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Polyharmonic potential theory on the Poincaré disk ☆

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ABSTRACT

We consider the open unit disk $\mathbb{D}$ equipped with the hyperbolic metric and the associated hyperbolic Laplacian $\mathfrak{L}$. For $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, a λ-polyharmonic function of order n is a function $f : \mathbb{D} \to \mathbb{C}$ such that $(\mathfrak{L} - \lambda I)^n f = 0$. If $n = 1$, one gets λ-harmonic functions. Based on a Theorem of Helgason on the latter functions, we prove a boundary integral representation theorem for λ-polyharmonic functions. For this purpose, we first determine n^{th}-order λ-Poisson kernels. Subsequently, we introduce the λ-polyspherical functions and determine their asymptotics at the boundary $\partial\mathbb{D}$, i.e., the unit circle. In particular, this proves that, for eigenvalues not in the interior of the L^2-spectrum, the zeroes of these functions do not accumulate at the boundary circle. Hence the polyspherical functions can be used to normalise the n^{th}-order Poisson kernels. By this tool, we extend to this setting several classical results of potential theory: namely, we study the boundary behaviour of λ-polyharmonic functions, starting

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with Dirichlet and Riquier type problems and then proceeding to Fatou type admissible boundary limits.

1. Introduction

The aim of this article is to initiate a detailed study of the potential theory associated with the polyharmonic, and more generally, λ-polyharmonic functions for the hyperbolic Laplacian $\mathfrak{L}$, that is, the solutions f of $(\mathfrak{L} - \lambda I)^n f = 0$, for $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$.

A *polyharmonic function of order* n on a Euclidean domain D is a complex-valued function f on D that belongs to the kernel of the n^{th} iterate of classical Euclidean Laplacian: $\Delta^n f \equiv 0$. The study of polyharmonic functions goes back to work in the 19$^{\text{th}}$ century, and continues to be a very active topic. See e.g. the books by Aronszajn, Creese and Lipkin [2] or by Gazzola, Grunau and Sweers [10]. A classical theorem of Almansi [1] says that if the domain D is star-like with respect to the origin, then every polyharmonic function of order n has a unique decomposition

$$f(z) = \sum_{k=0}^{n-1} |z|^{2k}\, h_k(z)\,,$$

where each h_k is harmonic on D, and $|z|$ is the Euclidean length of $z \in D$. In particular, let the domain be the unit disk

$$\mathbb{D} = \{z = x + \mathfrak{i}\, y \in \mathbb{C} : |z| = \sqrt{x^2 + y^2} < 1\}.$$

Assume for the moment that in Almansi's decomposition, each h_k is non-negative. Then it has an integral representation over the boundary $\partial\mathbb{D}$ of the disk, that is, the unit circle, with respect to the *Poisson kernel*

$$P(z,\xi) = \frac{1 - |z|^2}{|\xi - z|^2} \qquad (z \in \mathbb{D}\,,\ \xi = e^{\mathfrak{i}\,\phi} \in \partial\mathbb{D}) \tag{1}$$

Thus, Almansi's decomposition on the disk reads as

$$f(z) = \sum_{k=0}^{n-1} \int_{\partial\mathbb{D}} |z|^{2k}\, P(z,\xi)\, d\nu_k(\xi)\,, \tag{2}$$

where $\nu_0, \ldots, \nu_{n-1}$ are non-negative Borel measures on the unit circle. Without requiring non-negativity of the h_k, the result still remains true, taking *analytic functionals* (i.e., certain distributions) ν_k instead of Borel measures: this follows from results by Helgason [12], [13] which will be of crucial importance further below.

We now change our viewpoint and view $\mathbb{D}$ as the *hyperbolic* or *Poincaré disk* with the hyperbolic length element and resulting metric

$$ds = \frac{2\sqrt{dx^2 + dy^2}}{1 - |z|^2} \quad \text{and} \quad \rho(z, w) = \log \frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|}. \tag{3}$$

We note that the Poisson kernel can be written as

$$P(z, \xi) = e^{-\mathfrak{h}(z,\xi)} \quad \text{with} \quad \mathfrak{h}(z, \xi) = \lim_{w \to \xi} \Big(\rho(w, z) - \rho(w, 0)\Big), \tag{4}$$

the *Busemann function*. The hyperbolic Laplace (or Laplace-Beltrami) operator in the variable $z = x + \mathfrak{i}\, y$ is

$$\mathfrak{L} = \frac{(1 - |z|^2)^2}{4} \left(\partial_x^2 + \partial_y^2\right). \tag{5}$$

The harmonic functions for the two Laplacians on the disk clearly coincide, but this is no more true for polyharmonic functions of higher order. While there is an abundant ongoing literature on polyharmonic functions in the Euclidean setting, we are not aware of an extensive body of work for the hyperbolic Laplacian, or more generally, for Laplace-Beltrami operators on manifolds. A few references are, for example, Chung, Sario and Wang [6], Chung [5] as well as Schimming and Belger [31] plus some of the citations in the latter paper, and also Jaming [18].

The first main aim of this note is to provide an integral representation in the spirit of (2) for *hyperbolically* polyharmonic functions of order n. More generally, we consider λ-*polyharmonic* functions of order n, that is, solutions $f : \mathbb{D} \to \mathbb{C}$ of

$$(\mathfrak{L} - \lambda\, I)^n f = 0\,.$$

Here, I is the identity operator, and we are taking the n^{th} iterate of $\mathfrak{L} - \lambda\, I$, where $\lambda \in \mathbb{C}$. If $n = 1$, we speak of a λ-*harmonic function.* Considering λ as an "eigenvalue", one should be careful with respect to the space on which the operator acts. Indeed, we are *not* referring to the action of $\mathfrak{L}$ as a self-adjoint operator on $L^2(\mathbb{D}, \mathsf{area}_h)$, where area_h is the hyperbolic area measure of $\mathbb{D}$ and the corresponding spectrum $(-\infty\,, -\frac{1}{4}\,]$ is continuous. The mapping

$$\lambda(s) = s^2 - \frac{1}{4}\,, \quad s \in \mathbb{C}\,,\ \Re(s) \geqslant 0$$

maps the half-open half plane $\{s \in \mathbb{C} : \Re(s) \geqslant 0\} \setminus \{\mathfrak{i}\, t : t < 0\}$ bijectively onto $\mathbb{C}$. We write $s(\lambda) = \sqrt{\lambda + \frac{1}{4}}$ for the inverse mapping, where the square root of $re^{\mathfrak{i}\,\phi}$ is $\sqrt{r}\, e^{\mathfrak{i}\,\phi/2}$ for $r \geqslant 0$ and $\phi \in (-\pi\,,\, \pi]$.

Here is our first main result.

Theorem 1.1. *Every λ-polyharmonic function $f : \mathbb{D} \to \mathbb{C}$ of order n has a unique representation of the form*

$$f(z) = \begin{cases} \displaystyle\sum_{k=0}^{n-1} \int_{\partial\mathbb{D}} \mathfrak{h}(z,\xi)^k \, P(z,\xi)^{s(\lambda)+1/2} \, d\nu_k(\xi)\,, & \text{if } \lambda \neq -\frac{1}{4}\,, \\ \displaystyle\sum_{k=0}^{n-1} \int_{\partial\mathbb{D}} \mathfrak{h}(z,\xi)^{2k} \, P(z,\xi)^{1/2} \, d\nu_k(\xi)\,, & \text{if } \lambda = -\frac{1}{4}\,, \end{cases}$$

where $\nu_0, \dots, \nu_{n-1}$ are analytic functionals on $\partial\mathbb{D}$.

We postpone the precise definition of these functionals to §2, and we shall also rescale the kernels in the integrals in a suitable manner to get the (order $n+1$) λ-polyharmonic Poisson kernels $P_n(z, \xi \,|\, \lambda)$, $n \geqslant 0$; see Proposition 2.6.

Our proof of Theorem 1.1 in §2 is inspired by related results obtained in a discrete setting, mostly in recent work of Picardello and Woess [28] on polyharmonic functions on general trees, that was preceded by a long paper by Cohen et al. [7] who had used rather involved methods to obtain an integral representation for polyharmonic functions on a regular tree with respect to the standard graph Laplacian. (In [28], this is generalised and simplified.) Further motivation for the present work came from Sava-Huss and Woess [30], who studied the boundary behaviour of polyharmonic functions on regular trees. There are many profound analogies between the hyperbolic disk and regular trees. In the potential theoretic setting considered here, see the first part of the note by Boiko and Woess [3] for an exposition of those analogies.[1]

The natural next goal is to study the asymptoptic behaviour of λ-polyharmonic functions. For this purpose, but also by inherent interest and for further possible applications, in §3, we introduce the family of polyspherical functions $\Phi_n(z \,|\, \lambda)$, i.e., suitably normalised λ-polyharmonic functions of order $n+1$ which only depend on $r = |z|$. Here, the functions $\Phi_0(z \,|\, \lambda)$ for $\lambda \in \mathbb{C}$ are the classical spherical functions of the Poincaré disk. A major step, in itself of interest, is to determine the asymptotic behaviour of $\Phi_n(z \,|\, \lambda)$ near $\partial\mathbb{D}$, that is, as $r \to 1$ or equivalently, $R = \rho(z, 0) \to \infty$; see Theorem 3.4.

This is important because, for the study of the boundary behaviour of λ-polyharmonic functions one needs a suitable normalisation of the polyharmonic kernels $P_n(z, \xi \,|\, \lambda)$, in order to compensate for their growth or decay; this normalisation is then accomplished in §4; it extends the classical case $n = 0$ via the laborious computations of §3. Indeed, when $n = 0$, it is well-known that it is appropriate to normalise λ-harmonic functions by the λ-spherical function, see e.g. Michelson [27] and Sjögren [33], and for regular trees Korányi and Picardello [21]. This cannot be done for $\lambda \in (-\infty\,,\, -\frac{1}{4})$, because for these values of λ the zeroes of the λ-spherical functions accumulate at the boundary circle, while for all other values of λ, there are no zeroes at all; see Remark 3.3.

[1] In the formula for the hyperbolic Laplacian – which is (5) here – one of the two squares is missing in [3].

So for arbitrary n, our normalisation consists in dividing by $\Phi_n(z \mid \lambda)$, which is feasible since it follows from Theorem 3.4 that this function has no zeroes close to the boundary circle. In §4, we show that the resulting normalised kernels are good approximate identities at the boundary points, so that the classical convergence results hold for transforms of functions and measures on $\partial\mathbb{D}$; see Proposition 4.6.

§5 is dedicated to another important issue: continuous extensions from boundary data. We first limit attention to $n = 0$ and show that for the normalized kernel $P(z, \xi \mid \lambda)/\Phi_0(z \mid \lambda)$, the solution of the Dirichlet problem with continuous boundary data is unique for any (even complex) $\lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4})$ (Theorem 5.1). Then we extend the result to $n > 0$ by formulating a suitable version of the Riquier problem, adapted to the fact that the quotient of lower and higher order polyspherical functions tends to zero at the boundary, and provide such a solution (Corollary 5.4), which is inherently non-unique.

§6 answers another fundamental question on the asymptotic behaviour of λ-polyharmonic functions, the Fatou theorem. Theorem 6.5 yields admissible non-tangential convergence of the normalised transforms of measures on $\partial\mathbb{D}$ for $\lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4}\,]$. For the critical value $\lambda = -\frac{1}{4}$, we even have a wider approach region. Along the classical guidelines, the proofs are based on maximal inequalities.

The last §7 is devoted to related examples (in the standard case $\lambda = 0$), discussions and open questions. In particular, we provide all details of an example outlined to us by A. Borichev: a harmonic (indeed, analytic) function $h(z)$ such that $h(z)/R$ is bounded but has no radial limits at the boundary, as $R \to \infty$ where $R = \rho(z, 0) \sim \Phi_1(z|0)$, the biharmonic spherical function. In all the paper, for the reader's benefit, we give most of the details of the (sometimes lengthy) computations.

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2. Integral representation

For each $\delta > 0$, consider the space $\mathcal{H}(\mathbb{A}_\delta)$ of all holomorphic functions on the open annulus

$$\mathbb{A}_\delta = \{z \in \mathbb{C} : 1 - \delta < |z| < 1 + \delta\}\,.$$

The space is equipped with the topology of uniform convergence on compact sets. The space $\mathcal{H}(\partial\mathbb{D})$ of analytic functions on the unit circle consists of all functions $g : \partial\mathbb{D} \to \mathbb{C}$ which possess an extension in $\mathcal{H}(\mathbb{A}_\delta)$ for some $\delta = \delta(g) > 0$. The topology on $\mathcal{H}(\partial\mathbb{D})$ is the inductive limit of the topologies of $\mathcal{H}(\mathbb{A}_\delta)$ as $\delta \to 0$.

Definition 2.1. An *analytic functional* ν on $\partial\mathbb{D}$ is an element of the dual space of $\mathcal{H}(\partial\mathbb{D})$. We write

$$\int_{\partial\mathbb{D}} g\, d\nu := \nu(g)\,, \quad g \in \mathcal{H}(\partial\mathbb{D})\,.$$

A good way to understand the action of ν on $\mathcal{H}(\partial\mathbb{D})$ is described in [8, p. 114], see Köthe [23]: let $\nu_n = \nu(e^{-\mathrm{i}\, n\phi})$, $n \in \mathbb{Z}$ be the Fourier coefficients of ν. Then

$$\limsup_{|n|\to\infty} |\nu_n|^{1/|n|} \geqslant 1$$

(and this characterises the analytic functionals). If $g \in \mathcal{H}(\partial\mathbb{D})$ then the Fourier expansion $g(e^{\mathrm{i}\,\phi}) = \sum_{n\in\mathbb{Z}} g_n\, e^{\mathrm{i}\, n\phi}$ is such that

$$\limsup_{|n|\to\infty} |g_n|^{1/|n|} < 1.$$

Then

$$\int_{\partial\mathbb{D}} g\, d\nu = \sum_{n\in\mathbb{Z}} g_n\, \overline{\nu_n}\,. \tag{6}$$

For more on analytic functionals, resp. hyperfunctions, see e.g. Hörmander [16, Chapter IX] or Schlichtkrull [32].

It will be useful to write

$$P(z,\xi\,|\,\lambda) = P(z,\xi)^{s(\lambda)+1/2}.$$

The following results from an elementary and well-known computation.

Lemma 2.2. *For $\lambda \in \mathbb{C}$ and $\xi \in \partial\mathbb{D}$, the function $z \mapsto P(z,\xi\,|\,\lambda)$ satisfies*

$$\mathfrak{L}P(\,\cdot\,,\xi\,|\,\lambda) = \lambda\, P(\,\cdot\,,\xi\,|\,\lambda)\,.$$

We note here that we can write

$$P(z,\xi) = \frac{1-|z|^2}{(\xi - z)(1/\xi - \bar{z})}\,. \tag{7}$$

In this form, for fixed $z \in \mathbb{D}$ and any $\lambda \in \mathbb{C}$, the function $\xi \mapsto P(z,\xi\,|\,\lambda)$ is in $\mathcal{H}(\mathbb{A}_\delta)$ for $\delta = 1 - |z|$. Thus, as a function of ξ in the unit circle, it is in $\mathcal{H}(\partial\mathbb{D})$. We now recall an important result.

Theorem 2.3 *(Helgason [12], [13, Section V.6]). For any $\lambda \in \mathbb{C}$, every λ-harmonic function h for the hyperbolic Laplacian on the Poincaré disk has a unique representation*

$$h(z) = \int_{\partial\mathbb{D}} P(z,\xi\,|\,\lambda)\, d\nu(\xi)\,,$$

where $\nu = \nu^h$ is an analytic functional on $\partial\mathbb{D}$.

A very readable proof of this and several related results are contained in the beautiful expository paper by Eymard [8].

Remark 2.4. If the λ-harmonic function h is positive real, then $\lambda \geqslant -\frac{1}{4}$. Indeed, it is well known that positive λ-harmonic functions exist precisely when $\lambda \geqslant -\frac{1}{4}$; see e.g. Sullivan [35, Thm. 2.1].

For non-negative h, the functional ν^h is a non-negative Borel measure. This follows from general Martin boundary theory, see e.g. Karpelevič [19] or Taylor [36].

Before proving Theorem 1.1, we need to find suitable polyharmonic versions of the Poisson kernel. That is, for each $n \in \mathbb{N}_0$, we want to have a kernel of the form

$$P_n(z,\xi\,|\,\lambda) = g_{n,\lambda}\big(-\mathfrak{h}(z,\xi)\big)\,P(z,\xi\,|\,\lambda) \tag{8}$$

which satisfies

$$(\mathfrak{L} - \lambda\, I)^n P_n(z,\xi\,|\,\lambda) = P(z,\xi\,|\,\lambda)\,. \tag{9}$$

For this purpose, we shall use the following.

Lemma 2.5. *Let $f \in C^2(\mathbb{R})$ and set*

$$Q_f(z,\xi\,|\,\lambda) = f\big(-\mathfrak{h}(z,\xi)\big)\,P(z,\xi\,|\,\lambda)\,.$$

Then

$$(\mathfrak{L} - \lambda\, I)Q_f(z,\xi\,|\,\lambda) = Q_g(z,\xi\,|\,\lambda)\,,$$

where $g = f'' + 2s\,f'$ and $s = s(\lambda)$.

Proof. Here (and frequently also later) we shall use the fact that the Busemann function, hence also the Poisson kernel (as well as the Laplacian), are rotation invariant:

$$\mathfrak{h}(e^{\mathrm{i}\,\alpha}z, e^{\mathrm{i}\,\alpha}\xi) = \mathfrak{h}(z,\xi) \quad \text{for all} \;\; z \in \mathbb{D}\,,\ \xi \in \partial\mathbb{D}. \tag{10}$$

Thus, it is sufficient to consider $\xi = 1$. Furthermore, from the Poincaré disk model of the hyperbolic plane we can first pass to the upper half plane model via the inverse Cayley transform, where in the new coordinates (u,v) with $u \in \mathbb{R}$ and $v > 0$, the hyperbolic Laplacian transforms into $v^2(\partial^2 u + \partial^2 v)$ and the boundary point $1 \in \partial\mathbb{D}$ becomes $\mathfrak{i}\,\infty$. Then we make one more change of variables, setting $w = \log v$ to obtain the *logarithmic model,* where now $(u,w) \in \mathbb{R}^2$ and the hyperbolic Laplacian becomes

$$\mathfrak{L} = e^{2w}\partial_u^2 + \partial_w^2 - \partial_w \,. \tag{11}$$

In these coordinates, the Busemann function and Poisson kernel at $\mathfrak{i}\,\infty$ are

$$\mathfrak{h}\big((u,w),\mathfrak{i}\,\infty\big) = -w \quad \text{and} \quad P\big((u,w),\mathfrak{i}\,\infty\big) = e^w \,,$$

and

$$Q_f\big((u,w),\mathfrak{i}\,\infty|\lambda\big) = f(w)\, e^{(s+1/2)w} \,.$$

The statement now follows by applying the Laplacian in the form of (11). □

Proposition 2.6. *For $\lambda \neq -\frac{1}{4}$ and $s = s(\lambda)$, the kernel*

$$P_n(z,\xi\,|\,\lambda) = \frac{1}{n!(2s)^n}\big(-\mathfrak{h}(z,\xi)\big)^n\, P(z,\xi)^{s+1/2}$$

satisfies (9).

For $\lambda = -\frac{1}{4}$, where $s(\lambda) = 0$, identity (9) holds for

$$P_n(z,\xi| -\frac{1}{4}) = \frac{1}{(2n)!}\,\mathfrak{h}(z,\xi)^{2n}\, P(z,\xi)^{1/2}.$$

Proof. In order to find a function $g_{n,\lambda}$ as in (8), we start of course with $g_{0,\lambda} \equiv 1$. We proceed recursively, looking at each step for a function $f_n = f_{n,\lambda}$ such that

$$(\mathfrak{L} - \lambda\, I)\Big[f_{n,\lambda}\big(-\mathfrak{h}(z,\xi)\big)\, P(z,\xi\,|\,\lambda)\Big] = g_{n-1,\lambda}\big(-\mathfrak{h}(z,\xi\,|\,\lambda)\big)\, P(z,\xi\,|\,\lambda)\,. \tag{12}$$

The function f_n will then be replaced by the simpler $g_n = g_{n,\lambda}$ which satisfies (8) before proceeding to $n+1$. By Lemma 2.5, f_n must solve the differential equation

$$f_n'' + 2s\, f_n' = g_{n-1}\,, \quad s = s(\lambda). \tag{13}$$

The characteristic polynomial of (13) has roots 0 and $-2s$, when $\lambda \neq -\frac{1}{4}$. In the latter case, 0 is a double root.

We start with $\lambda \neq -\frac{1}{4}$ and $n = 1$. Since $g_0 = 1$, we are looking for a special solution of (13) of the form $f_1(w) = A_{1,1}w$, whence $A_{1,1} = 1/(2s)$. We get $g_1(w) = f_1(w) = w/(2s)$, and going back to the disc model, we obtain P_1 via (8), as proposed. We now prove by induction on n that by setting $g_n(w) = \dfrac{w^n}{n!(2s)^n}$ we obtain a solution for P_n.

Suppose this is true for all orders up to $n-1$. The right hand side of (13) is a polynomial of order $n-1$ in w. Hence there is a special solution of the form $f_n(w) = \sum_{k=1}^n A_{n,k}w^k$. The coefficients $A_{n,k}$ are obtained as solutions of a system of linear equations, yielding a solution of (12). However, by the induction hypothesis, the terms of order $k < n$ are annihilated when applying $(\mathfrak{L} - \lambda I)^n$ to $f_{n,\lambda}\big(-\mathfrak{h}(z,\xi)\big)\, P(z,\xi \,|\, \lambda)$, so that for (9) we only need $g_n(w) = A_{n,n} w^n$. Inserting f_n into (13), comparison of the highest order coefficients yields $2sn = \dfrac{w^{n-1}}{(n-1)!(2s)^{n-1}}$, which completes the induction step.

When $\lambda = -\frac{1}{4}$, the differential equation (13) simplifies to $f_n'' = g_{n-1}$. Here, we set $g_n = f_n$. Starting with $g_0 \equiv 1$, we integrate twice at each step and take just the highest appearing power: $g_1(w) = w^2/2$, $g_2(w) = w^4/4!$, and so on, so that $g_n(w) = w^{2n}/(2n)!$, as proposed. □

Note that when $\lambda = -\frac{1}{4}$, we even have $(\mathfrak{L} - \lambda I) P_n(z,\xi | -\frac{1}{4}) = P_{n-1}(z,\xi | -\frac{1}{4})$.

In the standard case $\lambda = 0$, we just write $P_n(z,\xi)$ for $P_n(z,\xi|0)$.

Proof of Theorem 1.1. Along with the Poisson kernel, also the function $\xi \mapsto P_n(z,\xi \,|\, \lambda)$ is in $\mathcal{H}(\mathbb{A}_\delta)$ for $\delta = 1 - |z|$, for every $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$ and $z \in \mathbb{D}$.

We claim that every λ-polyharmonic function f of order n has a unique representation of the form

$$f(z) = \sum_{k=0}^{n-1} f_k(z) \quad \text{with} \quad f_k(z) = \int_{\partial\mathbb{D}} P_k(z,\xi \,|\, \lambda)\, d\nu_k(\xi)\,, \tag{14}$$

where $\nu_0, \dots, \nu_{n-1}$ are analytic functionals on $\partial\mathbb{D}$. Furthermore, when $\lambda \geqslant -\frac{1}{4}$ is real, ν_k is a non-negative Borel measure if and only if

$$(\mathfrak{L} - \lambda I)^k g_k \geqslant 0\,, \quad \text{where} \quad g_k = f - (f_{n-1} + \cdots + f_{k+1})\,.$$

To prove this, we proceed by induction on n. For $n = 1$, this is Theorem 2.3. Suppose the statement is true for $n-1$. Let f be λ-polyharmonic of order n. Then $h = (\mathfrak{L} - \lambda I)^{n-1} f$ is λ-harmonic. By Theorem 2.3, there is a unique analytic functional ν_{n-1} on $\partial\mathbb{D}$ such that

$$h(z) = \int_{\partial\mathbb{D}} P(z,\xi \,|\, \lambda)\, d\nu_{n-1}(\xi)\,.$$

We set

$$f_{n-1}(z) = \int_{\partial\mathbb{D}} P_{n-1}(z,\xi \,|\, \lambda)\, d\nu_{n-1}(\xi)\,.$$

By Proposition 2.6,

$$(\mathfrak{L} - \lambda I)^{n-1} f_{n-1} = h = (\mathfrak{L} - \lambda I)^{n-1} f\,.$$

Thus, $f - f_{n-1}$ is λ-polyharmonic of order $n-1$, and we can apply the induction hypothesis to that function in order to get the representation of f. Uniqueness follows from

Helgason's Theorem 2.3. The statement on non-negativity for real $\lambda \geqslant -\frac{1}{4}$ is a consequence of the well known Poisson-Martin representation theorem for positive λ-harmonic functions. Since (14) differs from the proposed result only by the normalisation of the kernels P_n, the result is proved. □

The previous paper [28] on trees adopts a different method of proof, that could also be used here: it consists in differentiating $P(z,\xi\,|\,\lambda)$ with respect to λ instead of integrating a differential equation with respect to z, which is inherent in the proof of Proposition 2.6. The drawback is that this does not work at the critical value $\lambda = -\frac{1}{4}$, while on the other hand, it can be applied to more general domains as long as λ belongs to the L^2-resolvent set of the underlying Laplacian.

From now on, we use the representation formula (14) *for λ-polyharmonic functions, instead of the one of the statement of Theorem 1.1.*

3. Polyspherical functions

Definition 3.1. For $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$, the n^{th} λ-*polyspherical function* is

$$\Phi_n(z\,|\,\lambda) = \int_{\partial\mathbb{D}} P_n(z,\xi\,|\,\lambda)\,d\xi\,, \quad z \in \mathbb{D}\,,$$

where P_n is given by Proposition 2.6 and $d\xi = d\mathsf{m}(\xi)$ for the normalized Lebesgue measure m on the unit circle.

The function $\Phi_n(z\,|\,\lambda)$ is λ-polyharmonic of order $n+1$ and rotation invariant, i.e., it depends only on $|z|$. For $n=0$, we recover the classical spherical functions $\Phi(z\,|\,\lambda)$, where we omit the index 0: for $z = r\,e^{i\phi} \in \mathbb{D}$,

$$\begin{aligned}\Phi(z\,|\,\lambda) &= \int_{\partial\mathbb{D}} P(z,\xi\,|\,\lambda)\,d\xi = \int_{\partial\mathbb{D}} P(z,\xi)^{s(\lambda)+1/2}\,d\xi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1-r^2}{1+r^2-2r\cos\phi}\Big)^{s(\lambda)+1/2} d\phi\,.\end{aligned} \tag{15}$$

In particular, $\Phi(\,\cdot\,|\,0) \equiv 1$. The following is immediate from Proposition 2.6.

Lemma 3.2. *For any $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ and $\xi \in \partial\mathbb{D}$,*

$$(\mathfrak{L} - \lambda\, I)^n\, \Phi_n(z\,|\,\lambda) = \Phi(z\,|\,\lambda)\,.$$

In the specific case $\lambda = -\frac{1}{4}$ one even has

$$(\mathfrak{L} - \lambda\, I)\, \Phi_n(z\,|\,-\tfrac{1}{4}) = \Phi_{n-1}(z\,|\,-\tfrac{1}{4})\,.$$

Remark 3.3. For $\lambda \in \mathbb{C}\setminus(-\infty\,,\,-\frac{1}{4})$, the spherical function $\Phi(r\,|\,\lambda)$ has no zeroes in $[0\,,\,1)$, while for real $\lambda \in (-\infty\,,\,-\frac{1}{4})$, $\Phi(r\,|\,\lambda)$ has countably many zeroes which accumulate at 1.

For real $\lambda \in [-\frac{1}{4}\,,\,\infty)$, this is obvious because the integrand in (15) is positive. For the other values of λ, to the best of our knowledge, it seems that these facts are nowhere referred to in the relevant literature on the hyperbolic Laplacian and its spherical functions. Therefore, we add some explanations.

Expressing $\mathfrak{L}$ in polar coordinates leads to the differential equation

$$\frac{(1-r^2)^2}{4}\,\Phi''(r\,|\,\lambda) + \frac{(1-r^2)^2}{4r}\,\Phi'(r\,|\,\lambda) - \lambda\,\Phi(r\,|\,\lambda) = 0$$

in the variable $r \in [0\,,1)$. We now substitute $z = \dfrac{1+r^2}{1-r^2}$ and write $\Psi(z\,|\,\lambda) = \Phi(r\,|\,\lambda)$. Then the above differential equation transforms into

$$(z^2-1)\,\Psi''(z\,|\,\lambda) + 2z\,\Psi'(z\,|\,\lambda) - \lambda\,\Psi(z\,|\,\lambda) = 0\,.$$

One sees that the latter is Legendre's differential equation; see Hobson [17] and, in particular, Hille [14], [15, equations (1) and (2)]. Recall that the Legendre function P_a with parameter $a \in \mathbb{C}$ solves $(1-z^2)P_a''(z) - 2zP_a'(z) + a(a+1)P_a(z) = 0$, and

$$P_a(z) = F\left(a+1, -a; 1;\ \frac{1-z}{2}\right),$$

where F is Gauss' hypergeometric function, and in our case, $a(a+1) = \lambda$. We get

$$\Phi(r\,|\,\lambda) = P_a(z) \quad \text{with} \quad a = -\frac{1}{2} + s(\lambda) \quad \text{and} \quad z = \frac{1+r^2}{1-r^2} \in [1\,,\infty).$$

Compare with [8, identity (25)]. Note that the hyperbolic Laplacian in [8] is 4 times the one we are using here, and that $\Phi(z\,|\,\lambda)$ is the function $\varphi_0(z,\mu)$ of [8], with $\mu = \frac{1}{2} + s(\lambda)$. Note also that there are several known identities between hypergeometric functions with different parameters. For example, Grellier and Otal [11] use a different version, which coincides with the one given above, see e.g. Lebedev [24, p. 200].

It follows from old work of Mehler [26] and is explained in [14, pages 27–28, identity (35)] that for $a = -\frac{1}{2} + b\,\mathfrak{i}$ with $b \neq 0$, that is, for $\lambda \in (-\infty\,,\,-\frac{1}{4})$, the function $P_a(z)$ has countably many zeroes in $[1\,,\infty)$ which are such that the zeroes of $\Phi(r\,|\,\lambda)$ accumulate at 1. On the other hand, it is comprised in [15, Theorem V and the page preceding it], that with $\Re(a) \neq -1/2$ the function $P_a(z)$ has no zeroes in $[1\,,\infty)$.[2] □

[2] We thank Jean-Pierre Otal (Toulouse) for pointing us to the books by Lebedev and Hobson, which led us to the PhD thesis of Hille [14] and its follow-up [15]. We also acknowledge an exchange with Peter Sjögren (Göteborg) on the question how well this issue of zeroes of the spherical functions is known in the community – apparently not at all.

For any fixed $\lambda \in \mathbb{C} \setminus (-\infty\,,\,-\frac{1}{4})$, we shall need the asymptotic behaviour of $\Phi_n(z\,|\,\lambda)$ when $|z| \to 1$, that is, when

$$\rho(z,0) = \log \frac{1+|z|}{1-|z|} \to \infty\,.$$

In view of Remark 3.3, we do not consider real $\lambda \in (-\infty\,,\,-\frac{1}{4})$ because of the infinity of zeroes of $\Phi(r\,|\,\lambda)$.

In the sequel, we shall adopt the following convention: in the formulas where $0 < r < 1$ (resp. $z \in \mathbb{D}$ with $|z| = r$) appear, we always use capital $R = \rho(r,0) = \rho(z,0)$, and we do the same in case of subscripts like $r_k \leftrightarrow R_k$.

Theorem 3.4. *We have the following, as $R = \rho(z,0) \to \infty$.*
(A) *If $\lambda \in \mathbb{C} \setminus (-\infty\,,-\frac{1}{4}\,]$ then*

$$\Phi_n(z\,|\,\lambda) \sim \frac{c(\lambda)}{n!\,\big(2s(\lambda)\big)^n}\, R^n\, \exp\Big(\big(s(\lambda)-1/2\big)R\Big)\,,$$

where $c(\lambda) \neq 0$.
(B) *If $\lambda = -\frac{1}{4}$ then*

$$\Phi_n(z\,|\,\lambda) \sim \frac{2}{(2n+1)!\,\pi}\, R^{2n+1}\, \exp(-R/2)\,.$$

In particular, for every $n \geqslant 1$ and $\lambda \in \mathbb{C} \setminus (-\infty\,,\,-\frac{1}{4})$ there is $0 < r_{n,\lambda} < 1$ such that

$$\Phi_n(z\,|\,\lambda) \neq 0 \quad \textit{for all} \;\; z \in \mathbb{D} \;\; \textit{with} \;\; |z| \geqslant r_{n,\lambda}\,.$$

For $n = 0$, we set $r_{0,\lambda} = 0$.

Proof. For the parameter $0 < r < 1$, the even function

$$P_r(\phi) = P(r, e^{\mathrm{i}\,\phi}) = \frac{1-r^2}{1+r^2-2r\cos\phi}\,, \quad \phi \in [-\pi\,,\pi] \tag{16}$$

is strictly decreasing in $\phi \in [0\,,\pi]$. It attains its maximum in 0 with value $P_r(0) = (1+r)/(1-r) = e^R$. We write

$$\frac{P_r(\phi)}{P_r(0)} = \frac{1}{1+\tau^2\sin^2(\phi/2)} \quad \text{with} \quad \tau = \tau_r = \frac{2\sqrt{r}}{1-r} \sim P_r(0) = e^R \;\text{ as }\; r \to 1\,. \tag{17}$$

Proof of (A). In this case, $\Re(s) > 0$, where $s = s(\lambda)$. Then

$$\frac{n!\,(2s)^n\,\Phi_n(r\,|\,\lambda)}{e^{(s+1/2)R}\,R^n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{P_r(\phi)^{s+1/2}\,\big(\log P_r(\phi)\big)^n}{P_r(0)^{s+1/2}\,\big(\log P_r(0)\big)^n}\,d\phi$$

$$= \frac{2}{\tau\pi}\int_0^{\pi/2} \frac{\tau}{(1+\tau^2\sin^2\phi)^{s+1/2}}\left(\frac{R-\log\big(1+\tau^2\sin^2\phi\big)}{R}\right)^n d\phi\,.$$

We choose $0 < a < \dfrac{2\Re(s)}{2\Re(s)+1}$ and decompose the last integral into the two parts

$$\int_0^{\tau^{-a}} \text{ plus } \int_{\tau^{-a}}^{\pi/2}.$$

As for the second integral,

$$\left|\frac{R-\log\big(1+\tau^2\sin^2\phi\big)}{R}\right| \leqslant 1 \tag{18}$$

Furthermore,

$$\begin{aligned}(\tau^2\sin^2\phi)^{\Re(s)+1/2} &\geqslant (\tau^2\sin^2\tau)^{\Re(s)+1/2}\\ &\sim \tau^{(1-a)(2\Re(s)+1)} \quad \text{for}\ \ \phi\in[\tau^{-a}\,,\pi/2]\ \ \text{as}\ \ \tau\to\infty\,,\end{aligned} \tag{19}$$

and we see that the second integral tends to 0 by the choice of a.

As for the first integral in the decomposition, as τ, hence R, tend to infinity, we have

$$(1+\tau^2\sin^2\phi)^{s+1/2} \sim (1+\tau^2\phi^2)^{s+1/2} \quad \text{and} \quad \frac{R-\log\big(1+\tau^2\sin^2\phi\big)}{R} \sim 1-\frac{1+\tau^2\phi^2}{\log\tau}\,. \tag{20}$$

Therefore, with the substitution $x = \tau\phi$, by dominated convergence as $\tau\to\infty$, we obtain

$$\frac{2}{\pi}\int_0^{\tau^{-a}} \frac{\tau}{(1+\tau^2\sin^2\phi)^{s+1/2}}\left(\frac{R-\log\big(1+\tau^2\sin^2\phi\big)}{R}\right)^n$$

$$\sim \frac{2}{\pi}\int_0^{\tau^{-a}} \frac{\tau}{(1+\tau^2\phi^2)^{s+1/2}}\left(1-\frac{\log\big(1+\tau^2\phi^2\big)}{\log\tau}\right)^n d\phi$$

$$= \frac{2}{\pi}\int_0^{\tau^{1-a}} \frac{1}{(1+x^2)^{s+1/2}}\left(1-\frac{\log\big(1+x^2\big)}{\log\tau}\right)^n dx$$

$$\to \frac{2}{\pi} \int_0^\infty \frac{1}{(1+x^2)^{s+1/2}}\, dx = c(\lambda), \tag{21}$$

recalling that $s = s(\lambda)$. This proves **(A)**.

Proof of (B). In this case, $s = 0$, and

$$\frac{(2n)!\, \Phi_n(r| - \frac{1}{4})}{e^{R/2}\, R^{2n}} \sim \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{(1+\tau^2 \sin^2 \phi)^{1/2}} \left(1 - \frac{\log\bigl(1+\tau^2 \sin^2 \phi\bigr)}{\log \tau}\right)^{2n} d\phi\,.$$

This time, we decompose the last integral into the two parts

$$\int_0^{1/\log\tau} \text{ plus } \int_{1/\log\tau}^{\pi/2}.$$

By using the analogue of (20), the first integral is asymptotically equivalent to

$$\int_0^{1/\log\tau} \frac{1}{(1+\tau^2\,\phi^2)^{1/2}} \left(1 - \frac{\log\bigl(1+\tau^2\,\phi^2\bigr)}{\log \tau}\right)^{2n} d\phi \quad \text{as} \quad \tau \to \infty\,.$$

We now substitute $x = \dfrac{\log\bigl(1+\tau^2\,\phi^2\bigr)}{\log \tau}$, and observe that the upper integration limit $1/\log\tau$ transforms into

$$b_\tau = \frac{\log\bigl(1+(\tau/\log\tau)^2\bigr)}{\log \tau} \sim 2\,, \quad \text{as} \quad \tau \to \infty\,.$$

Thus, the latter integral becomes

$$\frac{\log\tau}{2\tau} \int_0^{b_\tau} (1-x)^{2n} \frac{dx}{\bigl(1-\tau^{-x}\bigr)^{1/2}} \sim \frac{\log\tau}{\tau} \cdot \frac{1}{2n+1}\,.$$

By (18), the second integral is bounded by

$$\int_{1/\log\tau}^{\pi/2} \frac{d\phi}{\tau \sin\phi} = \frac{1}{\tau}\Bigl(-\log\,\tan\frac{1}{2\log\tau}\Bigr) \sim \frac{\log\tau}{\tau} \cdot \frac{\log(2\log\tau)}{\log\tau}\,, \quad \text{as} \quad \tau \to \infty\,.$$

This leads to the asymptotic behaviour of **(B)**. □

For $\lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4}\,]$, we define the *associated real eigenvalue* $\lambda^* \in [-\frac{1}{4}\,, +\infty)$ by the equation

$$s(\lambda^*) = \Re\big(s(\lambda)\big). \tag{22}$$

We shall also need the function defined as

$$|\Phi|_n(z\,|\,\lambda) = \int_{\partial\mathbb{D}} \big|P_n(z,\xi\,|\,\lambda)\big|\,d\xi\,, \quad z \in \mathbb{D}\,. \tag{23}$$

The same computations as in the proof of Theorem 3.4 with this modified integrand lead to the following.

Proposition 3.5. *We have the following.*
(A) *If* $\lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4}\,]$ *and* $s = s(\lambda)$ *then, as* $R = \rho(z, 0) \to \infty$

$$|\Phi|_n(z\,|\,\lambda) \sim C_n(\lambda)\, R^n\, \exp\Big(\big(\Re(s) - 1/2\big)R\Big)\,,$$

as $R = \rho(z, 0) \to \infty$, *where* $C_n(\lambda) = \dfrac{c(\lambda^*)}{n!\,(2|s|)^n}$ *with* $c(\lambda^*)$ *according to* (21).
(B) *If* $\lambda = -\frac{1}{4}$ *then* $|\Phi|_n(z\,|\,\lambda) = \Phi_n(z\,|\,\lambda)$ *with asymptotics given by Theorem 3.4.B.*

Regarding the zeroes of the higher order polyspherical functions, we have $P_n(0, \xi\,|\,\lambda) = \Phi_n(0\,|\,\lambda) = 0$ for all $\lambda \in \mathbb{C}$ and $n \geqslant 1$. For the following, recall that

$$J_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos\phi)^n\,d\phi = \begin{cases} \dfrac{1}{2^n}\dbinom{n}{n/2}\,, & \text{if } n \text{ is even,} \\[2ex] 0\,, & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 3.6. *Let* $n \geqslant 1$. *Then we have the following as* $r \to 0$.
(A) *If* $\lambda \in \mathbb{C} \setminus \{-\frac{1}{4}\}$ *and* $s = s(\lambda)$,

$$\Phi_n(r\,|\,\lambda) \sim \begin{cases} \dfrac{1}{\big(\frac{n}{2}!\big)^2\,(2s)^n}\,r^n, & \textit{if } n \textit{ is even,} \\[3ex] \dfrac{1}{\frac{n+1}{2}!\;\frac{n-1}{2}!\,(2s)^{n-1}}\,r^{n+1}, & \textit{if } n \textit{ is odd.} \end{cases}$$

(B) *If* $\lambda = -\frac{1}{4}$ *then*

$$\Phi_n(r\,|\,\lambda) \sim \frac{1}{(n!)^2}\,r^{2n}.$$

Proof. In the sequel all $\mathfrak{o}(r^k)$ are uniform in $\phi \in (-\pi\,,\,\pi]$, as $r \to 0$. Short computations show that

$$P(r, e^{\mathrm{i}\,\phi}) = 1 + 2r\,\cos\phi + \mathfrak{o}(r) \quad \text{and}$$
$$\log P(r, e^{\mathrm{i}\,\phi}) = 2r\,\cos\phi + 2r^2(\cos^2\phi - 1) + \mathfrak{o}(r^2)\,,$$

so that

$$\begin{aligned} &P(r, e^{\mathrm{i}\,\phi})^{s+1/2}\left(\log P(r, e^{\mathrm{i}\,\phi})\right)^n \\ &\quad = \Big(1 + (2s+1)\,r\,\cos\phi + \mathfrak{o}(r)\Big) \\ &\qquad \times \Big(2^n r^n\,\cos^n\phi + n2^{n+1}r^{n+1}(\cos^{n+1}\phi - \cos^{n-1}\phi)\Big) + \mathfrak{o}(r^{n+1}) \\ &\quad = 2^n r^n\,\cos^n\phi + 2^n r^{n+1}\Big((2s+1+n)\cos^{n+1}\phi - n\,\cos^{n-1}\phi\Big) + \mathfrak{o}(r^{n+1}) \end{aligned} \tag{24}$$

Applying $\frac{1}{2\pi}\int_{-\pi}^{\pi}$ with respect to ϕ, using the above formula for J_n, and normalising as in Proposition 2.6, we obtain the proposed asymptotic behaviour near 0. □

While we do not know much about the zeroes of $\Phi_n(r\,|\,\lambda)$ in general, we have the following for real eigenvalues.

Proposition 3.7. *For real $\lambda \geqslant -\frac{1}{4}$ and all $n \in \mathbb{N}$, we have $\Phi_n(r\,|\,\lambda) > 0$ for $0 < r < 1$.*

Proof. The statement is clear for even n, as well as for all n when $\lambda = -\frac{1}{4}$. Fix $r > 0$ and consider

$$F_n(s) = \int_{\partial\mathbb{D}} \left(\log P(r,\xi)\right)^n P(r,\xi)^{s+1/2}\,d\xi\,,$$

a function of $s \geqslant 0$. For $s = s(\lambda)$, one has $\Phi_n(r\,|\,\lambda) = F_n(s)$ when $\lambda > -\frac{1}{4}$. Then for $s > 0$

$$\frac{d}{ds}F_{2n+1}(s) = F_{2n+2}(s) > 0\,,$$

so that $F_{2n+1}(s)$ is strictly increasing in $s \in [0\,,\,\infty)$. To complete the proof it is enough to show that $F_{2n+1}(0) = 0$, or equivalently, that

$$f_{2n+1}(z) = \int_{\partial\mathbb{D}} \left(\log P(z,\xi)\right)^{2n+1}\sqrt{P(z,\xi)}\,d\xi = 0 \quad \text{for all} \quad z \in \mathbb{D}\,.$$

We show this by induction on n. For $n = 1$, Lemma 2.5 yields that $(\mathfrak{L} + \frac{1}{4}I)f_1 = 0$ (recalling that $s = 0$). Since f_1 is radial, i.e., it depends only on $r = |z|$, it must be a constant multiple of the spherical function $\Phi(z\,|-\frac{1}{4})$. Moreover, as $f_1(0) = 0$, that constant factor must be 0, that is, $f_1 \equiv 0$ on $\mathbb{D}$. Now suppose that $n \geqslant 1$ and that we have already that $f_{2n-1} \equiv 0$ on $\mathbb{D}$. Once more, from Lemma 2.5, we get

$$(\mathfrak{L} + \tfrac{1}{4}I) f_{2n+1} = 2n(2n+1)\, f_{2n-1} \equiv 0$$

so that also f_{2n+1} must be a constant multiple of $\Phi(z \,|\, -\frac{1}{4})$. As above, we get $f_{2n+1} \equiv 0$ on $\mathbb{D}$. $\square$

For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, from Theorem 3.4 we know at least that $\Phi_n(z \,|\, \lambda) \neq 0$ when $|z|$ is sufficiently close to 1.

4. Polyharmonic kernels

Definition 4.1. Let $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. The n^{th} *generalised Poisson transform* of an analytic functional (or measure) ν on $\partial\mathbb{D}$ is the function

$$\Pi_{n,\lambda}\, \nu(z) = \int_{\partial\mathbb{D}} P_n(z, \xi \,|\, \lambda)\, d\nu(\xi)\,, \quad z \in \mathbb{D}\,,$$

where P_n is given by Proposition 2.6. If ν is an absolutely continuous measure with density function $g(\xi)$ with respect to the normalised Lebesgue measure m on the circle, then we write $\Pi_{n,\lambda}\, g(z)$ for the resulting transform. When $n = 0$, we just write

$$\Pi_\lambda = \Pi_{0,\lambda} \quad \text{and} \quad \Pi = \Pi_{0,0}\ .$$

Our aim is to consider boundary value problems for polyharmonic functions. The most basic one is the Dirichlet problem: given a continuous function g on $\partial\mathbb{D}$, look for a harmonic function h on $\mathbb{D}$ which provides a continuous extension of g to the closed disk $\mathbb{D} \cup \partial\mathbb{D}$, the hyperbolic compactification. The solution is unique and well known, $h(z) = \Pi\, g(z)$, the classical Poisson transform.

If we consider λ-harmonic functions with $0 \neq \lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4})$, then we cannot proceed in the same way, using $P(z, \xi)^{s(\lambda)+1/2}$. Indeed, if we take $g \equiv 1$ on $\partial\mathbb{D}$, then its λ-Poisson transform is $h(z) = \Phi(z \,|\, \lambda)$, which tends to 0 as $|z| \to 1$ when $s = 0$ or $0 < \Re(s) < 1/2$, and to ∞ in absolute value when $\Re(s) > 1/2$. It is well-known that in this case one should normalise, and the natural candidate is $\Phi(z \,|\, \lambda)$, see e.g. [27] and [21]. (Note that this normalisation is not feasible when $\lambda \in (-\infty\,, -\frac{1}{4})$ because of the zeroes of the spherical function which accumulate at $\partial\mathbb{D}$.)

Definition 4.2. For $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4})$, the *normalised polyharmonic kernel* is

$$\mathcal{K}_{n,\lambda}(z, \xi) = \frac{P_n(z, \xi \,|\, \lambda)}{\Phi_n(z \,|\, \lambda)}\,, \quad \xi \in \partial\mathbb{D}\,,\ z \in \mathbb{D}\,,\ |z| \geqslant r_{n,\lambda}\,,$$

where $r_{n,\lambda}$ is as in Theorem 3.4.

Thus,

$$\int_{\partial\mathbb{D}} \mathcal{K}_{n,\lambda}(z,\xi)\, d\nu(\xi) = \frac{\Pi_{n,\lambda}\,\nu(z)}{\Phi_n(z\,|\,\lambda)}\,.$$

Lemma 4.3. *There is a constant $\widetilde{C}(\lambda)$ for which the kernels of Definition 4.2 satisfy*

$$\big|\mathcal{K}_{n,\lambda}(z,\xi)\big| \leqslant \widetilde{C}(\lambda)\,\mathcal{K}_{0,\lambda^*}(z,\xi)$$

where $\lambda^* \geqslant -\frac{1}{4}$ is given by (22) and $|z| \geqslant r_{n,\lambda}$.

Proof. Note that $|\mathfrak{h}(re^{\mathrm{i}\,\alpha},\xi)| \leqslant \mathfrak{h}(r,1) = R$, where (recall) $r = |z|$ and $R = \rho(z,0) = \rho(r,0) = \log\dfrac{1+r}{1-r}$. By Theorem 3.4, as $r = |z| \to 1$,

$$\big|\Phi(z\,|\,\lambda)\big| \sim \frac{|c(\lambda)|}{c(\lambda^*)}\,\Phi(z\,|\,\lambda^*).$$

Note also that by Theorem 3.4,

$$\big|\Phi_n(z\,|\,\lambda)\big| \sim \begin{cases} \big|\Phi(z\,|\,\lambda)\big|\,R^n\,\dfrac{1}{n!\big(2|s(\lambda)|\big)^n}\,, & \text{if } \lambda\in\mathbb{C}\setminus(-\infty\,,\,-\frac{1}{4})\,,\\[2ex] \big|\Phi(z\,|\,\lambda)\big|\,R^{2n}\,\dfrac{1}{(2n+1)!}\,, & \text{if } \lambda = -\frac{1}{4}\ (=\lambda^*)\,, \end{cases}$$

as $r \to 1$. Therefore, there is $\widetilde{C}(\lambda) > 0$ (depending on n) such that

$$\widetilde{C}(\lambda)\big|\Phi_n(z\,|\,\lambda)\big| \geqslant \begin{cases} \Phi(z\,|\,\lambda^*)\,R^n\,, & \text{if } \lambda\in\mathbb{C}\setminus(-\infty\,,\,-\frac{1}{4})\,,\\[2ex] \Phi(z\,|\,\lambda^*)\,R^{2n}\,, & \text{if } \lambda = -\frac{1}{4}\,, \end{cases}$$

for all $r \geqslant r_{n,\lambda}$. It follows that $\big|\mathcal{K}_{n,\lambda}(z,\xi)\big|$ is bounded above by

$$\frac{P(z,\xi)^{\Re(s(\lambda))+1/2}R^n}{\big|\Phi_n(z\,|\,\lambda)\big|} \leqslant \widetilde{C}(\lambda)\,\frac{P(z,\xi)^{s(\lambda^*)+1/2}}{\Phi(z\,|\,\lambda^*)}, \quad \text{if } \lambda\in\mathbb{C}\setminus(-\infty\,,\,-\frac{1}{4}]\,,$$

and by

$$\frac{P(z,\xi)^{1/2}R^{2n}}{\Phi_n(z|-\frac{1}{4})} \leqslant \widetilde{C}(-\tfrac{1}{4})\,\frac{P(z,\xi)^{1/2}}{\Phi(z|-\frac{1}{4})}, \quad \text{if } \lambda = -\frac{1}{4}\,,$$

which proves our claim. □

Remark 4.4. (a) If $\lambda > -\frac{1}{4}$ then, by Lemma 3.6, $\mathcal{K}_{n,\lambda}(z,\xi)$ is well-defined for all $z \in \mathbb{D} \setminus \{0\}$. When n is even then, in view of (24), it extends continuously to $z = 0$, and we can set $r_{n,\lambda} = 0$ for even n.

However, when n is odd, again in view of (24), $\mathcal{K}_{n,\lambda}(z,\xi)$ has a pole at $z = 0$ unless $\xi = \pm \mathrm{i}$. Thus, for Lemma 4.3 we need to work with $r_{n,\lambda} = \varepsilon$ for odd n, where $\varepsilon > 0$ is arbitrary but fixed.
(b) If $\lambda = -\frac{1}{4}$ then $\mathcal{K}_{n,\lambda}(z,\xi)$ extends continuously to $z = 0$ for all n, and we can always use $r_{n,-\frac{1}{4}} = 0$.

Note that $\mathcal{K}_{0,\lambda^*}(z,\cdot)\, d\mathsf{m}$ is a probability measure on $\partial\mathbb{D}$ for each $z \in \mathbb{D}$.

Another look at the proof of Theorem 3.4 yields the following.

Lemma 4.5. *Let $\lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4})$, $s = s(\lambda)$ and $n \in \mathbb{N}_0$. Then*

$$\lim_{r \to 1} \mathcal{K}_{n,\lambda}(r, e^{\mathrm{i}\,\psi}) = 0 \qquad \text{uniformly for}$$

$$|\psi| \in \begin{cases} [2\tau^{-a}\,, \pi]\,, & \text{if } \lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4})\,, \quad \text{where } 0 < a < \dfrac{2\Re(s)}{2\Re(s)+1}\,, \\ [2(\log \tau)^{-a}\,, \pi]\,, & \text{if } \lambda = -\frac{1}{4}\,, \quad \text{where } 0 < a < 1\,. \end{cases}$$

Proof. In view of Lemma 4.3, we only need to prove this for $n = 0$ and real $\lambda \geqslant -\frac{1}{4}$. In this case, it is well known except maybe for the fact that usually the lower bound for $|\psi|$ is required to be a positive constant, while here it tends to 0 as $r \to 1$.

First, look at case **(A)** of Theorem 3.4. With τ as in (17) and $c(\lambda)$ as in (21), we have as $r \to 1$

$$\frac{P(r, e^{\mathrm{i}\,\psi}|\lambda)}{\Phi(r\,|\,\lambda)} \sim \frac{P_r(\psi)^{s+1/2}}{c(\lambda)\, P_r(0)^{s+1/2}\, e^{-R}} \sim \frac{1}{c(\lambda)} \frac{\tau}{\big(1 + \tau^2 \sin^2(\psi/2)\big)^{s+1/2}}.$$

By (19) and (20), this tends to 0 uniformly in the stated range.

In case **(B)** of Theorem 3.4, as $r \to 1$, that is, $\tau \to \infty$,

$$\frac{P(r, e^{\mathrm{i}\,\psi}| -\frac{1}{4})}{\Phi(r\,| -\frac{1}{4})} \sim \frac{\pi}{2} \frac{P_r(\psi)^{1/2}}{R\, e^{-R}\, P_r(0)^{1/2}} \sim \frac{\pi}{2} \frac{\tau/\log \tau}{\big(1 + \tau^2 \sin^2(\psi/2)\big)^{1/2}}.$$

Again, this tends to 0 uniformly in the stated range. □

The kernels are also rotation invariant: $\mathcal{K}_{n,\lambda}(e^{\mathrm{i}\,\alpha} z, e^{\mathrm{i}\,\alpha}\xi) = \mathcal{K}_{n,\lambda}(z,\xi)$. This fact and the last two lemmas yield the following by well-known methods.

Proposition 4.6. *Let $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4})$. For a measurable function $g : \partial\mathbb{D} \to \mathbb{C}$, resp. a complex Borel measure ν on $\partial\mathbb{D}$, let*

$$f(z) = \Pi_{n,\lambda}\, g(z)\,, \quad \text{resp.} \quad f(z) = \Pi_{n,\lambda}\, \nu(z)\,.$$

Then the following properties hold.

(i) *If* $g \in C(\partial\mathbb{D})$ *then* $\lim_{z\to\xi} \dfrac{f(z)}{\Phi_n(z\,|\,\lambda)} = g(\xi)$ *for all* $\xi \in \partial\mathbb{D}$ *and uniformly as* $|z| \to 1$.

(ii) *If* $g \in L^p(\partial\mathbb{D})$ $(1 \leqslant p < \infty)$ *then* $\lim_{r\to 1} \dfrac{f(r\xi)}{\Phi_n(r\,|\,\lambda)} = g(\xi)$ *for almost every* $\xi \in \partial\mathbb{D}$ *and in* $L^p(\partial\mathbb{D})$.

(iii) *If* $g \in L^\infty(\partial\mathbb{D})$ *then* $\lim_{r\to 1} \dfrac{f(r\xi)}{\Phi_n(r\,|\,\lambda)} = g(\xi)$ *in the weak*-topology of* $L^\infty(\partial\mathbb{D})$.

(iv) *If* ν *is a finite Borel measure on* $\partial\mathbb{D}$ *then the measures* $\dfrac{f(r\xi)}{\Phi_n(r\,|\,\lambda)}\,d\xi$ *converge to* ν *in the weak*-topology, as* $r \to 1$.

The classical reference is Zygmund [37, chapter 17, Theorems 1.20 & 1.23]. See also Stein and Weiss [34, §I.1, Theorems 1.18 & 1.25 and §II.2] and Garnett [9, Theorem 3.1]. In these references, the results are presented for the upper half space, resp. half plane and carry over to the disk model of the hyperbolic plane.

5. Dirichlet and Riquier problem at infinity

Reconsider statement (i) of Proposition 4.6. It says that for any $g \in \mathcal{C}(\partial\mathbb{D})$, the function

$$f(z) = \frac{\Pi_{n,\lambda}\, g(z)}{\Phi_n(z\,|\,\lambda)} \tag{25}$$

is λ-polyharmonic of order $n+1$ which provides a continuous extension of g to $\{z \in \overline{\mathbb{D}} : |z| \geqslant r_{n,\lambda}\}$. When $n \geqslant 1$ in (25) then we cannot expect that the given f is the unique function with this property. Indeed, for example also

$$\lim_{z\to\xi} \frac{f(z) + \Phi_k(z\,|\,\lambda)}{\Phi_n(z\,|\,\lambda)} = g(\xi) \quad \text{when} \;\; 0 \leqslant k < n\,.$$

However, when $n = 0$ and we are considering λ-harmonic functions, this is the solution of the λ-Dirichlet problem, valid on all of $\overline{\mathbb{D}}$ since $\Phi(\cdot\,|\,\lambda)$ has no zeroes in $\mathbb{D}$. It is well-known to be unique when $\lambda = 0$. The extension to *real* $\lambda \geqslant -\frac{1}{4}$ with normalisation by $\Phi(z\,|\,\lambda)$ is well understood via the maximum principle applied to the kernels $\mathcal{K}_{0,\lambda}(z,\xi)$ of Definition 4.2, which are probability kernels with respect to m for real λ. When $\lambda \in \mathbb{C} \setminus (-\infty\,,\,-\frac{1}{4})$ is *complex*, we can still prove uniqueness, via a different technique in place of the standard one.

Theorem 5.1. *Let* $\lambda \in \mathbb{C} \setminus (-\infty\,,\,-\frac{1}{4})$ *and* $g \in \mathcal{C}(\partial\mathbb{D})$. *Then* $h(z) = \Pi_\lambda\, g(z)$ *is the* unique *λ-harmonic function for which*

$$\lim_{\mathbb{D}\ni z\to\xi} \frac{h(z)}{\Phi(z\,|\,\lambda)} = g(\xi) \quad \textit{for every} \;\; \xi \in \partial\mathbb{D}. \tag{26}$$

Proof. We know that the given function h is a solution of the λ-Dirichlet problem, i.e. it satisfies (26). In order to show uniqueness, it is enough to prove that the constant function 0 is the only solution, when $g \equiv 0$ on $\partial\mathbb{D}$. In other words, we assume that h is a λ-harmonic function on $\mathbb{D}$ with

$$\lim_{\mathbb{D}\ni z\to\xi} \frac{h(z)}{\Phi(z\,|\,\lambda)} = 0 \quad \text{for every} \;\; \xi \in \partial\mathbb{D}\,,$$

and we have to show that $h \equiv 0$.

Let $\widehat{h}$ be the spherical average of h around 0,

$$\widehat{h}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(|z|e^{\mathrm{i}\,\phi})\,d\phi\,.$$

In particular, $\widehat{h}(0) = h(0)$. Then $\widehat{h}$ is also λ-harmonic, and it is rotation-invariant. Now, up to multiplication with constants, $\Phi(z\,|\,\lambda)$ is the unique λ-harmonic function which is rotation-invariant. See e.g. [8] for this well-known fact. Therefore

$$\widehat{h}(z) = h(0)\,\Phi(z\,|\,\lambda)\,.$$

On the other hand, the function on the closed disk which is $h/\Phi(\cdot\,|\lambda)$ in the interior and 0 on the boundary is continuous, whence uniformly continuous, so that

$$\lim_{|z|\to 1} \frac{h(z)}{\Phi(z\,|\,\lambda)} = 0 \quad \text{uniformly in} \;\; z.$$

Since $\Phi(z\,|\,\lambda)$ only depends on $|z|$, also

$$\lim_{|z|\to 1} \frac{\widehat{h}(z)}{\Phi(z\,|\,\lambda)} = 0.$$

We conclude that $h(0) = 0$.

Now let $z_0 \in \mathbb{D}$ be arbitrary. Then there is an isometry γ of the Poincaré disk (a Möbius transform) such that $\gamma\,0 = z_0$. The isometries commute with the hyperbolic Laplacian, whence also the function $h_\gamma(z) = h(\gamma z)$ is λ-harmonic. If $|z| \to 1$ then also $|\gamma z| \to 1$. Therefore also $h(\gamma z)/\Phi(\gamma z\,|\,\lambda) \to 0$ as $|z| \to 1$. Now Theorem 3.4 implies

$$\frac{\Phi(\gamma z\,|\,\lambda)}{\Phi(z\,|\,\lambda)} \sim \exp\Big((s-1/2)\big(\rho(\gamma\,z,o) - \rho(z,o)\big)\Big) \quad \text{as} \;\; |z| \to 1\,.$$

This is bounded, since

$$\big|\rho(\gamma\,z,o) - \rho(z,o)\big| = \big|\rho(z,\gamma^{-1}o) - \rho(z,o)\big| \leqslant \rho(\gamma^{-1}o,o).$$

We infer that

$$\frac{h_\gamma(z)}{\Phi(z\,|\,\lambda)} = \frac{h(\gamma z)}{\Phi(\gamma z\,|\,\lambda)}\,\frac{\Phi(\gamma z\,|\,\lambda)}{\Phi(z\,|\,\lambda)} \to 0 \quad \text{as} \quad |z| \to 1\,.$$

We can now apply the above argument to h_γ and its spherical average, and conclude that $h(z_0) = 0$. This is true for every $z_0 \in \mathbb{D}$. □

The next Lemma is in preparation of the Riquier problem for λ-polyharmonic functions.

Lemma 5.2. *For $\lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4})$, let f be λ-polyharmonic of order $n+1$ on $\mathbb{D}$ and such that the λ-harmonic function $h = (\mathfrak{L} - \lambda\, I)^n f$ satisfies*

$$\lim_{\mathbb{D}\ni z\to\xi} \frac{h(z)}{\Phi(z\,|\,\lambda)} = g(\xi) \quad \textit{for every} \;\; \xi \in \partial\mathbb{D}$$

where $g \in \mathcal{C}(\partial\mathbb{D})$. Then

$$f(z) = \Pi_{n,\lambda}\, g(z) + f_*(z)\,,$$

where f_* is λ-polyharmonic of order n.

Proof. This is very similar to the inductive argument in the proof of Theorem 1.1. It follows from Theorem 5.1 that $h = \Pi_\lambda\, g =: h_g$. Write $f_g = \Pi_{n,\lambda}\, g$. By Lemma 5.2,

$$(\mathfrak{L} - \lambda\, I)^n f_g = h_g = (\mathfrak{L} - \lambda\, I)^n f\,.$$

Therefore $(\mathfrak{L} - \lambda\, I)^n f_* = 0$. □

In the setting of the Euclidean Laplacian Δ on a bounded domain, the Riquier problem asks for solutions of $\Delta^n f = 0$ with prescribed boundary data $g_k = \Delta^k f$ for $k = 0, \dots, n-1$. This is not applicable to the hyperbolic setting of $(\mathfrak{L} - \lambda\, I)^n$, even when $\lambda = 0$, because in any case, the quotient of lower and higher order polyspherical functions tends to zero at the boundary. For this reason, we propose a different formulation.

Definition 5.3. Let $\lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4})$ and $g_0\,, \dots\,, g_{n-1} \in \mathcal{C}(\partial\mathbb{D})$. Then a solution of the associated *Riquier problem at infinity* is a polyharmonic function

$$f = f_0 + \cdots + f_{n-1}$$

of order n, where each f_k is λ-polyharmonic of order $k+1$ and

$$\lim_{z\to\xi} \frac{(\mathfrak{L} - \lambda\, I)^k f_k(z)}{\Phi(z\,|\,\lambda)} = g_k(\xi) \quad \text{for every} \;\; \xi \in \partial\mathbb{D}\,.$$

Note that by Remark 3.3, the denominator in the last quotient is always non-zero.

Corollary 5.4. *A solution of the Riquier problem as in Definition 5.3 is given by*

$$f_k(z) = \Pi_{k,\lambda}\, g_k(z)\,, \quad k = 0, \ldots, n-1\,.$$

One also has for every $\xi \in \partial\mathbb{D}$ and $k \in \{0\,, \ldots, n-1\}$

$$\lim_{z\to\xi} \frac{f_k(z)}{\Phi_k(z\,|\,\lambda)} = g_k(\xi) \quad \textit{and} \quad \lim_{z\to\xi} \frac{f_j(z)}{\Phi_k(z\,|\,\lambda)} = 0 \ \ \textit{for } j < k.$$

6. A Fatou theorem for polyharmonic functions

Definition 6.1. (i) For $0 < \delta \leqslant \pi$, consider the arc $B_\delta = \{e^{\mathrm{i}\,\phi} \colon |\phi| < \delta\} \subset \partial\mathbb{D}$, and for $\zeta = e^{\mathrm{i}\,\alpha}$ consider the rotated arc $B_\delta(\zeta) = \zeta B_\delta = \{e^{\mathrm{i}\,\phi} \colon |\phi - \alpha| < \delta\}$ with measure (normalised arc length) $\mathsf{m}\big(B_\delta(\zeta)\big) = \delta/\pi$. The *Hardy–Littlewood maximal operator* is defined on functions $g \in L^1(\partial\mathbb{D})$ as

$$\mathcal{M}g(\zeta) = \sup\left\{ \frac{1}{\mathsf{m}\big(B_\delta(\zeta)\big)} \int_{B_\delta(\zeta)} |g(\xi)|\, d\xi \ : \ 0 < \delta \leqslant \pi \right\}.$$

(ii) Let $[0, \zeta]$ be the line segment with endpoints 0 and $\zeta = e^{\mathrm{i}\,\alpha}$ in the unit disc $\mathbb{D}$, and for $a \geqslant 0$ consider the *admissible region*, or *tubular domain*

$$\Gamma_a(\zeta) = \Big\{ z \in \mathbb{D} \colon \rho\big(z, [0, \zeta]\big) \leqslant a \Big\}.$$

The *non-tangential (tubular) maximal operator of width* $a \geqslant 0$ is defined on functions $g \in L^1(\partial\mathbb{D})$ as

$$\mathfrak{M}_a^{(n,\lambda)} g(\zeta) = \sup\left\{ \left| \int_{\partial\mathbb{D}} \mathcal{K}_{n,\lambda}(z, \xi)\, g(\xi)\, d\xi \right| \ : \ z \in \Gamma_a(\zeta)\,, \ |z| \geqslant r_{n,\lambda} \right\},$$

where $\mathcal{K}_{n,\lambda}$ is the kernel introduced in Definition 4.2 and $r_{n,\lambda}$ is as in Theorem 3.4.
(iii) With the same ingredients as in (ii), the *enlarged-admissible region* is

$$\Gamma_{(a)}(\zeta) = \Big\{ z \in \mathbb{D} \colon \rho\big(z, [0, \zeta]\big) \leqslant a + \log \rho(z, 0) \Big\}.$$

The *extended maximal operator of width* a is defined on functions $g \in L^1(\partial\mathbb{D})$ as

$$\mathfrak{M}_{(a)}^{(n,\lambda)} g(\zeta) = \sup\left\{ \left| \int_{\partial\mathbb{D}} \mathcal{K}_{n,\lambda}(z, \xi)\, g(\xi)\, d\xi \right| \colon z \in \Gamma_{(a)}(\zeta)\,, \ |z| \geqslant r_{n,\lambda} \right\}.$$

Both $\Gamma_a(\zeta)$ and $\Gamma_{(a)}(\zeta)$ are (Euclidean) convex subsets of $\mathbb{D}$ which touch the boundary $\partial\mathbb{D}$ only at ζ. In the Euclidean metric, the "tube" $\Gamma_a(\zeta)$ is conical at ζ with Stolz angle $2\arctan(\sinh a)$. On the other hand, the boundary curve of the larger domain $\Gamma_{(a)}(\zeta)$ is tangent to $\partial\mathbb{D}$ at ζ.

The following fact is well-known (see, for instance, [22, Corollary of Lemma 1.1]).

Proposition 6.2. *The Hardy–Littlewood maximal operator is of weak type* $(1,1)$ *and strong type* (p,p) *for* $1 < p \leqslant \infty$.

Our aim is to prove the following.

Proposition 6.3. *For every* $a \geqslant 0$ *and* $n \in \mathbb{N}_0$, *there is a constant* $C(a,\lambda) > 0$ *such that for all* $g \in L^1(\partial\mathbb{D})$

$$\mathfrak{M}_a^{(n,\lambda)}\, g \leqslant C(a,\lambda)\, \mathcal{M}g\,, \quad \text{if } \lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4})\,,$$
$$\mathfrak{M}_{(a)}^{(n,\lambda)}\, g \leqslant C(a,\lambda)\, \mathcal{M}g\,, \quad \text{if } \lambda = -\frac{1}{4}\,.$$

Note that the inequality in the critical case is stronger, since the underlying domain is enlarged. By Lemma 4.3, it is sufficient to prove Proposition 6.3 for $n = 0$ (i.e., for $\mathfrak{L} - \lambda I$) and real $\lambda \geqslant -\frac{1}{4}$, showing that

$$\begin{aligned} \mathfrak{M}_a^{(0,\lambda)}\, g &\leqslant C(a,\lambda)\, \mathcal{M}g \quad \text{if } \lambda > -\frac{1}{4}\,, \quad \text{and} \\ \mathfrak{M}_{(a)}^{(0,\lambda)}\, g &\leqslant C(a,\lambda)\, \mathcal{M}g \quad \text{if } \lambda = -\frac{1}{4}\,. \end{aligned} \tag{27}$$

That is, we only need to work with the standard spherical functions $\Phi(z \,|\, \lambda)$.

In this case, the proof is practically folklore when $\lambda > -\frac{1}{4}$, see [37], [20] or [27], while the extended version for $\lambda = -\frac{1}{4}$ is contained in a note by Sjögren [33]. However, the extent to which the proof is folklore is such that it is hard to find a simple version for the hyperbolic Laplacian, and the note [33] is also not very easily accessible. Therefore, for the reader's convenience, we include a simple proof (more direct via partial integration than typical versions which decompose the unit circle in countably many arcs).

Proof of Proposition 6.3. 1) As mentioned, it is sufficient to work with $n = 0$ and λ real. Then $\mathcal{K}_{n,\lambda}(z,\xi)\, d\xi$ is a probability measure, and we have rotation invariance: $\mathcal{K}_{n,\lambda}(e^{\mathrm{i}\,\alpha} z, e^{\mathrm{i}\,\alpha}\xi) = \mathcal{K}_{n,\lambda}(z,\xi)$. Thus, it is enough to prove the inequality at $\zeta = 1$.
2) For the rest of this proof, we set

$$t = s(\lambda) + 1/2 \quad \text{and} \quad R^* = \begin{cases} 1\,, & \text{if } \lambda > -\frac{1}{4}\,, \\ R\,, & \text{if } \lambda = -\frac{1}{4}\,. \end{cases} \tag{28}$$

Thus $t > 1/2$ when $\lambda > -\frac{1}{4}$, and $t = 1/2$ when $\lambda = -\frac{1}{4}$. We can subsume both admissible regions $\Gamma_a(1)$ and $\Gamma_{(a)}(1)$ under

$$\left\{z = r\,e^{\mathrm{i}\,\alpha} \in \mathbb{D} : \rho\big(z, [0,1]\big) \leqslant a + \log R^*\right\},$$

where, as always, $R = \log \dfrac{1+r}{1-r}$. Elementary computations with hyperbolic distance yield for $z = r\,e^{\mathrm{i}\,\alpha}$ with $r < 1$ and $|\alpha| \leqslant \pi$ that for any $a \geqslant 0$,

$$\rho\big(z, [0,1]\big) \leqslant a + \log R^* \iff \underbrace{\frac{1-r^2}{2r}\,\sinh(a + \log R^*)}_{=:\,K_r} \geqslant \begin{cases} |\sin\alpha|\,, & \text{if } \ |\alpha| < \pi/2 \\ 1\,, & \text{if } \ |\alpha| \geqslant \pi/2\,. \end{cases} \tag{29}$$

If $|\alpha| \geqslant \pi/2$, this is the same as $R \leqslant a + \log R^*$. Thus, there is $r_a \in (0\,,\,1)$ such that when $r \geqslant r_a$ then the second of the above cases is excluded, and that $\sinh(a + R_a) > 1$, where $R_a = \rho(r_a, 0)$.

3) In view of (17), we need to show that for $z = r\,e^{\mathrm{i}\,\alpha}$ in the range of (29),

$$\frac{e^R}{2\pi\,R^*}\int_{-\pi}^{\pi} |g(e^{\mathrm{i}\,\phi})|\,\frac{1}{\left(1 + \frac{4r}{(1-r)^2}\sin^2\frac{\phi-\alpha}{2}\right)^t}\,d\phi \leqslant C(a,\lambda)\,\mathcal{M}g(1). \tag{30}$$

If $r \in [0\,,\,r_a)$ then the kernel of the above integral is bounded, so that the estimate is immediate with a suitable value of $C(a,\lambda)$. So we now consider the case $r \geqslant r_a$, where we know that $|\alpha| < \pi/2$, and $\alpha \to 0$ as $r \to 1$. For $\phi \in (-\pi\,,\,\pi]$, we have $|\phi - \alpha|/2 \leqslant 3\pi/4$, whence there is $\kappa \in (0\,,\,1)$ such that $|\sin\frac{\phi-\alpha}{2}| \geqslant \kappa\,|\frac{\phi-\alpha}{2}|$, as well as $|\sin\alpha| \geqslant \kappa\,|\alpha|$. We may assume that r_a is such that $2K_r \leqslant \pi$ for $r \geqslant r_a$. Now the left hand side of (30) is bounded above by

$$\frac{e^R}{2\pi\,R^*}\int_{-\pi}^{\pi} |g(e^{\mathrm{i}\,\phi})|\,\frac{1}{\left(1 + \frac{r}{\kappa^2(1-r)^2}(\phi-\alpha)^2\right)^t}\,d\phi\,.$$

We decompose the last integral into $\int_{|\phi|<2K_r/\kappa} + \int_{|\phi|\geqslant 2K_r/\kappa}$: for the first part,

$$\begin{aligned}
&\frac{e^R}{2\pi\,R^*}\int_{-2K_r/\kappa}^{2K_r/\kappa} |g(e^{\mathrm{i}\,\phi})|\,\frac{1}{\left(1 + \frac{r}{\kappa^2(1-r)^2}(\phi-\alpha)^2\right)^t}\,d\phi \\
&\qquad \leqslant \frac{e^R}{2\pi\,R^*}\int_{-2K_r/\kappa}^{2K_r/\kappa} |g(e^{\mathrm{i}\,\phi})|\,d\phi \leqslant \frac{4K_r\,e^R}{2\pi\,\kappa\,R^*}\,\mathcal{M}g(1) \leqslant \frac{4e^a}{2\pi\,\kappa\,r_a}\,\mathcal{M}g(1)\,.
\end{aligned}$$

For the second part, we set

$$H(\phi) = \int_0^{\phi} \big(|g(e^{\mathrm{i}\,\psi})| + |g(e^{-\mathrm{i}\,\psi})|\big)\, d\psi\,, \quad \phi \in [-\pi\,,\, \pi].$$

Then

$$H(\phi) \leqslant 2\phi\, \mathcal{M}g(1)\,.$$

Observe that, by (29) and the choices of r and κ, we have $|\alpha| \leqslant K_r/\kappa$, so that $|\phi| \geqslant 2K_r/\kappa$ implies $|\phi - \alpha| \geqslant |\phi|/2$. We get, with $c_\kappa = (2\kappa)^{2t}/(r_a\,\pi)$, and using partial integration

$$\begin{aligned}
&\frac{e^R}{2\pi\, R^*} \int\limits_{2K_r/\kappa \leqslant |\phi| \leqslant \pi} |g(e^{\mathrm{i}\,\phi})|\, \frac{1}{\big(1 + \frac{r}{\kappa^2(1-r)^2}(\phi-\alpha)^2\big)^t}\, d\phi \\
&\leqslant c_\kappa\, \frac{(1-r)^{2t-1}}{R^*} \int\limits_{2K_r/\kappa}^{\pi} \big(|g(e^{\mathrm{i}\,\phi})| + |g(e^{-\mathrm{i}\,\phi})|\big) \frac{1}{\phi^{2t}}\, d\phi \\
&= c_\kappa\, \frac{(1-r)^{2t-1}}{R^*} \left(\frac{H(\pi)}{\pi^{2t}} - \frac{H(2K_r/\kappa)}{(2K_r/\kappa)^{2t}} + 2t \int\limits_{2K_r/\kappa}^{\pi} \frac{H(\phi)}{\phi^{2t+1}}\, d\phi \right) \\
&\leqslant c_\kappa\, \frac{(1-r)^{2t-1}}{R^*} \left(\frac{2\pi}{\pi^{2t}} + 4t \int\limits_{2K_r/\kappa}^{\pi} \frac{1}{\phi^{2t}}\, d\phi \right) \mathcal{M}g(1) \\
&\leqslant \begin{cases} c_\kappa \left(\dfrac{2}{\pi^{2t-1}} + \dfrac{4t}{2t-1} \Big(\dfrac{\kappa}{\sinh a} \Big)^{2t-1} \right) \mathcal{M}g(1)\,, & \text{if } \ t > 1/2, \\ c_\kappa \left(\dfrac{2 + 2\log \pi}{R_a} + 2 \right) \mathcal{M}g(1)\,, & \text{if } \ t = 1/2\,, \end{cases}
\end{aligned}$$

because $-\log(2K_r/\kappa) \leqslant R$ by the choices of r_a and κ. This concludes the proof. □

As a consequence, we have the following convergence theorem (in the discrete setting of trees, see [21, Theorem 1, Theorem 3] for λ-harmonic functions, and [30, Theorem 4.6] for regular trees and λ-polyharmonic functions).

Definition 6.4. A function $f\colon \mathbb{D} \to \mathbb{C}$ *converges admissibly, (≡ non-tangentially)* resp. *enlarged-admissibly* at a boundary point $\xi \in \partial\mathbb{D}$ if, for every $a \geqslant 0$, the limit

$$\lim_{\Gamma_a(\xi) \ni z \to \xi} f(z)\,, \quad \text{resp.} \quad \lim_{\Gamma_{(a)}(\xi) \ni z \to \xi} f(z)$$

exists and is finite.

Theorem 6.5 *(Fatou theorem for λ-polyharmonic functions). Let $\lambda \in \mathbb{C} \setminus (-\infty\,, -\frac{1}{4})$ and ν a Borel measure on $\partial\mathbb{D}$. Then the normalised λ-polyharmonic function*

$$\frac{\Pi_{n,\lambda}\,\nu(z)}{\Phi_n(z\,|\,\lambda)} \qquad (\,|z| \geqslant r_{n,\lambda}\,)$$

converges admissibly at m-almost every point in $\partial\mathbb{D}$. The limit is the Radon–Nykodim derivative $d\nu^{ac}/d\mathsf{m}$, where (recall) m is the normalised arc length measure, and ν^{ac} is the absolutely continuous part of ν.

If $\lambda = -\frac{1}{4}$, convergence is even enlarged-admissible.

Proof. It is well-known that we can assume ν to be absolutely continuous with respect to m. We prove this fact for the sake of completeness: if ν is singular with respect to m, then there is an m-null set $E \subset \partial\mathbb{D}$ such that $\nu(\mathbb{D} \setminus E) = 0$. For every $\varepsilon > 0$, let $\{A_j \subset \partial\mathbb{D} : j = 1, \ldots, n\}$ be a finite collection of open arcs covering E with $\mathsf{m}\big(\bigcup_{j=1}^n A_j\big) < \varepsilon$. Let $U = \bigcup_{j=1}^n A_j$. Then

$$\left|\frac{f(z)}{\Phi_n(z\,|\,\lambda)}\right| \leqslant \int\limits_U \frac{\big|P(z,\xi\,|\,\lambda)\,\mathfrak{h}(z,\xi\,|\,\lambda)^n\big|}{|\Phi_n(z\,|\,\lambda)|}\,d\nu(\xi)\,,$$

which tends to 0 when $z \to \zeta \in \partial\mathbb{D} \setminus U$ by Lemma 4.5.

So we can assume that ν is absolutely continuous with respect to the normalised Lebesgue measure m, with $g = d\nu/d\mathsf{m} \in L^1(\partial\mathbb{D}, \mathsf{m})$. The rest of the proof is the continuous analogue of [21, Theorems 1 and 3] and [30, Theorem 4.6]. In brief, we can find a sequence (g_k) in $\mathcal{C}(\partial\mathbb{D})$ such that $\|g - g_k\| < 1/2^k$, and by Propositions 6.2 and 6.3,

$$\sum_k \mathsf{m}\big[\mathfrak{M}_a^{n,\lambda}(g - g_k) \geqslant \varepsilon\big] < \infty\,.$$

By the Borel-Cantelli Lemma,

$$\lim_{k\to\infty} \mathfrak{M}_a^{n,\lambda}(g - g_k)(\xi) = 0 \quad \text{for } \mathsf{m}\text{-almost every } \ \xi \in \partial\mathbb{D}.$$

We can now apply Proposition 4.6(i) to each of the g_k to get the proposed convergence at all those points $\xi \in \partial\mathbb{D}$. $\square$

Let us call a λ-polyharmonic function f *regular* if all the analytic functionals ν_k in the boundary representation of Theorem 1.1, or better (equivalently) formula (14), are complex Borel measures on $\partial\mathbb{D}$. The fact that $\Phi_k(r\,|\,\lambda)/\Phi_n(r\,|\,\lambda) \to 0$ for $k < n$, when $r \to 1$, yields the following.

Corollary 6.6. *For $\lambda \in \mathbb{C}\setminus(-\infty\,, -\frac{1}{4})$, let f be a regular λ-polyharmonic function of order $n+1$ and ν_n the highest-order representing measure of f in (14). Then $f(z)/\Phi_n(z\,|\,\lambda)$*

converges admissibly (resp. enlarged-admissibly, if $\lambda = -\frac{1}{4}$*) at almost every* $\xi \in \partial\mathbb{D}$*. The limit function is the Radon–Nykodim derivative* $d\nu_n^{ac}/d\mathsf{m}$*.*

Compare the following with [31].

Corollary 6.7. *For* $\lambda \in \mathbb{C} \setminus (-\infty, -1/4)$*, the only radial* λ*-polyharmonic functions of order* $n+1$ *on* $\mathbb{D}$ *are the linear combinations of the* λ*-polyspherical functions* $\Phi_0, \ldots, \Phi_n$*.*

7. Examples, complements, open problems

In this section, we study examples of (hyperbolically) harmonic and bi-harmonic functions which are not regular. We focus on the case $\lambda = 0$, where of course $s(0) + 1/2 = 1$.

First, for $0 \leqslant r < 1$ and $\xi \in \partial\mathbb{D}$,

$$P(r, \xi) = \frac{1 - r^2}{1 + r^2 - r\xi - r/\xi} = \sum_{n \in \mathbb{Z}} r^{|n|}\, \xi^n.$$

Thus, for any fixed $z = r\, e^{\mathrm{i}\,\alpha} \in \mathbb{D}$, the Fourier expansion of the Poisson kernel in the boundary variable $\xi = e^{\mathrm{i}\,\phi}$ is

$$P(r\, e^{\mathrm{i}\,\alpha}, e^{\mathrm{i}\,\phi}) = \sum_{n \in \mathbb{Z}} r^{|n|}\, e^{-\mathrm{i}\,n\alpha}\, e^{\mathrm{i}\,n\phi}.$$

Next, we want to determine the Fourier expansion of $-\mathfrak{h}(z, \xi) = \log P(z, \xi)$. By (7),

$$\log P(r, \xi) = \log(1 - r^2) - \log(\xi - r) - \log\Big(\frac{1}{\xi} - r\Big) = \log(1 - r^2) + \sum_{0 \neq n \in \mathbb{Z}} \frac{r^{|n|}}{|n|}\, \xi^n\,,$$

so that we have the Fourier expansion

$$\log P(r\, e^{\mathrm{i}\,\alpha}, e^{\mathrm{i}\,\phi}) = \log(1 - r^2) + \sum_{0 \neq n \in \mathbb{Z}} \frac{r^{|n|}}{|n|}\, e^{-\mathrm{i}\,n\alpha}\, e^{\mathrm{i}\,n\phi}.$$

Now we can write

$$P(r, \xi) \log P(r, \xi) = \sum_{n \in \mathbb{Z}} r^{|n|}\, d_{|n|}(r)\, \xi^n\,,$$

since the coefficients of ξ^n and ξ^{-n} must coincide. For $n \geqslant 0$,

$$r^n\, d_n(r) = r^n \log(1 - r^2) + \sum_{0 \neq k \in \mathbb{Z}} r^{|n-k|}\, \frac{r^{|k|}}{|k|}$$

$$= r^n \log(1-r^2) + \underbrace{\sum_{k=1}^{\infty} \frac{r^{n+2k}}{k}}_{= -r^n \log(1-r^2)} + \sum_{k=1}^{n} \frac{r^n}{k} + \sum_{k=n+1}^{\infty} \frac{r^{2k-n}}{k},$$

whence

$$d_n(r) = \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{\infty} \frac{r^{2k}}{k+n}. \tag{31}$$

Note that $d_n(r)$ is in fact a function of r^2. We have

$$d_n(r) - d_{n-1}(r) = \sum_{k=0}^{\infty} \frac{(1-r^2)r^{2k}}{k+n} \geqslant 0,$$

so that

$$d_n(r) = d_0(r) + (1-r^2) \sum_{k=0}^{\infty} \frac{r^{2k}}{k} \sum_{m=1}^{n} \frac{k}{k+n},$$

whence

$$\log \frac{1}{1-r^2} = d_0(r) \leqslant d_n(r) \leqslant \big(1 + n(1-r^2)\big) \log \frac{1}{1-r^2}. \tag{32}$$

We conclude that the Fourier expansion of the biharmonic Poisson kernel is

$$P_1(r\, e^{\mathrm{i}\,\alpha}, e^{\mathrm{i}\,\phi}) = P(r\, e^{\mathrm{i}\,\alpha}, e^{\mathrm{i}\,\phi}) \log P(r\, e^{\mathrm{i}\,\alpha}, e^{\mathrm{i}\,\phi}) = \sum_{n\in\mathbb{Z}} r^{|n|}\, d_{|n|}(r)\, e^{-\mathrm{i}\,n\alpha}\, e^{\mathrm{i}\,n\phi}.$$

Definition 7.1. Let $h(z)$ be a harmonic function on $\mathbb{D}$ and let ν^h be the analytic functional on $\partial\mathbb{D}$ in its Poisson representation. The *associated biharmonic function* is

$$f_h(z) = \int_{\partial\mathbb{D}} P_1(z,\xi)\, d\nu^h(\xi).$$

Now, let us start with an analytic, whence harmonic function

$$h(z) = \sum_{n=0}^{\infty} h_n\, z^n, \quad \limsup_{n\to\infty} |h_n|^{1/n} \leqslant 1 \tag{33}$$

on $\mathbb{D}$. We compute the Fourier coefficients $\nu_n^h = \nu^h(e^{-\mathrm{i}\,n\phi})$ of the corresponding analytic functional ν^h, that is

$$h(z) = \int_{\partial\mathbb{D}} P(z,\xi)\, d\nu^h(\xi) = \sum_{n\in\mathbb{Z}} r^{|n|}\, e^{-\mathrm{i}\,n\alpha}\, \overline{\nu_n^h},$$

where $z = r\,e^{-\mathrm{i}\,\alpha}$. Comparison with

$$h(z) = \sum_{n=0}^{\infty} h_n\, r^n\, e^{\mathrm{i}\,n\alpha}$$

yields

$$\nu_n^h = \begin{cases} \overline{h}_{-n}\,, & \text{if } n \leqslant 0\,, \\ 0\,, & \text{if } n > 0\,. \end{cases} \tag{34}$$

Then the associated biharmonic function is

$$f_h(z) = \sum_{n=0}^{\infty} h_n\, d_n(|z|)\, z^n\,.$$

Below we shall use the fact that (32) implies

$$\left| f_h(z) - \log\frac{1}{1-r^2} h(z) \right| \leqslant (1-r^2) \sum_{n=1}^{\infty} n\,|h_n|\, r^n \tag{35}$$

with $\log\frac{1}{1-r^2} \sim R$ as $r = |z| \to 1$. Moreover, the real part

$$\Re h(z) = \sum_{n=0}^{\infty} h_n\, r^n\, \frac{h_n e^{\mathrm{i}\,n\alpha} + \overline{h}_n e^{-\mathrm{i}\,n\alpha}}{2}$$

is harmonic, and the Fourier coefficients of the corresponding analytic functional $\nu^{\Re h}$ are

$$\nu_n^{\Re h} = \begin{cases} \Re h_0\,, & \text{if } n = 0, \\ h_n/2\,, & \text{if } n > 0, \\ \overline{h}_{-n}/2\,, & \text{if } n < 0\,. \end{cases}$$

(Indeed, every real harmonic function arises as the real part of an analytic function.) It follows that

$$f_{\Re h(z)} = -\int_{\partial\mathbb{D}} P_1(z,\xi)\, d\nu^{\Re h}(\xi) = \Re f_h(z)\,.$$

Discussion 7.2. Euclidean and hyperbolic harmonic functions coincide. We know that if a harmonic function h is bounded then its representing analytic functional is in fact a measure with bounded density with respect to the Lebesgue measure on $\partial\mathbb{D}$. Euclidean biharmonic functions are of the form $h_1(z) + r^2 h_2(z)$ with $h_1\,,h_2$ harmonic. The Euclidean biharmonic kernel is $r^2 P(z,\xi)$, and no normalisation is in place. In this context, Mazalov [25] provides an example of a *bounded* biharmonic function which has no radial limits at the boundary. The function is of the form $(1-r^2)\,\Re h(z)$ where $h(z)$ is as in (33) with a lacunary sequence of coefficients $h_n \geqslant 0$. It is bounded, while $h(z)$ is not bounded. What would be an analogue in the hyperbolic case? Since the biharmonic kernel needs to be normalised by $\Phi_1(z|0) \sim R \sim |\log(1-r)|$ as $R \to \infty$ (recall that $R = \rho(z,0)$ and $r = |z|$), we would look for a hyperbolically biharmonic function $f(z)$ such that $f(z)/|\log(1-r)|$ is bounded but has no radial limits.

Now suppose that $h(z)$ is such that $h_n \geqslant 0$ and such that $f(z)/|\log(1-r)|$ is bounded. Then we have by (32) for $|z| = r$

$$|h(z)| \leqslant h(r) \leqslant \sum_{n=0}^{\infty} h_n \frac{d_n(r)}{d_0(r)} r^n \leqslant \frac{f_h(r)}{|\log(1-r)|}\,.$$

Hence also the harmonic function $h(z)$ is bounded, the (common) representing analytic functional of $h(z)$ and $f_h(z)$ is a measure with bounded density and $f_h(z)/|\log(1-r)|$ has admissible limits almost everywhere on $\partial\mathbb{D}$ by Theorem 6.5. It is also worth mentioning that $f_h(z)/|\log(1-r)| - h(z)$ has admissible limit 0 almost everywhere. □

Question 7.3. Let ν be an analytic functional on $\partial\mathbb{D}$ and

$$f(z) = \Pi_{1,0}\,\nu(z) = \int_{\mathbb{D}} P_1(z,\xi)\,d\nu(\xi)$$

be such that $f(z)/R$ is bounded. Is it true that ν must be a measure with bounded density?

On an example of Borichev.[3]

The following interesting example is closely related to the results and methods in the paper by Borichev et al. [4].

Proposition 7.4. *There is a harmonic function $h(z)$ on $\mathbb{D}$ such that $h(z)/|\log(1-r)|$ is bounded, but has no radial limits at any point of $\partial\mathbb{D}$.*

Of course, this function is also $\mathfrak{L}$-biharmonic, so it is an example of a biharmonic function in the sense of Discussion 7.2.

We now provide the details of the proof of Proposition 7.4.

[3] We acknowledge literature hints of Fausto Di Biase (Pescara) which led us to the work of Borichev. We are particularly grateful to Alexander Borichev (Marseille) who indicated this clever example to us.

The set $K = \left\{ \frac{t+1}{6}\, e^{3\pi t \mathrm{i}} \;:\, t \in [0\,,\,1] \right\}$ is a spiral winding one and a half times around the origin with radius r varying from $1/6$ to $1/3$. Using Runge's approximation theorem as for example stated by Rudin [29, Thm. 13.7], one finds a homogeneous polynomial

$$p(z) = \sum_{j=1}^{s} p_j\, z^j \quad \text{with} \quad |p(z) - 5/3| < 2/3 \;\text{ for all }\; z \in K\,. \tag{36}$$

It is of course harmonic, and on $\mathbb{D}$,

$$|p(z)| \leqslant B\,|z|\,, \quad \text{where} \quad B = \sum_{j=1}^{s} |p_j|\,.$$

We construct the function

$$h(z) = \sum_{k=1}^{\infty} k!\, p\big(z^{2^{k!}}\big)\,, \quad z \in \mathbb{D}\,. \tag{37}$$

Note that $p(0) = 0$.

Lemma 7.5. *The series defining $h(z)$ converges absolutely in $\mathbb{D}$, and*

$$\sup_{z\in\mathbb{D}} \frac{|h(z)|}{\big|\log(1-|z|)\big|} < \infty\,.$$

Proof. Consider

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1-z).$$

For real $r \in [0\,,\,1)$, the sequence (r^n/n) is decreasing, whence

$$f(r) = \sum_{l=0}^{\infty} \sum_{n=2^l}^{2^{l+1}-1} \frac{r^n}{n} \geqslant \sum_{l=0}^{\infty} 2^l\, \frac{r^{2^{l+1}-1}}{2^{l+1}-1} \geqslant \frac{1}{2}\sum_{l=1}^{\infty} r^{2^l} = \frac{1}{2}\sum_{k=1}^{\infty} \sum_{l=k!}^{(k+1)!-1} r^{2^l}$$
$$\geqslant \frac{1}{2}\sum_{k=1}^{\infty}\big((k+1)!-k!\big) r^{2^{(k+1)!-1}} \geqslant \frac{1}{2}\sum_{k=1}^{\infty}\Big(1-\frac{1}{k+1}\Big)(k+1)!\, r^{2^{(k+1)!}} \geqslant \frac{1}{4}\sum_{k=2}^{\infty} k!\, r^{2^{k!}}\,.$$

Therefore

$$\sum_{k=1}^{\infty} k!\, |z|^{2^{k!}} \leqslant |z|^2 + 4\big|\log(1-|z|)\big| \leqslant c\,\big|\log(1-|z|)\big|\,.$$

Now

$$|h(z)| \leqslant \sum_{j=1}^{s} |p_j| \sum_{k=1}^{\infty} k!\,|z|^{j\,2^{k!}} \leqslant c \sum_{j=1}^{s} |h_j| \big|\log(1-|z|^j)\big| \leqslant c\,B\,\big|\log(1-|z|)\big|,$$

as stated. □

Lemma 7.6. *For every $\alpha \in (-\pi\,,\pi]$,*

$$\limsup_{r\to 1} \frac{|h(r\,e^{\mathrm{i}\,\alpha})|}{\big|\log(1-r)\big|} \geqslant 1\,.$$

Proof. We fix α and $N \in \mathbb{N}$ and can find $r_N = r_N(\alpha)$ such that

$$w_N = r_N\,e^{\mathrm{i}\,\alpha\,2^{N!}} \in K\,.$$

Thus,

$$1/6 \le r_N \leqslant 1/3\,, \quad \big|p(w_N) - 5/3\big| < 2/3\,, \quad \text{and} \quad \Re\big(p(w_N)\big) > 1.$$

Now we choose

$$z_N = r_N^{1/2^{N!}}\,e^{\mathrm{i}\,\alpha}\,,$$

so that $|z_N| \to 1$, and we consider

$$h(z_N) - N!\,p\big(z_N^{2^{N!}}\big) = \sum_{k=1}^{N-1} k!\,p\big(z_N^{2^{k!}}\big) + \sum_{k=N+1}^{\infty} k!\,p\big(z_N^{2^{k!}}\big).$$

We can estimate the first sum by

$$\left|\sum_{k=1}^{N-1} k!\,p\big(z_N^{2^{k!}}\big)\right| \leqslant B \sum_{k=1}^{N-1} k! \leqslant 2B\,(N-1)! = \frac{2B}{N} N!\,,$$

and, by setting $x_N = (1/3)^{N!}$ and using that $2^n \geqslant n$, the second sum is majorized as follows:

$$\begin{aligned}\left|\sum_{k=N+1}^{\infty} k!\,p\big(z_N^{2^{k!}}\big)\right| &\leqslant B\,N! \sum_{k=N+1}^{\infty} \frac{k!}{N!}\,x_N^{2^{(k!/N!)-1}} \leqslant B\,N! \sum_{k=N+1}^{\infty} \frac{k!}{N!}\,x_N^{(k!/N!)-1} \\ &\leqslant B\,N! \sum_{m=N+1}^{\infty} m\,x_N^{m-1} = B\,N!\,\underbrace{\frac{(N+1)x_N^N - N x_N^{N+1}}{(1-x_N)^2}}_{\to 0\,,\ \text{as}\ N\to\infty}\,.\end{aligned}$$

It follows that

$$\Big|h(z_N) - N!\,\underbrace{p\big(z_N^{2^{N!}}\big)}_{p(w_n)}\Big| \leqslant N!\,\epsilon_N\,, \quad \text{where } \epsilon_N \to 0 \text{ as } N \to \infty\,.$$

Now note that $1 < \log(1/r_N) < 2$. Using this and the fact that for $x > 0$ one has $x - x^2/2 \leqslant 1 - e^{-x} \leqslant x$, one obtains that

$$1 - |z_N| = 1 - e^{-2^{-N!}\log(1/r_N)} \begin{cases} < 2 \cdot 2^{-N!} \\ > \frac{1}{2} \cdot 2^{-N!}. \end{cases}$$

We deduce that $\big|\log(1 - |z_N|)\big| \sim N!$, whence

$$\left|\frac{h(z_N)}{\big|\log(1 - |z_N|)\big|} - p(w_N)\right| \to 0 \quad \text{as } N \to \infty\,.$$

Since $|p(w_N)| \geqslant \Re\big(p(w_N)\big) > 1$, the statement is proved. □

Lemma 7.7. $$\lim_{N\to\infty} \sup_{|z|=1-2^{-N!\sqrt{N}}} \frac{h(z)}{\big|\log(1 - |z|)\big|} = 0.$$

Proof. It is sufficient to show that

$$\lim_{N\to\infty} \frac{h_0\big(1 - 2^{-N!\sqrt{N}}\big)}{N!\sqrt{N}} = 0\,, \quad \text{where} \quad h_0(z) = \sum_{k=1}^{\infty} k!\, z^{2^{k!}}.$$

Similarly as above, using the fact that $(1 - 2^{-N!\sqrt{N}})^{2^{N!\sqrt{N}}} \leqslant 1/e$ and setting $y_N = e^{-N!}$, we find

$$\begin{aligned} h_0\big(1 - 2^{-N!\sqrt{N}}\big) &\leqslant \sum_{k=1}^{N} k! + \sum_{k=N+1}^{\infty} k!\, \Big(\frac{1}{e}\Big)^{2^{k!-N!\sqrt{N}}} \\ &\leqslant 2\,N! + N! \sum_{k=N+1}^{\infty} \frac{k!}{N!}\, (y_N)^{(k!/N!)-1} y_N^{1-\sqrt{N}} \\ &\leqslant 2\,N! + N! \sum_{m=N+1}^{\infty} m\, (y_N)^{m-1} y_N^{1-\sqrt{N}} \\ &= 2\,N! + N!\, \underbrace{\frac{(N+1)y_N^{N+1-\sqrt{N}} - N y_N^{N+2-\sqrt{N}}}{(1 - y_N)^2}}_{\to 0\,, \text{ as } N \to \infty}. \end{aligned}$$

Divided by $N!\sqrt{N}$, this tends to 0. □

Lemmas 7.5, 7.6 and 7.7 prove Proposition 7.4. With very small adaptations, one verifies that the same statement holds for the harmonic function $\Re h(z)$. We conclude with the following observation, which should not look too surprising. (Recall that for $\lambda = 0$, $\Phi_2(z|0) \sim R^2/2$, as $r = |z| \to 1$.)

Lemma 7.8. *The biharmonic function f_h associated with the function h of (37) is such that $f_h(z)/R^2$ is bounded and has no radial limits at any point of $\partial\mathbb{D}$.*

Proof. Similarly to Lemma 7.5, for $0 \leqslant r < 1$,

$$\frac{1}{1-r}\log\frac{1}{1-r} = \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\frac{1}{j}\right)r^n \geqslant \sum_{k=1}^{\infty}\left(\sum_{n=2^{k!}}^{2^{(k+1)!}-1}\sum_{j=1}^{n}\frac{1}{j}\right)r^{2^{(k+1)!}}$$
$$\overset{(*)}{\geqslant} \tilde{c}\sum_{k=1}^{\infty}(k+1)!\,2^{(k+1)!}\,r^{2^{(k+1)!}}.$$

For $(*)$, see below. We get via (35)

$$\left|f_h(z) - \log\frac{1}{1-r^2}\,h(z)\right| \leqslant (1-r^2)\sum_{j=1}^{N} j\,|p_j|\sum_{k=1}^{\infty} k!\,2^{k!}\,r^{j2^{k!}} \leqslant \widetilde{C}\log\frac{1}{1-r}$$

for a suitable constant $\widetilde{C} > 0$. The statement follows by dividing by $\left(\log\frac{1}{1-r}\right)^2$ and applying Proposition 7.4.

Let us prove $(*)$:

$$Q := \frac{1}{(k+1)!\,2^{(k+1)!}}\sum_{n=2^{k!}}^{2^{(k+1)!}-1}\sum_{j=1}^{n}\frac{1}{j}$$
$$= \frac{1}{(k+1)!\,2^{(k+1)!}}\,2^{k!}\sum_{j=1}^{2^{k!}}\frac{1}{j} + \frac{1}{(k+1)!\,2^{(k+1)!}}\sum_{j=2^{k!}+1}^{2^{(k+1)!}-1}\frac{2^{(k+1)!}-j}{j}.$$

As $k \to \infty$, the first term behaves like

$$\frac{2^{k!}\,k!\,\log 2}{(k+1)!\,2^{(k+1)!}} \to 0\,,$$

while the second term is

$$\frac{1}{(k+1)!}\sum_{j=2^{k!}+1}^{2^{(k+1)!}-1}\left(\frac{1}{j} - \frac{1}{2^{(k+1)!}}\right) \to \log 2\,.$$

Thus, the quotient Q is bounded below by some $\tilde{c} > 0$ for all k. □

Data availability

No data was used for the research described in the article.

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