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MEAN-FIELD CONTROL OF NON EXCHANGEABLE SYSTEMS

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Abstract. We study the optimal control of mean-field systems with heterogeneous and asymmetric interactions. This leads to considering a family of controlled Brownian diffusion processes with dynamics depending on the whole collection of marginal probability laws. We prove the well-posedness of such systems and define the control problem together with its related value function. We next prove a law invariance property for the value function which allows us to work on the set of collections of probability laws. We show that the value function satisfies a dynamic programming principle (DPP) on the flow of collections of probability measures. We also derive a chain rule for a class of regular functions along the flows of collections of marginal laws of diffusion processes. Combining the DPP and the chain rule, we prove that the value function is a viscosity solution of a Bellman dynamic programming equation in a L^2-set of Wasserstein space-valued functions.

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1. Introduction

The study of large population and complex systems is a major question in mathematical modelling with various applications in our society like *e.g.* social networks, power grid networks, financial markets, lightning networks. Classical models consider mean-field systems with symmetric particles and homogenous interaction: denoting by $X^{i,N}$ the state of the i-th particle (agent/player) in the N-population, it interacts with the other particles via the empirical measure: $\mu^N = \frac{1}{N}\sum_{j=1}^N \delta_{X^{j,N}}$, and therefore the system of N-particle $X^N = (X^{1,N}, \dots, X^{N,N})$ is exchangeable. The macroscopic behavior of the limiting mean-field system when the number N of agents goes to infinity leads to an equation with a representative state depending on its probability distribution, called McKean–Vlasov (MKV) equation, and has been extensively studied in the context of mean-field game (MFG where agents are in strategic interaction), of mean-field control (MFC with cooperative interaction among agents following a center of decision). We refer to the lectures of P.L. Lions [1] at Collège de France, and to the monographs by Bensoussan, Frehse and Yam [2] and Carmona and Delarue [3, 4] (and the references therein) for a comprehensive treatment of the mathematical tools (Itô's formula along flow of

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probability measures, maximum principle, forward backward SDE of MKV type, dynamic programming, Master Bellman equation in the Wasserstein space) used in the optimization problems from MFG and MFC.

In this paper, motivated by more realistic applications in complex networks, we deal with large systems of agents whose interactions are not necessarily symmetric and possibly heterogeneous, hence leading to non exchangeable systems as in [5]. For example, the theory of graphons (see [6]) provides a framework for such modelling, and has been used in [7–10] for the MFG problem with heterogeneous agents. In the context of cooperative interaction, the controlled dynamics of the particle $i \in [\![1, N]\!]$ in a graphon system with heterogeneous drift and volatility coefficients is driven by

$$\begin{aligned} \mathrm{d}X_t^{i,N} &= b\big(u_i, X_t^{i,N}, \alpha_t^{i,N}, \frac{1}{N_i}\sum_{j=1}^N G(u_i,u_j)\delta_{X_t^{j,N}}\big)\mathrm{d}t \\ &\quad + \ \sigma\big(u_i, X_t^{i,N}, \alpha_t^{i,N}, \frac{1}{N_i}\sum_{j=1}^N G(u_i,u_j)\delta_{X_t^{j,N}}\big)\mathrm{d}W_t^{u_i}, \end{aligned} \tag{1.1}$$

where $u_i = i/N \in U := [0,1]$ is the label of particle i in the N-system, G is a graphon, *i.e.*, a measurable function from $U \times U$ into U, measuring the weight of interaction between particles, $N_i = \sum_{j=1}^N G(u_i,u_j)$ is the degree of interaction of particle i, $W = (W^u)_{u\in U}$, is a family of i.i.d. Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\alpha^{i,N}$ is a control process valued in some action space A, followed by agent i. The aim of a center of decision in such a framework is to minimize over $(\alpha^{1,N}, \dots, \alpha^{N,N})$ a cost functional in the form

$$\begin{aligned} \frac{1}{R_N}\sum_{i=1}^N r(u_i)\mathbb{E}\Big[\int_0^T f\big(u_i, X_t^{i,N}, \alpha_t^{i,N}, \frac{1}{N_i}\sum_{j=1}^N G(u_i,u_j)\delta_{X_t^{j,N}}\big)\mathrm{d}t \\ + g\big(u_i, X_T^{i,N}, \frac{1}{N_i}\sum_{j=1}^N G(u_i,u_j)\delta_{X_T^{j,N}}\big)\Big], \end{aligned}$$

where $r(u_i)$ is the weight of particle i in the social cost criterion, $R_N := \sum_{i=1}^N r(u_i)$, and f, g are running/terminal costs, possibly heterogeneous (*i.e.* depending on the label of particles).

When the number of agents N goes to infinity, and in line with the convergence results in [11–14] for MFC with homogeneous interaction corresponding to $G \equiv 1$, and propagation of chaos for graphon mean-field systems in [15, 16], we formally expect to obtain the formulation of a graphon MFC as the problem of minimizing over a collection of control process $\alpha = (\alpha^u)_{u\in U}$ valued in A^U a cost functional in the form

$$\begin{aligned} \int_U \mathbb{E}\Big[\int_0^T f\big(u, X_t^u, \alpha_t^u, \int_0^1 \frac{G(u,v)}{\|G(u,.)\|_1}\mathbb{P}_{X_t^v}\mathrm{d}v\big)\mathrm{d}t \\ + \ g\big(u, X_T^u, \int_0^1 \frac{G(u,v)}{\|G(u,.)\|_1}\mathbb{P}_{X_T^v}\mathrm{d}v\big)\Big] r(u)\mathrm{d}u, \end{aligned} \tag{1.2}$$

where $\|G(u,.)\|_1 := \int_0^1 G(u,v)\mathrm{d}v$, $\mathbb{P}_Y$ denotes the probability law of a random variable Y on $(\Omega, \mathcal{F}, \mathbb{P})$, and $X = (X^u)_{u\in U}$ is a collection of controlled state process in $\mathbb{R}^d$ governed by

$$\begin{aligned} \mathrm{d}X_t^u = &\ b\big(u, X_t^u, \alpha_t^u, \int_0^1 \frac{G(u,v)}{\|G(u,.)\|_1}\mathbb{P}_{X_t^v}\mathrm{d}v\big)\mathrm{d}t \\ &+ \ \sigma\big(u, X_t^u, \alpha_t^u, \int_0^1 \frac{G(u,v)}{\|G(u,.)\|_1}\mathbb{P}_{X_t^v}\mathrm{d}v\big)\mathrm{d}W_t^u. \end{aligned} \tag{1.3}$$

Our work and contributions. Inspired by the above discussion, we extend the graphon MFC formulation in (1.2)–(1.3), and introduce a class of mean-field control for non exchangeable systems by considering a collection of controlled state process $X = (X^u)_{u\in U}$ governed by

$$dX_t^u = \ b\big(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_v, (\mathbb{P}_{\alpha_t^v})_v\big)dt + \sigma\big(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_v, (\mathbb{P}_{\alpha_t^v})_v\big)dW_t^u. \tag{1.4}$$

Then, the MFC problem is to minimize over the collection of control process $(\alpha^u)_{u\in U}$ the cost functional over a finite horizon

$$J(\alpha) = \ \int_U \mathbb{E}\Big[\int_0^T f\big(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_v, (\mathbb{P}_{\alpha_t^v})_v\big)dt + g\big(u, X_T^u, (\mathbb{P}_{X_T^v})_v\big)\Big]\lambda(du), \tag{1.5}$$

where λ is a positive finite measure on U that specifies the weight of the agents/particles in the social cost criterion. In this general modeling, we see from (1.4) that the state processes of the agents $u \in U$ are independent, but not identically distributed, and they interact through the whole collection of their probability laws on the state $(\mathbb{P}_{X_t^v})_{v\in U}$ and control $(\mathbb{P}_{\alpha_t^v})_{v\in U}$. Notice also the measurability issues $(u,\omega) \in U\times\Omega \mapsto X^u(\omega)$ due to the independence of the continuum of state process, see [17]. In particular, this probabilistic independence property prevents from proving a measurability property with respect to the product of σ-algebrae on U and Ω. As a consequence, we are not able to do computations and derive estimates involving the variables $u \in U$ and $\omega \in \Omega$ on the state process anymore. This issue leads us to focus on the collection of laws $(\mathbb{P}_{X_t^v})_{v\in U}$ as a state variable.

Our first task is a rigorous formulation of the MFC for coupled SDEs $(X^u)_{u\in U}$ with the admissible set of controls $\alpha = (\alpha^u)_{u\in U}$, and we specify in the next section the assumptions on the coefficients b, σ, f, g for ensuring the existence and uniqueness of a solution to (1.4) given initial conditions, and the well-posedness of the cost functional in (1.5). Our main goal is to provide an analytic characterization of the solution to this novel class of control problems by adopting a dynamic programming approach. This will be achieved through the following steps.

We define the value function associated to our MFC problem, and by a law invariance property, it is a function defined on $[0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, where $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ is the collection $(\mu^u)_{u\in U}$ of square integrable probability measures on $\mathbb{R}^d$ s.t. $\int_U\int_{\mathbb{R}^d} |x|^2\mu^u(dx)\lambda(du) < \infty$. From the flow property of the solution to (1.4), we then state directly the dynamic programming principle (DPP) for the value function.

Next, we aim to derive an Itô's formula for the collection of flow of probability measures $(\mathbb{P}_{X_t^u})_{t\in[0,T],u\in U}$ that extends the Itô formula for flow of measures in [18], see also Chapter 5 in [3]. This is obtained with the notion of linear (or flat) derivative of a function defined on $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, and by standard method of time discretization.

Once we have the DPP and the Itô's formula, we can derive as usual the associated Bellman equation that we express in a suitable and unified form that takes into consideration both dependence of the dynamics and cost on the collection of probability laws on state/control. A verification theorem is shown for classical solutions to the Bellman equation, and in general, we prove the (discontinuous) viscosity property of the value function to the Bellman equation. Uniqueness of viscosity solutions is beyond the scope of this paper and postponed for future research.

Outline. The plan of this paper is organized as follows. We formulate the mean-field control problem for non exchangeable systems in Section 2. In Section 3, we show the law invariance property of the value function and the dynamic programming principle. Section 4 is devoted to Itô formula along a collection of flow of probability measures with the notion of linear derivative on $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$. We then derive in Section 5 the dynamic programming Bellman equation for the value function, and its viscosity property. As a conclusion, a comparison between the approach adopted in this paper and an alternative label state approach is discussed. Some questions left for further research are also presented. Finally, we postpone some proofs to Appendix A, and collect in Appendix B some auxiliary results dealing with measurability questions that are needed in the proofs of some results.

Notations.

- We denote by $x \cdot y$ the scalar product between vectors x, y, and by $A : B = \mathrm{tr}(AB^\intercal)$ the inner product of two matrices A, B with compatible dimensions, where $B^\intercal$ is the transpose matrix of B.
- Throughout the paper, $T > 0$ denotes a fixed time horizon. For an integer $d \geq 1$ and $[a, b] \subset [0, T]$ the space $C([a, b]; \mathbb{R}^d)$ of continuous functions $[a, b] \to \mathbb{R}^d$ will be denoted simply $C^d_{[a,b]}$. It will be given the supremum norm and the corresponding distance and Borel sets. For $w \in C^d_{[a,b]}$ and $[a', b'] \subset [a, b]$, $w_{[a',b']} \in C^d_{[a',b']}$ stands for the restriction of w to $[a', b']$. We also denote by $\mathbb{W}_T$ the Wiener measure on $C^d_{[0,T]}$.
- For $a, b, c \in [0, T]$ such that $a \leq b \leq c$, $\hat{w} \in C^d_{[a,b]}$ and $\check{w} \in C^d_{[b,c]}$ we define the concatenation $\hat{w} \oplus \check{w} \in {C^d}_{[a,c]}$ by the formula

$$(\hat{w} \oplus \check{w})(s) = \begin{cases} \hat{w}(s) & \text{if } s \in [a, b], \\ \hat{w}(b) + \check{w}(s) - \check{w}(b) & \text{if } s \in [b, c]. \end{cases}$$

- For any generic Polish space E with a complete metric d, we denote by $\mathcal{P}_2(E)$ the Wasserstein space of Borel probability measures ρ on E satisfying $\int_E d(x, 0)^2 \, \rho(\mathrm{d}x) < \infty$, where 0 denotes an arbitrary fixed element in E (the origin when E is a vector space). $\mathcal{P}_2(E)$ is endowed with the 2-Wasserstein distance $\mathcal{W}_2$ corresponding to the quadratic transport cost $(x, y) \mapsto d(x, y)^2$. As a measurable space, $\mathcal{P}_2(E)$ is endowed with the corresponding Borel σ-algebra.
- Given a random variable Y on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $\mathbb{P}_Y$ the law of Y under $\mathbb{P}$. We shall also use the notation $\mathcal{L}(Y)$ for the law of Y (under $\mathbb{P}$) when there is no ambiguity.

2. Controlled mean field non exchangeable system and the optimization problem

2.1. Preliminaries

Let U be a Polish space endowed with its Borel σ-algebra $\mathcal{U}$, representing the continuum of heterogenous agents/particles, and λ be a positive finite measure on U. In particular, when λ is a discrete measure, this models the case of a finite number of class of heterogenous interacting agents.

We will often consider maps $u \mapsto \mu^u$ from U to $\mathcal{P}_2(E)$. When we need to check measurability of such maps we will use the fact that the Borel σ-algebra in $\mathcal{P}_2(E)$ coincides with the trace of the Borel σ-algebra corresponding to the weak topology, and that measurability holds if and only if the maps of the form

$$u \in U \mapsto \int_E \Phi(x) \, \mu^u(\mathrm{d}x) \in \mathbb{R}, \tag{2.1}$$

are measurable for every choice of bounded continuous function $\Phi : E \to \mathbb{R}$.

The space $L^2(U, \mathcal{U}, \lambda; \mathcal{P}_2(E))$, denoted $L^2_\lambda(\mathcal{P}_2(E))$ for short, consists of elements $\mu = (\mu^u)_{u \in U}$ that are measurable functions $U \to \mathcal{P}_2(E)$, $u \mapsto \mu^u$, satisfying

$$\int_U \int_{\mathbb{R}^d} d(x, 0)^2 \, \mu^u(\mathrm{d}x) \, \lambda(\mathrm{d}u) = \int_U \mathcal{W}_2(\mu^u, \delta_0)^2 \, \lambda(\mathrm{d}u) < \infty,$$

where δ_0 denotes the Dirac mass at the fixed element 0. $L^2_\lambda(\mathcal{P}_2(E))$ is endowed with the (complete) metric

$$\mathbf{d}(\mu, \nu) = \left(\int_U \mathcal{W}_2(\mu^u, \nu^u)^2 \, \lambda(\mathrm{d}u) \right)^{1/2}, \qquad \mu, \nu \in L^2_\lambda(\mathcal{P}_2(E)),$$

and the corresponding Borel σ-algebra (when we deal with measurability issues).

We will also often deal with the case when E is a space of continuous functions. For instance when $E = C^d_{[a,b]}$, for any $\mu = (\mu^u)_u \in L^2_\lambda(\mathcal{P}_2(C^d_{[a,b]}))$ and $s \in [a,b]$, we may consider $\mu_s := (\mu^u_s)_u$, where each μ^u_s is the image of the measure μ^u under the coordinate mapping $C_{[a,b]} \to \mathbb{R}^d$, $w \mapsto w(s)$. It is then easy to see that $\mu^u_s \in \mathcal{P}_2(\mathbb{R}^d)$, $\mu_s \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, $s \mapsto \mu_s$ is continuous $[a,b] \to L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, $u \mapsto \mu^u_s$ is measurable $U \to \mathcal{P}_2(\mathbb{R}^d)$, $(u,s) \mapsto \mu^u_s$ is also measurable $U \times [a,b] \to \mathcal{P}_2(\mathbb{R}^d)$ and finally, for any other $\nu = (\nu^u)_u \in L^2_\lambda(\mathcal{P}_2(C^d_{[a,b]}))$,

$$\mathcal{W}_2(\mu^u_s, \nu^u_s) \le \mathcal{W}_2(\mu^u, \nu^u), \qquad \mathbf{d}(\mu_s, \nu_s) \le \mathbf{d}(\mu, \nu),$$

for every $u \in U$, $s \in [a,b]$.

For a collection $\xi = (\xi^u)_{u \in U}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $\mathbb{P}_{\xi^\cdot}$ the collection of the probability laws $(\mathbb{P}_{\xi^u})_{u \in U}$.

We denote by A the set of control actions and we assume that it is a Polish space. As written above, we also denote by d and $\mathbf{d}$ the metrics on A and $L^2_\lambda(\mathcal{P}_2(A))$

2.2. Coupled controlled mean field SDEs

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. For every $u \in U$ we are given an $\mathbb{R}^\ell$-valued standard Brownian motion $W^u = (W^u_t)_{t \in [0,T]}$ and an independent real random variable Z^u having uniform distribution in $(0,1)$. We assume that $\{(W^u, Z^u) : u \in U\}$ is an independent family. For every $u \in U$ we denote by $(\mathcal{F}^{W^u}_t)_{t \in [0,T]}$ the natural filtration generated by W^u, by $\sigma(Z^u)$ the σ-algebra generated by Z^u and by $\mathbb{F}^u = (\mathcal{F}^u_t)_{t \in [0,T]}$ the filtration given by

$$\mathcal{F}^u_t = \mathcal{F}^{W^u}_t \vee \sigma(Z^u) \vee \mathcal{N}, \qquad t \in [0,T],$$

where $\mathcal{N}$ is the family of $\mathbb{P}$-null sets.

The coefficients of the control problem are functions b, σ, f, g satisfying suitable assumptions detailed later; b, σ, f are functions of $u \in U$, $x \in \mathbb{R}^d$, $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, $\nu \in L^2_\lambda(\mathcal{P}_2(A))$ $a \in A$, with values respectively in $\mathbb{R}^d$, $\mathbb{R}^{d \times \ell}$, $\mathbb{R}$; g is a real function of $u \in U$, $x \in \mathbb{R}^d$, $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$.

The dynamics of the controlled system is described as follows. For every starting time $t \in [0,T]$, we solve a system of controlled stochastic Itô differential equations, indexed by $u \in U$, of the form

$$\begin{cases} \mathrm{d}X^u_s &= b\big(u, X^u_s, \alpha^u_s, \mathbb{P}_{X^\cdot_s}, \mathbb{P}_{\alpha^\cdot_s}\big)\mathrm{d}s \\ & \quad + \sigma\big(u, X^u_s, \alpha^u_s, \mathbb{P}_{X^\cdot_s}, \mathbb{P}_{\alpha^\cdot_s}\big)\mathrm{d}W^u_s, \quad t \le s \le T, \\ X^u_t &= \xi^u, \quad u \in U. \end{cases} \tag{2.2}$$

Here, the initial condition is given by a collection $\xi = (\xi^u)_u$ of $\mathbb{R}^d$-valued random variable such that ξ^u is $\mathcal{F}^u_t$-measurable for each $u \in U$. From the definition of $\mathcal{F}^u_t$, ξ^u is also independent from $(W^u_s - W^u_t)_{s \ge t}$. We will also require that the map

$$u \longmapsto \mathcal{L}\big(W^u_{[0,t]}, Z^u, \xi^u\big), \tag{2.3}$$

is Borel measurable as a mapping from U to $\mathcal{P}_2(C^\ell_{[0,t]} \times (0,1) \times \mathbb{R}^d)$ and

$$\int_U \mathbb{E}[|\xi^u|^2] \lambda(\mathrm{d}u) < \infty.$$

This way we have $\mathbb{P}_{\xi^\cdot} \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ and even

$$\Big(\mathcal{L}\big(W^u_{[0,t]}, Z^u, \xi^u\big)\Big)_{u \in U} \in L^2_\lambda\Big(\mathcal{P}_2(C^\ell_{[0,t]} \times (0,1) \times \mathbb{R}^d)\Big).$$

When these conditions are met we say that ξ is an admissible initial condition at time t and we write $\xi \in \mathcal{I}_t$.

We recall that, for every $u \in U$, assuming that the random variable ξ^u is $\mathcal{F}_t^u$-measurable implies that it is $\mathbb{P}$-almost surely equal to a variable of the form $\underline{\xi}^u(W^u_{[0,t]}, Z^u)$ for a measurable function $\underline{\xi}^u : C^\ell_{[0,t]} \times (0,1) \to \mathbb{R}^d$. If $\underline{\xi}^u(w,z)$ is jointly measurable in (u,w,z) then the measurability condition (2.3) is satisfied. However in general $\underline{\xi}^u(w,z)$ is only measurable as a function of (w,z) and so (2.3) should be explicitly required in the definition of the set $\mathcal{I}_t$.

For every $u \in U$, the control processes $(\alpha_t^u)_{t\in[0,T]}$ are defined as follows. For an arbitrary Borel measurable function

$$\alpha : U \times [0,T] \times C^\ell_{[0,T]} \times (0,1) \to A$$

we define

$$\alpha_t^u = \alpha(u, t, W^u_{\cdot\wedge t}, Z^u), \quad t \in [0,T],\, u \in U,$$

where $W^u_{\cdot\wedge t}$ is the path $s \mapsto W^u_{s\wedge t}$, $s \in [0,T]$. We note that each process $(\alpha_t^u)_t$ is $\mathbb{F}^u$-predictable. We say that α is an admissible control policy (or simply a policy) if

$$\int_U \int_0^T \mathbb{E}[d(\alpha_s^u, 0)^2]\, \mathrm{d}s\, \lambda(\mathrm{d}u) \ < \ \infty. \tag{2.4}$$

We denote by $\mathcal{A}$ the class of all admissible policies α. We remark that, denoting $\mathbb{W}_T$ the Wiener measure on $C^\ell_{[0,T]}$, condition (2.4) is equivalent to

$$\int_U \int_0^T \int_{C^\ell_{[0,T]}} \int_0^1 \Big[\mathrm{d}\big(\alpha(u,s,w(\cdot\wedge s),z),0\big)^2\Big]\, \mathrm{d}z\, \mathbb{W}_T(\mathrm{d}w)\, \mathrm{d}s\, \lambda(\mathrm{d}u) \ < \ \infty,$$

which shows that the class $\mathcal{A}$ does not depend on the choice of Ω, $\mathcal{F}$, $\mathbb{P}$, W^u and Z^u.

A few explanations are in order. We note that (2.2) is a system of stochastic differential equations, indexed by $u \in U$, which is coupled due to occurrence of the terms $(\mathbb{P}_{X_s^u})_{u\in U}$. We will give conditions on b and σ implying that each equation in (2.2) has a unique $\mathbb{F}^u$-adapted continuous solution (up to indistinguishability), for λ-almost all $u \in U$.

In particular, $(X^u)_u$ will be an independent family of stochastic processes, because this holds for the family of Brownian motions $(W^u)_u$. As mentioned in the introduction, this raises an issue concerning the measurability with respect to the parameter $u \in U$. To overcome this issue, we shall work with the probability laws, and we will show that the obtained solution to (2.2) has the additional property that the map $u \mapsto \mathcal{L}(X^u, W^u, Z^u)$ is Borel measurable on $C^d_{[t,T]} \times C^\ell_{[0,T]} \times (0,1)$. Under some conditions to be precised later on the running and terminal costs, this will ensure that the gain functional in (1.5) is well defined.

A further comment concerns the introduction of the random variables Z^u as an additional source of noise besides the Brownian motions W^u. Recall that each Z^u has uniform distribution on $(0,1)$; we will use the well known property that any probability in $\mathbb{R}^d$ is the image of the uniform distribution under an appropriate Borel map $(0,1) \to \mathbb{R}^d$. For our purposes it is important that the initial conditions ξ^u (for the state Eq. (2.2) starting at time t) may have an arbitrary element of $\mathcal{P}_2(\mathbb{R}^d)$ as its law. Our requirement on ξ^u is that it should be square summable and $\mathcal{F}_t^u$-measurable, so we may define ξ^u as an appropriate function of Z^u and obtain the required distribution. We also let ξ^u depend on the trajectory of W^u up to time t to include initial conditions derived from the flow property of the considered processes. This will be helpful for establishing the dynamic programming principle.

Using the previous notation, it is convenient to recall that the index set U is a Polish space with a Borel finite positive measure λ, the space of control actions A is a Polish space, and $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, s.t. for every $u \in U$, W^u is an $\mathbb{R}^\ell$-valued standard Brownian motion, Z^u is a real random variable with uniform distribution in $(0,1)$, independent of W^u, and $\{(W^u, Z^u) : u \in U\}$ is an independent family. We next formulate the requirements that we need on the coefficients b, σ.

Assumption 2.1. The functions

$$b, \sigma : \ U \times \mathbb{R}^d \times A \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \times L^2_\lambda(\mathcal{P}_2(A)) \longrightarrow \mathbb{R}^d, \ \mathbb{R}^{d\times\ell}$$

are Borel measurable. There exist constants $L \geq 0$, $M \geq 0$ such that

(i)

$$|b(u,x,a,\mu,\nu) - b(u,x',a,\mu',\nu)| \leq L\left(|x-x'| + \mathbf{d}(\mu,\mu')\right)$$

(ii)

$$|\sigma(u,x,a,\mu,\nu) - \sigma(u,x',a,\mu',\nu)| \leq L\left(|x-x'| + \mathbf{d}(\mu,\mu')\right),$$

(iii)

$$|b(u,x,a,\mu,\nu)| + |\sigma(u,x,a,\mu,\nu)| \leq M\big(1 + |x| + d(a,0) + \mathbf{d}(\mu,\delta_0) + \mathbf{d}(\nu,\delta_0)\big)$$

for every $u \in U$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, $\nu \in L^2_\lambda(\mathcal{P}_2(A))$, $a \in A$.

We recall that 0 also denotes a fixed element of A and we note that in the previous expressions we write δ_0 to denote the collection $(\delta_0)_{u\in U}$.

Remark 2.2. We do not explicitly consider time-depending coefficients b, σ (and later f). However all our results have immediate extensions to this case, with almost identical proofs, provided b, σ, f are required to be measurable in time (jointly with the other arguments) and the assumptions hold for constants that do not depend on time.

Remark 2.3. Examples covered by Assumption 2.1 include the graphon interaction (1.3) discussed in the introduction. Many other examples can be considered, for instance a drift (or a volatility) of the form $b(\mu) = \int_U h(\bar{\mu}^u)\,\lambda(\mathrm{d}u)$ for some Lipschitz function $h : \mathbb{R}^d \to \mathbb{R}^d$, where $\bar{\mu}^u = \int_{\mathbb{R}^d} x\,\mu^u(\mathrm{d}x)$.

We are now able to state and prove the basic existence and uniqueness result for the controlled state equation. We first define a set of stochastic processes where a unique solution will be found.

Definition 2.4. Given $t \in [0,T]$, we say that a family $X = (X^u)_u$ of stochastic processes with values in $\mathbb{R}^d$ belongs to the space $\mathcal{S}_t$ if

1. the map $u \mapsto \mathcal{L}(X^u, W^u, Z^u)$ is Borel measurable from U to $\mathcal{P}_2(C^d_{[t,T]} \times C^\ell_{[0,T]} \times (0,1))$;
2. each process X^u is continuous and $\mathbb{F}^u$-adapted;
3. the following norm is finite:

$$\|X\| := \left(\int_U \mathbb{E}\left[\sup_{s\in[t,T]} |X^u_s|^2\right]\lambda(\mathrm{d}u)\right)^{1/2}.$$

We say that $(X^u)_u \in \mathcal{S}_t$ is a solution to (2.2) if the equations in (2.2) are satisfied for λ-almost every u. We say that the solution is unique if, whenever $(X^u)_u, (\tilde{X}^u)_u \in \mathcal{S}_t$ solve (2.2) then the processes X^u and $\tilde{X}^u$ coincide, up to a $\mathbb{P}$-null set, for λ-almost all $u \in U$.

Remark 2.5. As defined above, the space $\mathcal{S}_t$ endowed with $\|\cdot\|$ is not a Banach space, and even a vector space. As a matter of fact, $\|X - Y\|$ is not well defined for any $X, Y \in \mathcal{S}_t$ as the joint law of (X^u, Y^u) may not be a measurable function of $u \in U$. In particular, one cannot use a Picard iteration on $\mathcal{S}_t$ to construct solutions to (2.2). To overcome this issue, we shall work on the laws of processes which have the expected measurability property in $u \in U$.

Theorem 2.6. *Suppose that Assumption 2.1 holds. Let $t \in [0, T]$ and $\xi \in \mathcal{I}_t$ be an admissible initial condition. Let $\alpha \in \mathcal{A}$ be an admissible policy and define*

$$\alpha_t^u = \alpha(u, t, W_{\cdot\wedge t}^u, Z^u), \quad t \in [0, T],\, u \in U.$$

Then there exists a unique solution $X = (X^u)_u \in \mathcal{S}_t$ to the equation (2.2), in the sense of Definition 2.4.

Proof. We borrow some ideas from Proposition 2.1 in [15], but we need very different arguments because of the lack of time continuity of the coefficients in the stochastic equations due to the occurrence of the control process. The proof is postponed to Appendix A. □

We recall that $\xi \in \mathcal{I}_t$ requires the random variable ξ^u to be $\mathcal{F}_t^u$-measurable for every $u \in U$. Therefore it is $\mathbb{P}$-almost surely equal to a variable of the form $\underline{\xi}^u(W_{[0,t]}^u, Z^u)$ for a measurable function $\underline{\xi}^u : C_{[0,t]}^\ell \times (0, 1) \to \mathbb{R}^d$. The state equation (2.2) corresponding to a given admissible control $\alpha \in \mathcal{A}$ can be written

$$\begin{cases} \mathrm{d}X_s^u = b\big(u, X_s^u, \alpha_s^u, \mathbb{P}_{X_s^\cdot}, \mathbb{P}_{\alpha_s^\cdot}\big)\mathrm{d}s + \sigma\big(u, X_s^u, \alpha_s^u, \mathbb{P}_{X_s^\cdot}, \mathbb{P}_{\alpha_s^\cdot}\big)\mathrm{d}W_s^u, \quad s \in [t, T], \\ X_t^u = \underline{\xi}^u(W_{[0,t]}^u, Z^u), \\ \alpha_s^u = \alpha(u, s, W_{\cdot\wedge s}^u, Z^u) \quad u \in U. \end{cases} \tag{2.5}$$

We next present a result providing estimates on solution to those systems of SDEs.

Proposition 2.7. *Suppose that Assumption 2.1 holds. Let $t \in [0, T]$ and $\xi \in \mathcal{I}_t$ be an admissible initial condition. Let $\alpha \in \mathcal{A}$ be an admissible policy and define*

$$\alpha_t^u = \alpha(u, t, W_{\cdot\wedge t}^u, Z^u), \quad t \in [0, T],\, u \in U.$$

Then the unique solution $X = (X^u)_u \in \mathcal{S}_t$ to the equation (2.2) satisfies the following: there exists a constant $C \geq 0$, depending on T, $\lambda(U)$ and on the constants L, M in Assumption 2.1, such that

$$\int_U \mathbb{E}\left[\sup_{s\in[t,T]} |X_s^u|^2\right] \lambda(\mathrm{d}u) \leq C\left(1 + \int_U \mathbb{E}[|\xi^u|^2]\,\lambda(\mathrm{d}u) + \int_U \int_t^T \mathbb{E}[|\alpha_s^u|^2]\,\mathrm{d}s\,\lambda(\mathrm{d}u)\right). \tag{2.6}$$

Finally, if $(X^u)_u$, $(\bar{X}^u)_u$ are solutions corresponding to $\xi, \bar{\xi} \in \mathcal{I}_t$ and we assume that the map

$$u \mapsto \mathcal{L}\big(W_{[0,t]}^u, Z^u, \xi^u, \bar{\xi}^u\big), \tag{2.7}$$

is Borel measurable as a mapping from U to $\mathcal{P}_2(C_{[0,t]}^\ell \times (0, 1) \times \mathbb{R}^d \times \mathbb{R}^d)$ then we have

$$\int_U \mathbb{E}\left[\sup_{s\in[t,T]} |X_s^u - \bar{X}_s^u|^2\right] \lambda(\mathrm{d}u) \leq C \int_U \mathbb{E}[|\xi^u - \bar{\xi}^u|^2]\,\lambda(\mathrm{d}u) \tag{2.8}$$

for a constant C that only depends on the Lipschitz constant L, on T and on $\lambda(U)$.

Proof. We write the proof for the case $b = 0$. We only prove (2.8), the other assertion being proved by similar arguments.

We first note that the joint measurability condition (2.7) allows to apply Theorem 2.6 to the equation satified by the pair $(X^u, \bar{X}^u)$ and to conclude in particular that the map $u \mapsto \mathcal{L}\big(X^u, \bar{X}^u\big)$ is Borel measurable as a mapping from U to $\mathcal{P}_2(C^d_{[t,T]} \times C^d_{[t,T]})$. Subtracting the equations for X and $\bar{X}$, for some constant C (possibly different from line to line) we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{r\in[t,s]} |X^u_r - \bar{X}^u_r|^2\Big] \le C\,\mathbb{E}\Big[|\xi^u - \bar{\xi}^u|^2\Big] \\
&+ C\,\mathbb{E}\int_t^s \Big|\sigma\left(u, X^u_r, \alpha^u_r, \mathbb{P}_{X^\cdot_r}, \mathbb{P}_{\alpha^\cdot_r}\right) - \sigma\left(u, \bar{X}^u_r, \alpha^u_r, \mathbb{P}_{\bar{X}^\cdot_r}, \mathbb{P}_{\alpha^\cdot_r}\right)\Big|^2 \mathrm{d}r \\
&\le C\,\mathbb{E}\Big[|\xi^u - \bar{\xi}^u|^2\Big] + C\int_t^s \Big\{\mathbb{E}\,[|X^u_r - \bar{X}^u_r|^2] + \mathbf{d}\Big(\mathbb{P}_{X^\cdot_r}, \mathbb{P}_{\bar{X}^\cdot_r}\Big)^2\Big\}\,\mathrm{d}r.
\end{aligned}$$

Since

$$\mathbf{d}\Big(\mathbb{P}_{X^\cdot_r}, \mathbb{P}_{\bar{X}^\cdot_r}\Big)^2 \le \int_U \mathbb{E}\Big[|X^u_r - \bar{X}^u_r|^2\Big]\,\lambda(\mathrm{d}u)$$

integrating with respect to $\lambda(\mathrm{d}u)$ we obtain

$$\begin{aligned}
&\int_U \mathbb{E}\Big[\sup_{r\in[t,s]} |X^u_r - \bar{X}^u_r|^2\Big]\,\lambda(\mathrm{d}u) \\
&\le C\int_U \mathbb{E}\Big[|\xi^u - \bar{\xi}^u|^2\Big]\,\lambda(\mathrm{d}u) + C\int_t^s \int_U \mathbb{E}\,[\sup_{q\in[t,r]} |X^u_q - \bar{X}^u_q|^2]\,\lambda(\mathrm{d}u)\,\mathrm{d}r
\end{aligned}$$

and (2.8) follows from the Gronwall lemma. □

Remark 2.8. If the condition (iii) in Assumption 2.1 is strengthened to

$$|b(u,x,a,\mu,\nu)| + |\sigma(u,x,a,\mu,\nu)| \;\le\; M\big(1 + |x| + \mathbf{d}(\mu,\delta_0)\big) \tag{2.9}$$

for every $u \in U$, $x \in \mathbb{R}^d$, $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, $\nu \in L^2_\lambda(\mathcal{P}_2(A))$, $a \in A$, then we get a stronger estimate than (2.6), namely

$$\int_U \mathbb{E}\Big[\sup_{s\in[t,T]} |X^u_s|^2\Big]\,\lambda(\mathrm{d}u) \le\; C\Big(1 + \int_U \mathbb{E}[|\xi^u|^2]\,\lambda(\mathrm{d}u)\Big).$$

2.3. The control problem

We make the following assumptions on the running and terminal cost functions.

Assumption 2.9. The functions

$$f : U \times \mathbb{R}^d \times A \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \times L^2_\lambda(\mathcal{P}_2(A)) \to \mathbb{R}, \qquad g : U \times \mathbb{R}^d \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \to \mathbb{R}$$

are Borel measurable. In addition, we assume that Condition (2.9) holds and there exists a constant $M \ge 0$ such that

$$|f(u,x,a,\mu,\nu)| + |g(u,x,\mu)| \le\; M\,\big(1 + |x|^2 + \mathbf{d}(\mu,\delta_0)^2\big),$$

for every $u \in U$, $a \in A$, $x \in \mathbb{R}^d$, $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, $\nu \in L^2_\lambda(\mathcal{P}_2(A))$.

From the measurability of the map $u \mapsto \mathcal{L}(X^u, W^u, Z^u)$ for the solution $X = X^{t,\xi,\alpha} = (X^{t,\xi,\alpha,u})_u \in \mathcal{S}_t$ to (2.2), given $t \in [0,T]$, $\xi \in \mathcal{I}_t$, $\alpha \in \mathcal{A}$, and under Assumption 2.9 on f, g, together with the square integrability conditions of X in $\mathcal{S}_t$, we see that the map

$$u \mapsto \mathbb{E}\Big[\int_t^T f(u, X_s^u, \alpha_s^u, \mathbb{P}_{X_s^\cdot}, \mathbb{P}_{\alpha_s^\cdot})\,\mathrm{d}s + g(u, X_T^u, \mathbb{P}_{X_T^\cdot})\Big]$$

is Borel measurable, and we can then define the cost functional to be minimized

$$J(t,\xi,\alpha) = \int_U \mathbb{E}\Big[\int_t^T f\big(u, X_s^u, \alpha_s^u, \mathbb{P}_{X_s^\cdot}, \mathbb{P}_{\alpha_s^\cdot}\big)\,\mathrm{d}s + g\big(u, X_T^u, \mathbb{P}_{X_T^\cdot}\big)\Big]\,\lambda(\mathrm{d}u)$$

and the associated value function:

$$V(t,\xi) = \inf_{\alpha \in \mathcal{A}} J(t,\xi,\alpha), \qquad t \in [0,T],\ \xi \in \mathcal{I}_t.$$

Moreover using Proposition 2.7, and under condition (2.9) (see Rem. 2.8), we have

$$|V(t,\xi)| \le C\Big(1 + \int_U \mathbb{E}[|\xi^u|^2]\,\lambda(\mathrm{d}u)\Big), \quad t \in [0,T],\ \xi \in \mathcal{I}_t. \tag{2.10}$$

3. Law invariance of the value function and DPP

3.1. Law invariance

We show in this section the law invariance property of the value function, namely that $V(t,\xi)$ depends on ξ only through its law. We impose additional assumption on the functions f, g.

Assumption 3.1. There exist constants $K \ge 0$, $\gamma_i \in (0,1]$ $(i = 1,2,3,4)$ such that

$$\begin{aligned} &|f(u,x,a,\mu,\nu) - f(u,x',a,\mu,\nu)| \\ &\quad \le K\Big(|x-x'|^{\gamma_1}\,(1+|x|+|x'|)^{2-\gamma_1} + \mathbf{d}(\mu,\mu)^{\gamma_2}\,(1+\mathbf{d}(\mu,\delta_0)+\mathbf{d}(\mu,\delta_0))^{2-\gamma_2}\Big), \end{aligned}$$

$$\begin{aligned} &|g(u,x,\mu) - g(u,x',\mu)| \\ &\quad \le K\Big(|x-x'|^{\gamma_3}\,(1+|x|+|x'|)^{2-\gamma_3} + \mathbf{d}(\mu,\mu)^{\gamma_4}\,(1+\mathbf{d}(\mu,\delta_0)+\mathbf{d}(\mu,\delta_0))^{2-\gamma_4}\Big), \end{aligned}$$

for every $u \in U$, $x, x' \in \mathbb{R}^d$, $\mu, \mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, $\nu \in L^2_\lambda(\mathcal{P}_2(A))$, $a \in A$.

Remark 3.2. The above assumption on f, g are local Hölder dependence on x, μ, uniformly with respect to a, ν, u.

Lemma 3.3. *Let Assumptions 2.1, 2.9 hold. Suppose that $\xi = (\xi^u)_u$, $\bar\xi = (\bar\xi^u)_u \in \mathcal{I}_t$ for some $t \in [0,T]$ are such that*

$$\mathcal{L}(\xi^u, W^u_{[0,t]}, Z^u) = \mathcal{L}(\bar\xi^u, W^u_{[0,t]}, Z^u) \tag{3.1}$$

for λ-almost every $u \in U$. Then we have for any $\alpha \in \mathcal{A}$,

$$J(t,\xi,\alpha) = J(t,\bar\xi,\alpha).$$

Proof. We first notice by Proposition A.1 under condition (3.1) that

$$\mathcal{L}(X^u, W^u, Z^u) = \mathcal{L}(\bar{X}^u, W^u, Z^u)$$

for λ-almost every $u \in U$, where $(X^u)_u$ and $(\bar{X}^u)_u$ are the respective solutions to (2.2), with control α, initial time t and initial condition ξ and $\bar{\xi}$. In particular we get

$$\mathcal{L}(X^u, \alpha^u) = \mathcal{L}(\bar{X}^u, \alpha^u)$$

for λ-almost every $u \in U$. As $J(t, \xi, \alpha)$ and $J(t, \bar{\xi}, \alpha)$ are expectations of measurable functions of the (X, α) and $(\bar{X}, \alpha)$ respectively, we get the result. □

Theorem 3.4. *(law invariance). Let Assumptions 2.1, 2.9, 3.1 hold and fix $\xi = (\xi^u)_u$, $\bar{\xi} = (\bar{\xi}^u)_u \in \mathcal{I}_t$ for some $t \in [0,T]$. If $\mathbb{P}_{\xi^u} = \mathbb{P}_{\bar{\xi}^u}$ for λ-almost every $u \in U$ then $V(t, \xi) = V(t, \bar{\xi})$.*

Proof. We write the proof in the case $b \equiv 0$, the general case being completely similar. We first notice that from Lemma 3.3 and Proposition B.1, we can assume w.l.o.g. that the map

$$u \mapsto \mathcal{L}\big(W^u_{[0,t]}, Z^u, \xi^u, \bar{\xi}^u\big) \ \text{ is Borel measurable} \tag{3.2}$$

as a mapping from U to $\mathcal{P}_2(C^\ell_{[0,t]} \times (0,1) \times \mathbb{R}^d \times \mathbb{R}^d)$. As a matter of fact, we first use Proposition B.1 which gives Borel maps $\tilde{\xi}, \tilde{\bar{\xi}} : U \times C^\ell_{[0,t]} \times (0,1) \to \mathbb{R}^d$ such that

$$\begin{aligned}
\mathcal{L}\big(\tilde{\xi}^u(W^u_{[0,t]}, Z^u), W^u_{[0,t]}, Z^u\big) &= \mathcal{L}\big(\xi^u, W^u_{[0,t]}, Z^u\big),\\
\mathcal{L}\big(\tilde{\bar{\xi}}^u(W^u_{[0,t]}, Z^u), W^u_{[0,t]}, Z^u\big) &= \mathcal{L}\big(\bar{\xi}^u, W^u_{[0,t]}, Z^u\big)
\end{aligned}$$

for every $u \in U$. Then using Lemma 3.3, we get

$$\begin{aligned}
J(t, \xi, \alpha) &= J(t, (\tilde{\xi}^u(W^u_{[0,t]}, Z^u))_u, \alpha)\\
J(t, \bar{\xi}, \alpha) &= J(t, (\tilde{\bar{\xi}}^u(W^u_{[0,t]}, Z^u))_u, \alpha)
\end{aligned}$$

for any $\alpha \in \mathcal{A}$ and

$$\begin{aligned}
V(t, \xi,) &= V(t, (\tilde{\xi}^u(W^u_{[0,t]}, Z^u))_u)\\
V(t, \bar{\xi}) &= V(t, (\tilde{\bar{\xi}}^u(W^u_{[0,t]}, Z^u))_u).
\end{aligned}$$

We can therefore replace $(\xi, \bar{\xi})$ by $(\tilde{\xi}^u(W^u_{[0,t]}, Z^u))_u, \tilde{\bar{\xi}}^u(W^u_{[0,t]}, Z^u)_u)$ which satisfies (3.2) as $\tilde{\xi}$ and $\tilde{\bar{\xi}}$ are Borel measurable.

We consider, for fixed $t \in [0,T]$, $\xi \in \mathcal{I}_t$ and $\alpha \in \mathcal{A}$, the system:

$$\begin{cases}
\mathrm{d}X^u_s = \sigma\left(u, X^u_s, \alpha^u_s, \mathbb{P}_{X^{\cdot}_s}, \mathbb{P}_{\alpha^{\cdot}_s}\right) \mathrm{d}W^u_s, \ \ s \in [t,T],\\
X^u_t = \xi^u, \quad u \in U,\\
\alpha^u_s = \alpha(u, s, W^u_{\cdot\wedge s}, Z^u),
\end{cases}$$

and the corresponding cost

$$J(t, \xi, \alpha) = \int_U \mathbb{E}\Big[\int_t^T f(u, X^u_s, \alpha^u_s, \mathbb{P}_{X^{\cdot}_s}, \mathbb{P}_{\alpha^{\cdot}_s})\, \mathrm{d}s + g(u, X^u_T, \mathbb{P}_{X^{\cdot}_T})\Big]\, \lambda(\mathrm{d}u).$$

For every $\epsilon > 0$ we will find another control policy $\alpha^\epsilon \in \mathcal{A}$ such that $J(t, \alpha^\epsilon, \bar{\xi}) \to J(t, \alpha, \xi)$ as $\epsilon \to 0$. This way the required equality $V(t, \xi) = V(t, \bar{\xi})$ will be proved.

We first look for a convenient expression for $J(t, \alpha, \xi)$. We recall that for every u the random variable ξ^u is $\mathcal{F}^u_t$-measurable, so it is $\mathbb{P}$-almost surely equal to a variable of the form $\underline{\xi}^u(W^u_{[0,t]}, Z^u)$ for a function $\underline{\xi}^u(w, z) \in \mathbb{R}^d$ defined for $w \in C^\ell_{[0,t]}$, $z \in (0, 1)$ and measurable in (w, z) (not necessarily in u). Recalling the notation in (A.4)-(A.5) we rewrite the controlled equation in the equivalent way:

$$\begin{cases} \mathrm{d}X^u_s = \sigma\big(u, X^u_s, \alpha^u_s, \mathbb{P}_{X^\cdot_s}, \mathbb{P}_{\alpha^\cdot_s}\big)\mathrm{d}W^u_s, \quad s \in [t, T], \\ X^u_t = \underline{\xi}^u(W^u_{[0,t]}, Z^u), \\ \alpha^u_s = \tilde{\alpha}(u, s, W^u_{\cdot\wedge s}, W^u_{[0,t]}, Z^u). \end{cases}$$

For every u the random variable $\bar{\xi}^u$, being also $\mathcal{F}^u_t$-measurable, is $\mathbb{P}$-almost surely equal to $\underline{\bar{\xi}}^u(W^u_{[0,t]}, Z^u)$ for a measurable function $\underline{\bar{\xi}}^u : C^\ell_{[0,t]} \times (0, 1) \to \mathbb{R}^d$. By our assumptions we have, for λ-almost every u,

$$\mathcal{L}\left(\underline{\xi}^u(W^u_{[0,t]}, Z^u)\right) = \mathbb{P}_{\xi^u} = \mathbb{P}_{\bar{\xi}^u} = \mathcal{L}\left(\underline{\bar{\xi}}^u(W^u_{[0,t]}, Z^u)\right)$$

Since $\mathcal{L}(W^u_{[0,t]}, Z^u) = \mathbb{W}_T \otimes m$, it follows that $\underline{\xi}^u(w, z) = \underline{\bar{\xi}}^u(w, z)$ for almost all (w, z) with respect to $\mathbb{W}_T \otimes m$, which is a non-atomic measure on the Polish space $C^\ell_{[0,t]} \times (0, 1)$. By a classical result (see *e.g.* [3], Lem. 5.23) for every $\epsilon > 0$ there exists a Borel measurable map

$$\tau^{\epsilon,u} : C^\ell_{[0,t]} \times (0, 1) \to C^\ell_{[0,t]} \times (0, 1)$$

that preserves the measure $\mathbb{W}_T \otimes m$ and satisfies, for λ-almost all u,

$$|\underline{\xi}^u(\tau^{\epsilon,u}(w, z)) - \underline{\bar{\xi}}^u(w, z)| \le \epsilon, \qquad (w, z) \in C^\ell_{[0,t]} \times (0, 1),\ \mathbb{W}_T \otimes m - a.s. \tag{3.3}$$

Denote $(w, z) \mapsto \tau^{\epsilon,u}_1(w, z) \in C^\ell_{[0,t]}$ and $(w, z) \mapsto \tau^{\epsilon,u}_2(w, z) \in (0, 1)$ the coordinate maps of $\tau^{\epsilon,u} = (\tau^{\epsilon,u}_1, \tau^{\epsilon,u}_2)$. Using the independence of $(W^u_{[0,t]}, Z^u)$ and $W^u_{[t,T]} - W^u_t$, and the measure-preserving property of $\tau^{\epsilon,u}$ it is easy to check that the pair

$$W^{\epsilon,u} := \tau^{\epsilon,u}_1(W^u_{[0,t]}, Z^u) \oplus W^u_{[t,T]}, \quad Z^{\epsilon,u} := \tau^{\epsilon,u}_2(W^u_{[0,t]}, Z^u)$$

consists of a Wiener process on $[0, T]$ and an independent random variable with uniform distribution in $(0, 1)$. Then we consider the equation

$$\begin{cases} \mathrm{d}X^{\epsilon,u}_s = \sigma\big(u, X^{\epsilon,u}_s, \alpha^{\epsilon,u}_s, \mathbb{P}_{X^{\epsilon,\cdot}_s}, \mathbb{P}_{\alpha^{\epsilon,\cdot}_s}\big)\mathrm{d}W^u_s, \quad s \in [t, T], \\ X^{\epsilon,u}_t = \underline{\xi}^u(W^{\epsilon,u}_{[0,t]}, Z^{\epsilon,u}), \\ \alpha^{\epsilon,u}_s = \tilde{\alpha}(u, s, W^u_{\cdot\wedge s}, W^{\epsilon,u}_{[0,t]}, Z^{\epsilon,u}). \end{cases} \tag{3.4}$$

Since the increments of W^u and $W^{\epsilon,u}$ coincide on the interval $[t, T]$, in the above equations $\mathrm{d}W^u_s$ might be replaced by $\mathrm{d}W^{\epsilon,u}_s$ and $W^u_{\cdot\wedge s}$ by $W^{\epsilon,u}_{\cdot\wedge s}$. Then we see that the process $X^{\epsilon,u}$ is the trajectory corresponding to the control policy $\alpha \in \mathcal{A}$, the initial condition $\underline{\xi}^u(w, z)$ and the driving noise $(W^{\epsilon,u}, Z^{\epsilon,u})$. From Proposition A.1, it follows that $\mathcal{L}\left(X^{\epsilon,u}, W^{\epsilon,u}_{[0,T]}, Z^{\epsilon,u}\right) = \mathcal{L}\left(X^u, W^u_{[0,T]}, Z^u\right)$ which implies

$$\mathcal{L}\left(X^{\epsilon,u}_s, \alpha^{\epsilon,u}_s\right) = \mathcal{L}\left(X^u_s, \alpha^u_s\right)$$

for every $s \in [t,T]$ and $u \in U$ and finally

$$J(t,\xi,\alpha) = \int_U \mathbb{E}\Big[\int_t^T f\big(u, X_s^{\epsilon,u}, \alpha_s^{\epsilon,u}, \mathbb{P}_{X_s^{\epsilon,\cdot}}, \mathbb{P}_{\alpha_s^{\epsilon,\cdot}}\big)\,\mathrm{d}s + g\big(u, X_T^{\epsilon,u}, \mathbb{P}_{X_T^{\epsilon,\cdot}}\big)\Big]\,\lambda(\mathrm{d}u),$$

which is the expression we were looking for.

Next we consider the equation

$$\begin{cases} \mathrm{d}\bar{X}_s^{\epsilon,u} &= \sigma\big(u, \bar{X}_s^{\epsilon,u}, \alpha_s^{\epsilon,u}, \mathbb{P}_{\bar{X}_s^{\epsilon,\cdot}}, \mathbb{P}_{\alpha_s^{\varepsilon,\cdot}}\big)\mathrm{d}W_s^u, \quad s \in [t,T], \\ \bar{X}_t^{\epsilon,u} &= \bar{\xi}^u = \underline{\bar{\xi}}^u(W_{[0,t]}^u, Z^u), \\ \alpha_s^{\epsilon,u} &= \tilde{\alpha}(u, s, W_{\cdot\wedge s}^{\epsilon,u}, W_{[0,t]}^{\epsilon,u}, Z^{\epsilon,u}) \\ &= \tilde{\alpha}(u, s, W_{\cdot\wedge s}^u, \tau_1^{\epsilon,u}(W_{[0,t]}^u, Z^u), \tau_2^{\epsilon,u}(W_{[0,t]}^u, Z^u)). \end{cases} \tag{3.5}$$

The process $X^{\epsilon,u}$ starts at $\bar{\xi}^u$ and the equation contains the same control processes $\alpha^{\epsilon,u}$ as in (3.4), but it is now driven by the original noise (W^u, Z^u). We see that the process $X^{\epsilon,u}$ is the trajectory corresponding to the control policy

$$\alpha^\epsilon(u,s,w,z) := \alpha\Big(u, s, \tau_1^{\epsilon,u}(w_{[0,t]}, z) \oplus w_{[t,T]}, \tau_2^{\epsilon,u}(w_{[0,t]}, z)\Big), \qquad s \in [t,T],$$

while we may take both α and α^ϵ to be constant for $s \in [0,t)$, without loss of generality. To check that α^ϵ is indeed admissible, using the fact that $(W^{\epsilon,u}, Z^{\epsilon,u})$ and (W^u, Z^u) have the same law, we verify that

$$\begin{aligned} \int_U \int_t^T \mathbb{E}[|\alpha_s^{\epsilon,u}|^2]\,\mathrm{d}s\,\lambda(\mathrm{d}u) &= \int_U \int_t^T \mathbb{E}[|\tilde{\alpha}(u, s, W_{\cdot\wedge s}^{\epsilon,u}, W_{[0,t]}^{\epsilon,u}, Z^{\epsilon,u})|^2]\,\mathrm{d}s\,\lambda(\mathrm{d}u) \\ &= \int_U \int_t^T \mathbb{E}[|\tilde{\alpha}(u, s, W_{\cdot\wedge s}^u, W_{[0,t]}^u, Z^u)|^2]\,\mathrm{d}s\,\lambda(\mathrm{d}u) \\ &= \int_U \int_t^T \mathbb{E}[|\alpha_s^u|^2]\,ds\,\lambda(\mathrm{d}u), \end{aligned} \tag{3.6}$$

which is finite and even independent of ϵ. The corresponding cost is then

$$J(t,\bar{\xi},\alpha^\epsilon) = \int_U \mathbb{E}\Big[\int_t^T f\big(u, \bar{X}_s^{\epsilon,u}, \alpha_s^{\epsilon,u}, \mathbb{P}_{\bar{X}_s^{\epsilon,\cdot}}, \mathbb{P}_{\alpha_s^{\epsilon,\cdot}}\big)\,\mathrm{d}s + g\big(u, \bar{X}_T^{\epsilon,u}, \mathbb{P}_{\bar{X}_T^{\epsilon,\cdot}}\big)\Big]\,\lambda(\mathrm{d}u).$$

To conclude the proof it remains to prove that $J(t,\bar{\xi},\alpha^\epsilon) \to J(t,\xi,\alpha)$ as $\epsilon \to 0$. Comparing (3.4) and (3.5) and applying estimate (2.8) in Proposition 2.7 we see that there exists a constant $C \geq 0$, depending only on T and the Lipschitz constants of b, σ such that

$$\int_U \mathbb{E}\Big[\sup_{s\in[t,T]} |X_s^{\epsilon,u} - \bar{X}_s^{\epsilon,u}|^2\Big]\,\lambda(\mathrm{d}u) \leq C \int_U \mathbb{E}\Big[|\underline{\xi}^u(W_{[0,t]}^{\epsilon,u}, Z^{\epsilon,u}) - \bar{\xi}^u|^2\Big]\,\lambda(\mathrm{d}u).$$

But we have

$$\Big|\underline{\xi}^u(W_{[0,t]}^{\epsilon,u}, Z^{\epsilon,u}) - \bar{\xi}^u\Big| = \Big|\underline{\xi}^u(\tau^{\epsilon,u}(W_{[0,t]}^u, Z^u)) - \underline{\bar{\xi}}^u(W_{[0,t]}^u, Z^u)\Big| \leq \epsilon$$

which follows from (3.3) and the fact that $(W^u_{[0,t]}, Z^u)$ has law $\mathbb{W}_T \otimes m$. So we obtain

$$\int_U \mathbb{E}\Big[\sup_{s\in[t,T]} |X^{\epsilon,u}_s - \bar{X}^{\epsilon,u}_s|^2\Big]\,\lambda(\mathrm{d}u) \le C\epsilon^2\,\lambda(U). \tag{3.7}$$

Still using Proposition 2.7, we also note that

$$\begin{aligned}
&\int_U \mathbb{E}\Big[\sup_{s\in[t,T]} |X^{\epsilon,u}_s|^2\Big]\,\lambda(\mathrm{d}u) + \int_U \mathbb{E}\Big[\sup_{s\in[t,T]} |\bar{X}^{\epsilon,u}_s|^2\Big]\,\lambda(\mathrm{d}u) \\
&\le C\left(\int_U \mathbb{E}[|\xi^u|^2]\,ds\,\lambda(\mathrm{d}u) + \int_U \mathbb{E}[|\xi^u|^2]\,ds\,\lambda(\mathrm{d}u) + \int_U\int_t^T \mathbb{E}[|\alpha^{\epsilon,u}_s|^2]\,ds\,\lambda(\mathrm{d}u)\right) \\
&\le C,
\end{aligned} \tag{3.8}$$

for some constant C which is not dependent on ϵ, by (3.6). Using our assumptions on f, g we have

$$\begin{aligned}
&|J(t,\xi,\alpha) - J(t,\bar{\xi},\alpha^\epsilon)| \\
&\le C\int_U\int_t^T \mathbb{E}\Big[|X^{\epsilon,u}_s - \bar{X}^{\epsilon,u}_s|^{\gamma_1}\,(1 + |X^{\epsilon,u}_s| + |\bar{X}^{\epsilon,u}_s|)^{2-\gamma_1}\Big]\,\mathrm{d}s\,\lambda(\mathrm{d}u) \\
&\quad + C\int_U\int_t^T \Big(\mathbf{d}(\mathbb{P}_{X^{\epsilon,\cdot}_s}, \mathbb{P}_{\bar{X}^{\epsilon,\cdot}_s})^{\gamma_2}\,(1 + \mathbf{d}(\mathbb{P}_{X^{\epsilon,\cdot}_s}, \delta_0) + \mathbf{d}(\mathbb{P}_{\bar{X}^{\epsilon,\cdot}_s}, \delta_0))^{2-\gamma_2}\Big)\,\mathrm{d}s\,\lambda(\mathrm{d}u) \\
&\quad + C\int_U \mathbb{E}\Big[|X^{\epsilon,u}_T - \bar{X}^{\epsilon,u}_T|^{\gamma_3}\,(1 + |X^{\epsilon,u}_T| + |\bar{X}^{\epsilon,u}_T|)^{2-\gamma_3}\Big]\,\lambda(\mathrm{d}u) \\
&\quad + C\int_U \Big(\mathbf{d}(\mathbb{P}_{X^{\epsilon,\cdot}_T}, \mathbb{P}_{\bar{X}^{\epsilon,\cdot}_T})^{\gamma_4}\,(1 + \mathbf{d}(\mathbb{P}_{X^{\epsilon,\cdot}_T}, \delta_0) + \mathbf{d}(\mathbb{P}_{\bar{X}^{\epsilon,\cdot}_T}, \delta_0))^{2-\gamma_4}\Big)\,\lambda(\mathrm{d}u).
\end{aligned}$$

The Hölder inequality, with conjugate exponents $2/\gamma_1$ and $2/(2-\gamma_1)$, gives

$$\begin{aligned}
&\int_U\int_t^T \mathbb{E}\Big[|X^{\epsilon,u}_s - \bar{X}^{\epsilon,u}_s|^{\gamma_1}\,(1 + |X^{\epsilon,u}_s| + |\bar{X}^{\epsilon,u}_s|)^{2-\gamma_1}\Big]\,\mathrm{d}s\,\lambda(\mathrm{d}u) \\
&\le \int_U\int_t^T \left\{\Big(\mathbb{E}[|X^{\epsilon,u}_s - \bar{X}^{\epsilon,u}_s|^2]\Big)^{\frac{\gamma_1}{2}}\,\Big(\mathbb{E}[(1 + |X^{\epsilon,u}_s| + |\bar{X}^{\epsilon,u}_s|)^2]\Big)^{\frac{2-\gamma_1}{2}}\right\}\,\mathrm{d}s\,\lambda(\mathrm{d}u) \\
&\le T\int_U \left\{\Big(\mathbb{E}[\sup_{s\in[t,T]} |X^{\epsilon,u}_s - \bar{X}^{\epsilon,u}_s|^2]\Big)^{\frac{\gamma_1}{2}}\,\Big(\mathbb{E}[\sup_{s\in[t,T]} (1 + |X^{\epsilon,u}_s| + |\bar{X}^{\epsilon,u}_s|)^2]\Big)^{\frac{2-\gamma_1}{2}}\right\}\,\lambda(\mathrm{d}u) \\
&\le T\left\{\int_U \mathbb{E}[\sup_{s\in[t,T]} |X^{\epsilon,u}_s - \bar{X}^{\epsilon,u}_s|^2]\,\lambda(\mathrm{d}u)\right\}^{\frac{\gamma_1}{2}}\left\{\int_U \mathbb{E}[\sup_{s\in[t,T]} (1 + |X^{\epsilon,u}_s| + |\bar{X}^{\epsilon,u}_s|)^2]\,\lambda(\mathrm{d}u)\right\}^{\frac{2-\gamma_1}{2}} \\
&\le C\left\{\epsilon^2\,\lambda(U)\right\}^{\gamma_1/2}
\end{aligned}$$

by (3.7) and (3.8). Using the inequality

$$\mathbf{d}(\mathbb{P}_{X^{\epsilon,\cdot}_s}, \mathbb{P}_{\bar{X}^{\epsilon,\cdot}_s})^2 = \int_U \mathcal{W}_2(\mathbb{P}_{X^{\epsilon,u}_s}, \mathbb{P}_{\bar{X}^{\epsilon,u}_s})^2\,\lambda(\mathrm{d}u) \le \int_U \mathbb{E}[|X^{\epsilon,u}_s - \bar{X}^{\epsilon,u}_s|^2]\,\lambda(\mathrm{d}u)$$

similar passages and the Hölder inequality allow to treat the other terms and conclude that $J(t,\xi,\alpha) - J(t,\bar{\xi},\alpha^\epsilon) \to 0$. □

In view of Theorem 3.4 it is possible to define a function $\upsilon : [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \to \mathbb{R}$ as follows. For any $t \in [0,T]$ and $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, let us choose $\xi = (\xi^u)_u \in \mathcal{I}_t$ such that $\mathbb{P}_{\xi^u} = \mu^u$ for λ-almost every $u \in U$, and let us set

$$\upsilon(t,\mu) = V(t,\xi). \tag{3.9}$$

Finally we show how a required variable ξ can be constructed, given arbitrary t and μ. We note that the map $(u,A) \mapsto \mu^u(A)$ defined for $u \in U$ and any Borel set of $\mathbb{R}^d$ is a transition kernel. By a known extension of the Skorohod construction, there exists a (jointly) measurable function $j : U \times (0,1) \to \mathbb{R}^d$ such that, for every u, the image of the Lebesgue measure on $(0,1)$ under the map $z \mapsto j(u,z)$ equals μ^u; therefore we may define $\xi^u = j(u, Z^u)$. This way we obtain the required function $\underline{\xi}^u(w,z) = j(u,z)$ (that does not depend on w).

3.2. Dynamic programming principle

We note that for the solution $X_s^{t,\xi,\alpha,u}$ to equation (2.2) we have the following flow property: for $0 \le t \le \theta \le T$, $\alpha \in \mathcal{A}$, and $\xi \in \mathcal{I}_t$ we have, for every λ-a.e. $u \in U$, $\mathbb{P}$-a.s.

$$X_s^{t,\xi,\alpha,u} = X_s^{\theta, X_\theta^{t,\xi,\alpha},\alpha,u}, \qquad s \in [\theta, T]. \tag{3.10}$$

This follows immediately from the uniqueness statement in Theorem 2.6, since both terms are solution to the state equation (2.2) in the interval $[\theta,T]$. As a consequence, we can state the dynamic programming principle for the value function V.

Theorem 3.5. *For $t \in [0,T]$ and $\xi \in \mathcal{I}_t$, we have*

$$V(t,\xi) = \inf_{\alpha \in \mathcal{A}} \Big\{ \int_U \mathbb{E}\Big[\int_t^\theta f\big(u, X_s^{t,\xi,\alpha,u}, \mathbb{P}_{X_s^{t,\xi,\alpha,\cdot}}, \alpha_s^u, \mathbb{P}_{\alpha_s^\cdot}\big)\, \mathrm{d}s \Big] \lambda(\mathrm{d}u) + V\Big(\theta, (X_\theta^{t,\xi,\alpha,u})_u\Big) \Big\},$$

for any $\theta \in [t,T]$.

Proof. Fix $0 \le t \le \theta \le T$ and $\xi \in \mathcal{I}_t$. For $\alpha \in \mathcal{A}$ we have from (3.10) and the definitions of the cost functional J, and the value function V

$$\begin{aligned} J(t,\xi,\alpha) &= \int_U \mathbb{E}\Big[\int_t^\theta f(u, X_s^{t,\xi,\alpha,u}, \alpha_s^u, \mathbb{P}_{X_s^{t,\xi,\alpha,\cdot}}, \mathbb{P}_{\alpha_s^\cdot})\, \mathrm{d}s \Big] \lambda(\mathrm{d}u) \\ &\quad + \int_U \mathbb{E}\Big[\int_\theta^T f\big(u, X_s^{u,\theta,X_\theta^{t,\xi,\alpha},\alpha}, \alpha_s^u, \mathbb{P}_{X_s^{\theta,X_\theta^{t,\xi,\alpha},\alpha,\cdot}}, \mathbb{P}_{\alpha_s^\cdot}\big)\, \mathrm{d}s \\ &\quad + g\big(u, X_T^{\theta,X_\theta^{t,\xi,\alpha},\alpha,u}, \mathbb{P}_{X_T^{\theta,X_\theta^{t,\xi,\alpha},\alpha,\cdot}}\big)\Big] \lambda(\mathrm{d}u) \\ &\ge \int_U \mathbb{E}\left[\int_t^\theta f\big(u, X_s^{t,\xi,\alpha,u}, \alpha_s^u, \mathbb{P}_{X_s^{t,\xi,\alpha,\cdot}}, \mathbb{P}_{\alpha_s^\cdot}\big)\, \mathrm{d}s \right] \lambda(\mathrm{d}u) + V\left(\theta, (X_s^{t,\xi,\alpha,u})_u\right). \end{aligned}$$

Since the previous inequality holds for any $\alpha \in \mathcal{A}$, we get

$$V(t,\xi) \ge \inf_{\alpha \in \mathcal{A}} \left\{ \int_U \mathbb{E}\left[\int_t^\theta f\big(u, X_s^{t,\xi,\alpha,u}, \alpha_s^u, \mathbb{P}_{X_s^{t,\xi,\alpha,\cdot}}, \mathbb{P}_{\alpha_s^\cdot}\big)\, \mathrm{d}s \right] \lambda(\mathrm{d}u) + V\left(\theta, (X_\theta^{t,\xi,\alpha,u})_u\right) \right\}.$$

We turn to the reverse inequality. Fix $\alpha \in \mathcal{A}$ and $\varepsilon > 0$. From the definition of the value function V, there exists some $\alpha^\varepsilon = (\alpha^{\varepsilon,u})_u \in \mathcal{A}$ such that

$$J\big(\theta, (X_\theta^{t,\xi,\alpha,u})_u, \alpha^\varepsilon\big) \leq \; V\big(\theta, (X_\theta^{t,\xi,\alpha,u})_u\big) + \varepsilon \tag{3.11}$$

We next define the control $\bar{\alpha}^\varepsilon = (\bar{\alpha}^{\varepsilon,u})_u$ by

$$\bar{\alpha}_s^{\varepsilon,u} = \alpha_s^u \mathbb{1}_{[t,\theta)}(s) + \alpha_s^{\varepsilon,u}\mathbb{1}_{[\theta,T]}(s), \quad s \in [0,T],$$

for $u \in U$ and $s \in [t,T]$. We obviously have $\bar{\alpha}^\varepsilon \in \mathcal{A}$. Moreover, using the flow property (3.10), we have

$$J(t,\xi,\bar{\alpha}^\varepsilon) = \int_U \mathbb{E}\Big[\int_t^\theta f(u, X_s^{t,\xi,\alpha,u}, \alpha_s^u, \mathbb{P}_{X_s^{t,\xi,\alpha,\cdot}}, \mathbb{P}_{\alpha_s^\cdot})\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u) + J\big(\theta, (X_\theta^{t,\xi,\alpha,u})_u, \alpha^\varepsilon\big).$$

From (3.11), we get

$$\begin{aligned} J(t,\xi,\bar{\alpha}^\varepsilon) \leq& \int_U \mathbb{E}\Big[\int_t^\theta f(u, X_s^{t,\xi,\alpha,u}, \alpha_s^u, \mathbb{P}_{X_s^{t,\xi,\alpha,\cdot}}, \mathbb{P}_{\alpha_s^\cdot})\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u) \\ &+ V\left(\theta, (X_\theta^{t,\xi,\alpha,u})_u\right) + \varepsilon, \end{aligned}$$

and so

$$V(t,\xi) \leq \int_U \mathbb{E}\Big[\int_t^\theta f\big(u, X_s^{t,\xi,\alpha,u}, \alpha_s^u, \mathbb{P}_{X_s^{t,\xi,\alpha,\cdot}}, \mathbb{P}_{\alpha_s^\cdot}\big)\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u) + V\big(\theta, (X_\theta^{t,\xi,\alpha,u})_u\big) + \varepsilon.$$

Taking the infimum over $\alpha \in \mathcal{A}$ and sending ε to 0, we get the result. □

The following corollary is an immediate consequence of this result and Theorem 3.4.

Corollary 3.6. *For $t \in [0,T]$ and $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ we have*

$$\begin{aligned} v(t,\mu) = \inf_{\alpha\in\mathcal{A}} \Big\{ &\int_U \mathbb{E}\Big[\int_t^\theta f\big(u, X_s^{t,\xi,\alpha,u}, \alpha_s^u, \mathbb{P}_{X_s^{t,\xi,\alpha,\cdot}}, \mathbb{P}_{\alpha_s^\cdot}\big)\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u) \\ &+ v\big(\theta, \mathbb{P}_{X_\theta^{t,\xi,\alpha,\cdot}}\big)\Big\} \end{aligned}$$

for any $\theta \in [t,T]$, $\xi \in \mathcal{I}_t$ satisfying $\mathbb{P}_{\xi^u} = \mu^u$ for λ-almost every $u \in U$.

Proof. This follows from the definition of the function v in (3.9). □

4. Itô formula

4.1. Derivatives on square integrable measure maps

In our context, the Itô formula describes the time derivative of the composition of a real function $v(\mu)$ of $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ and a map $s \mapsto \mu_s$ corresponding to the law of a family of stochastic processes $(X^u)_u$, namely $\mu_s^u = \mathbb{P}_{X_s^u}$. It holds under regularity assumptions on v - that may also depend explicitly on time - and requires the definition of derivatives for functions $v : L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \to \mathbb{R}$ that we are now going to introduce. Given $\mu \in$

$L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, and a measurable function $(u,x) \in U \times \mathbb{R}^d \mapsto \varphi(u,x)$, with quadratic growth in x, uniformly in u, we define the duality product:

$$< \varphi, \mu > := \int_U \int_{\mathbb{R}^d} \varphi(u,x) \mu^u(\mathrm{d}x) \lambda(\mathrm{d}u).$$

Definition 4.1. Given a function $v : L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \to \mathbb{R}$, we say that a measurable function

$$\frac{\delta}{\delta m} v : L^2_\lambda(\mathbb{P}^2(\mathbb{R}^d)) \times U \times \mathbb{R}^d \ \ni \ (\mu, u, x) \ \longmapsto \ \frac{\delta}{\delta m} v(\mu)(u,x)$$

is the linear functional derivative of v if

1. for every compact set $K \subset L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ there exists a constant $C_K > 0$ such that
$$\left| \frac{\delta}{\delta m} v(\mu)(u,x) \right| \leq C_K \, (1 + |x|^2),$$
for every $u \in U$, $x \in \mathbb{R}^d$, $\mu \in K$;
2. for every $\mu, \nu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ we have
$$\begin{aligned} v(\nu) - v(\mu) &= \int_0^1 < \frac{\delta}{\delta m} v(\mu + \theta(\nu - \mu))(.), \nu - \mu > \mathrm{d}\theta \\ &= \int_0^1 \int_U \int_{\mathbb{R}^d} \frac{\delta}{\delta m} v(\mu + \theta(\nu - \mu)(u,x) \, (\nu^u - \mu^u)(\mathrm{d}x) \lambda(\mathrm{d}u) \mathrm{d}\theta. \end{aligned}$$

Remark 4.2. Under the additional condition that $\mu \mapsto \frac{\delta}{\delta m} v(\mu)(u,x)$ is continuous, the above definition is equivalent to the existence of the Gateaux-derivative

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{v(\mu + \varepsilon(\nu - \mu)) - v(\mu)}{\varepsilon} &= < \frac{\delta}{\delta m} v(\mu)(.), \nu - \mu > \\ &= \int_U \int_{\mathbb{R}^d} \frac{\delta}{\delta m} v(\mu)(u,x)(\nu^u - \mu^u)(\mathrm{d}x) \lambda(\mathrm{d}u). \end{aligned}$$

The following definition encompasses the minimal assumptions on v that are required to obtain the Itô formula.

Definition 4.3. We say that $v : [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \to \mathbb{R}$ is of class $\tilde{C}^{1,2}([0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ if

1. for every $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ the function $t \mapsto v(t,\mu)$ is continuously differentiable on $[0,T]$; we denote $\partial_t v(t,\mu)$ its time derivative;
2. for every $t \in [0,T]$ the derivative $\frac{\delta}{\delta m} v(t,\mu)(u,x)$ exists and it is measurable in all its arguments;
3. $\frac{\delta}{\delta m} v(t,\mu)(u,x)$ is twice continuously differentiable on $\mathbb{R}^d$ as a function of x and the gradient and the Hessian matrix
$$\partial_x \frac{\delta}{\delta m} v : [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \times U \times \mathbb{R}^d \to \mathbb{R}^d, \qquad \partial^2_x \frac{\delta}{\delta m} v : [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \times U \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$$
satisfy the following growth conditions: there exists a constant $C \geq 0$ such that
$$\left| \partial_x \frac{\delta}{\delta m} v(t,\mu)(u,x) \right| \leq C \left(1 + |x| + \mathbf{d}(\mu, \delta_0) \right), \tag{4.1}$$

$$\left|\partial_x^2 \frac{\delta}{\delta m} \upsilon(t,\mu)(u,x)\right| \le C, \tag{4.2}$$

for every $t \in [0,T]$, $u \in U$, $x \in \mathbb{R}^d$, $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$;

4. the function $\partial_t \upsilon(t,\mu)$ is continuous on $[0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$;
5. for every $u \in U$ and every compact set H of $\mathbb{R}^d$, the functions $\partial_x \frac{\delta}{\delta m}\upsilon(t,\mu)(u,x)$ and $\partial_x^2 \frac{\delta}{\delta m}\upsilon(t,\mu)(u,x)$ are continuous functions of $(t,\mu) \in [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, uniformly in $x \in H$; more precisely, whenever $t_n \to t$, $\mathbf{d}(\mu_n,\mu) \to 0$, $u \in U$ and $H \subset \mathbb{R}^d$ is compact we have

$$\sup_{x\in H}\left|\partial_x \frac{\delta}{\delta m}\upsilon(t_n,\mu_n)(u,x) - \partial_x \frac{\delta}{\delta m}\upsilon(t,\mu)(u,x)\right| \to 0,$$

$$\sup_{x\in H}\left|\partial_x^2 \frac{\delta}{\delta m}\upsilon(t_n,\mu_n)(u,x) - \partial_x^2 \frac{\delta}{\delta m}\upsilon(t,\mu)(u,x)\right| \to 0.$$

Remark 4.4. By standard arguments, if K is a compact subset of $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ then the functions $\partial_x \frac{\delta}{\delta m}\upsilon(t,\mu)(u,x)$ and $\partial_x^2 \frac{\delta}{\delta m}\upsilon(t,\mu)(u,x)$ are uniformly continuous functions of $(t,\mu) \in [0,T] \times K$, uniformly in $x \in H$; namely: for every $u \in U$ and compact sets $H \subset \mathbb{R}^d$, $K \subset L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, and every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\partial_x \frac{\delta}{\delta m}\upsilon(t',\mu')(u,x) - \partial_x \frac{\delta}{\delta m}\upsilon(t,\mu)(u,x)\right| + \left|\partial_x^2 \frac{\delta}{\delta m}\upsilon(t',\mu')(u,x) - \partial_x^2 \frac{\delta}{\delta m}\upsilon(t,\mu)(u,x)\right| < \epsilon$$

whenever $u \in U$, $x \in H$, $t,t' \in [0,T]$, $\mu,\mu' \in K$, $\mathbf{d}(\mu,\mu) < \delta$, $|t-t'| < \delta$.

We next give some examples of functions for which we compute the linear functional derivatives.

Example 4.5. Since we are mainly interested in showing the form of the indicated functions υ we do not spell precise conditions on φ, F, ϕ_i used below ensuring that υ is smooth. Detailed assumptions are easily determined.

(i) *Linear functions*

$$\upsilon(\mu) = \int_U \int_{\mathbb{R}^d} \varphi(u,x)\mu^u(\mathrm{d}x)\,\lambda(\mathrm{d}u),$$

where φ is a measurable function with quadratic growth in x. Then,

$$\frac{\delta}{\delta m}\upsilon(\mu)(u,x) = \varphi(u,x).$$

(ii) Collection of cylindrical functions:

$$\upsilon(\mu) = \int_U F\Big(\int_{\mathbb{R}^d} \phi_1(u,x)\,\mu^u(\mathrm{d}x), \dots, \int_{\mathbb{R}^d} \phi_k(u,x)\,\mu^u(\mathrm{d}x)\Big)\lambda(\mathrm{d}u).$$

Denoting the partial derivatives of F by $\partial_i F$, we have,

$$\frac{\delta}{\delta m}\upsilon(\mu)(u,x) = \sum_{i=1}^k \partial_i F\bigg(\int_{\mathbb{R}^d} \phi_1(u,x)\,\mu^u(\mathrm{d}x), \dots, \int_{\mathbb{R}^d} \phi_k(u,x)\,\mu^u(\mathrm{d}x)\bigg)\,\phi_i(u,x).$$

(iii) Cylindrical functions of measure collection:

$$v(\mu) = F\Big(\int_U \int_{\mathbb{R}^d} \phi_1(u,x)\,\mu^u(\mathrm{d}x)\lambda(\mathrm{d}u), \ldots, \int_U \int_{\mathbb{R}^d} \phi_k(u,x)\,\mu^u(\mathrm{d}x)\lambda(\mathrm{d}u)\Big),$$

where $F : \mathbb{R}^k \to \mathbb{R}$ and ϕ_i are real functions on $\mathbb{R}^d$. Then,

$$\frac{\delta}{\delta m} v(\mu)(u,x) = \sum_{i=1}^k \partial_i F\Big(\int_U \int_{\mathbb{R}^d} \phi_1(u,x)\,\mu^u(\mathrm{d}x)\lambda(\mathrm{d}u), \ldots, \int_U \int_{\mathbb{R}^d} \phi_k(u,x)\,\mu^u(\mathrm{d}x)\lambda(\mathrm{d}u)\Big)\,\phi_i(u,x).$$

(iv) k-interaction functions:

$$v(\mu) = \int_{U^k} \int_{(\mathbb{R}^d)^k} \varphi(u_1,\ldots,u_k,x_1,\ldots,x_k)\mu^{u_1}(\mathrm{d}x_1)\ldots\mu^{u_k}(\mathrm{d}x_k)\lambda(\mathrm{d}u_1)\ldots\lambda(\mathrm{d}u_k).$$

Then,

$$\frac{\delta}{\delta m} v(\mu)(u,x) = \sum_{i=1}^k \int_{U^{k-1}} \int_{(\mathbb{R}^d)^{k-1}} \varphi(u_1,\ldots,u_{i-1},u,u_{i+1},\ldots,u_k,x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,x_k)$$
$$\mu^{u_1}(\mathrm{d}x_1)\ldots\mu^{u_{1-1}}(\mathrm{d}x_{i-1})\mu^{u_{1+1}}(\mathrm{d}x_{i+1})\ldots\mu^{u_k}(\mathrm{d}x_k)\lambda(\mathrm{d}u_1)\ldots\lambda(\mathrm{d}u_{i-1})\lambda(\mathrm{d}u_{i+1})\ldots\lambda(\mathrm{d}u_k).$$

4.2. The chain rule

We are now ready to present the Itô formula (chain rule) that will be needed in the sequel. We suppose that the conditions (*1*) and (*2*) in Assumption 2.1 hold true.

Theorem 4.6. *Suppose that, for $u \in U$, b^u and σ^u are stochastic processes defined on $[0,T]$, with values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times\ell}$ respectively, progressively measurable with respect to $\mathbb{F}^u$; let X_0^u be an $\mathcal{F}_0^u$-measurable random variable in $\mathbb{R}^d$. Also assume that $(u,t) \mapsto \mathbb{P}_{(X_0^u,b_t^u,\sigma_t^u)}$ is Borel measurable and*

$$\int_U \Big\{\mathbb{E}[|X_0^u|^2] + \int_0^T \mathbb{E}\left[|b_t^u|^2 + |\sigma_t^u|^2\right]\,\mathrm{d}t\Big\}\,\lambda(\mathrm{d}u) \;<\; \infty.$$

Define

$$X_t^u = X_0^u + \int_0^t b_s^u\,\mathrm{d}s + \int_0^t \sigma_s^u\,\mathrm{d}W_s^u, \qquad t \in [0,T], \quad u \in U.$$

Suppose $v \in \tilde{C}^{1,2}([0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)))$. Then, denoting $\mu_t^u = \mathbb{P}_{X_t^u}$ and $\mu_t = (\mu_t^u)_u$, we have

$$v(t,\mu_t) - v(0,\mu_0) = \int_0^t \Big\{\partial_t v(s,\mu_s)$$
$$+ \int_U \mathbb{E}\Big[\partial_x \frac{\delta}{\delta m} v(s,\mu_s)(u,X_s^u)\cdot b_s^u + \frac{1}{2}\partial_x^2 \frac{\delta}{\delta m} v(s,\mu_s)(u,X_s^u) : \sigma_s^u(\sigma_s^u)^\intercal\Big]\,\lambda(\mathrm{d}u)\Big\}\,\mathrm{d}s$$

for every $t \in [0,T]$.

Remark 4.7. We note that the measurability assumption on $\mathbb{P}_{(X_0^u, b_t^u, \sigma_t^u)}$ implies that the maps

$$u \mapsto \mathbb{E}[|X_0^u|^2], \qquad (u,t) \mapsto \mathbb{E}\big[|b_t^u|^2 + |\sigma_t^u|^2\big]$$

$$(u,t) \mapsto \mathbb{E}\Big[\partial_x \frac{\delta}{\delta m} v(s,\mu_s)(u, X_s^u) \cdot b_s^u + \frac{1}{2}\partial_x^2 \frac{\delta}{\delta m} v(s,\mu_s)(u, X_s^u) : \sigma_s^u (\sigma_s^u)^\intercal\Big]$$

are measurable, even if there is no measurability condition imposed on the random elements X_0^u, b_t^u, σ_t^u as functions of $u \in U$: this is important for our future applications.

In the following lemma we collect some preliminary facts which are used in the proof of Theorem 4.6.

Lemma 4.8. *a) For $\nu_1, \nu_2, \mu_1, \mu_2 \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, $\theta \in [0,1]$ we have*

$$\mathbf{d}((1-\theta)\nu_1 + \theta\nu_2, (1-\theta)\mu_1 + \theta\mu_2)^2 \le (1-\theta)\mathbf{d}(\nu_1,\mu_1)^2 + \theta\mathbf{d}(\nu_2,\mu_2)^2. \tag{4.3}$$

In particular if $\nu_1 = \nu_2 =: \nu$ then

$$\mathbf{d}(\nu, (1-\theta)\mu_1 + \theta\mu_2)^2 \le (1-\theta)\mathbf{d}(\nu,\mu_1)^2 + \theta\mathbf{d}(\nu,\mu_2)^2. \tag{4.4}$$

b) We have $\int_U \mathbb{E}\big[\sup_{t\in[0,T]} |X_t^u|^2\big]\,\lambda(\mathrm{d}u) < \infty$, and the map $t \mapsto \mu_t$ is continuous from $[0,T]$ to $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$.
c) The set

$$K := \{(1-\theta)\mu_t + \theta\mu_s \,:\, \theta \in [0,1];\, s,t \in [0,T]\} \tag{4.5}$$

is compact in $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$.

Proof. (a) For every $u \in U$ and $i = 1,2$ let γ_i^u be an optimal coupling between ν_i^u and μ_i^u for the square distance cost used in the definition of the Wasserstein distance $\mathcal{W}_2$. Then $(1-\theta)\gamma_1^u + \theta\gamma_2^u$ is a coupling between $(1-\theta)\nu_1^u + \theta\nu_2^u$ and $(1-\theta)\mu_1^u + \theta\mu_2^u$. It follows that

$$\begin{aligned}\mathcal{W}_2\Big((1-\theta)\nu_1^u + \theta\nu_2^u, (1-\theta)\mu_1^u + \theta\mu_2^u\Big)^2 &\le \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\,[(1-\theta)\gamma_1^u + \theta\gamma_2^u](\mathrm{d}x\,\mathrm{d}y)\\ &= (1-\theta)\mathcal{W}_2(\nu_1^u,\mu_1^u)^2 + \theta\mathcal{W}_2(\nu_2^u,\mu_2^u)^2.\end{aligned}$$

Integrating with respect to $\lambda(\mathrm{d}u)$ and sending ε to 0 gives the required conclusion.

(b) We have

$$\sup_{t\in[0,T]} |X_t^u|^2 \le 3\,|X_0^u|^2 + 3\left(\int_0^T |b_s^u|\,\mathrm{d}s\right)^2 + 3\sup_{t\in[0,T]}\left|\int_0^t \sigma_s^u\,\mathrm{d}W_s^u\right|^2$$

and by the Hölder and Doob inequality and the Itô isometry we have, for some absolute constant $c > 0$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^u|^2\Big] \le c\,\mathbb{E}[|X_0^u|^2] + c\,T\,\mathbb{E}\Big[\int_0^T |b_s^u|^2\,\mathrm{d}s\Big] + c\,\mathbb{E}\Big[\int_0^T |\sigma_s^u|^2\,\mathrm{d}s\Big].$$

Integrating with respect to $\lambda(du)$, we obtain $\int_U \mathbb{E}[\sup_{t\in[0,T]} |X_t^u|^2]\,\lambda(\mathrm{d}u) < \infty$. This shows that $\mu_t \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ for every $t \in [0,T]$. To prove the required continuity we note that, by similar passages as before, for $0 \le s \le t \le T$,

$$\begin{aligned}\mathbf{d}(\mu_t,\mu_s)^2 &= \int_U \mathcal{W}_2(\mu_t^u,\mu_s^u)^2\,\lambda(\mathrm{d}u) \ \le\ \int_U \mathbb{E}[|X_t^u - X_s^u|^2]\,\lambda(\mathrm{d}u)\\ &\le\ c\int_U \Big\{(t-s)\,\mathbb{E}\Big[\int_s^t |b_r^u|^2\,\mathrm{d}r\Big] + \mathbb{E}\Big[\int_s^t |\sigma_r^u|^2\,\mathrm{d}r\Big]\Big\}\,\lambda(\mathrm{d}u),\end{aligned}$$

which tends to 0 as $t-s\to 0$.

(c) As a consequence of the previous point, the set $K_1 := \{\mu_t : t\in[0,T]\}$ is compact. We note that K is the image of the compact set $K_1\times K_1\times[0,1]$ under the mapping $(\mu,\nu,\theta)\mapsto(1-\theta)\mu+\theta\nu$. To prove compactness of K it is therefore enough to show that this mapping is continuous from $K_1\times K_1\times[0,1]$ to $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$. We will show that

(i) the map $(\mu,\nu)\mapsto(1-\theta)\mu+\theta\nu$ is continuous, uniformly in $\theta\in[0,1]$.
(ii) the map $\theta\mapsto(1-\theta)\mu+\theta\nu$ is continuous, for every fixed $(\mu,\nu)\in K_1\times K_1$.

These two statements easily imply the required continuity.

To prove (i) we note that if $\mathbf{d}(\mu_n,\mu)\to 0$, $\mathbf{d}(\nu_n,\nu)\to 0$, it follows from (4.3) that

$$\mathbf{d}((1-\theta)\mu+\theta\nu,(1-\theta)\mu_n+\theta\nu_n)^2 \le (1-\theta)\mathbf{d}(\mu,\mu_n)^2+\theta\mathbf{d}(\nu,\mu_n)^2 \le \mathbf{d}(\mu,\mu_n)^2+\mathbf{d}(\nu,\mu_n)^2$$

which tends to 0 uniformly in θ.

To prove (ii) we recall the Kantorovich duality: given $\eta,\rho\in\mathcal{P}_2(\mathbb{R}^d)$, the squared Wasserstein distance $\mathcal{W}_2(\eta,\rho)^2$ equals the supremum of

$$\int_{\mathbb{R}^d} f(x)\,\eta(\mathrm{d}x) + \int_{\mathbb{R}^d} g(y)\,\rho(\mathrm{d}y)$$

for f,g varying in the set of bounded Lipschitz function on $\mathbb{R}^d$ satisfying the constraint

$$f(x)+g(y)\le |x-y|^2, \qquad x,y\in\mathbb{R}^d. \tag{4.6}$$

Given $\mu,\nu\in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ and $\theta,\theta'\in[0,1]$ we apply the duality result to $\eta=(1-\theta')\mu^u+\theta'\nu^u$ and $\rho=(1-\theta)\mu^u+\theta\nu^u$ for fixed $u\in U$. We estimate the supremum of

$$\begin{aligned}&\int_{\mathbb{R}^d} f(x)\,[(1-\theta')\mu^u+\theta'\nu^u](\mathrm{d}x) + \int_{\mathbb{R}^d} g(y)\,[(1-\theta)\mu^u+\theta\nu^u](\mathrm{d}y)\\ &= (1-\theta)\int_{\mathbb{R}^d}(f+g)\,\mathrm{d}\mu^u + \theta'\int_{\mathbb{R}^d}(f+g)\,\mathrm{d}\nu^u + (\theta-\theta')\Big(\int_{\mathbb{R}^d} f\,\mathrm{d}\mu^u + \int_{\mathbb{R}^d} g\,\mathrm{d}\nu^u\Big).\end{aligned}$$

From (4.6) evaluated at $x=y$ it follows that $f+g\le 0$, so that the first two integrals are also ≤ 0. If $\theta>\theta'$ we also have

$$(\theta-\theta')\left(\int_{\mathbb{R}^d} f\,\mathrm{d}\mu^u + \int_{\mathbb{R}^d} g\,\mathrm{d}\nu^u\right) \le\ (\theta-\theta')\,\mathcal{W}_2(\mu^u,\nu^u)^2,$$

and by duality it follows that

$$\mathcal{W}_2\Big((1-\theta')\mu^u+\theta'\nu^u,(1-\theta)\mu^u+\theta\nu^u\Big)^2 \leq\ |\theta-\theta'|\,\mathcal{W}_2(\mu^u,\nu^u)^2.$$

Interchanging θ and θ' the same inequality also holds for $\theta<\theta'$ and so for every $\theta,\theta'\in[0,1]$.

Integrating with respect to $\lambda(\mathrm{d}u)$ we conclude that

$$\mathbf{d}\Big((1-\theta')\mu+\theta'\nu,(1-\theta)\mu+\theta\nu\Big)^2\leq|\theta-\theta'|\,\mathbf{d}(\mu,\nu)^2,$$

which gives the required conclusion. □

Proof of Theorem 4.6. We first verify that the terms in the Itô formula are well defined. By (4.1), and recalling that $s\mapsto\mu_s$ is continuous in $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$,

$$\begin{aligned}\int_0^T\Big|\partial_x\frac{\delta}{\delta m}v(s,\mu_s)(u,X_s^u)\cdot b_s^u\Big|\,\mathrm{d}s &\leq\ C\int_0^T\Big((1+|X_s^u|+\mathbf{d}(\mu_s,\delta_0))\ |b_s^u|\Big)\,\mathrm{d}s\\ &\leq\ C\Big(1+\sup_{s\in[0,T]}|X_s^u|+\sup_{s\in[0,T]}\mathbf{d}(\mu_s,\delta_0)\Big)\int_0^T|b_s^u|\,\mathrm{d}s.\end{aligned}$$

The right-hand side does not depend on θ and we have

$$\int_U\mathbb{E}\Big[\Big(1+\sup_{s\in[0,T]}|X_s^u|+\sup_{s\in[0,T]}\mathbf{d}(\mu_s,\delta_0)\Big)\int_0^T|b_s^u|\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u)<\ \infty$$

by the Hölder inequality, since

$$\int_U\mathbb{E}\Big[\sup_{s\in[0,T]}|X_s^u|^2\Big]\,\lambda(\mathrm{d}u)\ <\ \infty,\qquad\int_U\mathbb{E}\Big[\int_0^T|b_s^u|^2\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u)\ <\ \infty.$$

In a similar and simpler way, using the boundedness condition (4.2) instead of (4.1), we can check that

$$\int_U\mathbb{E}\Big[\int_0^T\Big|\frac12\partial_x^2\frac{\delta}{\delta m}v(s,\mu_s)(u,X_s^u):\sigma_s^u(\sigma_s^u)^\intercal\Big|\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u)<\infty.$$

In order to prove the formula we fix t and, for every positive integer n, we choose a subdivision of the interval $[0,t]$ by points $t_k^n:=kt/n$, $k=0,1,\ldots,n-1$. We evaluate the difference

$$\begin{aligned}v(t_{k+1}^n,\mu_{t_{k+1}^n})-v(t_k^n,\mu_{t_k^n}) &=\ v(t_{k+1}^n,\mu_{t_{k+1}^n})-v(t_k^n,\mu_{t_{k+1}^n})+v(t_k^n,\mu_{t_{k+1}^n})-v(t_k^n,\mu_{t_k^n})\\ &=\int_{t_k^n}^{t_{k+1}^n}\partial_t v(s,\mu_{t_{k+1}^n})\,\mathrm{d}s+v(t_k^n,\mu_{t_{k+1}^n})-v(t_k^n,\mu_{t_k^n}).\end{aligned}\tag{4.7}$$

We note that the last difference can be written, by the definition of the derivative $\frac{\delta}{\delta m}$, as

$$\begin{aligned}v(t_k^n,\mu_{t_{k+1}^n})-v(t_k^n,\mu_{t_k^n})\ &=\ \int_U\int_0^1\int_{\mathbb{R}^d}\frac{\delta}{\delta m}v\big(t_k^n,\mu_{\theta,k,n}\big)(u,x)\,(\mu_{t_{k+1}^n}^u-\mu_{t_k^n}^u)(\mathrm{d}x)\,\mathrm{d}\theta\,\lambda(\mathrm{d}u)\\ =\ \int_U\int_0^1\mathbb{E}\Big[&\frac{\delta}{\delta m}v\big(t_k^n,\mu_{\theta,k,n}\big)(u,X_{t_{k+1}^n}^u)-\frac{\delta}{\delta m}v\big(t_k^n,\mu_{\theta,k,n}\big)(u,X_{t_k^n}^u)\Big]\,\mathrm{d}\theta\,\lambda(\mathrm{d}u),\end{aligned}\tag{4.8}$$

where we have set $\mu_{\theta,k,n} = \mu_{t_k^n} + \theta(\mu_{t_{k+1}^n} - \mu_{t_k^n})$. Since we assume that $\frac{\delta}{\delta m}v$ is twice continuously differentiable in x we can apply the classical Itô formula to the process X^u on $[t_k^n, t_{k+1}^n]$:

$$\frac{\delta}{\delta m}v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X^u_{t_{k+1}^n}) - \frac{\delta}{\delta m}v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X^u_{t_k^n}) \;=\; \int_{t_k^n}^{t_{k+1}^n} \partial_x \frac{\delta}{\delta m}v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u)^\intercal\, \sigma_s^u \,\mathrm{d}W_s^u$$
$$+ \int_{t_k^n}^{t_{k+1}^n} \Big[\partial_x \frac{\delta}{\delta m}v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u)\cdot\, b_s^u + \frac{1}{2}\partial_x^2 \frac{\delta}{\delta m}v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u) \,:\, \sigma_s^u(\sigma_s^u)^\intercal\Big]\,\mathrm{d}s. \tag{4.9}$$

We next verify that the stochastic integral has zero expectation, by checking that the square root of its quadratic variation has finite expectation. Using (4.1),

$$\begin{aligned}
&\Big(\int_{t_k^n}^{t_{k+1}^n} \Big|\partial_x \frac{\delta}{\delta m}v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u)^\intercal\, \sigma_s^u\Big|^2\, ds\Big)^{1/2} \\
&\le C\Big(\int_{t_k^n}^{t_{k+1}^n} \big(1 + |X_s^u|^2 + \mathbf{d}(\mu_{\theta,k,n}, \delta_0)^2\big)\; |\sigma_s^u|^2\,\mathrm{d}s\Big)^{1/2} \\
&\le C\Big(1 + \sup_{s\in[t_k^n, t_{k+1}^n]} |X_s^u|^2 + \mathbf{d}(\mu_{\theta,k,n}, \delta_0)^2\Big)^{1/2}\Big(\int_{t_k^n}^{t_{k+1}^n} |\sigma_s^u|^2\, ds\Big)^{1/2}.
\end{aligned}$$

Both terms are square summable, by our assumptions, and the integrability property of the quadratic variation follows from the Hölder inequality. Taking expectation in the Itô formula (4.9) and replacing in (4.8) we obtain

$$\begin{aligned}
v(t_k^n, \mu_{t_{k+1}^n}) - v(t_k^n, \mu_{t_k^n}) \;=\; &\int_U \int_0^1 \int_{t_k^n}^{t_{k+1}^n} \mathbb{E}\Big[\partial_x \frac{\delta}{\delta m}v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u)\cdot\, b_s^u \\
&+ \frac{1}{2}\partial_x^2 \frac{\delta}{\delta m}v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u) \,:\, \sigma_s^u(\sigma_s^u)^\intercal\Big]\,\mathrm{d}s\,\mathrm{d}\theta\,\lambda(\mathrm{d}u).
\end{aligned}$$

Coming back to (4.7) and summing over k we arrive at

$$\begin{aligned}
v(t, \mu_t) - v(0, \mu_0) \;=\; &\sum_{k=0}^{n-1} \int_{t_k^n}^{t_{k+1}^n} \partial_t v(s, \mu_{t_{k+1}^n})\,\mathrm{d}s \\
&+ \int_U \int_0^1 \sum_{k=0}^{n} \int_{t_k^n}^{t_{k+1}^n} \mathbb{E}\Big[\partial_x \frac{\delta}{\delta m}v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u)\cdot\, b_s^u \\
&+ \frac{1}{2}\partial_x^2 \frac{\delta}{\delta m}v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u) \,:\, \sigma_s^u(\sigma_s^u)^\intercal\Big]\,\mathrm{d}s\,\mathrm{d}\theta\,\lambda(\mathrm{d}u).
\end{aligned}$$

We may write the Itô formula in similar way, decomposing the integral over $[0, t]$ into a sum of integrals over $[t_k^n, t_{k+1}^n]$, and we see that setting

$$I_1^n := \sum_{k=0}^{n-1} \int_{t_k^n}^{t_{k+1}^n} \Big(\partial_t v(s, \mu_{t_{k+1}^n}) - \partial_t v(s, \mu_s)\Big)\,\mathrm{d}s,$$

$$I_2^n := \int_U \int_0^1 \mathbb{E}\Big[\sum_{k=0}^{n} \int_{t_k^n}^{t_{k+1}^n} \Big\{\partial_x \frac{\delta}{\delta m}v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u)\cdot\, b_s^u - \partial_x \frac{\delta}{\delta m}v\big(s, \mu_s\big)(u, X_s^u)\cdot\, b_s^u\Big\}\,\mathrm{d}s\Big]\,\mathrm{d}\theta\,\lambda(\mathrm{d}u)$$

$$I_3^n := \int_U \int_0^1 \mathbb{E}\Big[\sum_{k=0}^{n} \int_{t_k^n}^{t_{k+1}^n} \Big\{ \frac{1}{2} \partial_x^2 \frac{\delta}{\delta m} v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u) : \sigma_s^u (\sigma_s^u)^\intercal - \frac{1}{2} \partial_x^2 \frac{\delta}{\delta m} v\big(s, \mu_s\big)(u, X_s^u) : \sigma_s^u (\sigma_s^u)^\intercal \Big\} \, \mathrm{d}s \Big] \, \mathrm{d}\theta \, \lambda(\mathrm{d}u),$$

it is enough to prove that $I_1^n + I_2^n + I_3^n \to 0$. It is immediate to see that $I_1^n \to 0$, since $\partial_t v$ is continuous, hence uniformly continuous, on $[0, T] \times K_1$, where $K_1 = \{\mu_s : s \in [0, T]\}$, because K_1 is compact in $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ by the continuity of $s \mapsto \mu_s$.

Next we consider I_2^n. We fix u, θ, ω. We note that for $\mathbb{P}$-almost all ω the set

$$H^{u,\omega} := \{X_s^u(\omega) : s \in [0, T]\}$$

is compact in $\mathbb{R}^d$ due to continuity of $s \mapsto X_s^u(\omega)$. By our assumptions and Remark 4.4, for every compact $K \subset L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ and $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \partial_x \frac{\delta}{\delta m} v(t', \mu)(u, x) - \partial_x \frac{\delta}{\delta m} v(t, \mu)(u, x) \right| < \epsilon$$

whenever $x \in H^{u,\omega}$, $t, t' \in [0, T]$, $\mu, \mu' \in K$, $\mathbf{d}(\mu, \mu') < \delta$, $|t - t'| < \delta$. As the set K, we choose the one defined in (4.5) and we note that points of the form (s, μ_s) and $(t_k^n, \mu_{\theta,k,n})$ belong to $[0, T] \times K$. It follows that, $\mathbb{P}$-a.s.,

$$\left| \partial_x \frac{\delta}{\delta m} v(t_k^n, \mu_{\theta,k,n})(u, X_s^u) - \partial_x \frac{\delta}{\delta m} v(s, \mu_s)(u, X_s^u) \right| < \epsilon$$

provided the inequalities

$$|s - t_k^n| < \delta, \quad \mathbf{d}(\mu_s, \mu_{\theta,k,n}) < \delta, \qquad s \in [t_k^n, t_{k+1}^n], \ k = 0, \dots, n-1,$$

are satisfied. We finally note that $|s - t_k^n| \le |t_{k+1}^n - t_k^n| \le 1/n$ and, by (4.4), the squared distance

$$\begin{aligned} \mathbf{d}(\mu_s, \mu_{\theta,k,n})^2 = \ & \mathbf{d}(\mu_s, (1-\theta)\mu_{t_k^n} + \theta \mu_{t_{k+1}^n})^2 \\ \le \ & (1-\theta)\mathbf{d}(\mu_s, \mu_{t_k^n})^2 + \theta \mathbf{d}(\mu_s, \mu_{t_{k+1}^n})^2 \ \le \ \mathbf{d}(\mu_s, \mu_{t_k^n})^2 + \mathbf{d}(\mu_s, \mu_{t_{k+1}^n})^2 \end{aligned}$$

can be made arbitrarily small by taking n sufficiently large, by the uniform continuity of $s \mapsto \mu_s$ on $[0, T]$. It follows that for all n large enough (depending on u, θ, ω)

$$\sum_{k=0}^{n} \int_{t_k^n}^{t_{k+1}^n} \left| \partial_x \frac{\delta}{\delta m} v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u) \cdot b_s^u - \partial_x \frac{\delta}{\delta m} v\big(s, \mu_s\big)(u, X_s^u) \cdot b_s^u \right| \mathrm{d}s \le \epsilon T.$$

We have thus proved that for fixed $\theta \in [0, 1]$ and $u \in U$ we have, $\mathbb{P}$-a.s.,

$$\sum_{k=0}^{n} \int_{t_k^n}^{t_{k+1}^n} \partial_x \frac{\delta}{\delta m} v\big(t_k^n, \mu_{\theta,k,n}\big)(u, X_s^u) \cdot b_s^u \, \mathrm{d}s \ \to \ \int_0^t \partial_x \frac{\delta}{\delta m} v\big(s, \mu_s\big)(u, X_s^u) \cdot b_s^u \, \mathrm{d}s.$$

In order to conclude that $I_2^n \to 0$ we wish to apply the dominated convergence theorem and pass to the limit under the expectation sign and under the integrals over $[0, 1]$ and U. To this end we consider the following estimates. By (4.1),

$$\sum_{k=0}^{n}\int_{t_k^n}^{t_{k+1}^n}\Big|\partial_x\frac{\delta}{\delta m}v\big(t_k^n,\mu_{\theta,k,n}\big)(u,X_s^u)\cdot\, b_s^u\Big|\,\mathrm{d}s \le\ C\sum_{k=0}^{n}\int_{t_k^n}^{t_{k+1}^n}\Big((1+|X_s^u|+\mathbf{d}(\mu_{\theta,k,n},\delta_0))\ |b_s^u|\Big)\,\mathrm{d}s$$
$$\le\ C\Big(1+\sup_{s\in[0,T]}|X_s^u|+\sup_{\mu\in K}\mathbf{d}(\mu,\delta_0)\Big)\int_0^t|b_s^u|\,\mathrm{d}s.$$

The right-hand side does not depend on n nor θ and satisfies

$$\int_U\mathbb{E}\Big[\Big(1+\sup_{s\in[0,T]}|X_s^u|+\sup_{\mu\in K}\mathbf{d}(\mu,\delta_0)\Big)\int_0^t|b_s^u|\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u)\ <\ \infty$$

by the Hölder inequality, since

$$\int_U\mathbb{E}\Big[\sup_{s\in[0,T]}|X_s^u|^2\Big]\,\lambda(\mathrm{d}u)\ <\ \infty,\qquad \int_U\mathbb{E}\Big[\int_0^T|b_s^u|^2\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u)<\infty.$$

So we can apply the dominated convergence theorem (three times) and we conclude that $I_2^n\to 0$.

Finally, the proof that $I_3^n\to 0$ is similar to the proof that $I_2^n\to 0$ and even simpler, since we apply the boundedness condition (4.2) instead of (4.1). □

Remark 4.9. We note that Theorem 4.6 applies to the process $(X^u)_u$ solution to the state equation (2.2). As a consequence, if $v\in\tilde{C}^{1,2}([0,T]\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)))$ we have, on the interval $[t,T]$,

$$\mathrm{d}v(s,\mu_s)=\ \partial_s v(s,\mu_s)\,\mathrm{d}s+\int_U\mathbb{E}\Big[\partial_x\frac{\delta}{\delta m}v(s,\mu_s)(u,X_s^u)\cdot b(u,X_s^u,\alpha_s^u,\mu_s,\mathbb{P}_{\alpha_s^\cdot})$$
$$+\ \frac{1}{2}\partial_x^2\frac{\delta}{\delta m}v(s,\mu_s)(u,X_s^u):\sigma\sigma^\intercal(u,X_s^u,\alpha_s^u,\mu_s,\mathbb{P}_{\alpha_s^\cdot})\Big]\,\lambda(\mathrm{d}u)\,\mathrm{d}s.$$

5. The Bellman equation

In this section, we make the standing Assumptions 2.1, 2.9, 3.1.

5.1. The equation

For $\pi=(\pi^u)_u\in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d\times A))$ we denote by $\pi_1^u\in\mathcal{P}_2(\mathbb{R}^d)$, $\pi_2^u\in\mathcal{P}_2(A)$ the marginals of $\pi^u\in\mathcal{P}_2(\mathbb{R}^d\times A)$ and we set $\pi_1=(\pi_1^u)_u\in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))=L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, $\pi_2=(\pi_2^u)_u\in L^2_\lambda(\mathcal{P}_2(A))$.

We next introduce the Hamiltonian $\mathcal{H}$ defined by

$$\mathcal{H}(u,t,\pi,\varphi)=\int_{\mathbb{R}^d\times A}\Big(\partial_x\frac{\delta}{\delta m}\varphi(t,\pi_1)(u,x)\cdot b(u,x,a,\pi_1,\pi_2)$$
$$+\ \frac{1}{2}\partial_x^2\frac{\delta}{\delta m}\varphi(t,\pi_1)(u,x):\sigma\sigma^\intercal(u,x,a,\pi_1,\pi_2)+f(u,x,a,\pi_1,\pi_2)\Big)\pi^u(\mathrm{d}x,\mathrm{d}a)$$

for $(u,t,\pi,\varphi)\in U\times[0,T]\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d\times A))\times\tilde{C}^{1,2}([0,T]\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)))$. In our framework, the Bellman equation is written as

$$-\partial_t v(t,\mu)-\inf_{\pi\in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d\times A)),\pi_1=\mu}\int_U\mathcal{H}(u,t,\pi,v)\lambda(\mathrm{d}u)=\ 0,\tag{5.1}$$

for $(t,\mu) \in [0,T) \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, together with the terminal condition

$$\upsilon(T,\mu) = \int_U \int_{\mathbb{R}^d} g(u,x,\mu)\, \mu^u(\mathrm{d}x)\, \lambda(\mathrm{d}u), \quad \mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)). \tag{5.2}$$

5.2. The regular case

In the case where the value function is smooth, we provide a verification result.

Theorem 5.1 (Verification)**.** *Let* $w \in \tilde{C}^{1,2}([0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)))$.

(i) Suppose that w *is solution to* (5.1)–(5.2) *Then we have* $w \le \upsilon$ *on* $[0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$.

(ii) Suppose further that there exists a Borel map $\hat{a} : U \times [0,T] \times \mathbb{R}^d \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \to A$ *such that for any* $(t,\mu) \in [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ *the infimum in* (5.1)*, with* w *in place of* υ*, is reached at a point* $\hat{\pi} = (\hat{\pi}^u)$ *which has the form*

$$\hat{\pi}^u = \mu^u \circ (Id \times \hat{a}(u,t,\cdot,\mu))^{-1} \quad u \in U,$$

i.e. such that its marginal π^u_1 *is* μ^u *and the marginal* π^u_2 *is the image of* μ^u *under a suitable map* $x \mapsto \hat{a}(u,t,x,\mu)$*, and the system*

$$\begin{cases} \mathrm{d}X^u_s = b\Big(u, X^u_s, \hat{a}(u,s,X^u_s,\mathbb{P}_{X^\cdot_s}), \mathbb{P}_{X^\cdot_s}, (\mathbb{P}_{\hat{a}(v,s,X^v_s,\mathbb{P}_{X^\cdot_s})})_v\Big)\mathrm{d}s \\ + \sigma\Big(u, X^u_s, \hat{a}(u,s,X^u_s,\mathbb{P}_{X^\cdot_s}), \mathbb{P}_{X^\cdot_s}, (\mathbb{P}_{\hat{a}(v,s,X^v_s,\mathbb{P}_{X^\cdot_s})})_v\Big)\mathrm{d}W^u_s, \quad s \in [t,T], \\ X^u_t = \xi^u, \quad u \in U, \end{cases}$$

admits a unique solution, in the sense of Definition 2.4, denoted by $(\hat{X}^{t,\xi,u})_u$ *such that* $(\hat{\alpha}^u)_u := (\hat{a}(u,s,\hat{X}^{t,\xi,u}_s, \mathbb{P}_{\hat{X}^{t,\xi,\cdot}_s}))_{u\in U, s\in[0,T]} \in \mathcal{A}$ *for any* $t \in [0,T]$ *and any* $\xi \in \mathcal{I}_t$*. Then* $w = \upsilon$ *on* $[0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ *and* $\hat{\alpha}$ *is an optimal Markov control:*

$$\upsilon(t,\mu) = J\big(t,\xi,(\hat{\alpha}^u)_u\big)$$

for $t \in [0,T]$*,* $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ *and* $\xi \in \mathcal{I}_t$ *such that* $(\mathbb{P}_{\xi_u})_u = \mu$.

Proof. (i) Fix $(t,\mu) \in [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$. Let $\xi \in \mathcal{I}_t$ such that $\mathbb{P}_{\xi^\cdot} = \mu$ and a control $\alpha \in \mathcal{A}$. Denote by $(X^u)_u$ the unique solution to the system

$$\begin{cases} \mathrm{d}X^u_s = b\big(u, X^u_s, \alpha^u_s, \mathbb{P}_{X^\cdot_s}, \mathbb{P}_{\alpha^\cdot_s}\big)\mathrm{d}s + \sigma\big(u, X^u_s, \alpha^u_s, \mathbb{P}_{X^\cdot_s}, \mathbb{P}_{\alpha^\cdot_s}\big)\mathrm{d}W^u_s, \; s \in [t,T], \\ X^u_t = \xi^u, \quad u \in U. \end{cases}$$

We next write μ_s, π_s, $b^u_s(x,a)$, $\sigma^u_s(x,a)$ and $f^u_s(x,a)$ for $\mathbb{P}_{X^\cdot_s}$, $\mathbb{P}_{(X^\cdot_s,\alpha^\cdot_s)}$, $b(u,x,a,\mathbb{P}_{X^\cdot_s},\mathbb{P}_{\alpha^\cdot_s})$, $\sigma(u,x,a,\mathbb{P}_{X^\cdot_s},\mathbb{P}_{\alpha^\cdot_s})$, and $f(u,x,a,\mathbb{P}_{X^\cdot_s},\mathbb{P}_{\alpha^\cdot_s})$ respectively. From Theorem 4.6 we have

$$\begin{aligned} w(t,\mu) \;=\; & w(T,\mu_T) - \int_t^T \Big\{\partial_t w(s,\mu_s)\,\mathrm{d}s \\ & + \int_U \int_{\mathbb{R}^d \times A} \Big[\partial_x \frac{\delta}{\delta m} w(s,\mu_s)(u,x) \cdot b^u_s(x,a) + \frac{1}{2}\partial^2_x \frac{\delta}{\delta m} w(s,\mu_s)(u,x) : \sigma^u_s(\sigma^u_s)^\intercal(x,a)\Big]\pi(\mathrm{d}x,\mathrm{d}a)\,\lambda(\mathrm{d}u)\Big\}\,\mathrm{d}s. \end{aligned}$$

Since w is solution to (5.1)–(5.2), we get

$$w(t,\mu) \le \int_U \Big[\int_{\mathbb{R}^d} g(u,x,\mu_T)\, \mu^u_T(\mathrm{d}x) + \int_t^T \int_{\mathbb{R}^d \times A} f^u_s(u,x)\pi_s(\mathrm{d}x,\mathrm{d}a)\mathrm{d}s\Big]\lambda(\mathrm{d}u) \;=\; J(t,\xi,\alpha).$$

Since this inequality holds for any $\alpha \in \mathcal{A}$, we get $w \leq \upsilon$.
(ii) We proceed as in (i) with the control $(\hat{\alpha}^u)_u$ instead of α. We then get

$$w(t,\mu) = w(T,\mu_T) - \int_t^T \Big\{ \partial_t w(s,\mu_s) + \inf_{\pi \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d \times A)), \pi_1 = \mu} \int_U \mathcal{H}(u,t,\pi,w)\lambda(\mathrm{d}u) \Big\} \mathrm{d}s$$
$$+ \int_t^T \int_U \int_{\mathbb{R}^d \times A} f_s(u,x,a,\mathbb{P}_{\hat{X}^{t,\xi,\cdot}_s}, \mathbb{P}_{\hat{\alpha}^{\cdot}_s}) \mathbb{P}_{(\hat{X}^{t,\xi,u}_s, \hat{\alpha}^u_s)}(\mathrm{d}x,\mathrm{d}a) \Big] \lambda(\mathrm{d}u)\mathrm{d}s.$$

Since w is solution to (5.1)–(5.2), we get

$$w(t,\mu) = \int_U \mathbb{E}\Big[g(u,\hat{X}^{t,\xi,u}_T, \mathbb{P}_{\hat{X}^{t,\xi,\cdot}_T}) + \int_t^T f(u,\hat{X}^{t,\xi,u}_s, \hat{\alpha}^u_s, \mathbb{P}_{\hat{X}^{t,\xi,\cdot}_s}, \mathbb{P}_{\hat{\alpha}^{\cdot}_s})\mathrm{d}s \Big] \lambda(\mathrm{d}u)$$
$$= J(t,\xi,\hat{\alpha}) \;\geq\; \upsilon(t,\mu).$$

Therefore, $w(t,\mu) = \upsilon(t,\mu) = J(t,\xi,\hat{\alpha})$. □

An example of such an optimal control problem for which the value function is regular is the so-called linear-quadratic case where the drift and diffusion coefficients are affine and the costs are quadratic. In this case, Theorem 5.1 in [19] states that the value function is also quadratic with coefficients satisfying some abstract Riccati equation.

Remark 5.2. In the case where the coefficients b, σ and f do not depend on the second marginal π_2 of π, the PDE (5.1) takes the following form

$$-\partial_t \upsilon(t,\mu) - \int_U \int_{\mathbb{R}^d} \inf_{a \in A} H(u,t,x,\mu,\upsilon,a)\mu(\mathrm{d}x)\lambda(\mathrm{d}u) = 0, \; (t,\mu) \in [0,T) \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)),$$

with

$$H(u,t,x,\mu,\varphi,a) = \partial_x \frac{\delta}{\delta m}\varphi(t,\mu)(u,x) \cdot b(u,x,a,\mu) + \frac{1}{2}\partial^2_x \frac{\delta}{\delta m}\varphi(t,\mu)(u,x) : \sigma\sigma^\intercal(u,x,a,\mu)$$
$$+ f(u,x,a,\mu),$$

for $(t,u,\mu,\varphi,a) \in [0,T] \times U \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \times \tilde{C}^{1,2}([0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))) \times A$.

In this case, using the same arguments, one can prove Theorem 5.1 (i). Moreover, if the Borel map $\hat{a}$ is replaced by a Borel map $\tilde{a}: \; U \times \mathbb{R}^d \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \to A$ such that

$$H(u,t,x,\mu,w,\tilde{a}(u,t,x,\mu)) = \inf_{a \in A} H(u,t,x,\mu,w,a)$$

for all $(t,u,x,\mu) \in [0,T] \times U \times \mathbb{R}^d \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ and the system

$$\begin{cases} dX^u_s = b\big(u, X^u_s, \tilde{a}(u,s,X^u_s,\mathbb{P}_{X^{\cdot}_s}), \mathbb{P}_{X^{\cdot}_s}, \big)\mathrm{d}s \\ \qquad\quad + \sigma\big(u, X^u_s, \tilde{a}(u,s,X^u_s,\mathbb{P}_{X^{\cdot}_s}), \mathbb{P}_{X^{\cdot}_s}, \big)\mathrm{d}W^u_s, \quad s \in [t,T], \\ X^u_t \;\; = \xi^u, \quad u \in U, \end{cases}$$

admits a unique solution, in the sense of Definition 2.4, denoted by $(\tilde{X}^{t,\xi,u})_u$ such that $(\tilde{\alpha}^u)_u := (\tilde{a}(u,.,\tilde{X}^{t,\xi,u}, \mathbb{P}_{\tilde{X}^{t,\xi,\cdot}}))_u \in \mathcal{A}$ for any $t \in [0,T]$ and any $\xi \in \mathcal{I}_t$. Then $w = \upsilon$ on $[0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ and $\tilde{\alpha}$ is an optimal Markov control:

$$\upsilon(t,\mu) = J\big(t,\xi,(\tilde{\alpha}^u)_u\big)$$

for $t \in [0,T]$, $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ and $\xi \in \mathcal{I}_t$ such that $\mathbb{P}_{\xi^\cdot} = \mu$. The proof follows the same lines as that of Theorem 5.1 (ii).

5.3. Viscosity properties

For a locally bounded function $w: [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \to \mathbb{R}$ (*i.e.* bounded on bounded sets), we define its lower and upper semicontinuous envelopes respectively by

$$w_*(t,\mu) = \liminf_{(s,\nu)\to(t,\mu),\ s<T} w(s,\nu), \qquad w^*(t,\mu) = \limsup_{(s,\nu)\to(t,\mu),\ s<T} w(s,\nu),$$

for $(t,\mu) \in [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$.

Definition 5.3. Let $w: [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \to \mathbb{R}$ be a locally bounded function.

(i) We say that w is a viscosity subsolution to (5.1)–(5.2) if

$$w^*(T,\mu) \le \int_U \int_{\mathbb{R}^d} g(u,x,\mu)\, \mu^u(\mathrm{d}x)\, \lambda(\mathrm{d}u), \quad \mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)),$$

and for any $\varphi \in \tilde{C}^{1,2}([0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)))$ and $(t,\mu) \in [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, such that

$$(w^* - \varphi)(t,\mu) = \max_{[0,T]\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))} (w^* - \varphi)$$

we have

$$-\partial_t \varphi(t,\mu) - \inf_{\pi \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d \times A)),\pi_1=\mu} \int_U \mathcal{H}(u,t,\pi,\varphi)\lambda(\mathrm{d}u) \le 0.$$

(ii) We say that w is a viscosity supersolution to (5.1)–(5.2) if

$$w_*(T,\mu) \ge \int_U \int_{\mathbb{R}^d} g(u,x,\mu)\, \mu^u(\mathrm{d}x)\, \lambda(\mathrm{d}u), \quad \mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)),$$

and for any $\varphi \in \tilde{C}^{1,2}([0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)))$ and $(t,\mu) \in [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, such that

$$(w_* - \varphi)(t,\mu) = \min_{[0,T]\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))} (w_* - \varphi)$$

we have

$$-\partial_t \varphi(t,\mu) - \inf_{\pi \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d \times A)),\pi_1=\mu} \int_U \mathcal{H}(u,t,\pi,\varphi)\, \lambda(\mathrm{d}u) \ge 0.$$

(iii) We say that w is a viscosity solution to (5.1)–(5.2) if w is both a viscosity subsolution and supersolution to (5.1)–(5.2).

Assumption 5.4. (i) There exist a constant $M \ge 0$ such that

$$|b(u,x,a,\mu,\nu)| + |\sigma(u,x,a,\mu,\nu)| \le M\big(1 + |x| + \mathbf{d}(\mu,\delta_0)\big)$$

for every $u \in U$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, $\nu \in L^2_\lambda(\mathcal{P}_2(A))$, $a \in A$.

(ii) The function

$$(t,\mu) \in [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)) \mapsto \inf_{\pi \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d \times A)), \pi_1 = \mu} \int_U \mathcal{H}(u,t,\pi,\varphi)\lambda(\mathrm{d}u)$$

is continuous for any $\varphi \in \tilde{C}^{1,2}([0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)))$.

The continuity condition appearing in Assumption 4.4 (ii) ensures to have the same PDE for the sub and super solutions. Otherwise, the supersolution property would involve the liminf of the nonlinear operator and the subsolution property would involve the limsup of the nonlinear operator.

Theorem 5.5. *Under Assumption 5.4(i), the value function υ is a viscosity subsolution to* (5.1)–(5.2). *Furthermore, if Assumption 5.4(ii) holds, then υ is a viscosity supersolution to* (5.1)–(5.2).

Proof. We first notice that the function υ is locally bounded by (2.10). We turn to the viscosity properties.
1. *Viscosity subsolution property on* $[0,T) \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$. Let $(t,\mu) \in [0,T) \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ and $(t_n,\mu_n)_n$ be a sequence of $[0,T) \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ such that

$$(t_n,\mu_n,\upsilon(t_n,\mu_n)) \xrightarrow[n\to+\infty]{} (t,\mu,\upsilon^*(t,\mu)).$$

Fix some $\pi \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d \times A))$ such that $\pi_1 = \mu$. Let $(\pi_n)_n$ be a sequence of $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d \times A))$ such that $\pi_{n,1} = \mu_n$ for $n \geq 1$ and $\pi_n \to \pi$ as $n \to +\infty$. Such a sequence can be constructed by decomposing π as $\pi^u(\mathrm{d}x,\mathrm{d}y) = \mu^u(\mathrm{d}x)\gamma^u(x,\mathrm{d}y)$ with γ a probability kernel and taking $\pi^u_n(\mathrm{d}x,\mathrm{d}y) = \mu^u_n(\mathrm{d}x)\gamma^u(x,\mathrm{d}y)$ for $u \in U$.

Applying Theorem B.3 with $t_n = t = 0$, there exists a sequence $(\underline{\xi}^n, \underline{\mathfrak{a}}^n)_n$ such that $\underline{\xi}^n$,$\mathfrak{a}^n$ are Borel maps from $U \times (0,1)$ to $\mathbb{R}^d$ and A, and $(\mathbb{P}_{(\xi^{n,u},\mathfrak{a}^{n,u})})_u = \pi_n$ with $\xi^{n,u} = \underline{\xi}^{n,u}(Z^u)$ and $\mathfrak{a}^{n,u} = \underline{\mathfrak{a}}^{n,u}(Z^u)$ for all $n \geq 1$. We define the control $\alpha \in \mathcal{A}$ by $\alpha^u_t = \mathfrak{a}(Z^u)$ for $t \in [0,T] \times U$. We consider the family of processes $(X^n)_n = (X^{n,u})_{n,u}$ as the unique solution to the SDE

$$\begin{cases} dX^{n,u}_s = b\big(u, X^{n,u}_s, \alpha^{n,u}_s, \mathbb{P}_{X^{n,\cdot}_s}, \mathbb{P}_{\alpha^{n,\cdot}_s}\big)\mathrm{d}s \\ \qquad\qquad + \sigma\big(u, X^{n,u}_s, \alpha^{n,u}_s, \mathbb{P}_{X^{n,\cdot}_s}, \mathbb{P}_{\alpha^{n,\cdot}_s}\big)\mathrm{d}W^u_s, \quad s \in [t_n,T], \\ X^{n,u}_{t_n} = \xi^{n,u}, \quad u \in U. \end{cases}$$

From the DPP given by Corollary 3.6, we have

$$\upsilon(t_n,\mu_n) \leq \int_U \mathbb{E}\Big[\int_{t_n}^{t_n+h} f\big(u, X^{n,u}_s, \alpha^{n,u}_s, \mathbb{P}_{X^{n,\cdot}_s}, \mathbb{P}_{\alpha^{n,\cdot}_s}\big)\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u) + \upsilon\big(t_n+h, \mathbb{P}_{X^{n,\cdot}_{t_n+h}}\big) \tag{5.3}$$

for $h > 0$ small enough. Let $\varphi \in \tilde{C}^{1,2}([0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)))$ such that $(\upsilon^* - \varphi)(t,\mu) = \max_{[0,T]\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))} (\upsilon^* - \varphi)$, so that with (5.3)

$$0 \leq \int_U \mathbb{E}\Big[\int_{t_n}^{t_n+h} f\big(u, X^{n,u}_s, \alpha^{n,u}_s, \mathbb{P}_{X^{n,\cdot}_s}, \mathbb{P}_{\alpha^{n,\cdot}_s}\big)\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u) + \varphi\big(t_n+h, \mathbb{P}_{X^{n,\cdot}_{t_n+h}}\big) - \varphi(t_n,\mu_n) + \gamma_n,$$

where we set $\gamma_n := \upsilon^*(t,\mu) - \upsilon(t_n,\mu_n) + \varphi(t_n,\mu_n) - \varphi(t,\mu) \to 0$ as n goes to infinity. By applying Itô formula in Theorem 4.6 to $\varphi(s, (\mathbb{P}_{X_s^{n,u}})_u)$ between t_n and $t_n + h$, and substituting into the above inequality, we then get

$$0 \leq \int_{t_n}^{t_n+h} \Big[\partial_t \varphi(s, \mu_{n,s}) + \int_U \mathcal{H}(u, s, \pi_{n,s}, \varphi)\lambda(\mathrm{d}u)\Big]\mathrm{d}s + \gamma_n,$$

where we set $\mu_{n,s}$, $\pi_{n,s}$ for $\mathbb{P}_{X_s^{n,\cdot}}$, $\mathbb{P}_{(X_s^{n,\cdot},\alpha_s^{n,\cdot})}$ to alleviate notations. By sending n to infinity, and using the continuity of all the involved functions, this implies

$$0 \leq \int_t^{t+h} \Big[\partial_t \varphi(s, \mu_s) + \int_U \mathcal{H}(u, s, \pi_s, \varphi)\lambda(\mathrm{d}u)\Big]\mathrm{d}s.$$

Dividing by h and sending h to 0, we get

$$-\partial_t \varphi(t,\mu) - \int_U \mathcal{H}(u, t, \pi, \varphi)\, \lambda(\mathrm{d}u) \leq 0.$$

Since this inequality holds for any $\pi \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d \times A))$ such that $\pi_1 = \mu$, we get the viscosity subsolution property on $[0,T) \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$.

2. *Viscosity subsolution property on* $\{T\} \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$. Let $(t_n,\mu_n)_n$ be a sequence of $[0,T) \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ such that

$$(t_n, \mu_n, \upsilon(t_n,\mu_n)) \xrightarrow[n\to+\infty]{} (T, \mu, \upsilon^*(T,\mu)).$$

Fix some $a \in A$ and define the control $\alpha \in \mathcal{A}$ by $\alpha_t^u = a$ for $t \in [0,T] \times U$. From Theorem B.3, there exists $\xi \in \mathcal{I}_T$ such that $\mathbb{P}_{\xi^\cdot} = \mu$ and $(\xi^n)_n$ such that $\xi^n \in \mathcal{I}_{t_n}$ $\mathbb{P}_{\xi^{n,\cdot}} = \mu_n$ for $n \geq 1$ and

$$\int_0^T \mathbb{E}\big[|\xi^{n,u} - \xi^u|^2\big]\lambda(\mathrm{d}u) \xrightarrow[n\to+\infty]{} 0. \tag{5.4}$$

Define the family of processes $(X^n)_n = (X^{n,u})_{n,u}$ as the unique solution to the SDE

$$\begin{cases} dX_s^{n,u} = b\big(u, X_s^{n,u}, \alpha_s^u, \mathbb{P}_{X_s^{n,\cdot}}, \mathbb{P}_{\alpha_s^\cdot}\big)\mathrm{d}s \\ \qquad\qquad + \sigma\big(u, X_s^{n,u}, \alpha_s^u, \mathbb{P}_{X_s^{n,\cdot}}, \mathbb{P}_{\alpha_s^\cdot}\big)\mathrm{d}W_s^u, \quad s \in [t_n, T], \\ X_{t_n}^{n,u} = \xi^{n,u}, \quad u \in U. \end{cases}$$

By definition of the value function υ, we have

$$\upsilon(t_n,\mu_n) \leq \int_U \mathbb{E}\Big[\int_{t_n}^T f\big(u, X_s^{n,u}, a, \mathbb{P}_{X_s^{n,\cdot}}, \delta_a\big)\,\mathrm{d}s + g\big(u, X_T^{n,u}, \mathbb{P}_{X_T^{n,\cdot}}\big)\Big]\,\lambda(\mathrm{d}u).$$

From (5.4) and Proposition B.4 and Assumption 3.1, we get

$$\int_U \mathbb{E}\Big[\int_{t_n}^T f\big(u, X_s^{n,u}, a, \mathbb{P}_{X_s^{n,\cdot}}, \delta_a\big)\,\mathrm{d}s + g\big(u, X_T^{n,u}, \mathbb{P}_{X_T^{n,\cdot}}\big)\Big]\,\lambda(\mathrm{d}u)$$
$$\xrightarrow[n\to+\infty]{} \quad \int_U \int_{\mathbb{R}^d} g(u,x,\mu)\,\mu^u(\mathrm{d}x)\,\lambda(\mathrm{d}u)$$

and so

$$v^*(T,\mu) \le \int_U \int_{\mathbb{R}^d} g(u,x,\mu)\,\mu^u(\mathrm{d}x)\,\lambda(\mathrm{d}u).$$

3. *Viscosity supersolution property on* $[0,T)\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$. We argue by contradiction and suppose that there exist $(t,\mu)\in[0,T)\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$, $\eta>0$ and $\varphi\in \hat{C}^{1,2}([0,T]\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)))$ such that

$$(v_* - \varphi)(t,\mu) = \min_{[0,T]\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))}(v_* - \varphi) \tag{5.5}$$

and

$$\partial_t\varphi(t,\mu) + \inf_{\pi\in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d\times A)),\ \pi_1=\mu} \int_U \mathcal{H}(u,t,\pi,\varphi)\,\lambda(\mathrm{d}u) \quad =: \quad 2\eta > 0.$$

From Assumption 5.4 (ii), there exists some $\varepsilon>0$ such that

$$\partial_t\varphi(s,\nu) + \inf_{\pi\in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d\times A)),\ \pi_1=\nu} \int_U \mathcal{H}(u,s,\pi,\varphi)\,\lambda(\mathrm{d}u) \quad \ge \quad \eta. \tag{5.6}$$

for $s\in[t,(t+\varepsilon)\wedge T]$ and $\nu\in B_{L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))}(\mu,\varepsilon)$. Let $(t_n,\mu_n)_n$ be a sequence of $[t,(t+\varepsilon)\wedge T)\times B_{L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))}(\nu,\varepsilon)$ such that

$$(t_n,\mu_n,v(t_n,\mu_n)) \xrightarrow[n\to+\infty]{} (t,\mu,v_*(t,\mu)). \tag{5.7}$$

From Theorem B.3, there exists ξ such that $\xi\in\mathcal{I}_t$ and $(\mathbb{P}_{\xi^u})_u=\mu$, and a sequence $(\xi^n)_n$ such that $\xi^n\in\mathcal{I}_{t_n}$, $(\mathbb{P}_{(\xi^{n,u})})_u=\mu_n$ for all $n\ge1$, and

$$\int_0^T \mathbb{E}\big[|\xi^{n,u}-\xi^u|^2\big]\lambda(\mathrm{d}u) \xrightarrow[n\to+\infty]{} 0.$$

We next define the sequence $(\rho_n)_n$ by

$$\rho_n = v(t_n,\mu_n) - \varphi(t_n,\mu_n) - \big(v_*(t,\mu)-\varphi(t,\mu)\big)$$

for $n\ge1$. From (5.5) and (5.7), we have $\rho_n\ge0$ for $n\ge1$, and $\rho_n\to0$, as n goes to infinity. We take a sequence $(h_n)_n$ of $(0,+\infty)$ such that

$$h_n \xrightarrow[n\to+\infty]{} 0 \ \text{ and }\ \frac{\rho_n}{h_n} \xrightarrow[n\to+\infty]{} 0.$$

From the DPP given by Corollary 3.6, there exists $\alpha^n\in\mathcal{A}$ such that

$$\begin{aligned} v(t_n,\mu_n) + \frac{\eta h_n}{2} \ge & \int_U \mathbb{E}\Big[\int_{t_n}^{\theta_n} f\big(u,X_s^{n,u},\alpha_s^{n,u},\mathbb{P}_{X_s^{n,\cdot}},\mathbb{P}_{\alpha_s^{n,\cdot}}\big)\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u) \\ & +\ v\big(\theta_n,\mathbb{P}_{X^{n,\cdot}_{\theta_n}}\big), \end{aligned} \tag{5.8}$$

where the family of processes $(X^n)_n = (X^{n,u})_{n,u}$ stands for the unique solution to the SDE

$$\begin{cases} dX_s^{n,u} = b\big(u, X_s^{n,u}, \alpha_s^{n,u}, \mathbb{P}_{X_s^{n,\cdot}}, \mathbb{P}_{\alpha_s^{n,\cdot}}\big)\mathrm{d}s \\ \qquad\qquad + \sigma\big(u, X_s^{n,u}, \alpha_s^{n,u}, \mathbb{P}_{X_s^{n,\cdot}}, \mathbb{P}_{\alpha_s^{n,\cdot}}\big)\mathrm{d}W_s^u, \quad s \in [t_n, T], \\ X_{t_n}^{n,u} = \xi^{n,u}, \quad u \in U, \end{cases}$$

for $n \geq 1$, and the sequence $(\theta_n)_n$ is defined by

$$\theta_n = \tau_n \wedge (t_n + h_n), \qquad \text{with} \qquad \tau_n \;=\; \inf\big\{s \geq t_n \;:\; \mathbb{P}_{X_s^{n,\cdot}} \notin B_{L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))}(\mu, \varepsilon)\big\} \wedge T.$$

From (5.8) and the definition of the sequence $(\rho_n)_n$, we have

$$\varphi(t_n, \mu_n) + \rho_n + \frac{\eta h_n}{2} \geq \int_U \mathbb{E}\Big[\int_{t_n}^{\theta_n} f\big(u, X_s^{n,u}, \alpha_s^{n,u}, \mathbb{P}_{X_s^{n,\cdot}}, \mathbb{P}_{\alpha_s^{n,\cdot}}\big)\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u) + \varphi\big(\theta_n, \mathbb{P}_{X_{\theta_n}^{n,\cdot}}\big),$$

and thus by applying Itô formula in Theorem 4.6, we get

$$\frac{1}{h_n}\int_{t_n}^{\theta_n} \Big\{\partial_t\varphi(s, \mu_{n,s}) + \int_U \mathcal{H}(u, s, \pi_{n,s}, \varphi)\lambda(\mathrm{d}u)\Big\}\,\mathrm{d}s \leq \frac{\rho_n}{h_n} + \frac{\eta}{2}.$$

where we set again $\mu_{n,s}$, $\pi_{n,s}$ for $\mathbb{P}_{X_s^{n,\cdot}}$, $\mathbb{P}_{(X_s^{n,\cdot}, \alpha_s^{n,\cdot})}$. From (5.6), this yields

$$\frac{\rho_n}{h_n} + \eta\Big(\frac{1}{2} - \frac{\theta_n - t_n}{h_n}\Big) \geq 0.$$

Now, we notice that $\tau_n \geq \gamma_n$ for n large enough, where

$$\gamma_n := \inf\Big\{s \geq t_n \;:\; \int_U \mathbb{E}\Big[|X_s^{n,u} - \xi^{u,n}|^2\Big]\lambda(\mathrm{d}u) \geq \frac{\varepsilon^2}{2}\Big\} \wedge T.$$

From Assumption 5.4 (i) and classical estimates on diffusion processes, we have $\inf_{n\geq 1}(\gamma_n - t_n) > 0$, and thus

$$\frac{\theta_n - t_n}{h_n} \xrightarrow[n\to+\infty]{} 1.$$

Therefore, we get a contradiction by sending n to ∞ in (5.9).

4. *Viscosity supersolution property on* $\{T\} \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$. Let $(t_n, \mu_n)_n$ be a sequence of $[0, T) \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ such that

$$(t_n, \mu_n, \upsilon(t_n, \mu_n)) \xrightarrow[n\to+\infty]{} (T, \mu, \upsilon_*(T, \mu)).$$

By definition of the function υ, there exists a control $\alpha^n \in \mathcal{A}$ such that

$$\upsilon(t_n, \mu_n) + \frac{1}{n} \geq \int_U \mathbb{E}\Big[\int_{t_n}^T f\big(u, X_s^{n,u}, \alpha_s^{n,u}, \mathbb{P}_{X_s^{n,\cdot}}, \mathbb{P}_{\alpha_s^{n,\cdot}}\big)\,\mathrm{d}s + g\big(u, X_T^{n,u}, \mathbb{P}_{X_T^{n,\cdot}}\big)\Big]\,\lambda(\mathrm{d}u).$$

From Assumption 5.4 (i) and classical estimates on diffusion processes we get

$$\int_U \mathbb{E}\Big[\int_{t_n}^T f\big(u, X_s^{n,u}, \alpha_s^{n,u}, \mathbb{P}_{X_s^{n,\cdot}}, \mathbb{P}_{\alpha_s^{n,\cdot}}\big)\,\mathrm{d}s\Big]\,\lambda(\mathrm{d}u) \xrightarrow[n\to+\infty]{} 0.$$

Therefore, using the continuity of g, we get by sending n to $+\infty$

$$\upsilon_*(T,\mu) \geq \int_U \int_{\mathbb{R}^d} g(u,x,\mu)\,\mu^u(\mathrm{d}x)\,\lambda(\mathrm{d}u).$$

□

Remark 5.6. From Definition 5.3, we notice that if $w \in \tilde{C}^{1,2}([0,T]\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)))$ is a viscosity solution to (5.1)–(5.2), then w is a classical solution as one can take $\varphi = w$ as a test function. In particular, under the additional assumption $\upsilon \in \tilde{C}^{1,2}([0,T]\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d)))$, we get from Theorem 5.5 that υ is a classical solution to (5.1)–(5.2).

6. Conclusion: discussion on the results and perspectives

The results presented in this paper concern the optimal control of the system governed by the dynamics in (2.2). The main issue related to such a system is the presence of an uncountable family of independent random processes and variables which induces a lack of measurability in the variable $u \in U$. Our approach allows to overcome this issue and to derive a related dynamic programming equation.

Following [10], an alternative approach would be to consider a randomized version of the label $u \in U$. More precisely, it consists to suppose that $u \in U$ is also a component of the state of the system with an initial distribution given by λ. This leads to consider a $U\times\mathbb{R}^d$-valued controlled process $\tilde{X} = (U, X)$ solution to the SDE

$$\begin{cases} \mathrm{d}\tilde{X}_s = \tilde{b}\big(\tilde{X}_s, \tilde{\alpha}_s, \mathbb{P}_{\tilde{X}_s}, \mathbb{P}_{\tilde{\alpha}_s}\big)\mathrm{d}s + \tilde{\sigma}\big(\tilde{X}_s, \tilde{\alpha}_s, \mathbb{P}_{\tilde{X}_s}, \mathbb{P}_{\tilde{\alpha}_s}\big)\mathrm{d}W_s, \quad s\in[t,T], \\ \tilde{X}_t = (Y, \underline{\xi}^Y(W_{[0,t]}, Z)), \\ \tilde{\alpha}_s = \alpha(Y, s, W_{\cdot\wedge s}, Z), \end{cases} \tag{6.1}$$

with Y independent of (W,Z) such that $\mathbb{P}_Y = \lambda$, $\tilde{b}(\tilde{x}, a, \tilde{\mu}) = (0, b(u,x,a,\mu_u))$ and $\tilde{\sigma}(\tilde{x}, a, \tilde{\mu}) = (0, \sigma(u,x,a,\mu_u))$ for $\tilde{x} = (u,x) \in U\times\mathbb{R}^d$ and $(\mu^u)_u$ the desintegration of $\tilde{\mu} \in \mathcal{P}_2^\lambda(U\times\mathbb{R}^d)$ w.r.t. λ. Here, $\mathcal{P}_2^\lambda(U\times\mathbb{R}^d)$ is the set of measures in $\mathcal{P}_2(U\times\mathbb{R}^d)$ with first marginal equal to λ, equipped with the standard Wasserstein distance, W is a standard Brownian motion and Z is an independent uniformly distributed random variable.

The space $\mathcal{P}_2^\lambda(U\times\mathbb{R}^d)$ is actually isometric to the space $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ considered throughout this paper. To see this, define the isometry in the following way: $\mu \in L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ is mapped to $\tilde{\mu} \in \mathcal{P}_2^\lambda(U\times\mathbb{R}^d)$ where $\tilde{\mu}$ is defined by $\int_{U\times\mathbb{R}^d} f(u,x)\tilde{\mu}(\mathrm{d}u,\mathrm{d}x) = \int_{U\times\mathbb{R}^d} f(u,x)\mu^u(\mathrm{d}x)\lambda(\mathrm{d}u)$, for all bounded measurable functions f; the inverse of such map is provided by the disintegration theorem, and we can easily verify that this map preserves distances.

We also note that the standard functional derivative for functions defined on $\mathcal{P}_2^\lambda(U\times\mathbb{R}^d)$ coincides with the flat derivative introduced in Definition 4.1 and that the Itô formula for smooth functions defined on $[0,T]\times\mathcal{P}_2^\lambda(U\times\mathbb{R}^d)$ is equivalent to the one presented in Theorem 4.6.

One can show under mild assumptions that the system $\tilde{X}$, solution to (6.1), identifies in law with the system $(X^u)_{u\in U}$ solution to (2.5) in the following sense

$$\mathbb{E}[g(\tilde{X})] = \int_U \mathbb{E}[g(u, X^u)]\lambda(\mathrm{d}u)$$

for any measurable function $g : U \times C^d_{[t,T]} \to \mathbb{R}$. In particular, the functional cost J can be written as follows

$$J(t, \xi, \alpha) = \tilde{J}(t, \tilde{\xi}, \alpha) := \mathbb{E}[g(\tilde{X}_T)] + \int_t^T \mathbb{E}[f(\tilde{X}_s, \tilde{\alpha}_s)]\mathrm{d}s,$$

and the control problem becomes a standard mean-field control problem.

However, considering the label as a component of the state variable has some disadvantages. The first one is that one has to impose continuity conditions on the coefficients with respect to the variable u, which is not natural in our original problem. On the contrary, our approach avoids to impose such conditions as we only suppose that the coefficients are measurable with respect to the variable $u \in U$. Moreover, to derive a PDE related to our control problem, one embeds the original problem into the space of probability measures. In particular, one needs to consider derivatives in the space of probability measures on $U \times \mathbb{R}^d$. Therefore, the possible probability distributions for the variable $u \in U$ might be any probability measure on U, which is not natural, regarding our original problem. Our approach avoids this problem by fixing the distribution of the variable u to λ and working on each X^u for $u \in U$. The choice between the two formulations has possible analogies. For instance, when dealing with time-dependent coefficients, one may remove explicit dependence on time by treating the time variable as an additional label state variable. However, as in the present case, this reformulation often leads to unnecessary assumptions or to less precise results.

We also notice that, even if it induces measurability issues, the formulation (2.5) has its own interest as it is an original and interesting problem which requires a specific study. This study differs from the classical approach for (mean field) SDEs. In particular, the proof of the existence involves new arguments compared to the classical mean field case.

Concerning the research perspectives on the subject, several questions remain open. The first question related to the previous discussion concerns the system (2.5). We indeed expect it to be the strong limit of the particle system (1.1). Such a result has not been proved yet, but would stress the importance of the study of (2.5) itself, compared to the system (6.1) which would be the weak limit of (1.1).

A second related question is the propagation of chaos for the value function. We indeed expect the convergence of the value function related to (1.1) to the value function V defined in this paper. Such a result opens the door for setting numerical methods for the approximation of the value function and possible optimal strategies.

Finally, a natural question that remains open is the uniqueness of the solution to the HJB equation. This question deserves an own study as it is a strong issue which has been solved only in the recent period for the classical mean field case. The heterogeneity of the agents makes this question more challenging and needs a novel approach to be studied.

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Data availability statement

We did not use data

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Appendix A. Some results on the collection of state equations

A.1 Proof of Theorem 2.6

We write the proof in the case $b \equiv 0$, the general case being completely similar.

We fix $\xi \in \mathcal{I}_t$ and $\alpha \in \mathcal{A}$. Adapting an idea in Proposition 2.1 [15], the existence for equation (2.2) will be obtained by a fixed point argument in the space $L^2_\lambda(\mathcal{P}_2(C_{[t,T]}))$. For any ν in this space and for any $u \in U$ let us denote $X^{\nu,u}$ the solution to

$$\begin{cases} dX^{\nu,u}_s = \sigma\left(u, X^{\nu,u}_s, \alpha^u_s, (\nu^v_s)_v, \mathbb{P}_{\alpha^\cdot_s}\right) \mathrm{d}W^u_s, \\ \alpha^u_s = \alpha(u, s, W^u_{\cdot\wedge s}, Z^u), \quad s \in [t,T], \\ X^{\nu,u}_t = \xi^u, \quad u \in U. \end{cases} \tag{A.1}$$

These equations differ from the original ones because the given $(\nu^v_s)_v$ replaces $\left(\mathbb{P}_{X^v_s}\right)_v$. The equations are no longer coupled so that they can be solved individually. Under our assumptions each of them satisfies standard Lipschitz conditions and it has a continuous $\mathbb{F}^u$-adapted solution, unique up to a $\mathbb{P}$-null set. Let us define $\mathbf{\Psi}(\nu) = (\mathbf{\Psi}(\nu)^u)_u$ setting $\mathbf{\Psi}(\nu)^u = \mathbb{P}_{X^{\nu,u}}$, the law of the process $X^{\nu,u}$ seen as a probability on $C^d_{[t,T]}$. We will prove the following claim:

$$\begin{gathered} \text{the map } u \mapsto \mathcal{L}(X^{\nu,u}, W^u_{[0,T]}, Z^u) \text{ is Borel measurable} \\ \text{from } U \text{ to } \mathcal{P}_2(C^d_{[t,T]} \times C^\ell_{[0,T]} \times (0,1)). \end{gathered} \tag{A.2}$$

Admitting this for a moment, standard estimates on the stochastic equation in (A.1) show that

$$\mathbb{E}\Big[\sup_{s\in[t,T]} |X^{\nu,u}_s|^2\Big] \le c\left(1 + \mathbb{E}[|\xi^u|^2] + \int_0^T \Big\{\mathbb{E}[d_A(\alpha^u_s, 0)^2] + \mathcal{W}_2(\nu^u_s, \delta_0)^2 + \mathcal{W}_2(\mathbb{P}_{\alpha^v_s}, \delta_0)^2\Big\}\,\mathrm{d}s\right)$$

for some constant $c > 0$ (that does not depend on ν). Noting that $\mathcal{W}_2(\mathbb{P}_{\alpha^u_s}, \delta_0)^2 \le \mathbb{E}[d_A(\alpha^u_s, 0)^2]$ and recalling the admissibility condition (2.4) we find $\|X^\nu\|^2 = \int_U \mathbb{E}[\sup_{s\in[t,T]} |X^{\nu,u}_s|^2]\,\lambda(\mathrm{d}u) < \infty$ and so $X^\nu \in \mathcal{S}_t$. This implies that $(\mathbb{P}_{X^u})_u$ belongs to $L^2_\lambda(\mathcal{P}_2(C^d_{[t,T]}))$, so that the map $\mathbf{\Psi} : L^2_\lambda(\mathcal{P}_2(C^d_{[t,T]})) \to L^2_\lambda(\mathcal{P}_2(C^d_{[t,T]}))$ is well defined.

We will next prove the following claim:

$$\mathbf{\Psi} \text{ has unique fixed point } \bar\nu \text{ in } L^2_\lambda(\mathcal{P}_2(C^d_{[t,T]})). \tag{A.3}$$

This leads immediately to the required conclusion. Indeed, the process $X^{\bar\nu}$ corresponding to the fixed point is clearly a solution. Moreover, if $(\tilde X^u)_u \in \mathcal{S}_t$ is another solution then its law $(\mathbb{P}_{\tilde X^u})_u$ also belongs to $L^2_\lambda(\mathcal{P}_2(C^d_{[t,T]}))$ and it is a fixed point of $\mathbf{\Psi}$; it must therefore coincide with $\bar\nu$ and it follows that $X^u = \tilde X^u$ for λ-almost all $u \in U$ since they are both solutions to the same equations (A.1).

In order to conclude the proof we have to prove the two claims.

Step I: proof of claim (A.2).

We note that $\hat w \oplus \check w$ depends on $\check w$ only through its increments, namely denoting $\check w - \check w_t$ the function $(\check w(s) - \check w(t))_{s\in[t,T]}$ we have $\hat w \oplus \check w = \hat w \oplus (\check w - \check w_t)$.

Given $t \in [0,T]$ and a Borel measurable function $\alpha : U \times [0,T] \times C^\ell_{[0,T]} \times (0,1) \to A$, we define a Borel measurable function $\tilde\alpha : U \times [0,T] \times C^\ell_{[t,T]} \times C^\ell_{[0,t]} \times (0,1) \to A$ setting

$$\tilde\alpha(u, s, \check w, \hat w, z) = \alpha(u, s, \hat w \oplus \check w, z) \tag{A.4}$$

for $u \in U$, $s \in [0,T]$, $\hat w \in C^\ell_{[0,t]}$, $\check w \in C^\ell_{[t,T]}$, $z \in (0,1)$. We note that this formula establishes a bijection between those classes of functions whose inverse is

$$\alpha(u, s, w, z) = \tilde\alpha(u, s, w_{[0,t]}, w_{[t,T]}, z),$$

where we recall that $w_{[0,t]}$, $w_{[t,T]}$ denote the restrictions of $w \in C^\ell_{[0,T]}$ to the indicated intervals. Finally we note that, for $s \in [t,T]$,

$$\alpha(u,s,w_{s\wedge\cdot},z) = \tilde{\alpha}(u,s,w_{s\wedge\cdot},w_{[0,t]},z). \tag{A.5}$$

where we write $w_{s\wedge\cdot}$ instead of the more cumbersome $\big(w_{[t,T]}\big)_{s\wedge\cdot}$.

Below we need a similar notation for functions defined on spaces of paths. Given $\Phi : C^d_{[t,T]} \times C^\ell_{[0,T]} \times (0,1) \to \mathbb{R}$ it is convenient to consider the function $\tilde{\Phi} : C^d_{[t,T]} \times C^\ell_{[t,T]} \times C^\ell_{[0,t]} \times (0,1) \to \mathbb{R}$ defined by

$$\tilde{\Phi}(x,\check{w},\hat{w},z) = \Phi(x,\hat{w}\oplus\check{w},z), \quad x \in C^d_{[t,T]}, \check{w} \in C^\ell_{[t,T]}, \hat{w} \in C^\ell_{[0,t]}, z \in (0,1). \tag{A.6}$$

Using these notations we write equation (A.1) in a different way. Recalling (A.4) and (A.5) we first have

$$\alpha^u_s = \tilde{\alpha}(u,s,W^u_{\cdot\wedge s},W^u_{[0,t]},Z^u) = \tilde{\alpha}(u,s,W^u_{\cdot\wedge s}-W^u_t,W^u_{[0,t]},Z^u), \quad s \in [t,T].$$

We note that $(W^u_{[0,t]},Z^u)$ is independent of the increments $W^u_{\cdot\wedge s}-W^u_t$ and the law of (W^u,Z^u) is the product $\mathbb{W}_T \otimes m$ of the Wiener measure $\mathbb{W}_T$ on $C^\ell_{[0,T]}$ and the Lebesgue measure m on $(0,1)$. Given a bounded continuous $\phi : A \to \mathbb{R}$ we have

$$\begin{aligned}
\mathbb{E}\Big[\phi(\alpha^u_s)\Big] &= \mathbb{E}\Big[\phi\Big(\tilde{\alpha}(u,s,W^u_{\cdot\wedge s},W^u_{[0,t]},Z^u)\Big)\Big] \\
&= \int_{C^\ell_{[0,T]}\times(0,1)} \mathbb{E}\Big[\phi\Big(\tilde{\alpha}(u,s,W^u_{\cdot\wedge s},w'_{[0,t]},z')\Big)\Big]\,\mathcal{L}(W^u,Z^u)(\mathrm{d}w'\,\mathrm{d}z') \\
&= \int_{C^\ell_{[0,T]}\times(0,1)} \mathbb{E}\Big[\phi\Big(\tilde{\alpha}(u,s,W^u_{\cdot\wedge s},w'_{[0,t]},z')\Big)\Big]\,(\mathbb{W}_T\otimes m)(\mathrm{d}w'\,\mathrm{d}z').
\end{aligned}$$

Setting $\alpha^{u,w,z}_s = \tilde{\alpha}(u,s,W^u_{\cdot\wedge s},w_{[0,t]},z)$ this shows that the laws $\mathbb{P}_{\alpha^u_s}$ may be written

$$\mathbb{P}_{\alpha^u_s} = \int_{C^\ell_{[0,T]}\times(0,1)} \mathbb{P}_{\alpha^{u,w',z'}_s}\,(\mathbb{W}_T\times m)(\mathrm{d}w'\,\mathrm{d}z').$$

Equation (A.1) then becomes

$$\begin{cases}
dX^{\nu,u}_s = \sigma\Big(u,X^{\nu,u}_s,\alpha^u_s,(\nu^v_s)_v,\Big(\int_{C^\ell_{[0,T]}\times(0,1)}\mathbb{P}_{\alpha^{v,w',z'}_s}\,(\mathbb{W}_T\times m)(\mathrm{d}w'\,\mathrm{d}z')\Big)_v\Big)\,\mathrm{d}W^u_s,\ s\in[t,T],\\
X^{\nu,u}_t = \xi^u,\\
\alpha^u_s = \tilde{\alpha}(u,s,W^u_{\cdot\wedge s},W^u_{[0,t]},Z^u).
\end{cases} \tag{A.7}$$

Let us consider the analogue of this equation where the random elements ξ^u, $W^u_{[0,t]}$, Z^u are "freezed" at given points $x \in \mathbb{R}^d$, $w \in C^\ell_{[0,t]}$, $z \in (0,1)$, namely

$$\begin{cases}
\mathrm{d}X^{\nu,u,x,w,z}_s = \sigma\Big(u,X^{\nu,u,x,w,z}_s,\alpha^{u,w,z}_s,(\nu^v_s)_v,\Big(\int_{C^\ell_{[0,T]}\times(0,1)}\mathbb{P}_{\alpha^{v,w',z'}_s}\,(\mathbb{W}_T\times m)(\mathrm{d}w'\,\mathrm{d}z')\Big)_v\Big)\,\mathrm{d}W^u_s\\
X^{\nu,u,x,w,z}_t = x,\\
\alpha^{u,w,z}_s = \tilde{\alpha}(u,s,W^u_{\cdot\wedge s},w,z).
\end{cases} \tag{A.8}$$

For fixed $u \in U$, this is a stochastic equation depending measurably on the parameters x,w,z and it admits as a solution a measurable function $(\omega,s,x,w,z) \mapsto X^{\nu,u,x,w,z}_s(\omega)$. Measurability is understood in the following sense. Since the equation is driven by the increments of the Brownian motion $(W^u_s - W^u_t)_{s\ge t}$ the solution is predictable with respect to the corresponding filtration, *i.e.* measurable for the corresponding σ-algebra on $[t,T]$, say $\mathcal{P}^t$; measurability of $(\omega,s,x,w,z) \mapsto X^{\nu,u,x,w,z}_s(\omega)$ is understood with respect to the σ-algebra $\mathcal{P}^t \otimes \mathcal{B}(\mathbb{R}^d \times C^\ell_{[t,T]} \times (0,1))$: see [20] and the references therein.

Therefore we may consider the composition $X^{\nu,u,\xi^u,W^u_{[0,t]},Z^u}$ obtained substituting (x,w,z) with $(\xi^u, W^u_{[0,t]}, Z^u)$. Using the fact that the latter is independent of $X^{\nu,u,x,w,z}$ we may see that this is well defined; indeed, if $\tilde X^{\nu,u,x,w,z}$ is another solution to (A.8) with the same measurability properties we have

$$G(x,w,z) := \mathbb{E}[\sup_{s\in[t,T]} |X^{\nu,u,x,w,z}_s - \tilde X^{\nu,u,x,w,z}_s|^2] = 0$$

for every x, w, z and by independence

$$\mathbb{E}[\sup_{s\in[t,T]} |X_s^{\nu,u,\xi^u,W^u_{[0,t]},Z^u} - \tilde X_s^{\nu,u,\xi^u,W^u_{[0,t]},Z^u}|^2] = \mathbb{E}[G(\xi^u, W^u_{[0,t]}, Z^u)] = 0$$

so that $X^{\nu,u,\xi^u,W^u_{[0,t]},Z^u}$ and $\tilde X^{\nu,u,\xi^u,W^u_{[0,t]},Z^u}$ are indistinguishable. By similar arguments one concludes that the process $X^{\nu,u,\xi^u,W^u_{[0,t]},Z^u}$ satisfies equation (A.7) and therefore it coincides with $X^{\nu,u}$, up to a $\mathbb{P}$-null set, for λ-almost all u.

Our aim is to prove that the law $\mathcal{L}(X^{\nu,u}, W^u_{[0,T]}, Z^u)$ (a measure on $C^d_{[t,T]} \times C^\ell_{[0,T]} \times (0,1)$) depends in a measurable way on $u \in U$. To this end we use the criterion (2.1) and we consider the integral of an arbitrary bounded continuous function $\Phi : C^d_{[t,T]} \times C^\ell_{[0,T]} \times (0,1) \to \mathbb{R}$ that we write in the form

$$\begin{aligned}\int_{C^d_{[t,T]}\times C^\ell_{[0,T]}\times(0,1)} &\Phi(x,w,z)\, \mathcal{L}(X^{\nu,u}, W^u_{[0,T]}, Z^u)(\mathrm{d}x\,\mathrm{d}w\,\mathrm{d}z) = \mathbb{E}\Big[\Phi\left(X^{\nu,u}, W^u_{[0,T]}, Z^u\right)\Big]\\ &= \mathbb{E}\Big[\Phi\Big(X^{\nu,u,\xi^u,W^u_{[0,t]},Z^u}, W^u_{[0,T]}, Z^u\Big)\Big] = \mathbb{E}\Big[\tilde\Phi\Big(X^{\nu,u,\xi^u,W^u_{[0,t]},Z^u}, W^u_{[t,T]}, W^u_{[0,t]}, Z^u\Big)\Big],\end{aligned}$$

using the notation $\tilde\Phi$ introduced in (A.6). We recall that we may replace $W^u_{[t,T]}$ by its increments $W^u_{[t,T]} - W^u_t$, which are independent of $(W^u_{[0,t]}, Z^u)$. Noting that equation (A.8) is driven by $W^u_{[t,T]} - W^u_t$ we conclude that the pair $(X^{\nu,u,x,w,z}, W^u_{[t,T]} - W^u_t)$ is independent of $(W^u_{[0,t]}, Z^u, \xi^u)$. It follows that

$$\begin{aligned}&\mathbb{E}\Big[\Phi\left(X^{\nu,u}, W^u_{[0,T]}, Z^u\right)\Big]\\ &\quad = \int_{C^\ell_{[0,t]}\times(0,1)\times\mathbb{R}^d} \mathbb{E}\Big[\tilde\Phi\left(X^{\nu,u,x,w,z}, W^u_{[t,T]}, w, z\right)\Big]\, \mathcal{L}(W^u_{[0,t]}, Z^u, \xi^u)(\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}x).\end{aligned}$$

Now let us take an arbitrary complete probability space $(\hat\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ with an $\mathbb{R}^\ell$-valued standard Brownian motion $\hat W$ and denote $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t\in[0,T]}$ the corresponding completed Brownian filtration. For every $u \in U$, $x \in \mathbb{R}^d$, $w \in C^\ell_{[0,t]}$, $z \in (0,1)$ we consider the equation

$$\begin{cases} d\hat X^{\nu,u,x,w,z}_s = \sigma\Big(u, \hat X^{\nu,u,x,w,z}_s, \hat\alpha^{u,w,z}_s, (\nu^v_s)_v, \Big(\int_{C^\ell_{[0,T]}\times(0,1)} \hat{\mathbb{P}}_{\hat\alpha^{v,w',z'}_s}\, (\mathbb{W}_T\times m)(\mathrm{d}w'\,\mathrm{d}z')\Big)_v\Big)\, \mathrm{d}\hat W_s\\ \hat X^{\nu,u,x,w,z}_t = x,\\ \hat\alpha^{u,w,z}_s = \tilde\alpha(u, s, \hat W_{\cdot\wedge s}, w, z).\end{cases}$$

Since $\hat W$ does not depend on u, this is a stochastic equation depending measurably on all the parameters u, x, w, z (including u) and it admits as its solution a measurable function $(\omega, s, u, x, w, z) \mapsto \hat X^{\nu,u,x,w,z}_s(\omega)$. Similar as before, measurability is understood with respect to the σ-algebra $\mathcal{P}^t \otimes \mathcal{B}(U \times \mathbb{R}^d \times C^\ell_{[t,T]} \times (0,1))$: see [20]. Comparing this equation with (A.8) we see that they have the same coefficients and are both driven by the increments of Brownian motions - W^u and $\hat W$ respectively - on the interval $[t,T]$. As we have strong uniqueness to the considered SDEs, we conclude that on the space $C^d_{[t,T]} \times C^\ell_{[t,T]}$ the law of $(X^{\nu,u,x,w,z}, W^u - W^u_t)$ under $\mathbb{P}$ is the same as the law of $(\hat X^{\nu,u,x,w,z}, \hat W - \hat W_t)$ under $\hat{\mathbb{P}}$. It follows that

$$\mathbb{E}\Big[\Phi\left(X^{\nu,u}, W^u_{[0,T]}, Z^u\right)\Big] \tag{A.9}$$
$$= \int_{C^\ell_{[0,t]}\times(0,1)\times\mathbb{R}^d} \hat{\mathbb{E}}\Big[\tilde{\Phi}\left(\hat{X}^{\nu,u,x,w,z}, \hat{W}_{[t,T]}, w, z\right)\Big]\, \mathcal{L}(W^u_{[0,t]}, Z^u, \xi^u)(\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}x).$$

Now what occurs under the sign $\hat{\mathbb{E}}$ is a measurable function of u, x, w, z. Since we are assuming that $\mathcal{L}(W^u_{[0,t]}, Z^u, \xi^u)$ depends measurably on u we conclude that $\mathcal{L}(X^{\nu,u}, W^u_{[0,T]}, Z^u)$ is also a measurable function of u (as a measure on $C^d_{[t,T]} \times C^\ell_{[0,T]} \times (0,1)$). This concludes the proof of the claim (A.2).

We note for later use that, when the function Φ only depends on its first argument, equation (A.9) becomes a formula for the map $\boldsymbol{\Psi}(\nu) = (\boldsymbol{\Psi}(\nu)^u)_u$ introduced at the beginning of the proof. Indeed, recalling that $\boldsymbol{\Psi}(\nu)^u = \mathbb{P}_{X^{\nu,u}}$, it follows from (A.9) that for every continuous bounded $\phi : C^d_{[t,T]} \to \mathbb{R}$,

$$\int_{C^d_{[t,T]}} \phi(x)\, \boldsymbol{\Psi}(\nu)^u(\mathrm{d}x) = \mathbb{E}\Big[\phi\left(X^{\nu,u}\right)\Big]$$
$$= \int_{C^\ell_{[0,t]}\times(0,1)\times\mathbb{R}^d} \hat{\mathbb{E}}\Big[\phi\left(\hat{X}^{\nu,u,x,w,z}\right)\Big]\, \mathcal{L}(W^u_{[0,t]}, Z^u, \xi^u)(\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}x).$$

Step II: proof of claim (A.3).

For any $r \in [t,T]$ we consider the space $L^2_\lambda(\mathcal{P}_2(C^d_{[t,r]}))$ with the corresponding distance, that will be denoted $\mathbf{d}^r_t$. Given $\nu, \mu \in L^2_\lambda(\mathcal{P}_2(C^d_{[t,T]}))$, let $(X^{\nu,u})_u$, $(X^{\mu,u})_u$ denote the corresponding solutions to (A.1). For suitable constants C_1, C_2 we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[t,r]} |X^{\nu,u}_s - X^{\mu,u}_s|^2\Big] \\
&\le C_1\, \mathbb{E}\int_t^r \Big|\sigma\left(u, X^{\nu,u}_s, \alpha^u_s, (\nu^v_s)_v, \left(\mathbb{P}_{\alpha^v_s}\right)_v\right) - \sigma\left(u, X^{\mu,u}_s, \alpha^u_s, (\mu^v_s)_v, \left(\mathbb{P}_{\alpha^v_s}\right)_v\right)\Big|^2 \mathrm{d}s \\
&\le C_2 \int_t^r \Big\{\mathbb{E}\left[|X^{\nu,u}_s - X^{\mu,u}_s|^2\right] + \mathbf{d}(\nu_s, \mu_s)^2\Big\}\, \mathrm{d}s \\
&\le C_2 \int_t^r \Big\{\sup_{q\in[t,s]} \mathbb{E}\left[|X^{\nu,u}_q - X^{\mu,u}_q|^2\right] + \mathbf{d}^s_t(\nu,\mu)^2\Big\}\, \mathrm{d}s.
\end{aligned}$$

Next we note that $(X^{\nu,u}, X^{\mu,u})_u$ (a collection of $\mathbb{R}^{d\times d}$-valued processes) satisfies a stochastic equation to which the assumptions of Theorem 2.6 apply. In particular, it starts at time t from the initial condition $(\xi^u, \xi^u)_u$, which is admissible, since the map $u \mapsto \mathcal{L}(\xi^u, \xi^u, W_{[0,t]}, Z^u)$ is Borel measurable. So we can apply the already proved claim (A.2) and conclude in particular that $u \mapsto \mathcal{L}(X^{\nu,u}, X^{\mu,u})$ is Borel measurable. It follows that both sides of the previously displayed inequality are measurable functions of u. Integrating with respect to $\lambda(\mathrm{d}u)$ and applying the Gronwall lemma yields

$$\mathbf{d}^r_t(\Psi(\nu), \Psi(\mu))^2 \le \int_U \mathbb{E}\Big[\sup_{s\in[t,r]} |X^{\nu,u}_s - X^{\mu,u}_s|^2\Big]\, \lambda(\mathrm{d}u) \le C \int_t^r \mathbf{d}^s_t(\nu,\mu)^2\, \mathrm{d}s, \qquad r \in [t,T], \tag{A.10}$$

for some constant $C > 0$ that only depends on the Lipschitz constants of b, σ, on T and on $\lambda(U)$. Setting $r = T$ we obtain

$$\mathbf{d}^T_t(\Psi(\nu), \Psi(\mu))^2 \le \|X^\nu - X^\mu\|^2 \le C \cdot (T-t)\, \mathbf{d}^T_t(\nu,\mu)^2$$

which proves in particular the continuity of Ψ. Iterating (A.10) one proves that

$$\mathbf{d}^r_t(\boldsymbol{\Psi}^{(k+1)}(\nu), \boldsymbol{\Psi}^{(k+1)}(\mu))^2 \le \frac{C^{k+1}}{k!} \int_t^r \mathbf{d}^s_t(\nu,\mu)^2 (r-s)^k\, \mathrm{d}s. \tag{A.11}$$

Choosing an arbitrary $\nu^{(0)} \in L^2_\lambda(\mathcal{P}_2(C^d_{[t,T]}))$ and setting $\nu^{(k+1)} = \boldsymbol{\Psi}(\nu^{(k)})$ for $k \geq 0$, it follows that

$$\mathbf{d}_t^T(\nu^{(k+1)}, \nu^{(k)})^2 \leq \frac{C^k (T-t)^k}{k!} \mathbf{d}_t^T(\nu^{(1)}, \nu^{(0)})^2.$$

Now standard arguments allow to conclude that the sequence $(\nu^{(k)})_k$ is Cauchy for $\mathbf{d}_t^T$ and it converges in $L^2_\lambda(\mathcal{P}_2(C^d_{[t,T]}))$ to a limit, denoted $\bar{\nu}$, which is a fixed point of the map $\boldsymbol{\Psi}$. The uniqueness of the fixed point follows from (A.11). The claim (A.3) is proved. □

A.2 Uniqueness in law

Proposition A.1. *Let $t \in [0,T]$ and $\alpha : U \times [0,T] \times C^\ell_{[0,T]} \times (0,1) \to A$ a Borel measurable function Fix $\xi = (\xi^u)_u \in \mathcal{I}_t$ and denote by $X = (X^u)_u$ the unique solution to (2.2).*

For $u \in U$, we consider an $\mathbb{R}^\ell$-valued random process $(\tilde{W}^u_t)_{t\geq 0}$, a real random variable $\tilde{Z}^u$ and an $\mathbb{R}^d$-valued random variable $\tilde{\xi}^u$ (possibly defined on a different probability space) such that

$$\mathcal{L}(\xi^u, W^u_{[0,t]}, Z^u) = \mathcal{L}(\tilde{\xi}^u, \tilde{W}^u_{[0,t]}, \tilde{Z}^u) \tag{A.12}$$

for all $u \in U$. We define $(\tilde{X}^u)_u$ as the unique solution to the SDE

$$\begin{cases} \mathrm{d}\tilde{X}^u_s = b\left(u, \tilde{X}^u_s, \tilde{\alpha}^u_s, \mathbb{P}_{\tilde{X}^\cdot_s}, \mathbb{P}_{\tilde{\alpha}^\cdot_s}\right) \mathrm{d}s \\ \qquad\quad + \sigma\left(u, \tilde{X}^u_s, \tilde{\alpha}^u_s, \mathbb{P}_{\tilde{X}^\cdot_s}, \mathbb{P}_{\tilde{\alpha}^\cdot_s}\right) \mathrm{d}\tilde{W}^u_s, \quad s \in [t,T], \\ \tilde{X}^u_t = \tilde{\xi}^u, \quad u \in U, \end{cases}$$

where $\tilde{\alpha}$ is defined by

$$\tilde{\alpha}^u_t = \alpha(u, t, \tilde{W}^u_{\cdot\wedge t}, \tilde{Z}^u), \quad t \in [0,T],\, u \in U.$$

Then we have

$$\mathcal{L}(X^u, W^u_{[0,T]}, Z^u) = \mathcal{L}(\tilde{X}^u, \tilde{W}^u_{[0,T]}, \tilde{Z}^u)$$

for all $u \in U$.

Proof. We only sketch the proof. In the proof of Theorem 2.6 the solution was obtained *via* a fixed point for the map $\nu \mapsto \Psi(\nu)$ introduced there. The fixed point can be obtained by a Picard iteration scheme $\nu^{n+1} = \Psi(\nu^n)$ starting from $\nu^0 = \mathcal{L}(\xi) = \mathcal{L}(\tilde{\xi})$. In view of (A.12), formula (A.9) makes it clear that at each iteration we have

$$\mathcal{L}(X^{\nu^n,u}, W^u_{[0,T]}, Z^u) = \mathcal{L}(\tilde{X}^{\nu^n,u}, \tilde{W}^u_{[0,T]}, \tilde{Z}^u)$$

for all $u \in U$. We know that the sequence (ν^n) converges in $L^2_\lambda(\mathcal{P}_2(C^d_{[t,T]}))$ to the fixed point. This allows to pass to the limit in (A.9) and conclude that $\mathcal{L}(X^u, W^u_{[0,T]}, Z^u) = \mathcal{L}(\tilde{X}^u, \tilde{W}^u_{[0,T]}, \tilde{Z}^u)$ as required. □

Appendix B. Some auxiliary results

Proposition B.1. *Let $t \in [0,T]$ and $\xi \in \mathcal{I}_t$. Then, there exists a Borel map $\tilde{\xi} : U \times C^\ell_{[0,t]} \times (0,1) \to \mathbb{R}^d$ such that such that*

$$\mathcal{L}\big(\tilde{\xi}^u(W_{[0,t]}, Z), W_{[0,t]}, Z\big) = \mathcal{L}\big(\xi^u, W^u_{[0,t]}, Z^u\big)$$

for every $u \in U$ and for any choice of the random pair (W, Z) (defined on an arbitrary probability space), where $W = (W_t)_{t\in[0,T]}$ is an $\mathbb{R}^\ell$-valued standard Brownian motion and Z is a real random variable having uniform distribution in $(0,1)$ and independent of W.

We first need to prove the following result.

Lemma B.2. *For any family $(Y^u)_u$ of random variables uniformly distributed on $(0,1)$ and any family $(\Phi^u)_u$ of Borel maps from $(0,1)$ to some Polish space S such that $u \mapsto \mathcal{L}(\Phi^u(Y^u))$ is Borel measurable, there exists a Borel map $\tilde{\Phi} : U \times (0,1) \to S$ such that $\mathcal{L}(\Phi^u(Y^u)) = \mathcal{L}(\tilde{\Phi}(u,Y))$ for every $u \in U$ where Y is any random variable uniformly distributed on $(0,1)$ (defined on an arbitrary probability space).*

Proof. To prove this claim note that setting $Q(u,A) = \mathcal{L}(\Phi^u(Y^u))(A)$, for $u \in U$ and any Borel set $A \subset U$, we define a transition kernel from U to S, by the measurability assumption. It is a classical result of Skorohod (used in the proof of the Skorohod representation theorem) that there exists a Borel function $\tilde{\Phi}(u,\cdot) : (0,1) \to S$ carrying the Lebesgue measure on $(0,1)$ to the measure $Q(u,\cdot)$ and therefore satisfying $\mathcal{L}(\tilde{\Phi}(u,Y)) = Q(u,\cdot) = \mathcal{L}(\Phi^u(Y^u))$. The function $\tilde{\Phi}(u,\cdot)$ is obtained from $Q(u,\cdot)$ in a constructive way which shows that, since Q is a kernel, the function $\tilde{\Phi}(u,y)$ is in fact Borel measurable in $(u,y) \in U \times (0,1)$: for a detailed proof see for instance the proof of Theorem 3.1.1 in [21]. □

Proof of Proposition B.1. Fix a bijection $\psi : C^\ell_{[0,t]} \times (0,1) \to (0,1)$ such that ψ and ψ^{-1} are Borel measurable (such a map exists since $C^\ell_{[0,t]} \times (0,1)$ is an uncountable Polish space: see *e.g.* Corollary 7.16.1 in [22]). Then set $X^u = \psi(W^u_{[0,t]}, Z^u)$, for $u \in U$. Then $(X^u)_{u\in U}$ is a family of identically distributed random variables with common c.d.f. denoted by F that is continuous as ψ is one to one and $\mathcal{L}(W^u_{[0,t]}, Z^u)$ has no atom. In particular, the random variables $Y^u := F(X^u)$, $u \in U$, are uniformly distributed on $[0,1]$ and $(W^u_{[0,t]}, Z^u) = \psi^{-1}(F^{-1}(Y^u))$ for $u \in U$, where F^{-1} stands for the generalized inverse of F. Then we have

$$\mathcal{L}\big(\underline{\xi}^u(W^u_{[0,t]}, Z^u), W^u_{[0,t]}, Z^u\big) = \mathcal{L}\big(\Phi^u(Y^u)\big)$$

where $\Phi^u(y) \in S := \mathbb{R}^d \times C^\ell_{[0,t]} \times (0,1)$ is defined as $\Phi^u(y) = \big(\underline{\xi}^u(\psi^{-1}(F^{-1}(y))), \psi^{-1}(F^{-1}(y))\big)$ for $y \in (0,1)$ and $u \in U$. Define $Y = F(\psi(W_{[0,t]}, Z))$. Then, Y is uniformly distributed and there exists some Borel map $\tilde{\Phi} :\ U \times (0,1) \to S$ such that $\mathcal{L}(\Phi^u(Y^u)) = \mathcal{L}(\tilde{\Phi}(u,Y))$ for every $u \in U$. If we denote by $\tilde{\Phi}_1$ the first component of $\tilde{\Phi}$, the map $\tilde{\xi}^u = \tilde{\Phi}_1(u, F(\psi(\cdot)))$ is a solution to our initial problem. □

Proposition B.3. *Let $(t_n, \mu_n)_n$ be a sequence of $[0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ and $(t,\mu) \in [0,T] \times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$ such that $(t_n,\mu_n) \to (t,\mu)$ as $n \to +\infty$. There exist Borel maps $\underline{\xi}$ and $(\underline{\xi}_n)_n$ from $U \times C^\ell_{[0,t]} \times (0,1)$ to $\mathbb{R}^d$ s.t. $(\mathbb{P}_{\underline{\xi}^u(W^u_{[0,t]}, Z^u)})_u = \mu$, $(\mathbb{P}_{\underline{\xi}^{n,u}(W^u_{[0,t_n]}, Z^u)})_u = \mu_n$ for all $n \geq 1$ and*

$$\int_0^T \mathbb{E}\Big[\big|\underline{\xi}^{n,u}(W^u_{[0,t_n]}, Z^u) - \underline{\xi}^u(W^u_{[0,t]}, Z^u)\big|^2\Big]\lambda(\mathrm{d}u) \xrightarrow[n\to+\infty]{} 0.$$

Proof. As in the proof of the Proposition B.1, we simply need to prove the following statement:

For any family $(Y^u)_u$ of random variables uniformly distributed on $(0,1)$ and any Polish space S, there exists Borel maps Φ and Φ_n, $n \geq 1$, from $U \times (0,1)$ to S such that $\mathcal{L}(\Phi(u,Y^u)) = \mu^u$, $\mathcal{L}(\Phi_n(u,Y^u)) = \mu^u_n$ for every $u \in U$ and

$$\int_0^T \mathbb{E}\Big[\big|\mathcal{L}(\Phi(u,Y^u)) - \mathcal{L}(\Phi_n(u,Y^u))\big|^2\Big]\lambda(\mathrm{d}u) \xrightarrow[n\to+\infty]{} 0.$$

Let $(\pi^n)_n$ be a sequence $L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d))$ such that $\pi^u_{n,1} = \mu^u$ and $\pi^u_{n,2} = \mu^u_n$ for all $u \in U$, and

$$\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \pi^u_n(\mathrm{d}x,\mathrm{d}y) \leq \int_0^T \mathcal{W}_2^2(\mu^u_n, \mu^u)\lambda(\mathrm{d}u) + \frac{1}{n}.$$

We now disintegrate the measure π_n by writing

$$\pi^u_n(\mathrm{d}x,\mathrm{d}y) = \mu^u(\mathrm{d}x)\gamma^u_n(x,\mathrm{d}y)$$

for $n \geq 1$. From Lemma B.2, there exists a Borel function $\tilde{\Phi} :\ U \times (0,1) \to S$ such that $\mathcal{L}(\Phi(u,Y^u)) = \mu^u$ for $u \in U$. Still using Lemma B.2 with $U \times \mathbb{R}^d$ in place of U, there exist a Borel functions $\psi_n :\ U \times S \times (0,1) \to S$ and $\mathcal{L}(\psi(u,x,Y^u)) =$

$\gamma_n^u(x,.)$ for $x \in S$ and $u \in U$. Now take a Borel map $\zeta:\ (0,1) \to (0,1)\times(0,1)$ such that $\zeta = (\zeta_1, \zeta_2) \sim \mathcal{U}_{(0,1)\times(0,1)}$ and define the Borel maps Φ and Φ_n by

$$\Phi(u,x) \;=\; \tilde{\Phi}(u,\zeta_1(x)), \qquad \Phi_n(u,x) \;=\; \psi_n(u,\tilde{\Phi}(\zeta_1(x)),\zeta_2(x))$$

for $x \in (0,1)$ and $u \in U$ and $n \geq 1$. Then Φ and Φ_n are solutions to the problem. □

Proposition B.4. *Let $\alpha \in \mathcal{A}$, $t \in [0,T]$, $(t_n)_n$ a sequence of $[0,T]$, $\xi \in \mathcal{I}_t$ and $(\xi^n)_n$ a sequence such that $\xi^n \in \mathcal{I}_{t_n}$ for all $n \geq 1$. Suppose that*

$$\big(t_n, \mathbb{P}_{\xi^{n,\cdot}}\big) \xrightarrow[n\to+\infty]{} \big(t, \mathbb{P}_{\xi^{\cdot}}\big)$$

in $\mathbb{R}\times L^2_\lambda(\mathcal{P}_2(\mathbb{R}^d))$. Let X and (X^n) be the respective solutions to (2.2) with initial conditions ξ and ξ^n at time t and t_n and control α. Then we have

$$(\mathbb{P}_{X^{n,u}_{\cdot\vee t_n}})_u \xrightarrow[n\to+\infty]{} (\mathbb{P}_{X^u_{\cdot\vee t}})_u$$

in $L^2_\lambda(\mathcal{P}_2(C^d_{[0,T]}))$.

Proof. We take W a $\mathbb{R}^d$ valued brownian motion and Z an independent $(0,1)$-uniformly distributed random variable. From Proposition B.3, there are Borel maps $\underline{\xi}$ and $(\underline{\xi}_n)_n$ from $U\times C^\ell_{[0,t]}\times(0,1)$ to $\mathbb{R}^d$ such that $(\mathbb{P}_{\underline{\xi}^u(W_{[0,t]},Z)})_u = (\mathbb{P}_{\xi^u})_u$, $(\mathbb{P}_{\underline{\xi}^{n,u}(W_{[0,t_n]},Z)})_u = (\mathbb{P}_{\xi^{n,u}})_u$ for all $n\geq 1$ and

$$\int_0^T \mathbb{E}\Big[\big|\underline{\xi}^{n,u}(W_{[0,t_n]},Z) - \underline{\xi}^u(W_{[0,t]},Z)\big|^2\Big]\lambda(\mathrm{d}u) \xrightarrow[n\to+\infty]{} 0.$$

We define $(\tilde{X}^u)_u$ as the unique solution to the SDE

$$\begin{cases} d\tilde{X}^u_s &=& b\Big(u,\tilde{X}^u_s,\tilde{\alpha}^u_s,\mathbb{P}_{\tilde{X}^{\cdot}_s},\mathbb{P}_{\tilde{\alpha}^{\cdot}_s}\Big)\,\mathrm{d}s \\ & & +\sigma\Big(u,\tilde{X}^u_s,\tilde{\alpha}^u_s,\mathbb{P}_{\tilde{X}^{\cdot}_s},\mathbb{P}_{\tilde{\alpha}^{\cdot}_s}\Big)\,\mathrm{d}W_s, \quad s\in[t,T], \\ \tilde{X}^u_t &=& \underline{\xi}^u(W_{[0,t]},Z), \quad u \in U, \end{cases}$$

and $(\tilde{X}^{n,u})_u$ as the unique solution to the SDE

$$\begin{cases} d\tilde{X}^{n,u}_s &=& b\Big(u,\tilde{X}^{n,u}_s,\tilde{\alpha}^u_s,\big(\mathbb{P}_{\tilde{X}^{n,v}_s}\big)_v,\big(\mathbb{P}_{\tilde{\alpha}^{n,v}_s}\big)_v\Big)\,\mathrm{d}s \\ & & +\sigma\Big(u,\tilde{X}^{n,u}_s,\tilde{\alpha}^u_s,\big(\mathbb{P}_{\tilde{X}^{n,v}_s}\big)_v,\big(\mathbb{P}_{\tilde{\alpha}^{v}_s}\big)_v\Big)\,\mathrm{d}W_s, \quad s\in[t,T], \\ \tilde{X}^{n,u}_{t_n} &=& \underline{\xi}^{n,u}(W_{[0,t_n]},Z), \quad u \in U, \end{cases}$$

where $\tilde{\alpha}$ is defined by

$$\tilde{\alpha}^u_t = \tilde{\alpha}(u,t,W_{\cdot\wedge t},Z), \quad t\in[0,T],\, u\in U.$$

Using Proposition A.1 with $\tilde{W}^u = W$ and $\tilde{Z}^u = Z$ for $u \in U$, we get

$$\mathcal{L}(X^u,W^u,Z^u) \;=\; \mathcal{L}(\tilde{X}^u,W,Z), \qquad \text{and} \qquad \mathcal{L}(X^{n,u},W^u,Z^u) \;=\; \mathcal{L}(\tilde{X}^{n,u},W,Z),$$

for all $n\geq 1$ and $u \in U$. Now, using (B.1) we get from classical estimates on diffusion processes that

$$\int_0^T \mathbb{E}\Big[\sup_{s\in[t\vee t_n,T]}\big|\tilde{X}^{n,u}_s - \tilde{X}^u_s\big|^2\Big]\lambda(\mathrm{d}u) \xrightarrow[n\to+\infty]{} 0,$$

which gives the result. □