denoting $y_{1,2}$ the holonomies of $USp(4)$ and $u_{1,2}$ the holonomies of $SO(5)$ we can make the change of variables

$$u_1=y_1+y_2\,,\quad u_2=y_1-y_2 \tag{5.46}$$

and show that $\mathcal{I}^{SO(5)}=\mathcal{I}^{USp(4)}$.

• $SO(6)$**:** In this case we can consider the holonomies of $SU(4)$ and enforce the SU constraint explicitly on their definition:

$$\pm(x_i - x_j)\,, \quad i<j\,; \quad \pm(x_i + x_j + 2x_k)\,, \quad i \neq j \neq k\,, \tag{5.47}$$

with $i,j = 1,2,3$. The holonomies of $SO(6)$, denoted u_i with $i = 1,2,3$, can be mapped to the $SU(4)$ ones by the change of variables

$$u_1 = x_2 + x_3\,, \quad u_2 = x_3 + x_1\,, \quad u_3 = x_1 + x_2\,, \tag{5.48}$$

thus showing that $\mathcal{I}^{SO(6)} = \mathcal{I}^{SU(4)}$.

For all SU and USp cases (computed in [106] and here respectively) we see that the leading contribution always has a logarithmic correction compatible with the formula $\log|\text{center}(G)|$, where by center(G) we mean the center of the gauge group G, i.e. $\mathbb{Z}_{N_c}$ and $\mathbb{Z}_2$ respectively. As discussed in the introduction this correction is generically smaller if there are fields charged under the center symmetry (which is not the case for SYM).

Motivated by the above discussion, in this subsection we study the leading contribution to the Cardy-like limit of the SCI for both the $SO(2N_c+1)$ and the $SO(2N_c)$ case. In the $SO(2N_c+1)$ case we find the same result obtained for the leading contribution of the symplectic case, as predicted by S-duality. Nevertheless the matching is nontrivial because we have a different number of solutions to the saddle point equations. Only after a careful evaluation of the three-dimensional Chern–Simons partition function we will have a proper matching of the two indices including the finite logarithmic corrections.

The $SO(2N_c+1)$ case

We start by studying $SO(2N_c+1)$. In this case the matrix integral is given by formula (5.21). We can then study the saddle point equations:

$$\begin{aligned}\sum_{a=1}^{3}\sum_{j=1}^{N_c}\Big(&B_2(\{u_{ij}^{(\pm)}+\Delta_a\}_\tau) - B_2(\{-u_{ij}^{(\pm)}+\Delta_a\}_\tau) + \\ &+ B_2(\{u_i+\Delta_a\}_\tau) - B_2(\{-u_i+\Delta_a\}_\tau)\Big) = 0\,,\end{aligned} \tag{5.49}$$

for $i = 1,\ldots,N_c$. Here we focus only on the solutions that have been studied in [76] in the Cardy-like limit. In this case the leading saddle corresponds to solution at $u_{*j} = 0$. We expand the holonomies around this solution as in (5.19), i.e. $u_j = 0 + \overline{u}_j \equiv v_j\tau$. Expanding the effective action around this saddle point we find

$$\begin{aligned}S_{\text{eff}}|_{\vec{u}=\{0\}_{N_c}} =& -\frac{(i\pi\eta(2N_c-1))\sum_{i=1}^{N_c}\overline{u}_i^2}{\tau^2} + \sum_{j\neq k}\log\left(2\sin\left(\frac{\pi\overline{u}_{jk}^{(\pm)}}{\tau}\right)\right) \\ &+ \sum_{j=1}^{N_c}\log\left(2\sin\left(\frac{\pm\pi\overline{u}_j}{\tau}\right)\right) - \frac{i\pi N_c(2N_c+1)}{\tau^2}\prod_{a=1}^{3}\left(\Delta_a - \frac{\eta+1}{2}\right) \\ &+ \frac{1}{12}i\pi(6-5\eta)\left(2N_c^2+N_c\right) - i\pi N_c^2 - N_c\log(\tau)\,.\end{aligned} \tag{5.50}$$

Upon changing variables as $-i\sigma_j \equiv v_j\tau$, the superconformal index becomes

$$\mathcal{I}^{SO(2N_c+1)} = \tau^{N_c} e^{-i\pi\frac{N_c(2N_c+1)}{2}} \mathcal{I}_0^{SO(2N_c+1)} Z_{S^3}^{SO(2N_c+1)_{-\eta(2N_c-1)}} \tag{5.51}$$

where the last contribution corresponds to the three-sphere partition function of a 3d $SO(2N_c+1)$ pure Chern–Simons theory at level $-\eta(2N_c-1)$. We also defined

$$\begin{aligned}\mathcal{I}_0^{SO(2N_c+1)} \equiv \exp\Bigg[&- \frac{i\pi N_c(2N_c+1)}{\tau^2} \prod_{a=1}^{3} \left(\Delta_a - \frac{\eta+1}{2}\right) \\ &+ \frac{1}{12} i\pi(6-5\eta)\left(2N_c^2+N_c\right) - i\pi N_c^2 - N_c\log(\tau)\Bigg] .\end{aligned} \tag{5.52}$$

We can evaluate $Z_{S^3}^{SO(2N_c+1)_{-\eta(2N_c-1)}}$ exactly as done in formula (A.18). We finally arrive at

$$\mathcal{I}^{SO(2N_c+1)} = 2\exp\left[-\frac{i\pi N_c(2N_c+1)}{\tau^2}\prod_{a=1}^{3}\left(\Delta_a - \frac{\eta+1}{2}\right) + \mathcal{O}\left(e^{-\frac{1}{|\tau|}}\right) + \ldots\right], \tag{5.53}$$

where the ellipsis represents the contributions from other saddles ignored here.
We observe the appearance of the expected $\log 2$ correction to $\log \mathcal{I}^{SO(2N_c+1)}$, which is *not* due to the degeneracy of the saddles as in the $USp(2N_c)$ case but rather to the extra factor of 2 in the evaluation of the partition function for the pure Chern–Simons theory; see again (A.18).

The $SO(2N_c)$ case

We now turn to $SO(2N_c)$. In this case the matrix integral is given by formula (5.22). We can then study the saddle point equations. We have:

$$\sum_{a=1}^{3}\sum_{j=1}^{N_c} B_2(\{u_{ij}^{(\pm)} + \Delta_a\}_\tau) - B_2(\{-u_{ij}^{(\pm)} + \Delta_a\}_\tau) = 0\,, \tag{5.54}$$

for $i = 1, \ldots, N_c$. Again, we focus only on the solutions that have been studied in [76] in the Cardy-like limit. They are given by the ansatz

$$\vec{u} = \left\{u_j^{(m)} = \frac{m}{2} + \overline{u}_j \equiv \frac{m}{2} + v_j\tau\right\} \quad \text{with} \quad m = 0, 1\,. \tag{5.55}$$

Expanding the effective action around the saddle point (5.55) we find

$$\begin{aligned} S_{\text{eff}}|_{\vec{u}=\{\frac{m}{2}\}_{N_c}} =& -\frac{(2i\pi\eta(N_c-1))\sum_{i=1}^{N_c}\overline{u}_i^2}{\tau^2} + \sum_{j\neq k}\log\left(2\sin\left(\frac{\pi\overline{u}_{jk}{}^{(\pm)}}{\tau}\right)\right) \\ &- \frac{i\pi N_c(2N_c-1)}{\tau^2}\prod_{a=1}^{3}\left(\Delta_a - \frac{\eta+1}{2}\right) + \frac{1}{12}i\pi(6-5\eta)N_c\left(2N_c-1\right) - i\pi N_c(N_c-1) - N_c\log(\tau)\,. \end{aligned} \tag{5.56}$$

Upon changing variables as $-i\sigma_j \equiv v_j\tau$, the SCI becomes

$$\mathcal{I}^{SO(2N_c)} = 2\tau^{N_c} e^{-i\pi\frac{N_c(2N_c-1)}{2}} \mathcal{I}_0^{SO(2N_c)} Z_{S^3}^{SO(2N_c)_{-2\eta(N_c-1)}}\,, \tag{5.57}$$

where the last contribution corresponds to the three-sphere partition function of a 3d $SO(2N_c)$ pure CS theory at level $-2\eta(N_c-1)$. We also defined

$$\mathcal{I}_0^{SO(2N_c)} \equiv \exp\left[-\frac{i\pi N_c(2N_c-1)}{\tau^2} \prod_{a=1}^{3}\left(\Delta_a - \frac{\eta+1}{2}\right) \right. \\ \left. + \frac{1}{12} i\pi(6-5\eta)N_c\,(2N_c-1) - i\pi N_c(N_c-1) - N_c\log(\tau)\right] . \quad (5.58)$$

Evaluating $Z_{S^3}^{SO(2N_c)_{-2\eta(N_c-1)}}$ exactly as done in formula (A.21) and multiplying it by (5.58) we obtain

$$\mathcal{I}^{SO(2N_c)} = 4\exp\left[-\frac{i\pi N_c(2N_c-1)}{\tau^2} \prod_{a=1}^{3}\left(\Delta_a - \frac{\eta+1}{2}\right) + \mathcal{O}\left(e^{-\frac{1}{|\tau|}}\right) + \dots\right] \quad (5.59)$$

where the ellipsis represents the contributions from other saddles ignored here.
We observe the appearance of the expected $\log 4$ correction to $\log \mathcal{I}^{SO(2N_c)}$, which is partly due to the degeneracy of the saddles (5.55) counted by m and partly due to the extra factor of 2 in the evaluation of the partition function of the pure Chern–Simons theory. The final result is consistent with the fact that the center is either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ depending on the parity of N_c.

5.3 A non-toric example: the Leigh–Strassler fixed point

In this section we study the SCI of the so-called $\mathcal{N}=1^*$ theory of [114], i.e. the theory obtained by turning on a complex mass for one of the $\mathcal{N}=1$ adjoint chirals in $\mathcal{N}=4$ $SU(N_c)$ Super Yang-Mills, and flowing to the fixed point. We integrate out the massive field Φ_3 after deforming the superpotential of $\mathcal{N}=4$ accordingly:

$$\mathcal{W}^{\text{mass}}_{\mathcal{N}=4} \sim \operatorname{Tr}\Phi_3[\Phi_1,\Phi_2] + \operatorname{Tr}\Phi_3^2 \quad \longrightarrow \quad \mathcal{W}_{\mathcal{N}=1^*} \sim \operatorname{Tr}[\Phi_1,\Phi_2]^2\,. \quad (5.60)$$

It is interesting to study this case because this $\mathcal{N}=1$ theory is non-toric, and so far such models have not been discussed in the literature.[9]

5.3.1 Cardy-like limit of the index

The N_c-1 saddle point equations read

$$\sum_{j=1}^{N_c} B_2(\{u_{ij}+\Delta_a\}_\tau) - B_2(\{u_{Nj}+\Delta_a\}_\tau) - B_2(\{u_{ij}+\Delta_a\}_\tau) + B_2(\{-u_{Nj}+\Delta_a\}_\tau) = 0, \quad (5.61)$$

[9] In effect, [106] deals only with *toric* $\mathcal{N}=1$ $SU(N_c)$ quivers and gives leading contribution and logarithmic correction of the SCI in the Cardy-like limit; [79] gives only the *leading* contribution for a *general* (i.e. not necessarily toric) $\mathcal{N}=1$ gauge theory with gauge group G in terms of its central charges a, c (and flavor central charges if present). Partial progress for general theories has also been made in [78, 104]. We are grateful to D. Cassani for comments on this point.

and they have the same solutions as those discussed in [106], that we report here:

$$u_i = \frac{m}{N_c} + \frac{I - \frac{C-1}{2}}{C} + v_i \tau \quad \text{with} \quad \sum_{i=1}^{N_c} v_i = 0\,, \tag{5.62}$$

where $I = \lfloor \frac{i-1}{N_c/C} \rfloor$, with $i = 1, \dots, N_c$ and $m = 0, \dots, \frac{N_c}{C} - 1$, with C an integer divisor of N_c. This corresponds to the K-gon solution of [78], and it can be visualized as C sets each containing N_c/C holonomies, uniformly distributed along the unit interval.

Leading saddle: $C = 1$

The leading saddle corresponds to the ansatz with $C = 1$. In the following we discuss this case explicitly. The fugacities associated with the adjoints Φ_1 and Φ_2 are denoted Δ_1 and Δ_2 respectively, and the superpotential $\mathcal{W}_{\mathcal{N}=1^*}$ in (5.60) imposes the constraint $\Delta_1 + \Delta_2 = 1$. It follows that in this case the constraint on the quantities $\{\Delta_a\}_\tau$ is given by

$$\{\Delta_1\}_\tau + \{\Delta_2\}_\tau = \tau + 1 + \frac{\eta}{2} \tag{5.63}$$

where $\eta = \pm 1$. By expanding the index at small $|\tau|$ (and fixed $\arg \tau \in (0,1)$) we obtain:

$$\begin{aligned} S_{\text{eff}}^{\text{LS}} = & -\frac{i\pi\eta}{\tau^2} N_c \left(\sum_{i=1}^{N_c} \overline{u}_i - \frac{1}{N_c} \sum_{j=1}^{N_c} \overline{u}_j \right)^2 + \sum_{i \neq j} \log \left(2 \sin \frac{\pi \overline{u}_{ij}}{\tau} \right) \\ & - \frac{i\pi (N_c^2 - 1) \left(\Delta_1 - \frac{1+\eta}{2} \right) \left(\Delta_2 - \frac{1+\eta}{2} \right) \left(\Delta_1 + \Delta_2 - (1+\eta) \right)}{\tau^2} \\ & + \frac{i\pi}{12} (6 - 5\eta)(N_c^2 - 1) - \frac{i\pi}{2} (N_c^2 - N_c) - (N_c - 1) \log \tau\,, \end{aligned} \tag{5.64}$$

where we defined $\overline{u}_i \equiv v_i \tau$. Once again, upon the change of variables $i\sigma_j \equiv \frac{m}{N} + v_j \tau$, we recognize a 3d pure CS partition function. By evaluating the latter on the different N_c saddles the final result is

$$\mathcal{I}_{\text{sc}}^{\text{LS}} = N_c \, e^{-\frac{\pi i (N_c^2 - 1)}{\tau^2} \left(\Delta_1 - \frac{1+\eta}{2} \right) \left(\Delta_2 - \frac{1+\eta}{2} \right) \left(\Delta_1 + \Delta_2 - (1+\eta) \right) + \mathcal{O}(e^{-1/|\tau|})} + \dots \tag{5.65}$$

where the ellipsis refers to the contribution of other saddles we ignored. Notice that the index has the functional structure of the 4d central charge a, that in this case is given by

$$a_{\text{LS}} = \frac{27}{32} \Delta_1 \Delta_2 \left(\Delta_1 + \Delta_2 \right)\,. \tag{5.66}$$

Furthermore we observe the appearance of the expected $\log N_c$ correction to $\log \mathcal{I}^{\text{LS}}$, which is inherited from the parent $\mathcal{N} = 4$ $SU(N_c)$ Super Yang–Mills.

Subleading saddles: C**-center solutions**

A similar analysis can be carried out for the C-center solutions introduced in [101, 106]. Here we redefine (5.62) as

$$u_i = \frac{m}{N_c} + \frac{I - \frac{C-1}{2}}{C} + u_{I, i-(N_c/C)I}\,, \tag{5.67}$$

by introducing the quantity $u_{I,i-(N_c/C)I}$. The action for the C-center solution is given by

$$S_{\text{eff}}^{\text{LS},C} = \sum_{a=1}^{2}\sum_{I,J=0}^{C-1}\sum_{i,j=1}^{N_c/C} 2\pi i Q\left(\left\{\frac{I-J}{C}+\Delta_a\right\}_\tau + u_{I,i}-u_{J,j};\tau\right) \tag{5.68}$$
$$+\sum_{I,J=0}^{C}\sum_{i,j=0}^{N_c/C}\log\left(\theta_0\left(\frac{I-J}{C}+u_{I,i}-u_{J,j};\tau\right)\right)+2(N_c-1)\log(q;q)_\infty\ .$$

Using the relations

$$\left\{\frac{J}{C}+\Delta_1\right\}_\tau+\left\{\frac{J}{C}+\Delta_2\right\}_\tau=\tau+1+\frac{\xi_J}{2}\ ,\quad \{C\Delta_1\}_\tau+\{C\Delta_2\}_\tau=C\tau+1+\frac{\eta_C}{2}\ , \tag{5.69}$$

with $\xi_0=\eta_1$ and $\eta_C=\sum_{J=0}^{C-1}\xi_J$, we can expand the action for $|\tau|\to 0$ (and fixed $\arg\tau$) obtaining for the leading terms

$$S_{\text{eff}}^{\text{LS},C} = -\frac{\pi i}{2\tau^2}\frac{N_c^2}{C^2}\sum_{I,J=0}^{C-1}\xi_{I-J}\left(\sum_{i=1}^{N_c/C}u_{I,i}-\sum_{j=1}^{N_c/C}u_{J,j}\right)^2$$
$$+\sum_{I=0}^{C-1}\left(-\frac{\pi i\eta_c N_c}{C\tau^2}\sum_{i=1}^{N_c/C}\left(u_{I,i}-\frac{C}{N_c}\sum_{j=1}^{N_c/C}u_{I,j}\right)^2+\sum_{i\neq j}^{N_c/C}\log\left(\left(2\sin\frac{\pi(u_{I,i}-u_{I,j})}{\tau}\right)\right)\right)$$
$$-\frac{\pi i N_c^2}{C^3\tau^2}\left(\{C\Delta_1\}_\tau-\frac{1+\eta_c}{2}\right)\left(\{C\Delta_2\}_\tau-\frac{1+\eta_c}{2}\right)\cdot\left(\{C\Delta_1\}_\tau+\{C\Delta_2\}_\tau-(1+\eta_c)\right)$$
$$+\frac{\pi i}{\tau^2}\left(\{\Delta_1\}_\tau-\frac{1+\eta_1}{2}\right)\left(\{\Delta_2\}_\tau-\frac{1+\eta_1}{2}\right)\left(\{\Delta_1\}_\tau+\{\Delta_2\}_\tau-(1+\eta_1)\right)$$
$$-\frac{5\pi i\eta_C N_c^2}{12C}+\frac{\pi i N_c}{2}-\frac{\pi i(6-5\eta_1)}{12}-(N_c-1)\log\tau\ . \tag{5.70}$$

The calculation of the Chern–Simons integrals is identical to the one performed in [106] for the C-center solution of $\mathcal{N}=4$ $SU(N_c)$ SYM.
The final result is:

$$\mathcal{I}^{\text{LS},C} = \frac{N_c}{C}\,e^{-\frac{\pi i N_c^2}{C^3\tau^2}\left(\{C\Delta_1\}_\tau-\frac{1+\eta_c}{2}\right)\left(\{C\Delta_2\}_\tau-\frac{1+\eta_c}{2}\right)\left(\{C\Delta_1\}_\tau+\{C\Delta_2\}_\tau-(1+\eta_c)\right)} \tag{5.71}$$
$$\cdot\, e^{\frac{\pi i}{\tau^2}\left(\{\Delta_1\}_\tau-\frac{1+\eta_1}{2}\right)\left(\{\Delta_2\}_\tau-\frac{1+\eta_1}{2}\right)\left(\{\Delta_1\}_\tau+\{\Delta_2\}_\tau-(1+\eta_1)\right)+\frac{5\pi i(\eta_1-C\eta_C)}{12}}\cdot Z_{S^3}^{U(1)}+\dots\ ,$$

where $Z_{S^3}^{U(1)}$ denotes the CS partition function of the abelian factors as in [106].

5.3.2 Entropy function and dual black hole entropy

We conclude the analysis of the LS fixed point by studying the entropy function S_E that represents the log of the number of states and corresponds to the Legendre transform of the index. In the holographic dictionary the Legendre transform of S_E gives the entropy of the dual black hole. The entropy function can be read off of the logarithm of the SCI, and is thus given by

$$S_E=-\kappa\frac{i\pi\left(\Delta_1-\frac{\eta+1}{2}\right)\left(\Delta_2-\frac{\eta+1}{2}\right)\left(\Delta_1+\Delta_2-(\eta+1)\right)}{\tau^2} \tag{5.72}$$

with the constraint $\Delta_1 + \Delta_2 - \tau - 1 - \frac{\eta}{2} = 0$ (which is derived from (5.63)). The overall constant κ is fixed as $\kappa = \frac{1}{8}$ (see the discussion in [105]).
The entropy is computed in terms of the charges $Q_{1,2}$ and angular momentum J of the dual black hole. (Observe that since we are identifying σ and τ, we only have one angular momentum $J_1 = J_2 \equiv J$.) The Legendre transform of the entropy function S_E is given by the formula

$$S = S_E + 2\pi i(Q_1\Delta_1 + Q_2\Delta_2 + J\tau) + 2\pi i\Lambda\left(\Delta_1 + \Delta_2 - \tau - 1 - \frac{\eta}{2}\right) , \tag{5.73}$$

where Λ is a Lagrange multiplier that enforces the above constraint between the chemical potentials. The entropy function satisfies the simple equation

$$S_E = \Delta_1 \frac{\partial S_E}{\partial \Delta_1} + \Delta_2 \frac{\partial S_E}{\partial \Delta_2} + \tau \frac{\partial S_E}{\partial \tau} , \tag{5.74}$$

implying that the entropy can be extracted from the Lagrange multiplier Λ as $S = -2\pi i\Lambda$. In order to find an expression for Λ we first write down the equations $\partial_{\Delta_{1,2}} S = 0$ and $\partial_\tau S = 0$. These three equations allow to express the quantities $\Lambda + Q_{1,2}$ and $\Lambda - J$ in terms of $\Delta_{1,2}$ and τ. They read

$$\begin{aligned}
\frac{\Lambda + Q_1}{\kappa} &= -\frac{\left(\Delta_2 - \frac{\eta+1}{2}\right)\left(2\Delta_1 + \Delta_2 - \frac{3}{2}(\eta+1)\right)}{2\tau^2} , \\
\frac{\Lambda + Q_2}{\kappa} &= -\frac{\left(\Delta_1 - \frac{\eta+1}{2}\right)\left(\Delta_1 + 2\Delta_2 - \frac{3}{2}(\eta+1)\right)}{2\tau^2} , \\
\frac{\Lambda - J}{\kappa} &= \frac{\left(\Delta_1 - \frac{\eta+1}{2}\right)\left(\Delta_2 - \frac{\eta+1}{2}\right)\left(\Delta_1 + \Delta_2 - \eta - 1\right)}{\tau^3} .
\end{aligned} \tag{5.75}$$

Using these relations we can find an identity involving Λ, $Q_{1,2}$, and J. In the case of $\mathcal{N} = 4$ SYM this is a cubic equation in Λ; here instead we found a fifth-order equation in Λ, which to the best of our knowledge appears for the first time in such a calculation. It reads:

$$\begin{aligned}
&\frac{1}{2}(\Lambda - J)^2(\Lambda + 2Q_1 - Q_2)(2\Lambda + Q_1 + Q_2)(\Lambda - Q_1 + 2Q_2) \\
&\qquad + \frac{27}{32}\kappa(\Lambda - J)^4 - \frac{2(Q_1 - Q_2)^2(\Lambda + Q_1)^2(\Lambda + Q_2)^2}{\kappa} = 0 .
\end{aligned} \tag{5.76}$$

Solving this equation in Λ yields the entropy S as a function of the charges as explained above. In order to obtain a sensible result we should also impose that Λ is purely imaginary. In general a fifth order equation with two imaginary solutions can be written as

$$\Lambda^5 + c_2\Lambda^4 + (c_1 + c_3)\Lambda^3 + (c_1 c_2 + c_4)\Lambda^2 + c_1 c_3 \Lambda + c_1 c_4 = 0 , \tag{5.77}$$

with solutions $\Lambda = \pm i\sqrt{c_1}$. The coefficients c_i can be expressed in terms $Q_{1,2}$ and J. (This is a reality condition on the entropy, which also imposes a constraint among the charges.) The black hole entropy is then given by the following relation:

$$S = -2\pi i\Lambda = 2\pi\sqrt{c_1} = 2\pi\sqrt{\frac{\alpha - \sqrt{\alpha^2 + 32\,\kappa\,\beta}}{16\kappa}} , \tag{5.78}$$

with

$$\begin{aligned}\alpha &\equiv \kappa J(27\kappa - 8J) + 8(Q_1 + Q_2)(3\kappa J + 4(Q_1 - Q_2)^2)\\ &\quad + 12\kappa(Q_1^2 - 4Q_2Q_1 + Q_2^2)\,,\\ \beta &\equiv 27\kappa^2 J^3 + 12\kappa J^2(Q_1^2 - 4Q_2Q_1 + Q_2^2)\\ &\quad + 8(Q_1 + Q_2)(\kappa J(Q_2 - 2Q_1)(Q_1 - 2Q_2) + 4Q_1Q_2(Q_1 - Q_2)^2)\,.\end{aligned} \tag{5.79}$$

5.4 Expansion of the index: the general formula

In this section we give a formula capturing all the contributions for generic $\mathcal{N} = 1$ SCFTs, which only depends on the central charges a and c, gauge algebra, and matter representations.
In the holographic case, that is for theories with a dual, the Cardy-like limit of the SCI introduced above reproduces the Legendre transform of the entropy of the dual rotating BH. Here we propose a general formula for the index in this limit at finite order both in rk_G and τ, regardless of the existence of a gravity dual.

Our main result is that the index takes the form

$$\log \mathcal{I}(\tau, \Delta) \underset{|\tau|\to 0}{=} \frac{4\pi i(\eta - 6\tau + 12\eta\tau^2 + \dots)(3c - 2a)}{27\tau^2} + \frac{8\pi i(2 - 5\eta\tau + \dots)(c - a)}{6\tau} + \log \Gamma_Z\,, \tag{5.80}$$

where we use the same functions a and c that, when evaluated on the R-charges, reproduce $\mathrm{Tr}\, R$ and $\mathrm{Tr}\, R^3$ via $\mathrm{Tr}\, R = 16(a - c)$ and $\mathrm{Tr}\, R^3 = \frac{16}{9}(5a - 3c)$ [120]. Here we evaluate these functions on a new set of *charges* $\hat{\Delta}_I$ (for the matter fields) defined as

$$\hat{\Delta}_I \equiv \frac{2}{2\tau - \eta}\{\Delta_I\}_\tau\,, \tag{5.81}$$

with Δ_I defined in (5.5), the τ-modded value $\{\cdot\}_\tau$ given in (5.10), and $\eta = \pm 1$. This latter choice has been used before to study the Cardy-like limit of the SCI [78,79] and match it against the dual black hole entropy when available (see also the discussion in [13, Sec. 5]).
We stress, to avoid any confusion on the interpretation of formula (5.80), that it has to be read as follows: the central charges a and c are computed as in a generic SCFT by considering the charges of the fermions in the matter multiplets and in the vector multiplet. While for the former we use the *new* charges $\hat{\Delta}_I$ defined above (instead of the R-charges), for the latter the redefinition does *not* apply (and we use their R-charge).

Our aim is to support the validity of (5.80) for 4d $\mathcal{N} = 1$ SCFTs with a generic amount of gauge groups, each with algebra of type ABCD, without specifying the ranks. Before proceeding we still need to define the positive integer Γ_Z in (5.80).
Let us begin by observing that the superconformal index cannot capture the global aspects of the gauge group. This implies that, once the representations of the matter fields charged under the gauge group are specified, the index of the theory with gauge group G is equivalent to the index of the theory with gauge group G/H, where H is a discrete subgroup of the center $Z(G)$.

Here we observe, in all the examples under investigation, that the logarithmic correction to the index in the Cardy-like limit is given by $\log \Gamma_Z$, where Γ_Z is the minimal charge of the matter fields under the center $Z(G)$ of the gauge group. Alternatively, as noted in [112], Γ_Z can be interpreted as the order of the one-form symmetry of the theory.
For example for $\mathcal{N} = 4$ theories the matter fields are all in the adjoint representation, i.e. they have the same charge as the dimension of the center, and in this case indeed $\Gamma_Z = \dim Z(G)$ (see [106] and section 5.2). On the other hand if we consider $SU(N_c)$ SQCD, the matter fields in the fundamental representation have charge one under the gauge group, and in this case indeed we find $\Gamma_Z = 1$ (see sections 5.6.1 and 5.6.2). Whenever we consider models with a center symmetry given by the product $\prod_{a=1}^{n_G} \mathbb{Z}_{k_a}$, we will refer to the sum of the charges under each single $\mathbb{Z}_{k_a}$ factor as "charge". With this convention in mind we can see that our definition of Γ_Z is also consistent with what was found for toric quivers in [106], where the center symmetry is $\mathbb{Z}_{N_c}^{n_G}$ and each field Φ_{ij} is in the fundamental representation of $SU(N_c)_i$ and in the anti-fundamental representation of $SU(N_c)_j$, and thus has total charge N_c. In this case it was indeed found that $\Gamma_Z = N_c$.

5.4.1 Derivation

We are now ready to proceed with a derivation of (5.80). We start our analysis by focusing on the contribution to the index of the matter fields. For a generic field Φ we have to consider the contribution $2\pi i Q(\{\rho_\Phi(\vec{u}) + \Delta_\Phi\}_\tau; \tau)$. The function $Q(u;\tau)$ was defined in (5.15).

Vanishing holonomies. Let us first focus on the contribution of the field Φ at vanishing holonomies. In this work we are only interested in a set of charges $\hat{\Delta}_\Phi$ for the matter fields Φ that satisfy the constraint

$$\sum_{\Phi \in \mathcal{W}} \hat{\Delta}_\Phi = 2 \quad \Rightarrow \quad \sum_{\Phi \in \mathcal{W}} \{\Delta_\Phi\}_\tau = 2\tau - \eta \,, \tag{5.82}$$

where the notation $\Phi \in \mathcal{W}$ means that we sum over the fields in *each* superpotential term, i.e. (5.82) represent a set of $n_{\mathcal{W}}$ (redundant) equations, where $n_{\mathcal{W}}$ corresponds to the number of superpotential terms.
A field Φ then contributes to the index as

$$\begin{aligned} 2\pi i Q(\{\Delta_\Phi\}_\tau;\tau) &= 2\pi i Q\left(\frac{2\tau-\eta}{2}\hat{\Delta}_\Phi + \frac{1+\eta}{2};\tau\right) \\ &= 4\pi i \frac{(\eta - 6\tau + 12\eta\tau^2 + \dots)}{27\tau^2}(3c(\hat{\Delta}_\Phi) - 2a(\hat{\Delta}_\Phi)) \\ &+ \frac{8\pi i}{6\tau}(2 - 5\eta\tau + \dots)(c(\hat{\Delta}_\Phi) - a(\hat{\Delta}_\Phi)) \equiv J(\hat{\Delta}_\Phi) \,, \end{aligned} \tag{5.83}$$

consistently with (5.80).

Non-vanishing holonomies. Next, we focus on the contribution at non-vanishing holonomies. To do that, we make the following observation. As discussed in various papers (see e.g. [78,79]), the index evaluated at zero holonomies reproduces the leading contribution to the Legendre transform of the entropy of the dual black hole. Furthermore there are other saddle point solutions of (5.18) corresponding to packages of coincident holonomies placed homogeneously along the unitary circle, that reproduce the BH entropy as well. We have checked in many concrete examples that the number of these inequivalent solutions corresponds to the integer Γ_Z defined above. (For instance, for an $SU(N_c)$ theory with adjoint matter fields there are N_c solutions as in [78, 106], while in presence of fundamental matter only the solution with all the holonomies at the origin is allowed. There are N_c solutions also for toric quivers [78, 106] because the bifundamental matter fields imply that the gauge group is $\prod_{a=1}^{n_G} SU(N_c)_a/\mathbb{Z}_{N_c}^{\text{diag}}$.) A crucial observation is that in each example considered here $\rho_\Phi(\vec{u}_*) \in \mathbb{Z}$, with $\vec{u}_*$ defined in (5.19). In this way we can simplify the expansion of $2\pi i Q(\{\rho_\Phi(\vec{u}) + \Delta_\Phi\}_\tau; \tau)$ using that, on the saddles (5.19), we have $\{\rho_\Phi(\vec{u}) + \Delta_\Phi\}_\tau = \{\tau\rho_\Phi(\vec{v}) + \Delta_\Phi\}_\tau$. Even if we do not have an analytic proof we expect that this result holds in general for any $\mathcal{N} = 1$ SCFT. Thanks to this, we can expand the Bernoulli polynomials in terms of the holonomies as follows:

$$
\begin{aligned}
B_3(\{v\tau + \Delta_\Phi\}_\tau) &\underset{|\tau|\to 0}{=} B_3(\{\Delta_\Phi\}_\tau) + 3B_2(\{\Delta_\Phi\}_\tau)v\tau + 3B_1(\{\Delta_\Phi\}_\tau)v^2\tau^2 \,, \\
B_2(\{v\tau + \Delta_\Phi\}_\tau) &\underset{|\tau|\to 0}{=} B_2(\{\Delta_\Phi\}_\tau) + 2B_1(\{\Delta_\Phi\}_\tau)v\tau \,, \\
B_1(\{v\tau + \Delta_\Phi\}_\tau) &\underset{|\tau|\to 0}{=} B_1(\{\Delta_\Phi\}_\tau) \,.
\end{aligned}
\tag{5.84}
$$

It follows that, in the expansion of $2\pi i Q(\{\rho_\Phi(\vec{u}) + \Delta_\Phi\}_\tau; \tau)$:

- the linear term in the holonomies vanishes for all ABCD cases;
- the quadratic term in the holonomies corresponds to the partition function of three-dimensional pure Chern–Simons theory at level $-\eta T(G)$, where $T(\mathcal{R})$ is the Dynkin index of the representation $\mathcal{R}$, and $T(G)$ refers to the adjoint representation. (See e.g. appendix A for the relevant notation.)
  This calculation is done as follows. We first plug the explicit form of the Bernoulli polynomials into $2\pi i Q(\{\rho_\Phi(\vec{u}) + \Delta_\Phi\}_\tau; \tau)$ and expand them around the saddle points as in (5.84). In this way we obtain the quadratic contributions in the variables $\vec{v}$, altogether amounting to
  $$
  \frac{\pi i}{2} \sum_{I=1}^{n_\chi} \rho_I^2(\vec{v})(2\tau - \eta - \{\Delta_I\}_\tau) \,, \tag{5.85}
  $$
  which is valid for both $\eta = \pm 1$. The quantity $\rho_I^2(\vec{v})$ represents the sum of the squares of the weights of each field Φ_I in the representation $\mathcal{R}_I$ parameterized by the holonomies $\vec{v}$, as explained in (5.8). This can then be expressed in terms of the Dynkin index $T(\mathcal{R}_I)$. Using this observation and the relation between $\{\Delta_\Phi\}_\tau$ and $\hat{\Delta}_\Phi$ given in (5.81) we finally find
  $$
  \begin{aligned}
  \frac{\pi i}{2} \sum_{I=1}^{n_\chi} \rho_I^2(\vec{v})(2\tau - \eta - \{\Delta_I\}_\tau) &= \frac{\pi i(2\tau - \eta)}{2} \sum_{I=1}^{n_\chi} \rho_I^2(\vec{v})(1 - \hat{\Delta}_I) \\
  &= \frac{\pi i(2\tau - \eta)}{2} \sum_{a=1}^{n_G} \sum_{i_a=1}^{\text{rk}G_a} v_{i_a}^2 \left(\sum_{\Phi \in G_a} T(\mathcal{R}_\Phi)(1 - \hat{\Delta}_\Phi) \right) .
  \end{aligned}
  \tag{5.86}
  $$

Here the notation $\Phi \in G_a$ means that we consider the sum over *all* fields Φ that are charged under the a-th gauge group G_a. The leading order in τ thus reads:

$$\frac{\eta\pi}{2} \sum_{a=1}^{n_G} \sum_{i_a=1}^{\mathrm{rk}_{G_a}} \lambda_{i_a}^2 \left(\sum_{\Phi \in G_a} T(\mathcal{R}_\Phi)(1 - \hat{\Delta}_\Phi) \right) , \tag{5.87}$$

where we also defined $\lambda_{i_a} \equiv i v_{i_a}$ for future convenience.
In the toric case the constraint $\sum_{\Phi \in \mathcal{W}} \hat{\Delta}_\Phi = 2$ automatically ensures that

$$T(G) + \sum_{\Phi \in G_a} T(\mathcal{R}_\Phi)(\hat{\Delta}_\Phi - 1) = 0 , \tag{5.88}$$

reflecting the fact that the anomaly freedom of the R-symmetry coincides with the requirement $R(\mathcal{W}) = 2$. However the relation between the constraints imposed by the superpotential and by the requirement of a non-anomalous R-symmetry does not hold in general, and we will **assume** that the Cardy-like limit has to be taken by imposing the *anomaly* cancellation for the $\hat{\Delta}_I$ variables as well, namely condition (5.88) above.

Evaluating the CS integral,[10] i.e. the partition function of a pure 3d Chern–Simons theory with gauge group G and CS level $-\eta T(G)$, and summing this result to the contribution from the vector multiplets, coming from the terms in $\theta_0(\alpha_a(\vec{u}); \tau)$ and $(q;q)_\infty$ in (5.7), we arrive at the result

$$\begin{aligned} &4\pi i \frac{(\eta - 6\tau + 12\eta\tau^2 + \dots)}{27\tau^2}(3c(2) - 2a(2)) \\ &+ \frac{8\pi i}{6\tau}(2 - 5\eta\tau + \dots)(c(2) - a(2)) = -\frac{\pi i(2 - 5\eta\tau + \dots)}{12\tau} \end{aligned} \tag{5.89}$$

for each vector multiplet.
In addition there is a contribution $\log \Gamma_Z$ coming for the degeneration of the saddle points, as discussed above.

5.4.2 The examples

In this analysis we have made an educated guess regarding the solutions of the saddle point equations for generic matter content and gauge group, and this cannot be regarded as a rigorous proof of the formula (5.80). For this reason and for the sake of clarity in the next section we will study some explicit examples, supporting the result claimed in this section. We have chosen examples that do *not* belong to the vast family of toric quiver gauge theories, that have been thoroughly investigated in this context e.g. in [76,78,82,85,98,105,106].

[10] By inspection we found that the integral is given by the formula

$$Z_{S^3}^{G_{-\eta T(G)}} = \exp\left(i\pi \frac{(|G| - \mathrm{rk}_G)}{2} - \frac{1}{12} i\pi(6 - 5\eta)|G| \right) ,$$

where $|G|$ is the dimension of the gauge group $G \equiv \mathrm{Lie}(\mathfrak{g})$, with $\mathfrak{g}$ of ABCD type. It would be interesting to check the validity of such a general formula for the exceptional Lie algebras as well. See [121–124] for some results in this direction.

We kick off our analysis with an exception though, namely by studying a toric quiver gauge theory engineered by a stack of N_c D3-branes probing the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity. The reason is that in this case we can perform a Seiberg duality that lands us on a so-called *non-toric* phase. We show that matching the Cardy-like limit of the superconformal index across the dual phases requires some care in the correct identification of the matter charges under the duality map.
We then move to a fully non-toric example, namely the quiver gauge theory corresponding to a stack of N_c D3-branes probing the cone over the dP$_4$ singularity. We choose the phase of the theory with all but one equal ranks [125]. We show that formula (5.80) is valid in this case as well. Another non-toric example that we tackle is Laufer's theory, introduced in [126–129].
All of these examples admit a large-N_c limit with $\mathrm{Tr}\, R = \mathcal{O}(1)$, and they are conjectured to have a weakly-coupled gravity dual. However our formula (5.80) goes beyond this requirement, providing a result that should be valid also for theories without a large-N_c limit dual to classical gravity. We test this conjecture by studying the case of $SU(N_c)$ SQCD and adjoint $SU(N_c)$ SQCD. In the latter theory we discuss the modification of our formalism in presence of accidental symmetries as well. We conclude with the case of $USp(2N_c)$ SQCD and the Intriligator–Pouliot duality it enjoys.
We then move to cases with $\mathcal{N} = 2$ supersymmetry. As a first example we study a family of $\mathcal{N} = 1$ Lagrangians that enhance in the infrared (IR) to the (A_1, A_{2n-1}) Argyres–Douglas (AD) fixed points [130–134]. We conclude our analysis with a fully Lagrangian $\mathcal{N} = 2$ SCFT with matter fields in tensor representations: the gauge group is $SU(N_c)$ and we have a symmetric and an antisymmetric hypermultiplet. This theory is interesting both because it has a known supergravity dual description and because the matter fields force different logarithmic corrections for the even N_c and odd N_c case, in perfect agreement with the logic we explained below (5.80).

5.5 $\mathcal{N} = 1$ examples with $\mathrm{Tr}\, R = \mathcal{O}(1)$

First we study examples of quivers with different ranks for the various gauge groups. Namely we study the toric/non-toric Seiberg duality for the quiver associated with the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity; the dP$_4$ quiver theory; Laufer's theory [126–129, 135].

5.5.1 $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$: a toric/non-toric duality

We will start our analysis with $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. This model has a toric holographic dual description, i.e. it is obtained from a stack of N_c D3-branes probing a Calabi–Yau threefold cone over a Sasaki–Einstein five-manifolds with $U(1)^3$ isometry. This corresponds to a toric SCFT, described by a quiver gauge theory with four $SU(N_c)$ gauge nodes, and pairs of bifundamental and anti-bifundamental fields connecting each pair of nodes. There are many equivalent ways to translate the toric condition of the metric on the dual quiver. For instance a possibility consists of *planarizing* the quiver on a torus. We refer the reader to [48] for further details. Even if the Cardy-like limit of toric quiver gauge theories has been thoroughly analyzed in the literature, we still find it useful to consider this model because by applying the rules of Seiberg duality one obtains a dual SCFT without an explicit toric description (i.e. it is not possible to represent the Seiberg-dual quiver on a two-torus). Furthermore the non-toric dual phase is instructive because the ranks of the dual gauge groups are not all equal to N_c. This will be a generic feature for some other models holographically dual to non-toric manifolds that we will consider below.

Toric phase

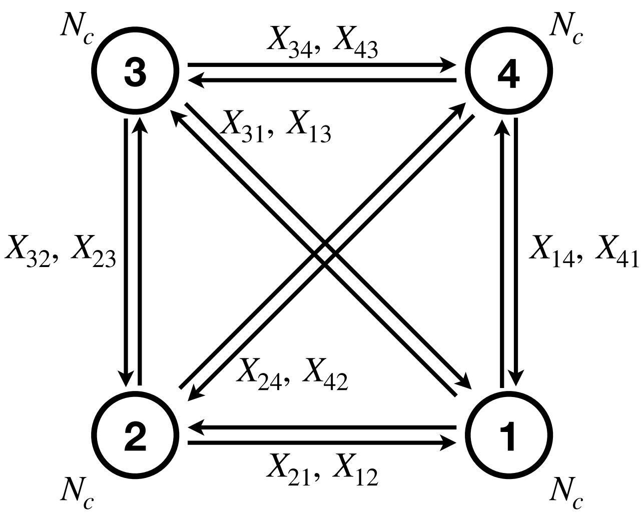


Figure 5.1: Quiver for the toric phase of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$.

The toric phase is a quiver gauge theory with four gauge groups $SU(N_c)$ (see figure 5.1), with all matter fields in the bifundamental representation of $SU(N_c) \times SU(N_c)$. The superpotential reads

$$\begin{aligned}\mathcal{W} = &X_{12}X_{23}X_{31} - X_{12}X_{24}X_{41} + X_{13}X_{34}X_{41} - X_{13}X_{32}X_{21} \\ &+ X_{14}X_{42}X_{21} - X_{14}X_{43}X_{31} + X_{24}X_{43}X_{32} - X_{23}X_{34}X_{42} \,.\end{aligned} \tag{5.90}$$

The SCI of this theory[11] is given by

$$\mathcal{I}_{\text{sc}}(\tau,\Delta) = \frac{(q;q)_\infty^{8N_c}}{(N_c!)^4}\int \prod_{a=1}^{4}\prod_{i_a=1}^{N_c} du_{i_a}\, \frac{\prod_{a\neq b}^{4}\prod_{i,j=1}^{N_c}\tilde{\Gamma}(u_{ij}^{ab}+\Delta_{ab})}{\prod_{a=1}^{4}\prod_{i\neq j}^{N_c}\tilde{\Gamma}(u_{ij}^{a})}\,, \tag{5.91}$$

where $u_{ij}^{ab} \equiv u_{i_a} - u_{j_b}$ and $u_{ij}^{a} \equiv u_{i_a} - u_{j_a}$. The effective action S_{eff} (5.7) in this case becomes:

$$\begin{aligned}S_{\text{eff}}(\vec{u};\tau,\Delta) = &\sum_{a\neq b}^{4}\sum_{i,j=1}^{N_c}\log\tilde{\Gamma}\big(u_{ij}^{ab}+\Delta_{ab}\big) + \sum_{a=1}^{4}\sum_{i\neq j}^{N_c}\log\theta_0\big(u_{ij}^{a};\tau\big) \\ &+ 8(N_c-1)\log(q;q)_\infty \,.\end{aligned} \tag{5.92}$$

The charges $\{\Delta_{ab}\}_\tau$ are related by constraints that can be read off of the superpotential (5.90), i.e. $\sum_{\Phi\in\mathcal{W}}\{\Delta_\Phi\}_\tau = 2\tau - \eta$. These constraints are equivalent to those that can be derived using the

[11] The SCI for the $\mathbb{C}^3/\mathbb{Z}_k \times \mathbb{Z}_l$ orbifold theory has been computed in certain limits (albeit different from the Cardy-like one) in [136].

anomaly cancellation for the variables $\hat{\Delta}_{ab} \equiv \frac{2}{2\tau-\eta}\{\Delta_{ab}\}_\tau$. The gauge anomaly constraints on the variable $\hat{\Delta}_{ab}$ imply the constraints

$$\sum_{\Phi \in G_a} \{\Delta_\Phi\}_\tau = 4\tau - 2\eta \,, \quad a = 1, \dots, 4 \tag{5.93}$$

on the variables $\{\Delta_{ab}\}_\tau$. It is straightforward to prove that, in general, in the toric case the constraints (5.93) are implied by the superpotential constraints. Taking the Cardy-like limit, it can be expanded as:

$$\begin{aligned} N_c^2 S_{\text{eff}}(\vec{u};\tau,\Delta) &= -\frac{\pi i}{3\tau^2}\sum_{a\neq b=1}^{4}\Big[\sum_{i_a,j_b=1}^{N_c} B_3\big(\{u^{ab}_{i_a j_b}+\Delta_{ab}\}_\tau)\big)\Big] \\ +\frac{\pi i}{\tau}\sum_{a\neq b=1}^{4}\Big[\sum_{i_a,j_b=1}^{N_c} & B_2\big(\{u^{ab}_{i_a j_b}+\Delta_{ab}\}_\tau)\big)\Big] + \frac{\pi i}{\tau}\sum_{a=1}^{4}\sum_{i_a\neq j_a=1}^{N_c}\{u^a_{i_a j_a}\}_\tau(1-\{u^a_{i_a j_a}\}_\tau) \\ -\frac{5\pi i}{6}\sum_{a\neq b=1}^{4}\Big[\sum_{i_a,j_b=1}^{N_c} & B_1\big(\{u^{ab}_{i_a j_b}+\Delta_{ab}\}_\tau\big)\Big] + \pi i\sum_{a=1}^{4}\sum_{i_a\neq j_a=1}^{N_c}\{u^a_{i_a j_a}\}_\tau \\ &+\pi i N_c^2\,\frac{4\tau^2-6\tau-2}{3\tau} + 4\pi i N_c + \pi i\,\frac{2\tau^2-6\tau+2}{3\tau} - 4(N_c-1)\log\tau \\ &+\sum_{a=1}^{4}\sum_{i_a\neq j_a=1}^{N_c}\log\Big(1-e^{-\frac{2\pi i}{\tau}\left(1-\{u^a_{i_a j_a}\}_\tau\right)}\Big)\Big(1-e^{-\frac{2\pi i}{\tau}\{u^a_{i_a j_a}\}_\tau}\Big) + \mathcal{O}(\dots) \end{aligned} \tag{5.94}$$

In order to apply the saddle point approach, we need to solve the saddle point equations $\frac{\partial}{\partial u_{i_a}} S_{\text{eff}}(\vec{u};\tau,\Delta) = 0$ at leading order, which for the theory at hand take the form

$$-\frac{\pi i}{\tau^2}\sum_{a\neq b}^{4}\sum_{j=1}^{N_c}\Big(B_2\big(\{u^{ab}_{ij}+\Delta_{ab}\}_\tau\big) - B_2\big(\{u^{ab}_{N_c j}+\Delta_{ab}\}_\tau\big)\Big) = 0 \,, \tag{5.95}$$

keeping $a = 1, \dots, 4$ and $i_a = 1, \dots, N_c$ fixed.
Here we consider only the solution that reproduces the BH entropy of the holographic dual description. This solution has already been discussed in the literature [78, 106]. Other possible solutions, subleading in the regime of charges that we focus on, have been discussed in [101, 106]. The solution to (5.95) is given by

$$\vec{u}_a = \Big\{ u^{(m)}_{j_a} = \frac{m}{N_c} + \bar{u}_{j_a} \equiv \frac{m}{N_c} + v_{j_a}\tau \;\Big|\; v_{j_a} \sim \mathcal{O}(|\tau|^0) \,,\; \sum_{j_a=1}^{N_c} v_{j_a} = 0 \Big\}, \tag{5.96}$$

with $m = 0, \dots, N_c - 1$ and $a = 1, \dots, 4$. We can now evaluate the effective action around its saddle points by analyzing individually the three terms in (5.92): the first is due to the matter fields, the second due to the gauge fields, and the last coming from the q-Pochhammer symbol.

Matter fields. The contribution to the index of the matter fields is given by the Bernoulli polynomials in (5.17), plus a term proportional to τ, that is negligible in our expansion. The

Bernoulli polynomials can be simplified using the relations (5.84).
We notice that the terms proportional to $3B_2(\{\Delta\}_\tau)\bar{u}$ and $2B_1(\{\Delta\}_\tau)\bar{u}$ cancel, due to the $SU(N_c)$ constraint on the holonomies, $\sum_{i_a=1}^{N_c}\bar{u}_{i_a} = 0 \mod \mathbb{Z}$. The term proportional to $3B_1(\{\Delta\}_\tau)\bar{u}^2$ gives instead

$$\begin{aligned} &-\frac{\pi i}{\tau^2}\sum_{a\neq b}^{4}\sum_{i,j=1}^{N_c} B_1(\{\Delta_{ab}\}_\tau)(\bar{u}^{ab}_{ij})^2 \\ &= -\frac{\pi i}{\tau^2}\sum_{a\neq b=1}^{4}\sum_{i_a,j_b=1}^{N_c} B_1(\{\Delta_{ab}\}_\tau)(\bar{u}_{i_a}^2+\bar{u}_{j_b}^2-2\bar{u}_{i_a}\bar{u}_{j_b}) \\ &= -\frac{\pi i}{\tau^2}N_c\sum_{a=1}^{4}\sum_{i_a=1}^{N_c}\bar{u}_{i_a}^2\sum_{b\neq a=1}^{4}(\{\Delta_{ab}\}_\tau+\{\Delta_{ba}\}_\tau-3\eta) \\ &= -\frac{\pi i}{\tau^2}N_c\sum_{a=1}^{4}\sum_{i_a=1}^{N_c}\bar{u}_{i_a}^2(2\tau+\eta) = -\frac{\pi i}{\tau^2}\eta N_c\sum_{a=1}^{4}\sum_{i_a=1}^{N_c}\bar{u}_{i_a}^2+\mathcal{O}(|\tau|), \end{aligned} \tag{5.97}$$

where we used both the SU constraint and the constraints from the superpotential. Finally, the parts without holonomies give the contribution

$$\begin{aligned} &-\frac{\pi i}{\tau^2}N_c^2\sum_{a\neq b}^{4}\left[\frac{1}{3}B_3\big(\{\Delta_{ab}\}_\tau\big)-\tau B_2\big(\{\Delta_{ab}\}_\tau+\frac{5}{6}\tau^2B_1\big(\{\Delta_{ab}\}_\tau\big)\right]+\mathcal{O}(|\tau|) \\ &= N_c^2\sum_{a\neq b}^{4}J(\hat{\Delta}_{ab})+\mathcal{O}(|\tau|)\,, \end{aligned} \tag{5.98}$$

where $J(\hat{\Delta}_\Phi)$ was defined in (5.83).

Gauge fields & q-Pochhammer. The contribution of the gauge fields is

$$\begin{aligned} &\sum_{a=1}^{4}\sum_{i\neq j}^{N_c}\log\theta_0\big(u^a_{ij};\tau\big) \\ &\qquad = \sum_{a=1}^{4}\sum_{i\neq j}^{N_c}\log\Big(2\sin\frac{\pi\bar{u}^a_{ij}}{\tau}\Big)-\frac{2\pi i N_c(N_c-1)}{3\tau}+\mathcal{O}(|\tau|) \end{aligned} \tag{5.99}$$

while the q-Pochhammer symbol gives the contribution

$$\begin{aligned} &8(N_c-1)\log(q;q)_\infty \\ &\quad = -4(N_c-1)\log\tau-\frac{2\pi i(N_c-1)}{3\tau}+2\pi i(N_c-1)+\mathcal{O}(|\tau|)\,. \end{aligned} \tag{5.100}$$

Effective action & index. Therefore, the Cardy-like limit of the effective action evaluated at the saddle points can be written as:

$$S_{\text{eff}}(\vec{u};\tau,\Delta) = -\frac{\pi i \eta}{\tau^2} N_c \sum_{a=1}^{4}\sum_{i_a=1}^{N_c} \bar{u}_{i_a}^2 + \sum_{a=1}^{4}\sum_{i\neq j} \log\left(2\sin\frac{\pi \bar{u}_{ij}^a}{\tau}\right) \tag{5.101}$$
$$+ N_c^2 \sum_{a\neq b}^{4} J(\hat{\Delta}_{ab}) - \frac{2\pi i (N_c^2-1)}{3\tau} + 2\pi i (N_c-1) - 4(N_c-1)\log\tau\,,$$

with $J(\hat{\Delta}_\Phi)$ defined in (5.83). The SCI is thus

$$\mathcal{I}(\tau,\Delta) \underset{|\tau|\to 0}{=} \sum_{m=0}^{N_c-1} \frac{\mathcal{A}}{(N_c!)^4} \tag{5.102}$$
$$\cdot \int \prod_{a=1}^{4}\prod_{i_a=1}^{N_c} du_{i_a}\, e^{-\frac{\pi i\eta}{\tau^2}N_c \sum_{i_a=1}^{N_c}(\bar{u}_{i_a}^a)^2 + \sum_{i\neq j}^{N_c}\log\left(2\sin\frac{\pi\bar{u}_{ij}^a}{\tau}\right)}\,,$$

where the prefactor $\mathcal{A}$ is given by

$$\mathcal{A} = e^{N_c^2\sum_{a\neq b}^4 J(\hat{\Delta}_{ab}) - \frac{2\pi i(N_c^2-1)}{3\tau} + 2\pi i(N_c-1) - 4(N_c-1)\log\tau}\,. \tag{5.103}$$

The change of variables

$$\bar{u}_j = -i\lambda_j\tau\,, \quad \sum_{j=1}^{N_c}\lambda_j = 0\,, \tag{5.104}$$

modifies both the measure of the integral in (5.102) and the contribution in $\sin\frac{\pi\bar{u}_{ij}^a}{\tau}$. In these new variables, the index becomes:

$$\mathcal{I}(\tau,\Delta) \underset{|\tau|\to 0}{=} N_c\, e^{-2\pi i(N_c^2-1)}\tau^{4(N_c-1)} \tag{5.105}$$
$$\cdot \frac{\mathcal{A}}{(N_c!)^4}\int \prod_{a=1}^{4}\prod_{i_a=1}^{N_c} d\lambda_{i_a}\, e^{\pi i\eta N_c\sum_{ja=1}^{N_c}\lambda_{ja}^2 + \sum_{j\neq k}^{N_c}\log\left(2\sinh(\pi\lambda_{jk}^a)\right)}\,.$$

Recall that the three-sphere partition function of 3d supersymmetric $SU(N_c)_\kappa$ Chern–Simons theory is given by

$$Z^{\text{CS}}_{SU(N_c)_\kappa} = \frac{1}{N_c!}\int\prod_{i=1}^{N_c} d\lambda_i\, e^{-\pi i\kappa\sum_{j=1}^{N_c}\lambda_j^2 + \sum_{j\neq k}^{N_c}\log\left(2\sinh(\pi\lambda_{jk})\right)}\,, \tag{5.106}$$

with the constraint $\sum_{j=1}^{N_c}\lambda_j = 0$. For $\kappa = -\eta N_c$ we have

$$Z^{\text{CS}}_{SU(N_c)_{-\eta N_c}} = e^{\frac{5}{12}i\pi\eta\left(N_c^2-1\right)+\frac{1}{2}i\pi(N_c-1)N_c} \tag{5.107}$$

and thus

$$\prod_{a=1}^{n_G} Z^{\text{CS}}_{SU(N_c)_{-\eta N_c}} = e^{n_G\left(\frac{5}{12}i\pi\eta\left(N_c^2-1\right)+\frac{1}{2}i\pi(N_c-1)N_c\right)}\,. \tag{5.108}$$

Using (5.108), the index (5.105) can be rewritten as

$$\log \mathcal{I}(\tau, \Delta) \underset{|\tau|\to 0}{=} \log \left(N_c \, \mathcal{A} \, e^{-2\pi i (N_c^2 - 1)} \tau^{4(N_c - 1)} e^{\pi i \frac{N_c^2 - 6N_c + 5}{3}} \right) . \tag{5.109}$$

The N_c contribution is due to the saddle point degeneracy in (5.96), counted by m. By inspection we see that (5.109) coincides with the result proposed in formula (5.80).

Non-toric phase

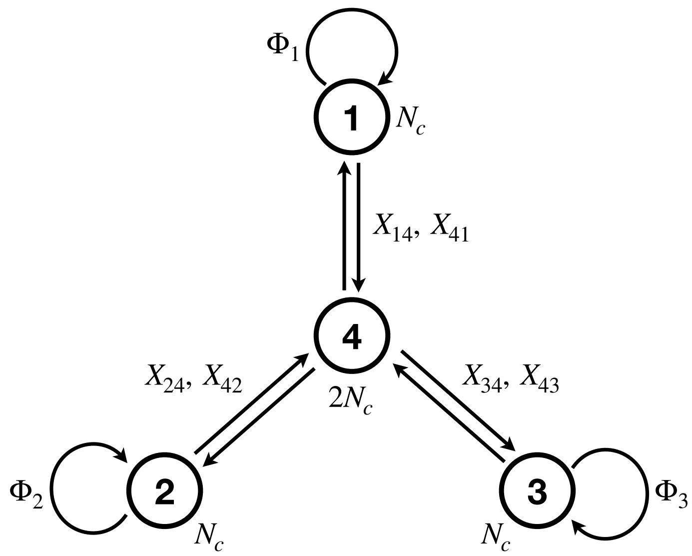


Figure 5.2: Quiver for the non-toric phase of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$. Observe that the fields $\Phi_{1,2,3}$ in the figure are associated with bifundamental reducible representations.

The non-toric phase of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ that we are considering (figure 5.2) is obtained by performing Seiberg duality at node 4 of the original toric quiver (figure 5.1). The superpotential of the dual theory is

$$\begin{aligned} \mathcal{W} = &\, X_{41}\Phi_1 X_{14} + X_{41}\Phi_2 X_{24} + X_{43}\Phi_3 X_{34} \\ &+ X_{41} X_{14} [X_{42} X_{24}, X_{43} X_{34}] \, , \end{aligned} \tag{5.110}$$

where all the fields are in the bifundamental representation.

Seiberg duality implies also a mapping between the charges Δ_Φ of the electric theory (the toric phase discussed above) and the charges δ_Φ of the magnetic theory (the non-toric phase discussed here). The explicit map is given by the following identifications:

$$\begin{aligned}
&\delta_1 = \Delta_{14} + \Delta_{41}\,, \quad \delta_2 = \Delta_{24} + \Delta_{42}\,, \quad \delta_3 = \Delta_{34} + \Delta_{43}\,, \\
&\delta_{14} = -\frac{\Delta_{41}}{2} + \frac{\Delta_{42}}{2} + \frac{\Delta_{43}}{2}\,, \quad \delta_{41} = -\frac{\Delta_{14}}{2} + \frac{\Delta_{24}}{2} + \frac{\Delta_{34}}{2}\,, \\
&\delta_{24} = +\frac{\Delta_{41}}{2} - \frac{\Delta_{42}}{2} + \frac{\Delta_{43}}{2}\,, \quad \delta_{42} = +\frac{\Delta_{14}}{2} - \frac{\Delta_{24}}{2} + \frac{\Delta_{34}}{2}\,, \\
&\delta_{34} = +\frac{\Delta_{41}}{2} + \frac{\Delta_{42}}{2} - \frac{\Delta_{43}}{2}\,, \quad \delta_{43} = +\frac{\Delta_{14}}{2} + \frac{\Delta_{24}}{2} - \frac{\Delta_{34}}{2}\,.
\end{aligned} \tag{5.111}$$

The constraints imposed by the superpotential and the anomalies on the charges Δ_Φ automatically imply the correct constraints in the dual theory on the charges δ_Φ. The SCI of the theory is given by:

$$\begin{aligned}
\mathcal{I}(\tau, \delta) = &\frac{(q;q)_\infty^{10N_c}}{(N_c!)^3 (2N_c)!} \int \prod_{a=1}^{3} \prod_{i_a=1}^{N_c} \prod_{r=1}^{2N_c} dv_{i_a} dw_r \prod_{a=1}^{3} \left(\frac{\prod_{i,j=1}^{N_c} \tilde{\Gamma}(v_{ij}^a + \delta_a)}{\prod_{i\neq j}^{N_c} \tilde{\Gamma}(v_{ij}^a)} \right) \\
&\cdot \frac{\prod_{a=1}^{3} \prod_{i_a=1}^{N_c} \prod_{r=1}^{2N_c} \tilde{\Gamma}(v_{i_a} - w_r + \delta_{a4}) \tilde{\Gamma}(w_r - v_{i_a} + \delta_{4a})}{\prod_{r\neq s}^{2N_c} \tilde{\Gamma}(w_{rs})}\,,
\end{aligned} \tag{5.112}$$

where $v_{ij}^a \equiv v_{i_a} - v_{j_a}$. Notice that the index $i_a = 1, \ldots, N_c$ runs over the gauge groups with rank N_c, while the index $r = 1, \ldots, 2N_c$ runs over the gauge group with rank $2N_c$ (see figure 5.1). Using the relation (5.7), the index can be expressed in terms of the effective action, which is given by:

$$\begin{aligned}
S_{\text{eff}}(\vec{u}; \tau, \delta) = &\sum_{a=1}^{3} \sum_{i_a=1}^{N_c} \sum_{r=1}^{2N_c} \Big(\log \tilde{\Gamma}(v_{i_a} - w_r + \delta_{a4}) + \log \tilde{\Gamma}(w_r - v_{i_a} + \delta_{4a}) \Big) \\
&+ \sum_{a=1}^{3} \sum_{i\neq j}^{N_c} \log \tilde{\Gamma}(v_{ij}^a + \delta_a) + \sum_{a=1}^{3} \sum_{i\neq j}^{N_c} \log \theta_0(v_{ij}^a; \tau) + \sum_{r\neq s}^{2N_c} \log \theta_0(w_{rs}; \tau) \\
&+ 6(N_c - 1) \log(q;q)_\infty + 2(2N_c - 1) \log(q;q)_\infty\,.
\end{aligned} \tag{5.113}$$

The constraints among the chemical potentials can be read off of the superpotential (5.110), with the usual relation $\sum_{\Phi\in\mathcal{W}} \hat{\delta}_\Phi = 2$ that implies $\sum_{\Phi\in\mathcal{W}} \{\delta_\Phi\}_\tau = 2\tau - \eta$. Equivalently, they can be derived using the anomaly cancellation equation for the variables $\hat{\delta}_\Phi$. For example, for the central node of the quiver we obtain:

$$2N_c + \frac{1}{2} N_c \sum_{I=1}^{3} (\hat{\delta}_{I4} + \hat{\delta}_{4I} - 2) = 0\,, \tag{5.114}$$

and thus

$$\sum_{I=1}^{3} (\hat{\delta}_{I4} + \hat{\delta}_{4I}) = 2 \quad \Rightarrow \quad \sum_{I=1}^{3} (\{\delta_{I4}\}_\tau + \{\delta_{4I}\}_\tau) = 2\tau - \eta\,. \tag{5.115}$$

Similar relations hold for the other gauge nodes:

$$\begin{aligned}
\{\delta_{41}\}_\tau + \{\delta_1\}_\tau + \{\delta_{14}\}_\tau = 2\tau - \eta, \\
\{\delta_{42}\}_\tau + \{\delta_2\}_\tau + \{\delta_{24}\}_\tau = 2\tau - \eta, \\
\{\delta_{43}\}_\tau + \{\delta_3\}_\tau + \{\delta_{34}\}_\tau = 2\tau - \eta.
\end{aligned} \tag{5.116}$$

Given the effective action (5.113), the saddle point equations read:

$$
\begin{aligned}
-\frac{\pi i}{\tau^2}\Bigg[\sum_{r=1}^{2N_c}\Big(& B_2\big(\{v_{i_a}-w_r+\delta_{a4}\}_\tau\big)-B_2\big(\{w_r-v_{i_a}+\delta_{4a}\}_\tau\big) \\
& -B_2\big(\{v_{(N_c)_a}-w_r+\delta_{a4}\}_\tau\big)+B_2\big(\{w_r-v_{(N_c)_a}+\delta_{4a}\}_\tau\big)\Big) \\
+\sum_{j_a=1}^{N_c}\Big(& B_2\big(\{v^a_{ij}+\delta_a\}_\tau\big)-B_2\big(\{-v^a_{ij}+\delta_a\}_\tau\big) \\
& -B_2\big(\{v^a_{N_cj}+\delta_a\}_\tau\big)+B_2\big(\{-v^a_{N_cj}+\delta_a\}_\tau\big)\Big)\Bigg]=0\,, \qquad (5.117)
\end{aligned}
$$

for $a=1,\dots,3,\ i_a=1,\dots,N_c-1$, and

$$
\begin{aligned}
-\frac{\pi i}{\tau^2}\sum_{a=1}^{3}\sum_{i_a=1}^{N_c}\Big(& -B_2\big(\{v_{i_a}-w_r+\delta_{a4}\}_\tau\big)+B_2\big(\{w_r-v_{i_a}+\delta_{4a}\}_\tau\big) \\
& +B_2\big(\{v_{i_a}-w_{2N_c}+\delta_{a4}\}_\tau\big)-B_2\big(\{w_{2N_c}-v_{i_a}+\delta_{4a}\}_\tau\big)\Big)=0\,, \qquad (5.118)
\end{aligned}
$$

for $r=1,\dots,2N_c-1$. The leading solutions to these equations are given by

$$
\vec{v}_a=\left\{v^{(m)}_{j_a}=\frac{m}{N_c}+\bar{v}_{j_a}\equiv\frac{m}{N_c}+\sigma_{j_a}\tau\;\Big|\;\sigma_{j_a}\sim\mathcal{O}(|\tau|^0),\ \sum_{j_a=1}^{N_c}\sigma_{j_a}=0\right\}, \qquad (5.119)
$$

with $a=1,\dots,3$ and $m=0,\dots,N_c-1$, and

$$
\vec{w}=\left\{w^{(m)}_{r}=\frac{m}{N_c}+\bar{w}_{r}\equiv\frac{m}{N_c}+\sigma_{r}\tau\;\Big|\;\sigma_{r}\sim\mathcal{O}(|\tau|^0),\ \sum_{r=1}^{2N_c}\sigma_{r}=0\right\}, \qquad (5.120)
$$

with $m=0,\dots,N_c-1$. In fact, in (5.120) we find $w_r^{(n)}=\frac{n}{2N_c}+\bar{w}_r$, with $n=2m$, and thus we can rewrite these solutions as $w_r^{(m)}=\frac{m}{N_c}$. As in the toric case, in order to evaluate the effective action around these saddles, we can separate the contributions coming from matter fields, gauge fields, and q-Pochhammer symbol.

Matter fields. The relevant contribution to the index of the matter fields is given by the first three terms in (5.113). As in the toric case, the terms with a linear dependence from the holonomies vanish due to the $SU(N_c)$ constraint. The terms with a quadratic dependence from the holonomies become

$$
\begin{aligned}
-\frac{\pi i}{\tau^2}\sum_{a=1}^{3}\Bigg[&\sum_{i_a=1}^{N_c}\sum_{r=1}^{2N}\Big(B_1(\{\delta_{a4}\}_\tau)(\bar{v}_{i_a}-\bar{w}_r)^2+B_1(\{\delta_{4a}\}_\tau)(\bar{w}_r-\bar{v}_{i_a})^2\Big) \\
&+\sum_{i_a,j_a=1}^{N_c}B_1(\{\delta_a\}_\tau)(\bar{v}_{i_a}-\bar{v}_{j_a})^2\Bigg]\,, \qquad (5.121)
\end{aligned}
$$

which is equal to

$$-\frac{2\pi i\eta N_c}{\tau^2}\sum_{r=1}^{2N_c}\bar{w}_r^2-\frac{\pi i\eta N_c}{\tau^2}\sum_{a=1}^{3}\sum_{i_a=1}^{N_c}(\bar{v}_{i_a})^2+\mathcal{O}(|\tau|)\ , \tag{5.122}$$

where we have exploited the relations among the chemical potentials coming from the superpotential and the $SU(N_c)$ condition on the holonomies. The term without holonomies becomes:

$$\begin{aligned}
&-\frac{\pi i N_c^2}{\tau^2}\sum_{a=1}^{3}\Bigg[\frac{1}{3}\Big(2B_3\big(\{\delta_{a4}\}_\tau\big)+2B_3\big(\{\delta_{4a}\}_\tau\big)+B_3\big(\{\delta_a\}_\tau\big)\Big)\\
&+\tau\Big(2B_2\big(\{\delta_{a4}\}_\tau\big)+2B_2\big(\{\delta_{4a}\}_\tau\big)+B_2\big(\{\delta_a\}_\tau\big)\Big)\\
&+\frac{5}{6}\tau^2\Big(2B_2\big(\{\delta_{a4}\}_\tau\big)+2B_2\big(\{\delta_{4a}\}_\tau\big)+B_2\big(\{\delta_a\}_\tau\big)\Big)\Bigg]+\mathcal{O}(|\tau|)
\end{aligned} \tag{5.123}$$

equal to

$$N_c^2\sum_{a=1}^{3}\Big(2J(\hat{\delta}_{a4})+2J(\hat{\delta}_{4a})+J(\hat{\delta}_a)\Big)+\mathcal{O}(|\tau|)\ , \tag{5.124}$$

with the definition of $J(\hat{\Delta}_\Phi)$ given in (5.83).

Gauge fields & q-Pochhammer. The gauge fields contribute to the index as:

$$\begin{aligned}
&\sum_{a=1}^{3}\sum_{i\neq j}^{N_c}\log\theta_0\big(v_{ij}^a;\tau\big)+\sum_{r\neq s}^{2N_c}\log\theta_0\big(w_{rs};\tau\big)\\
&=\sum_{a=1}^{3}\sum_{i\neq j}^{N_c}\log\Big(2\sin\frac{\pi\bar{v}_{ij}^a}{\tau}\Big)+\sum_{r\neq s}^{2N_c}\log\Big(2\sin\frac{\pi\bar{w}_{rs}}{\tau}\Big)-\frac{\pi i(7N_c^2-5N_c)}{6\tau}+\mathcal{O}(|\tau|)\ ,
\end{aligned} \tag{5.125}$$

while the contribution from the q-Pochhammer symbol is

$$10(N_c-1)\log(q;q)_\infty=-(5N_c-4)\log\tau-\frac{\pi i(5N_c-4)}{6\tau}+\frac{\pi i(5N_c-4)}{2}\ . \tag{5.126}$$

Effective action & index. Therefore, the Cardy-like limit of the effective action evaluated at the saddle points can be written as:

$$\begin{aligned}
S_{\text{eff}}(\vec{u};\tau,\delta)=&-\frac{\pi i\eta N_c}{\tau^2}\sum_{a=1}^{3}\sum_{i_a=1}^{N_c}\bar{v}_{i_a}^2-\frac{2\pi i\eta N_c}{\tau^2}\sum_{r=1}^{2N_c}\bar{w}_r^2+\sum_{a=1}^{3}\sum_{i\neq j}^{N_c}\log\left(2\sin\frac{\pi\bar{v}_{ij}^a}{\tau}\right)\\
&+\sum_{r\neq s}^{2N_c}\log\Big(2\sin\frac{\pi\bar{w}_{rs}}{\tau}\Big)N_c^2\sum_{a=1}^{3}\Big(2J(\hat{\delta}_{a4})+2J(\hat{\delta}_{4a})+J(\hat{\delta}_a)\Big)\\
&-\frac{7\pi i N_c^2}{6\tau}+\frac{5}{2}\pi i N_c+\pi i\left(\frac{2}{3\tau}-2\right)-(5N_c-4)\log\tau\ .
\end{aligned} \tag{5.127}$$

We can write the SCI of the theory expanded in the Cardy like limit as

$$\begin{aligned}\mathcal{I}(\tau,\delta) \underset{|\tau|\to 0}{=} & \sum_{m=0}^{N_c-1} \frac{\mathcal{A}}{(2N_c!)(N_c!)^3} \\ & \cdot \int \prod_{a=1}^{3} \prod_{i_a=1}^{N_c} d\bar{v}_{i_a}\, e^{-\frac{\pi i \eta N_c}{\tau^2}\sum_{i_a=1}^{N_c}(\bar{v}_{i_a})^2+\sum_{i\neq j}^{N_c}\log\left(2\sin\frac{\pi \bar{v}_{ij}^a}{\tau}\right)} \\ & \cdot \prod_{r=1}^{2N_c} d\bar{w}_r\, e^{-\frac{2\pi i\eta N_c}{\tau^2}\sum_{r=1}^{2N_c}\bar{w}_r^2+\sum_{r\neq s}^{2N_c}\log\left(2\sin\frac{\pi \bar{w}_{rs}}{\tau}\right)}\,,\end{aligned} \tag{5.128}$$

where $\sum_{r=0}^{2N_c}\overline{w}_r = 0$ and $\sum_{i_a=0}^{N_c}\overline{v}_{i_a} = 0$ for $a = 1,2,3$. The prefactor $\mathcal{A}$ is given by

$$\mathcal{A} = e^{N_c^2\sum_{a=1}^{3}\left(2J(\hat{\delta}_{a4})+2J(\hat{\delta}_{4a})+J(\hat{\delta}_a)\right)+\frac{5}{2}\pi i N_c+\pi i\left(\frac{2}{3\tau}-2\right)-(5N_c-4)\log\tau}\,. \tag{5.129}$$

Using the change of variables in (5.104) and the definition of the CS partition function in (5.106), we obtain

$$\begin{aligned}\log\mathcal{I}(\tau,\delta) \underset{|\tau|\to 0}{=} \log & \Bigg[N_c\, e^{-\frac{3\pi i}{2}(N_c^2-1)}e^{-\frac{\pi i}{2}(4N_c^2-1)}\tau^{5N_c-4}\mathcal{A} \\ & \cdot\left(\prod_{a=1}^{3} Z^{\text{CS}}_{SU(N_c)_{-\eta N_c}}\right) Z^{\text{CS}}_{SU(2N_c)_{-2\eta N_c}}\Bigg]\,.\end{aligned} \tag{5.130}$$

Performing the integral $Z^{\text{CS}}_{SU(N_c)_\kappa}$ as in (5.107) and using the duality map in (5.111) (generalizing it to the hatted charges) we checked that the index in (5.130) coincides with the one computed in (5.109). This is a non-trivial check of the validity of our calculation in the dual non-toric phase, where the ranks of the gauge groups are not coincident.

5.5.2 Cone over dP$_4$

In this section we study the Cardy-like limit of a fully non-toric quiver gauge theory, engineered by a stack of D3-branes probing a cone over the dP$_4$ singularity. In this case the theory has one exact R-symmetry and four non-anomalous baryonic symmetries, while there are no other flavor symmetries, reflecting the non-toricity of the model. The quiver is reported in figure 5.3 and the superpotential is [125]:

$$\begin{aligned}\mathcal{W} = \; & a_{3,1}x_{1,4}x_{4,3} + c_{3,1}x_{1,4}x_{4,3} + a_{1,2}a_{3,1}x_{2,4}x_{4,3} + a_{1,2}c_{3,1}x_{2,4}x_{4,3} \\ & + b_{1,2}c_{3,1}x_{2,4}x_{4,3} + x_{2,4}x_{3,2}x_{4,3} + c_{3,1}x_{1,5}x_{5,3} - a_{3,1}b_{1,2}x_{2,5}x_{5,3} \\ & - a_{1,2}b_{3,1}x_{2,5}x_{5,3} - b_{1,2}c_{3,1}x_{2,5}x_{5,3} + x_{2,5}x_{3,2}x_{5,3} - \frac{1}{2}a_{3,1}x_{1,6}x_{6,3} \\ & - \frac{1}{2}b_{3,1}x_{1,6}x_{6,3} - \frac{1}{2}c_{3,1}x_{1,6}x_{6,3} - \frac{1}{2}a_{1,2}a_{3,1}x_{2,6}x_{6,3} - \frac{1}{2}a_{1,2}b_{3,1}x_{2,6}x_{6,3} \\ & - \frac{1}{2}a_{1,2}c_{3,1}x_{2,6}x_{6,3} + x_{2,6}x_{3,2}x_{6,3} - \frac{1}{2}b_{3,1}x_{1,7}x_{7,3} - \frac{1}{2}c_{3,1}x_{1,7}x_{7,3} \\ & - \frac{1}{2}a_{1,2}b_{3,1}x_{2,7}x_{7,3} + \frac{1}{2}a_{1,2}c_{3,1}x_{2,7}x_{7,3} + x_{2,7}x_{3,2}x_{7,3}\,.\end{aligned} \tag{5.131}$$

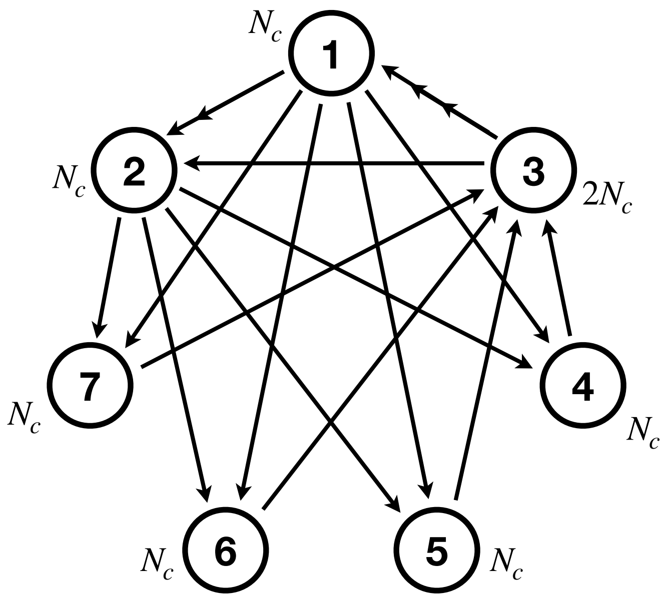


Figure 5.3: Quiver for cone over dP_4.

The table of non-anomalous global charges is reported in table 5.1. Let us call q^i are the charges under the four non-anomalous baryonic symmetries $U(1)_{B_i}$ appearing in the table. At the superconformal fixed point the central charges are $a = \frac{27}{20}N_c^2 - \frac{21}{16}$ and $c = \frac{27}{20}N_c^2 - \frac{7}{8}$.
We can now study the Cardy-like limit of the SCI, though in less detail w.r.t. the example in section 5.5.1. Imposing the anomaly cancellation and superpotential constraints on the charges $\hat{\Delta}_\Phi$, we can read the charges $\{\Delta_\Phi\}_\tau$ from (5.1):

$$\{\Delta_\Phi\}_\tau = \frac{2\tau - \eta}{2}\left(\sum_{i=1}^{4} q^i_\Phi \hat{B}_i + R_\Phi \hat{v}_R\right) , \tag{5.132}$$

where the superpotential constraints imply $\hat{v}_R = 1$. (This is because the model does not have any further flavor symmetry that can mix with the R-symmetry.)
We label the holonomies as u_{i_a} where $i_a = 1, \ldots, N_c$ for all $a = 1, \ldots, 7$ but 3, and $i_3 = 1, \ldots, 2N_c$. These holonomies are constrained as $\sum_{i_a=1}^{N_c} u_{i_a} = 0 \mod \mathbb{Z}$ for all a's but 3, and $\sum_{i_3=1}^{2N_c} u_{i_3} = 0 \mod \mathbb{Z}$. We then study the saddle point equations, finding the solutions

$$\vec{u} = \left\{ u^{(m)}_{j_a} = \frac{m}{N_c} + \bar{u}_{j_a} \equiv \frac{m}{N_c} + v_{j_a}\tau \;\Big|\; v_{j_a} \sim \mathcal{O}(|\tau|^0) \, , \; \sum_{j_a=1}^{N_c} v_{j_a} = 0 \right\} \tag{5.133}$$

for all a's but 3, and

$$\vec{u} = \left\{ u^{(m)}_{j_3} = \frac{m}{N_c} + \bar{u}_{j_3} \equiv \frac{m}{N_c} + v_{j_3}\tau \;\Big|\; v_{j_3} \sim \mathcal{O}(|\tau|^0) \, , \; \sum_{j_3=1}^{2N_c} v_{j_3} = 0 \right\} . \tag{5.134}$$

	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	$U(1)_{B_3}$	$U(1)_{B_4}$
$a_{1,2}$	$4/5$	0	0	0	4
$b_{1,2}$	$4/4$	0	0	0	4
$a_{3,1}$	$3/5$	0	0	0	-2
$b_{3,1}$	$3/5$	0	0	0	-2
$c_{3,1}$	$3/5$	0	0	0	-2
$x_{3,2}$	$7/5$	0	0	0	2
$x_{1,4}$	$6/5$	0	0	1	1
$x_{1,5}$	$6/5$	0	1	-1	1
$x_{1,6}$	$6/5$	1	-1	0	1
$x_{1,7}$	$6/5$	-1	0	0	1
$x_{2,4}$	$2/5$	0	0	1	-3
$x_{2,5}$	$2/5$	0	1	-1	-3
$x_{2,6}$	$2/5$	1	-1	0	-3
$x_{2,7}$	$2/5$	-1	0	0	-3
$x_{4,3}$	$1/5$	0	0	-1	1
$x_{5,3}$	$1/5$	0	-1	1	1
$x_{6,3}$	$1/5$	-1	1	0	1
$x_{7,3}$	$1/5$	1	0	0	1

Table 5.1: Non anomalous global charges of dP_4.

In both cases $m = 0, \dots, N_c - 1$.
We then expand the effective action up to finite order in τ. In order to compute the finite-order terms we need to perform seven integrals, corresponding to the three-sphere partition function of pure CS theory $SU(2N_c)_{-2\eta N_c} \times SU(N_c)^6_{-\eta N_c}$. Evaluating these integrals we end up with the expected result (5.80). The final result is

$$\begin{aligned}&\log \mathcal{I}^{\mathrm{dP}_4}(\tau, \mathcal{B}) \\ &\underset{|\tau|\to 0}{=} \log N_c + \frac{i\pi}{5}\left(\frac{\mathcal{B}^2\eta}{2\tau^2} + \frac{18\mathcal{B}^2+35}{6\tau} + \frac{\eta(82\mathcal{B}^2+175)}{12}\right)\end{aligned} \tag{5.135}$$

where $\mathcal{B}^2 \equiv N_c^2(5(\hat{B}_1^2 - \hat{B}_2\hat{B}_1 + \hat{B}_2^2 + \hat{B}_3^2 - \hat{B}_2\hat{B}_3 + 10\hat{B}_4^2) - 2)$. Observe that the N_c contribution to $\mathcal{I}$ is due to the degeneration of the solutions (5.134) of the saddle point equations.

5.5.3 Laufer's theory

This model is a quiver gauge theory with product gauge group $SU(N_c) \times SU(2N_c)$ and matter content

$$\begin{array}{c|cc|cc} & SU(2N_c) & SU(N_c) & U(1)_R & U(1)_B \\ \hline X & \mathbf{Adj} & \mathbf{1} & 1/2 & 0 \\ Y & \mathbf{Adj} & \mathbf{1} & 3/4 & 0 \\ a & \overline{\mathbf{2N}_c} & \mathbf{N}_c & 1/2 & 1 \\ b & \mathbf{2N}_c & \overline{\mathbf{N}_c} & 1/2 & -1 \end{array} \tag{5.136}$$

The superpotential is

$$\mathcal{W} = X^4 + X^2Y + Xab + (ab)^2 \; . \tag{5.137}$$

The central charges are $a = \frac{567}{512}N_c^2 - \frac{831}{2048}$ and $c = \frac{567}{512}N_c^2 - \frac{671}{2048}$. It follows that $\mathrm{Tr}\, R = \mathcal{O}(1)$. For this reason a weakly-coupled holographic dual is possible. The metric is anyway unknown and a comparison with the gravitational result in this case is not possible at the moment. For an expanded discussion about this theory see [128, 129].
The Cardy-like limit of the SCI in this case can be studied by solving the saddle point equations for the holonomies u_{i_1} $(i_1 = 1, \ldots, N_c)$ for the $SU(N_c)$ gauge factor, and u_{i_2} $(i_1 = 1, \ldots, 2N_c)$ for the $SU(2N_c)$ gauge factor, with the constraints

$$\sum_{i_1=1}^{N_c} u_{i_1} = \sum_{i_2=1}^{2N_c} u_{i_2} = 0 \mod \mathbb{Z} \,. \tag{5.138}$$

We find that the leading contribution in the region that we are interested in, which corresponds to imposing the constraint

$$\sum_{\Phi \in \mathcal{W}} \hat{\Delta}_\Phi = 2 \,, \tag{5.139}$$

is given by

$$\vec{u} = \left\{ u_{j_a}^{(m)} = \frac{m}{N_c} + \bar{u}_{j_a} \equiv \frac{m}{N_c} + v_{j_a}\tau \,\middle|\, v_{j_a} \sim \mathcal{O}(|\tau|^0) \,,\, \sum_{j_a=1}^{aN_c} v_{j_a} = 0 \right\} \tag{5.140}$$

with $a = 1, 2$ and $m = 0, \ldots, N_c - 1$. Imposing the constraints (5.139) and (5.88) on the charges $\hat{\Delta}_\Phi$ we have $\{\Delta_\Phi\}_\tau = \frac{2\tau - \eta}{2}(q_\Phi \hat{B} + R_\Phi \hat{v}_R)$, or more explicitly:

$$\begin{aligned}
\{\Delta_X\}_\tau &= \frac{2\tau - \eta}{4}\hat{v}_R \,, & \{\Delta_a\}_\tau &= \frac{2\tau - \eta}{2}\left(\frac{\hat{v}_R}{2} + \hat{B}\right) \,, \\
\{\Delta_Y\}_\tau &= \frac{3(2\tau - \eta)}{8}\hat{v}_R \,, & \{\Delta_b\}_\tau &= \frac{2\tau - \eta}{2}\left(\frac{\hat{v}_R}{2} - \hat{B}\right) \,.
\end{aligned} \tag{5.141}$$

where again the superpotential constraints imply $\hat{v}_R = 1$. Using these constraints we expand the effective action up to finite order in τ.
In order to compute the finite-order terms we need to perform two integrals, corresponding to the three-sphere partition function of pure CS theory $SU(2N_c)_{-2\eta N_c} \times SU(N_c)_{-\eta N_c}$. Evaluating these integrals we end up with

$$\begin{aligned}
\log \mathcal{I}^{\text{Laufer}}(\tau, B) \underset{|\tau| \to 0}{=} \frac{1}{128} i\pi \Bigg(& \frac{4(32\hat{B}^2 - 21)\eta N_c^2 + 13\eta}{4\tau^2} + \frac{36(32\hat{B}^2 - 21)N_c^2 + 277}{6\tau} \\
& + \frac{4}{3}N_c((288\hat{B}^2 + 11)\eta N_c + 144) - \eta - 128 \Bigg) + \log N_c \,,
\end{aligned} \tag{5.142}$$

which matches with the expected result in (5.80). Once again, the $\log N_c$ contribution is recovered because of the degeneration of the solutions (5.140) to the saddle point equations.

We conclude this analysis with an observation about the solutions of the saddle point equations. In this case the constraint $\hat{v}_R = 1$ allows for another solution of the type

$$\vec{u} = \left\{ u_{j_a}^{(m_a)} = \frac{m_a}{aN_c} + \bar{u}_{j_a} \equiv \frac{m_a}{aN_c} + v_{j_a}\tau \,\middle|\, v_{j_a} \sim \mathcal{O}(|\tau|^0) \,, \sum_{j_a=1}^{aN_c} v_{j_a} = 0 \right\} \tag{5.143}$$

with $2m_1 - m_2 = \pm N_c$. This solution can be visualized on the unitary circle as follows. On the $SU(N_c)$ gauge group the solution corresponds to placing N_c holonomies at the same point $\frac{m_1}{N_c}$ with $m_1 = 0, \dots, N_c - 1$, i.e. it is the same solution studied above. On the other hand each value of m_1 fixes $m_2 = 2m_1 \pm N_c$, where the sign is chosen such that $0 < m_2 < 2N_c - 1$. We have checked that the index expanded around this saddle is subleading with respect to the one evaluated around (5.140).

5.6 $\mathcal{N} = 1$ examples with $\mathrm{Tr}\, R = \mathcal{O}(N_c)$

In this section we go beyond the set of theories studied so far: we analyze SCFTs with $\mathrm{Tr}\, R = \mathcal{O}(N_c)$. For this reason they are not expected to have a weakly-coupled gravity dual and in this sense we refer to their Cardy-like limit as a generalization of the one studied above, without any reference to the dual rotating black hole.

5.6.1 $SU(N_c)$ SQCD

The simplest SCFT that we study is $SU(N_c)$ SQCD in the conformal window, with N_f pairs of fundamental and antifundamental flavors denoted Q and $\tilde{Q}$ respectively, with $\frac{3}{2}N_f \leq N_c \leq 3N_f$. The effective action in this case is given by the formula

$$
\begin{aligned}
S_{\text{eff}}^{\text{SQCD}}(\vec{u};\tau,\Delta) = N_f \sum_{i=1}^{N_c} \Big(\log \widetilde{\Gamma}(u_i + \Delta_Q) + \log \widetilde{\Gamma}(-u_i + \Delta_{\tilde{Q}}) \Big) \\
+ \sum_{i \neq j} \log \theta_0(u_{ij};\tau) + 2N_c \log(q;q)_\infty \, .
\end{aligned} \tag{5.144}
$$

There are $N_c - 1$ saddle point equations,

$$
\begin{aligned}
& B_2(\{u_j + \Delta_Q\}_\tau) - B_2(\{u_{N_c} + \Delta_Q\}_\tau) \\
& \quad - B_2(\{-u_j + \Delta_{\tilde{Q}}\}_\tau) + B_2(\{-u_{N_c} + \Delta_{\tilde{Q}}\}_\tau) = 0 \, ,
\end{aligned} \tag{5.145}
$$

labeled by the index $j = 1, \dots, N_c - 1$. In order to solve these equations we impose only the relation (5.88) on the charges $\hat{\Delta}_Q$ and $\hat{\Delta}_{\tilde{Q}}$, obtaining

$$
\begin{aligned}
\{\Delta_Q\}_\tau &= \frac{2\tau - \eta}{2} \left(1 - \frac{N_c}{N_f} + \frac{\hat{B}}{N_c} \right) \, , \\
\{\Delta_{\tilde{Q}}\}_\tau &= \frac{2\tau - \eta}{2} \left(1 - \frac{N_c}{N_f} - \frac{\hat{B}}{N_c} \right) \, .
\end{aligned} \tag{5.146}
$$

There are no further constraints to impose because the superpotential vanishes.
The leading solution of the saddle point equations (5.145) for the holonomies is $u_i = 0$ for $i = 1, \dots, N_c$. Indeed in this case we cannot package the holonomies on the unit circle as in the cases discussed above. As expected it reflects into the fact that the fields are charged under the $\mathbb{Z}_{N_c}$ center of the gauge group with charge 1. We then expand the effective action evaluated

around this saddle point up to finite order in τ, under the constraints (5.146) with $\eta = \pm 1$.

$$S_{\text{eff}}^{\text{SQCD}}(\vec{u};\tau,\Delta) = -\frac{\pi i \eta N_c}{\tau^2}\sum_{i=1}^{N_c}\bar{v}_i^2 + \sum_{i\neq j}^{N_c}\log\left(2\sin\left(\frac{\pi \bar{v}_{ij}}{\tau}\right)\right) + N_f N_c (J(\hat{\Delta}_Q) + 2J(\hat{\Delta}_{\tilde{Q}}))$$
$$-\frac{i\pi\left(\tau^2+1\right)N_c^2}{6\tau} + \frac{1}{2}i\pi N_c + \frac{i\pi\left(\tau^2-3\tau+1\right)}{6\tau} - 2(N_c-1)\log\tau\,. \qquad (5.147)$$

In order to compute the finite-order terms we need to perform the matrix integral corresponding to the three-sphere partition function of pure 3d Chern–Simons theory $SU(N_c)_{-\eta N_c}$. Evaluating the integral we find

$$S_{\text{eff}}^{\text{SQCD}}(\vec{u};\tau,\Delta) = \frac{i\pi}{6}\left(\frac{\eta(N_f^2(N_c^2-3\hat{B}^2)-N_c^4)}{2\tau^2 N_f^2} + \frac{9\hat{B}^2N_f^2 - 2N_c^2N_f^2 + 3N_c^4 + N_f^2}{\tau N_f^2}\right.$$
$$\left. + 18\hat{B}^2\eta - N_c\left(\eta N_c\left(1-\frac{6N_c^2}{N_f^2}\right)+3\right)-3\right)\,. \qquad (5.148)$$

Observe the absence of corrections of the form $\log N_c$ to $\log \mathcal{I}^{\text{SQCD}}(\tau,\Delta)$ in this case: this is because we only have a single solution (the one with vanishing holonomies) to the saddle point equations. Also in this case it is possible to check that the expression (5.148) matches with the general one in (5.80) in terms of the central charges a and c evaluated on the charges $\hat{\Delta}_Q$ and $\hat{\Delta}_{\tilde{Q}}$.

We can study the Seiberg dual phase as well. The model is $SU(N_f - N_c)$ gauge theory with N_f pairs of dual fundamentals and anti-fundamentals denoted q and $\tilde{q}$ respectively, as well as N_f^2 singlets M, identified with the $Q\tilde{Q}$ mesons of the electric theory. There is also a superpotential $\mathcal{W} = Mq\tilde{q}$. Imposing the constraint from the superpotential and from the anomaly cancellation on the charges $\hat{\Delta}_\Phi$ we have

$$\begin{aligned}
\{\Delta_q\}_\tau &= \frac{2\tau-\eta}{2}\left(\frac{N_f}{N_c}+\frac{\hat{B}}{N_f-N_c}\right)\,,\\
\{\Delta_{\tilde{q}}\}_\tau &= \frac{2\tau-\eta}{2}\left(\frac{N_f}{N_c}-\frac{\hat{B}}{N_f-N_c}\right)\,,\\
\{\Delta_M\}_\tau &= (2\tau-\eta)\left(1-\frac{N_f}{N_c}\right)\,.
\end{aligned} \qquad (5.149)$$

Using these relations, that can be regarded as the duality map, it is straightforward to match the magnetic index in the Cardy-like limit with the one obtained in the electric phase.

5.6.2 $SU(N_c)$ adjoint SQCD and accidental symmetries

In this section we study another SCFT with $\text{Tr}\, R = \mathcal{O}(N_c^2)$, namely adjoint SQCD with a power-law superpotential for the adjoint field. The model consists of an $\mathcal{N}=2$ $SU(N_c)$ gauge theory with N_f pairs of fundamental Q and anti-fundamentals $\tilde{Q}$ chiral multiplets and an adjoint chiral multiplet X. The superpotential is

$$\mathcal{W} = \text{Tr}\, X^{k+1} \qquad (5.150)$$

with $k \in \mathbb{N}$ and $k < N_c$. The constraints on the charges $\hat{\Delta}_\Phi$ imply

$$\begin{aligned}
\{\Delta_X\}_\tau &= \frac{2\tau - \eta}{k+1} \,, \\
\{\Delta_Q\}_\tau &= \frac{2\tau - \eta}{2} \left(1 - \frac{2N_c}{(k+1)N_f} + \frac{\hat{B}}{N_c} \right) \,, \\
\{\Delta_{\tilde{Q}}\}_\tau &= \frac{2\tau - \eta}{2} \left(1 - \frac{2N_c}{(k+1)N_f} - \frac{\hat{B}}{N_c} \right) \,.
\end{aligned} \tag{5.151}$$

As in the case of SQCD the saddle point equations are solved by $u_i = 0$ for $i = 1, \ldots, N_c$. Performing the expansion of the effective action around this solution we obtain

$$\begin{aligned}
S_{\text{eff}}^{\text{SQCD}_{\text{adj}}} = \frac{i\pi}{3(k+1)^3 N_f^2} \Bigg(& \frac{\eta(N_f^2((2k^2+k+1)N_c^2 - 3\hat{B}^2(k+1)^2 + (1-k)k) - 4N_c^4)}{2\tau^2} \\
+ & \frac{N_f^2(9\hat{B}^2(k+1)^2 - (5k^2+k+2)N_c^2 + 4k^2 - k + 1) + 12N_c^4}{\tau} \\
- & \frac{\eta(N_f^2(36\hat{B}^2(k+1)^2 - (k(19k+2)+7)N_c^2 + 17k^2 - 2k + 5) + 48N_c^4)}{2} \Bigg) .
\end{aligned} \tag{5.152}$$

Again there is no $\log N_c$ correction and it is possible to check that the expression (5.152) matches with the general one in (5.80) in terms of the central charges a and c evaluated on the charges $\hat{\Delta}_\Phi$.

Also in this case we can study the Seiberg-dual theory, derived in [137]. This is a $SU(kN_f - N_c)$ gauge theory with N_f pairs of fundamental q and anti-fundamental $\tilde{q}$ chiral multiplets, an adjoint chiral multiplet Y and kN_f^2 singlets $M_j \equiv QX^j\tilde{Q}$ with superpotential

$$\mathcal{W} = \operatorname{Tr} Y^{k+1} + \sum_{j=0}^{k-1} M_j q Y^{k-1-j} \tilde{q} \,. \tag{5.153}$$

In this case the constraints imposed on the charges $\hat{\Delta}_\Phi$ by the anomaly cancellations and by the superpotential translate into the following constraints on the $\{\Delta_\Phi\}_\tau$ variables:

$$\begin{aligned}
\{\Delta_Y\}_\tau &= \frac{2\tau - \eta}{k+1} \,, \\
\{\Delta_q\}_\tau &= \frac{2\tau - \eta}{2} \left(\frac{2N_c - (k-1)N_f}{(k+1)N_f} - \frac{\hat{B}}{kN_f - N_c} \right) \,, \\
\{\Delta_{\tilde{q}}\}_\tau &= \frac{2\tau - \eta}{2} \left(\frac{2N_c - (k-1)N_f}{(k+1)N_f} + \frac{\hat{B}}{kN_f - N_c} \right) \,, \\
\{\Delta_M\}_\tau &= (2\tau - \eta) \left(\left(1 - \frac{2N_c}{(k+1)N_f} \right) + \frac{j}{k+1} \right) .
\end{aligned} \tag{5.154}$$

Using such constraints and the fact that the saddle point equations are solved only by $u_i = 0$ for $i = 1, \ldots, kN_f - N_c$, we can match the dual magnetic index in the Cardy-like limit at finite order in τ.

We conclude the analysis of this model by discussing the modification of the above formulae in presence of accidental symmetries associated with gauge-invariant operators $\mathcal{O}_i$ in the chiral ring that violate the unitarity bound, i.e. $R_{\mathcal{O}_i} < \frac{2}{3}$ in 4d. In this case one has to modify a and c accordingly [138], by adding the contribution of a singlet and subtracting the contribution of the operator that violates the bound. In terms of the charge $\hat{\Delta}_{\mathcal{O}}$, the variations $\boldsymbol{\Delta}$ underwent by the central charges are:

$$\boldsymbol{\Delta}a(\hat{\Delta}_{\mathcal{O}}) = a\left(\frac{2}{3}\right) - a(\hat{\Delta}_{\mathcal{O}}) \, , \quad \boldsymbol{\Delta}c(\hat{\Delta}_{\mathcal{O}}) = c\left(\frac{2}{3}\right) - c(\hat{\Delta}_{\mathcal{O}}) \, . \tag{5.155}$$

This translates into the modification of formula (5.80) by a term

$$\begin{aligned} &\frac{4\pi i(\eta - 6\tau + 12\eta\tau^2 + \dots)(3\boldsymbol{\Delta}c(\hat{\Delta}_{\mathcal{O}}) - 2\boldsymbol{\Delta}a(\hat{\Delta}_{\mathcal{O}}))}{27\tau^2} \\ &+ \frac{8\pi i(2 - 5\eta\tau + \dots)(\boldsymbol{\Delta}c(\hat{\Delta}_{\mathcal{O}})) - \boldsymbol{\Delta}a(\hat{\Delta}_{\mathcal{O}}))}{6\tau} \, . \end{aligned} \tag{5.156}$$

We should now modify the calculation of the SCI, according to the discussion in [139], by adding the contribution

$$\boldsymbol{\Delta}Q(\mathcal{O};\tau) = \frac{i\pi(3\eta\tau^2 + 4\eta - 6\tau)}{324\tau^2} - Q(\{\Delta_{\mathcal{O}}\}_\tau;\tau) \, , \tag{5.157}$$

with $\hat{\Delta}_{\mathcal{O}} = \frac{2}{2\tau-\eta}\{\Delta_{\mathcal{O}}\}_\tau$ (and $Q(u;\tau)$ defined in (5.15)). It is straightforward to prove that (5.156) and (5.157) coincide at leading order in τ for $\eta = \pm 1$.
These formulae can be applied to the case of adjoint SQCD, where the mesons M_j hit the bound of unitarity if we vary the values of k, N_c and N_f. For example the meson M_0 hits the bound if $\frac{N_f}{N_c} < \frac{6}{k+1}$. In such a case we must modify the central charges and the index as above in order to correctly reproduce its behavior in the Cardy-like limit. On the field theory side the presence of the meson hitting the unitarity bound corresponds to adding a superpotential term $\mathrm{Tr}\, N(M_0 + Q\tilde{Q})$ in the electric theory, where N is a gauge singlet. The first term in this superpotential is irrelevant and its coupling flows to zero in the IR while the second term becomes exactly marginal at the fixed point. We are then left with a free singlet of R-charge $\frac{2}{3}$ and an interacting one with R-charge $2 - R_Q - R_{\tilde{Q}}$. This procedure, that modifies the central charge as in formula (5.155), clarifies also the behavior of the dual theory, where the coupling $M_0 q Y^{k-1}\tilde{q}$ becomes irrelevant and one is left with a free singlet M_0. This analysis can be applied to match the Cardy-like limit of the electric index with the one of the dual phase if the meson M_0 hits the unitarity bound. The discussion can then be generalized to other singlets, in this case the other mesons $M_{j>0}$.

5.6.3 $USp(2N_c)$ SQCD and Intriligator–Pouliot duality

We wish to conclude our analysis with the case of an $\mathcal{N} = 1$ duality involving a real gauge group. We focus on one of the simplest cases, i.e. the generalization of Seiberg duality to $USp(2N_c)$ SQCD, originally derived in [140].
In this case the $USp(2N_c)$ electric SQCD has $2N_f$ fundamentals Q and vanishing superpotential

and it is dual to $USp(2\tilde{N}_c \equiv 2(N_f - N_c - 2))$ with $2N_f$ dual fundamentals q and an antisymmetric meson $M = Q \cdot Q$ with superpotential $\mathcal{W} = M \cdot q \cdot q$, where $\cdot$ represents the symplectic product. In this case we find that only the solution at vanishing holonomies is allowed in both the electric and magnetic theory. The charges $\{\Delta_\Phi\}_\tau$ are given by

$$\begin{aligned} \{\Delta_Q\}_\tau &= \frac{2\tau - \eta}{2} \frac{N_f - N_c - 1}{N_f} , \\ \{\Delta_q\}_\tau &= \frac{2\tau - \eta}{2} - \{\Delta_Q\}_\tau , \\ \{\Delta_M\}_\tau &= 2\{\Delta_Q\}_\tau . \end{aligned} \tag{5.158}$$

By performing the calculation we find that, both in the electric and magnetic theory, the effective action is given by

$$\begin{aligned} S_{\text{eff}}^{USp(2N_c)}(\vec{u};\tau,\Delta) = -\frac{i\pi}{6N_f^2}\Bigg(& \frac{\eta(N_c+1)N_c((N_c+1)^2 - N_f^2)}{\tau^2} \\ - \frac{N_c(6(N_c+1)^3 - (4N_c+3)N_f^2)}{\tau} & + \frac{\eta N_c(24(N_c+1)^3 - (14N_c+9)N_f^2)}{2}\Bigg) , \end{aligned} \tag{5.159}$$

and it can be checked that $\log \mathcal{I}^{USp(2N_c)}(\tau,\Delta)$ extracted from this action coincides with the general result in (5.80).

5.7 $\mathcal{N} = 2$ examples

5.7.1 The (A_1, A_{2n-1}) Argyres–Douglas $\mathcal{N} = 1$ Lagrangians

In this section we study a family of $\mathcal{N} = 1$ Lagrangian field theories that enhance in the IR to the $\mathcal{N} = 2$ (A_1, A_{2n-1}) AD fixed points. The models consist of an $\mathcal{N} = 1$ $SU(n)$ gauge theory with a fundamental q, an anti-fundamental $\tilde{q}$, and an adjoint ϕ with superpotential

$$\mathcal{W} = \sum_{i=0}^{n-2} \alpha_i \operatorname{Tr} q\phi^i\tilde{q} + \sum_{j=2}^{n} \beta_j \operatorname{Tr} \phi^j , \tag{5.160}$$

where α_i and β_j are gauge singlets. The table of charges is obtained by performing a-maximization after imposing the constraints from the anomalies and from the superpotential:

	$U(1)_R$	$U(1)_T$	$U(1)_B$
q	$\frac{n+3}{3(n+1)}$	$\frac{2}{3(n+1)}$	1
$\tilde{q}$	$\frac{n+3}{3(n+1)}$	$\frac{2}{3(n+1)}$	-1
ϕ	$\frac{2}{3(n+1)}$	$\frac{2}{3(n+1)}$	0
α_i	$\frac{4n-2i}{3(n+1)}$	$\frac{4n-2i}{3(n+1)}$	0
β_j	$2-\frac{2j}{3(n+1)}$	$-\frac{2j}{3(n+1)}$	0

(5.161)

with $U(1)_{T,B}$ two flavor symmetries. The central charges are

$$a = \frac{1}{2}(n+1) + \frac{1}{2(n+1)} - \frac{29}{24} , \quad c = \frac{1}{2}(n+1) + \frac{1}{2(n+1)} - \frac{7}{6} . \tag{5.162}$$

The Cardy-like limit is studied in terms of charges that satisfy the following constraints

$$\begin{aligned}
\hat{\Delta}_q &= \frac{n+3}{3n+3} + \frac{2\hat{T}}{3\,(n+1)} + \hat{B}\,, \\
\hat{\Delta}_{\tilde{q}} &= \frac{n+3}{3n+3} + \frac{2\hat{T}}{3\,(n+1)} - \hat{B}\,, \\
\hat{\Delta}_\phi &= \frac{2}{3\,(n+1)} + \frac{2\hat{T}}{3\,(n+1)}\,, \\
\hat{\Delta}_{\alpha_i} &= \frac{4n-2i}{3\,(n+1)} + \frac{2\hat{T}(2n-i)}{3\,(n+1)}\,, \\
\hat{\Delta}_{\beta_j} &= 2 - \frac{2j}{3\,(n+1)} - \frac{2j\hat{T}}{3\,(n+1)}\,.
\end{aligned} \tag{5.163}$$

The saddle point equations are solved by $u_i = 0$, consistently with what has been found in [79]. Expanding the effective action around this vacuum and using the constraints (5.163) we obtain the following result:

$$\begin{aligned}
S_{\text{eff}}^{\text{AD}} = -\frac{i\pi\eta}{162\tau^2(n+1)^3}\Big[&\hat{T}^3(n(10-n(n+1)(7n-3))+2) \\
&-3\hat{T}^2(n+1)(n((n-6)n+2)+1) \\
&-3\hat{T}(n(n((n-6)n+6)+6-9\hat{B}^2(n+1)^2)+1) \\
&+(n+1)^2(n(3(9\hat{B}^2-4)n+2)+2)\Big] \\
+\frac{i\pi}{27\tau(n+1)^3}\Big[&\hat{T}^3(n(10-n(n+1)(7n-3))+2) \\
&-3\hat{T}^2(n+1)(n((n-6)n+2)+1) \\
&-3\hat{T}n((n-1)^2(3n+1)-9\hat{B}^2(n+1)^2) \\
&+(n+1)^2(n(3(9\hat{B}^2-4)n+5)+5)\Big] \\
-\frac{i\pi\eta}{54(n+1)^3}\Big[&4\hat{T}^3(n(10-n(n+1)(7n-3))+2) \\
&-12\hat{T}^2(n+1)(n((n-6)n+2)+1) \\
&+3\hat{T}(n(36\hat{B}^2(n+1)^2+(19-14n)n^2+n+1)+1) \\
&+(n+1)^2(n(12(9\hat{B}^2-4)n+23)+23)\Big]\,,
\end{aligned} \tag{5.164}$$

and the associated $\log \mathcal{I}^{\text{AD}}(\tau, \Delta)$ coincides with what is expected from the general formula (5.80).

5.7.2 An $\mathcal{N}=2$ orbi-orientifold and its dual black hole entropy

The last model that we study is an $\mathcal{N}=2$ SCFT with gauge group $SU(N_c)$, plus a symmetric and an antisymmetric hypermultiplet. This model has been studied in [141] (model A5 there) and can be engineered by N_c D3-branes in the background of an O7 where a further orbifold

acts on the internal spacetime [142, Sec. 3.5].
In $\mathcal{N}=1$ language we have an adjoint chiral X, a symmetric chiral multiplet S with its conjugate $\widetilde{S}$, and an antisymmetric chiral multiplet A with its conjugate $\widetilde{A}$. The $\mathcal{N}=1$ superpotential is

$$\mathcal{W} = AX\widetilde{A} + SX\widetilde{S}\,. \tag{5.165}$$

The table of charges is:

$$\begin{array}{c|cccc} & U(1)_R & SU(2)_R & U(1)_\ell & U(1)_{\tilde{\ell}} \\ \hline X & 2 & 0 & 0 & 0 \\ A & 0 & 1 & 1 & 1 \\ \widetilde{A} & 0 & 1 & -1 & -1 \\ S & 0 & 1 & 1 & -1 \\ \widetilde{S} & 0 & 1 & -1 & 1 \end{array} \tag{5.166}$$

where $U(1)_R$ and $SU(2)_R$ are the $\mathcal{N}=2$ R-symmetries (with generators $T_{\mathcal{N}=2}$ and J_3 respectively), and we have used the notation of [142] to identify the two non-R global symmetries. The $\mathcal{N}=1$ $U(1)_R$ R-symmetry is given by $T_{\mathcal{N}=1} = \frac{1}{3}T_{\mathcal{N}=2} + \frac{4}{3}J_3$. We refer the reader to [142] for a more complete discussion of this model and for the holographic dual construction.
Let us make the following redefinitions: $\delta_1 \equiv \Delta_A$, $\delta_2 \equiv \Delta_S$, $\delta_3 \equiv \Delta_{\widetilde{A}}$, $\delta_4 \equiv \Delta_{\widetilde{S}}$. The SCI for this model reads:

$$\begin{aligned}\mathcal{I}^{\mathcal{N}=2}(\tau,\Delta) = {} & \frac{(q;q)_\infty^{2(N_c-1)}\tilde{\Gamma}(\Delta_X)^{N_c-1}}{N_c!}\int\prod_{i=1}^{N_c}du_i\,\frac{\prod_{a=1}^{2}\prod_{i<j}^{N_c}\tilde{\Gamma}(u_{ij}^{+}+\delta_a)}{\prod_{i\neq j}^{N_c}\tilde{\Gamma}(u_{ij}^{-})}\\ & \cdot\prod_{b=3}^{4}\prod_{i<j}^{N_c}\tilde{\Gamma}(-u_{ij}^{+}+\delta_b)\prod_{i=1}^{N_c}\tilde{\Gamma}(2u_i+\delta_2)\tilde{\Gamma}(-2u_i+\delta_4)\prod_{i\neq j}^{N_c}\tilde{\Gamma}(u_{ij}^{-}+\Delta_X)\,,\end{aligned} \tag{5.167}$$

where $u_{ij}^{\pm} \equiv u_i \pm u_j$. Using the definition (5.6), we can write down the effective action at leading order in $|\tau|$:

$$\begin{aligned}S_{\text{eff}}^{\mathcal{N}=2}(\vec{u};\tau,\Delta) = {} & -\frac{i\pi}{3\tau^2}\Bigg(\sum_{a=1}^{2}\sum_{i<j}^{N_c}B_3(\{u_{ij}^{+}+\delta_a\}_\tau)+\sum_{a=3}^{4}\sum_{i<j}^{N_c}B_3(\{-u_{ij}^{+}+\delta_a\}_\tau)\\ & +\sum_{i=1}^{N_c}B_3(\{2u_i+\delta_2\}_\tau)+B_3(\{-2u_i+\delta_4\}_\tau)+\sum_{i\neq j}^{N_c}B_3(\{u_{ij}^{-}+\Delta_X\}_\tau)\Bigg)\,.\end{aligned} \tag{5.168}$$

We now compute $\frac{\partial}{\partial u_k}S_{\text{eff}}^{\mathcal{N}=2} = 0$ for $k = 1,\dots,N_c-1$ upon imposing the $SU(N_c)$ constraint $\sum_{i=1}^{N_c}u_i = 0 \mod \mathbb{Z}$ on the holonomies in (5.168). We obtain:

$$\begin{aligned}&\sum_{a=1}^{2}\left(\sum_{i\neq k}^{N_c}B_2(\{u_{ik}^{+}+\delta_a\}_\tau)-\sum_{i=1}^{N_c-1}B_2(\{u_{iN_c}^{+}+\delta_a\}_\tau)\right)\\ &-\sum_{a=3}^{4}\left(\sum_{i\neq k}^{N_c}B_2(\{-u_{ik}^{+}+\delta_a\}_\tau)-\sum_{i=1}^{N_c-1}B_2(\{-u_{iN_c}^{+}+\delta_a\}_\tau)\right)\end{aligned}$$

$$
\begin{aligned}
&+2\Big(B_2(\{2u_k+\delta_2\}_\tau)-B_2(\{-2u_k+\delta_4\}_\tau) \\
&-B_2(\{2u_{N_c}+\delta_2\}_\tau)+B_2(\{-2u_{N_c}+\delta_4\}_\tau)\Big) \\
&+\sum_{i=1}^{N_c}\Big(B_2(\{u_{ki}^-+\Delta_X\}_\tau)-B_2(\{-u_{ki}^-+\Delta_X\}_\tau) \\
&-B_2(\{u_{N_ci}^-+\Delta_X\}_\tau)+B_2(\{-u_{N_ci}^-+\Delta_X\}_\tau)\Big)=0\,. \qquad (5.169)
\end{aligned}
$$

Given the fact that the holonomies live on a torus with modular parameter τ, i.e. $u_i \sim u_i+1$, we immediately see that for even N_c we can solve the above equations identically by taking all holonomies equal to $u_i=0$ or $u_i=\frac{1}{2}$ (indeed $u_{N_c}=-\sum_{i=1}^{N_c-1}u_i=0$ or $\frac{1}{2}(N_c-1)=\frac{1}{2} \mod 1$ respectively); for odd N_c we only have the $u_i=0$ solution (with $u_{N_c}=0$). These saddle points are again consistent with the fact that in the odd case only the gauge group $SU(N_c)$ is allowed by the matter content (i.e. it is charged under the full $\mathbb{Z}_{N_c}$ center) while in the even case we could also have $SU(N_c)/\mathbb{Z}_2$ gauge group (i.e. the matter is uncharged under a $\mathbb{Z}_2$ subgroup of the center). We will see that this reflects into the $\log\Gamma_Z$ correction in (5.80) specialized to the current gauge group and matter content.
In order to study the Cardy-like limit of the SCI we impose the superpotential and anomaly constraints on the charges $\hat{\Delta}_\Phi$.[12] These translate into the following relations on the charges $\{\Delta_\Phi\}_\tau$:

$$
\{\Delta_\Phi\}_\tau=\frac{2\tau-\eta}{2}\left(R_1\hat{\Delta}_1+R_2\hat{\Delta}_2+q_\ell\hat{\Delta}_\ell+q_{\tilde{\ell}}\hat{\Delta}_{\tilde{\ell}}\right)\,, \qquad (5.170)
$$

where $\hat{\Delta}_{1,2,\ell,\tilde{\ell}}$ are the (hatted) chemical potentials of the symmetries in table (5.166), and $R_{1,2}$, $q_{\ell,\tilde{\ell}}$ the matter field charges under the latter. More explicitly:

$$
\begin{aligned}
\{\Delta_X\}_\tau&=\frac{2\tau-\eta}{2}\hat{\Delta}_1\,,\\
\{\Delta_A\}_\tau&=\frac{2\tau-\eta}{4}(\hat{\Delta}_2+\hat{\Delta}_\ell+\hat{\Delta}_{\tilde{\ell}})\,,\\
\{\Delta_{\tilde{A}}\}_\tau&=\frac{2\tau-\eta}{4}(\hat{\Delta}_2-\hat{\Delta}_\ell-\hat{\Delta}_{\tilde{\ell}})\,,\\
\{\Delta_S\}_\tau&=\frac{2\tau-\eta}{4}(\hat{\Delta}_2+\hat{\Delta}_\ell-\hat{\Delta}_{\tilde{\ell}})\,,\\
\{\Delta_{\tilde{S}}\}_\tau&=\frac{2\tau-\eta}{4}(\hat{\Delta}_2-\hat{\Delta}_\ell+\hat{\Delta}_{\tilde{\ell}})\,,
\end{aligned}
\qquad (5.171)
$$

where the superpotential and the anomaly cancellation impose the same constraint, namely $\hat{\Delta}_1+\hat{\Delta}_2=2$. We thus find: We thus find:

$$
\begin{aligned}
S_{\text{eff}}^{\mathcal{N}=2}(\vec{u};\tau,\Delta)=&-\frac{i\pi(12\eta\tau^2-6\tau+\eta)}{32\tau^2}\Big(\hat{\Delta}_1(\hat{\Delta}_2^2-\hat{\Delta}_\ell^2-\hat{\Delta}_{\tilde{\ell}}^2)N_c^2\\
&-2\hat{\Delta}_1\hat{\Delta}_\ell\hat{\Delta}_{\tilde{\ell}}N_c+\frac{4}{9}\hat{\Delta}_1(3\hat{\Delta}_1(\hat{\Delta}_2+1)-8)\Big)\\
&+\frac{i\pi\eta(\hat{\Delta}_1-\hat{\Delta}_2+2)(1-3\tau^2)}{72\tau^2}+\log\frac{3+(-1)^{N_c}}{2}\,. \qquad (5.172)
\end{aligned}
$$

[12] We also found other saddle point solutions to (5.169), which are however subleading in the BH region specified by these constraints on the charges $\hat{\Delta}_\Phi$, and for this reason are not discussed here.

Again we find that $\log \mathcal{I}^{\mathcal{N}=2}(\tau, \Delta)$ in the Cardy-like limit is given by (5.80) where the logarithmic correction is $\log 2$ in the even N_c case and it vanishes in the odd N_c one.

We conclude by observing that formula (5.172) reproduces the *universal* result of [90] at leading order in N_c for $\hat{\Delta}_\ell = \hat{\Delta}_{\tilde{\ell}} = 0$ and up to finite order terms in $\sigma = \tau$:

$$S_{\text{eff}}^{\mathcal{N}=2}(\vec{u}; \tau, \Delta) = -\frac{i\pi N_c^2 (12\eta\tau^2 - 6\tau + \eta)\hat{\Delta}_1 \hat{\Delta}_2^2}{32\tau^2} + \mathcal{O}(|\tau|) \; . \tag{5.173}$$

We can also compute the entropy of the dual Kerr–Newman black hole that is expected from the holographic duality, by considering only the leading contribution in N_c^2 and distinguishing the two fugacities τ and σ for the rotations:

$$S_{\text{BH}}^{\mathcal{N}=2} = 2\pi \sqrt{Q_2^2 - Q_\ell^2 - Q_{\tilde{\ell}}^2 + 2Q_1(Q_2 - Q_\ell - Q_{\tilde{\ell}}) - \frac{a}{4}(J_1 + J_2)} \; , \tag{5.174}$$

where $a = \frac{1}{4} N_c^2$ is the central charge of the 4d theory and the other quantities are the electric charges $Q_{1,2,\ell,\tilde{\ell}}$ and the angular momenta $J_{1,2}$ of the dual black hole. By turning off Q_ℓ and $Q_{\tilde{\ell}}$ this reduces to the result of [90], as expected.

Part III

In the bulk

Chapter 6

Matter-coupled $\mathcal{N} = 2$ gauged supergravity in $d = 4, 5$

In this chapter we present a review of the main features of supergravity theories with $\mathcal{N} = 2$ supersymmetry in four and five dimensions, coupled to matter fields, in which some of the global symmetries are gauged.
As well known, lower-dimensional supergravities arise in a natural way when supersymmetry is made local. Nevertheless, part of the interest in these theories comes from the fact that some of them can be obtained also from consistent truncations of ten- or eleven-dimensional supergravities. Hence, they allow to study simpler models that can be eventually uplifted in higher dimensions, leading to important applications even in the framework of the AdS/CFT correspondence.

The simplest supergravity theory that one can consider has $\mathcal{N} = 1$ supersymmetry and does not contain matter. When $\mathcal{N} \geq 2$ we talk about *extended supergravity* in contrast with minimal supergravity. In the rest of this work we will always consider $\mathcal{N} = 2$. On the other hand, when the supergravity theory contains also other multiplets besides the supergravity one is called *matter-coupled supergravity*. When matter is not included the theory is often called pure supergravity.
An important property on which we will expand later is that matter-coupled supergravity enjoys a series of global symmetries closely connected to the scalar fields of the theory. Some of these symmetries can be made local, or in other words can be gauged, enriching the structure of the starting system. As we will see, many different gaugings are possible, giving rise to a plethora of different theories, called *gauged supergravities*.

The main references we have followed for writing this chapter are the books [28,38,143] and the papers [144–148], while the structure is largely inspired by [34,35]. Further references are given in each section.
The content is organized as follows. In section 6.1 we list the gravity and matter multiplets, both in four and five dimensions, while in section 6.2 we describe the moduli spaces of the two theories. In section 6.3 we present a short review on the isometries and their possible gaugings, and finally in section 6.4 we write the Lagragians for both the theories and we report the supersymmetry variations for the five-dimensional case.

6.1 Gravity and matter multiplets

As in any supersymmetric theory, the supergravity fields are organized in supermultiplets. Here we restrict ourselves to supergravity theories with $\mathcal{N} = 2$ supersymmetry coupled to n_V vector multiplets and n_H hypermultiplets, in four and five dimensions. It is common to denote vector multiplets and hypermultiplets collectively as matter multiplets.

6.1.1 $d = 4$ multiplets

The supermultiplets of the four-dimensional theory are the following.

- **Supergravity multiplet**:
$$\{e_\mu^a, \psi_\mu^i, A_\mu\}\,, \tag{6.1}$$
containing the vierbein e_μ^a (the *graviton*), an $SU(2)_R$-doublet of spin-3/2 fermions ψ_μ^i (two *gravitinos*) and a vector field A_μ (the *graviphoton*). The index $\mu = 0, \ldots, 4$ parametrizes local spacetime, $a = 0, \ldots, 4$ is the tangent space index and $i = 1, 2$ is the $SU(2)$ index, coming from $U(2)_R \cong U(1)_R \times SU(2)_R$.

- **Vector multiplets**:
$$\{A_\mu^\alpha, \lambda^{\alpha\, i}, z^\alpha\}\,, \tag{6.2}$$
where $\alpha = 1, \ldots, n_V$ labels the n_V vector multiplets, each of them containing a vector field A_μ^α, an $SU(2)$ doublet of Majorana spinors $\lambda^{\alpha\, i}$ (two *gauginos*) and a complex scalar z^α.

- **Hypermultiplets**:
$$\{q^X, \zeta^A\}\,, \tag{6.3}$$
where $X = 1, \ldots, 4n_H$ and $A = 1, \ldots, 2n_H$. Each of the n_H hypermultiplets contains four scalars q^X (*hyperscalars*) and two spin-1/2 fermions (two *hyperinos*). Notice that hyperinos do not carry $SU(2)$ indices.

It is convenient to group all the vector fields of the theory together as $A_\mu^\Lambda = (A_\mu^0, A_\mu^x)$, with $\Lambda = 0, 1, \ldots, n_V$, where A_μ^0 denotes the graviphoton.

6.1.2 $d = 5$ multiplets

The supermultiplets of the five-dimensional theory are the following.

- **Supergravity multiplet**:
$$\{e_\mu^a, \psi_\mu^i, A_\mu\}\,, \tag{6.4}$$
containing the fünfbein e_μ^a (the *graviton*), an $SU(2)_R$-doublet of spin-3/2 fermions ψ_μ^i (two *gravitinos*) and a vector field A_μ (the *graviphoton*). The index $\mu = 0, \ldots, 4$ parametrizes local spacetime, $a = 0, \ldots, 4$ is the tangent space index and $i = 1, 2$ is the $SU(2)_R$ index.

- **Vector multiplets**:
$$\{A_\mu^x, \lambda^{x\, i}, \phi^x\}\,, \tag{6.5}$$
where $x = 1, \ldots, n_V$ labels the n_V vector multiplets, each of them containing a vector field A_μ^x, an $SU(2)_R$-doublet of symplectic Majorana spinors $\lambda^{x\, i}$ (two *gauginos*) and a real scalar ϕ^x.

- **Hypermultiplets**:

$$\{q^X, \zeta^A\}, \tag{6.6}$$

where $X = 1, \ldots, 4n_H$ and $A = 1, \ldots, 2n_H$. Each of the n_H hypermultiplets contains four reals scalars q^X (*hyperscalars*) and two spin-1/2 fermions (two *hyperinos*). Notice that hyperinos do not carry $SU(2)$ indices.

It is convenient to group all the vector fields of the theory together as $A^I_\mu = (A^0_\mu, A^x_\mu)$, with $I = 0, 1, \ldots, n_V$, where A^0_μ denotes the graviphoton.

6.2 Moduli spaces

The matter multiplets parametrize a manifold $\mathcal{M}$, called moduli space, which is given by the direct product

$$\mathcal{M} = \mathcal{S} \otimes \mathcal{Q}, \tag{6.7}$$

where $\mathcal{S}$ and $\mathcal{Q}$ specify the manifolds parametrized by the scalars that belong to the vector multiplets and to the hypermultiplets, respectively.
In particular:

- the manifold $\mathcal{S}$ is
  - a special Kähler manifold in $d = 4$,
  - a very special real manifold in $d = 5$,
- the manifold $\mathcal{Q}$ is a quaternionic-Kähler manifold, both in $d = 4$ and in $d = 5$.

In the following we give a short review of the main features of these spaces.

6.2.1 Special Kähler manifold

The complex scalar z^α of the vector multiplets in $d = 4$ $\mathcal{N} = 2$ gauged supergravity parametrize a target space called *special Kähler manifold* [144, 149–154]. An n_V-dimensional special Kähler manifold is a Kähler–Hodge manifold $\mathcal{M}$ with holomorphic coordinates z^α that satisfy the two following properties:

- There exists a holomorphic $Sp(2n_V + 2, \mathbb{R})$-bundle $\mathcal{H}$ over $\mathcal{M}$ and a *holomorphic section* $v(z)$ of $\mathcal{L} \otimes \mathcal{H}$ such that the Kähler potential is

$$\mathcal{K} = -\log\left(i\langle v, \bar{v}\rangle\right). \tag{6.8}$$

  where $\langle V, W\rangle = V_\Lambda W^\Lambda - V^\Lambda W_\Lambda$ is the symplectic-invariant antisymmetric form and $\mathcal{L}$ denotes a line bundle (a holomorphic vector bundle of rank 1) over $\mathcal{M}$ such that its first Chern class equals the cohomology class of the Kähler potential.

- The section $v(z)$ satisfies

$$\langle v, \partial_\alpha v\rangle = 0\,. \tag{6.9}$$

Special Kähler manifolds are a class of Kähler manifolds and thus they are completely characterized by a real function $\mathcal{K}$ called Kähler potential, that defines the metric of the manifold as

$$G_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}(z, \bar{z})\,. \tag{6.10}$$

The line bundle $\mathcal{L}$ can be associated with the Kähler transformation

$$\mathcal{K} \to \mathcal{K} + f(z) + \bar{f}(\bar{z})\,, \tag{6.11}$$

that can be thought as a $U(1)$ gauge transformation with connection

$$\mathcal{A}_\mu = -\frac{i}{2}(\partial_\alpha \mathcal{K}\, \partial_\mu z^\alpha - \partial_{\bar{\alpha}} \mathcal{K}\, \partial_\mu \bar{z}^{\bar{\alpha}})\,. \tag{6.12}$$

An alternative definition of a special Kähler manifold can be obtained by constructing a flat $(2n_V + 2)$-dimensional symplectic bundle over the Kähler-Hodge manifold whose *symplectic sections*

$$\mathcal{V} = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix}, \qquad \Lambda = 0, \ldots, n_V\,, \tag{6.13}$$

are covariantly holomorphic

$$D_{\bar{\alpha}} \mathcal{V} = 0 \tag{6.14}$$

and obey the further constraints

$$i\langle \mathcal{V}, \bar{\mathcal{V}} \rangle \equiv i\big(M_\Lambda \bar{L}^\Lambda - L^\Lambda \bar{M}_\Lambda\big) = 1 \tag{6.15}$$

and

$$\langle \mathcal{V}, D_\alpha \mathcal{V} \rangle = 0\,. \tag{6.16}$$

The operators used in equations (6.15) and (6.16) are the Kähler-covariant derivatives, defined as

$$D_\alpha \mathcal{V} \equiv \partial_\alpha \mathcal{V} - \frac{1}{2}(\partial_\alpha \mathcal{K})\mathcal{V}\,, \qquad D_{\bar{\alpha}} \mathcal{V} \equiv \partial_{\bar{\alpha}} \mathcal{V} - \frac{1}{2}(\partial_{\bar{\alpha}} \mathcal{K})\mathcal{V}\,. \tag{6.17}$$

The above holomorphic sections v can be related to the symplectic sections $\mathcal{V}$ as

$$v = e^{-\mathcal{K}/2} \mathcal{V} = e^{-\mathcal{K}/2} \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \equiv \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \tag{6.18}$$

and thus from (6.8) we can rewrite the Kähler potential as

$$\mathcal{K} = -\log\big[i\big(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda\big)\big]. \tag{6.19}$$

The upper and lower components of $\mathcal{V}$ are in principle independent. However, up to symplectic transformations, there always exists a frame in which it is possible to express the lower components as functions of the upper ones

$$F_\Lambda = F_\Lambda(X) \tag{6.20}$$

and then the constraint (6.16) implies the integrability condition

$$\frac{\partial F_\Sigma}{\partial X^\Lambda} - \frac{\partial F_\Lambda}{\partial X^\Sigma} = 0\,. \tag{6.21}$$

Therefore, the sections F_Λ are the derivatives of a holomorphic homogeneous function $F(X)$ of degree 2 called prepotential, i.e.:

$$F_\Lambda = \frac{\partial F(X)}{\partial X^\Lambda}\,. \tag{6.22}$$

In this frame the geometry is completely specified by the prepotential and the X^Λ are called *homogeneous coordinates*. The physical scalars can be expressed as

$$z^\alpha \equiv \frac{X^\alpha}{X^0}, \qquad \alpha = 1, \ldots, n_V, \tag{6.23}$$

known as *special coordinates*.

The coupling between the scalars z^α and the vector fields is determined by the *period matrix* $\mathcal{N}$, defined via the relations

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma, \qquad D_{\bar\alpha} \bar M_\Lambda = \mathcal{N}_{\Lambda\Sigma} D_{\bar\alpha} \bar L^\Sigma. \tag{6.24}$$

When a prepotential exists, the period matrix is obtained from

$$\mathcal{N}_{\Lambda\Sigma} = \bar F_{\Lambda\Sigma} + 2i \frac{(\mathrm{Im} F_{\Lambda\Gamma}) X^\Gamma (\mathrm{Im} F_{\Sigma\Delta}) X^\Delta}{X^\Psi (\mathrm{Im} F_{\Psi\Omega}) X^\Omega}, \tag{6.25}$$

where $F_{\Lambda\Sigma} = \partial_\Lambda \partial_\Sigma F$.

6.2.2 Very special real manifold

The scalars ϕ^x of the vector multiplets in $d = 5$ $\mathcal{N} = 2$ gauged supergravity parametrize a moduli space that is a *very special real manifold*, as originally described in [155, 156]. We also refer to [146, 157], besides the books we cited at the beginning of this chapter.
A very special real manifold $\mathcal{S}$ is an n_V-dimensional hypersurface that can be defined introducing an $(n_V + 1)$-dimensional embedding manifold spanned by the homogeneous coordinates $h^I(\phi^x)$, $I = 0, \ldots, n_V$; the hypersurface is identified by the cubic polynomial

$$\mathcal{V}(h) \equiv \left\{ C_{IJK} h^I(\phi) h^J(\phi) h^K(\phi) = 1 \right\} \subset \mathbb{R}^{n_V+1}, \tag{6.26}$$

specified by the constant real totally symmetric tensor C_{IJK}. Thus, the n_V real scalars ϕ^x are coordinates that live in a hypersurface that is embedded in an (n_V+1)-dimensional Riemannian space defined by the constraint (6.26). From the tensor C_{IJK} and the sections $h^I(\phi^x)$ it is possible to derive all the quantities needed to describe the kinetic and interaction terms.
In fact, on the very special manifold we can introduce a metric g_{xy} as the pull-back of the metric a_{IJ} on the ambient space,

$$g_{xy} = \frac{3}{2} \partial_x h^I \partial_y h^J a_{IJ} \quad \text{with} \quad a_{IJ} = -\frac{1}{3} \frac{\partial}{\partial h^I} \frac{\partial}{\partial h^J} \log \mathcal{V}(h) \Big|_{\mathcal{V}=1}. \tag{6.27}$$

The coupling between scalars ϕ^x and vectors A^I_μ is realized through the metric a_{IJ}, which can be alternatively defined as

$$\begin{aligned} a_{IJ} &= -2 C_{IJK} h^K + 3 h_I h_J\,, \\ \text{where} \quad h_I &\equiv a_{IJ} h^J = C_{IJK} h^J h^K\,. \end{aligned} \tag{6.28}$$

We also report the standard definition

$$h^I_x \equiv -\sqrt{\frac{3}{2}}\partial_x h^I(\phi) \quad \Rightarrow \quad h_{Ix} \equiv a_{IJ} h^J_x(\phi) = \sqrt{\frac{3}{2}}\partial_x h_I(\phi) \tag{6.29}$$

and some useful relations

$$\begin{aligned} &h^I h_I = 1\,, \qquad h^I_x h_I = h_{Ix} h^I = 0\,,\\ &a_{IJ} = h_I h_J + h_{xI} h^x_J\,. \end{aligned} \tag{6.30}$$

It is interesting noting that it is possible to dimensionally reduce a very special real manifold to four dimensions, giving rise to a theory with special Kähler manifold. The map between these two theories is called the **r**-map [157]. More specifically, the **r**-map leads to a subclass of special Kähler manifolds, known as very special Kähler manifolds, in which the prepotential can be expressed as a function of the totally symmetric tensor C_{IJK}. We give an explicit realization of the **r**-map in appendix B.

6.2.3 Quaternionic-Kähler manifold

The real scalars in the hypermultiplets span the *quaternionic-Kähler manifold*, common both to four- and to five-dimensional theories. Good references are [144,158] and again [28,38].
A quaternionic-Kähler manifold $\mathcal{Q}$ can be defined as a $4n_H$-dimensional Riemannian manifold with coordinates q^X and metric g_{XY} endowed with a locally defined triplet $\vec{J}_X{}^Y$ of almost complex structures satisfying the relations:

$$\begin{aligned} &(J^r)_X{}^Z (J^s)_Z{}^Y = -\delta^{rs}\delta^Y_X + \epsilon^{rst}(J^t)_X{}^Y\,,\\ &(J^{(r)})_X{}^Z (J^{(r)})_Y{}^W g_{ZW} = g_{XY}\,,\\ &\widetilde{\nabla}_Z \vec{J}_X{}^Y = \nabla_Z \vec{J}_X{}^Y + 2\,\vec{\omega}_Z \times \vec{J}_X{}^Y = 0\,, \end{aligned} \tag{6.31}$$

where $\{r,s,t\} = 1,2,3$ are indices of the vectorial representation of $SU(2)$[1]. The first expression in (6.31) is the defining relation of a quaternionic structure, thus making $\vec{J}_X{}^Y$ an almost hyper-complex structure. The second one implies that the metric g_{XY} is Hermitian with respect to each of the three complex structures (notice that there is no sum over r). Thus, for each of the complex structures, we can define a Kähler two-form

$$\vec{K} := \frac{1}{2}\vec{J}_{XY} dq^X \wedge dq^Y \quad \text{with} \quad \vec{J}_{XY} := g_{YZ}\vec{J}_X{}^Z\,. \tag{6.33}$$

The triplet of Kähler forms is known as the hyper-Kähler two-form.
Finally, the last equation in (6.31) indicates that $\vec{J}$ is covariantly constant w.r.t. the connection $\vec{\omega} = \vec{\omega}_X(q)dq^X$ associated with an $SU(2)$-bundle defined over the quaternionic manifold. The

[1] In this work we use also the doublet notation instead of the vector (or triplet) one,

$$J_X{}^Y{}_i{}^j \equiv i\,\vec{J}_X{}^Y \cdot \vec{\sigma}_i{}^j\,, \tag{6.32}$$

where $\vec{\sigma}_i{}^j$ are the three Pauli matrices. Of course, this transition between triplet and doublet notation holds also for other quantities in the adjoint representation of $SU(2)$.

corresponding $SU(2)$-curvature $\vec{\mathcal{R}}$ has to be proportional to the hyper-Kähler two-form as it follows

$$\vec{\mathcal{R}} = d\vec{\omega} + 2\,\vec{\omega} \wedge \vec{\omega} = \frac{1}{2}\,\nu \vec{K} \qquad \text{with} \qquad \nu \equiv \frac{1}{4n_H(n_H+2)}\,R, \tag{6.34}$$

where R denotes the curvature scalar of the manifold.

From the above structure it follows that the holonomy of the Levi-Civita connection of quaternionic-Kähler spaces is $SU(2) \times USp(2n_H)$. Thus, it is useful to split the tangent indices into a pair of flat indices: one fundamental $SU(2)$ index $i = 1, 2$ and one $USp(2n_H)$ index $A = 1, \ldots, 2n_H$. Using these indices, one can find a vielbein one-form for this space

$$f^{iA} = f^{iA}{}_X(q)dq^X \tag{6.35}$$

such that

$$g_{XY} = f^{iA}{}_X\, \varepsilon_{ij}\, C_{AB} f^{jB}{}_Y\,, \tag{6.36}$$

where $\varepsilon_{ij} = -\varepsilon_{ji}$ and $C_{AB} = -C_{BA}$ are the invariant symplectic metrics in $USp(2) \sim SU(2)$ and $USp(2n_H)$. Other useful relations defining the inverse vielbein are given by

$$f^{iA}{}_X f^Y{}_{iA} = \delta^Y_X\,, \qquad f^{iA}{}_X f^X{}_{jB} = \delta^i_j \delta^A_B\,. \tag{6.37}$$

The vielbein f^{iA} is covariantly constant with respect to the full connection

$$\widetilde{\nabla}_Y f^{iA}{}_X = \partial_Y f^{iA}{}_X + f^{iB}{}_X \omega_{YB}{}^A + f^{jA}{}_X \omega_{Yj}{}^i - \Gamma^Z_{XY} f^{iA}{}_Z = 0, \tag{6.38}$$

where Γ^Z_{XY} is the Levi-Civita connection, $\omega_{Xi}{}^j$ is the $SU(2)$-connection and $\omega_{XA}{}^B$ is the $USp(2n_H)$-connection. Finally, it is possible to express each almost complex structure in terms of the vielbeins as

$$(J^r)_X{}^Y = -i\, f^{iA}{}_X f^Y{}_{jA} (\sigma^r)_i{}^j, \tag{6.39}$$

from which follows

$$i\,\vec{J}^Y_X \cdot \vec{\sigma}_i{}^j = 2 f^{jA}{}_X f^Y{}_{iA} - \delta^j_i \delta^Y_X. \tag{6.40}$$

6.3 Isometries and gauging

As we mentioned in the introduction of this chapter, the structure of $\mathcal{N} = 2$ pure supergravities that we have described so far can be modified by gauging some of the global symmetries of the theories.

The ungauged supergravity theory determines the kinetic terms for the scalars, that define the metric. The isometries of the metric are global symmetries for the theory.
Gauged supergravities can be obtained by gauging a subgroup of either these global symmetries or the R-symmetry, as we will recap later in this section. We start by introducing the possible isometries of the various target spaces and then we proceed with the gaugings.

6.3.1 Isometries

A useful tool to describe the isometries of the special and quaternionic-Kähler manifolds is given by the so-called *momentum map*. The momentum map is a geometric construction that can be used on any manifold with a symplectic structure, such as Kähler manifolds.

Let us consider a Kähler manifold M_K with metric g_{ij^*}. If g_{ij^*} has a non trivial group $\mathcal{G}$ of continuous isometries generated by the Killing vectors k^i_Λ, $\Lambda = 1, \ldots, \mathrm{dim}\mathcal{G}$, then $\mathcal{G}$ is a group of global spacetime symmetries for the kinetic Lagrangian $\mathcal{L}_{\text{kin}}$, i.e. $\mathcal{L}_{\text{kin}}$ under the infinitesimal variation

$$z^i \to z^i + \delta z^i \equiv z^i + \epsilon^\Lambda k^i_\Lambda(z), \tag{6.41}$$

where ϵ^Λ are constant infinitesimal parameters. The Killing vectors generate the Lie algebra associated with symmetry group $\mathcal{G}$:

$$[k_\Lambda, k_\Sigma] = f_{\Lambda\Sigma}{}^\Gamma k_\Gamma\,, \tag{6.42}$$

where $f_{\Lambda\Sigma}{}^\Gamma$ are the structure constants of the algebra.

If all the scalar fields appear only through the metric, the curvature, the period matrix and the Christoffel symbol in the covariant derivative, then the isometry of g_{ij^*} extends to a symmetry of the whole Lagrangian.

The generic Killing equation

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0 \tag{6.43}$$

can be written using holomorphic indices as

$$\nabla_i k_j + \nabla_j k_i = 0\,, \quad \nabla_{i^*} k_j + \nabla_j k_{i^*} = 0\,, \tag{6.44}$$

with $k_j = g_{ji^*} k^{i^*}$.

The Killing vectors k^i_Λ are generators of infinitesimal holomorphic coordinate transformations that leave the metric invariant. In a Kähler manifold, in the same way as the metric is the derivative of the Kähler potential, the Killing vectors are the derivatives of suitable prepotentials, called *Killing prepotentials* or momentum maps. In fact, if we consider holomorphic vectors, the first equation in (6.44) is automatically satisfied, while the second equation can be written as

$$k^i_\Lambda = i\, g^{ij^*} \partial_{j^*} P_\Lambda, \quad P^*_\Lambda = P_\Lambda\,, \tag{6.45}$$

where P_Λ indicates the momentum map.

In order to preserve the Kähler structure, we have to require the Kähler potential to be invariant under the action of a Killing vector, up to a Kähler transformation:

$$(k^i_\Lambda \partial_i + k^{\bar{\imath}}_\Lambda \partial_{\bar{\imath}})\mathcal{K} = f_\Lambda(z) + \bar{f}_\Lambda(\bar{z})\,. \tag{6.46}$$

This provides an expression for the momentum maps as functions of the Killing vectors,

$$P_\Lambda = (k^i_\Lambda \mathcal{A}_i + k^{\bar{\imath}}_\Lambda \mathcal{A}_{\bar{\imath}}) + \frac{i}{2}\big[f_\Lambda(z) - \bar{f}_\Lambda(\bar{z})\big], \tag{6.47}$$

where $\mathcal{A}$ is the $U(1)$ Kähler connection, from which we can see that the momentum maps are defined up to a real constant.

Special Kähler manifold Since special Kähler manifolds form a subclass of Kähler manifolds, all the properties presented in the previous part must hold also in this case. The additional presence of a symplectic structure has to be taken into account as well. This requirement imposes extra conditions on the Killing vectors and prepotentials. However, in this work we will not gauge the isometries of the special Kähler target manifold and thus we will not go deeper into this topic. We refer the interested reader to [28, 144] for a detailed analysis.

Quaternionic-Kähler manifold Here we move to the study of the target manifold generated by the hypermultiplets, expanding the discussion presented in the general case of Kähler manifolds. The symmetries in this sector of the theory must be isometries of the metric g_{XY} that preserve the quaternionic-Kähler structure and the hyper-Kähler structure K^r. The hyper-Kähler structure has to be invariant under the variation

$$\delta_\epsilon q^X = \epsilon^\Lambda k_\Lambda^X(q)\,, \tag{6.48}$$

where k_Λ^X are isometries of g_{XY} only up to q-dependent $SU(2)$ transformations characterized by certain parameters λ_Λ^r. These $SU(2)$ parameters are related to the Killing vectors, the $SU(2)$-connection $\vec{\omega}_X$ and the *triholomorphic momentum maps* $\vec{P}_\Lambda$ through the equation

$$\lambda_\Lambda^r \equiv k_\Lambda^X \omega_X^r - P_\Lambda^r. \tag{6.49}$$

The triholomorphic maps play the same role in the gauging of the symmetries of the quaternionic-Kähler manifold that the holomorphic momentum maps do for the special Kähler manifold and they perform the same geometrical roles. Furthermore, in both cases the Killing vectors can be obtained as derivatives of the momentum maps, using (6.45) and

$$\vec{J}_{XY} k_\Lambda^Y = \widetilde{\nabla}_X \vec{P}_\Lambda \equiv \partial_X \vec{P}_\Lambda + 2\,\vec{\omega}_X \times \vec{P}_\Lambda\,, \tag{6.50}$$

respectively.
Other useful relations between Killing vectors and prepotentials are given by

$$2\,n_H \nu \vec{P}_\Lambda = \vec{J}_X{}^Y \nabla_Y k_\Lambda^X\,, \tag{6.51}$$

$$k_\Lambda^X \vec{K}_{XY} k_\Sigma^Y = f_{\Lambda\Sigma}{}^\Gamma \vec{P}_\Gamma + \nu \vec{P}_\Lambda \times \vec{P}_\Sigma\,, \tag{6.52}$$

where $f_{\Lambda\Sigma}{}^\Gamma$ are the structure constants introduced in (6.42).

There is one last particular situation that we have to mention. Even in the absence of hypermultiplets, and hence when there is no quaternionic-Kähler manifold, there are two possible cases in which the momentum map P_Λ^r can still be defined:

- The gauge group contains an $SU(2)$ and

$$P_\Lambda^r = e_\Lambda^r \xi\,, \tag{6.53}$$

where ξ is an arbitrary constant and the e_Λ^r are constants satisfying

$$\epsilon^{rst} e_\Lambda^s e_\Sigma^t = f_{\Lambda\Sigma}^\Omega e_\Omega^r \tag{6.54}$$

and being non-zero for Λ corresponding to the $SU(2)$ factor.

- The gauge group contains a $U(1)$ factor and

$$e_\Lambda^r = e^r \xi_\Lambda\,, \tag{6.55}$$

where ξ_Λ is an arbitrary constant that is non-zero when Λ is the $U(1)$ factor, while e_Λ^r is an arbitrary $\mathfrak{su}(2)$ vector.

In these cases the momentum maps are equivalent to a set of constant Fayet–Iliopoulos terms.

Very special manifold In this case the target manifold is real, and thus we cannot apply the momentum map construction. As in the previous situations, the gauging procedure relies on the set of Killing vectors $k_I^x(\phi)$ generating the isometries of the manifold. Preserving the very special structure imposes constraints on the Killing vectors, on the same line as it happened in the previous cases due to the presence of a complex structure. We do not expand further since we will not deal with this gauging procedure and we refer again to [28] for more details.

6.3.2 Gauging of global symmetries

Once we have identified the global symmetries of the theories, we can proceed to present the possible gaugings in four and five dimensions.
The gauging of matter-coupled $\mathcal{N}=2$ supergravity theories is obtained by identifying the gauge group G as a subgroup, at most of dimension n_V+1 (corresponding to the number of gauge fields), of the isometries of the space $\mathcal{M}$. We refer to [28, 144–146] for accurate discussions on the topic.

Gauging of the 4d theory

We first focus on the four-dimensional theory. The global symmetry group G can be split as a direct product as

$$G = G_{\text{bos}} \times SU(2)_R \times U(1)_R\,, \tag{6.56}$$

where G_{bos} acts on the bosonic fields, while the R-symmetry group acts only on fermions. All the symmetries in G_{bos} act on scalars and we can split the group as $G_{\text{bos}} = G_v \times G_h$ depending on which scalars the symmetries of the subgroup act on: we denote by G_v the symmetries acting on the complex scalar fields, while we denote by G_h the ones acting the hyperscalars. However, notice that the vector fields must transform under the adjoint representation of the symmetries that we are going to gauge. Therefore, if we gauge a subgroup of the R-symmetry group or symmetries of hypermultiplet sector, the same symmetries must be in the vector multiplet, acting in the adjoint representation.

There are two main possibilities of gauging the bosonic group. On the one hand, if a non-abelian subgroup $G_{na} \subset G_v$ is gauged, G_{na} has to be also a subgroup of G_h, since as we mentioned G_v and G_h are gauged by the same vectors. On the other hand, if an abelian subgroup of G_h, is gauged, such for example $U(1)^{n_V+1}$, the isometries of G_v are not gauged. This latter is the gauging we will focus on in the remainder of this work.

For what concerns the fermions, as we introduced above, the gauging of the R-symmetry can be considered as a special case of the gauging of the isometries of the quaternionic-Kähler manifold and it is referred to as Fayet–Iliopoulos gauging. It can be abelian or non-abelian, depending whether we are gauging the $U(1) \subset SU(2)$ or the $SU(2)$ factor. In the latter case, a subgroup of G_v with the same $SU(2)$ factor must be gauged as well.

The gauging of the isometries requires the modification of some elements of the theory. First of all, the standard derivatives and the vector field strengths must be replaced by their gauge covariant versions. For the bosonic fields we have

$$\begin{aligned} \mathcal{D}_\mu z^\alpha &= \partial_\mu z^\alpha + g\, A_\mu^\Lambda k_\Lambda^\alpha\,, \\ \mathcal{D}_\mu q^X &= \partial_\mu q^X + g\, A_\mu^\Lambda k_\Lambda^X\,, \\ F_{\mu\nu}^\Lambda &= 2\partial_{[\mu} A_{\nu]}^\Lambda + g\, f_{\Sigma\Omega}{}^\Lambda A_{[\mu}^\Sigma A_{\nu]}^\Omega\,. \end{aligned} \tag{6.57}$$

Furthermore, while it does not affect the supersymmetry variations for the bosonic fields, it modifies the ones for the bosonic fields, since they now have to include fermion shifts. We report these fermionic supersymmetry transformation rules in the next section. To make the action invariant under the new supersymmetry variations, a scalar potential must be added, having the form

$$V = -P_\Lambda^r P_\Sigma^r \Big(\big(\mathrm{Im}\,\mathcal{N}\big)^{-1|\Lambda\Sigma} + 8\, e^{\mathcal{K}} X^\Lambda \bar{X}^\Sigma \Big) + 4\, e^{\mathcal{K}} g_{XY} k_\Lambda^X k_\Sigma^Y X^\Lambda \bar{X}^\Sigma. \tag{6.58}$$

Gauging of the 5d theory

The discussion on the gauging of the five-dimensional theory is very similar to the one in four dimensions. The global symmetry group G can be decomposed in the same way, except for the absence of the $U(1)_R$ factor.

The covariant derivatives of the scalars and vector field strengths are given by

$$\begin{aligned} \mathcal{D}_\mu \phi^x &= \partial_\mu \phi^x + g\, A_\mu^I k_I^x\,, \\ \mathcal{D}_\mu q^X &= \partial_\mu q^X + g\, A_\mu^I k_I^X\,, \\ F_{\mu\nu}^I &= 2\partial_{[\mu} A_{\nu]}^I + g\, f_{JK}{}^I A_{[\mu}^J A_{\nu]}^K\,, \end{aligned} \tag{6.59}$$

while the scalar potential reads

$$V(\phi, q) = P_I^r P_J^r \left(4 h^I h^J - 3 g^{xy} \partial_x h^I \partial_y h^J\right) - \frac{3}{4} g_{XY} k_I^X k_J^Y h^I h^J\,. \tag{6.60}$$

6.4 Bosonic four- and five-dimensional Lagrangians

We conclude this chapter by presenting the bosonic Lagrangians of $\mathcal{N} = 2$ matter-coupled gauged supergravity theories in four and five dimensions.

The four-dimensional theory

The bosonic part of the four-dimensional Lagrangian is given by

$$\begin{aligned} e^{-1}\mathscr{L} =& \frac{R}{2} - G_{\alpha\bar{\beta}}(z,\bar{z})\partial_\mu z^\alpha \partial^\mu z^{\bar{\beta}} - \frac{1}{2} g_{XY}(q)\mathcal{D}_\mu q^X \mathcal{D}^\mu q^Y \\ &+\frac{1}{8}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} - \frac{e^{-1}}{16}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}(z,\bar{z})F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} - g^2 V(z,\bar{z},q)\,, \end{aligned} \tag{6.61}$$

where the covariant derivatives are written in (6.57).

The five-dimensional theory

The bosonic five-dimensional Lagrangian reads

$$\begin{aligned} e^{-1}\mathscr{L} =& \frac{1}{2}R - \frac{1}{2}g_{xy}(\phi)\mathcal{D}_\mu\phi^x\mathcal{D}^\mu\phi^y - \frac{1}{2}g_{XY}(q)\mathcal{D}_\mu q^X\mathcal{D}^\mu q^Y \\ &- \frac{1}{4}a_{IJ}(\phi)F^I_{\mu\nu}F^{J\,\mu\nu} + \frac{e^{-1}}{12}\sqrt{\frac{2}{3}}C_{IJK}\epsilon^{\mu\nu\rho\sigma\tau}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\tau - g^2V(\phi,q), \end{aligned} \tag{6.62}$$

where the covariant derivatives are reported in (6.59).

Finally, we report the $\mathcal{N}=2$ supersymmetry variations using the conventions of [148, 159, 160], which we will use in the last chapter of the work. Given a $\mathcal{N}=2$ Killing spinor ϵ^i, the fermionic variations have the following form

$$\delta\psi^i_\mu = \left[D_\mu + \frac{i}{4\sqrt{6}}h_I(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho)F^I_{\nu\rho}\right]\epsilon^i - \frac{i}{\sqrt{6}}g\,\gamma_\mu h^I (P_I)^{ij}\epsilon_j, \tag{6.63}$$

$$\delta\lambda^{x\,i} = \left(-\frac{i}{2}\gamma^\mu\partial_\mu\phi^x + \frac{1}{4}\gamma^{\mu\nu}F^I_{\mu\nu}h^x_I\right)\epsilon^i - g\,P^{x\,ij}\epsilon_j, \tag{6.64}$$

$$\delta\zeta^A = \frac{i}{2}\gamma^\mu\mathcal{D}_\mu q^X f^{iA}_X\epsilon_i - \frac{\sqrt{6}\,g}{4}h^I\,k^X_I\,f^{iA}_X\epsilon_i, \tag{6.65}$$

where we have introduced the supercovariant derivative

$$D_\mu\epsilon^i = \partial_\mu\epsilon^i + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\epsilon^i - \partial_\mu q^X\omega^{ij}_X\epsilon_j - gA^I_\mu P^{ij}_I\epsilon_j\,, \tag{6.66}$$

acting on the supersymmetry parameter ϵ^i.

Chapter 7

Kerr–Newman black holes from Leigh–Strassler

To show an example of one of the many applications of the AdS/CFT correspondence, in this chapter we compute the entropy of the (putative) black hole solution of the supergravity theory dual to the SCFT that we have presented in section 5.3. Whilst in the latter we performed the computation from a field theory analysis, starting from the superconformal index, here we arrive to the same result through a computation in supergravity, based on the so-called *attractor mechanism*.

Due to the attractor mechanism [17], the entropy of an extremal four-dimensional black hole and the values of the scalar fields at the event horizon depend only on its electric and magnetic charges, and not on the asymptotic values of the scalars. The scalars on the horizon are constrained by supersymmetry and they can be obtained as the values that extremize an effective potential. The black hole entropy is given by the value of this potential at its extremum. The attractor mechanism has been extended over the years to different classes of black holes in various dimensions. However a formulation for rotating black holes in five-dimensional gauged supergravity is still absent.

Attractor mechanisms in supergravity are related through the AdS/CFT correspondence to extremization problems in the dual superconformal field theories. Many explicit cases have been worked out in the recent past.

An early attempt was provided in [161] where it was shown that the maximization of the conformal anomaly a_{4d} [162] corresponded to the minimization of the scalar potential in the AdS_5 supergravity dual. In this case the R-charges were related to the scalar fields in the vector multiplets, and the role of the hypermultiplets was discussed as well (see [163] for an explicit example). In a related paper [164] it was shown that a similar relation holds between the attractor mechanism and the coefficient of the two-point function for the R-current τ_{RR} (see [165] for an explicit derivation of τ_{RR}-minimization).

The latter result is interesting because it can be extended to other dimensions. For example the extremization of τ_{RR} in $2d$, corresponding to the central charge, led to the principle of c-extremization [21]. The supergravity dual mechanism was discussed in [20, 166–169]. It has been shown that c-extremization can be reformulated in terms of an attractor mechanism in AdS_3. Furthermore in three dimensions τ_{RR} coincides with the free energy on S^3 for holographic theories [170], and this allows to relate the attractor mechanism to localization [171, 172] (see also [173] for a discussion on τ_{RR} minimization in AdS_4 gauged supergravity).

By generalizing this idea in [174–176] it was shown that the extremization of the topologically twisted index [94] can be associated at large N to the entropy of AdS_4 BPS black holes. This mechanism has been then generalized to other dimensions and associated with the attractor mechanism [81,177–185].

A more recent extremization problem on the field theory side regards the extremization of the entropy function and its relation with the entropy of 5d rotating black holes. It has been shown in [13] that the Legendre transform of such function, for the case of $\mathcal{N} = 4$ Super Yang–Mills, gives rise to the entropy of the electrically charged rotating BPS black holes in $\mathrm{AdS}_5 \times S^5$ [10,14,72–74]. On the gravitational side this quantity has been shown to originate from the on-shell action of the Euclidean black hole [113], while on the field theory side the entropy function has then been shown to originate from the superconformal index [11,12].
However, the dual attractor mechanism for rotating five-dimensional black holes in gauged supergravity is unknown and to obtain the gravitational function a more sophisticated construction has been necessary [13], based on the general reduction of BPS attractors in gauged supergravity performed in [186]. In this case, after fixing the two angular momenta to be equal, it is possible to dimensionally reduce the five-dimensional solution down to four dimensions along the $U(1)$ Hopf fiber of the enhanced $SU(2) \times U(1)$ isometry of the black hole metric on the squashed sphere. In this framework the dual extremization problem was formulated in terms of the four-dimensional BPS black hole attractor mechanism [18].
More recently this result has been extended to the case of truncations with hypermultiplets, and checked for the case of $T^{1,1}$ studied in [105] and of M5-branes in [90]. Observe that in these last cases the results are obtained by conjecturing the existence of a BPS Kerr–Newman black hole, that has nevertheless never been directly studied from a 5d analysis, differently from the case of S^5. By using this assumption on the existence of the five-dimensional black hole the final results on its entropy, obtained from the $\mathrm{AdS}_2 \times S^2$ attractor, have been matched with the expectations form the superconformal index.

From the discussion above a general framework emerges. With the current techniques, few five-dimensional supersymmetric rotating black holes can and have been obtained, and all of them require the absence of hypermultiplets. Only vector multiplets are allowed and the only holographic dual example that fits in this class corresponds to $\mathcal{N} = 4$ SYM. On the other hand, the recent results obtained from the evaluation of the Bekenstein–Hawking entropy from the dual field theory approach have enlarged the class of models for which a dual black hole is expected. For example, for toric quiver gauge theories, the entropy function can be worked out in full generality in terms of the 10d geometry of type IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{SE}_5$. The general expectation is then that there is a dual mechanism that reproduces the entropy function from some consistent truncation and the case of $T^{1,1}$ discussed above is of this type. The results following from the field theoretical analysis further predict the possibility of finding similar black holes for truncations that do not correspond to SE_5 manifolds.

Motivated by the above discussion, in this chapter we study the relation between the superconformal index of the $\mathcal{N} = 4$ theory with superpotential $W = \epsilon_{ijk}\Phi_i\Phi_j\Phi_k$ perturbed by the $\mathcal{N} = 1$ Leigh–Strassler (LS) deformation, $\Delta W = m\Phi_3^2$, and the entropy of the holographic dual BPS Kerr–Newman rotating black hole.
We consider the truncation of [187] corresponding to a model with one vector multiplet and one hypermultiplet, and a $U(1) \times U(1)$ gauging of the isometries of the scalar manifold. This

truncation is less rich that the one studied in [105], because only the R-symmetry current is captured, but it has other interesting properties that require a detailed analysis. The first non trivial aspect is that we have to apply a local rotation that aligns the Killing prepotentials. This allows us to use the approach of [13,105] in the framework of general matter coupled to $\mathcal{N} = 2$ gauged supergravity. A second aspect is related to the different conventions in the original truncations of [187] with respect to the ones used in [13,105]. This amounts to a slightly different choice of the Kaluza–Klein (KK) ansatz in our case. Motivated by this difference, we perform a general analysis of the KK ansatz.
In this setup we then reduce the 5d gauged supergravity to 4d and study the AdS_2 attractor of the conjectured black hole solution. This gives rise to an extremization problem that corresponds to the one found in section 5.3 from the saddle point analysis of the superconformal index, once restricted to the fugacities visible in the truncation of [187].

The chapter is based on [188] and it is organized as follows. In section 7.1 we review the basic aspects of the 5d truncation dual to the LS fixed point discussed in [187]. In section 7.2 we review the 5d/4d reduction along the lines of [13,105] and we apply the construction to the truncation of the LS fixed point. We further find the extremization problem that originates from the AdS_2 attractor. This problem is then matched with the field theory expectations in section 7.3. As anticipated we also give a detailed analysis of the KK ansatz used in the bulk of the chapter in appendix B.

7.1 A consistent truncation dual to the LS fixed point

In this section we review the relevant aspects of the AdS_5 gauged supergravity dual of the 4d SCFT LS fixed point.
The flow to such an IR fixed point was first reproduced from the holographic perspective in [189] within $\mathcal{N} = 8$ supergravity. It was then shown that it could be obtained in $\mathcal{N} = 2$ gauged supergravity in [187]. The starting point in the analysis of [187] is a model with one vector multiplet and one hypermultiplet, and a $U(1) \times U(1)$ gauging of the isometries of the scalar manifold $\mathcal{M} = \mathcal{SM} \times \mathcal{QM}$, where $\mathcal{SM}$ is a very special real manifold and $\mathcal{QM}$ is a quaternionic-Kähler manifold. The UV and the IR fixed points are connected by an R-symmetric flow along the quaternionic-Kähler manifold $\mathcal{QM}$. The IR fixed point corresponds to the LS fixed point and it is the starting point of our analysis.

In the following we briefly review the relevant aspects of such IR fixed point. We also discuss an useful manipulation of the results of [187], corresponding to an $USp(2)$ rotation[1] of the Killing prepotentials, that becomes relevant in the study of the 5d/4d reduction performed below.

[1] This $Sp(1)$ is the one appearing in the $USp(2n_H) \times USp(2)$ holonomy group of $\mathcal{QM}$. It is often referred to as an $SU(2)_R$ symmetry.

The model

The model considered in [187] to reproduce the FGPW flow consists of 5d $\mathcal{N} = 2$ gauged supergravity with one vector ($n_V = 1$) and one hypermultiplet ($n_H = 1$), and scalar manifold

$$\mathcal{M} = O(1,1) \times \frac{SU(2,1)}{SU(2) \times U(1)}. \tag{7.1}$$

The scalar manifold of the very special geometry is specified by the totally symmetric tensor C_{IJK} as

$$\mathcal{SM} = \{\mathcal{V}(h) \equiv C_{IJK} h^I h^J h^K = 1\}, \tag{7.2}$$

where h^I $(I = 1, 2)$[2] are coordinates on $\mathbb{R}^2$ and $h^I(\phi)$ represent the sections of the special geometry. In this truncation we have

$$C_{122} = 3\sqrt{3}, \tag{7.3}$$

and thus

$$\mathcal{V}(h) = 9\sqrt{3} h^1 (h^2)^2. \tag{7.4}$$

The quaternionic-Kähler manifold $\mathcal{QM}$ is parametrized by four real scalars $q^u = \{V, \sigma, \theta, \tau\}$, with $V > 0$. The metric is given by

$$ds^2 = \frac{dV^2}{2V^2} + \frac{1}{2V^2}\big(d\sigma - 2\tau\, d\theta + 2\theta\, d\tau\big)^2 + \frac{2}{V}(d\theta^2 + d\tau^2). \tag{7.5}$$

The gauging

The Killing vectors are given by a linear combination of the generators of the $SU(2)$ and $U(1)$ subgroups of the $SU(2,1)$ isometry group of the metric (7.5), denoted as T_3 and T_8 in [187]. The $U(1) \times U(1)$ gauging corresponds to consider the two Killing vectors

$$K_1 = \frac{3}{\sqrt{2}}\Big(T_3 + \sqrt{3}\, T_8\Big), \qquad K_2 = \sqrt{3}\Big(\sqrt{3}\, T_3 - T_8\Big), \tag{7.6}$$

where

$$T_3 = \frac{1}{4}(k_1 + k_6 - 3k_4), \qquad T_8 = \frac{\sqrt{3}}{4}(k_4 + k_1 + k_6) \tag{7.7}$$

are the Abelian generators of $SU(2) \times U(1)$, constructed from the eight Killing vectors of the $SU(2,1)$ isometry group of the metric (7.5). Explicitly, the relevant generators in (7.7) are

$$\vec{k}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{k}_4 = \begin{pmatrix} 0 \\ 0 \\ -\tau \\ 0 \end{pmatrix}, \quad \vec{k}_6 = \begin{pmatrix} 2V\sigma \\ \sigma^2 - (V + \theta^2 + \tau^2)^2 \\ \sigma\theta - \tau(V + \theta^2 + \tau^2) \\ \sigma\tau + \theta(V + \theta^2 + \tau^2) \end{pmatrix}. \tag{7.8}$$

The Killing prepotentials associated with (7.6) are given by

$$\vec{P}_1 = \begin{pmatrix} -\frac{3(-V\theta + \theta^3 + \sigma\tau + \theta\tau^2)}{\sqrt{2V}} \\ \frac{3(V\tau - \theta^2\tau + \theta\sigma - \tau^3)}{\sqrt{2V}} \\ -\frac{3(1 + V^2 + \sigma^2 - 6V(\theta^2 + \tau^2) + (\theta^2 + \tau^2)^2)}{4\sqrt{2}V} \end{pmatrix}, \quad \vec{P}_2 = \begin{pmatrix} \frac{3\theta}{\sqrt{V}} \\ \frac{3\tau}{\sqrt{V}} \\ \frac{3(-V + \theta^2 + \tau^2)}{2V} \end{pmatrix}. \tag{7.9}$$

[2] For reasons related to the 5d/4d analysis here we shift the index $I = 0, \ldots, 1$ by one unit, i.e. $I = 1, \ldots, n_V + 1$.

The FGPW flow

By choosing the parametrization

$$\sigma = 0, \quad V = 1 - \theta^2 - \tau^2 \equiv 1 - \zeta^2 \equiv 1 - \tanh^2\chi \tag{7.10}$$

with $0 < V \leq 1$, $-1 < \zeta < 1$ and $-\infty < \chi < \infty$ one can reproduce the FGPW flow from the truncation discussed so far. The fields θ and τ can be parameterized as $\theta = \zeta \cos\phi$ and $\tau = \zeta \sin\phi$.
The UV vacuum corresponds, without loss of generality, to

$$\sigma = \zeta^2 = 0, \quad V = 1, \quad \rho = 1, \tag{7.11}$$

where ρ is the vector modulus. This is the starting point of the flow, holographically dual to $\mathcal{N} = 4$ SYM. In this UV fixed point the $U(1)$ symmetry gauged by the graviphoton is generated by the Killing vector $\frac{1}{\sqrt{3}}(K_1+K_2)$. The other massless vector gauges the $U(1)$ isometry associated with $\frac{1}{\sqrt{3}}(K_1 - K_2)$. The first $U(1)$ is associated with the $U(1)_R \subset SU(4)_R$ R-symmetry group of $\mathcal{N} = 4$ assigning charges $R = \frac{2}{3}$ to the three chiral adjoints in the field theory dual. The other $U(1)$ corresponds to one of the two Abelian flavor symmetries in the Cartan of $SU(4)_R$.
The IR vacuum corresponds, without loss of generality, to

$$\sigma = 0, \quad \zeta^2 = \frac{1}{4}, \quad V = \frac{3}{4}, \quad \rho = 2^{1/6}. \tag{7.12}$$

The flat direction associated with ϕ is a marginal deformation from the SCFT dual. In this IR fixed point the $U(1)$ symmetry gauged by the graviphoton is generated by the Killing vector $\frac{2^{2/3}}{\sqrt{6}}(\sqrt{2}K_1 + K_2)$. The other vector in this case is massive, and the broken isometry associated is generated by $\frac{2^{2/3}}{\sqrt{6}}(\frac{1}{\sqrt{2}}K_2 - 2K_1)$.

Summarizing, the spectrum of vector fields and Killing vectors they couple to, around IR fixed point, is given by

$$\begin{aligned} A^R_\mu &\equiv \frac{2^{1/3}}{\sqrt{6}}\left(\frac{A^1_\mu}{\sqrt{2}} + 2\,A^2_\mu\right) \,:\, m^2 = 0, \quad K_R = \frac{2^{2/3}}{\sqrt{6}}(\sqrt{2}K_1 + K_2), \\ A^W_\mu &\equiv \frac{2^{1/3}}{\sqrt{6}}\left(\sqrt{2}A^2_\mu - A^1_\mu\right) \,:\, m^2 = 6\cdot 2^{4/3}g^2, \quad K_W = \frac{2^{2/3}}{\sqrt{6}}\left(\frac{K_2}{\sqrt{2}} - 2K_1\right). \end{aligned} \tag{7.13}$$

The vector A^W_μ acquires a mass eating the Stückelberg scalar χ. The mass eigenstates are

$$\mathbb{B}^I{}_J A^J_\mu, \qquad \text{where} \qquad \mathbb{B} = \frac{2^{1/3}}{\sqrt{6}}\begin{pmatrix} \frac{1}{\sqrt{2}} & 2 \\ -1 & \sqrt{2} \end{pmatrix} \tag{7.14}$$

is the matrix that diagonalizes them.

Rotating the Killing prepotentials

The Killing prepotentials P_I^r in (7.9) are aligned along $r = 3$ at the UV vacuum, while they are misaligned in the IR. In order to simplify the analysis below here we consider an $USp(2)$ rotation of these Killing prepotentials such that they are aligned around the IR fixed point. Following [190], the $SU(2)$ matrix

$$\begin{aligned} U(\zeta, \phi) &= \exp\left(-\frac{i}{2}\sigma^1 f(\zeta)\right) \cdot \exp\left(-\frac{i}{2}\sigma^3 g(\phi)\right), \\ \text{with } f(\zeta) &= \arctan\left(\frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1}\right), \quad g(\phi) = -\phi + \frac{\pi}{2}, \end{aligned} \tag{7.15}$$

rotates the moment maps (7.9) into a form such that the prepotentials become

$$\vec{P}_1 = \begin{pmatrix} -\frac{3}{\sqrt{2}}\sigma \sinh\chi \\ \frac{3}{2\sqrt{2}}\sigma^2 \sinh\chi \\ \frac{3}{4\sqrt{2}}\left(\sigma^2+2\right)\left(2-\cosh^2\chi\right) \end{pmatrix}, \quad \vec{P}_2 = \begin{pmatrix} 0 \\ 0 \\ \frac{3}{2}\cosh^2\chi \end{pmatrix} \tag{7.16}$$

where we used (7.10) and parameterized $\theta = \tanh\chi\cos\phi$ and $\tau = \tanh\chi\sin\phi$.
The absence of a ϕ dependence in the $\vec{P}_I$ signals the fact that there this field is related to a marginal direction, both at the UV fixed point and at the IR one. Furthermore observe that around the IR vacuum, where $\sigma = 0$ both $\vec{P}_1$ and $\vec{P}_2$ are in the form $P_I^r = \delta_{3r}P_I^3$. This last observation is valid also if we keep the χ dependence explicit. As observed in similar AdS_5 and AdS_4 context indeed the field χ can be interpreted as a Lagrange multiplier in the superpotential, enforcing the constraints from the massive vectors on the special geometry. We will see its explicit role also in our case studied in detail below.

7.2 5d/4d reduction and attractor mechanism

Here we briefly illustrate the procedure introduced in [13]. This amounts of studying a putative 5d black hole by relating the near horizon region through a Kaluza–Klein dimensional reduction along the Hopf fiber of the S^3 horizon. By considering the limit with equal angular momenta the problem can be reformulated as the study of a 4d static black hole, and the entropy is obtained through the attractor mechanism.
In the original construction of [13] only vector multiplets where considered. Afterwards hypermultiplets have been added to the analysis in [105].

The first step consists in reducing the 5d theory on a circle corresponding to the Hopf fiber of S^3. This can be achieved by employing the KK reduction ansatz [13, 105, 191, 192]

$$\begin{aligned} ds^2_{(5)} &= e^{2\widetilde{\phi}} ds^2_{(4)} + e^{-4\widetilde{\phi}}(dy - A^0_{(4)})^2, \\ h^I &= -\sqrt{\frac{2}{3}} e^{2\widetilde{\phi}} \,\mathrm{Im}\, z^I \qquad (I = 1, \ldots, n_V + 1), \end{aligned} \tag{7.17}$$

where y is the direction of the circular fiber and $A^0_{(4)}$ is the KK vector. Using the constraint on $\mathcal{SM}$ in (7.2), the field $\widetilde{\phi}$ can be eliminated with $e^{-6\widetilde{\phi}} = -\frac{4}{3\sqrt{6}}\mathcal{V}(\mathrm{Im}\, z^I)$.

Notice that our ansatz differs from the one in [105] by a factor $\sqrt{\frac{2}{3}}$ in the reduction of the coordinates h^I. This is because the 5d Lagrangian of [187] we start with (see (6.62)) is written with different conventions with respect to the ones used in [13, 105]. Such difference requires a modification of the KK ansatz in order to produce a 4d Lagrangian suitable for the analysis of the AdS_2 attractor of [18]. The detailed computation is quite long and is very similar to the one performed in [105]. For this reason we report it in appendix B.

The coordinate y is compactified on a circle of length $4\pi/g$ where g is the gauge coupling. In this way the relation between the gravity coupling constants assumes the standard form

$$\frac{1}{G_N^{(4)}} = \frac{4\pi}{g G_N^{(5)}}. \tag{7.18}$$

In addition to the usual KK ansatz, a Scherk–Schwarz twist [193, 194] for the gravitino in needed in order to satisfy the BPS conditions in 4d, as noted in [13, 105]. Thus, to complete the ansatz, we turn on flat gauge connections ξ^I along y:

$$A^I_{(5)} = A^I_{(4)} + \mathrm{Re}\, z^I (dy - A^0_{(4)}) + \xi^I dy. \tag{7.19}$$

This twist will bring the extra Killing vector in the 4d reduced theory.

Once the reduction is performed one is left with a 4d gauged supergravity with the following salient features. The special Kähler manifold is specified by the prepotential

$$F(X) = \frac{4}{3\sqrt{6}} C_{IJK} \frac{\check{X}^I \check{X}^J \check{X}^K}{X^0} \qquad \text{with} \qquad \check{X}^I = X^I + \xi^I X^0 \tag{7.20}$$

and the scalars fields are identified with the special coordinates $z^I = X^I/X^0$. The quaternionic-Kähler manifold $\mathcal{QM}$ remains the same as in 5d.
In the 4d theory there are three Killing vectors: two of them are inherited from the 5d theory, while the additional one is given by

$$K_0^u = \xi^I K_I^u, \tag{7.21}$$

which is gauged by the KK vector field $A^0_{(4)}$ in (7.19). Similarly, the third Killing prepotential of the theory is given by $\vec{P}_0 = \xi^I \vec{P}_I$.
The 4d electric and magnetic charges are

$$\begin{aligned} p^0 &= 1, \qquad & q_0 &= 4G_N^{(4)} g^2 J + \frac{2}{9} C_{IJK}\xi^I \xi^J \xi^K, \\ p^I &= 0, \qquad & q_{\mathfrak{T}} &= 4G_N^{(4)} g^2 Q_{\mathfrak{T}} + \frac{1}{3} C_{\mathfrak{T}JK}\xi^J \xi^K, \end{aligned} \tag{7.22}$$

where the index $\mathfrak{T}$ runs only over the massless vectors $\mathbb{B}^{\mathfrak{T}}{}_J A^J_\mu$ in (7.14). The corresponding conserved charges are $Q_{\mathfrak{T}} \equiv Q_J (\mathbb{B}^{-1})^J{}_{\mathfrak{T}}$.
Electric and magnetic charges form a symplectic vector:

$$\mathcal{Q} = (p^\Lambda, q_\Lambda), \qquad \Lambda = 0, 1, \ldots, n_V + 1. \tag{7.23}$$

Other useful definitions are

$$\vec{\mathcal{P}} = (0, \vec{P}_\Lambda), \qquad \vec{\mathcal{Q}} = \langle \vec{\mathcal{P}}, \mathcal{Q} \rangle, \tag{7.24}$$

where $\langle V, W \rangle = V_\Lambda W^\Lambda - V^\Lambda W_\Lambda$ is the symplectic-invariant antisymmetric form and the vector are triplets.
Following [105], we impose the ansatz $\vec{\mathcal{Q}} \cdot \vec{\mathcal{Q}} = 1$ and choose a gauge in which

$$\mathcal{Q}^1 = \mathcal{Q}^2 = 0 \quad \text{and} \quad \mathcal{Q}^3 = -1. \tag{7.25}$$

Maxwell's equations at the horizon give the condition

$$\mathcal{K}^X g_{XY} \langle \mathcal{K}^Y, \mathcal{Q} \rangle = 0, \tag{7.26}$$

where $\mathcal{K}^X = (0, K^X_\Lambda)$ because we work in a purely electric duality frame.
To make the hyperino variation vanish we have to impose

$$\langle \mathcal{K}^u, \mathcal{V} \rangle = 0, \tag{7.27}$$

where $\mathcal{V}(z, \bar{z}) = e^{\mathcal{K}/2}(X^\Lambda, F_\Lambda)$ (see (6.13) and (6.18)) and $F_\Lambda = \frac{\partial F}{\partial X^\Lambda}$.
The attractor equations for the near-horizon limit of 4d BPS static black hole solution are [18]

$$\frac{\partial}{\partial z^I}\left(\frac{\mathcal{Z}}{\mathcal{L}}\right) = 0 \quad \text{with} \quad \frac{\mathcal{Z}}{\mathcal{L}} = 2ig^2 L_S^2, \quad \mathcal{Z} = \langle \mathcal{Q}, \mathcal{V} \rangle,\ \mathcal{L} = \langle \mathcal{P}^3, \mathcal{V} \rangle. \tag{7.28}$$

Notice that only $\mathcal{P}^3$ contributes to the superpotential $\mathcal{L}$ after an opportune $SU(2)$ rotation is performed on the prepotentials.
The attractor equations are equivalent to

$$\partial_\Lambda \left[e^{-\mathcal{K}/2} \Big(\mathcal{Z}(X) - 2ig^2 L_S^2 \mathcal{L}(X) \Big) \right] = 0, \tag{7.29}$$

which is the key formula to reproduce the holographic dual extremization problem that allows to extract the black hole entropy from the superconformal index.

7.2.1 5d/4d reduction for the LS fixed point

We now focus on the model reviewed in section 7.1. Rewriting the symplectic vector of electric and magnetic charges as

$$\vec{\mathcal{Q}} = \langle \vec{\mathcal{P}}, \mathcal{Q} \rangle = \vec{\mathcal{P}}_\Lambda \mathcal{Q}^\Lambda - \vec{\mathcal{P}}^\Lambda \mathcal{Q}_\Lambda = \vec{P}_\Lambda p^\Lambda, \tag{7.30}$$

from (7.25) we obtain the conditions

$$P^3_0 = P^3_\Lambda p^\Lambda = -1, \qquad P^1_0 = P^2_0 = 0, \tag{7.31}$$

while Maxwell's equations in (7.26) give

$$p^\Lambda K^X_\Lambda = 0 \qquad \Rightarrow \qquad K^X_0 = 0. \tag{7.32}$$

Solving (7.31) and (7.32) (using the rotated prepotentials in (7.16)[3]) we obtain the following conditions on the fields and the gauge connections:

$$\sigma = 0, \quad V = 1 - \theta^2 - \tau^2 = 1 - \tanh^2\chi, \quad \xi_2 = \frac{\xi_1}{\sqrt{2}}, \quad \xi_1 = -\frac{1}{3\sqrt{2}}. \tag{7.33}$$

Finally, from the hyperino variation in (7.27), we have

$$\langle \mathcal{K}^X, \mathcal{V} \rangle = \mathcal{K}^X_\Lambda \mathcal{V}^\Lambda - \mathcal{L}^{X\Lambda}\mathcal{V}_\Lambda = e^{\mathcal{K}/2} K^X_\Lambda\, X^\Lambda = 0 \quad \Rightarrow \quad \sqrt{2}X^2 - X^1 = 0. \tag{7.34}$$

that corresponds to the condition imposed on the sections X^I appearing in the massive vector A^W_μ in (7.13).
Using (7.3), the prepotential (7.20) of our 4d theory becomes

$$F(X) = \sqrt{2}\, \frac{\check{X}^1(\check{X}^2)^2}{X^0}, \qquad \text{where} \qquad \check{X}^I = X^I + \xi^I X^0. \tag{7.35}$$

Furthermore

$$\begin{aligned} e^{-\mathcal{K}/2}\mathcal{Z}(X) &= q_\Lambda\, X^\Lambda - F_0 = \hat{q}_\Lambda\, X^\Lambda + \sqrt{2}\, \frac{X^1(X^2)^2}{(X^0)^2} \\ e^{-\mathcal{K}/2}\mathcal{L}(X) &= P^3_\Lambda\, X^\Lambda = -X^0 + \frac{3}{\sqrt{2}}\,(2 - \cosh^2\chi)\, X^1 + 3\cosh^2\chi\, X^2, \end{aligned} \tag{7.36}$$

where

$$\begin{aligned} \hat{q}_I &= q_I - \frac{1}{3} C_{IJK}\xi^J\xi^K, \\ \hat{q}_0 &= q_0 - \frac{2}{9} C_{IJK}\xi^I\xi^J\xi^K \end{aligned} \tag{7.37}$$

and in the last line we have introduced the new coordinate χ defined in (7.10). Thus, from (7.29) we obtain the following set of equations:

$$\begin{aligned} \partial_\Lambda \left[\sqrt{2}\, \frac{X^1(X^2)^2}{(X^0)^2} + \hat{q}_\Lambda\, X^\Lambda - 2ig^2 L_s^2 \left(3\sqrt{2}\, X^1 - X^0 - 3\cosh^2\chi \left(\frac{X^1}{\sqrt{2}} - X^2 \right) - \alpha \right) \right] &= 0, \\ \frac{\partial}{\partial L_s^2} \left[\sqrt{2}\, \frac{X^1(X^2)^2}{(X^0)^2} + \hat{q}_\Lambda\, X^\Lambda - 2ig^2 L_s^2 \left(3\sqrt{2}\, X^1 - X^0 - 3\cosh^2\chi \left(\frac{X^1}{\sqrt{2}} - X^2 \right) - \alpha \right) \right] &= 0. \end{aligned} \tag{7.38}$$

In the first line we rewrote (7.29), adding the constant α that does not modify the equations, while the second line fixes the gauge $\mathcal{L} = \alpha$.
The variation in (7.38) with respect to X^2 gives the equation

$$2\sqrt{2}\, \frac{X^1 X^2}{(X^0)^2} + \hat{q}_2 - 6ig^2 L_s^2 \cosh^2\chi = 0, \tag{7.39}$$

[3] During the computation we are actually using $P^r_\Lambda \to \tilde{P}^r_\Lambda = 2P^r_\Lambda$ (see (B.46) and the related notes).

that determines χ and q_2 in terms of the sections and of L_s.
We can now use the hyperino condition (7.34) to eliminate X^2 from the equations in (7.38). The remaining equations are equivalent to the conditions of extremization of the function

$$\mathcal{S} = \beta \left[\frac{\sqrt{2}}{2} \frac{(X^1)^3}{(X^0)^2} + \hat{q}_0 X^0 - \sqrt{3} \hat{q}_R X^1 - 2ig^2 L_s^2 \left(3\sqrt{2} X^1 - X^0 - \alpha \right) \right] \tag{7.40}$$

w.r.t. the variables X^0, X^1 and L_s. Here β is a constant that will be useful later and $\hat{q}_R$ is the charge with respect to the massless vector A_R in (7.14):

$$\hat{q}_R = \frac{1}{\sqrt{6}} (\sqrt{2} \hat{q}_1 + \hat{q}_2) = 4g^2 G_N^{(4)} Q_R. \tag{7.41}$$

Notice that we have divided by $2^{2/3}$ the coefficients of K_R, in order to have a proper normalization of the AdS scale. In fact, the relation between L and the coupling constant g can be red from the value of the scalar potential at the IR critical point $V|_{\text{IR}} = -6 \cdot 2^{4/3}$, from which

$$L = \frac{1}{2^{2/3} g}. \tag{7.42}$$

$\mathcal{S}$ is an homogeneous function of degree 1 in X^Λ, except for the term involving α. Therefore

$$\mathcal{S}\Big|_{\text{crit}} = 2i\alpha\beta g^2 L_s^2 \tag{7.43}$$

at the critical point and choosing

$$\alpha\beta = \frac{\pi}{2iG_N^{(4)} g^2}, \tag{7.44}$$

we obtain that $\mathcal{S}\big|_{\text{crit}}$ corresponds to the black hole entropy. Using (7.22) and (7.37), the extremization problem (7.40) becomes

$$\mathcal{S} = \frac{1}{\alpha} \left[\frac{\pi}{2ig^2 G_N^{(4)}} \frac{\sqrt{2}}{2} \frac{(X^1)^3}{(X^0)^2} - 2\pi i \left(J X^0 + \sqrt{3} Q_R X^1 \right) - 2\pi i \Lambda \left(3\sqrt{2} X^1 - X^0 - \alpha \right) \right], \tag{7.45}$$

where we have redefined the Lagrange multiplier $L_s^2 = 2iG_N^{(4)} \Lambda$.

7.3 Holographic matching

The last step consists in pointing out the AdS/CFT dictionary between the charges in gravity and field theory and check the agreement between the results in subsection 7.2.1 and the ones in section 5.3. First, the number of colors N_c of the gauge group in field theory is related to the Newton constant $G_N^{(5)}$. In fact

$$G_N^{(5)} = \frac{L^3 \text{Vol}(Y_5)}{2\pi^2 N_c^2}, \qquad \text{with} \qquad \text{Vol}(Y_5) = \frac{\pi^3 N_c^2}{4a}, \tag{7.46}$$

where L is the AdS$_5$ length scale. For the 4d SCFT LS fixed point

$$a = \frac{27}{128} N_c^2. \tag{7.47}$$

Using (7.42) and (7.47) we obtain

$$G_N^{(5)} = \frac{4\pi}{27g^3 N_c^2} \tag{7.48}$$

and, using (7.18), we can relate the Newton constant in 4d to N_c:

$$G_N^{(4)} = \frac{1}{27g^2 N_c^2}. \tag{7.49}$$

The angular momentum J on the gravity side corresponds to the one on the field theory side, while the electric charge r is related to Q_R by

$$r = \frac{Q_R}{\gamma}. \tag{7.50}$$

The coefficient γ can be inferred by comparing the 't Hooft anomalies for global currents in the boundary field theory with the Chern–Simons couplings for gauge fields in the bulk 5d gravity:

$$\frac{g^3}{24\pi^2}\,\mathrm{Tr}(Q_I Q_J Q_K) = \frac{1}{8\pi G_N^{(5)}}\frac{2}{3\sqrt{6}}\cdot\frac{1}{3}C_{IJK}. \tag{7.51}$$

Inserting $C_{RRR} = 6$, we obtain $\gamma = \sqrt{6}$. Finally, after the change of coordinates $X^0 \to 2\alpha\omega$ and $X^1 \to \frac{2\sqrt{2}}{3}\alpha X_r$, our entropy exactly matches the one obtained from the large N_c limit of the superconformal index of the dual theory given in formula (5.73).

We conclude this section with an observation on formula (7.51). While here we are focusing on a specific truncation, the relation between the 5d Chern–Simons and the 4d anomalies holds true in general. For the SE_5 case the C_{IJK} Chern–Simons are constructed starting from the self dual five form flux F_5 and the three forms of the SE_5 that appear by considering the fluctuations of F_5. Beyond the toric case, even if the interpretation of (7.51) remains the same, the details of the truncation are necessary to give a 10d (or 11d) interpretation. Anyway, large classes of 5d rotating black holes have been constructed [74] (in absence of hypermultiplets) where the coefficients C_{IJK} parametrize a symmetric space. Here we stress that among the SE_5 models only the coefficients C_{IJK} of S^5 respect the condition of symmetric space.

Part IV

Compactifications on spindles

Chapter 8

Compactification on curved spaces

The compactification of superconformal field theories on curved spaces provides a way of defining new interacting fixed points in lower dimensions. When such compactification preserves some supersymmetry there is a high control on the lower-dimensional physics, because some observables can be traced through anomaly inflow and/or localization. For example, in the case of even-dimensional field theories compactified on complex manifolds one can integrate the anomaly polynomial over the compact sub-space to obtain the anomalies of the lower-dimensional field theory, as we will show later in this chapter. In other dimensions similar results have been proven, with the help of supersymmetric localization. This provides a strong tool to claim the existence of a lower-dimensional interacting fixed point.
In many cases the lower-dimensional complex manifold does not preserve any supersymmetry and it is necessary to turn on opportune background fluxes to restore the cancellations of the spin connection in the fermionic variations. This idea, generically referred to as (partial) topological twist [195–197], allows for the analysis of large classes of models. Furthermore, the constructions discussed above have an interesting dual counterpart in the AdS/CFT correspondence. The original case was discussed in [19] for branes wrapped on Riemann surfaces. This mechanism, denoted as flow across dimensions, was further studied in [20] and associated with the dual c-extremization principle formulated in [21].
The parallel treatment of the topological twist and of the flow across dimensions has been checked in many examples, and it provides a further check of the existence of the new superconformal fixed point in the lower-dimensional theory.
Recently a new type of compactification has been considered [22]. The starting point consists of considering the compact space as an orbifold, instead of a manifold. If this orbifold is a spindle $\mathbb{\Sigma}$, topologically a two sphere with deficit angles at the poles, it is possible to show that some supersymmetry of the higher-dimensional theory is preserved in a new way, as we will see in the second part of this chapter.

In the following we present the most salient aspects of compactifications of superconformal field theories on curved spaces, both on manifolds and on spindles. In section 8.1 we review the original construction by Maldacena and Nuñez, in which new classes of SCFTs are obtained wrapping D3- and M5-branes on a Riemann surface of constant curvature. In section 8.2 we introduce compactifications on spindles, focusing on the way supersymmetry is preserved and describing the procedure that allows to extract the central charge and the superconformal R-symmetry of the lower-dimensional SCFT from the higher-dimensional one.

8.1 Compactification on curved manifolds

New classes of superconformal field theories can be engineered in string theory and M-theory by wrapping branes on a compact space Ω and taking the decoupling limit, i.e. flowing to the IR. The first supergravity solutions corresponding to branes wrapping a Riemann surface Σ_g were found in the seminal work of Maldacena and Nuñez [19], which we briefly recap in the following.

Let us consider a supersymmetric field theory in flat space having a Killing spinor ϵ such that $\partial_\mu \epsilon = 0$. Typically, putting the theory on a curved manifold Ω breaks supersymmetry because the Killing spinor equation is modified by the appearance of a spin connection ω_μ thus becoming $(\partial_\mu + \omega_\mu)\epsilon$, which is in general non-zero. However, if the field theory has a global R-symmetry, we can introduce an external gauge field that couples to the R-symmetry current such that it "compensates" the presence of the spin connection, i.e. if we choose $A_\mu = \omega_\mu$ then we recover $(\partial_\mu + \omega_\mu - A_\mu)\epsilon = \partial_\mu \epsilon = 0$. We thus obtain a constant Killing spinor and the resulting theory is called a *twisted theory*, due to the coupling to the external field that effectively changes the spins of the fields of the theory. This idea is usually referred to as *topological twist*.
The above mechanism is analogous to the one with which branes wrapped on non-trivial cycles in compactifications of string theory or M-theory can preserve some supersymmetry. In these latter cases, Ω is the worldvolume geometry of the cycle and the external field A_μ is the connection on the non-trivial bundle normal to the cycle. In order to preserve supersymmetry, the spin connection must be equal to the gauge connection. If we then take the decoupling limit $\ell_s \to 0$ while keeping the volume of the cycle fixed, we get a field theory on the brane that is twisted, due to the non-trivial embedding of the cycle in the ambient space.

More concretely, we can start from $(d+2)$-dimensional field theories living on $\mathbb{R}^d \times \Sigma_g$, where Σ_g is a Riemann surface of genus g. At small energies in comparison with the inverse size of Σ_g, these theories reduce to d-dimensional field theories and one can study the supergravity solutions that describe the flow between them. In the cases considered in [19], they start from SCFTs in $d+2$ dimensions that have the maximum amount of supersymmetry, which give rise to d-dimensional theories preserving $1/2$ or $1/4$ of the original SUSY, depending on how the Riemann surface is embedded in the higher-dimensional manifold. In fact, different embeddings correspond to different normal bundles, which affect the external gauge fields that can be coupled to the R-symmetry current. Moreover, in the cases where the resulting d-dimensional field theories in the IR are conformal, one can also find a dual AdS$_{d+1}$ geometry and the mechanism can be viewed as a compactification of a $(d+3)$-dimensional gauged supergravity on AdS$_{d+1} \times \Sigma_g$ with magnetic fluxes on Σ_g. From the gravitational side, this is usually referred to as *flow across dimensions*.

The examples presented in [19] regard N D3- and M5-branes wrapping Riemann surfaces Σ_g with constant curvature and genus g, in the large-N limit. As we said, the field theories on the branes are twisted theories [197], i.e. there is a coupling to an external $SO(n)$ gauge field, where n is the number of transverse directions to the branes. From the dual gravitational point of view, this coupling corresponds to boundary conditions at the boundary of AdS$_{d+1}$, for both the metric and the $SO(n)$ gauge fields. In these examples, only Riemann surfaces with constant curvature are considered and the spin connection is embedded into the $SO(n)$ connection.

In the first case, on the worldvolume of a stack of N D3-branes wrapped on $\mathbb{R}^2 \times \Sigma_g$ lives a $\mathcal{N} = 4$ Super Yang–Mills theory in 3+1 dimensions. There are different possible twisting of this field theory, as considered in [198], that depend on the way in which the spin connection is embedded in the $SU(4) \cong SO(6)$ R-symmetry group. Two embeddings are presented. Breaking $SO(6) \to SO(2) \times SO(4)$, the first one corresponds to picking a $U(1) \subset SO(2)$ factor in $SO(6)$. This twisting gives rise to a two-dimensional field theory with $(4, 4)$ supersymmetry. The second embedding consists in taking the gauge connection in $U(1) \subset SU(2) \subset SO(4)$ from the splitting of $SO(6)$ considered above, which leads to an IR two-dimensional theory preserving $(2, 2)$ supersymmetry. The holographic dual of this mechanism is represented by a supergravity solution interpolating between a compactification of Type IIB supergravity to AdS_5 and a compactification of the same theory to $\text{AdS}_3 \times \Sigma_g$, with different asymptotic boundary conditions reflecting the different couplings.
On the other hand, in the second case, if we consider N M5-branes wrapped on $\mathbb{R}^4 \times \Sigma_g$ and we send $\ell_s \to 0$ while keeping the size of Riemann surface fixed, we obtain a six-dimensional superconformal field theory on $\mathbb{R} \times \Sigma_g$ with $(0, 2)$ supersymmetry. Depending on the embedding of the spin connection in the $SO(5)$ R-symmetry group, this theory gives rise to new $\mathcal{N} = 2$ or $\mathcal{N} = 1$ superconformal field theories in four dimensions, for $g > 1$, i.e. when the Riemann surface has negative curvature. From the gravity side, this corresponds to a solution that interpolates between AdS_7 and $\text{AdS}_5 \times \Sigma_{g>1}$, both coming from compactifications of eleven-dimensional supergravity.

The flows presented above are some examples of application of the topological twist. Recently, it has been observed that one can extend the notion of the topological twist on manifolds with orbifold singularities [22], such for example the spindles. The compactification of higher dimensional theories on spindles will be the main topic of the remainder of this work.

8.2 Compactification on spindles

As we mentioned at the beginning of this chapter, when the compactification is performed on a spindle, it is possible to show that some supersymmetry of the higher-dimensional theory is preserved in an unusual way. The Killing spinors are indeed not constant on the spindle and have definite chirality at the poles. It has been shown that there are two possible ways to preserve supersymmetry by turning on background fluxes for the R-symmetry on the spindle. These two ways have been denoted as the twist and the anti-twist [199].

More in detail, a spindle is a weighted projective space $\mathbb{WCP}^1_{[n_N, n_S]}$ with conical deficit angles at the north and the south pole. The geometry is specified by the two co-prime integers n_N and n_S, associated with the deficit angles $2\pi \left(1 - \frac{1}{n_{N,S}}\right)$ at the poles.

Several $\text{AdS}_{d-2} \times \mathbb{Z}$ supersymmetric solutions of gauged supergravity theories in various dimensions have been constructed, showing very non-trivial holographic matchings, and allowing to conjecture the existence of vast new families of SCFTs [199–221]. In the following chapters we will focus on solutions in $d = 5$, associated with D3- and M5-branes wrapping spindles.
As we mentioned above, superconformal field theories can be realized as fixed points of the RG flows arising from compactifications of higher-dimensional field theories on curved spaces,

whose gravitational dual counterpart is given, under proper conditions, by a generalized flow across dimensions in supergravity. Thus, the existence of both an AdS_5 and an $\mathrm{AdS}_3 \times \mathbb{\Sigma}$ solution to the same gravitational model suggests that one can compactify the four-dimensional superconformal field theory on the spindle, and the compactified theory flows to a two-dimensional SCFT in the IR, with less supersymmetry.

More in detail, in the examples involving D3-branes, after uplifting the $\mathrm{AdS}_3 \times \mathbb{\Sigma}$ solutions on specific five-dimensional Sasaki–Einstein manifolds, one obtains $\mathrm{AdS}_3 \times Y_7$ solutions of Type IIB supergravity that are completely smooth. The dual 2d SCFT has $\mathcal{N} = (0, 2)$ supersymmetry and it comes from the compactification on $\mathbb{\Sigma}$ of a 4d $\mathcal{N} = 1$ SCFT, which is dual to a specific $\mathrm{AdS}_5 \times \mathrm{SE}_5$ solution of Type IIB supergravity.
On the other hand, in the examples involving M5-branes, after the uplift one obtains solutions of $d = 11$ supergravity that are still singular. Nevertheless, strong evidence has been provided [199, 203] that they holographically describe M5-branes wrapping spindles.

8.2.1 Twist and anti-twist

The twist and the anti-twist are determined by the R-symmetry flux through the spindle [199] :

$$\frac{1}{2\pi}\int_{\Sigma} F^R = \pm\frac{n_N + n_S}{n_N n_S} \qquad \text{twist}\,, \tag{8.1}$$

$$\pm\frac{n_S - n_N}{n_N n_S} \qquad \text{anti-twist}\,. \tag{8.2}$$

In the case of the twist the integrated R-symmetry flux on the spindle corresponds to the Euler characteristic of the spindle. This is the reason for using the name twist for such a case, being this property shared with the usual partial topological twist as well. Furthermore there is a second possibility, denoted as the anti-twist, that does not have any counterpart in the standard partial topological twist.

As we mentioned above, there are some crucial differences between how supersymmetry is preserved in the usual topological twist for Riemann surfaces or in the twist/anti-twist for spindles. The preserved Killing spinors in the latter cases depend on (some of) the coordinates on the spindle, while in the first case they are constant on the Riemann surfaces. Indeed, in the twist/anti-twist the spinors are sections of non-trivial bundles over the spindles, whereas in the topological twist the spinors are sections of trivial bundles over the Riemann surface, and thus constant.

8.2.2 Results from field theory

From the field theory side of the duality, the central charge and the superconformal R-symmetry of the lower-dimensional theory can be extracted from the higher-dimensional one integrating the *anomaly polynomial*.

In general, the anomaly polynomial of a $2n$-dimensional theory is a formal $(2n + 2)$-form characteristic class [1] constructed from a fiber bundle and the tangent bundle, that contains the anomalies of the theory (see, for example, appendix A of [20] for a more exhaustive review). In the cases of our interest the fiber bundle corresponds to the global symmetry group. More explicitly, if $\mathcal{S}$ is the set of all chiral fields ψ_i of the theory, the total anomaly polynomial is defined as [222]

$$I_{2n+2}^{\text{total}} = \left(\sum_{\psi_i \in \mathcal{S}} c(\psi_i) \right) P_{n+1} = \sum_{\psi_i \in \mathcal{S}} I_{2n+2}^{(i)} \,, \tag{8.3}$$

where $c(\psi_i)$ are coefficient depending on the chiral field content of the theory and P_{n+1} is the characteristic class in the case where P is a gauge invariant polynomial of F and we are considering

$$P_m(F) = \operatorname{Tr} F^m \equiv \operatorname{Tr} \underbrace{F \wedge \cdots \wedge F}_{m \text{ times}} \,. \tag{8.4}$$

Furthermore, since $P_m(F)$ are closed $2m$-forms, they must be locally exact, i.e. there must exist suitable $(2m-1)$-forms $Q_{2m-1}(A_{(i)}, F_{(i)})$, with $F_{(i)}$ =d$A_{(i)}$, such that

$$P_m(F) = \mathrm{d}Q_{2m-1}(A_{(i)}, F_{(i)}) \quad \text{locally on each } U_i \,. \tag{8.5}$$

$Q_{2m-1}(A, F)$ is called the Chern–Simons $(2m-1)$-form. Thus, characteristic classes and Chern–Simons classes are related through descent equations.

To give a concrete example, in four-dimensional $\mathcal{N}=1$ SCFTs the anomaly polynomial, related to the anomalous divergence of the R-current by the descent procedure, is given by [223]

$$I_6 = \frac{\mathrm{Tr} R^3}{6} c_1(R_{4d})^3 - \frac{\mathrm{Tr} R}{24} c_1(R_{4d})\, p_1(T_4) \,, \tag{8.6}$$

where TrR and TrR^3 are the linear and cubic 't Hooft anomalies of the superconformal R-symmetry, respectively, $c_1(R_{4d})$ denotes the first Chern class of the 4d superconformal $U(1)_R$ symmetry bundle and $p_1(T_4)$ is the first Pontryagin class of the bundle tangent to the 4d space-time manifold on which the theory is defined.

From a more physical point of view, the anomaly polynomial of a theory is a quantity that describes how the phase of the partition function of the theory changes under gauge transformations of the background metric and/or gauge fields (see, e.g. [224]).
If the theory that we are considering comes from a dimensional reduction on a compactified manifold $\mathcal{M}_d$ of a D-dimensional theory for which the anomaly polynomial I_{D+2} is known, one can compute the anomaly polynomial I_{D-d+2} of the resulting $(D-d)$-dimensional theory by integrating I_{D+2} over $\mathcal{M}_d$.

Integration of the anomaly polynomial

Following [224], we start by reviewing the case in which we neglect the isometry of $\mathcal{M}_d$. We will mention how the presence of the isometry group of $\mathcal{M}_d$ modifies the current analysis at

[1] A characteristic class P is a local form on a compact manifold that is constructed from the curvature or field strength F and such that its integral over the manifold is sensitive to non-trivial topology [222]. The latter property follows from the fact that P is closed but not exact.

the end of this section. The lower-dimensional anomaly polynomial I_{D-d+2} is related to the higher-dimensional one I_{D+2} by

$$I_{D-d+2} = \int_{\mathcal{M}_d} I_{D+2} \,. \tag{8.7}$$

To see how this relation is derived, let X_D be the spacetime on which the theory lives and let $Y_{D+1} = X_D \times S^1$. We introduce the background metric and the background gauge fields on Y_{D+1}, such that Y_{D+1} is obtained starting from $X_D \times [0,1]$ and gluing the two boundaries $X_D|_0$ and $X_D|_1$ by the aforementioned diffeomorphism and gauge transformation that modify the phase of the partition function of the theory. Such phase is given by

$$\int_{Y_{D+1}} \mathcal{CS}_{D+1}, \quad \text{where} \quad \mathrm{d}\mathcal{CS}_{D+1} = I_{D+2} \,. \tag{8.8}$$

Compactifying on $\mathcal{M}_d$ means taking $X_D = X_{D-d} \times \mathcal{M}_d$, that implies $Y_{D+1} = Y_{D-d+1} \times \mathcal{M}_d$. Thus:

$$\int_{Y_{D+1}} \mathcal{CS}_{D+1} = \int_{Y_{D-d+1}} \mathcal{CS}_{D-d+1} \,, \quad \text{where} \quad \mathcal{CS}_{D-d+1} = \int_{\mathcal{M}_d} \mathcal{CS}_{D+1} \,. \tag{8.9}$$

If we take a manifold Z_{D-d+2} such that $\partial Z_{D-d+2} = Y_{D-d+1}$ and set $Z_{D+2} = Z_{D-d+2} \times \mathcal{M}_d$, then the relation (8.9) implies

$$\int_{Z_{D+2}} I_{D+2} = \int_{Z_{D-d+2}} I_{D-d+2} \,, \quad \text{which means} \quad I_{D-d+2} = \int_{\mathcal{M}_d} I_{D+2} \,. \tag{8.10}$$

From this analysis we can observe that, in order to include the background gauge field for the isometry group G of $\mathcal{M}_d$, we need to take Z_{D+2} to be a non-trivial $\mathcal{M}_d$ bundle over Z_{D-d+2} with a non-trivial G connection, i.e.:

$$\mathcal{M}_d \hookrightarrow Z_{D+2} \to Z_{D-d+2} \,. \tag{8.11}$$

Chapter 9

$T^{1,1}$ truncation on the spindle

In the previous chapter we gave a flavor of how compactifications on spindles work. In order to illustrate a more concrete example, in this chapter we present a supersymmetric $\mathrm{AdS}_3 \times \mathbb{\Sigma}$ solution asymptotic to the AdS_5 $\mathcal{N} = 2$ truncation of the conifold with a Betti vector multiplet found in [225]; the model consists of gauged supergravity with two vector multiplets and two hypermultiplets. The vector fields gauge a subgroup of the quaternionic manifold and one gauge field becomes massive via Higgs mechanism. In the low energy spectrum there are then two massless fields, the graviphoton and the Betti vector. One is associated with the R-symmetry and the other one to the baryonic symmetry of the dual Klebanov-Witten field theory [54]. When this model is compactified on the spindle many of the scalars in the hypermultiplet can be further truncated. A crucial aspect of this compactification is that we need to include some of the scalars from the hypermultiplet in the analysis.

The motivation beyond our choice is related to the fact that in the last few years compactifications on spindles have been studied in many setups, in various dimensions, showing very non trivial holographic matchings, and allowing to conjecture the existence of vast new families of SCFTs [199–221]. In four-dimensional SCFTs many predictions have been further made by the field theory analysis corresponding to the integration of the anomaly polynomial. Many of these predictions have been holographically checked, both from a 10 or 11d perspective, from the analysis of consistent truncations in gauged supergravity. In this last case for example it has been possible to check the behaviour of the universal twist and anti-twist and in general the study of the $U(1)^3$ STU gauged supergravity has allowed to match the expected result for the case of $\mathcal{N} = 4$ SYM. In this case the anti-twist was studied in [201, 202], while a general analysis of all possible twists was given in [199].
Very recently a truncation with hypermultiplets has been considered as well. It corresponds to the Leigh–Strassler $\mathcal{N} = 1^*$ fixed point and it has been shown that also in this case the expected dual results can be reproduced from the supergravity dual description. The case of the topological twist in this case was studied from the supergravity perspective in [226]. One of the most remarkable results of [227] was that the central charge of the theory compactified on the spindle can be obtained without the knowledge of the full solution of the BPS equation. It has been shown indeed that the correct central charge can be obtained by solving these equations only at the poles of the spindle, i.e. by specifying the boundary conditions on the fields and the conserved magnetic charges in terms of the data at the poles of the spindle. This analysis at the poles is also a necessary step for constructing the numerical solution when the magnetic charge for the flavor symmetry is turned on, because it fixes most of the boundary conditions when

solving the BPS equation.

Therefore, motivated by the results of [227], in this chapter we study another five-dimensional $\mathcal{N} = 2$ consistent truncation with two vector multiplets and two hypermultiplets originally found in [159, 225]. This truncation is associated with the Klebanov Witten theory [54] and, due to the Higgs mechanism triggered by a scalar in a hypermultiplet, one vector field becomes massive. The two remaining massless vector fields are the graviphoton and the so called Betti vector. This structure of massless vector fields allows non-trivial comparisons with the field theory results in terms of the magnetic fluxes for the R-symmetry and the baryonic symmetry. The role of baryonic symmetries in the case of the topological twist was then exploited in [168, 169]. Here we study the compactification of this model on the spindle, along the lines of the analysis of [227]. We find that also in this case the central charge can be extracted simply from the pole data and then we solve numerically the BPS equation in order to construct the full AdS_3 solution. As a consistency check we also show that our results are in agreement with the ones expected from the dual field theory for the anti-twist class.

This chapter, based on [228], is organized as follows. In section 9.1 we review the 5d setup corresponding to the 5d $\mathcal{N} = 2$ Betti vector truncation found in [225] for the conifold. In section 9.2 we study the BPS equations and the Maxwell equations for the AdS_3 ansatz on the spindle, turning on suitable magnetic fluxes for the gauge fields. Then in section 9.3 we solve these equations at the poles of the spindle. This analysis fixes the boundary conditions for many of the scalars and it imposes the necessary constraints on the fluxes. Then we show that these solutions are enough to compute the central charge from the Brown–Henneaux formula. In section 9.4 we provide a complete solution of the BPS equations. First we turn off the magnetic charge for the baryonic symmetry reducing to minimal supergravity. In this case we recover the analytic results of [200] for the universal anti-twist and we match it with the result obtained in section 9.3 from the analysis of the BPS equations at the poles of the spindle. Then we provide the numerical solution of the BPS equations in presence of non-vanishing baryonic magnetic charge, again finding an agreement with the result obtained from the pole data. In section 9.5 we then compare our findings with the calculation of the central charge for the conifold obtained from the dual field theory analysis. In this case we match the result by turning off the magnetic charges associated with the mesonic symmetries of the dual field theories, that are indeed invisible in the 5d truncation considered here. Finally, in appendix C we discuss further details of the quaternionic geometry of the specific 5d supergravity model we consider in this chapter.

9.1 The supergravity set-up

In this section we introduce the $\mathcal{N} = 2$ supergravity set-up in five dimensions. We provide a short summary on $\mathcal{N} = 2$ consistent truncations of Type IIB supergravity on squashed Sasaki-Einstein manifolds and we then focus on compactifications defined by the conifold as the internal manifold. In this regards, we introduce the $\mathcal{N} = 2$ 5d gauged supergravity model associated to this truncation and we study its scalar manifold with particular focus on the gauging. Finally we discuss a further truncation of fields in 5d restricting to those moduli which capture the physics of $AdS_3 \times \mathbb{\Sigma}$ backgrounds.

9.1.1 Type IIB on the conifold and $\mathcal{N} = 2$ supergravity

We start with a (very) brief summary on $\mathcal{N} = 2$ consistent truncations of Type IIB supergravity over the (squashed) conifold

$$T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)} . \tag{9.1}$$

Such compactifications belong to a general class of consistent truncations on 5d squashed Sasaki-Einstein manifolds, which have been extensively studied in the literature (see for instance [159, 225, 229–235]). More specifically in [230, 231] reductions over squashed Sasaki-Einstein manifolds to the $\mathcal{N} = 4$ 5d universal sector were constructed, then in [225, 234] this class of truncations was extended to the non-trivial second cohomology forms on $T^{1,1}$. The resulting lower-dimensional theory is a $\mathcal{N} = 4$ 5d gauged supergravity coupled to two vector multiplets, coming from the universal sector, plus a third vector multiplet. The latter is called the Betti multiplet and it is associated with left-invariant modes acting on the conifold.

The bosonic field content of this $\mathcal{N} = 4$ supergravity includes the 5d gravitational field, the graviphoton A^0_μ, 8 vectors and 16 real scalar fields. We need now to impose a further truncation to select the $\mathcal{N} = 2$ sector. As it was showed in [225], truncating to the $\mathcal{N} = 2$ sector is not trivial since it requires the truncation either of the $\mathcal{N} = 2$ Betti-vector multiplet or of the Betti-hypermultiplet. In this way one obtains two inequivalent theories.

We will focus on the $\mathcal{N} = 2$ 5d supergravity retaining in its spectrum the Betti vector .[1] Such a theory is described by the coupling to two vector multiplets and two hypermultiplets whose scalar geometry is encoded in the following manifold [225],

$$\mathcal{M} = SO(1,1)^2 \times \frac{SO(4,2)}{SO(4) \times SO(2)} . \tag{9.2}$$

Let us thus explore with more detail the supergravity model defined by the scalar manifold (9.2). To this aim we will follow the notation of [160]. The $\mathcal{N} = 2$ matter multiplets include 10 scalar fields and 3 vector fields

$$\begin{aligned} \{u_1, u_2, u_3, k, a, \phi,\, b^i, \bar{b}^i\} \quad &\text{with} \quad i = 1, 2 \\ \text{and} \quad A^I_\mu \quad &\text{with} \quad I = 0, 1, 2\,. \end{aligned} \tag{9.3}$$

Apart from the graviphoton A^0_μ, the above fields are organized into two vector multiplets, defined by the real scalars $\{u_2,\, u_3\}$ and the two vectors A^x_μ with $x = 1, 2$, and two hypermultiplets, parametrized by the scalars $\{u_1, k, a,\, \phi, b^i, \bar{b}^i\}$ where b^1, b^2 are written in complex notation.

We first analyze the vector multiplet sector. The two real scalars parametrize the very special real manifold $SO(1,1)^2$. We can organize the moduli fields following the general analysis of chapter 6,

$$\phi^x = (u_2, u_3) \qquad \text{and} \qquad g_{xy} = \begin{pmatrix} 4 & 0 \\ 0 & 12 \end{pmatrix} , \tag{9.4}$$

[1] The "twin" $\mathcal{N} = 2$ theory is defined by truncating away the Betti-vector [225]. The matter content of this theory is featured by one vector multiplets and three hypermultiplets. In this case the coset manifold is given by $\mathcal{M} = SO(1,1) \times \frac{SO(4,3)}{SO(4)\times SO(3)}$.

where g_{xy} is metric on the manifold $SO(1,1)^2$. As explained in chapter 6, a very special real manifold can be defined through the embedding relation $C_{IJK}h^I h^J h^K = 1$, where $h^I(\phi)$ are homogeneous coordinates. For our model these are given by

$$h^0 = e^{4u_3}, \qquad h^1 = e^{2u_2 - u_3}, \qquad h^2 = e^{-2u_2 - 2u_3}\,, \tag{9.5}$$

while the symmetric tensor C_{IJK} has a unique non-vanishing component given by $C_{012} = 1/6$. Through the general relation written in (6.28) we can thus derive the metric a_{IJ} on the embedding manifold. Such quantity defines the coupling in the action of vector-scalars with gauge fields,

$$a_{IJ} = \begin{pmatrix} \frac{1}{3}e^{-8u_3} & 0 & 0 \\ 0 & \frac{1}{3}e^{-4u_2+4u_3} & 0 \\ 0 & 0 & \frac{1}{3}e^{4u_2+4u_3} \end{pmatrix}. \tag{9.6}$$

Let us consider now the hypermultiplet sector. This is defined by the quaternionic manifold $\frac{SO(4,2)}{SO(4)\times SO(2)}$ which is spanned by the fields $\{u_1, k, a,\ \phi, b^1, \bar{b}^1, b^2, \bar{b}^2\}$. We point out that the scalars a and ϕ can be also written in complex notation as $\tau = a + ie^{-\phi}$, where the complex modulus τ results directly from the reduction of the axio-dilaton of Type IIB supergravity. As discussed in chapter 6, the scalars parametrizing a quaternionic manifold are organized in quadruples q^X. Following the notation of [159, 160] we can write for our model

$$q^X = (u_1, k, a, \phi,\ b^1,\ \bar{b}^1,\ b^2,\ \bar{b}^2)\,. \tag{9.7}$$

Then the line element takes the following form

$$\begin{aligned} g_{XY}dq^X dq^Y =& -2e^{-4u_1}M_{ij}\left(b^i d\bar{b}^j + \bar{b}^i db^j\right) - 4du_1^2 - \frac{1}{4}d\phi^2 - \frac{1}{4}e^{2\phi}da^2 \\ & - \frac{1}{4}e^{-8u_1}\left[dk + 2\varepsilon_{ij}\left(b^i d\bar{b}^j + \bar{b}^i db^j\right)\right]^2, \end{aligned} \tag{9.8}$$

where

$$M_{ij} = e^{\phi}\begin{pmatrix} a^2 + e^{-2\phi} & -a \\ -a & 1 \end{pmatrix}. \tag{9.9}$$

We point out that the matrix M_{ij} is covariant under $SL(2,\mathbb{R})$ symmetry inherited from Type IIB supergravity. This scalar geometry was already studied in [159, 160, 236]. We refer to appendix C for the derivation of quaternionic structures $\vec{J}$ and the $SU(2)$ spin connections $\vec{\omega}$.

On the quaternionic manifold we have gauged symmetries defined by the following set of abelian Killing vectors $k_I = k_I^X\partial_X$,

$$\begin{aligned} k_0 &= -3ib^1\partial_{b^1} - 3ib^2\partial_{b^2} + 3i\bar{b}^1\partial_{\bar{b}^1} + 3i\bar{b}^2\partial_{\bar{b}^2} - Q\partial_k, \\ k_1 &= 2\partial_k, \\ k_2 &= 2\partial_k, \end{aligned} \tag{9.10}$$

where Q is a constant. Given the Killing vectors on the quaternionic manifold we can also introduce the associated Killing prepotentials P_I^r,

$$\begin{aligned} P_0^r &= \begin{pmatrix} \frac{3}{2}e^{-2u_1-\frac{\phi}{2}}\left(b^1 - iae^{\phi}(b^1-\bar{b}^1) + ie^{\phi}(b^2-\bar{b}^2) + \bar{b}^1\right) \\ \frac{3}{2}e^{-2u_1-\frac{\phi}{2}}\left(ib^1 + ae^{\phi}(b^1+\bar{b}^1) - e^{\phi}(b^2+\bar{b}^2) - i\bar{b}^1\right) \\ -\frac{3}{2} + e^{-4u_1}\left(\frac{Q}{4} - 3i(b^1\bar{b}^2 - b^2\bar{b}^1)\right) \end{pmatrix}, \\ P_1^r &= \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}e^{-4u_1} \end{pmatrix}, \qquad P_2^r = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}e^{-4u_1} \end{pmatrix}. \end{aligned} \tag{9.11}$$

The full bosonic Lagrangian of this model is given in [159, 160]. Specifically, the scalar potential can be obtained by specifying the general expression (6.60) with the data on the scalar geometry given in this section. We point out that the Killing vectors (9.10) satisfy abelian commutation relations and generate the gauge group $U(1)^2 \times \mathbb{R}$. After such a gauging the scalars turn out to be charged under the subgroup $U(1) \times \mathbb{R}$ through the gauge vectors A^0_μ and $QA^0_\mu - 2A^1_\mu - 2A^2_\mu$ [159, 160].

9.1.2 The model

In this section we specify us to a further truncation of the 5d supergravity model introduced in 9.1.1. Our aim is to retain the minimum set of fields needed to capture the oscillations of Type IIB supergravity described by a warped product of an AdS$_3$ factor with the spindle. In this regards we may firstly truncate the fields

$$a = 0 \qquad \text{and} \qquad \phi = 0\,. \tag{9.12}$$

This condition is equivalent to restrict to Type IIB systems with trivial axio-dilaton profile. Then we can also impose that

$$b^1 = \bar{b}^1 = b^2 = \bar{b}^2 = 0\,, \tag{9.13}$$

which is equivalent to exclude those scalar fields associated with three-form fluxes in Type IIB, namely 5-brane contributions. It follows that with this truncation we focus only on 3-brane systems. Summarizing we look at those solutions featured only by the hyperscalar u_1, the vector multiplet-scalars u_2, u_3 and the three vectors A^I_μ. The remaining scalar k is a flat direction of the potential.

Given the above truncation, the Killing vectors (9.10) boil down to $k_0 = -Q\,\partial_k$, $k_1 = 2\partial_k$, $k_2 = 2\partial_k$. From this expression one can observe that the scalar k gets charged under the vector $QA^0_\mu - 2A^1_\mu - 2A^2_\mu$, which in turns becomes massive. As far as the moment maps (9.11) are concerned, only the $r = 3$ $SU(2)$-components survive, leading to

$$P_0'^3 = -\frac{3}{2} + \frac{Q}{4}\,e^{-4u_1}, \qquad P_1^3 = P_2^3 = -\frac{1}{2}e^{-4u_1}. \tag{9.14}$$

For such supergravity model we can introduce a superpotential as it follows,

$$\begin{aligned} W &= \sqrt{\frac{2}{3}}h^I P_I^3 \\ &= \sqrt{\frac{2}{3}}\left(\frac{1}{4}e^{-4u_1-2u_3}\left(Qe^{6u_3} - 4\cosh(2u_2)\right) - \frac{3}{2}e^{4u_3}\right). \end{aligned} \tag{9.15}$$

The scalar potential can be thus derived by the general formula $V = 2g^{\Lambda\Sigma}\partial_\Lambda W \partial_\Sigma W - \frac{4}{3}W^2$ [159] where Λ, Σ include vector multiplet and hypermultiplet fields. This truncation contains the following AdS$_5$ vacuum for $u_{1,2,3}$,

$$u_1 = -\frac{1}{4}\log\frac{4}{Q}, \quad u_2 = 0, \quad u_3 = \frac{1}{6}\log\frac{4}{Q} \tag{9.16}$$

with all the other scalars in the hypermultiplets have been set to zero.

9.2 AdS$_3 \times \Sigma$ geometry and BPS equations

In this section we introduce the AdS$_3\times\Sigma$ ansatz and we present the corresponding BPS equations (for details on the derivation of BPS equations see appendix D). For the space Σ we will take a compact spindle with conical singularities at the poles. Once presented the Ansatz and BPS equations, we will also derive Maxwell equations for vector fields and study the corresponding conserved charge. In what follows we will adapt to the case of the conifold the analysis presented in [227] on AdS$_3 \times \Sigma$ geometries dual to Leigh–Strassler SCFT compactified on a spindle.

9.2.1 The ansatz and Maxwell equations

Let's start with the following AdS$_3 \times \Sigma$ geometry [227]

$$ds^2 = e^{2V(y)}ds^2_{\mathrm{AdS}_3} + f(y)^2dy^2 + h(y)^2dz^2 \tag{9.17}$$

together with the gauge fields

$$A^{(I)} = a^{(I)}(y)dz\,, \tag{9.18}$$

where $ds^2_{\mathrm{AdS}_3}$ is the metric of AdS$_3$ with unit radius. We suppose that the scalars u_1, u_2, u_3 are functions of y, while we take the hyperscalar k linear along the z-direction, i.e. $k = \bar{k}z$. This prescription, originally given in [227], follows from Maxwell equations which imply that $k = k(z)$. Then in order to reproduce a set of Maxwell equations which are ODE along the y-direction (avoid terms as $\partial_z k$) we need k to be linear.

The space Σ in (9.17) is a compact spindle, with azimuthal symmetry parameterized by ∂_z, where the coordinate z is periodic with period $\Delta z = 2\pi$. The coordinate y is compact, bounded by y_N and y_S (with $y_N < y_S$), i.e. finite values at the north and the south pole of the spindle. The general analysis of [199] allows to conclude that the function $h(y)$ vanishes at the poles and a crucial problem consists of finding the boundary conditions for the other fields at the poles of the spindle. In the analysis below we will mostly work in the conformal gauge.[2]

[2]Observe that as discussed in [227] this choice differs from the one of [199]

$$f = e^V \tag{9.19}$$

As we will see below the boundary conditions for f will be fixed from the ones of V. These last will follow from the pole analysis.

In order to study the Killing spinor equations and the equations of motion of the gauge fields, it will be useful to work in the orthonormal frame

$$e^a = e^V \bar{e}^a , \qquad e^3 = f dy , \qquad e^4 = h dz , \tag{9.20}$$

where $\bar{e}^a$ is an orthonormal frame for AdS_3. In this basis the field strengths takes the following form

$$f h F_{34}^{(I)} = \partial_y a^{(I)} . \tag{9.21}$$

We can thus derive Maxwell equations specified to our Ansatz (9.17) and (9.18). We noticed that imposing that the scalars u_1, u_2, u_3 are functions of y and $k = \bar{k} z$, Maxwell equations can be easily integrated. Thus we can write them in the orthonormal frame as

$$\frac{1}{3} e^{3V+4u_3} \left(e^{-4u_2} F_{34}^{(1)} - e^{4u_2} F_{34}^{(2)} \right) = \mathcal{E}_1 , \tag{9.22}$$

$$\frac{1}{3} e^{3V} \left(e^{-8u_3} F_{34}^{(0)} + \frac{Q}{4} e^{4u_3} \left(e^{-4u_2} F_{34}^{(1)} + e^{4u_2} F_{34}^{(2)} \right) \right) = \mathcal{E}_2 , \tag{9.23}$$

$$\partial_y \left(\frac{1}{3} e^{3V-4u_2+4u_3} F_{34}^{(0)} \right) = - e^{3V-8u_1} f h^{-1} g D_z k , \tag{9.24}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are constants of motion and we defined $D_z k = \bar{k} - g Q a^{(0)} + 2 g a^{(1)} + 2 g a^{(2)}$. From the last equations we immediately notice that the scalar k is charged under the vector $Q A_z^0 - 2 A_z^1 - 2 A_z^2$, as we mentioned in previous section.

9.2.2 The BPS equations

In order to derive the BPS equations for the $AdS_3 \times \mathbb{Σ}$ geometry described in previous section we need to give a prescription on Killing spinors. We start by factorizing the spinor as it follows [227],

$$\epsilon = \psi \otimes \chi , \tag{9.25}$$

where χ is a two-component spinor on the spindle and ψ is a two-component spinor on AdS_3 satisfying

$$\nabla_m \psi = - \frac{\kappa}{2} \Gamma_m \psi , \tag{9.26}$$

where $\kappa = \pm 1$ specifies the two chiral cases with $\mathcal{N} = (2, 0)$ or $\mathcal{N} = (0, 2)$ supersymmetry. We outline the derivation of BPS equations in appendix D. The Killing spinor analysis starts with the decomposition of the 5d gamma matrices that we choose as it follows

$$\gamma^m = \Gamma^m \otimes \sigma^3 , \qquad \gamma^3 = \mathbb{I}_2 \otimes \sigma^1 , \qquad \gamma^4 = \mathbb{I}_2 \otimes \sigma^2 , \tag{9.27}$$

with $\Gamma^m = \left(-i\sigma^2, \sigma^3, \sigma^1 \right)$. From gravitino SUSY variations it turns out that the Killing spinor ϵ is defined in terms of a function $\xi(y)$ and it has the following form

$$\epsilon = \left[\cos\left(\frac{\xi}{2} \right) \mathbb{I} - \sin\left(\frac{\xi}{2} \right) \gamma^4 \right] \eta \qquad \text{with} \qquad \eta = e^{V/2} \, e^{isz} \, \eta_0 \, , \tag{9.28}$$

where s is a constant and η_0 is a constant spinor. The above structure of the Killing spinor characterizes the spindle geometry, for instance it was already obtained in [227] for $AdS_3 \times \mathbb{Σ}$

geometries in Leigh-Strassler compactification.[3]
We can thus write the BPS equations obtained by specifying the $\mathcal{N}=2$ SUSY variations of fermionic fields (6.63), (6.64) and (6.65) to the AdS_3 ansatz (9.17) and (9.18) with a spinor ϵ of the form (9.28),

$$\begin{aligned}
\xi' &= 3g\, f\, W \cos\xi + 2\kappa\, f\, e^{-V}\,,\\
V' &= g\, f\, W \sin\xi\,,\\
h' &= h\, f\, \sin^{-1}\xi\big(2\kappa e^{-V}\cos\xi + gW(1+2\cos^2\xi)\big)\,,\\
u_1' &= -\frac{3}{8} g\, f\, \partial_{u_1} W\, \sin^{-1}\xi\,,\\
u_2' &= -\frac{3}{4} g\, f\, \partial_{u_2} W \sin\xi\,,\\
u_3' &= -\frac{1}{4} g\, f\, \partial_{u_3} W \sin\xi\,,
\end{aligned} \tag{9.29}$$

where W is the superpotential defined in (9.15). In addition to the first-order equations the analysis of SUSY variations leads to two algebraic constraints

$$(s - Q_z) = -\frac{h'}{2f}\cos\xi + \frac{h}{\sqrt{6}} H_{34}\sin\xi, \tag{9.30}$$

$$\frac{3g}{2}\partial_{u_1} W \cos\xi = h^{-1}\partial_{u_1} Q_z \sin\xi. \tag{9.31}$$

where $Q_\mu dx^\mu = Q_z\, dz$ is the connection associated with the supercovariant derivative $D_\mu\epsilon = (\nabla_\mu - iQ_\mu)\epsilon$ appearing in the gravitino variation (D.5). The tensor $H_{\mu\nu} \equiv h_I F^I_{\mu\nu}$ is introduced in (D.5) and its non-zero components (in the flat basis (9.20)) are given by

$$\begin{aligned}
H_{34} &= \frac{1}{3}\left(e^{-4u_3} F^{(0)}_{34} + e^{2u_3}\left(e^{-2u_2} F^{(1)}_{34} + e^{2u_2} F^{(2)}_{34}\right)\right)\\
&= -\sqrt{6}\big(\kappa\, e^{-V} + g\, W \cos\xi\big)\,.
\end{aligned} \tag{9.32}$$

Finally from the variations of gauginos and hyperinos (D.17), (D.19) and (D.22) we can obtain the non-zero components of the field strengths F^I_{34},

$$\begin{aligned}
e^{-4u_3} F^{(0)}_{34} &= -\sqrt{\frac{3}{2}} g \cos\xi\,(2W + \partial_{u_3} W) - \sqrt{6}\,\kappa e^{-V},\\
e^{-2u_2+2u_3} F^{(1)}_{34} &= -\frac{g}{2}\sqrt{\frac{3}{2}}\cos\xi\,(4W - \partial_{u_3} W + 3\partial_{u_2} W) - \sqrt{6}\,\kappa e^{-V},\\
e^{2u_2+2u_3} F^{(2)}_{34} &= -\frac{g}{2}\sqrt{\frac{3}{2}}\cos\xi\,(4W - \partial_{u_3} W - 3\partial_{u_2} W) - \sqrt{6}\,\kappa e^{-V}.
\end{aligned} \tag{9.33}$$

As it is discussed in [227], the analysis of BPS equation is simplified observing that we can integrate one out the first three differential equations in (9.29) obtaining

$$h = \ell\, e^V \sin\xi \tag{9.34}$$

[3] Observe that the general form of Killing spinors on spindles was studied in [199].

where ℓ is a constant that has to be determined. The BPS equation for ξ' can be further simplified by plugging in (9.34) and we obtain

$$f^{-1}\xi' = -2\ell^{-1}(s - Q_z)e^{-V}. \tag{9.35}$$

Similarly the constraint (9.30) can be simplified to

$$(s - Q_z) = -\frac{3}{2}g\ell e^V W \cos\xi - \kappa\ell. \tag{9.36}$$

Finally we can use the field strengths (9.33) to get a fully explicit form of the conserved charges $\mathcal{E}_{1,2}$ obtained after having integrated out Maxwell equations in (9.22) and (9.23),

$$\begin{aligned}
\mathcal{E}_1 &= 3ge^{3V}\cos\xi - \frac{1}{\sqrt{6}}\kappa e^{2V-4u_3}\left(Qe^{6u_3}\cosh(2u_2) + 2\right), \\
\mathcal{E}_2 &= 2\sqrt{\frac{2}{3}}\kappa e^{2(u_3+V)}\sinh(2u_2)\,,
\end{aligned} \tag{9.37}$$

where we used the superpotential W written in (9.15).

We observe that with the redefinitions

$$g \to \frac{g}{3}\,, \quad u_2 \to \beta\,, \quad u_3 \to \alpha\,, \quad u_1 \to 2\sqrt{2}\varphi \tag{9.38}$$

our equations take the form of (3.10) in [227]. This relation between the BPS equations obtained here and the ones of [227] is dictated by the similarity between the special geometries and by the fact that only one hyperscalar (u_1 in our notations and φ in the notations of [227]) can be consistently considered. Despite this fact, the Type IIB interpretation of our 5d fields is completely different from that one of [227]. Indeed the analysis of the equations requires a proper quantization of the magnetic charges, as we will discuss later in this chapter.

9.3 Central charge from the pole data

In this section we compute the central charge of the dual field theory obtained by the twist and the anti-twist of the truncation of $T^{1,1}$ without solving the BPS equations.
The relevant result is indeed that this value can be predicted by specifying the boundary conditions at the poles of the spindle solution. This does not guarantee the existence of the solutions, that requires to solve the BPS equation from the north to the south pole of the spindle and that will be the subject of the analysis in section 9.4. Nevertheless, it is a notable result, already noticed in [227] for the Leigh–Strassler truncation.

9.3.1 Simplifications at the poles

At the poles $\ell\sin\xi \to 0$, then if $\ell \neq 0$ it follows that $\cos\xi_{N,S} = (-1)^{t_{N,S}}$ where $t_{N,S} = 0$ or $t_{N,S} = 1$. The poles are identified with $y_{N,S}$, where without loss of generality we can choose $y_N \le y \le y_S$. We have to impose also that

$$|h'|_{N,S} = |\ell \sin'\xi|_{N,S} = \frac{1}{n_{N,S}}. \tag{9.39}$$

This follows from the metric and from the assumption that the deficit angles at the poles are $2\pi\left(1-\frac{1}{n_{N,S}}\right)$, where $n_{N,S}>1$.
From the BPS equations, we observe that the $\mathbb{Z}_2$ symmetry acts on $\{h, a^{(i)}, Q_z, s, \ell\}$ by inverting their sign. This transformation leaves the frame invariant and it can be used to restrict the analysis to the region $h \geq 0$ and then, since $V \in \mathbb{R}$, also $\ell \sin\xi \geq 0$.
The combination $\ell \sin\xi$ is thus positive and it vanishes at the two poles, with $y_N < y_S$. Its derivative is then positive at y_N and negative at y_S. This can be formalized by introducing two further constants, $l_N = 0$ and $l_S = 1$ and requiring

$$\ell \sin' \xi|_{N,S} = \frac{(-1)^{l_{N,S}}}{n_{N,S}}. \tag{9.40}$$

As we mentioned before, the twist and the anti-twist are determined by the R-symmetry flux through the spindle [199] and they are distinguished by the relation between the chiralities of the preserved spinors at the two poles, coincident and opposite respectively. Then, among the four choices of (t_N, t_S), the cases $(0,0)$ and $(1,1)$ correspond to the twist and the other two options $(1,0)$ and $(0,1)$ correspond to the anti-twist. Then the complete set of pole data that we have to specify correspond to $\{l_{N,S}, n_{N,S}, t_{N,S}\}$. The simplification occurred in (9.36) allows to express the quantity $(s - Q_z)$ at the poles in term of these data as

$$s - Q_Z|_{N,S} = \frac{1}{2n_{N,S}}(-1)^{t_{N,S}+l_{N,S}+1}. \tag{9.41}$$

As noted in (2.36), (2.37) of [199] and revisited in (3.25) of [227], this relation is also obtained by requiring regularity of the spinor at the poles.
By looking at the BPS equation obtained in formula (D.22), we observe that it is necessary that $\partial_{u_1} W|_{N,S} = 0$ at the poles, otherwise u_1 does not stay finite. Another consequence of this constraint, combined with (9.36), is that also $\partial_{u_1} Q_z|_{N,S} = 0$. Furthermore, the two reals scalars in the special geometry are constrained at the poles as $Qe^{6u_3} - 4\cosh(2u_2)|_{N,S} = 0$. A further assumption (*a posteriori* motivated by the numerical analysis in sub-section 9.4.2) is that $u_1|_{N,S} \neq 0$. It has been shown in [218] that this assumption implies $D_z k|_{N,S} = 0$.
It is then useful to use these relations to re-consider the expressions obtained above for the conserved charges $\mathcal{E}_{1,2}$, using some of the simplifications occurred at the poles. By defining

$$M_{(1)} = g e^{4u_3+V}, \quad M_{(2)} = -\kappa + 3\sqrt{\frac{3}{2}} M_{(1)} \cos(\xi) \tag{9.42}$$

the charges can be written as

$$\begin{aligned} \mathcal{E}_1 = \frac{M_{(1)}^2}{g^2} \left(\sqrt{\frac{2}{3}} M_{(2)} e^{-12u_3} - \frac{\kappa Q^2}{4\sqrt{6}} \right) \\ + \frac{\kappa Q}{4\sqrt{6}} e^{2(u_3+V)} \left(Q e^{6u_3} - 4\cosh(2u_2) \right), \end{aligned} \tag{9.43}$$

$$\begin{aligned} \mathcal{E}_2^2 = \frac{M_{(1)}^4}{6g^4} \left(Q^2 - 16 e^{-12u_3} \right) \\ + \frac{M_{(1)}^4}{6g^4} \left(16 e^{-12u_3} \cosh^2(2u_2) - Q^2 \right) \end{aligned} \tag{9.44}$$

and hence last terms in (9.43) and (9.44) vanish at the poles. From (9.36) and $W|_{N,S} = -\sqrt{\frac{3}{2}}e^{4u_{3N,S}}$ we also have

$$M_{(1)}|_{N,S} = \frac{\sqrt{6}}{9}\left(2\kappa(-1)^{-t_{N,S}} - \frac{(-1)^{l_{N,S}}}{\ell\, n_{N,S}}\right), \qquad (9.45)$$
$$M_{(2)}|_{N,S} = \kappa - \frac{(-1)^{l_{N,S}-t_{N,S}}}{\ell\, n_{N,S}}.$$

From the definition of $M_{(1)}$ in (9.42), we see that requiring $u_3, V \in \mathbb{R}$ and $g > 0$ imposes that this $M_{(1)}$ is positive. This reflects into the constraints $M_{(1)}|_{N,S} > 0$.
The fact that $\mathcal{E}_{1,2}$ are constant, and then equal at the poles, implies the two following equations for u_3 at the poles:

$$\begin{bmatrix} -\frac{16}{Q^2}M^4_{(1)}|_N & \frac{16}{Q^2}M^4_{(1)}|_S \\ M^2_{(1)}|_N M_{(2)}|_N & -M^2_{(1)}|_S M_{(2)}|_S \end{bmatrix}\begin{bmatrix} e^{-12u_{3N}} \\ e^{-12u_{3S}} \end{bmatrix} = \begin{bmatrix} M^4_{(1)}|_S - M^4_{(1)}|_N \\ \frac{\kappa Q^2}{8}\left(M^2_{(1)}|_N - M^2_{(1)}|_S\right) \end{bmatrix}.$$

9.3.2 Magnetic fluxes

Here we express the magnetic fluxes in terms of the pole data and of the fields evaluated at such poles. This will allow us to express the constant ℓ in terms of the spindle data, without solving the BPS equations. We start observing that

$$F^{(I)}_{yz} = (a^{(I)})' = (\mathcal{I}^{(I)})' \qquad (9.46)$$

with [4]

$$\mathcal{I}^{(I)} \quad = \quad \sqrt{\frac{3}{2}}\,\ell\, e^V h^I \cos\xi. \qquad (9.47)$$

This relation can be worked out by looking at the BPS equations studied above. A similar formula has been found in [218]. It has been further observed in [218] that (9.47) can be obtained by combining the BPS equations of the hyperscalars. It would be interesting to check if a similar relation holds in our case as well.

It follows that the (still not quantized!) fluxes can be expressed in terms of the pole data as

$$\frac{p_I}{n_N n_S} = \frac{1}{2\pi}\int_\Sigma gF^{(I)} = g\mathcal{I}^{(I)}|_N^S \qquad (9.48)$$

with

$$\mathcal{I}^{(0)}|_{N,S} = \sqrt{\frac{3}{2}}\frac{\ell}{g}M_{(1)}|_{N,S}(-1)^{t_{N,S}},$$
$$\mathcal{I}^{(2)}|_{N,S} - \mathcal{I}^{(1)}|_{N,S} = \pm\sqrt{\frac{3}{2}}\frac{\ell(-1)^{t_{N,S}}M_{(1)}|_{N,S}}{2g}\sqrt{Q^2 - 16e^{-12u_{3N,S}}}$$

[4] We hope that the notation h^I for the sections does not generate confusion with respect to the scalar function $h(y)$ in (9.17).

where $\pm$ depends on the sign of u_3. Observe that we are not claiming yet that the fluxes $p_{0,1,2}$ are correctly quantized and we will come back to this problem in a few, reading the correct normalization from the AdS/CFT correspondence.

In order to work with properly quantized charges we now fix $Q = 4$ and consider the following (quantized) charges

$$\begin{aligned}
p_R &\equiv \frac{1}{2}(p_0 + p_1 + p_2) = \frac{1}{2}\left(n_S(-1)^{t_N} + n_N(-1)^{t_S}\right),\\
p_F &\equiv \frac{3}{4}(p_2 - p_1) = \frac{\text{sign}(u_3) g\, n_N n_S}{2}(\mathcal{I}^{(3)}|_N^S - \mathcal{I}^{(2)}|_N^S),\\
p_M &\propto 2\, p_0 - p_1 - p_2 = 0
\end{aligned} \tag{9.49}$$

where the coefficient of p_F is chosen to match with the one of the baryonic symmetry in the holographic dual description. This coefficient can be extracted from the 't Hooft anomaly TrRB^2 along the lines of the procedure discussed in [105]. Requiring the quantization of this charge then imposes a constraint on the constant ℓ and no further constraints on the spindle data.
The charge p_R is quantized if $n_N(-1)^{t_S} + n_S(-1)^{t_N} \in 2\,\mathbb{Z}$. The quantization of p_F is obtained as follows. First we use the fact that $2p_0 = p_1 + p_2$ that gives the relation $2p_F = 3p_2 - 2p_R$. This is a crucial relation to determine the quantization of p_F. We indeed have that $p_R = \frac{3}{4}(p_1 + p_2) \in \mathbb{Z}$ and we must also impose that $p_F = \frac{3}{4}(p_1 - p_2) \in \mathbb{Z}$. As anticipated above, the proportionality coefficient in this last relation comes from anomaly matching and it contains the information on how the 5d solution is uplifted on $T^{1,1}$ to give a Type IIB solution, thanks to the AdS/CFT correspondence. Indeed as explained in [105] the comparison with the field theory to fix the factor of 3/4 is equivalent to the quantization of the five-form flux of Type IIB supergravity in the uplifted solution.
Then the relations $p_R + p_F = \frac{3}{2}p_2 \in \mathbb{Z}$ and $p_R - p_F = \frac{3}{2}p_1 \in \mathbb{Z}$ imply $\frac{3}{2}p_{1,2} \in \mathbb{Z}$. This tells us the correct normalization to impose in (9.48), i.e. the definition of the fluxes $p_{0,1,2}$ should be modified by multiplying it by the factor 3/2, in order to have integer fluxes, say $\hat{p}_{0,1,2} \equiv \frac{3}{2}p_{0,1,2} \in \mathbb{Z}$. Furthermore $p_R \pm p_F \in \mathbb{Z}$ tell us that $p_{R,F}$ must have the same parity.

Then from (9.44) we have

$$\begin{aligned}
|\mathcal{E}_2| &= \frac{4M_{(1)}^2|_{N,S}}{\sqrt{6}g^2}\sqrt{1 - e^{-12u_3|_{N,S}}}\\
&\to \mathcal{I}^{(2)}|_{N,S} - \mathcal{I}^{(1)}|_{N,S} = \frac{3}{2}\frac{g\ell}{M_{(1)}|_{N,S}}|\mathcal{E}_2|(-1)^{t_{N,S}}.
\end{aligned} \tag{9.50}$$

From this relation we have to require $e^{-12u_3|_{N,S}} \in (0, 1]$. This constraint becomes a restriction on the allowed values of the constant ℓ that will be computed below. Then using the fact that $\mathcal{E}_2$ is constant and equal at the poles we can write

$$\mathcal{I}^{(2)}|_N^S - \mathcal{I}^{(1)}|_N^S = \frac{3}{2}g\ell|\mathcal{E}_2|\left(\frac{(-1)^{t_S}}{M_{(1)}|_S} - \frac{(-1)^{t_N}}{M_{(1)}|_N}\right). \tag{9.51}$$

We can simplify this expression using the relation

$$(-1)^{t_S}M_{(1)}|_N - (-1)^{t_N}M_{(1)}|_S = (-1)^{t_S+t_N+1}\sqrt{\frac{2}{3}}\frac{n_S(-1)^{t_N} + n_N(-1)^{t_S}}{3\,\ell\, n_N\, n_S}$$

where we used the fact that the possible values taken by t_S and t_N are all the possible combinations of 0 and 1. Then we arrive at

$$\begin{aligned}\frac{p_F}{g n_S n_N} &= \frac{3}{4}\left(\mathcal{I}^{(2)}\big|_N^S - \mathcal{I}^{(1)}\big|_N^S\right) \\ &= \frac{3}{4\sqrt{6}}\frac{g|\mathcal{E}_2|(-1)^{t_S+t_N+1}}{M_{(1)}|_S M_{(1)}|_N}\frac{n_S(-1)^{t_N}+n_N(-1)^{t_S}}{n_N n_S}\end{aligned} \tag{9.52}$$

while

$$\begin{aligned}\frac{p_R}{g n_S n_N} &= \frac{1}{2}\left(\mathcal{I}^{(0)}\big|_N^S + \mathcal{I}^{(1)}\big|_N^S + \mathcal{I}^{(2)}\big|_N^S\right) \\ &= \frac{1}{2g}\left(\frac{n_S(-1)^{t_N}+n_N(-1)^{t_S}}{n_N n_S}\right).\end{aligned} \tag{9.53}$$

Comparing these last two expressions we have

$$\frac{p_F^2}{p_R^2} = \frac{3g^4\mathcal{E}_2^2}{8M_{(1)}^2|_S M_{(1)}^2|_N}. \tag{9.54}$$

This equation allows to determine the constant ℓ in terms of the integers $n_{N,S}$, $t_{N,S}$, $l_{N,S}$ and p_F. Solving (9.54) for ℓ we obtain

$$\begin{aligned}\ell = (-1)^{t_N+1}\frac{n_N^4 + n_N n_S^3 + n_N^3 n_S + n_S^4 - 4p_F^2 n_N n_S}{\kappa n_N n_S(n_N - n_S)(3(n_N+n_S)^2 + 4p_F^2)}, \\ \text{for} \quad (t_N, t_S) = (0,0) \quad \text{or} \quad (1,1)\end{aligned} \tag{9.55}$$

corresponding to the case of the twist, and

$$\begin{aligned}\ell = (-1)^{t_N}\frac{n_N^4 - n_N n_S^3 - n_N^3 n_S + n_S^4 + 4p_F^2 n_N n_S}{\kappa n_N n_S(n_N + n_S)(3(n_S-n_N)^2 + 4p_F^2)}, \\ \text{for} \quad (t_N, t_S) = (1,0) \quad \text{or} \quad (0,1)\end{aligned} \tag{9.56}$$

corresponding to the case of the anti-twist.

9.3.3 Central charge from the pole data

We are now ready to compute the central charge from the pole data. These last correspond to the integers $\{l_{N,S}, n_{N,S}, t_{N,S}\}$ and in addition to the constant p_F. The central charge is obtained from the Brown-Henneaux formula,

$$c_{2d} = \frac{3R_{\mathrm{AdS}_3}}{2G_3} \tag{9.57}$$

where the ratio between R_{AdS_3} and the three dimensional Newton constant is

$$\frac{R_{\mathrm{AdS}_3}}{G_3} = \frac{1}{G_5}\Delta z\int_{y_N}^{y_S} e^{V(y)}|f(y)h(y)|dy. \tag{9.58}$$

The five dimensional Newton constant for the conifold truncation and the R_{AdS_5} radius are

$$G_5 = \frac{8\pi}{27N_c^2(gW)^3}\,, \quad R_{\mathrm{AdS}_5} = gW\,. \tag{9.59}$$

This can be verified by computing the 4d central charge, related to G_5 through the holographic relation

$$a_{T^{1,1}} = \frac{\pi R_{\mathrm{AdS}_5^3}}{8G_5} = \frac{27N_c^2}{64} \tag{9.60}$$

where the last equality holds by plugging (9.59) in (9.60), and it corresponds to the central charge for the dual $T^{1,1}$ SCFT. Then we must compute the integral in (9.57). In this case we observe that

$$e^{V(y)}f(y)h(y) = -\frac{\ell}{2\kappa}(e^{3V(y)}\cos\xi(y))' \tag{9.61}$$

and this justifies the fact that the central charge can be obtained only from the knowledge of the fields at the poles.

Furthermore in the conformal gauge (9.19) the integral in the central charge becomes $e^{V(y)}|h(y)|$ and the absolute value can be removed observing as above that we can restrict to the region $h \geq 0$. We arrive at the expression

$$c_{2d} \;= \frac{243g^3\ell N_c^2}{32\kappa}\sqrt{\frac{3}{2}}(e^{3V(y_S)}\cos\xi(y_S) - e^{3V(y_N)}\cos\xi(y_N)). \tag{9.62}$$

Plugging the values of the fields evaluated at the poles of the spindle in terms of the pole data the central charge becomes

$$c_{2d} = (-1)^{t_N}\frac{3N_c^2(n_N+n_S)((n_N+n_S)^2-4p_F^2)(3(n_N+n_S)^2+4p_F^2)}{16\kappa n_N n_S((n_N^4+n_N n_S^3+n_N^3 n_S+n_S^4)-4p_F^2 n_N n_S)} \tag{9.63}$$

for the twist, and

$$c_{2d} = (-1)^{t_N+1}\frac{3N_c^2(n_S-n_N)((n_S-n_N)^2-4p_F^2)(3(n_S-n_N)^2+4p_F^2)}{16\kappa n_N n_S((n_S^4-n_N n_S^3-n_N^3 n_S+n_N^4)-4p_F^2 n_N n_S)} \tag{9.64}$$

for the anti-twist.

The case of the twist is completely ruled out by this analysis because $c_{2d} > 0$ is not compatible with the requirements $M_1|_{N,S} > 0$ and $e^{-12u_3|_{N,S}} \in (0,1]$. On the other hand, the solution in the case of the anti-twist exists in the following cases:

$$\begin{aligned}
&\text{for} \quad t_N = 0 \;\&\; \kappa > 0 \quad \text{or} \quad t_N = 1 \;\&\; \kappa < 0 \quad \text{if} \quad n_N - n_S > 2|p_F| > 0\,,\\
&\text{for} \quad t_N = 0 \;\&\; \kappa < 0 \quad \text{or} \quad t_N = 1 \;\&\; \kappa > 0 \quad \text{if} \quad n_S - n_N > 2|p_F| > 0\,.
\end{aligned}$$

9.4 Solving the BPS equations

In this section we present the solutions to the BPS equations. We have solved them analytically when $p_F = 0$ and numerically when $p_F \neq 0$.

9.4.1 Analytic solution for the R-symmetry anti-twist

Here we study the solutions of the BPS equations for the case of the anti-twist by turning off the charge for the baryonic symmetry p_F. From the supergravity sid,e this implies a further truncation to the massless graviton sector. Such a truncation always exists for a five-dimensional Sasaki–Einstein manifold [237,238].
The truncation requires to fix the field u_1 to its AdS$_5$ vacuum, i.e. $u_1 = 0$ when fixing $Q = 4$ (see (9.16)). Furthermore the analysis at the poles shows that in this case the other two scalars $u_{2,3}$ are both set to zero, compatibly with (9.16) for $Q = 4$. Observe also that this is in contrast with the assumption that the scalar u_1 is non vanishing in order to have a solution of the BPS equations, but it is the case only for the analytic solution that corresponds to the universal one, i.e. a further truncation to minimal gauged supergravity. We will see that the other BPS equations can be then analytically solved in the case of the anti-twist by also fixing $p_F = 0$, corresponding to the universal R-symmetry anti-twist.

The metric and the gauge fields are (see [22] for the original derivation of this solution)

$$\begin{aligned} ds^2 &= \frac{1}{g^2W^2}\left(\frac{4y}{9}ds^2_{\mathrm{AdS}_3} + \frac{y}{q(y)}dy^2 + \frac{c_0^2 q(y)}{36y^2}dz^2\right), \\ A^{(0)} &= A^{(1)} = A^{(2)} = -\frac{1}{12}\left(\frac{c_0\kappa}{4g}\left(1-\frac{a}{y}\right)+\frac{s}{g}\right)dz \end{aligned} \tag{9.65}$$

and the function $\xi(y)$ can be expressed in terms of $q(y)$ by the relations

$$\sin\xi = -\frac{\sqrt{q(y)}}{2y^{3/2}}, \quad \cos\xi = \frac{\kappa(3y-a)}{2y^{3/2}} \tag{9.66}$$

where $q(y)$ is

$$q(y) = 4y^3 - 9y^2 + 6ay - a^2. \tag{9.67}$$

The constants a and c_0 can be read from the analysis of the BPS equations from the pole data discussed above and they are

$$\begin{aligned} a &= \frac{(2n_N + n_S)^2\left(n_Nn_S + n_N^2 - 2n_S^2\right)^2}{4\left(n_Nn_S + n_N^2 + n_S^2\right)^3}, \\ c_0 &= \frac{2\left(n_Nn_S + n_N^2 + n_S^2\right)}{3n_Nn_S\left(n_N + n_S\right)}. \end{aligned} \tag{9.68}$$

Then taking $n_S > n_N$ the poles y_N and y_S are

$$y_N = \frac{(n_S - n_N)^2(2n_N + n_S)^2}{4(n_N^2 + n_Nn_S + n_S^2)^2}, \tag{9.69}$$

$$y_S = \frac{(n_S - n_N)^2(n_N + 2n_S)^2}{4(n_N^2 + n_Nn_S + n_S^2)^2} \tag{9.70}$$

and they correspond to the two lowest roots of the polynomial $q(y)$.
Armed with these results we can compute the central charge by evaluating the integral (9.58) between the two poles $y_{N,S}$, reproducing the central charge (9.64) obtained from the pole data as in (9.65) and by setting $p_F = 0$.

9.4.2 Numerical solution

The solution found by turning off p_F for the anti-twist is a consistency check of the analysis, because in this case we are truncating to minimal gauged supergravity, where a solution is expected [22]. The analysis of the BPS equations from the pole data in the conformal gauge (9.19) suggested the existence of a more general solution for non vanishing p_F. Indeed the central charge (9.64) is positive for suitable choices of the fluxes in the anti-twist class. Here we want to find this solution numerically for various numbers of n_S, n_N and p_F in the case of the anti-twist in the conformal gauge (9.19).

The solution is constructed by solving the BPS equations by fixing the initial conditions at one pole, for example at $y = y_N$, for $u_{2,3}$ and V. Such conditions can be read from the analysis at the pole data, that also sets the value of ℓ necessary to find the profile for the $h(y)$ function. At this pole we further have $\sin\xi = 0$ by assumption. On the other hand, the initial value of u_1 is unfixed and indeed finding its initial value at $y = y_N$ is the task of the analysis.
By ranging over various choices of $u_1(y_N)$ indeed the numerical solutions must interpolate the values of the other fields from y_N to y_S. Finding the correct and unique value of $u_1(y_N)$, up to a numerical approximation, leads then to the finite value of $y = y_S$ for which $\sin\xi = 0$, recovering the compact geometry of the spindle.

In the following we list a series of values of n_S, n_N and p_F for which we have obtained a solution. In each case we have extracted the boundary values of the hyperscalar u_1. Furthermore, from the numerical analysis, we have extracted also the location of the pole y_S, by fixing $y_N = 0$.

n_S	n_N	p_F	$u_1(y_N)$	$u_1(y_S)$	$y_S - y_N$
7	1	1	0.0134835	0.00528031	2.3878
9	1	2	0.0289877	0.012056	2.56658
11	1	1	0.00569605	0.00144584	2.73857
11	1	3	0.0398161	0.0179648	2.72743
9	3	1	0.0105465	0.00768971	1.85913
11	5	1	0.00973604	0.0084572	1.72323
13	1	2	0.0153541	0.00410856	2.87009
13	1	4	0.0471909	0.0229531	2.87991
13	3	1	0.00441848	0.00230355	2.07173
13	3	3	0.0335495	0.023278	2.04402

(9.71)

Observe that $n_S - n_N$ is even as discussed above and we required also that p_F and $\frac{1}{2}(n_S - n_N)$ have the same parity and that $2p_F < n_S - n_N$. The explicit numerical solutions for the functions $u_{1,2,3}$, $e^V(y)$ and $h(y) = \ell e^{V(y)} \sin\xi(y)$ for such values are given in figure 9.1 and 9.2. In these figures we have depicted the solutions from the north pole y_N of the spindle to the south pole y_S and then we have continued the integration until $y = 2(y_S - y_N)$. In this way we have shown explicitly a consistency check of these solutions, indeed once the south pole is reached the equations reach the boundary condition that allows us to find the solutions from y_S to y_N.

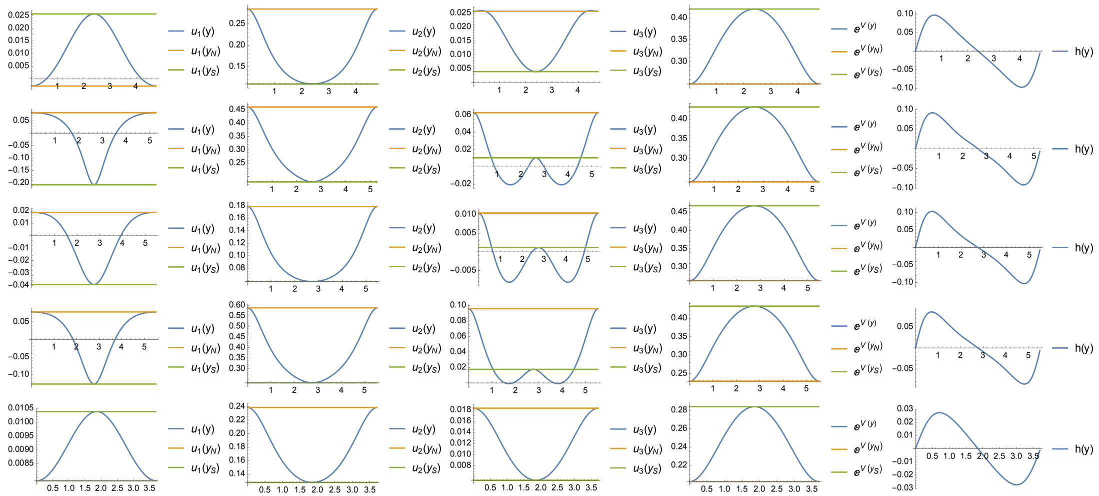


Figure 9.1: Numerical solutions for the scalar fields $u_{1,2,3}(y)$ and the scalar functions $e^V(y)$ and $h(y)$ interpolating between the north pole at $y = y_N = 0$ and $y = 2(y_S - y_N)$, where y_S is the south pole. Each line in the plot is associated with one of the first five lines in the table (9.71), where the values of n_S, n_N and p_F are specified. These values are explicitly $(n_S, n_N, p_F) = (7, 1, 1)$ for the first line, $(9, 1, 2)$ for the second line, $(11, 1, 1)$ for the third line, $(11, 1, 3)$ for the fourth line, and $(9, 3, 1)$ for the fifth line.

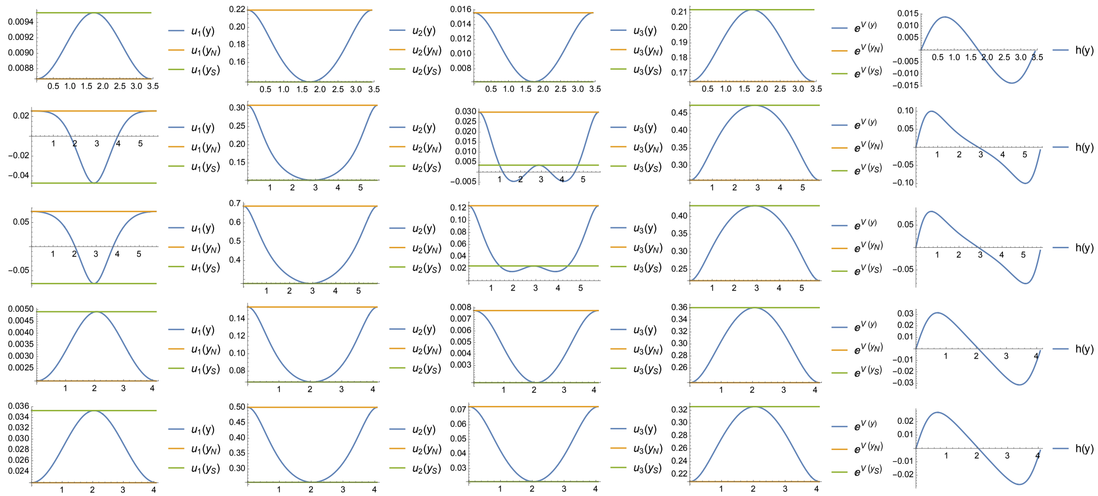


Figure 9.2: Numerical solutions for the scalar fields $u_{1,2,3}(y)$ and the scalar functions $e^V(y)$ and $h(y)$ interpolating between the north pole at $y = y_N = 0$ and $y = 2(y_S - y_N)$, where y_S is the south pole. Each line in the plot is associated with one of the last five lines in the table (9.71), where the values of n_S, n_N and p_F are specified. These values are explicitly $(n_S, n_N, p_F) = (11, 5, 1)$ for the first line, $(13, 1, 2)$ for the second line, $(13, 1, 4)$ for the third line, $(13, 3, 1)$ for the fourth line, and $(13, 3, 3)$ for the fifth line.

9.5 Comparison with the field theory results

In this section we compare the results found above for the central charge in the anti-twist class for the conifold truncation with respect to the calculation performed in the dual field theory. As we introduced above, such a dual field theory, also referred as the Klebanov–Witten theory, corresponds to a stack of N D3 branes probing the tip of the Calabi–Yau threefold with a five dimensional Sasaki–Einstein base, that, in this case, corresponds to the conifold singularity [54]. The model can be represented in terms of two unitary $SU(N)_{1,2}$ gauge groups with bifundamentals $a_{1,2}$ and anti-bifundamentals $b_{1,2}$ interacting through a quartic superpotential $W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1$. The global symmetry group is $U(1)_R \times SU(2)^2 \times U(1)_B$. The massless vectors in the truncation that we have used here are associated only with the R-symmetry $U(1)_R$ and with the baryonic symmetry $U(1)_B$.

The calculation of the central charge for the conifold on the spindle, from the field theory side, has been performed originally in [201] by integrating the anomaly polynomial over the geometry of the spindle,[5] by considering magnetic charges for the whole the global symmetry. Here we will survey the results of [201] and then we will restrict to the magnetic charges of $U(1)_R \times U(1)_B$ in order to compare with the supergravity results obtained above.
The integration of the anomaly polynomial has been pursued thanks to the observation that it can be written as a gluing formula in terms of the four-dimensional conformal anomaly, formally expressed in terms of the R-charges, the deficit angles and the flavor fluxes. By c-extremization the final expression, in terms of the pole data and of the quantized fluxes, is [218]

$$c_{2d} = \frac{3(m_- + \sigma m_+)^2 \sum_{a<b, c\neq a,b} \mathbf{p}_a \mathbf{p}_b \mathbf{p}_c^2 \cdot \sum_{a<b<c} \mathbf{p}_a \mathbf{p}_b \mathbf{p}_c}{(m_-^2 - \sigma m_- m_+ + m_+^2) \prod_{a<b} (\mathbf{p}_a + \mathbf{p}_b) - \sigma m_+ m_- \Theta_{KW}} N^2 \tag{9.72}$$

with

$$\Theta_{KW} = \sum_{a<b, c\neq a,b} \mathbf{p}_a \mathbf{p}_b \mathbf{p}_c^4 - 2 \sum_{a<b} \mathbf{p}_a \mathbf{p}_b \prod_c \mathbf{p}_c . \tag{9.73}$$

The comparison with the gravitational calculation requires a further restriction on the values of the fluxes. The fluxes $\mathbf{p}_{1,2,3,4}$ are associated with the Cartan of the global $U(1)_r \times SU(2)_L \times SU(2)_R \times U(1)_B$ of the conifold. They are constrained by the relation $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = -\frac{m_- + \sigma m_+}{m_- m_+}$ ensuring the correct quantization for the magnetic flux associated with the R-symmetry for the twist $\sigma = 1$ and the anti-twist $\sigma = -1$. The baryonic flux is instead associated with the combination $\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_4$. The other global symmetries associated with the flavor symmetries are set to zero in the 5d supergravity model, that is indeed realized by truncating over the Reeb vector. It follows that the actual comparison requires to fix the fluxes as

$$\mathbf{p}_1 = \mathbf{p}_3 = -\frac{m_- + \sigma m_+}{4 m_- m_+} + \frac{p_b}{2 m_- m_+} \tag{9.74}$$

and

$$\mathbf{p}_2 = \mathbf{p}_4 = -\frac{m_- + \sigma m_+}{4 m_- m_+} - \frac{p_b}{2 m_- m_+} . \tag{9.75}$$

Furthermore we choose $\sigma = -1$ because we have to consider the anti-twist. Eventually we fix the pole data in the notation of [218] and the ones here by identifying $m_+ = n_N$ and $m_- = n_S$.

[5] The original understanding of how to integrate anomaly polynomials on a spindle, suitably accounting for the mixing of the global symmetries and the $U(1)$ isometry of the spindle, was provided in [22].

The central charge in formula (9.72) then becomes

$$c_{2d} = \frac{3N_c^2(n_N - n_S)(3(n_S - n_N)^2 + p_b^2)((n_S - n_N)^2 - p_b^2)}{16 n_N n_S (n_N^4 - n_N n_S^3 - n_N^3 n_S + n_S^4 - p_b^2 n_N n_S)}. \tag{9.76}$$

This expression matches (9.64) upon the identification $p_F = p_b$.

Let us conclude this section with an observation related to the relation between our results and the general discussion that appeared in [218] regarding the geometries of Type IIB constructed from a 5d SE and a spindle. It has been observed that such geometries are constructed by fixing only the fluxes associated with the flavor symmetries, i.e. one can always turn off the baryonic symmetry to obtain the internal Gauntlett–Kim 7d geometry [239]. We expect that the solutions found here, when uplifted to 10d are in the class of [239], with just five-form flux turned on. Furthermore we expect that, working with properly quantized fluxes, the 10d uplifted solutions will be regular, similarly to the cases discussed in [22, 200] for D3- and M2-branes. This is consistent with the discussion on [227] as well.

Chapter 10

BBBW on the spindle

To present another interesting example of reductions on spindle of a gravitational theory containing hypermultiplets, in this chapter we focus on the case of M5-branes wrapped on a complex curve Σ_g in a Calabi–Yau three-fold X, constructed by Bah, Beem, Bobev and Wecht (*BBBW*) in [223, 240]. These models are a generalization of the ones obtained in [19], where M5-branes wrapped on a Riemann surface were considered. The construction of [223, 240] generates an infinite family of 4d SCFTs obtained by gluing T_N theories [37]. The setup is specified by two integers that depend on the local geometry of X, corresponding to a decomposable $\mathbb{C}^2$ bundle over Σ_g. The (non-negative) integers, denoted as p and q, are the Chern numbers of the line bundles $\mathcal{L}_{1,2}$ that specify $\mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{C}_g$. For $p = q$ the $\mathcal{N} = 1$ case studied in [19] is recovered, while $p = 0$ (or $q = 0$) corresponds to the $\mathcal{N} = 2$ case of [19]. For other choices of p and q the 4d SCFT corresponds to a different $\mathcal{N} = 1$ SCFT.

While M5-branes and the theories of [19] have been already studied on the spindle in various setups [202, 203, 211, 213], a general analysis for the models introduced in [223, 240] has not been pursued so far. Here we are interested in generic choices of p and q from the supergravity perspective. Our starting point are the 5d consistent truncations obtained in full generality by [241] (see also [163, 242–244] for earlier results in this direction). Such truncations have the advantage to hold for any choice of p and q, but the price to pay in this case is the presence of hypermultiplets. Anyway, by exploiting the general recipe of [227], we can analyze the reduction on the spindle of the consistent truncations of [241] even in presence of hypermultiplets. The reason is that in this case one hyperscalar triggers an Higgs mechanism that gives a mass to one of the vector multiplets. The Higgsing simplifies the analysis of the BPS equations and of the fluxes at the poles of the spindle, allowing to find the boundary conditions that most of the scalars have to satisfy at the poles in order to compute the central charges in the twist and in the anti-twist class. While this analysis makes the calculation of the central charges possible, it does not guarantee the existence of a solution. Furthermore, it does not fix the boundary condition for the hyperscalar.
However, by restricting to the graviton sector, the universal analytic solution of the type discussed in [22, 200] is found. In this case the scalars are fixed to their AdS$_5$ value. Observe that the universal twist is consistent only if the 4d superconformal R-charge is rational, and this limits the amount of accessible truncations. For more general twists, beyond the universal one, we solved numerically the BPS equations for various values of the hyperscalar at one of the poles of the spindle. When the (unique) value of the hyperscalar that solves the BPS equation, at such pole of the spindle, is found, the existence of the solution is guaranteed. The procedure

fixes also the boundary condition for the hyperscalar at the other pole and the finite distance between the poles.
In the following we will exploit such procedure for the consistent truncations of [241] and we will compare our results with the one found on the field theory side by integrating the anomaly polynomial.

This chapter is based on [245] and it is organized as follows. In section 10.1 we study the spindle compactification of the four-dimensional non-Lagrangian theories obtained in [223]. First, in subsection 10.1.1, we review the relevant aspects of the construction of [223] focusing on the 't Hooft anomalies and on the distinction between the trial R-symmetry emerging from the higher dimensional picture and the exact one due to a-maximization. This distinction indeed plays a crucial role in the analysis. Then in subsection 10.1.2 we study the compactification on the spindle and we compute the central charge of the emerging two dimensional theory. In the computation of the exact two-dimensional R-symmetry we observe that the result can be formulated (when the conditions of integerness on the fluxes are satisfied) in terms of the four-dimensional trial R-symmetry or in terms of the four-dimensional exact one. As a bonus we also study in subsection 10.1.3 the case of the spindle compactification of four-dimensional models associated with negative degree bundles, corresponding to the models obtained in [246]. The in section 10.2 we review the supergravity truncation of [241] in order to fix the notations and the conventions that we use in subsequent sections of the paper. In section 10.3 we study the compactification of the spindle of these five-dimensional $\mathcal{N} = 2$ gauged supergravities, obtaining the relevant BPS and Maxwell equations. In section 10.4 we focus on the calculation of the conserved charges and of the integer fluxes. In this way we can fix most of the scalars at their boundary values on the spindle and from these results we extract the exact central charges form the gravitational perspective. We eventually observe that these results agree with the ones obtained from the field theoretical analysis. In section 10.5 we complete our analysis by studying the gravitational solution. First, in subsection 10.5.1 we look for an analytical solution, finding that it exists for the universal twist, for choices of p and q that correspond to a rational 4d R-symmetry. Then in subsection 10.5.2 we look for numerical solutions for more generic values of p and q, by turning on also the magnetic charge associated with the flavor symmetry. We find numerical solutions only in the case of the anti-twist class for Riemann surfaces of positive curvature.

10.1 The 4d SCFT on the spindle

In subsection 10.1.1 we are going to review the M-theory construction of $\mathcal{N} = 1$ SCFTs in 4d of [223], which is going to be the starting point for our effective 2d theories compactified on the spindle. These models turn out to be dual to $\mathcal{N} = 1$ SCFT built by opportunely gluing T_N blocks [37]. Then in subsection 10.1.2 we construct the theory compactified on the spindle $\mathbb{\Sigma}$, closely following [22, 227] mutatis mutandis. In subsection 10.1.3 we study the case of negative degree bundles, obtained in [246], on the spindle.

10.1.1 The 4d model

The worldvolume theory of stack of N M5-branes is well known to be a $6d\,\mathcal{N}=(2,0)$ SCFT. One can construct effective 4d theories by wrapping the branes on some specific geometry. In this particular case, we are interested in effective 4d theories obtained by wrapping the M5-branes on a complex Riemann curve of genus $\mathfrak{g}$ $\mathcal{C}_\mathfrak{g}$ in a Calabi-Yau three-fold. This geometric construction gives rise to an infinite family of 4d effective theories which are parametrized by two integers depending on the local geometry of the Calabi–Yau three-fold X, which in the case of interest is just a holomorphic $\mathbb{C}^2$ bundle over $\mathcal{C}_\mathfrak{g}$

$$\mathbb{C}^2 \hookrightarrow X \xrightarrow{\pi} \mathcal{C}_\mathfrak{g}. \tag{10.1}$$

Crucially, when X is decomposable it will take the simpler form $X = \mathcal{L}_1 \oplus \mathcal{L}_2$. This structure has a manifest $\mathrm{U}(1)^2$ isometry, one factor for each fiber in the line bundle. The two isometries give rise to two abelian symmetries, one being the R-symmetry $\mathrm{U}(1)_R$ and the other being an additional flavor symmetry $\mathrm{U}(1)_F$.
The integers describing the families of IR $\mathcal{N}=1$ SCFTs are just the Chern numbers labelling the possible bundle decomposition

$$c_1(\mathcal{L}_1) = p, \qquad c_1(\mathcal{L}_2) = q, \tag{10.2}$$

subject to the condition $p+q = 2(\mathfrak{g}-1)$. Depending on the choices of these two integers, the fields in the M5-brane theory transform in different representations of the $\mathrm{U}(1)_F$ symmetry, leading to different IR fixed points. A solution to the constraint of the Chern numbers is given by the following parametrization

$$p = (1+\mathbf{z})(\mathfrak{g}-1), \qquad q = (1-\mathbf{z})(\mathfrak{g}-1) \tag{10.3}$$

where $\mathbf{z}(\mathfrak{g}-1) \in \mathbb{Z}$.
From the class-$\mathcal{S}$ point of view, these theories can be built from opportune gluing of $2(\mathfrak{g}-1)$ T_N building blocks to create a Riemann surface with no punctures.
In this setup the key observables are the central charges c and a, determined by the following combinations of R-symmetry anomalies

$$\begin{aligned} c &= \frac{1}{32}\left(9\,\mathrm{Tr}\,R^3 - 5\,\mathrm{Tr}\,R\right)\,, \\ a &= \frac{3}{32}\left(3\,\mathrm{Tr}\,R^3 - \mathrm{Tr}\,R\right)\,. \end{aligned} \tag{10.4}$$

Note that in the large N limit, for holographic SCFTs $a=c$. The central charges can be recovered from the known anomaly polynomial of the M5-brane theory integrated over $\mathcal{C}_\mathfrak{g}$, assuming that no accidental symmetries are generated along the flow. Since the abelian symmetries $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_F$ mix together, the exact superconformal R-symmetry is found by a-maximization [162].
One finds that the 't Hooft anomalies of the trial R-charge, for theories of type $G = A_N, D_N, E_N$, are given by

$$\begin{aligned} \mathrm{Tr}\,R^3 &= (\mathfrak{g}-1)[(r_G + d_G h_G)(1+\mathbf{z}\epsilon^3) - d_G h_G(\epsilon^2 + \mathbf{z}\epsilon)]\,, \\ \mathrm{Tr}\,R &= (\mathfrak{g}-1) r_G (1+\mathbf{z}\epsilon)\,, \end{aligned} \tag{10.5}$$

where r_G, d_G and h_G are the rank, dimension and Coxeter number of G respectively, while ϵ is the mixing parameter.
We are interested in the A_{N-1} case. The mixed 't Hooft anomalies between the trial R-symmetry R and the flavor symmetry F can be computed from (10.5) and they read

$$
\begin{aligned}
k_{RRR} &= (\mathfrak{g}-1)N^3, & k_{RRF} &= -\frac{1}{3}(\mathfrak{g}-1)\mathbf{z}N^3, \\
k_{RFF} &= -\frac{1}{3}(\mathfrak{g}-1)N^3, & k_{FFF} &= (\mathfrak{g}-1)\mathbf{z}N^3.
\end{aligned}
\tag{10.6}
$$

On the other hand, by considering $R^* = R + \epsilon^* F$, a-maximization yields

$$
\epsilon^* = \frac{1+\mathbf{k}\sqrt{1+3\mathbf{z}^2}}{3\mathbf{z}}\,, \tag{10.7}
$$

where $\mathbf{k}$ is the curvature of $\mathcal{C}_{\mathfrak{g}}$. Choosing $\mathbf{k}=-1$ for later purposes, the 't Hooft anomalies for the superconformal R-symmetry R^* read

$$
\begin{aligned}
k_{R^*R^*R^*} &= \frac{2(\mathfrak{g}-1)}{27\mathbf{z}^2}\left[9\mathbf{z}^2-1+(3\mathbf{z}^2+1)^{3/2}\right]N^3, & k_{R^*R^*F} &= \frac{(\mathfrak{g}-1)}{9}\mathbf{z}N, \\
k_{R^*FF} &= -\frac{(\mathfrak{g}-1)}{3}\sqrt{3\mathbf{z}^2+1}N^3, & k_{FFF} &= (\mathfrak{g}-1)\mathbf{z}N^3.
\end{aligned}
\tag{10.8}
$$

10.1.2 BBBW on the spindle

Consider the 4d SCFT reviewed above, whose anomaly polynomial in the large N limit reads

$$
I_6 = \frac{1}{6}\sum_{i,j,k=R,F} k_{ijk}\, c_1(F_i)c_1(F_j)c_1(F_k) \tag{10.9}
$$

where the coefficients k_{ijk} are given by the mixed 't Hooft anomalies (10.6) and the $c_1(F_{R,F})$ are the first Chern-classes for the $\mathrm{U}(1)$-bundles over the total space X_4 with gauge curvature R and F.
We proceed to compactify further the 4d theory over the spindle $\Sigma \equiv \mathbb{WCP}^1_{[n_N,n_S]}$, where n_N, n_S label the deficit angles at the north and south pole of the orbifold respectively, with background magnetic fluxes for the two abelian $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_F$ symmetries of the 4d theory. In order to do that, we need to take into account the azimuthal $\mathrm{U}(1)_J$ isometry of the spindle which is generated by rotations about the axis passing through the poles. Geometrically, this is given by considering the total space X_4 as a X_2 orbibundle fibered over Σ. In the field theory, this can be achieved by turning on a connection A_J for the $\mathrm{U}(1)_J$ isometry, so that we can write the following gauge connections

$$
A^{(I)} = \rho_I(y)(dz + A_J) \qquad I = R, F \tag{10.10}
$$

where $\rho_I(y)$ are the background fluxes for the abelian symmetries, and $(y,\, z)$ are respectively the longitudinal and azimuthal coordinates over Σ, with $y \in [y_N,\, y_S]$ and $z \sim z + 2\pi$. The curvatures for the fields (10.10) are given by

$$
F^{(I)} = \rho_I'(y)dy \wedge (dz + A_J) + \rho_I(y)F_J \qquad I = R, F \tag{10.11}
$$

where $F_J = dA_J$. These fields are consistent with the flux condition

$$\frac{1}{2\pi}\int_\Sigma F^{(I)} = [\rho_I]^{y_S}_{y_N} = \frac{p_I}{n_S n_N}. \tag{10.12}$$

The curvature forms $F^{(I)}$ define a $\mathrm{U}(1)$-line bundle $\mathcal{L}_I$ over X_4, and the associated first Chern classes are[1]

$$c_1(\mathcal{L}_I) \equiv \left[\frac{F^{(I)}}{2\pi}\right] \in \mathrm{H}^2(X_4,\mathbb{R}), \qquad c_1(J) \equiv \left[\frac{F_J}{2\pi}\right] \in \mathrm{H}^2(X_2,\mathbb{R}). \tag{10.13}$$

To obtain the 2d anomaly polynomial, we make the following substitution

$$c_1(R) \to c_1(R) + \frac{1}{2}c_1(\mathcal{L}_R), \qquad c_1(F) \to c_1(F) + c_1(\mathcal{L}_F) \tag{10.14}$$

where $c_1(R)$ and $c_1(F)$ are the pull-back of the $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_F$ bundles over X_2 respectively. The choice of normalization is such that the R-symmetry generators give charge 1 to the supercharges. Thus, we shift the curvatures in equation (10.11) accordingly, compute the anomaly polynomial in (10.9) and integrate it over Σ. The result is a combination of the four non-zero mixed 't Hooft anomalies given in sec. 10.1.1. In the following, as a working example we show only the computation for the terms proportional to k_{RRR}

$$\int_\Sigma \left(c_1(R) + \frac{1}{2}c_1(\mathcal{L}_R)\right)^3 = \int_\Sigma \left(\frac{3}{2}c_1(R)^2 c_1(\mathcal{L}_R) + \frac{3}{4}c_1(R)c_1(\mathcal{L}_R)^2 + \frac{1}{8}c_1(\mathcal{L}_\mathcal{R})^3\right)\,, \tag{10.15}$$

where the product of forms is understood. Notice that the $c_1(R)$ does not depend on the spindle, so they can be factorized out of the integral. Let us consider the first term in (10.15)

$$\int_\Sigma \frac{3}{2}c_1(R)^2 c_1(\mathcal{L}_R) = \frac{3}{2}c_1(R)^2\int_\Sigma \frac{F^{(R)}}{2\pi} = \frac{3}{2}c_1(R)^2[\rho_R]^{y_S}_{y_N}. \tag{10.16}$$

The second term reads

$$\begin{aligned}
\int_\Sigma \frac{3}{4}c_1(R)c_1(\mathcal{L}_R)^2 &= \frac{3}{4}c_1(R)\int_\Sigma \frac{1}{4\pi^2}F^{(R)}\wedge F^{(R)}\\
&= \frac{3}{4}c_1(R)\int_\Sigma \frac{2}{4\pi^2}\rho_R(y)\rho_R'(y)dy\wedge(dz + A_J)\wedge F_J\\
&= \frac{3}{4}c_1(R)\int_\Sigma \frac{1}{4\pi^2}d\rho_R^2\wedge(dz\wedge F_J + A_J\wedge F_J)\\
&= \frac{3}{4}c_1(R)c_1(J)\int_\Sigma \frac{1}{2\pi}d\rho_R^2\wedge dz\\
&= \frac{3}{4}c_1(R)c_1(J)[\rho_R^2]^{y_S}_{y_N}
\end{aligned} \tag{10.17}$$

where we used the fact that $A_J\wedge F_J$ is just a total derivative and that F_J does not depend on the spindle as stated in (10.13). In the second to last step we went back from forms to cohomology classes. The last term in (10.15) evaluates to

$$\frac{1}{8}\int_\Sigma c_1(\mathcal{L}_R)^3 = \frac{1}{8}\int_\Sigma \frac{1}{(2\pi)^3}F^{(R)}\wedge F^{(R)}\wedge F^{(R)} = \frac{1}{8}c_1(J)^2[\rho_R^3]^{y_S}_{y_N}. \tag{10.18}$$

[1]Note that the gauge curvature of J is only defined on X_2. It's Chern class will not contribute in the integral.

The complete anomaly four-form of the 2d theory reads

$$\begin{aligned}
I_4 =& \left(\frac{3}{2}k_{RRR}[\rho_R]_{y_N}^{y_S} + k_{RRF}[\rho_F]_{y_N}^{y_S}\right) c_1(R)^2 + \left(\frac{1}{2}k_{RFF}[\rho_R]_{y_N}^{y_S} + 3\,k_{FFF}[\rho_F]_{y_N}^{y_S}\right) c_1(F)^2 \\
&+ \left(\frac{1}{8}k_{RRR}[\rho_R^3]_{y_N}^{y_S} + k_{FFF}[\rho_F^3]_{y_N}^{y_S} + \frac{1}{4}k_{RRF}[\rho_F\rho_R^2]_{y_N}^{y_S} + \frac{1}{2}k_{RFF}[\rho_R\rho_F^2]_{y_N}^{y_S}\right) c_1(J)^2 \\
&+ \left(k_{RRF}[\rho_R]_{y_N}^{y_S} + 2\,k_{RFF}[\rho_F]_{y_N}^{y_S}\right) c_1(F)c_1(R) \\
&+ \left(\frac{3}{4}k_{RRR}[\rho_R^2]_{y_N}^{y_S} + k_{RRF}[\rho_R\rho_F]_{y_N}^{y_S} + k_{RFF}[\rho_F^2]_{y_N}^{y_S}\right) c_1(J)c_1(R) \\
&+ \left(3\,k_{FFF}[\rho_F^2]_{y_N}^{y_S} + \frac{1}{4}k_{RRF}[\rho_R^2]_{y_N}^{y_S} + k_{RFF}[\rho_R\rho_F]_{y_N}^{y_S}\right) c_1(J)c_1(F)
\end{aligned} \tag{10.19}$$

To compute the exact central charge we allow a mixing between the various $\mathrm{U}(1)$ factors $c_1(J) = \epsilon c_1(R)$ and $c_1(F) = x c_1(R)$, extremizing the function

$$c_{\text{trial}}^{2d}(\epsilon, x) = \frac{6I_4}{c_1(R)^2}. \tag{10.20}$$

The background magnetic fluxes are fixed to be

$$\int \frac{F^{(R)}}{2\pi} = \frac{p_R}{n_S n_N}, \qquad \int \frac{F^{(F)}}{2\pi} = \frac{p_F}{n_S n_N} \tag{10.21}$$

where $p_R, p_F \in \mathbb{Z}$. For the R-symmetry, we have two possible choices of fluxes consistent with supersymmetry

$$\rho_R(y_N) = \frac{(-1)^{t_N}}{n_N}, \qquad \rho_R(y_S) = \frac{(-1)^{t_S+1}}{n_S} \tag{10.22}$$

where $t_N = 0, 1$, while t_S is fixed by the twisting procedure, namely $t_S = t_N$ for the twist, while $t_S = t_N + 1$ for the anti-twist. For the flavor symmetry, the flux can be fixed to

$$\rho_F(y_N) = \mathbf{z}_0, \qquad \rho_F(y_S) = \frac{p_F}{n_S n_N} + \mathbf{z}_0 \tag{10.23}$$

where $\mathbf{z}_0$ is an arbitrary constant.
Let us consider the following parametrization of the on-shell central charge

$$c_{2d} = \frac{f(n_S, n_N, p_F; \mathbf{z})}{g(n_S, n_N, p_F; \mathbf{z})} a_{4d} \tag{10.24}$$

where $a_{4d} = (\mathfrak{g} - 1)N^3$. In the case of the twist we have

$$
\begin{aligned}
f(n_S, n_N, p_F; \mathbf{z}) = & \left(4p_F^2 - (n_N + n_S)^2\right) \left(2\mathbf{z}p_F + (-1)^{t_N} (n_N + n_S)\right) \\
& \times \Big((-1)^{t_N} (n_N + n_S) \left(16\mathbf{z}p_F + \left(\mathbf{z}^2 + 3\right) (-1)^{t_N} (n_N + n_S)\right) \\
& + 4\left(3\mathbf{z}^2 + 1\right) p_F^2 \Big), \\
g(n_S, n_N, p_F; \mathbf{z}) = & \, 2n_N n_S \Big(8p_F^2 \left(-2n_N n_S + 3\mathbf{z}^2 n_S^2 + 3\mathbf{z}^2 n_N^2\right) - 32\mathbf{z}p_F^3 (-1)^{t_N} (n_N + n_S) \\
& + 8\mathbf{z}p_F (-1)^{t_N} (n_N + n_S) \left(3n_N^2 - 2n_N n_S + 3n_S^2\right) \\
& - 48\mathbf{z}^2 p_F^4 + (n_N + n_S)^2 \big(-2\left(\mathbf{z}^2 + 2\right) n_N n_S + \left(\mathbf{z}^2 + 4\right) n_S^2 \\
& + \left(\mathbf{z}^2 + 4\right) n_N^2 \big) \Big).
\end{aligned}
\tag{10.25}
$$

The central charge is extremized by the mixing ϵ^*, x^* for which we give the exact, albeit quite cumbersome, result

$$
\epsilon^* = \frac{\varepsilon(n_S, n_N, p_F; \mathbf{z})}{d(n_S, n_N, p_F; \mathbf{z})}, \qquad x^* = \frac{\chi(n_S, n_N, p_F; \mathbf{z})}{d(n_S, n_N, p_F; \mathbf{z})} - \mathbf{z}_0 \epsilon^* \tag{10.26}
$$

where

$$
\begin{aligned}
\varepsilon(n_S, n_N, p_F; \mathbf{z}) = & \, 4n_N n_S (-1)^{t_N} (n_N - n_S)(2n_N (-1)^{t_N} (8\mathbf{z}p_F + (\mathbf{z}^2 + 3) n_S (-1)^{t_N}) \\
& + 16\mathbf{z}p_F n_S (-1)^{t_N} + 4(3\mathbf{z}^2 + 1)p_F^2 + (\mathbf{z}^2 + 3)n_S^2 + (\mathbf{z}^2 + 3)n_N^2) \\
\chi(n_S, n_N, p_F; \mathbf{z}) = & -2n_S^2 \left(2\left(\mathbf{z}^2 - 3\right) p_F n_N (-1)^{t_N} - 20\mathbf{z}p_F^2 + 3\mathbf{z}n_N^2\right) \\
& - 4n_S^3 (-1)^{t_N} \left(\mathbf{z}n_N (-1)^{t_N} - 2p_F\right) \\
& - 4\mathbf{z}n_S (-1)^{t_N} \left(n_N^2 - 4p_F^2\right) \left(2\mathbf{z}p_F + n_N (-1)^{t_N}\right) \\
& - 16\left(\mathbf{z}^2 + 1\right) p_F^3 n_N (-1)^{t_N} - 4\left(\mathbf{z}^2 + 1\right) p_F n_N^3 (-1)^{t_N} \\
& - 24\mathbf{z}p_F^2 n_N^2 - 16\mathbf{z}p_F^4 - \mathbf{z}n_S^4 - \mathbf{z}n_N^4 \\
d(n_S, n_N, p_F; \mathbf{z}) = & \, 24\mathbf{z}^2 p_F^2 n_S^2 + 4n_N^3 (-1)^{t_N} \left(6\mathbf{z}p_F + (-1)^{t_N} n_S\right) \\
& + 2\mathbf{z}n_N^2 \left(4p_F n_S (-1)^{t_N} + 12\mathbf{z}p_F^2 - \mathbf{z}n_S^2\right) \\
& + 4n_N (-1)^{t_N} \left(n_S^2 - 4p_F^2\right) \left(2\mathbf{z}p_F + n_S (-1)^{t_N}\right) \\
& + 24\mathbf{z}p_F n_S^3 (-1)^{t_N} - 32\mathbf{z}p_F^3 n_S (-1)^{t_N} \\
& - 48\mathbf{z}^2 p_F^4 + \left(\mathbf{z}^2 + 4\right) n_S^4 + \left(\mathbf{z}^2 + 4\right) n_N^4
\end{aligned}
\tag{10.27}
$$

Notice that there is no explicit $\mathbf{z}_0$ dependence in the central charge.
We can check the validity of the result, by considering the S^2 limiting case, where $n_S = n_N = 1$ and comparing with the result of [20]. As expected, the two results match[2].

[2] From the result of [20], one fixes $\eta_1 = 2(\mathfrak{g} - 1), \eta_2 = 2, \kappa_1 = -1, \kappa_2 = 1, z_1 = z_2 = \mathbf{z}$ to find the matching.

Instead, for the anti-twist case the on-shell central charge is given by

$$\begin{aligned}
f(n_S, n_N, p_F; \mathbf{z}) = & \left((n_S - n_N)^2 - 4p_F^2\right)\left(2\mathbf{z}p_F + (-1)^{t_N}(n_N - n_S)\right) \\
& \times \Big((-1)^{t_N}(n_N - n_S)\left(16\mathbf{z}p_F + \left(\mathbf{z}^2 + 3\right)(-1)^{t_N}(n_N - n_S)\right) \\
& + 4\left(3\mathbf{z}^2 + 1\right)p_F^2\Big) \\
g(n_S, n_N, p_F; \mathbf{z}) = & \, 2n_N n_S \Big(8p_F^2\left(2n_N n_S + 3\mathbf{z}^2 n_S^2 + 3\mathbf{z}^2 n_N^2\right) + 32\mathbf{z}p_F^3(-1)^{t_N}(n_S - n_N) \\
& - 8\mathbf{z}p_F(-1)^{t_N}(n_S - n_N)\left(3n_N^2 + 2n_N n_S + 3n_S^2\right) \\
& - 48\mathbf{z}^2 p_F^4 + (n_S - n_N)^2\left(2\left(\mathbf{z}^2 + 2\right)n_N n_S + \left(\mathbf{z}^2 + 4\right)n_S^2\right. \\
& \left. + \left(\mathbf{z}^2 + 4\right)n_N^2\right)\Big)
\end{aligned} \tag{10.28}$$

where the extremum, using the same parametrization as in (10.26), is reached for the following mixing

$$\begin{aligned}
\varepsilon(n_S, n_N, p_F; \mathbf{z}) = & -4n_N n_S(-1)^{t_N}(n_N + n_S)\Big(2n_N(-1)^{t_N}\left(8\mathbf{z}p_F - (-1)^{t_N}\left(\mathbf{z}^2 + 3\right)n_S\right) \\
& - 16\mathbf{z}p_F n_S(-1)^{t_N} + 4\left(3\mathbf{z}^2 + 1\right)p_F^2 + \left(\mathbf{z}^2 + 3\right)\left(n_S^2 + n_N^2\right)\Big) \\
\chi(n_S, n_N, p_F; \mathbf{z}) = & -2n_S^2\left(2(-1)^{t_N}\left(\mathbf{z}^2 - 3\right)p_F n_N - 20\mathbf{z}p_F^2 + 3\mathbf{z}n_N^2\right) \\
& + 4(-1)^{t_N}n_S^3\left((-1)^{t_N}\mathbf{z}n_N - 2p_F\right) \\
& + 4(-1)^{t_N}\mathbf{z}n_S\left(n_N^2 - 4p_F^2\right)\left(2\mathbf{z}p_F + (-1)^{t_N}n_N\right) \\
& - 16(-1)^{t_N}\left(\mathbf{z}^2 + 1\right)p_F^3 n_N - 4(-1)^{t_N}\left(\mathbf{z}^2 + 1\right)p_F n_N^3 \\
& - 24\mathbf{z}p_F^2 n_N^2 - 16\mathbf{z}p_F^4 - \mathbf{z}(n_S^4 + n_N^4) \\
d(n_S, n_N, p_F; \mathbf{z}) = & \, 24\mathbf{z}^2 p_F^2 n_S^2 + 4(-1)^{t_N}n_N^3\left(6\mathbf{z}p_F - (-1)^{t_N}n_S\right) \\
& + 2\mathbf{z}n_N^2\left(-4(-1)^{t_N}p_F n_S + 12\mathbf{z}p_F^2 - \mathbf{z}n_S^2\right) \\
& + 4(-1)^{t_N}n_N\left(n_S^2 - 4p_F^2\right)\left(2\mathbf{z}p_F - (-1)^{t_N}n_S\right) \\
& - 24(-1)^{t_N}\mathbf{z}p_F n_S^3 + 32(-1)^{t_N}\mathbf{z}p_F^3 n_S \\
& - 48\mathbf{z}^2 p_F^4 + \left(\mathbf{z}^2 + 4\right)\left(n_S^4 + n_N^4\right)
\end{aligned} \tag{10.29}$$

Once again, the on-shell central charge does not depend on $\mathbf{z}_0$ as expected.
The central charge calculated from the R^*, F anomalies (10.8) instead of R, can be computed in the same manner as just described. The two exact central charges will then match as follows

$$\begin{aligned}
& c_{2d}^*\Big(\varepsilon_1^*, x_1^*; R, F, n_S(-1)^{t_N} + n_N(-1)^{t_S}, p_F\Big) \\
= & \, c_{2d}^*\Big(\varepsilon_2^*, x_2^*; R^*, F, n_S(-1)^{t_N} + n_N(-1)^{t_S}, p_F + \epsilon^* \frac{n_S(-1)^{t_N} + n_N(-1)^{t_S}}{2}\Big)
\end{aligned} \tag{10.30}$$

where ϵ^* is the 4d mixing parameter found in (10.7) with $\mathbf{k} = -1$ and we specified which symmetries we are considering as well as their fluxes. Namely, the former is obtained from the anomaly polynomial considering the 't Hooft anomalies (10.6) and their fluxes, while the latter

is obtained considering the anomalies (10.8), while their fluxes are related to the first ones by a shift.
Observe that the universal twist is consistent only if the exact 4d R-symmetry is rational. From the second line in (10.30) it follows that this choice requires to set the combination $p_F + \epsilon^* \frac{n_S(-1)^{t_N}+n_N(-1)^{t_S}}{2}$ to zero. The integerness conditions on p_F, n_S and n_N then restrict the allowed values of p and q admitting the universal twist.

10.1.3 Negative degree bundles

Here we further generalize the construction of [223,240] by gluing $2(\mathfrak{g}-1)$ together copies of $T_N^{(m)}$ theories [247]. This construction reproduces the model of [223,240] when $m=0$ [246] and generalizes it for generic m. The construction of [223,240] in fact allows only for positive $p,q \geq 0$, while in the construction of [246], one can allow also for negative degree bundles. Although these theories have no known supergravity description at this time, we give the field theory calculation for completeness.
The cubic anomalies of the model of [223,240] can be recovered from the ones of the $T_N^{(m)}$ blocks by linear combination of the $\mathrm{U}(1)_i$ isometries of the line bundles. Namely $R=(J_+ + J_-)/2$ and $F=(J_- - J_+)/2$, following the naming convention of [246]. Therefore, in the large-N limit

$$\begin{aligned} k_{RRR} &= \frac{N^3}{2}, & k_{RRF} &= -\frac{1}{6}(1+2m)N^3, \\ k_{RFF} &= -\frac{N^3}{6}, & k_{FFF} &= \frac{1}{2}(1+2m)N^3 \end{aligned} \tag{10.31}$$

where the integer m parametrizes the degree of the line bundles $p=m+1$ and $q=-m$.
Following the same arguments as before, we can compactify these theories on the spindle and find the central charge of a family of theories parametrized by m. By taking the anomaly polynomial constructed from the anomalies (10.31), we find the following central charge in the case of the twist

$$\begin{aligned} f(n_S,n_N,p_F;m) = {} & 2\left((n_N+n_S)^2-4p_F^2\right)\left(2(2m+1)p_F+(-1)^{t_N}(n_S+n_N)\right) \\ & \times\Big((-1)^{t_N}(n_N+n_S)\left(4(2m+1)p_F+\left(m^2+m+1\right)(-1)^{t_N}(n_N+n_S)\right) \\ & +4(3m(m+1)+1)p_F^2\Big) \end{aligned} \tag{10.32}$$

$$\begin{aligned} g(n_S,n_N,p_F;m) = {} & n_N n_S\Big((-1)^{t_N}\big(4n_S^3\left(6(2m+1)p_F+(-1)^{t_N}n_N\right) \\ & +2(-1)^{t_N}(2m+1)n_S^2\big(12(2m+1)p_F^2 \\ & -(-1)^{t_N}n_N\left((-1)^{t_N}(2m+1)n_N-4p_F\right)\big) \\ & +4n_S\left(n_N^2-4p_F^2\right)\left(2(2m+1)p_F+(-1)^{t_N}n_N\right) \\ & +n_N\big((-1)^{t_N}n_N\big((-1)^{t_N}n_N\left(24(2m+1)p_F+(-1)^{t_N}(4m(m+1)+5)n_N\right) \\ & +24(2m+1)^2p_F^2\big)-32(2m+1)p_F^3\big)+(-1)^{t_N}(4m(m+1)+5)n_S^4\big) \\ & -48(2m+1)^2p_F^4\Big) \end{aligned} \tag{10.33}$$

where we used the parametrization (10.24). The mixing is given by

$$\begin{aligned}
\varepsilon(n_S,n_N,p_F;m) &= 16n_Nn_S(-1)^{t_N}\Big(4(-1)^{t_N}(2m+1)p_F\left(n_N^2-n_S^2\right) \\
&+4(3m(m+1)+1)p_F^2\left(n_N-n_S\right) \\
&+\left(m^2+m+1\right)\left(n_N-n_S\right)\left(n_N+n_S\right)^2\Big) \\
\chi(n_S,n_N,p_F;m) &= -4n_N^3(-1)^{t_N}\left(2\left(2m^2+2m+1\right)p_F+(-1)^{t_N}(2m+1)n_S\right) \\
&-4n_N(-1)^{t_N}\Big(2\left(2m^2+2m-1\right)p_Fn_S^2+8\left(2m^2+2m+1\right)p_F^3 \\
&-4(-1)^{t_N}(2m+1)p_F^2n_S+(-1)^{t_N}(2m+1)n_S^3\Big) \\
&-2(2m+1)n_N^2\left(4(-1)^{t_N}(2m+1)p_Fn_S+12p_F^2+3n_S^2\right) \\
&+32(-1)^{t_N}(2m+1)^2p_F^3n_S+40(2m+1)p_F^2n_S^2 \\
&-16(2m+1)p_F^4+8(-1)^{t_N}p_Fn_S^3-(2m+1)n_S^4-(2m+1)n_N^4 \\
d(n_S,n_N,p_F;m) &= -32(-1)^{t_N}(2m+1)p_F^3\left(n_N+n_S\right) \\
&+8p_F^2\left(3(2m+1)^2n_N^2+3(2m+1)^2n_S^2-2n_Nn_S\right) \\
&+8(-1)^{t_N}(2m+1)p_F\left(n_N+n_S\right)\left(-2n_Nn_S+3n_N^2+3n_S^2\right) \\
&-48(2m+1)^2p_F^4+\left(n_N+n_S\right)^2\Big(-2(4m(m+1)+3)n_Nn_S \\
&+(4m(m+1)+5)(n_N^2+n_S^2)\Big)
\end{aligned} \tag{10.34}$$

For the anti-twist case we get

$$\begin{aligned}
f(n_S,n_N,p_F;m) &= 2\left(\left(n_N-n_S\right)^2-4p_F^2\right)\left(2(2m+1)p_F+(-1)^{t_N}(n_S+n_N)\right) \\
&\times\Big(\left(-1\right)^{t_N}\left(n_N-n_S\right)\left(4(2m+1)p_F+\left(m^2+m+1\right)(-1)^{t_N}\left(n_N-n_S\right)\right) \\
&+4(3m(m+1)+1)p_F^2\Big)
\end{aligned} \tag{10.35}$$

$$\begin{aligned}
g(n_S,n_N,p_F;m) &= -n_Nn_S\Big((-1)^{t_N}\big(-4n_S^3\left(6(2m+1)p_F+(-1)^{t_N}n_N\right) \\
&+2(-1)^{t_N}(2m+1)n_S^2\big(12(2m+1)p_F^2 \\
&-(-1)^{t_N}n_N\left((-1)^{t_N}(2m+1)n_N-4p_F\right)\big) \\
&-4n_S\left(n_N^2-4p_F^2\right)\left(2(2m+1)p_F+(-1)^{t_N}n_N\right) \\
&+n_N\big((-1)^{t_N}n_N\big((-1)^{t_N}n_N\left(24(2m+1)p_F+(-1)^{t_N}(4m(m+1)+5)n_N\right) \\
&+24(2m+1)^2p_F^2\big)-32(2m+1)p_F^3\big)+(-1)^{t_N}(4m(m+1)+5)n_S^4\big) \\
&-48(2m+1)^2p_F^4\Big)
\end{aligned} \tag{10.36}$$

where we used the parametrization (10.24). The mixing is given by

$$
\begin{aligned}
\varepsilon(n_S,n_N,p_F;m) &= -16n_N n_S(-1)^{t_N}\Big(4(-1)^{t_N}(2m+1)p_F\left(n_N^2-n_S^2\right)\\
&+4(3m(m+1)+1)p_F^2\left(n_N+n_S\right)\\
&+\left(m^2+m+1\right)\left(n_N+n_S\right)\left(n_N-n_S\right)^2\Big)\\
\chi(n_S,n_N,p_F;m) &= -4n_N^3(-1)^{t_N}\left(2\left(2m^2+2m+1\right)p_F-(-1)^{t_N}(2m+1)n_S\right)\\
&-4n_N(-1)^{t_N}\Big(2\left(2m^2+2m-1\right)p_F n_S^2+8\left(2m^2+2m+1\right)p_F^3\\
&+4(-1)^{t_N}(2m+1)p_F^2 n_S-(-1)^{t_N}(2m+1)n_S^3\Big)\\
&-2(2m+1)n_N^2\left(-4(-1)^{t_N}(2m+1)p_F n_S+12p_F^2+3n_S^2\right)\\
&-32(-1)^{t_N}(2m+1)^2 p_F^3 n_S+40(2m+1)p_F^2 n_S^2\\
&-16(2m+1)p_F^4-8(-1)^{t_N}p_F n_S^3-(2m+1)n_S^4-(2m+1)n_N^4\\
d(n_S,n_N,p_F;m) &= -32(-1)^{t_N}(2m+1)p_F^3\left(n_N-n_S\right)\\
&+8p_F^2\left(3(2m+1)^2n_N^2+3(2m+1)^2n_S^2+2n_N n_S\right)\\
&+8(-1)^{t_N}(2m+1)p_F\left(n_N-n_S\right)\left(2n_N n_S+3n_N^2+3n_S^2\right)\\
&-48(2m+1)^2p_F^4+\left(n_N-n_S\right)^2\Big(2(4m(m+1)+3)n_N n_S\\
&+(4m(m+1)+5)(n_N^2+n_S^2)\Big)
\end{aligned}
\tag{10.37}
$$

In in the limit of $m\to 0$ one recovers the same result of the compactified model of [223, 240], as expected.

10.2 The 5d supergravity truncation

The five-dimensional supergravity model we are working with is a consistent truncation from eleven-dimensional supergravity studied in [241]. It contains two vector multiplets and one hypermultiplet and it has gauge group $\mathrm{U}(1)\times\mathbb{R}$.

As we mentioned before, this truncation generalizes the structure associated with the solutions of [223, 240] and it completes the consistent truncation of seven-dimensional $\mathcal{N}=4$ SO(5) gauged supergravity reduced on a Riemann surface $\mathcal{C}_\mathfrak{g}$ analyzed in [163]. There, the 5d model was obtained truncating the 7d supergravity to the $\mathrm{U}(1)^2$ sector, corresponding to the Cartan of SO(5). Besides enclosing the two U(1) gauge fields and the two scalars belonging to the vector multiplets, the bosonic sector of the construction made in [241] also includes all the scalar fields in the hypermultiplet, and furthermore it gives a direct derivation of the gauging. In the following we outline the construction made in [241]. The eleven-dimensional metric is

$$
ds_{11}^2=e^{2\Delta}ds^2_{\mathrm{AdS}_5}+ds_6^2, \tag{10.38}
$$

which corresponds to a warped product $\mathrm{AdS}_5\times_{\mathrm{w}}\mathcal{M}$ with warp factor $e^{2\Delta}\ell^2=e^{2f_0}\bar{\Delta}^{1/3}$, where ℓ is the AdS radius and $\bar{\Delta}$ and f_0 are constants. $\mathcal{M}_6$ is a six-dimensional manifold given by a fibration of a squashed-sphere $\mathcal{M}_4$ over the Riemann surface $\mathcal{C}_\mathfrak{g}$ and has metric

$$ds_6^2 = \bar{\Delta}^{1/3} e^{2g_0} ds^2_{\mathcal{C}_\mathfrak{g}} + \frac{1}{4}\bar{\Delta}^{-2/3} ds_4^2, \tag{10.39}$$

where g_0 is a constant. The Riemann surface has Ricci scalar curvature $\mathbf{k}$ as discussed after formula (10.7) and the metric on $\mathcal{M}_4$ is

$$ds_4^2 = X_0^{-1} d\mu_0^2 + \sum_{i=1,2} X_i^{-1}\big(d\mu_i^2 + \mu_i^2 (d\varphi_i + A^{(i)})^2\big), \tag{10.40}$$

with

$$\mu_0 = \cos\zeta, \qquad \mu_1 = \sin\zeta\cos\frac{\theta}{2}, \qquad \mu_2 = \sin\zeta\sin\frac{\theta}{2}. \tag{10.41}$$

The angles φ_1, φ_2 are in $[0, 2\pi]$, while ζ, θ are in $[0, \pi]$. $A^{(1)}$ and $A^{(2)}$ gauge two $\mathrm{U}(1)$ isometries of the squashed S^4. Furthermore

$$\bar{\Delta} = \sum_{I=0}^{2} X_I \mu_I^2, \quad e^{f_0} = X_0^{-1}, \quad e^{2g_0} = -\frac{1}{8}\mathbf{k} X_1 X_2 \big[(1-\mathbf{z})X_1 + (1+\mathbf{z})X_2\big], \tag{10.42}$$

where $\mathbf{z}$, that can be read from (10.3) as

$$\mathbf{z} = \frac{p-q}{p+q}, \tag{10.43}$$

is a discrete parameter related to the Chern numbers p and q and

$$\begin{aligned} X_0 &= (X_1X_2)^{-2}, \\ X_1 X_2^{-1} &= \frac{1+\mathbf{z}}{2\mathbf{z} - \mathbf{k}\sqrt{1+3\mathbf{z}^2}}, \\ X_1^5 &= \frac{1 + 7\mathbf{z} + 7\mathbf{z}^2 + 33\mathbf{z}^3 + \mathbf{k}(1+4\mathbf{z}+19\mathbf{z}^2)\sqrt{1+3\mathbf{z}^2}}{4\mathbf{z}(1-\mathbf{z})^2}. \end{aligned} \tag{10.44}$$

There is also a four-form flux but we address the interested reader to [241] for its explicit form.

Notice that the $\mathcal{N}=1$ and $\mathcal{N}=2$ twistings studied in [19] can be recovered as special cases from this model: the first one arises from setting $p=q$ (corresponding to $\mathbf{z}=0$), while the second one from $p=0$ or $q=0$ ($\mathbf{z}=\pm 1$).

10.2.1 $\mathcal{N}=2$ supergravity structure

The reduction described above gives rise to an infinite family of $\mathcal{N}=2$ gauged supergravity theories in five dimensions. Here we summarize the most salient features of the model[3].

[3]The Lagrangian in (B.10) of [241] that we are using here can be obtained from the one used here by rescaling the gauge fields and the coupling constant as

$$A^I_{\text{there}} = -\sqrt{\frac{3}{2}} A^I_{\text{here}}, \qquad g_{\text{there}} = -\sqrt{\frac{2}{3}} g_{\text{here}}. \tag{10.45}$$

Focusing on the vector multiplet sector, the two real scalars Σ and ϕ parametrize the Very Special Real Manifold

$$\mathcal{M}_V = \mathbb{R}_+ \times \mathrm{SU}(1,1) \tag{10.46}$$

that has metric

$$g_{xy} = \begin{pmatrix} \frac{3}{\Sigma^2} & 0 \\ 0 & 1 \end{pmatrix}. \tag{10.47}$$

The homogeneous coordinates $h^I(\Sigma, \phi)$ (from now on we will omit the explicit dependence of the sections from the two real scalars Σ and ϕ) are given by

$$h^0 = \frac{1}{\Sigma^2}, \qquad h^1 = -\Sigma H^1, \qquad h^2 = -\Sigma H^2, \tag{10.48}$$

where

$$H^1 = \sinh\phi, \qquad H^2 = \cosh\phi \tag{10.49}$$

parametrize the unit hyperboloid $\mathrm{SO}(1,1)$, while Σ parametrizes $\mathbb{R}^+$. The non-zero components of the totally symmetric tensor C_{IJK} are

$$C_{0\bar{I}\bar{J}} = C_{\bar{I}0\bar{J}} = C_{\bar{I}\bar{J}0} = \frac{1}{3}\eta_{\bar{I}\bar{J}}, \qquad \text{for } \bar{I}, \bar{J} = 1, 2, \tag{10.50}$$

with $\eta = \mathrm{diag}(-1, 1)$.
Moving to the hypermultiplet sector, the quaternionic manifold

$$\mathcal{M}_H = \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2) \times \mathrm{U}(1)} \tag{10.51}$$

is spanned by the scalars $q^X = \{\varphi, \Xi, \theta_1, \theta_2\}$ with line element[4]

$$g_{XY} dq^X dq^Y = -d\varphi^2 - \frac{1}{2} e^{2\varphi}(d\theta_1^2 + d\theta_2^2) - \frac{1}{4} e^{4\varphi}(d\Xi - \theta_1 d\theta_2 + \theta_2 d\theta_1)^2. \tag{10.52}$$

Only the hypermultiplet sector is gauged and the corresponding Killing vectors $k_I = k_I^X \partial_X$ read

$$\begin{aligned} k_0 &= \partial_\Xi, \\ k_1 &= \mathbf{zk}\partial_\Xi, \\ k_2 &= -\mathbf{k}\partial_\Xi + 2(\theta_2 \partial_{\theta_1} - \theta_1 \partial_{\theta_2}), \end{aligned} \tag{10.53}$$

with associated Killing prepotentials

$$\begin{aligned} P_0^r &= \{0, 0, \frac{1}{4} e^{2\varphi}\}, \\ P_1^r &= \{0, 0, \frac{\mathbf{zk}}{4} e^{2\varphi}\}, \\ P_2^r &= \{\sqrt{2} e^{\varphi}\theta_1, \sqrt{2} e^{\varphi}\theta_2, -1 + \frac{1}{4} e^{2\varphi}(2\theta_1^2 + 2\theta_2^2 - \mathbf{k})\}. \end{aligned} \tag{10.54}$$

[4]We are using a different normalization w.r.t. [241]. This allows us to obtain a simplified version of the hyperino variation, as it was pointed out in the previous chapter.

10.2.2 The model

In the remainder of this paper we will work with a further truncation of the 5d supergravity model introduced above, which is obtained by setting

$$\theta_1 = \theta_2 = 0, \tag{10.55}$$

consistently with the AdS_5 vacuum of the model we started from. In this truncation, the Killing vectors (10.53) simplify to

$$k_0 = \partial_\Xi, \qquad k_1 = \mathbf{z}\mathbf{k}\partial_\Xi, \qquad k_2 = -\mathbf{k}\partial_\Xi. \tag{10.56}$$

Notice that from (10.56) we can see that the field Ξ gets charged under the vector $A_\mu^{(0)} + \mathbf{z}\mathbf{k}A_\mu^{(1)} - \mathbf{k}A_\mu^{(2)}$, that becomes massive. Furthermore, only the third $\mathrm{SU}(2)$-components of the Killing prepotentials (10.54) survive and they reduce to

$$P_0^3 = \frac{1}{4}e^{2\varphi}, \qquad P_1^3 = \frac{\mathbf{z}\mathbf{k}}{4}e^{2\varphi}, \qquad P_2^3 = -1 - \frac{\mathbf{k}}{4}e^{2\varphi}. \tag{10.57}$$

We can thus introduce a superpotential as

$$W = h^I P_I^3 = \frac{\Sigma^3\left(\left(\mathbf{k}e^{2\varphi}+4\right)\cosh\phi - \mathbf{z}\mathbf{k}e^{2\varphi}\sinh\phi\right) + e^{2\varphi}}{4\Sigma^2}. \tag{10.58}$$

Furthermore, the following AdS_5 vacuum is also a vev for the scalars Σ, ϕ, φ in this truncation:

$$\begin{aligned}
\varphi &= \frac{1}{2}\log\left(\frac{4}{\sqrt{3\mathbf{z}^2+1}-2\mathbf{k}}\right), \\
\phi &= \operatorname{arctanh}\left(\frac{1+\mathbf{k}\sqrt{1+3\mathbf{z}^2}}{3\mathbf{z}}\right), \\
\Sigma^3 &= \frac{\sqrt{2\left(3\mathbf{z}^2-1-\mathbf{k}\sqrt{1+3\mathbf{z}^2}\right)}}{\mathbf{z}\left(\sqrt{1+3\mathbf{z}^2}-2\mathbf{k}\right)}.
\end{aligned} \tag{10.59}$$

10.3 The 5d truncation on the spindle

In this section we briefly review the geometric construction used to split the five-dimensional background as the warped product $AdS_3 \times \Sigma$, where the space Σ is a compact spindle with azimuthal symmetry and conical singularities at the poles. As in the previous chapter, once introduced the ansatz on the geometry and on the gauge fields, we present the corresponding BPS equations and Maxwell equations of motion.

10.3.1 The ansatz and Maxwell equations

We begin by considering the $AdS_3 \times \Sigma$ ansatz made in [227][5]:

$$\begin{aligned}
ds^2 &= e^{2V(y)}ds^2_{AdS_3} + f(y)^2dy^2 + h(y)^2dz^2, \\
A^{(I)} &= a(y)^{(I)}dz,
\end{aligned} \tag{10.60}$$

[5]We are using the mostly plus signature.

where $ds^2_{\mathrm{AdS}_3}$ is the metric on unitary AdS$_3$, while (y,z) are the coordinates on Σ, which is a compact spindle with an azimuthal symmetry generated by ∂_z. The azimuthal coordinate z has periodicity $\Delta z = 2\pi$. The longitudinal coordinate y is compact, bounded by y_N and y_S (with $y_N < y_S$), implying that the function $h(y)$ vanishes at the poles of the spindle.

We assume that the scalars Σ, ϕ, φ depend on the y coordinate only, while the hyperscalar Ξ is linear in z, i.e. $\Xi = \overline{\Xi} z$ (with $\overline{\Xi}$ a constant).
Following [227], we will use an orthonormal frame to simplify the analysis of the Killing spinor equations and of the equations of motion of the gauge fields:

$$e^a = e^V \bar{e}^a, \qquad e^3 = f\, dy, \qquad e^4 = h\, dz, \tag{10.61}$$

where $\bar{e}^a$ is an orthonormal frame for AdS$_3$. In this basis, the field strengths read

$$f\, h\, F^{(I)}_{34} = \partial_y a^{(I)}. \tag{10.62}$$

Given that Σ, ϕ, φ are functions of y only and $\Xi = \overline{\Xi} z$, two out of the three gauge equations of motion specified to our ansatz can be easily integrated, and they can be written in the orthonormal frame as

$$\frac{2e^{3V}}{3\Sigma^2}\Big[(\cosh 2\phi - \mathbf{z}\sinh 2\phi)F^{(1)}_{34} + (\mathbf{z}\cosh 2\phi - \sinh 2\phi)F^{(2)}_{34}\Big] = \mathcal{E}_1, \tag{10.63}$$

$$\frac{2e^{3V}}{3\Sigma^2}\Big[\mathbf{z}\mathbf{k}\Sigma^6 F^{(0)}_{34} - (\cosh 2\phi + \mathbf{z}\sinh 2\phi)F^{(1)}_{34} + (\mathbf{z}\cosh 2\phi + \sinh 2\phi)F^{(2)}_{34}\Big] = \mathcal{E}_2, \tag{10.64}$$

$$\partial_y\left(\frac{1}{3}e^{3V}\Sigma^4 F^{(0)}_{34}\right) = \frac{1}{4}e^{4\psi+3V} g\, f\, h^{-1} D_z\Xi, \tag{10.65}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are constants and we defined $D_z\Xi \equiv \overline{\Xi} + g(a^{(0)} + \mathbf{z}\mathbf{k}a^{(1)} - \mathbf{k}a^{(2)})$.

10.3.2 The BPS equations

To derive the BPS equations for the geometry introduced above, we need to factorize the Killing spinor [227]:

$$\epsilon = \psi \otimes \chi, \tag{10.66}$$

where χ is a two-component spinor on the spindle and ϕ is a two-component spinor on AdS$_3$ such that

$$\nabla_m \psi = -\frac{\kappa}{2}\Gamma_m \psi, \tag{10.67}$$

with $\kappa = \pm 1$ depending on the $\mathcal{N} = (2,0)$ or $\mathcal{N} = (0,2)$ supersymmetry chirality of the dual 2d SCFT.
We then decompose the 5d gamma matrices as

$$\gamma^m = \Gamma^m \otimes \sigma^3, \qquad \gamma^3 = \mathbb{I}_2 \otimes \sigma^1, \qquad \gamma^4 = \mathbb{I}_2 \otimes \sigma^2. \tag{10.68}$$

with $\Gamma^m = (-i\sigma^2, \sigma^3, \sigma^1)$.
The analysis of the BPS equations is similar to the one in appendix D (or to the original of [227]). Here again the spinor χ can be written as

$$\chi = e^{V/2} e^{isz} \begin{pmatrix} \sin\frac{\xi}{2} \\ \cos\frac{\xi}{2} \end{pmatrix}, \tag{10.69}$$

with s a constant. Notice that, as expected, the spinor is not constant on the spindle.
In the following we summarize the differential relations coming from the BPS equations

$$
\begin{aligned}
\xi' - 2f(gW\cos\xi + \kappa e^{-V}) &= 0 \\
V' - \frac{2}{3}fgW\sin\xi &= 0 \\
\Sigma' + \frac{2}{3}fg\,\Sigma^2\sin\xi\,\partial_\Sigma W &= 0 \\
\phi' + 2fg\sin\xi\,\partial_\phi W &= 0 \\
\varphi' + \frac{fg}{\sin\xi}\partial_\varphi W &= 0 \\
h' - \frac{2fh}{3\sin\xi}(gW(1+2\cos^2\xi) + 3\kappa e^{-V}\cot\xi) &= 0,
\end{aligned}
\tag{10.70}
$$

where W is the superpotential defined in (10.58). Besides the first-order equations, there are also two algebraic constraints that can be derived from the supersymmetry variations

$$
\begin{aligned}
\sin\xi(s - Q_z) &= -h(gW\cos\xi + \kappa e^{-V}) \\
gh\partial_\varphi W\cos\xi &= \partial_\varphi Q_z\sin\xi,
\end{aligned}
\tag{10.71}
$$

where Q_z can be read from the supercovariant derivative $D_\mu\epsilon = \nabla_\mu\epsilon - iQ_\mu\epsilon$ that appears in the gravitino variation and for our model takes the form

$$
Q_z = \frac{e^{2\varphi}}{4}D_z\Xi - ga^{(2)}. \tag{10.72}
$$

We can also reduce the differential system by observing that

$$
h = ke^V\sin(\xi) \tag{10.73}
$$

where k is an arbitrary constant that needs to be determined. Finally, we can take advantage of the BPS equations to express the field strengths in terms of the scalar fields as

$$
\begin{aligned}
F^{(0)}_{34} &= \frac{6\kappa e^{-V} + 4gW\cos\xi - 4g\Sigma\,\partial_\Sigma W\cos\xi}{3\Sigma^2}, \\
F^{(1)}_{34} &= -\frac{2\Sigma}{3}\left[\sinh\phi\Big(g\cos\xi\big(2W + \Sigma\,\partial_\Sigma W\big) + 3\kappa e^{-V}\Big) + 3g\partial_\phi W\cos\xi\cosh\phi\right], \\
F^{(2)}_{34} &= -\frac{2\Sigma}{3}\left[\cosh\phi\Big(g\cos\xi\big(2W + \Sigma\,\partial_\Sigma W\big) + 3\kappa e^{-V}\Big) + 3g\partial_\phi W\cos\xi\sinh\phi\right].
\end{aligned}
\tag{10.74}
$$

10.4 Analysis at the poles

In this section we study the solutions of the BPS equations derived above and we show how to obtain the 2d central charge from the pole analysis. The procedure follows the one originally described in [227] and then applied in our previous chapter and in [221] for the case of the conifold. We start by summarizing the BPS equations, the constraints and the Maxwell equations. Then we derive the explicit expressions of the conserved charges and the magnetic fluxes. The

charge conservation imposes the constraints that allow us to fix the boundary conditions at the poles for the scalars that enter in the calculation of the central charge. We then compute the central charge from the Brown-Henneaux formula and discuss its relation with the calculation done on the field theory side.
Before starting our analysis let us stress that, differently from the previous discussions, we have not found from the pole analysis immediate reasons to exclude the possibility of having solutions in the twist class. We will further comment on this issue in the next section where we provide numeric and analytical solutions of the BPS equations.

10.4.1 Conserved charges and restriction to the poles

From the expressions of the fields strengths in (10.74) we can study the Maxwell equations using the two conserved charges $\mathcal{E}_{1,2}$ in (10.63) and (10.64). In order to keep the hyperscalar $\varphi(y)$ finite we require that $\partial_\varphi W|_{N,S} = 0$. This constraint gives rise to

$$\mathbf{k}\,\Sigma|_{N,S}^3 + \frac{1}{\cosh\phi|_{N,S} - \mathbf{z}\sinh\phi|_{N,S}} = 0\,. \tag{10.75}$$

Using (10.75) and the fact that $\mathcal{E}_1$ and $\mathcal{E}_2$ are conserved we found simpler expressions by working with the following linear combinations

$$\begin{aligned} Q_1|_{N,S} &= \mathcal{E}_1|_{N,S} = \frac{4}{3}e^{2V|_{N,S}}\left(\frac{\kappa(\sinh(\phi|_{N,S}) - \mathbf{z}\cosh(\phi|_{N,S}))}{\Sigma|_{N,S}} - \mathbf{z}g e^{V|_{N,S}}\cos(\xi|_{N,S})\right), \\ Q_2|_{N,S} &= \mathcal{E}_1|_{N,S} - \mathcal{E}_2|_{N,S} = \frac{4\kappa e^{2V|_{N,S}}}{3\Sigma|_{N,S}}\left(2\sinh(\phi|_{N,S}) - \mathbf{z}\mathbf{k}\Sigma|_{N,S}^3\right). \end{aligned} \tag{10.76}$$

At the north and at the south poles we have $k\sin\xi \to 0$. For non vanishing k this gives $\cos\xi_{N,S} = (-1)^{t_{N,S}}$ with $t_{N,S} = 0$ or $t_{N,S} = 1$. Denoting the poles as $y_{N,S}$ we can work with $y_N \leq y \leq y_S$. Furthermore

$$|h'|_{N,S} = |k\sin'\xi|_{N,S} = \frac{1}{n_{N,S}}. \tag{10.77}$$

This relation is due to the metric and to the deficit angles at the poles $2\pi\left(1 - \frac{1}{n_{N,S}}\right)$ where $n_{N,S} > 1$. From the $\mathbb{Z}_2$ symmetry of the BPS equations acting on $h, a^{(I)}, s, Q_z$ and k we can restrict to $h \geq 0$ and $k\sin\xi \geq 0$. We have then $k\sin\xi \geq 0$ and this quantity is vanishing at the poles, with a positive derivative at y_N and a negative one at y_S. Formally we introduce two constants, $l_N = 0$ and $l_S = 1$ such that

$$k\sin'\xi|_{N,S} = \frac{(-1)^{l_{N,S}}}{n_{N,S}}. \tag{10.78}$$

Then the cases $(t_N, t_S) = (0,0)$ and $(1,1)$ correspond to the twist while $(t_N, t_S) = (1,0)$ and $(0,1)$ correspond to the anti-twist. The quantity $(s - Q_z)$ at the poles becomes

$$s - Q_Z|_{N,S} = \frac{1}{2n_{N,S}}(-1)^{t_{N,S}+l_{N,S}+1}. \tag{10.79}$$

Furthermore, the relation $\partial_\varphi W|_{N,S} = 0$ imposes from the second relation in (10.71) that $\partial_\varphi Q_z|_{N,S} = 0$. Another assumption (justified a posteriori by the numerical results) is that is that $\psi|_{N,S} \neq 0$. Such assumption implies also that $D_z\Xi|_{N,S} = 0$.

10.4.2 Fluxes

Here we introduce the magnetic fluxes for the reduction of this truncation on the spindle. This will be necessary in order to find the constant k introduced in (10.73) in terms of the data of the spindle. First we observe that

$$F_{yz}^{(I)} = (a^{(I)})' = \left(\mathcal{I}^{(I)}\right)' \quad \text{with} \quad \mathcal{I}^{(I)} \equiv -ke^{V}\cos\xi\, h^{I}\ . \tag{10.80}$$

At this point we need to define the fluxes starting from (10.80). Let us start by defining the integer fluxes p_I from the relations

$$\frac{p_I}{n_N n_S} = \frac{1}{2\pi}\int_\Sigma gF^{(I)} = g\mathcal{I}^{(I)}|_N^S\ . \tag{10.81}$$

The magnetic charge associated with the R-symmetry is

$$-gn_N n_S \mathcal{I}^{(2)}|_N^S = \frac{1}{2}\left(n_S(-1)^{t_N} + n_N(-1)^{t_S}\right). \tag{10.82}$$

This expression is quantized if $n_S(-1)^{t_N} + n_N(-1)^{t_S}$ is even. Observe also that

$$\mathcal{I}^{(0)} + \mathbf{zk}\mathcal{I}^{(1)} - \mathbf{k}\mathcal{I}^{(2)} = 0 \tag{10.83}$$

that implies also that the combination $p_0 + \mathbf{zk}p_1 - \mathbf{k}p_2$ does not give rise to a conserved magnetic flux. The last flux that we need to discuss is the one associated with the flavor symmetry. The integer flavor flux is given by

$$p_F = gn_N n_S \mathcal{I}^{(1)}|_N^S\ . \tag{10.84}$$

It is important to observe that the relation $p_0 = \mathbf{k}(\mathbf{z}p_F + p_2) \in \mathbb{Z}$ requires that for $\mathbf{z} \in \mathbb{Q}\setminus\mathbb{Z}$ we have the further constraint $\mathbf{z}p_F \in \mathbb{Z}$. Furthermore, we also found useful to use the substitution

$$\tanh(\phi) \equiv \frac{1-\delta}{\mathbf{z}} \tag{10.85}$$

such that the charges evaluated at the poles simplify to

$$\begin{aligned} Q_{1N,S} \;&=\; \frac{\mathbf{k}\delta_{N,S}((-1)^{l_{N,S}} - 2\kappa k n_{N,S}(-1)^{t_{N,S}})^2}{6\mathbf{z}g^2k^3n_{N,S}^3} \\ &\times\;\; (2\kappa k n_{N,S}(\delta_{N,S}-1)\delta_{N,S} - (-1)^{l_{N,S}-t_{N,S}}((\delta_{N,S}-1)^2 - \mathbf{z}^2)), \end{aligned} \tag{10.86}$$

$$Q_{2N,S} = \frac{\mathbf{k}\kappa((-1)^{l_{N,S}} - 2\kappa k n_{N,S}(-1)^{t_{N,S}})^2}{3\mathbf{z}g^2k^2n_{N,S}^2}(\mathbf{z}^2 - 1 + \delta_{N,S}(4-3\delta_{N,S})). \tag{10.87}$$

It follows that we have three equations: the first one is (10.84), that after the substitution (10.85) becomes

$$p_F = \frac{(\delta_N - 1)n_S(-1)^{-t_N} + n_N(-1)^{-t_S}(\delta_S - 1) - 2\kappa k n_N n_S(\delta_N - \delta_S)}{2\mathbf{z}} \tag{10.88}$$

while the other two equations correspond to $Q_1|_N = Q_1|_S$, i.e.

$$\frac{(1+2\kappa k n_S(-1)^{t_S})^2}{(1-2\kappa k n_N(-1)^{t_N})^2}\cdot\frac{\delta_S n_N^3}{\delta_N n_S^3}\cdot\frac{2\kappa k n_S(-1)^{t_S}(\delta_S-1)\delta_S+(\delta_S-1)^2-\mathbf{z}^2}{2\kappa k n_N(-1)^{t_N}(\delta_N-1)\delta_N-(\delta_N-1)^2+\mathbf{z}^2} = (-1)^{t_S+t_N}$$

and $Q_2|_N = Q_2|_S$, i.e.

$$\frac{n_N^2}{n_S^2}\cdot\frac{\mathbf{z}^2-1+\delta_S(4-3\delta_S)}{\mathbf{z}^2-1+\delta_N(4-3\delta_N)}\cdot\frac{(1+2\kappa k n_S(-1)^{t_S})^2}{(1-2\kappa k n_N(-1)^{t_N})^2} = 1 \tag{10.89}$$

for the three variables, k, δ_S and δ_N. By solving these three equations we obtain then the boundary conditions to impose for the scalars V, h, ϕ, Σ in terms of the integers n_S, n_N and p_F of the spindle for generic values of the parameters $\mathbf{z} \in \mathbb{Q}$ and $\mathbf{k} = \pm 1$ in both the twist and the anti-twist class. The requirement of reality for these fields imposes further constraints on the allowed values of the integers $n_{S,N}$ and p_F. The only field that is not involved in this analysis is the hyperscalar φ, that we are assuming as non vanishing at the poles.

10.4.3 Central charge from the pole data

Once the boundary data for $\delta_{N,S}$ and the constant k are specified we can read the central charge of the putative 2d CFT from the pole analysis. The central charge is obtained from the formula

$$c_{2d} = \frac{3R_{AdS_3}}{2G_3} = \frac{3}{2G_5}\Delta z\int_{y_N}^{y_S} e^{V(y)}|f(y)h(y)|dy. \tag{10.90}$$

The relation

$$e^{V(y)}f(y)h(y) = -\frac{k}{2\kappa}(e^{3V(y)}\cos\xi(y))' \tag{10.91}$$

implies that the central charge can be computed from the value of the fields at the poles that we have computed above, without specifying the value of the hyperscalar. The consistency of this analysis represents just a necessary condition for the existence of a solution. Nevertheless, when a solution exists, the central charge computed here is the correct one.
In the conformal gauge the integrand in (10.90) is $e^{V(y)}|h(y)|$, where we remove the absolute value here and consider $h(y) > 0$ thanks to the symmetries of the BPS equations as discussed above. The central charge becomes $c_{2d} = c_S - c_N$ where

$$c_{N,S} = \frac{3\pi\mathbf{k}\delta_{N,S}}{2\mathbf{z}^2 g^3 G_5\kappa k^2}\left(\kappa k - \frac{(-1)^{l_{N,S}-t_{N,S}}}{2n_{N,S}}\right)^3((\delta_{N,S}-1)^2-\mathbf{z}^2). \tag{10.92}$$

The central charge in the case of the anti-twist is given by

$$c_{2d}^{A.T.} = \frac{3\pi\mathbf{k}\kappa((4p_F^2-n_-^2)(2\mathbf{z}p_F(-1)^{t_N}+n_-))(n_-(16\mathbf{z}p_F(-1)^{t_N}+(\mathbf{z}^2+3)n_-)+4(3\mathbf{z}^2+1)p_F^2}{4g^3G_5 n_*(8\mathbf{z}p_F(-1)^{t_N}n_-(3n_{2+}+2n_*-4p_F^2)+16p_F^2 n_*+4n_-n_{3-}+\mathbf{z}^2(24p_F^2 n_{2+}-48p_F^4+n_{2-}^2))}.$$

where we have defined the auxiliary variables $n_S n_N \equiv n_*$, $n_S \pm n_N \equiv n_\pm$, $n_s^2 \pm n_N^2 \equiv n_{2\pm}$, $n_s^3 \pm n_N^3 \equiv n_{3\pm}$. The central charge in the case of the twist is given by

$$c_{2d}^{T.} = \frac{3\pi\mathbf{k}\kappa(4p_F^2-n_-^2)(2\mathbf{z}p_F(-1)^{t_N}+n_-)(n_+(16\mathbf{z}p_F(-1)^{t_N}+(\mathbf{z}^2+3)n_+)+4(3\mathbf{z}^2+1)p_F^2)}{4g^3G_5 n_*(8\mathbf{z}p_F(-1)^{t_N}n_+(3n_{3+}-2n_*-4p_F^2)-16p_F^2 n_*+4n_+n_{3+}+\mathbf{z}^2(24p_F^2 n_{2+}-48p_F^4+n_{2-}^2))}.$$

The five dimensional Newton constant can be read from the holographic dictionary. Indeed from the general relation $a_{4d} = \frac{\pi R^3_{AdS_5}}{8G_5}$ and from the explicit values of the central charge and of the AdS$_5$ radius, given by

$$a_{4d} = \frac{(g-1)\left((1-9\mathbf{z}^2)\mathbf{k} + \left(3\mathbf{z}^2+1\right)^{3/2}\right)}{48\mathbf{k}\mathbf{z}^2}\,, \qquad (10.93)$$
$$R^3_{AdS_5} = \frac{(1-9\mathbf{z}^2)\mathbf{k} + \left(3\mathbf{z}^2+1\right)^{3/2}}{4\mathbf{z}^2}$$

we can extract $G_5 = \frac{3\pi\mathbf{k}}{2(g-1)}$. Substituting this expression in the 2d central charge computed above we can then recover the result obtained from the field theory calculation in section 10.1.2.

Some comments are in order. First we have checked in many cases if the various constraints, imposed by the quantization of the fluxes, by the reality condition on the scalars and by the positivity of the central charge, are enough to exclude the existence of some solutions. While in many cases the answer is affirmative, we have not been able to exclude whole families of solutions. In general there are four main families of possible solutions, identified by the value of $\mathbf{k} = \pm 1$ and by the fact that they can be in the twist or in the anti-twist class. Anyway, anticipating the results of next section, we have found solutions only in the anti-twist class for $\mathbf{k} = -1$.

10.5 The solution

In this section we obtain the AdS$_3 \times \Sigma$ solution for the model discussed above. We separate the analysis in two parts. In the first part we discuss the analytic solution for the universal truncation. This corresponds to a further truncation of the model to the graviton sector. In this case we found the explicit solution corresponding to the general one found in [22,200]. Similarly to the cases discussed above and in [221,227] in presence of hypermultiplets, here we found an analytic solution only in the anti-twist class. Furthermore, we have found such solution only for $\mathbf{k} = -1$. We have also checked that the 2d central charge matches the general expectation

$$c_{2d} = \frac{4}{3}\frac{a_{4d}(n_S - n_N)^3}{n_N n_S(n_N^2 + n_N n_S + n_S^2)}\,. \qquad (10.94)$$

In the second part of this section we study the solution turning on a generic flux p_F. In this case we have obtained the solution numerically. Again we found solutions only in the anti-twist class for $\mathbf{k} = -1$ and for generic values of $\mathbf{z}$.

10.5.1 Analytic solution for the graviton sector

Here we study the AdS$_3 \times \Sigma$ solution by restricting to the graviton sector. This requires to fix $A^{(1)} + \epsilon^* A^{(2)} = 0$ (with ϵ^* defined in (10.7)) and to identify $A^{(R)} = -A^{(2)}$. This further fixes $2p_F = \epsilon^*(n_S - n_N)$. We have found a solution in this case for the anti-twist class and $\mathbf{k} = -1$ by fixing the scalars $\Sigma(y)$, $\phi(y)$ and $\varphi(y)$ at their AdS$_5$ value (10.59). Observe that $\phi_{N,S} = \phi_{AdS_5}$ and $\Sigma_{N,S} = \Sigma_{AdS_5}$ when $p_F = \epsilon^*(n_S - n_N)/2$.

Before continuing the discussion a comment is in order. The choice of p_F that allows to study the universal twist is, for generic values of $\mathbf{z}$, in contrast with the requirement that $\mathbf{z}p_F$ is an integer. The only cases that are allowed correspond to the ones that give rise to a rational exact R-symmetry. In these cases a solution exists when (the even quantity) $n_S - n_N$ gives rise to an integer $\mathbf{z}p_F$. This analysis restricts the possible truncations to the graviton sector that can be placed on the spindle. This is the counterpart of the field theory argument that we made after formula (10.30). The discussion fits with similar ones appeared in the literature of the spindle (see for example footnote 20 of [201] for an analogous behavior in the case of toric SE_5). Having this caveat in mind, the scalar functions $V(y)$, $f(y)$ and $h(y)$ in (10.60) are

$$e^{V(y)} = \frac{\sqrt{y}}{W}, \quad f(y) = \frac{3}{2W}\sqrt{\frac{y}{q(y)}}, \quad h(y) = \frac{c_0\sqrt{q(y)}}{4Wy} \tag{10.95}$$

while the gauge field is

$$A^{(R)} = \left(\frac{c_0\kappa(a-y)}{4y} - s\right) dz\ . \tag{10.96}$$

We also found that

$$\sin\xi(y) = \frac{\sqrt{q(y)}}{2y^{3/2}}, \quad \cos\xi(y) = \frac{\kappa(3y-a)}{2y^{3/2}} \tag{10.97}$$

with

$$q(y) = 4y^3 - 9y^2 + 6ay - a^2\ . \tag{10.98}$$

The constants a and c_0 are obtained from the solutions of the BPS equations at the poles. We found

$$c_0 = \frac{2\left(n_N^2 + n_N n_S + n_S^2\right)}{3n_N n_S\left(n_N + n_S\right)} \tag{10.99}$$

while the constant a is

$$a = \frac{\left(n_N - n_S\right)^2\left(2n_N + n_S\right)^2\left(n_N + 2n_S\right)^2}{4\left(n_N n_S + n_N^2 + n_S^2\right)^3}\ . \tag{10.100}$$

From here it follows that

$$y_N = \frac{\left(-2n_N^2 + n_N n_S + n_S^2\right)^2}{4\left(n_N^2 + n_N n_S + n_S^2\right)^2}, \quad y_S = \frac{\left(n_N - n_S\right)^2\left(n_N + 2n_S\right)^2}{4\left(n_N^2 + n_N n_S + n_S^2\right)^2}\ . \tag{10.101}$$

The central charge becomes

$$c_{2d} = \frac{9\pi\left(n_S - n_N\right)^3}{16G_5 W_{\text{crit}}^3 n_N n_S\left(n_N^2 + n_N n_S + n_S^2\right)}\ . \tag{10.102}$$

Using then $a_{4d} = \frac{\pi R_{AdS_5}^3}{8G_5}$ and $R_{AdS_5} = \frac{3}{2W_{\text{crit}}}$ we arrive at the expected universal result (10.94).

10.5.2 Numerical solution for generic p_F

Here we look for more generic solutions of the BPS equations interpolating among the poles of the spindle. From the analysis above we have observed that the only possible analytic solutions (i.e. with $p_F = \frac{\epsilon^*}{2}(n_S - n_N)$) are in the anti-twist class with $\mathbf{k} = -1$. Here we search for numerical solutions for a generic integer $\mathbf{z}p_F$. We have scanned over large regions of parameters and again we have only found solutions with $\mathbf{k} = -1$ in the anti-twist class.

The solutions are found along the lines of the analysis of [221,227] and of our previous chapter. First we specify the values of $\mathbf{z}$, n_S, n_N and p_F. Then we fix the initial conditions imposed by the analysis at the poles. In this way we are left with one unknown initial condition for the hyperscalar φ. Finding the initial condition of φ corresponds to find the solution for the BPS equations on the spindle. There is just (up to the numerical approximation) a single value φ_S (here we are fixing the south pole at $y_S = 0$) that allows to integrate the BPS equation giving rise to a finite spindle in the y direction. Once this value is found a good sanity check consists of running the numerics until $2\Delta y$, that corresponds to solve the BPS equations from the north to the south pole as well. We have scanned over various values of the parameters and here we present some of the solutions that we found.

n_S	n_N	p_F	$\mathbf{z}$	φ_S	φ_N	Δy
1	3	0	2	-0.285076	-0.274493	1.83241
1	7	-1	2	-0.172372	-0.170589	2.39707
1	3	0	3	-0.555814	-0.542721	1.82303
1	5	-1	3	-0.300428	-0.300346	2.16012
1	9	3	$\frac{1}{3}$	0.463989	0.363277	2.57446
1	5	0	$\frac{1}{3}$	0.126802	0.124497	2.16392
1	7	2	$\frac{1}{2}$	0.484886	0.347516	2.3322
3	7	0	$\frac{1}{2}$	0.104192	0.103447	1.74866

(10.103)

In each case we have fixed $\mathbf{k} = -1$ and chosen $\kappa = 1$ (corresponding to the choice $n_N > n_S$). The explicit solutions are plot in figure 10.1. Observe that the solutions for the cases at $p_F = 0$ do not correspond to the universal twist (at least for $\mathbf{z} \neq 0$).

The cases at $p_F = 0$ correspond to a twist along a trial R-symmetry, obtained from a linear combination (with irrational coefficients) of the (irrational) exact R-symmetry and the flavor symmetry.

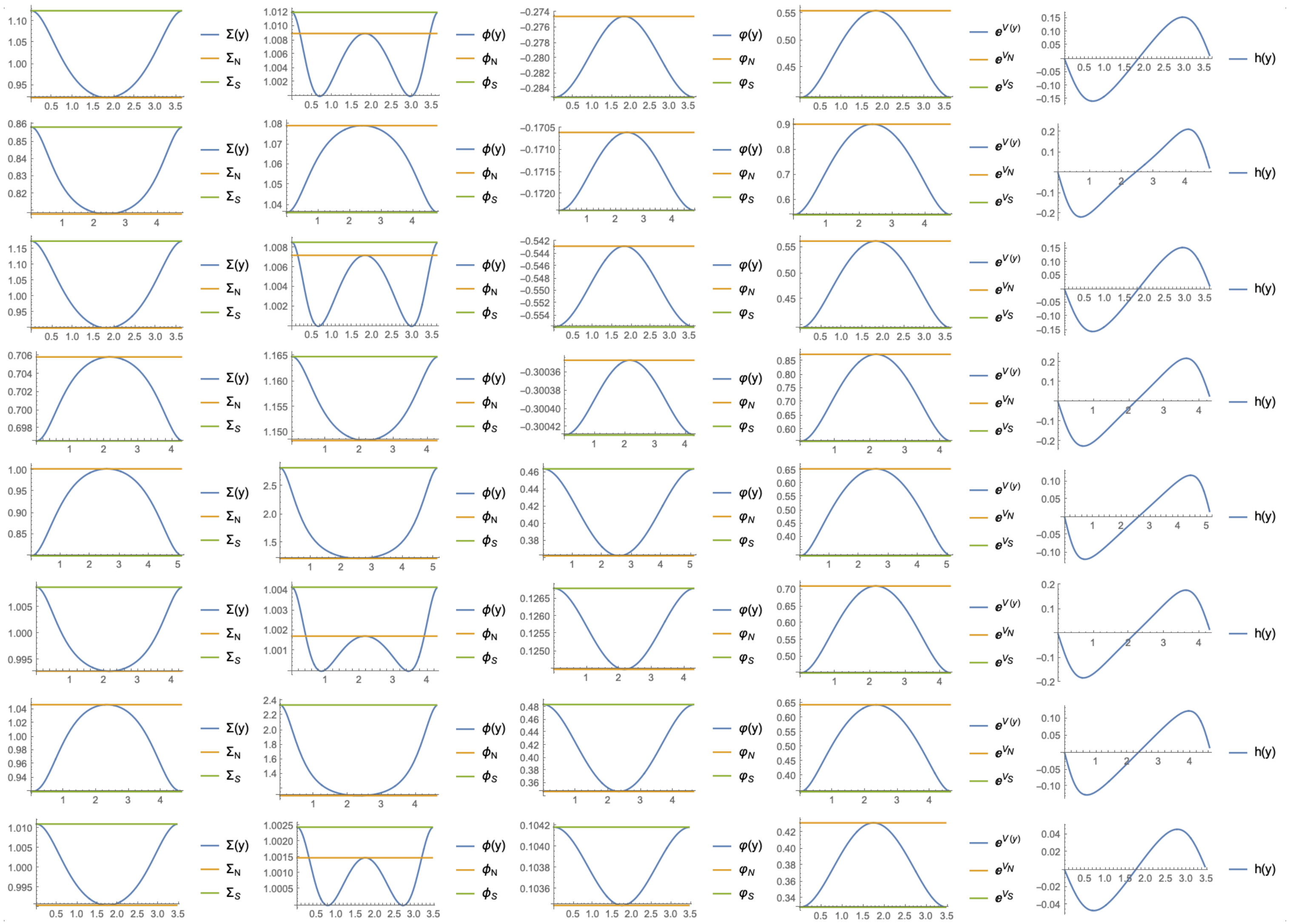


Figure 10.1: Numerical solutions for the scalar fields $\Sigma(y), \phi(y), \varphi(y)$ and the scalar functions $e^{V}(y)$ and $h(y)$ interpolating between $y = y_S = 0$ and $y = 2(y_N - y_S)$. The values of n_S, n_N, p_F and $\mathbf{z}$ are the ones fixed in (10.103).The values of the fields at y_S are the green lines and at y_N are the orange ones.

Conclusions

This thesis collects the results on holographic black objects obtained during the three years of doctoral studies. It is focused on recent frontiers in the framework of the AdS/CFT correspondence, investigating both sides of the duality.

A first example is the analysis of the saddle-point expansion in the Cardy-like limit of the index of different superconformal field theories presented in the second part. This can be a tool for shedding some light on the dual gravitational interpretation of the expansion, in terms both of black hole solutions and of quantum corrections to the entropy.
The remainder of the work has been devoted to the study of extremization problems in holography. An important case is represented by the extremization procedure performed on the field theory side that allows to extract the entropy of the dual black hole from the superconformal index, which has a counterpart in the attractor mechanism in supergravity. Other interesting instances come from the compactification of superconformal field theories on curved spaces, which allows to define new interacting fixed points in lower dimensions. The c-extremization procedure performed in field theory to derive the central charge of the lower-dimensional theory has a gravitational counterpart in the flow across dimensions. When the original theory is compactified on an orbifold, e.g. a spindle, instead of a manifold, this mechanism exhibits new peculiar features.

We conclude by presenting a recapitulation of the various topics covered in the thesis and addressing some possible further directions of research.

In the first part of chapter 5 we have studied the superconformal index of $4d\,\mathcal{N}=4\,USp(2N_c)$ and $SO(N_c)$ SYM from a matrix model perspective. We have focused on the Cardy-like limit of the index. Both in the symplectic and orthogonal case we have found that the index is dominated by a saddle point solution which we identify, reducing the calculation to a matrix integral of a pure Chern–Simons theory on the three-sphere. We have further computed the subleading logarithmic corrections, which are of the order of the center of the gauge group. In the $USp(2N_c)$ case we have also studied other subleading saddles of the matrix integral. Finally, we have discussed the case of the Leigh–Strassler fixed point with $SU(N_c)$ gauge group, and we have computed the entropy of the dual black hole from the Legendre transform of the entropy function.
There are some open questions that should be further explored. First, the analysis for the orthogonal gauge group has been carried on in [117], where the relation with the $USp(2N_c)$

case is also discussed. Moreover, it should be possible to apply the analysis of [101] to classify the saddle-point solutions of the $USp(2N_c)$ case via its center symmetry, and to relate them to the massive and Coulomb vacua of $\mathcal{N} = 1^*$ $USp(2N_c)$ SYM on $\mathbb{R}^3 \times S^1$ [248–253]. This can be helpful also for the analysis of the SCI for both orthogonal and symplectic gauge group from the BAE approach of [12, 80]. This analysis should provide a useful check of our results. Another open question regards the identification of the holographic dual to the finite-order logarithmic corrections we found. It would be very interesting to obtain this result from the supergravity side. The problem is very similar to the 3d one recently discussed in [111].

For what concerns the second part of chapter 5, we have studied the Cardy-like limit of the superconformal index of generic $\mathcal{N} = 1$ SCFTs with ABCD gauge algebra, providing strong evidence for a universal formula that captures the behavior of the index at finite order in the rank and in the fugacities associated with angular momenta, anticipating the results of [112] from the second sheet. The formula extends previous results valid at lowest order, and generalizes them to generic SCFTs. We have corroborated the validity of our proposal by studying several examples, beyond the well-understood toric class. We have computed the index also for models without a weakly-coupled gravity dual, whose gravitational anomaly is not of order one.
We have left open some questions that may deserve a further analysis. First, the validity of our formula (5.80) has been claimed (and checked) only for non-exceptional gauge algebras, i.e. the ABCD cases. In the exceptional EFG cases we did not make any claim because we are not aware of an exact evaluation of the three-sphere partition function for the associated 3d pure CS theory at level $-\eta T(G)$. (For some results in this direction see [121–124].) Once this integral is performed we expect our result (5.80) to hold in general for all semi-simple gauge algebras. Another obvious extension of the result consists in finding the generalization of (5.80) to the case where $\sigma \neq \tau$, so as to fully extend the result of [78] to finite order in both angular momenta. Further investigations should be performed to obtain a general analysis of the saddle point equations as well. Here we have observed through a case-by-case analysis that the number of solutions (leading to the dominant contribution to the index at $\eta = \pm 1$) is equal to the logarithm of the minimal value among the sums of the charges of each matter field under the centers of the factors of the product gauge group (or, more formally, to the order of the character lattice of the gauge algebra modulo the action of the Weyl symmetry). It would be desirable to have a proof of such a statement. (For recent progress in this direction see [104].) Furthermore we did not investigate other subleading solutions, such as the C-center ones discussed in [106], and a general analysis of the subleading structure of the index would be certainly an interesting subject.

In chapter 7 we have studied the entropy function for a conjectured BPS Kerr-Newman black hole originating from the 5d truncation of the LS fixed point proposed in [187]. After an opportune local rotation on the Killing prepotential and a generalization of the KK ansatz we have obtained the entropy function and we have matched such results from the one expected on the field theory side. It should be interesting to apply the procedure of [105] to other consistent truncation with a dual interpretation in order to validate the results obtained from the field theory side. For example the truncation of the LS fixed point discussed in [226] is very interesting because it involves two massless vectors, i.e. the full global symmetry is visible in such a truncation. Such truncation has however a non abelian gauging and a further truncation is needed to get an abelian gauging in order to study it in the formalism of [105]. It should be interesting to recast such a truncation in the language of [146] and then apply the procedure discussed here in order to obtain an entropy function from supergravity matching the one with the global symmetry turned on. It would be also interesting to investigate on the possible interpretation of the flow relating the UV and the IR fixed points studied in [187] after the KK reduction (see also related ideas in [254], where a field theoretical interpretation of the flow across dimensions has been proposed). Another interesting open question regards the analysis of higher derivatives corrections, in order to reproduce similar corrections depending on the gravitational anomaly, obtained from the field theory side (see [255, 256]). A last comment regards the difficulties of finding truncations where the conjecture, motivated by the index calculation, on the existence of 5d rotating black holes can be tested from the gravity side. The role played by the 5d Chern–Simons in the entropy function suggests that a universal behavior is possible. It would be worth to deeply explore such universality directly from the 10d perspective.

In chapter 9 we studied a supersymmetric $\text{AdS}_3 \times \Sigma$ asymptotic to the AdS_5 $\mathcal{N} = 2$ truncation of the conifold with a Betti vector multiplet found in [225]. The model consists of gauged supergravity with two vector multiplets and two hypermultiplets. The vector fields gauge a subgroup of the quaternionic manifold and one gauge field becomes massive via Higgs mechanism. In the low energy spectrum there are then two massless fields, the graviphoton and the Betti vector. One is associated with the R-symmetry and the other one with the baryonic symmetry of the dual Klebanov-Witten field theory. When this model is compactified on the spindle many of the scalars in the hypermultiplet can be further truncated. A crucial aspect of this compactification is that we need to include some of the scalars from the hypermultiplet in the analysis. After a suitable ansatz on the dependence of the spindle coordinates from the scalar fields, we have computed the BPS and the gauge fields equations. This has helped us in discussing the properly quantized fluxes through the spindle. We have shown that the analysis of the fluxes can be performed by studying the BPS equations at the poles of the spindle and that this analysis, in the conformal gauge (9.19), fixes the proper boundary conditions for all the scalars except the ones in the hypermultiplet. Furthermore thanks to this analysis it has been possible to compute the central charge of the would be AdS_3 solution from the Brown–Henneaux formula. By inspection we have observed that only the anti-twist class is consistent with the unitarity bound requiring a positive central charge. This analysis is however not enough to claim the existence of the AdS_3 solution and for this reason it is necessary to solve the BPS equations explicitly from the north to the south pole of the spindle. By turning off the baryonic magnetic charge we have observed that the universal solution of [22] is recovered. Furthermore we have provided a numerical analysis for the case with the magnetic baryonic charge turned on. In the numerical analysis we have first imposed the boundary conditions obtained from the analysis of the BPS equations at the north pole. Then we have scanned the boundary value of the scalar in the hypermultiplet that we cannot turn off. Solving the BPS equations for various initial data we have looked for the unique solution corresponding to the existence of a compact spindle. Once this value has been found we have checked that indeed the other fields take the value expected from the analysis of the pole data at the south pole, where this last value has been extracted numerically. We have eventually compared our general expression of central charge for the anti-twist with the one obtained in [201, 218] from the dual field theory, restricted to the baryonic anti-twist, and we have found an exact agreement.
Many interesting directions in the study of supergravity truncations on the spindle deserve

further investigation. For example one can study other consistent 5d $\mathcal{N} = 2$ truncations with vector multiplets and hypermultiplets where an holographic dual field theory is available. Another interesting aspect discussed in [227] consists of reformulating the BPS equations in a $d = 4$ Janus form in order to interpret the conserved charges from a different perspective. We expect that similar results can be provided for our model as well. A more complicated question regards the interpretation of the solutions found here from a flow across dimensions along the radial direction. This requires a modification of the ansatz (9.17) and requires to solve the BPS equations also for the radial profiles of the scalars. Finding such solutions is a necessary step to obtain a supergravity attractor mechanism dual to c-extremization on the spindle.

In chapter 10 we studied the reduction of the consistent truncations found in [241] on the spindle. These truncations are associated with M5-branes wrapping holomorphic curves in a CY_3 and the dual field theories have been obtained in [223, 240]. Using these results we matched the 2d central charge obtained from the field theoretical analysis with the one predicted in gauged supergravity from the analysis at the poles of the spindle. We have also studied the full solution, showing its existence for consistent choices of the parameters, analytically for the universal anti-twist and numerically after including the magnetic charge of the flavor symmetry.
There are many interesting aspects that we did not investigate. A first open question regards the uplift of our solutions to 7d and 11d supergravity. An interesting limit corresponds to set $\mathbf{z} = \pm 1$ and consider $p_F = 2\mathbf{z}\big(q - \frac{1}{4}(n_S - n_N)\big)$. In this case we reproduce the results obtained in [202] for the $\mathcal{N} = 2$ Maldacena-Nuñez theory. Observe that the matching works when p_F and $(n_S - n_N)/2$ have the same parity. Another open question regards the existence of solutions for $\mathbf{k} = 1$ and $|\mathbf{z}| > 1$ in both the twist and the anti-twist class and for $\mathbf{k} = -1$ in the twist class. Even if we have not been able to exclude these possibilities (for generic values of $\mathbf{z}$) we have not found any solution of this type neither in the analytical nor in the numerical analysis carried out in section 10.5. Nevertheless we observe that by choosing $\mathbf{z} = 0$ we can simplify the problem (for $\mathbf{k} = -1$) and we obtain results similar to the one studied in the previous chapter. This limit corresponds to the $\mathcal{N} = 1$ Maldacena-Nuñez theory and in this case the pole analysis completely excludes the existence of solutions in the twist class. The reason is that in this case we can impose further reality constraints on the conserved charges against the existence of such solutions. Our analysis has been performed at leading order in N, i.e. the central charge here is scales with N^3. There is a subleading contribution of order N, proportional to the gravitational anomaly of the SCFT, that we have computed from the field theoretical side. It would be interesting to match this contribution from the holographic analysis. A similar calculation was carried out for the case of the topological twist in [257]. It would also be interesting to consider M5 branes wrapping other geometries. For example by considering a disc, an holographic dual of an $\mathcal{N} = 2$ SCFT of AD type was proposed in [258–260] (see also [212]). As then observed in [261, 262] indeed the disc and spindle geometries are different global completions of the same local solution. Finally, it would be possible to study the models discussed here from the 11d perspective along the lines of the recent discussions of [263–266] from the theory of equivariant localization.

Appendices

Appendix A

Pure Chern–Simons three-sphere partition function

Here we collect some results on the calculation of three-sphere partition functions for the pure Chern-Simons theories encountered in the main text.

A.1 $USp(2N_c)$

Pick two complexes ω_1, ω_2 in the upper half-plane and define $\omega \equiv \frac{1}{2}(\omega_1 + \omega_2)$. When localizing on the squashed three-sphere S^3_b with one squashing parameter b (to preserve $\mathcal{N} = 2$ supersymmetry in three dimensions), we set $\omega_1 = ib,\ \omega_2 = \frac{i}{b}$; therefore $\omega_1\omega_2 = -1$ and $\omega = \frac{i}{2}(b + \frac{1}{b})$. For the round three-sphere, which we will focus on hereafter, $\omega_1 = \omega_2 = \omega = i$. The localization procedure produces hyperbolic Gamma functions

$$\Gamma_h(z;\omega_1,\omega_2) \equiv \prod_{m,n=1}^{\infty} \frac{(n+1)\omega_1 + (m+1)\omega_2 - z}{n\omega_1 + m\omega_2} \tag{A.1}$$

from the one-loop determinants for the vector (and matter) multiplets. Then the partition function of pure CS theory (i.e. without matter) with gauge group $USp(2N_c)_k$ and CS level $k = \frac{t}{2}$ is given by a matrix integral $J_{N_c,0,t}$ which has been studied in the mathematical literature. The exact evaluation is given by [267, Prop. 5.3.18]:

$$J_{N_c,0,t} \equiv \frac{1}{2^{N_c} N_c!} \int \frac{\prod_{j=1}^{N_c} d\sigma_i\, c(2t \sum_{j=1}^{N_c} \sigma_j^2)}{\prod_{1\le i<j\le N_c} \Gamma_h(\pm\sigma_i \pm \sigma_j) \prod_{j=1}^{N_c} \Gamma_h(\pm\sigma_j)} \tag{A.2}$$

$$= \frac{e\left(-\frac{(2+\mathrm{sign}(t))N_c}{8}\right)}{(t\ \mathrm{sign}(t))^{\frac{N_c}{2}}}\, c\left(-\frac{N_c(N_c+1)(2N_c+1)(\omega_1^2+\omega_2^2)}{3t}\right)$$

$$\cdot \prod_{1\le i<j\le N_c} 4\sin\left(\frac{\pi(i\pm j)}{t}\right) \prod_{j=1}^{N_c} 2\sin\left(\frac{2\pi j}{t}\right)\,, \tag{A.3}$$

where

$$c(z) \equiv e^{\frac{\pi i z}{2\omega_1\omega_2}}\,, \quad e(z) \equiv e^{2\pi i z}\,. \tag{A.4}$$

From (5.31) we see that in our case $2k = t = 2\eta(N_c+1)$ with $\eta = \pm 1$; thus

$$\prod_{1\le i<j\le N_c} 4\sin\left(\frac{\pi(i\pm j)}{2\eta(N_c+1)}\right) \prod_{j=1}^{N_c} 2\sin\left(\frac{2\pi j}{2\eta(N_c+1)}\right) \tag{A.5}$$

evaluates to

$$e^{-\frac{i\pi}{2}(N_c^2-N_c)}(2(N_c+1))^{\frac{N_c}{2}}(\text{sign}(\eta))^{N_c^2}\ . \tag{A.6}$$

Notice that this evaluation is nontrivial, so it is interesting to observe that for the nongeneric value of $2k=t$ extracted from the 4d calculation it can be carried out explicitly. For other values of t one may use the (large-N_c) topological string techniques of [268] to evaluate $Z_{S^3}^{USp(2N_c)_k}$. Alternatively, one can work with standard trigonometric functions by exploiting the relation [269, Eq. (A.18)]

$$\frac{1}{\Gamma_h(\pm x)} = -4\sin\left(\frac{\pi x}{\omega_1}\right)\sin\left(\frac{\pi x}{\omega_2}\right) = -4\sinh(\pm\pi x)\ . \tag{A.7}$$

Substituting it back into the integrand of (A.2) we gain a factor of $e^{i\pi N_c^2}$. When the dust settles we are left with:

$$\begin{aligned} Z_{S^3}^{USp(2N_c)_{\eta(N_c+1)}} &\equiv \frac{e^{i\pi N_c^2}}{2^{N_c}N_c!}\int\prod_{j=1}^{N_c} d\sigma_j e^{-2\eta\pi i(N_c+1)\sigma_j^2}(4\sinh(\pm\pi\sigma_j)) \\ &\quad\cdot \prod_{1\le i<j\le N_c} 4\sinh(\pi(\pm\sigma_i\pm\sigma_j)) \\ &= e^{i\pi N_c^2-\frac{5}{12}i\pi\eta N_c(2N_c+1)}\ . \end{aligned} \tag{A.8}$$

A.2 $SO(N_c)$

Here we evaluate the (round) three-sphere partition function of pure CS theory with gauge group $SO(N_c)_k$.

The partition function $Z_{S_b^3}^{SO(2N_c+1)_k}$ of the $SO(2N_c+1)_k$ pure CS theory on the squashed sphere is given by the integral

$$Z_{S_b^3}^{SO(2N_c+1)_k} \equiv \frac{1}{2^{N_c}N_c!}\int\frac{\prod_{j=1}^{N_c} d\sigma_i\, e^{\frac{i\pi k\sigma_j^2}{\omega_1\omega_2}}}{\prod_{i<j}^{N_c}\Gamma_h(\pm\sigma_i\pm\sigma_j)\prod_{i=1}^{N_c}\Gamma_h(\pm\sigma_i)}\ . \tag{A.9}$$

In the even $Z_{S_b^3}^{SO(2N_c)_k}$ case, the integral reads instead:

$$Z_{S_b^3}^{SO(2N_c)_k} \equiv \frac{1}{2^{N_c-1}N_c!}\int\frac{\prod_{j=1}^{N_c} d\sigma_i\, e^{\frac{i\pi k\sigma_j^2}{\omega_1\omega_2}}}{\prod_{i<j}^{N_c}\Gamma_h(\pm\sigma_i\pm\sigma_j)}\ . \tag{A.10}$$

(In both cases, the dependence on b is through ω_1,ω_2.)

We start our analysis by computing the integral (A.9) with $k>0$. The key formula in order to compute such integral is the generalization of the Weyl character formula for the B_{N_c} algebra, that is

$$\det\left\{2\sin\left(\frac{\pi(2j-1)\sigma_\ell}{z}\right)\right\}_{1\le j,\ell\le N_c}$$
$$=\prod_{1\le j<\ell\le N_c} 4\sin\left(\frac{\pi(\sigma_j\pm\sigma_\ell)}{z}\right)\prod_{j=1}^{N_c} 2\sin\left(\frac{\pi\sigma_j}{z}\right)\;. \tag{A.11}$$

Upon using the first relation in (A.7), the integrand of (A.9) becomes

$$\frac{\prod_{j=1}^{N_c} e^{\frac{i\pi k\sigma_j^2}{\omega_1\omega_2}}}{\prod_{1\le j<\ell\le N_c}\Gamma_h(\pm\sigma_j\pm\sigma_\ell)\prod_{j=1}^{N_c}\Gamma_h(\pm\sigma_j)} \tag{A.12}$$
$$=\prod_{a=1}^{2}\det\left\{\frac{\pi(2j-1)\sigma_\ell}{\omega_a}\right\}_{1\le j,\ell\le N_c} e^{-\frac{i\pi k\sum_{j=1}^{N_c}\sigma_j^2}{\omega_1\omega_2}}\;.$$

We further simplify the integral using the relation [267, Eq. (5.3.20)], i.e.

$$\int\det\{f_j(\sigma_\ell)\}_{1\le j,\ell\le N_c}\det\{g_j(\sigma_\ell)\}_{1\le j,\ell\le N_c}\prod_{\ell=1}^{N_c}h(\sigma_\ell)d\sigma_\ell=$$
$$N_c!\det\left\{\int f_j(\sigma)g_\ell(\sigma)h(\sigma)d\sigma\right\}_{1\le j,\ell\le N_c}\;. \tag{A.13}$$

The last integral in (A.13) can be explicitly computed:

$$\int\sin\left(\frac{\pi(2j-1)\sigma}{\omega_1}\right)\sin\left(\frac{\pi(2\ell-1)\sigma}{\omega_2}\right)e^{\frac{i\pi k\sigma^2}{\omega_1\omega_2}}= \tag{A.14}$$
$$2\sqrt{\frac{\omega_1\omega_2}{ik}}e^{-\frac{i\pi((2j-1)^2\omega_1^2+(2\ell-1)^2\omega_2^2)}{2k\omega_1\omega_2}}\sin\left(\frac{\pi(2j-1)(2\ell-1)}{2k}\right)\;.$$

By further using the relation

$$\det\left\{2\sin\left(\frac{\pi(2j-1)\ell}{2k}\right)\right\}_{1\le j<\ell\le N_c} \tag{A.15}$$
$$=\prod_{1\le j<\ell\le N_c}4\sin\left(\frac{\pi(j\pm\ell)}{2k}\right)\prod_{j=1}^{N_c}2\sin\left(\frac{\pi j}{2k}\right)$$

we can compute (A.9), which gives

$$Z_{S_b^3}^{SO(2N_c+1)_{k>0}}=\frac{e^{-\frac{i\pi N_c(4N_c^2-1)(\omega_1^2+\omega_2^2)}{12k\omega_1\omega_2}-\frac{3i\pi N_c}{4}}}{k^{N_c/2}} \tag{A.16}$$
$$\cdot\prod_{1\le j<\ell\le N_c}4\sin\left(\frac{\pi(j+\ell-1)}{k}\right)\sin\left(\frac{\pi(j-\ell)}{k}\right)\prod_{j=1}^{N_c}2\sin\left(\frac{\pi(2j-1)}{2k}\right)\;.$$

An analogous computation can be performed in the case of $k<0$, obtaining

$$Z_{S_b^3}^{SO(2N_c+1)_{k<0}} = \frac{e^{-\frac{i\pi N_c(4N_c^2-1)(\omega_1^2+\omega_2^2)}{12k\omega_1\omega_2}-\frac{i\pi N_c}{4}}}{(-k)^{N_c/2}} \tag{A.17}$$
$$\cdot \prod_{1\leq j<\ell\leq N_c} 4\sin\left(\frac{\pi(j+\ell-1)}{k}\right)\sin\left(\frac{\pi(j-\ell)}{k}\right)\prod_{j=1}^{N_c} 2\sin\left(\frac{\pi(2j-1)}{2k}\right) .$$

Fixing $k=-\eta(2N_c-1)$ and $\omega_1=\omega_2=i$, and substituting (A.7) into the integrand of (A.9), the partition function evaluates to

$$Z_{S^3}^{SO(2N_c+1)_{-\eta(2N_c-1)}} = 2\,e^{\frac{5i\pi\eta N_c}{12}(2N_c+1)+i\pi N_c^2} . \tag{A.18}$$

An analogous computation can be performed for $SO(2N_c)_k$. In this case the generalized Weyl character formula for the D_{N_c} algebra is

$$\frac{1}{2}\det\left\{2\cos\frac{\pi(2j-1)\sigma_\ell}{z}\right\}_{1\leq j,\ell\leq N_c} \tag{A.19}$$
$$= \prod_{1\leq j<\ell\leq N_c} 4\sin\left(\frac{\pi(\sigma_j\pm\sigma_\ell)}{z}\right) .$$

By following the steps discussed above the computation is straightforward, and we obtain

$$Z_{S_b^3}^{SO(2N_c)_k} = \frac{e^{-\frac{i\pi N_c}{4}(2-sign(k))}}{|k|^{N_c/2}} c\left(-\frac{N_c(N_c-1)(2N_c-1)(\omega_1^2+\omega_2^2)}{3k}\right)$$
$$\cdot \prod_{1\leq j<\ell<N_c} 4\sin\left(\frac{\pi(j+\ell-2}{|k|}\right)\sin\left(\frac{\pi(j-\ell)}{|k|}\right) . \tag{A.20}$$

In this case for $k=-2\eta(N_c-1)$, $\omega_1=\omega_2=i$, and substituting (A.7) into the integrand of (A.10), we have:

$$Z_{S^3}^{SO(2N_c)_{-2\eta(N_c-1)}} = 2\,e^{\frac{5i\pi\eta N_c}{12}(2N_c-1)+i\pi(N_c^2-N_c)} . \tag{A.21}$$

Appendix B

Revisiting Kaluza–Klein reduction

In this appendix we revisit the Kaluza–Klein reduction from the 5d Lagrangian (6.62) to the 4d one (6.61). Our analysis generalizes the one performed in the appendix D of [105], giving a recipe that allows to construct a 4d Lagrangian in the formalism of [13, 105] starting from a 5d Lagrangian written in generic conventions. As remarked in chapter 7, such an analysis has been necessary in our case in order to obtain a 4d Lagrangian suitable for the computation of the AdS_2 attractor in the formalism of [18].

We start from a 5d theory with n_V vector multiplets and n_H hypermultiplets and we indicate with an hat the fields that live in five dimensions. We use indices

$$I, J = 1, \dots, n_V + 1, \qquad \Lambda, \Sigma = 0, \dots, n_V + 1, \qquad u, v = 1, \dots, 4n_H, \tag{B.1}$$

$$\mu, \nu = 1, \dots, 4, \qquad M, N = 0, \dots, 4. \tag{B.2}$$

and we put hats on the five-dimensional quantities. We indicate the 5d vector fields as $\hat{A}^I_M$. We reduce the five-dimensional theory on a circle, using the following KK ansatz [192]:

$$\begin{aligned} d\hat{s}^2 &= e^{2\alpha\widetilde{\phi}} ds^2 + e^{2\beta\widetilde{\phi}} (dy - A^0)^2, \\ h^I &= -k\, e^{2\alpha\widetilde{\phi}} z_2^I, \\ \hat{A}^I_M &= \big(A^I_\mu - c\, z_1^I A^0_\mu,\; c\, z_1^I + \xi^I\big), \end{aligned} \tag{B.3}$$

where α, β, c and k are constants, y is the direction of the circular fiber with range $4\pi/g$, A^0 is the KK vector and ξ^I are flat gauge connections. All the fields are independent of y. We are using the notation

$$z_1^I \equiv \text{Re}\, z^I, \qquad z_2^I \equiv \text{Im}\, z^I. \tag{B.4}$$

Notice that in the last line of (B.3) we are also performing a Scherk–Schwarz twist [13, 105, 193, 194]. This is necessary to satisfy the 4d BPS equations and it will bring the extra Killing vector in the 4d theory.

Because of the constraint $\mathcal{V}(h) = 1$ in (6.26), the field $\widetilde{\phi}$ is redundant:

$$e^{-6\alpha\widetilde{\phi}} = -k^3 C_{IJK} z_2^I z_2^J z_2^K. \tag{B.5}$$

We will use this relation during the computation.

We can fix the first constant by requiring that the dimensionally-reduced Lagrangian is of the Einstein–Hilbert form $\mathscr{L} = -\frac{1}{2}e\,R + \ldots$ In fact, the five-dimensional Ricci scalar reduces to

$$\hat{R} = e^{-2\alpha\widetilde{\phi}}\big(R - 6\alpha^2\partial_\mu\widetilde{\phi}\,\partial^\mu\widetilde{\phi} - \frac{1}{4}e^{-6\alpha\widetilde{\phi}}F^0_{\mu\nu}F^{0\,\mu\nu}\big) + \text{ total derivatives} \tag{B.6}$$

and the determinant of the metric gives

$$\hat{e} = e^{(\beta+4\alpha)\widetilde{\phi}}\,e. \tag{B.7}$$

Therefore, we have to impose $\beta = -2\alpha$.
We can thus rewrite the 5d metric and its inverse as

$$\begin{aligned}
\hat{g}_{MN} &= \begin{pmatrix} e^{2\alpha\widetilde{\phi}}g_{\mu\nu} + e^{-4\alpha\widetilde{\phi}}A^0_\mu A^0_\nu & -e^{-4\alpha\widetilde{\phi}}A^0_\mu \\ -e^{-4\alpha\widetilde{\phi}}A^0_\nu & e^{-4\alpha\widetilde{\phi}} \end{pmatrix}, \\
\hat{g}^{MN} &= \begin{pmatrix} e^{-2\alpha\widetilde{\phi}}g^{\mu\nu} & e^{-2\alpha\widetilde{\phi}}A^{0\mu} \\ e^{-2\alpha\widetilde{\phi}}A^{0\nu} & e^{4\alpha\widetilde{\phi}} + e^{-2\alpha\widetilde{\phi}}A^0_\rho A^{0\rho} \end{pmatrix}, \\
\hat{e} &= e^{2\alpha\widetilde{\phi}}e.
\end{aligned} \tag{B.8}$$

The reduction of the Einstein term follows from (B.6) and (B.7):

$$\hat{\mathscr{L}}_1 = \frac{\hat{e}\hat{R}}{2} = e\left(\frac{R}{2} - 3\alpha^2\partial_\mu\widetilde{\phi}\,\partial^\mu\widetilde{\phi} - \frac{e^{-6\alpha\widetilde{\phi}}}{8}F^0_{\mu\nu}F^{0\,\mu\nu}\right) + \text{ total derivatives.} \tag{B.9}$$

The reduction of the kinetic term of the scalars in the vector multiplet gives

$$\begin{aligned}
\hat{\mathscr{L}}_2 &= -\hat{e}\,\frac{1}{2}g_{xy}\,\partial_M\phi^x\,\partial^M\phi^y = -\frac{3}{4}e\,a_{IJ}\,\hat{g}^{MN}\partial_M h^I\partial_N h^J \\
&= -\frac{3k^2}{4}e\,a_{IJ}\,g^{\mu\nu}\partial_\mu\big(e^{2\alpha\widetilde{\phi}}z_2^I\big)\,\partial_\nu\big(e^{2\alpha\widetilde{\phi}}z_2^J\big) = e\left[-\frac{3k^2}{4}e^{4\alpha\widetilde{\phi}}a_{IJ}\,\partial_\mu z_2^I\,\partial^\mu z_2^J + 3\alpha^2\,\partial_\mu\widetilde{\phi}\partial^\mu\widetilde{\phi}\right],
\end{aligned} \tag{B.10}$$

where in the first line we used the relations (6.27) and (6.29), and the simplifications in the last line occur due to (6.30), which implies

$$h_I\,\partial_\mu\big[h^I(\phi)\big] = h_I\,\partial_x h^I\partial_\mu\phi^x = 0, \tag{B.11}$$

giving the condition on the 4d scalars

$$2\alpha\,\partial_\mu\widetilde{\phi}\,z_2^I + \partial_\mu z_2^I = 0. \tag{B.12}$$

Notice that the last term in $\hat{\mathscr{L}}_2$ exactly cancels the second term in $\hat{\mathscr{L}}_1$. For simplicity we fix $\alpha = 1$.
The reduction of the kinetic term of the scalars in the hypermultiplet gives

$$\begin{aligned}
\hat{\mathscr{L}}_3 &= -\hat{e}\,\frac{1}{2}g_{XY}\,\hat{g}^{MN}\hat{\mathcal{D}}_M q^X\,\hat{\mathcal{D}}_N q^Y \\
&= e\left[-\frac{1}{2}g_{XY}\mathcal{D}_\mu q^X\mathcal{D}^\mu q^Y - \frac{1}{2}e^{6\widetilde{\phi}}g^2(k_0^X + c\,z_1^I k_I^X)g_{XY}(k_0^Y + c\,z_1^J k_J^Y)\right],
\end{aligned} \tag{B.13}$$

where $\hat{\mathcal{D}}_M q^X$ is the 4d covariant derivative defined in (6.57), while

$$\mathcal{D}_\mu q^X = \partial_\mu q^X + g\, A^I_\mu k^X_I + g\, A^0_\mu \xi^I k^X_I = \partial_\mu q^X + g\, A^\Lambda_\mu k^X_\Lambda \tag{B.14}$$

is the 4d covariant derivative, which contains the extra Killing vector defined as

$$k^X_0 := \xi^I k^X_I\,. \tag{B.15}$$

The reduction of the gauge kinetic term gives

$$\begin{aligned}\hat{\mathscr{L}}_4 &= -e\,\frac{1}{4} a_{IJ} \hat{F}^I_{MN} \hat{F}^{J\,MN} \\ &= e\left[-\frac{1}{4} e^{-2\tilde{\phi}} a_{IJ} (F^I_{\mu\nu} - c\, z^I_1 F^0_{\mu\nu})(F^{J\mu\nu} - c\, z^J_1 F^{0\,\mu\nu}) - \frac{c^2}{2} e^{4\tilde{\phi}} a_{IJ}\, \partial_\mu z^I_1\, \partial^\mu z^J_1\right],\end{aligned} \tag{B.16}$$

where $\hat{F}_{MN}$ and $F_{\mu\nu}$ are the 5d and 4d field strengths. The first one can be obtained from the 5d vector fields in (B.3), i.e.

$$\hat{A}^I = A^I - c\, z^I_1 A^0 + (c\, z^I_1 + \xi^I) dy. \tag{B.17}$$

Thus

$$\hat{F}^I = d\hat{A}^I = F^I - c\, dz^I_1 \wedge A^0 - c\, z^I_1 F^0 + c\, dz^I_1 \wedge dy, \tag{B.18}$$

from which we can read off the components

$$\begin{aligned}\hat{F}^I_{\mu 4} &= -\hat{F}^I_{4\mu} = c\, \partial_\mu z^I_1, \\ \hat{F}^I_{\mu\nu} &= F^I_{\mu\nu} - c\, z^I_1 F^0_{\mu\nu} + c\, A^0_\mu \partial_\nu z^I_1 - c\, A^0_\nu \partial_\mu z^I_1.\end{aligned} \tag{B.19}$$

To perform the reduction of the Chern–Simons term it is convenient to extend the geometry (B.3) to a 6d bulk having the original 5d space as boundary. This can be obtained by extending the circle parametrized by the y coordinate to a unit disk with radius $\rho \in [0,1]$ and the 5d connections $\hat{A}^I$ in (B.3) to the following 6d ones:

$$\tilde{A}^I = A^I + \xi^I A^0 + \rho^2 (z^I_1 + \xi^I)(dy - A^0). \tag{B.20}$$

We can thus rewrite the Chern–Simons action term as

$$\int_{5d} \hat{\mathscr{L}}_5 = \int_{5d} \frac{2}{3\sqrt{6}} C_{IJK} \hat{F}^I \wedge \hat{F}^J \wedge \hat{A}^K = \int_{6d} \frac{2}{3\sqrt{6}} C_{IJK} \tilde{F}^I \wedge \tilde{F}^J \wedge \tilde{F}^K, \tag{B.21}$$

with $\tilde{F}^I = d\tilde{A}^I$. Integrating over $d\rho^2 \wedge (dy - A^0)$ we can extract the 4d Lagrangian:

$$\begin{aligned}\hat{\mathscr{L}}_5 = \frac{1}{2\sqrt{6}} C_{IJK} \epsilon^{\mu\nu\rho\sigma} \Big[&(c\, z^I_1 + \xi^I) F^J_{\mu\nu} F^K_{\rho\sigma} - (c^2\, z^I_1 z^J_1 - \xi^I \xi^J) F^K_{\mu\nu} F^0_{\rho\sigma} \\ &+ \frac{c^3\, z^I_1 z^J_1 z^K_1 + \xi^I \xi^J \xi^K}{3} F^0_{\mu\nu} F^0_{\rho\sigma}\Big].\end{aligned} \tag{B.22}$$

Finally, the reduction of the scalar potential gives

$$\begin{aligned}\hat{\mathscr{L}}_6 &= -\hat{e}\, g^2 \Big[P^r_I P^r_J \big(3 g^{xy} \partial_x h^I \partial_y h^J - 4 h^I h^J\big) + \frac{3}{4} g_{XY} k^X_I k^Y_J h^I h^J\Big] \\ &= -e^{2\tilde{\phi}}\, e\, g^2 \Big[P^r_I P^r_J \big(3 g^{xy} \partial_x h^I \partial_y h^J - 4k^2\, e^{4\tilde{\phi}} z^I_2 z^J_2\big) + \frac{3k^2}{4} e^{4\tilde{\phi}} g_{XY} k^X_I k^Y_J\, z^I_2 z^J_2\Big].\end{aligned} \tag{B.23}$$

We now rearrange the various pieces of the reduced Lagrangian to reproduce the general form of 4d $\mathcal{N} = 2$ gauged supergravity with $n_V + 1$ vector multiplets and n_H hypermultiplets in (6.61).

The Einstein term descends from $\hat{\mathscr{L}}_1$:

$$\mathscr{L}_1 = \frac{e\,R}{2}. \tag{B.24}$$

The kinetic term of the scalars in the vector multiplet receives contributions from $\hat{\mathscr{L}}_2$ and $\hat{\mathscr{L}}_4$:

$$\mathscr{L}_2 = -e\, e^{4\widetilde{\phi}} a_{IJ}\Big(-\frac{3k^2}{4}\partial_\mu z_2^I\,\partial^\mu z_2^J + \frac{c^2}{2}\partial_\mu z_1^I\,\partial^\mu z_1^J\Big) = -e\,G_{I\bar{J}}\,\partial_\mu z^I\,\partial^\mu \bar{z}^{\bar{J}}, \tag{B.25}$$

where we defined the Hermitian metric

$$G_{I\bar{J}} := \frac{3k^2}{4} e^{4\widetilde{\phi}} a_{I\bar{J}}. \tag{B.26}$$

Notice that this recasting imposes a constraint between the parameters of the KK ansatz in (B.3), which we can fix as

$$c = \sqrt{\frac{3}{2}}k. \tag{B.27}$$

The kinetic term of the scalars in the hypermultiplet is contained in $\hat{\mathscr{L}}_3$,

$$\mathscr{L}_3 = -\frac{e}{2} g_{XY}\mathcal{D}_\mu q^X\,\mathcal{D}^\mu q^Y, \tag{B.28}$$

where the covariant derivative $\mathcal{D}_\mu q^u$ is defined in (B.14) and (B.15).

The scalar potential receives contributions from $\hat{\mathscr{L}}_3$ and $\hat{\mathscr{L}}_6$:

$$\begin{aligned}\mathscr{L}_6 = {}& e^{-2\widetilde{\phi}}\, g^2\Big[P_I^r P_J^r\big(3e^{2\widetilde{\phi}} g^{xy}\partial_x h^I\partial_y h^J - 4k^2 e^{6\widetilde{\phi}} z_2^I z_2^J\big) \\ & + e^{6\widetilde{\phi}} g_{XY}\big(\frac{3k^2}{4}k_I^X\,k_J^Y\,z_2^I z_2^J + \frac{1}{2}(k_0^X + c\,z_1^I k_I^X)(k_0^Y + c\,z_1^J k_J^Y)\big)\Big].\end{aligned} \tag{B.29}$$

If we impose the constraint (B.27) and we fix

$$k = \sqrt{\frac{2}{3}}, \tag{B.30}$$

the second line of (B.29) can be rewritten as

$$\frac{1}{2}e^{6\widetilde{\phi}} g_{XY} k_\Lambda^X k_\Sigma^Y X^\Lambda \bar{X}^\Sigma, \tag{B.31}$$

in which we are using special coordinates $z^I = X^I/X^0$ in the Kähler frame $|X^0|^2 = 1$. (Recall that the indices Λ, Σ run over 0 and then the values of I, J.) We will rearrange the first part of the scalar potential in a few lines.

Instead, we now focus on the gauge kinetic term, that gets contributions from $\hat{\mathscr{L}}_1$ and $\hat{\mathscr{L}}_4$:

$$\begin{aligned}\mathscr{L}_4 &= -e\,\frac{e^{-6\widetilde{\phi}}}{8}\Big[F_{\mu\nu}^o F^{0\,\mu\nu} + 4G_{IJ}\big(F_{\mu\nu}^I - z_1^I F_{\mu\nu}^0\big)\big(F^{J\,\mu\nu} - z_1^J F^{0\,\mu\nu}\big)\Big] \\ &= \frac{e}{8}\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\,F_{\mu\nu}^\Lambda F^{\Sigma\,\mu\nu}.\end{aligned} \tag{B.32}$$

In the last line we recast the field-dependent pieces in the matrix

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} = -e^{-6\tilde{\phi}}\begin{pmatrix} 1+4\,G_{MN}z_1^M z_1^N & -4\,G_{MJ}z_1^M \\ -4\,G_{IM}z_1^M & 4\,G_{IJ}\end{pmatrix}, \tag{B.33}$$

which we will show later to be actually the imaginary part of the period matrix that descends from a proper prepotential.
Finally, we can rewrite $\hat{\mathscr{L}}_5$ as

$$\mathscr{L}_5 = \hat{\mathscr{L}}_5 = \frac{1}{12\sqrt{6}}\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\epsilon^{\mu\nu\rho\sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma}, \tag{B.34}$$

where

$$\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma} = \frac{4}{3\sqrt{6}}\begin{pmatrix} 2\,C_{KLM}\big(z_1^K z_1^L z_1^M + \xi^K\xi^L\xi^M\big) & -3\,C_{JKL}\big(z_1^K z_1^L - \xi^K\xi^L\big) \\ -3\,C_{IKL}\big(z_1^K z_1^L - \xi^K\xi^L\big) & 6\,C_{IJK}\big(z_1^K + \xi^K\big)\end{pmatrix}. \tag{B.35}$$

We now show that $G_{I\bar{J}}$ and $\mathcal{N}_{\Lambda\Sigma}$ come from the following prepotential:

$$\begin{aligned} F(X) &= \frac{4}{3\sqrt{6}}C_{IJK}\frac{\check{X}^I\check{X}^J\check{X}^K}{X^0} \qquad\qquad \text{with } \check{X}^I \equiv X^I + \xi^I X^0 \\ &= \frac{4}{3\sqrt{6}}C_{IJK}\Big(\frac{X^IX^JX^K}{X^0} + 3\,\xi^I X^J X^K + 3\,\xi^I\xi^J X^K X^0 + 2\,\xi^I\xi^J\xi^K (X^0)^2\Big). \end{aligned} \tag{B.36}$$

Using special coordinates $z^I = X^I/X^0$, in the Kähler frame $|X^0|^2 = 1$, the Kähler potential (6.19) becomes

$$\mathcal{K} = -\log\Big(-\frac{32}{3\sqrt{6}}C_{IJK}z_2^I z_2^J z_2^K\Big) = -\log\big(8e^{-6\tilde{\phi}}\big). \tag{B.37}$$

from which one can derive the Kähler metric (6.10)

$$\begin{aligned} G_{i\bar{\jmath}} &= \sqrt{\frac{2}{3}}e^{6\tilde{\phi}}\Big(C_{IJK}z_2^K + \sqrt{\frac{2}{3}}e^{6\tilde{\phi}}C_{IKL}C_{JMN}z_2^K z_2^L z_2^M z_2^N\Big) \\ &= 8\sqrt{\frac{2}{3}}e^{\mathcal{K}}\Big(C_{IJK}z_2^K + 8\sqrt{\frac{2}{3}}e^{\mathcal{K}}C_{IKL}C_{JMN}z_2^K z_2^L z_2^M z_2^N\Big), \end{aligned} \tag{B.38}$$

that corresponds to (B.26) with (B.30).
On the other hand, from the prepotential (B.36) we obtain

$$F_{\Lambda\Sigma} = \frac{4}{3\sqrt{6}}\begin{pmatrix} 2\,C_{KLM}\big(z^K z^L z^M + \xi^K\xi^L\xi^M\big) & -3\,C_{JKL}\big(z^K z^L - \xi^K\xi^L\big) \\ -3\,C_{IKL}\big(z^K z^L - \xi^K\xi^L\big) & 6\,C_{IJK}\big(z^K + \xi^K\big)\end{pmatrix}, \tag{B.39}$$

from which we can derive the period matrix (6.25), whose real and imaginary parts are in agreement with (B.35) and (B.33). In the computation of the period matrix we used the following relations:

$$\begin{aligned} (X^0)^{-2}X^{\Lambda}(\mathrm{Im}F_{\Lambda\Sigma})X^{\Sigma} &= \frac{8}{3\sqrt{6}}C_{IJK}\Big(\mathrm{Im}(z^Iz^Jz^K) - 3z^I\mathrm{Im}(z^Jz^K) + 3z^Iz^J\mathrm{Im}(z^K)\Big) \\ &= -\frac{32}{3\sqrt{6}}C_{IJK}z_2^Iz_2^Jz_2^K = 8e^{-6\tilde{\phi}} = e^{-\mathcal{K}}, \end{aligned} \tag{B.40}$$

$$(X^0)^{-1}(\mathrm{Im}F_{I\Sigma})X^\Sigma = \frac{8i}{\sqrt{6}} C_{IKL} z_2^K z_2^L, \tag{B.41}$$

$$(X^0)^{-1}(\mathrm{Im}F_{0\Sigma})X^\Sigma = 2e^{-6\widetilde{\phi}} - \frac{8i}{\sqrt{6}} C_{IJK} z_1^I z_2^J z_2^K, \tag{B.42}$$

$$C_{IKL}C_{JMN} z_2^K z_2^L z_2^M z_2^N = \sqrt{\frac{3}{2}} e^{-6\widetilde{\phi}} \Big(\sqrt{\frac{3}{2}} e^{-6\widetilde{\phi}} G_{IJ} - C_{IJK} z_2^K \Big). \tag{B.43}$$

Finally, we return to the 4d scalar potential in (B.29). To manipulate its first line, we notice that

$$\big(\mathrm{Im}\,\mathcal{N}\big)^{-1|\Lambda\Sigma} = -2e^{\mathcal{K}} \begin{pmatrix} 4 & 4\,z_1^K \\ 4\,z_1^J & 4\,z_1^J z_1^K + G^{JK} \end{pmatrix}, \tag{B.44}$$

and thus

$$\big(\mathrm{Im}\,\mathcal{N}\big)^{-1|\Lambda\Sigma} + 8\,e^{\mathcal{K}} X^{(\Lambda}\bar{X}^{\Sigma)} = -2\,e^{\mathcal{K}} \begin{pmatrix} 0 & 0 \\ 0 & G^{IJ} - 4\,z_2^I z_2^J \end{pmatrix}. \tag{B.45}$$

Using this last relation, the scalar potential becomes

$$\mathscr{L}_6 = -eg^2 \Big[-\tilde{P}_\Lambda^r \tilde{P}_\Sigma^r \Big(\big(\mathrm{Im}\mathcal{N}\big)^{-1|\Lambda\Sigma} + 8\,e^{\mathcal{K}} X^\Lambda \bar{X}^\Sigma \Big) + 4\,e^{\mathcal{K}} h_{uv} k_\Lambda^u k_\Sigma^v X^\Lambda \bar{X}^\Sigma \Big], \tag{B.46}$$

where $\tilde{P}_\Lambda^r = 2\,P_\Lambda^r$[1].
Notice that we cannot extract $\vec{P}_0$ directly from the potential, because it is multiplied by the null components of the matrix in (B.45). Nevertheless, being related to the Killing vector, it can be read from (B.15) and it is determined as $\vec{P}_0 = \xi^I \vec{P}_I$.

The 4d Lagrangian that we have obtained performing this KK reduction exactly reproduces (6.61).

[1]The extra factor 2 is due to the difference of conventions between [105] and [187].

Appendix C

Details on the quaternionic geometry

In this appendix we derive the quaternionic structures $\vec{J}$ and the $SU(2)$ spin connections $\vec{\omega}$ for the quaternionic manifold

$$\frac{SO(4,2)}{SO(4)\times SO(2)}. \tag{C.1}$$

This scalar geometry was already studied in detail in [159,160,236] and it is described by $n_H = 2$ hypermultiplets, whose real scalars can be organized as $q^X\,,\ X = 1,\dots,8.$
Following [159,160] we use the coordinates $q^X = \{u_1, k, a, \phi, b^1, \bar{b}^1, b^2, \bar{b}^2\}$, with line element given by

$$\begin{aligned} g_{XY}dq^X dq^Y =& -2e^{-4u_1}M_{ij}\left(b^i d\bar{b}^j + \bar{b}^i db^j\right) - 4du_1^2 - \frac{1}{4}d\phi^2 - \frac{1}{4}e^{2\phi}da^2 \\ & - \frac{1}{4}e^{-8u_1}\left[dk + 2\varepsilon_{ij}\left(b^i d\bar{b}^j + \bar{b}^i db^j\right)\right]^2, \end{aligned} \tag{C.2}$$

where

$$M_{ij} = e^{\phi}\begin{pmatrix} a^2 + e^{-2\phi} & -a \\ -a & 1 \end{pmatrix}. \tag{C.3}$$

We have chosen the normalization of the metric in order to have the curvature scalar $R = 64$ and thus $\vec{\mathcal{R}} = \vec{J}$ (see (6.34)). This last equality is necessary to obtain the simplified version of the hyperino variation in (D.21).
To display the quaternionic structure of the coset, we follow the explicit construction made in Appendix E of [270],[1] adapting it to our coordinates, and we introduce the quaternionic vielbeins

$$\begin{aligned} f^1 &= \frac{i}{2}d\phi\,, \qquad f^2 = 2i\,du_1\,, \qquad f^3 = \frac{i}{2}e^{\phi}da\,, \\ f^4 &= ie^{-4u_1}\left(\frac{1}{2}dk - \bar{b}^2 db^1 + \bar{b}^1 db^2 - b^2 d\bar{b}^1 + b^1 d\bar{b}^2\right), \\ f^5 &= \frac{i}{\sqrt{2}}e^{-2u_1-\frac{\phi}{2}}\left(db^1 + d\bar{b}^1\right), \qquad f^6 = \frac{1}{\sqrt{2}}e^{-2u_1-\frac{\phi}{2}}\left(db^1 - d\bar{b}^1\right), \\ f^7 &= \frac{i}{\sqrt{2}}e^{-2u_1+\frac{\phi}{2}}\left(-a\,db^1 + db^2 - a\,d\bar{b}^1 + d\bar{b}^2\right), \\ f^8 &= -\frac{1}{\sqrt{2}}e^{-2u_1+\frac{\phi}{2}}\left(a\,db^1 - db^2 + a\,d\bar{b}^1 - d\bar{b}^2\right). \end{aligned} \tag{C.4}$$

[1] See also [236] for a similar analysis.

We can thus obtain the triplet of almost complex structures $\vec{J}$ as

$$
\begin{aligned}
J^1 &= \frac{1}{\sqrt{2}}\Big(f^{15}+f^{18}+f^{25}-f^{28}-f^{36}+f^{37}-f^{46}-f^{47}\Big)\,,\\
J^2 &= \frac{1}{\sqrt{2}}\Big(f^{16}-f^{17}+f^{26}+f^{27}+f^{35}+f^{38}+f^{45}-f^{48}\Big)\,,\\
J^3 &= -\Big(f^{13}+f^{24}+f^{56}+f^{78}\Big)\,,
\end{aligned}
\tag{C.5}
$$

where $f^{ij} \equiv f^i \wedge f^j$ and $J^r = \frac{1}{2} J^r{}_{XY}\, dq^X \wedge dq^Y$ for $r = 1, 2, 3$. One can check that these structures satisfy the quaternionic relations in (6.31). In this setting, the $SU(2)$ connections take the form

$$
\begin{aligned}
\omega^1 &= \frac{1}{2}e^{-2u_1-\frac{\phi}{2}}(db^1+d\bar{b}^1) - \frac{i}{2}e^{-2u_1+\frac{\phi}{2}}\left[a(db^1-d\bar{b}^1)-(db^2-d\bar{b}^2)\right]\,,\\
\omega^2 &= -\frac{i}{2}e^{-2u_1-\frac{\phi}{2}}(db^1-d\bar{b}^1) - \frac{1}{2}e^{-2u_1+\frac{\phi}{2}}\left[a(db^1+d\bar{b}^1)-(db^2+d\bar{b}^2)\right]\,,\\
\omega^3 &= \frac{1}{4}e^{\phi}da - \frac{1}{2}e^{-4u_1}\left(\frac{1}{2}dk - \bar{b}^2 db^1 + \bar{b}^1 db^2 - b^2 d\bar{b}^1 + b^1 d\bar{b}^2\right).
\end{aligned}
\tag{C.6}
$$

Appendix D

Derivation of the BPS equations

Following the analysis made in [227], in order to construct $AdS_3 \times \Sigma$ solutions to the BPS equations, we first decompose the Clifford algebra via

$$\gamma^m = \Gamma^m \otimes \sigma^3, \qquad \gamma^3 = \mathbb{I}_2 \otimes \sigma^1, \qquad \gamma^4 = \mathbb{I}_2 \otimes \sigma^2, \tag{D.1}$$

where $\Gamma^m = \left(-i\sigma^2, \sigma^3, \sigma^1\right)$[1] are the gamma matrices in $d = 3$ and σ^i, $i = 1, 2, 3$, are the Pauli matrices. We can thus write the Killing spinor as

$$\epsilon = \psi \otimes \chi, \tag{D.2}$$

where χ is a two-component spinor on the spindle and ψ is a two-component spinor on AdS_3 satisfying

$$\nabla_m \psi = -\frac{\kappa}{2} \Gamma_m \psi\,, \tag{D.3}$$

where $\kappa = \pm 1$ specifies the chirality of the supersymmetry of the dual 2d SCFT. In this section we will analyze the BPS equations in order to determine the structure of the spinor χ, that is given by

$$\chi = e^{V/2} e^{isz} \begin{pmatrix} \sin\frac{\xi}{2} \\ \cos\frac{\xi}{2} \end{pmatrix} \tag{D.4}$$

as we will see in more detail in a few.

D.1 Gravitino variation

The supersymmetry variation for the gravitino in (6.63) splits into two decoupled equations if we impose the projection condition on the symplectic-Majorana ϵ^i.[2] This decoupling is due to the fact that in our model only the $r = 3$ $SU(2)$-components of the moment maps survive (see (9.14)), which are related to the doublet notation through the third Pauli matrix via (6.32). Thus, one of the two BPS equations obtained from the gravitino variation can be written as

$$\delta\psi_\mu = \left[\nabla_\mu - iQ_\mu + \frac{i}{4\sqrt{6}} H_{\nu\rho}\left(\gamma^{\nu\rho}\gamma_\mu + 2\gamma^\nu \delta^\rho_\mu\right) + \frac{1}{2} g W \gamma_\mu\right]\epsilon = 0\,, \tag{D.5}$$

[1] We are using the mostly plus signature, while in [227] they use the mostly minus one.

[2] See appendix A.2.1 of [257] for a more general overview on the projections of $SU(2)$ symplectic-Majorana fermions.

where ϵ is now a Dirac spinor, $\nabla_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}$ and $Q_\mu \equiv \partial_\mu q^X \omega_X^3 + g A_\mu^I P_I^3$. We also introduced the superpotential $W \equiv \sqrt{\frac{2}{3}} h^I P_I^3$ and $H_{\mu\nu} \equiv h_I F_{\mu\nu}^I$.
The components of (D.5) that are tangent to the directions along AdS_3 give:

$$\left[-\left(\kappa e^{-V} + \frac{1}{\sqrt{6}} H_{34}\right)\gamma^{34} + i\, V' f^{-1} \gamma^3\right]\epsilon = -igW\epsilon\,. \tag{D.6}$$

In order to have non-trivial solutions to this equation, we have to impose that the two coefficients on the left hand side live on a circle, i.e.

$$\left[\cos\xi \gamma^{34} + i \sin\xi \gamma^3\right]\epsilon = -i\epsilon\,. \tag{D.7}$$

This projection condition is solved by

$$\epsilon = e^{-i\frac{\xi}{2}\gamma^4}\eta\,, \qquad \gamma^3\eta = i\gamma^4\eta \tag{D.8}$$

and it allows us to split (D.6) in

$$-\kappa e^{-V} - \frac{1}{\sqrt{6}} H_{34} = gW\cos\xi\,, \qquad V' f^{-1} = gW\sin\xi\,. \tag{D.9}$$

If we now focus on the component of the gravitino variation in the longitudinal direction of the spindle, i.e. $\mu = y$, we can rewrite it as

$$\left[\partial_y - \frac{1}{2}V' - \frac{i}{2}\left(\partial_y \xi + \sqrt{\frac{3}{2}} f H_{34} + \kappa f e^{-V}\right)\gamma^4\right]\eta = 0\,. \tag{D.10}$$

One can notice that this expression is in the form $(a_1 + a_2\gamma^4)\eta = 0$, which implies that $a_1^2 + a_2^2 = 0$. Therefore, from the first part of (D.10) we can infer that η has the structure

$$\eta = e^{V/2} e^{isz}\eta_0\,, \tag{D.11}$$

where s is a constant and η_0 is independent from y and z. From the second part of (D.10) we obtain

$$\partial_y \xi + \sqrt{\frac{3}{2}} f H_{34} + \kappa f e^{-V} = 0\,. \tag{D.12}$$

Similarly, the component along the azimuthal direction of the spindle ($\mu = z$) gives

$$\begin{aligned}\Big[\partial_z - iQ_z + \frac{i}{2} f^{-1} h' \cos\xi - \frac{i}{\sqrt{6}} H_{34} h \sin\xi \\ + \left(-\frac{1}{2} f^{-1} h' \sin\xi + \frac{1}{2} gWh - \frac{1}{\sqrt{6}} H_{34} h\cos\xi\right)\gamma^4\Big]\eta = 0\,,\end{aligned} \tag{D.13}$$

from which

$$(s - Q_z) + \frac{1}{2} f^{-1} h' \cos\xi - \frac{1}{\sqrt{6}} H_{34} h \sin\xi = 0\,, \tag{D.14}$$

$$-\frac{1}{2} f^{-1} h' \sin\xi + \frac{gWh}{2} - \frac{1}{\sqrt{6}} H_{34} h \cos\xi = 0\,. \tag{D.15}$$

D.2 Gaugino variation

Using some relations of the Very Special Real geometry and the definition of the superpotential, the variation of the gaugino (6.64) gives

$$\delta\lambda^x = \Big[-\frac{i}{2}\gamma^\mu\partial_\mu\phi^x + \frac{1}{4}\sqrt{\frac{3}{2}}g^{xy}\partial_y h_I\gamma^{\mu\nu}F^I_{\mu\nu} + i\sqrt{\frac{3}{2}}g\, g^{xy}\partial_y W\Big]\epsilon = 0. \tag{D.16}$$

From the first component ($x = 1$), imposing again the projection condition in (D.7), we obtain

$$f^{-1}u_2' + \frac{3g}{4}\partial_{u_2}W\sin\xi = 0\,, \tag{D.17}$$

$$3g\,\partial_{u_2}W\cos\xi + \sqrt{\frac{2}{3}}e^{2u_3}\big(e^{-2u_2}F^{(1)}_{34} - e^{2u_2}F^{(2)}_{34}\big) = 0\,, \tag{D.18}$$

where we have used the explicit expressions for the sections h^I and for the field strengths. Similarly, from the component $x = 2$, we have

$$f^{-1}u_3' + \frac{g}{4}\partial_{u_3}W\sin\xi = 0\,, \tag{D.19}$$

$$\begin{aligned} &3g\,\partial_{u_3}W\cos\xi \\ &\quad + \sqrt{\frac{2}{3}}\big(2e^{-4u_3}F^{(0)}_{34} - e^{-2u_2+2u_3}F^{(1)}_{34} - e^{2u_2+2u_3}F^{(2)}_{34}\big) = 0\,. \end{aligned} \tag{D.20}$$

D.3 Hyperino variation

In order to simplify the BPS equation coming from the hyperino variation in (6.65), we can multiply its expression for f_{jAY}. Using (6.40), after some calculations we obtain the relation

$$\Big(-i\gamma^\mu\partial_\mu u_1 + \frac{3}{8}ig\partial_{u_1}W + \frac{1}{4}\partial_{u_1}Q_\mu\gamma^\mu\Big)\epsilon = 0\,. \tag{D.21}$$

Notice that to single out the vector Q_μ, introduced in (D.5), it is necessary to make a precise choice of the normalization of the metric of the quaternionic manifold, as we have pointed out in appendix C.

Finally, imposing the projection condition (D.7), this last equation gives

$$f^{-1}u_1' = -\frac{3g}{8}\frac{\partial_{u_1}W}{\sin\xi}\,, \tag{D.22}$$

$$\frac{3g}{2}\partial_{u_1}W\cos\xi = h^{-1}\partial_{u_1}Q_z\sin\xi\,. \tag{D.23}$$

Publications

The content of this thesis is based on the following (mostly published) works:

[1] A. Amariti, S. Mancani, D. Morgante, N. Petri and A. Segati, *BBBW on the spindle*, 2309.11362.

[2] A. Amariti, N. Petri, A. Segati, *$T^{1,1}$ truncation on the spindle*, *JHEP* **07** (2023) 087 [2304.03663].

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