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Prym map and second Gaussian map for Prym-canonical line bundles[☆]

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Abstract

We show that the second fundamental form of the Prym map lifts the second Gaussian map μ_A of the Prym-canonical bundle. We prove, by degeneration to binary curves, that μ_A is surjective for the general point $[C, A]$ of $\mathcal{R}_g$ for $g \geq 20$.

MSC: 14H10; 14K12

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1. Introduction

Similarly to the period map $P_g : \mathcal{M}_g \to \mathcal{A}_g$, the Prym map $Pr_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$ provides a way to link the geometry of moduli spaces of curves to the geometry of moduli spaces of principally polarized abelian varieties. Recall that $\mathcal{R}_g$ denotes the moduli space which parametrizes isomorphism classes of pairs $[C, A]$, where C is a smooth curve of genus g and $A \in Pic^0(C)[2] - \{\mathcal{O}_C\}$ is a torsion point of order 2, or equivalently isomorphism classes of

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unramified double coverings $\pi : \tilde{C} \to C$. The Prym map associates to a point $[(C, A)] \in \mathcal{R}_g$ the isomorphism class of the connected component of zero, $P(C, A)$ of the kernel of the norm map $Nm_\pi : J\tilde{C} \to JC$, with its principal polarization. Both the period map and the Prym map have been extensively studied since a long time, but also recently there have been important developments on the birational geometry of $\mathcal{R}_g$ [11].

In this paper we focus on the study of the second fundamental form of the Prym map analogously to what it has been done for the second fundamental form of the period map and its link with the second Gaussian map. In fact in [10] it is shown that the second fundamental form of the period map lifts the second Gaussian map of the canonical line bundle, as stated in an unpublished paper by Green and Griffiths (cf. [13]). With this geometrical motivation, in [7] we investigated curvature properties of $\mathcal{M}_g$ endowed with the Siegel metric. In fact, we computed the holomorphic sectional curvature of $\mathcal{M}_g$ along the tangent directions given by the Schiffer variations in terms of the second Gaussian map. This also suggested that the second Gaussian map itself could give interesting information on the geometry of the curves, hence its rank properties have been investigated in a series of papers (see [4,6,8,9]).

Here we first generalize the lifting result of [10] to the Prym map, namely at the point $[C, A] \in \mathcal{R}_g$ we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{N}^*_{\mathcal{R}^0_g/\mathcal{A}_{g-1},[C,A]} \cong I_2(K_C \otimes A) & \xrightarrow{II} & S^2\Omega^1_{\mathcal{R}^0_g,[C,A]} \cong S^2H^0(K_C^{\otimes 2}) \\ \Big\downarrow -\frac{1}{2\pi i}\mu_A & \swarrow \mathbf{m} & \\ H^0(K_C^{\otimes 4}) & & \end{array}$$

where $I_2(K_C \otimes A)$ is the kernel of the multiplication map $S^2H^0(K_C \otimes A) \to H^0(K_C^{\otimes 2})$, II is the second fundamental form of the Prym map, m is the multiplication map and μ_A is the second Gaussian map associated to the Prym canonical bundle $K_C \otimes A$. This also allows us to generalize the results of [7] on the holomorphic sectional curvature of $\mathcal{R}_g$ with the Siegel metric induced by $\mathcal{A}_{g-1}$ via the Prym map.

In the second part of the paper we concentrate on the study of the second Gaussian map μ_A, $A \in Pic^0(C)[2]$ non trivial. The main result is the proof of the surjectivity of μ_A for the general curve $[C, A] \in \mathcal{R}_g$ of genus $g \geq 20$, generalizing analogous results on the surjectivity of the second Gaussian map of the canonical line bundle for the general curve in $\mathcal{M}_g$ for $g \geq 18$. For the canonical line bundle this surjectivity for general curves of high genus was proved in [8] using curves on K3 surfaces, then the sharp result for genus $\geq$18 has been shown in [4] using degeneration to binary curves, i.e. stable curves which are the union of two rational curves meeting transversally at $g + 1$ points. Here we generalize these degeneration techniques to prove the surjectivity of second Gaussian maps μ_A, for the Prym-canonical bundles $K_C \otimes A$.

In particular, this shows that the locus of curves $[C, A] \in \mathcal{R}_g$ ($g \geq 20$) for which the map μ_A is not surjective is a proper subscheme of $\mathcal{R}_g$ and one observes that for $g = 20$ it is an effective divisor in $\mathcal{R}_{20}$ of which we compute the cohomology class both in $\mathcal{R}_{20}$ and in a partial compactification $\tilde{\mathcal{R}}_{20}$ following computations developed in [11].

The paper is organized as follows: in Section 2 we describe the second fundamental form, we prove that it is a lifting of the second Gaussian map μ_A and we compute the holomorphic sectional curvature along the Schiffer variations. In Section 3 we construct the Prym-canonical binary curves that we use for the degeneration. In Section 4 we explicitly describe the ideal of the quadrics containing the Prym-canonical binary curve. In Section 5 we prove, by induction on the genus, the surjectivity of μ_A. In Section 6 we compute the cohomology class in $\tilde{\mathcal{R}}_{20}$ of the degeneracy locus of the second Gaussian map. Finally in the Appendix we list the Maple scripts used in the computations.

Finally we observe that the results and the techniques of Section 2 are of different nature from the rest of the paper, which can be read separately, once one has looked at the definition of the Gaussian maps in Section 2.2.

We also notice that in the proof of the surjectivity of μ_A, in particular in Sections 4 and 5, we follow the lines of the proof of the surjectivity of the second Gaussian map of the canonical line bundle given in [4].

2. The second fundamental form and the second Gaussian map

2.1. *The second fundamental form of the Prym map*

We start by recalling the definition of the Prym map

$$Pr : \mathcal{R}_g \to \mathcal{A}_{g-1},$$

which associates to a point $[(C, A)] \in \mathcal{R}_g$ its Prym variety $P(C, A)$ with its principal polarization. If $\pi : \tilde{C} \to C$ is the unramified double covering associated to the pair (C, A), the Prym variety of the double covering is the principally polarized abelian variety of dimension $g-1$ defined as the connected component of zero of the kernel of the norm map $Nm_\pi : J\tilde{C} \to JC$,

$$P(C, A) = Ker(Nm_\pi)^0 \subset J\tilde{C}.$$

We recall that the Prym map is generically an embedding for $g \geq 7$ ([12,15]). Hence there exists an open set $\mathcal{R}_g^0 \subset \mathcal{R}_g$ where Pr is an embedding and such that there exists the universal family $f : \mathcal{X} \to \mathcal{R}_g^0$. If $b \in \mathcal{R}_g^0$, we have $f^{-1}(b) = (C_b, A_b)$ where C_b is a smooth irreducible curve of genus g and $A_b \in Pic^0(C_b)[2]$ is a line bundle of order 2 on C_b. Denote by $\mathcal{P} \in Pic(\mathcal{X})$ the corresponding Prym bundle and by $\mathcal{F}^{Pr} := f_*(\omega_{\mathcal{X}/\mathcal{R}_g^0} \otimes \mathcal{P})$. Observe that $\mathcal{F}^{Pr}$ is the pullback of the Hodge bundle on $\mathcal{A}_{g-1}$ to $\mathcal{R}_g^0$. More precisely, if $\psi : \mathcal{P}r(\mathcal{X}) \to \mathcal{R}_g^0$ is the universal family of Prym varieties, so $\psi^{-1}(b) = P(C_b, A_b)$ is the Prym variety associated to the pair (C_b, A_b), then $\mathcal{F}^{Pr}$ is the Hodge bundle $\mathcal{H}^{1,0} \subset R^1\psi_*\mathbb{C}$ of the family ψ.

On the local system $R^1\psi_*\mathbb{C}$ we have the flat Gauss–Manin connection ∇^{GM}, and a non degenerate bilinear form defined as follows. At the point $P(C, A)$, the fiber of $R^1\psi_*\mathbb{C}$ is isomorphic to the vector space $H^1(\tilde{C}, \mathbb{C})^-$, where $\tilde{C} \xrightarrow{\pi} C$ is the double covering associated to (C, A) and $H^1(\tilde{C}, \mathbb{C})^-$ is the anti-invariant part of the cohomology under the covering involution on $\tilde{C}$. If $[\omega_1], [\omega_2] \in H^1(\tilde{C}, \mathbb{C})^-$, we have the following non degenerate bilinear form $\langle [\omega_1], [\omega_2] \rangle = i \int_{\tilde{C}} \omega_1 \wedge \overline{\omega_2}$ and the Gauss–Manin connection ∇^{GM} is compatible with it, so it induces a metric connection $\nabla^{1,0}$ on $\mathcal{H}^{1,0}$, hence a connection on $\mathcal{F}^{Pr}$ (still denoted by $\nabla^{1,0}$) and on its second symmetric power, $S^2\mathcal{F}^{Pr}$. Observe that this metric on $S^2\mathcal{F}^{Pr}$ is the pullback via the Prym map of the metric on $\mathcal{A}_{g-1}$ induced by the unique (up to scalar) $Sp(2g-2, \mathbb{R})$-invariant metric on the Siegel space H_{g-1}. So we will call this metric the Siegel metric.

Consider the tangent bundle exact sequence of the Prym map

$$0 \to T_{\mathcal{R}_g^0} \to T_{\mathcal{A}_{g-1}|\mathcal{R}_g^0} \to \mathcal{N}_{\mathcal{R}_g^0/\mathcal{A}_{g-1}} \to 0. \quad (1)$$

Its dual becomes

$$0 \to \mathcal{I}_2 \xrightarrow{i} S^2 f_*(\omega_{\mathcal{X}/\mathcal{R}_g^0} \otimes \mathcal{P}) \xrightarrow{m} f_*(\omega^{\otimes 2}_{\mathcal{X}/\mathcal{R}_g^0}) \to 0 \tag{2}$$

where m is fibrewise the multiplication map and we denote by $\mathcal{I}_2$ the conormal bundle $\mathcal{N}^*_{\mathcal{R}_g^0/\mathcal{A}_{g-1}}$. Recall that the second fundamental form of the exact sequence (2) is defined as follows

$$II : \mathcal{I}_2 \to f_*(\omega^{\otimes 2}_{\mathcal{X}/\mathcal{R}_g^0}) \otimes \Omega^1_{\mathcal{R}_g^0}, \qquad II(s) = m(\nabla(i(s))),$$

where ∇ is the metric connection on $S^2 f_*(\omega_{\mathcal{X}/\mathcal{R}_g^0} \otimes \mathcal{P}) = S^2\mathcal{F}^{Pr}$ defined above. At the point $(C, A) \in \mathcal{R}_g^0$ the exact sequence (2) becomes

$$0 \to I_2(K_C \otimes A) \to S^2H^0(K_C \otimes A) \xrightarrow{m} H^0(K_C^{\otimes 2}) \to 0.$$

Hence, if we identify $T_{\mathcal{R}_g^0,b_0} \xrightarrow{\cong} H^1(T_C)$ via the Kodaira–Spencer map of the family $\mathcal{X} \xrightarrow{f} \mathcal{R}_g^0$, the second fundamental form II at $[C]$ can be seen as a map $II : I_2(K_C \otimes A) \to H^0(2K_C) \otimes H^0(2K_C)$.

2.2. *Gaussian maps*

Let Y be a smooth complex projective variety and let $\Delta_Y \subset Y \times Y$ be the diagonal. Let L and M be line bundles on Y. For a non-negative integer k, the *k-th Gaussian map* associated to these data is given by restriction to the diagonal

$$\begin{aligned} \gamma^k_{L,M} &: H^0(Y \times Y, I^k_{\Delta_Y} \otimes L \boxtimes M) \to H^0(Y \times Y, I^k_{\Delta_Y} \otimes L \boxtimes M \otimes \mathcal{O}_{\Delta_Y}) \\ &\cong H^0(Y, S^k\Omega^1_Y \otimes L \otimes M). \end{aligned} \tag{3}$$

The exact sequence

$$0 \to I^{k+1}_{\Delta_Y} \to I^k_{\Delta_Y} \to S^k\Omega^1_Y \to 0, \tag{4}$$

twisted by $L \boxtimes M$, shows that the domain of the k-th Gaussian map is the kernel of the previous one:

$$\gamma^k_{L,M} : ker\gamma^{k-1}_{L,M} \to H^0(Y, S^k\Omega^1_Y \otimes L \otimes M).$$

In this paper, we will exclusively deal with the second Gaussian map for curves C, assuming also that $L = M$.

The map γ^0_L is the multiplication map of global sections

$$H^0(C, L) \otimes H^0(C, L) \to H^0(C, L^{\otimes 2})$$

which obviously vanishes identically on $\wedge^2 H^0(L)$. Consequently, $H^0(C \times C, I_{\Delta_C} \otimes L \boxtimes L)$ decomposes as $\wedge^2 H^0(L) \oplus I_2(L)$, where $I_2(L)$ is the kernel of $S^2H^0(C, L) \to H^0(C, L^{\otimes 2})$. Since γ^1_L vanishes on symmetric tensors, one writes

$$\gamma^1_L : \wedge^2 H^0(L) \to H^0(K_C \otimes L^{\otimes 2}).$$

Again, $H^0(C \times C, I^2_{\Delta_C} \otimes L \boxtimes L)$ decomposes as the sum of $I_2(L)$ and the kernel of γ^1_L. Since γ^2_L vanishes identically on skew-symmetric tensors, one usually writes

$$\gamma^2_L : I_2(L) \to H^0(K_C^{\otimes 2} \otimes L^{\otimes 2}).$$

Assume now that the line bundle L is $K_C \otimes A$, with $A \in Pic^0(C)[2]$, and denote by

$$\mu_A := \gamma^2_{K_C \otimes A} : I_2(K_C \otimes A) \to H^0(K_C^{\otimes 4}) \tag{5}$$

the second Gaussian map. It is useful to provide also a local description of it. Fix a basis $\{\omega_i\}$ of $H^0(K_C \otimes A)$ and write it in a local coordinate z as $\omega_i = f_i(z)dz \otimes l$, where l is a local generator of the line bundle A. For a quadric $Q \in I_2(K_C \otimes A)$ we have $Q = \sum_{i,j} a_{ij}\omega_i \otimes \omega_j$, where $a_{ij} = a_{ji}$ and $\sum_{i,j} a_{ij} f_i f_j \equiv 0$, hence we have $\sum_{i,j} a_{ij} f_i' f_j \equiv 0$. The local expression of $\mu_A(Q)$ is

$$\mu_A(Q) = \sum_{i,j} a_{ij} f_i'' f_j (dz)^4 = -\sum_{i,j} a_{ij} f_i' f_j' (dz)^4.$$

The maps μ_A glue together to give a map of vector bundles on $\mathcal{R}_g^0$,

$$\mu : \mathcal{I}_2 \to f_*((\omega_{\mathcal{X}/\mathcal{R}_g^0} \otimes \mathcal{P})^{\otimes 2} \otimes \omega_{\mathcal{X}/\mathcal{R}_g^0}{}^{\otimes 2}) \cong f_*(\omega^{\otimes 4}_{\mathcal{X}/\mathcal{R}_g^0}), \tag{6}$$

where $\mathcal{I}_2$ is as in (2).

2.3. The theorem

In this subsection we show that the second fundamental form II of the Prym map is a lifting of the second Gaussian map μ_A as it happens for the second fundamental form of the period map and the second Gaussian map of the canonical line bundle (see Theorems 2.1 and 4.5 of [10]).

To this purpose, recall that given a holomorphic line bundle A of degree zero on a curve C, there exists a unique (up to constant) hermitian metric H on A and a unique connection D_H on A which is compatible both with the holomorphic structure and with the metric and which is flat (see e.g. [14]). If moreover $A^{\otimes 2} = \mathcal{O}_C$ and we denote by $\pi : \tilde{C} \to C$ the associated unramified double covering, we can take an atlas $\{(U_\alpha, s_\alpha)\}$ of A such that the sections s_α have values in $\tilde{C}$, hence the cocycle $g_{\alpha,\beta} = s_\alpha/s_\beta$ has values in $\{\pm 1\}$, so it induces a flat structure on A and a compatible flat hermitian metric on A, which is then equal to H up to scalar (see [16]).

So we can write $D_H = D_H' + \overline{\partial}$, where D_H' is the $(1,0)$ component. Such a pair (A, H) is also called a harmonic line bundle and we have the following properties (see [17]):

- The Kähler identities.
- The associated harmonic decomposition

  $$\mathcal{A}^\bullet(A) = \mathcal{H}^\bullet(A) \oplus im(D_H) \oplus im(D_H^*) = \mathcal{H}^\bullet(A) \oplus im(\overline{\partial}) \oplus im(\overline{\partial}^*),$$

  where $\mathcal{H}(A)$ is the kernel of the laplacian operator $\Delta = D_H D_H^* + D_H^* D_H = 2(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})$.
- The principle of two types

  $$ker(D_H') \cap ker(\overline{\partial}) \cap (im(D_H') + im(\overline{\partial})) = im(D_H'\overline{\partial}).$$

Theorem 2.1. *The diagram*

$$\begin{array}{ccc} I_2(K_C \otimes A) & \xrightarrow{II} & S^2H^0(K_C^{\otimes 2}) \\ \downarrow{\scriptstyle -\frac{1}{2\pi i}\mu_A} & \swarrow{\scriptstyle \mathbf{m}} & \\ H^0(K_C^{\otimes 4}) & & \end{array} \tag{7}$$

is commutative.

Proof. The proof follows the lines of the proof of Theorem 2.1 of [10]. First of all we take $v \in H^1(C, T_C)$ and we compute $II(Q)(v)$ for every $Q \in I_2(K_C \otimes A)$. Using the Kodaira–Spencer map k we can assume that $v = k(\frac{\partial}{\partial t})$, where t is the local coordinate of the unit disk $\Delta = \{|t| < 1\}$ parametrizing a one dimensional deformation $\mathcal{X} \xrightarrow{f} \Delta$ where $(C, A) = f^{-1}(0)$. Take Y a $\mathcal{C}^\infty$ lifting of the holomorphic vector field $\frac{\partial}{\partial t}$ on Δ, so we have a $\mathcal{C}^\infty$ trivialization $\tau : \Delta \times (C, A) \to \mathcal{X}$, $\tau(t, x) := \Phi_{tY}(1)$, where $\Phi_Y(t)$ is the flow of the vector field Y. Then $\theta := \overline{\partial} Y_{|(C,A)}$ is a closed form in $A^{0,1}(T_C)$ such that $[\theta] = v \in H^1(C, T_C)$. Denote by (C_t, A_t) the fiber of f over t, where A_t is a holomorphic line bundle in $Pic^0(C_t)[2]$ endowed with the flat structure induced by the double covering $\pi_t : \tilde{C}_t \to C_t$. We denote by H_t the flat hermitian metric and by $D_{H_t} = D'_{H_t} + \overline{\partial}_t$ the flat Chern connection.

Let $\omega(t)$ be a section of $\mathcal{F}^{Pr}$, hence $\forall t \in \Delta$, $\omega(t) \in H^0(K_{C_t} \otimes A_t) \cong H^{1,0}(A_t)$. Denote by $\tau_t : C \to C_t$ and by $\sigma_t : \tilde{C} \to \tilde{C}_t$ the diffeomorphisms induced by τ, where $\tilde{C}$ and $\tilde{C}_t$ are the unramified double coverings induced by A and by A_t. We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\sigma_t} & \tilde{C}_t \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_t} \\ C & \xrightarrow{\tau_t} & C_t \end{array}$$

so we have an induced map by pullback $\tau_t^* : A^1(A_t) \to A^1(A)$, and since $\omega_t \in A^{1,0}(A_t)$ is D_{H_t}-closed, then also $\tau_t^*(\omega_t)$ is D_H-closed because we have $\tau_t^*(D_{H_t}) = D_H$. In fact by the commutativity of the diagram one immediately sees that the flat structure on A_t given by the covering $\pi_t : \tilde{C}_t \to C_t$ induces by pullback the flat structure on A given by the covering $\pi : \tilde{C} \to C$.

So $\tau_t^*(\omega_t)$ is D_H-closed, hence it has a power series expansion at $t = 0$

$$\tau_t^*(\omega_t) = \omega + (\alpha + D_H h)t + o(t),$$

where $\omega := \omega(0)$, $\alpha \in A^1(A)$ is harmonic and h is a $\mathcal{C}^\infty$ section of A (by the harmonic decomposition for D_H). So we have $\nabla^{GM}_{\frac{\partial}{\partial t}}[\omega(t)]_{t=0} = [\alpha]$, $\theta \cdot \omega = \alpha^{0,1} + \overline{\partial} h$, so $k(\frac{\partial}{\partial t}) \cdot [\omega] = [\alpha^{0,1}]$, where $\alpha^{0,1}$ is the $(0, 1)$ component of α.

Now assume that $\{\omega_i\}_{i=1,\dots,g-1}$ is a basis of $H^0(K_C \otimes A)$. Take a quadric $Q \in I_2(K_C \otimes A)$, $Q = \sum_{i,j} a_{i,j} \omega_i \otimes \omega_j$, with $a_{i,j} = a_{j,i}$, then $\forall i$ we have $\nabla^{1,0}_{\frac{\partial}{\partial t}}[\omega_i(t)]_{t=0} = [\alpha_i{}^{1,0}]$, so if $\tilde{Q}(t) = \sum_{i,j} a_{i,j}(t)\omega_i(t) \otimes \omega_j(t)$ is a section of $\mathcal{I}_2$ such that $\tilde{Q}(0) = Q$, we have $II(Q)(v) = m(\nabla_{\frac{\partial}{\partial t}} \tilde{Q}_{|t=0}) = \sum_{i,j} a'_{i,j}(0)\omega_i\omega_j + 2\sum_{i,j} a_{i,j}\alpha_i^{1,0}\omega_j$. Since $\sum_{i,j} a_{i,j}(t)\omega_i(t)\omega_j(t) \equiv 0$, also its derivative with respect to t at $t = 0$ must be zero, i.e. $2\sum_{i,j} a_{i,j}(\alpha_i + D_H h_i)\omega_j + \sum_{i,j} a'_{i,j}(0)\omega_i\omega_j \equiv 0$, and if we take the $(1, 0)$ part we have $2\sum_{i,j} a_{i,j}(\alpha_i^{1,0} + D'_H h_i)\omega_j + \sum_{i,j} a'_{i,j}\omega_i\omega_j \equiv 0$, so $II(Q)(v) = -2\sum_{i,j} a_{i,j}\omega_j D'_H h_i$.

Now we observe that $\sum_{i,j} a_{i,j}\omega_j D'_H h_i = \rho(Q)(v)$ where ρ is the map defined in Theorem 4.4 of [10]. So we conclude by Theorem 4.5 of [10] that asserts that ρ is a lifting of μ_A. $\square$

Corollary 2.2. *Let $\xi_P \in H^1(T_C)$ be a Schiffer variation at a point $P \in C$. Then we have*

$$\mu_A(Q)(P) = -\frac{1}{2\pi i} II(Q)(\xi_p \odot \xi_P) = -\frac{1}{2\pi i} \xi_P(II(Q)(\xi_P)).$$

Proof. Recall that given a point $P \in C$, a Schiffer variation $\xi_P \in H^1(T_C)$ is a generator of the image of the coboundary map $H^0(T_C(P)_{|P}) \to H^1(T_C)$. Given a local coordinate in a neighborhood of P, under the Dolbeault isomorphism $H^1(T_X) \cong H^{0,1}(T_X)$, ξ_P is represented by the form $\theta_P = \frac{1}{z-z(P)}\overline{\partial} b_P \otimes \frac{\partial}{\partial z}$, where b_P is a bump function around P. Notice that if we choose b_P to be one in a neighborhood of P, ξ_P depends only on the choice of z. The choice of the local coordinate also allows us to see the evaluation val_P in P as an element of $H^0(K_C^{\otimes 4})^*$ and it holds: $m^*(val_P) = \frac{1}{(2\pi i)^2}\xi_P \odot \xi_P$ (see [10] p. 139). In fact, if $\{\lambda_i\}$ is a basis of $H^0(K_C^{\otimes 2})$, such that locally $\lambda_i = g_i(z)(dz)^2$, we have $(\xi_P \odot \xi_P)(\lambda_i \odot \lambda_j) = \xi_P(\lambda_i)\xi_P(\lambda_j)$ and by Serre duality

$$\begin{aligned}\xi_P(\lambda_i) &= \int_C g_i(z)dz \wedge \frac{\overline{\partial} b_P}{z-z(P)} = -\int_C d\left(\frac{b_P g_i(z)}{z-z(P)}dz\right) \\ &= \int_\Gamma \frac{g_i(z)}{z-z(P)}dz = (2\pi i) g_i(P),\end{aligned} \tag{8}$$

where Γ is a small circle around P. Hence by Theorem 2.1 we have $\mu_A(Q)(P) = -\frac{1}{2\pi i} II(Q)(\xi_p \odot \xi_P) = -\frac{1}{2\pi i}\xi_P(II(Q)(\xi_P))$. □

2.4. Curvature

In this subsection we would like to give an explicit formula for the holomorphic sectional curvature of the Siegel metric on $\mathcal{R}_g$ along the tangent directions given by the Schiffer variations. This formula is analogous to the formula of the holomorphic sectional curvature of the Siegel metric on $\mathcal{M}_g$ induced by the Period map, given in Cor.3.8 [7].[1]

Assume that $\{Q_i\}$ is an orthonormal basis of $I_2(K_C \otimes A)$, $\{\omega_i\}$ an orthonormal basis of $H^0(K_C \otimes A)$ and choose a local coordinate z at P and a local generator a of A such that locally $\omega_i = f_i(z)dz \otimes a$.

Proposition 2.3. *The holomorphic sectional curvature H of $T_{\mathcal{R}_g^0}$ at $[C, A] \in \mathcal{R}_g^0$ computed at the tangent vector ξ_P given by a Schiffer variation in P is given by:*

$$H(\xi_P) = -1 - \frac{1}{16\alpha_P^4\pi^2}\sum_i |\mu_A(Q_i)(P)|^2$$

where $\alpha_P = \sum_i |f_i(P)|^2$.

Proof. Also the proof follows the lines of [7]. We start recalling that

$$H(\xi_P) = \frac{\langle R(\xi_P), \xi_P\rangle(\xi_P, \overline{\xi_P})}{\langle \xi_P, \xi_P\rangle^2},$$

and by the Gauss formula we have $\langle R(\xi_P), \xi_P\rangle(\xi_P, \overline{\xi_P}) = \langle \tilde{R}(\xi_P), \xi_P\rangle(\xi_P, \overline{\xi_P}) - \langle \sigma(\xi_P), \sigma(\xi_P)\rangle(\xi_P, \overline{\xi_P})$, where R is the curvature of the Siegel metric, $\tilde{R}$ is the curvature of $\mathcal{A}_{g-1}$ and σ is the second fundamental form of $\mathcal{R}_g^0$ in $\mathcal{A}_{g-1}$ (see the tangent bundle exact sequence (1)).

First of all observe that the holomorphic sectional curvature of $\mathcal{A}_{g-1}$ along the Schiffer variations is equal to -1 (see the argument below Corollary 3.8 of [7]). In fact a Schiffer

[1] Notice that between the formula of Cor 3.8 of [7] and the formula given in Proposition 2.3 there is a difference by a factor 4, due to a small mistake in Cor 3.8 of [7] where we identified ρ with II, while $II = -2\rho$ ([10] p.136).

variation ξ_P, seen as a symmetric homomorphism $H^0(K_C \otimes A) \to H^0(K_C \otimes A)^* \cong H^1(A)$ has rank 1, since its kernel is $H^0(K_C \otimes A(-P))$. To compute $\langle \sigma(\xi_P), \sigma(\xi_P)\rangle(\xi_P, \overline{\xi_P})$, recall that $\sigma(\xi_P)(Q) = II(Q)(\xi_P)$, for all $Q \in I_2(K_C \otimes A)$, hence $\sigma(\xi_P) = \sum_i II(Q_i)(\xi_P) \otimes Q_i^*$, where $\{Q_i\}$ is an orthonormal basis of $I_2(K_C \otimes A)$. Hence $\langle \sigma(\xi_P), \sigma(\xi_P)\rangle = \sum_i \langle II(Q_i)(\xi_P), II(Q_i)(\xi_P)\rangle$. Now recall that the Schiffer variations at $3g-3$ general points of C give a basis of $H^1(T_C)$, so we can write $II(Q_i)(\xi_P) = \sum_S \xi_S(II(Q_i)(\xi_P))\xi_S^*$, therefore $\langle \sigma(\xi_P), \sigma(\xi_P)\rangle(\xi_P, \overline{\xi_P}) = \sum_i \xi_P(II(Q_i)(\xi_P))\overline{\xi_P(II(Q_i)(\xi_P))}$. By Corollary 2.2 we have $\sum_i \xi_P(II(Q_i)(\xi_P))\overline{\xi_P(II(Q_i)(\xi_P))} = 4\pi^2 \sum_i |\mu_A(Q_i)(P)|^2$. Now it remains to show that $\langle \xi_P, \xi_P\rangle = 8\pi^2\alpha_P^2$ as in Lemma 2.2 of [7]. To do this we write ξ_P as an element of $S^2(H^0(K_C \otimes A)^*)$ as $\xi_P = \sum_{i,j} \xi_P(\omega_i)(\omega_j)(\omega_j^* \odot \omega_i^*)$, where $\{\omega_i^*\}$ is the dual basis of the orthonormal basis $\{\omega_i\}$. One computes $\xi_P(\omega_i)(\omega_j) = \xi_P(\omega_i\omega_j) = 2\pi i f_i(P) f_j(P)$ by (8), then the proof follows exactly as in Lemma 2.2 of [7]. □

3. Prym-canonical binary curves

3.1. *Strategy of the proof of surjectivity*

The rest of the paper is devoted to the proof of the surjectivity of the second Gaussian map μ_A for the general point $[C, A] \in \mathcal{R}_g$. We will do it by degeneration to binary curves following the method used in [4] for the second Gaussian map of the canonical line bundle. We recall that $\mathcal{R}_g$ admits a suitable compactification $\overline{\mathcal{R}}_g$, which is isomorphic to the coarse moduli space of the stack $\mathbf{R}_g$ of Beauville admissible double covers ([3,1]) and to the coarse moduli space of the stack of Prym curves ([2]).

Consider the partial compactification $\tilde{\mathcal{R}}_g$ of $\mathcal{R}_g$ introduced in [11]. Denote by $f : \mathcal{X} \to \tilde{\mathbf{R}}_g$ the universal family and by $\mathcal{P} \in Pic(\mathcal{X})$ the corresponding Prym bundle as in [11] 1.1. The map of vector bundles over $\mathcal{R}_g^0$, $\mu : \mathcal{I}_2 \to f_*((\omega_{\mathcal{X}/\mathcal{R}_g^0} \otimes \mathcal{P})^{\otimes 2} \otimes \omega_{\mathcal{X}/\mathcal{R}_g^0}{}^{\otimes 2}) \cong f_*(\omega_{\mathcal{X}/\mathcal{R}_g^0}^{\otimes 4})$ defined in (6), extends to a map

$$\tilde{\mu} : \tilde{\mathcal{I}}_2 \to f_*((\omega_f \otimes \mathcal{P})^{\otimes 2} \otimes S^2(\Omega_f^1)) \cong f_*(\omega_f^{\otimes 4} \otimes \mathcal{P}^{\otimes 2} \otimes \mathcal{I}_Z^{\otimes 2}), \tag{9}$$

where $\tilde{\mathcal{I}}_2$ is the kernel of the multiplication map $S^2 f_*(\omega_f \otimes \mathcal{P}) \to f_*(\omega_f^{\otimes 2} \otimes \mathcal{P}^{\otimes 2})$, and Z is the locus of nodes of fibers of f, so $\Omega_f^1 \cong \omega_f \otimes \mathcal{I}_Z$.

If $[C, A] \in$ is a point in $\tilde{\mathcal{R}}_g$, the local expression of

$$\mu_A : I_2(\omega_C \otimes A) \to H^0((\omega_C \otimes A)^{\otimes 2} \otimes S^2(\Omega_C^1)) \tag{10}$$

is as follows. Let $\{\omega_i\}$ be a basis of $H^0(\omega_C \otimes A)$ and write it in a local coordinate as $\omega_i = f_i(z)\xi \otimes l$, where ξ and l are local generators of the line bundles ω_C, respectively A. For a quadric $Q = \sum_{i,j} a_{ij}\omega_i \otimes \omega_j \in I_2(\omega_C \otimes A)$, $\mu_A(Q)$ is locally defined as

$$\mu_A(Q) = -\sum_{i,j} a_{ij}(df_i)(df_j)\xi^{\otimes 2} \otimes l^{\otimes 2}. \tag{11}$$

To prove by semicontinuity the surjectivity of μ_A for the general point in $\mathcal{R}_g$ in the following we will exhibit a Prym-canonical binary curve (C, A) for which μ_A is surjective.

3.2. Construction of Prym-canonical binary curves

Recall that a binary curve of genus g is a stable curve consisting of two rational components C_j, $j = 1, 2$ meeting transversally at $g + 1$ points. Moreover one can check that if $A \in Pic^0(C)$ then $H^0(C, \omega_C \otimes A)$ has dimension $g - 1$ and the restriction of $\omega_C \otimes A$ to the component C_j is $K_{C_j}(D_j)$ where D_j is the divisor of nodes on C_j. Since $K_{C_j}(D_j) \cong \mathcal{O}_{\mathbb{P}^1}(g-1)$ we observe that the components are embedded by a linear subsystem of $\mathcal{O}_{\mathbb{P}^1}(g - 1)$, hence they are projections from a point of rational normal curves in $\mathbb{P}^{g-1}$. Vice versa, let us take 2 rational curves embedded in $\mathbb{P}^{g-2}$ by non complete linear systems of degree $g-1$ intersecting transversally at $g+1$ points. Then their union C is a binary curve of genus g embedded either by a linear subsystem of ω_C or by a complete linear system $|\omega_C \otimes A|$, where $A \in Pic^0(C)$ is non trivial (see e.g. [5], Lemma 10). In this section we will construct a binary curve C embedded in $\mathbb{P}^{g-2}$ by a linear system $|\omega_C \otimes A|$ with $A^{\otimes 2} \cong \mathcal{O}_C$, and A is non trivial.

Assuming that the first $g - 1$ nodes, $P_1, \ldots, P_{g-1}$ are in general position, up to projective transformations we will take $P_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the i-th place. Then we can assume that C_j is the image of the map

$$\begin{aligned} &\phi_j : \mathbb{P}^1 \to \mathbb{P}^{g-2}, \quad j = 1, 2 \\ &\phi_j(t, u) := \left[M_j(t, u)\frac{(\delta_{1,j}t - c_{1,j}u)}{(t - a_{1,j}u)}, \ldots, M_j(t, u)\frac{(\delta_{g-1,j}t - c_{g-1,j}u)}{(t - a_{g-1,j}u)} \right] \end{aligned} \tag{12}$$

with $M_j(t, u) := \prod_{r=1}^{g-1}(t - a_{r,j}u)$, $j = 1, 2$ and $\phi_j([a_{l,j}, 1]) = P_l$, $l = 1, \ldots, g - 1$.

We will also impose that the remaining two nodes $P_g := [t_1, \ldots, t_{g-1}]$ and $P_{g+1} := [s_1, \ldots, s_{g-1}]$ are the images of $[0, 1]$ and $[1, 0]$ through the maps ϕ_j, $j = 1, 2$. This is equivalent to

$$c_{i,j} = \frac{d_j t_i a_{i,j}}{A_j}, \qquad \delta_{i,j} = \mu_j s_i,$$

where μ_j, d_j are non zero scalars and $A_j = \prod_{k=1}^{g-1} a_{k,j}$, $j = 1, 2$.

Lemma 3.1. *Let us choose $s_i = 1$ for $i = 1, \ldots, [\frac{g}{2}]$, $s_i = 0$, for $i = [\frac{g}{2}] + 1, \ldots, g - 1$, while $t_i = 0$ for $i = 1, \ldots, [\frac{g}{2}]$, $t_i = 1$, for $i = [\frac{g}{2}]+1, \ldots, g-1$, $\mu_1 = \mu_2 =: \mu$, $d_1 = -\frac{d_2 A_1}{A_2}$. Then, for a general choice of $a_{i,j}$'s, $C = C_1 \cup C_2$ is a binary curve embedded in $\mathbb{P}^{g-2}$ by a linear system $|\omega_C \otimes A|$ with $A^{\otimes 2} \cong \mathcal{O}_C$ and A non trivial.*

Proof. One can easily check that if we choose the elements $a_{k,j}$ general ($j = 1, 2$, $k = 1, \ldots, g - 1$), C_j are smooth rational curves and C has exactly $g + 1$ nodes at the points P_k, $k = 1, \ldots, g + 1$, and no other singularity. Then, by the above discussion we know that C is a binary curve embedded in $\mathbb{P}^{g-2}$ by a linear system of $\omega_C \otimes A$, with $deg(A) = 0$. We will now show that A is a 2-torsion non trivial element in $Pic^0(C)$. In fact, recall that $Pic^0(C) \cong \mathbb{C}^{*g}$ and if we denote by $\alpha : N \to C$ the normalization map, we have an exact sequence

$$0 \to (\omega_C \otimes A) \to \alpha_*(\alpha^*(\omega_C \otimes A)) \to \bigoplus_{i=1}^{g+1} \mathbb{C}_{P_i} \to 0. \tag{13}$$

If we set $\{q_i, r_i\} = \alpha^{-1}(P_i)$, with $q_i \in C_1$, $r_i \in C_2$, $i = 1, \ldots, g + 1$, $D_1 := \sum_{i=1}^{g+1} q_i$, $D_2 := \sum_{i=1}^{g+1} r_i$ we have $\alpha^*(\omega_C \otimes A) = K_N(D_1 + D_2)$. So if we take the long exact sequence in cohomology associated to (13), we have

$$0 \to H^0(\omega_C \otimes A) \to H^0(K_{C_1}(D_1)) \oplus H^0(K_{C_2}(D_2)) \xrightarrow{e} \mathbb{C}^{g+1} \to 0. \tag{14}$$

Clearly $H^0(K_{C_1}(D_1)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(g-1)) \cong \mathbb{C}^g \cong H^0(K_{C_2}(D_2))$. Recall that the line bundle A corresponds to an element in $\mathbb{C}^{*g}$ as follows. Consider the natural isomorphisms $f_i : (\alpha^*(A)_{|C_1})_{q_i} \to (\alpha^*(A)_{|C_2})_{r_i}$, and choose local trivializations $(\alpha^*(A)_{|C_1})_{q_i} \cong \mathbb{C}$, $(\alpha^*(A)_{|C_2})_{r_i} \cong \mathbb{C}$, $\forall i = 1, \dots, g+1$. Then f_i is given by multiplication by an element $h_i \in \mathbb{C}^*$, $\forall i = 1, \dots, g+1$, and we associate to A the element $(h_1, \dots, h_{g+1})$ modulo the diagonal action of $\mathbb{C}^*$.

Notice that if $\sigma \in H^0(\omega_C \otimes A)$ and $\alpha^*(\sigma) = (\sigma_1, \sigma_2) \in H^0(K_{C_1}(D_1)) \oplus H^0(K_{C_2}(D_2))$, then we have

$$Res_{q_i}(\sigma_1) - h_i Res_{r_i}(\sigma_2) = 0, \quad \forall i = 1, \dots, g+1.$$

We claim that with our assumptions the line bundle A corresponds to the element $[(h_1, \dots, h_{g+1})] \in \mathbb{C}^{*g+1}/\mathbb{C}^*$, where $h_i = 1$, for $i < [\frac{g}{2}] + 1$, $h_i = -1$, for $i = [\frac{g}{2}] + 1, \dots, g-1$, $h_g = -1$, $h_{g+1} = 1$, so A is of 2-torsion. In fact, consider the hyperplane $x_i = 0$, $i = 1, \dots, g-1$ in $\mathbb{P}^{g-2}$, and set $\sigma_{i,1} := \phi_1^*(x_i) \in H^0(\mathcal{O}_{\mathbb{P}^1}(g-1)) \cong H^0(K_{C_1}(D_1))$, $\sigma_{i,2} := \phi_2^*(x_i) \in H^0(\mathcal{O}_{\mathbb{P}^1}(g-1)) \cong H^0(K_{C_2}(D_2))$. We have

$$\sigma_{i,j} = \frac{(\delta_{i,j} t - c_{i,j})}{(t - a_{i,j})t} dt, \quad j = 1, 2.$$

Notice that, with our assumptions, we have

$$\begin{aligned} &\delta_{i,1} = \delta_{i,2} = \mu, \quad i < \left[\frac{g}{2}\right] + 1, \qquad \delta_{i,1} = \delta_{i,2} = 0, \quad i \geq \left[\frac{g}{2}\right] + 1, \\ &c_{i,1} = c_{i,2} = 0, \quad i < \left[\frac{g}{2}\right] + 1, \\ &c_{i,1} = \frac{d_1 a_{i,1}}{A_1} = -\frac{d_2 a_{i,1}}{A_2}, \qquad c_{i,2} = \frac{d_2 a_{i,2}}{A_2}, \quad i \geq \left[\frac{g}{2}\right] + 1, \end{aligned} \tag{15}$$

and for simplicity we shall choose $d_2 = \mu = 1$, so the only parameters are the $a_{i,j}$'s. Hence, for $j = 1, 2$, we have $Res_{q_i}(\sigma_{i,1}) = \delta_{i,1} - c_{i,1}/a_{i,1} = \mu = \delta_{i,2} - c_{i,2}/a_{i,2} = Res_{r_i}(\sigma_{i,2})$ for $i < [\frac{g}{2}] + 1$, so

$$h_i = \frac{Res_{q_i}(\sigma_{i,1})}{Res_{r_i}(\sigma_{i,2})} = 1, \quad \text{for } i < \left[\frac{g}{2}\right] + 1,$$

$$h_i = \frac{Res_{q_i}(\sigma_{i,1})}{Res_{r_i}(\sigma_{i,2})} = \frac{\frac{c_{i,1}}{a_{i,1}}}{\frac{c_{i,2}}{a_{i,2}}} = -1, \quad \text{for } i = \left[\frac{g}{2}\right] + 1, \dots, g-1,$$

$$h_g = \frac{Res_{q_g}(\sigma_{g-1,1})}{Res_{r_g}(\sigma_{g-1,2})} = \frac{\frac{c_{g-1,1}}{a_{g-1,1}}}{\frac{c_{g-1,2}}{a_{g-1,2}}} = -1,$$

$$h_{g+1} = \frac{Res_{q_{g+1}}(\sigma_{1,1})}{Res_{r_{g+1}}(\sigma_{1,2})} = \frac{Res_0\left(\frac{(\delta_{1,1} - uc_{1,1})}{(1-\delta_{1,1}u)}\left(-\frac{1}{u}\right)du\right)}{Res_0\left(\frac{(\delta_{1,2} - uc_{1,2})}{(1-\delta_{1,2}u)}\left(-\frac{1}{u}\right)du\right)} = \frac{\delta_{1,1}}{\delta_{1,2}} = 1. \quad \square$$

4. Quadrics

In this section we explicitly describe the ideal $I_2(C) := I_2(\omega_C \otimes A)$ of the quadrics containing the Prym-canonical binary curve C embedded in $\mathbb{P}^{g-2}$ by $\omega_C \otimes A$ as in the previous section for a general choice of the $a_{i,j}$'s. Similarly as in Proposition 7 of [4], the ideal $I_2(C)$ is described as the space of solutions of the linear system given in Proposition 4.3 which has maximal rank $2g-2$, so the curve is quadratically normal.

Observe that, since the curves C_1 and C_2 pass through the coordinate points, the equation of a quadric $Q \subset \mathbb{P}^{g-2}$ containing C_k has the form

$$\sum_{1 \le i < j \le g-1} s_{ij} x_i x_j = 0. \tag{16}$$

In the next lemma we give a set of generators of $I_2(C_k)$ of the above form.

Lemma 4.1. *Set*

$$Q_{n,k} := \sum_{1 \le i < j \le g-1} \tilde{q}_{g-1-n,k;i,j} \cdot s_{ij}, \quad n = 0, \dots, g-1, \ k = 1, 2 \tag{17}$$

with

$$\begin{aligned} \tilde{q}_{0,k;i,j} &:= q_{0,k;i,j} \delta_{i,k} \delta_{j,k} \\ \tilde{q}_{1,k;i,j} &:= q_{1,k;i,j} \delta_{i,k} \delta_{j,k} - q_{0,k;i,j} (\delta_{i,k} c_{j,k} + c_{i,k} \delta_{j,k}) \\ \tilde{q}_{r,k;i,j} &:= q_{r,k;i,j} \delta_{i,k} \delta_{j,k} - q_{r-1,k;i,j} (\delta_{i,k} c_{j,k} + c_{i,k} \delta_{j,k}) + q_{r-2,k;i,j} c_{i,k} c_{j,k}, \ (r \ge 2), \end{aligned} \tag{18}$$

where

$$q_{h,k;i,j} := \sum_{m=0}^{h} a_{i,k}^m a_{j,k}^{h-m} \tag{19}$$

and $\delta_{i,k}$, $c_{i,k}$, $a_{i,k}$ are as in (12).

Then the quadrics in $I_2(C_k)$ ($k = 1, 2$) are the solutions of the linear system:

$$Q_{n,k}(s_{ij}) = 0, \quad n = 0, \dots, g-1. \tag{20}$$

Proof. The quadrics of the form (16) containing C_k are the quadrics which satisfy the equations:

$$P_k(t) = \sum_{1 \le i < j \le g-1} M_k(t, 1) \frac{(\delta_{i,k} t - c_{i,k})(\delta_{j,k} t - c_{j,k})}{(t - a_{i,k})(t - a_{j,k})} s_{ij} = \sum_{n=0}^{g-1} P_{n,k}(s_{ij}) t^n \equiv 0, \tag{21}$$

$k = 1, 2$, where the coefficients $P_{n,k}(s_{ij})$ of the polynomial $P_k(t)$ are linear in the s_{ij}'s. We will show that the linear system $P_{n,k}(s_{ij}) = 0$ is equivalent to the system (20).

By expanding the product $M_k(t, 1)$ one sees that the coefficients $p_{h,k;i,j}$ of s_{ij} in $P_{g-1-h,k}$ are

$$p_{0,k;i,j} = \delta_{i,k} \delta_{j,k}, \qquad p_{1,k;i,j} = -\sum_{i_1 \ne i,j} a_{i_1,k} \delta_{i,k} \delta_{j,k} - (\delta_{i,k} c_{j,k} + c_{i,k} \delta_{j,k}) \tag{22}$$

$$\begin{aligned} p_{h,k;i,j} &= (-1)^h \left(\sum_{\substack{1 \le i_1 < i_2 < \cdots < i_h \le g-1 \\ \text{all} \neq i,j}} a_{i_1,k} \cdots a_{i_h,k} \right) \delta_{i,k}\delta_{j,k} \\ &\quad + (-1)^h \left(\sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{h-1} \le g-1 \\ \text{all} \neq i,j}} a_{i_1,k} \cdots a_{i_{h-1},k} \right) (\delta_{i,k} c_{j,k} + c_{i,k}\delta_{j,k}) \\ &\quad + (-1)^h \left(\sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{h-2} \le g-1 \\ \text{all} \neq i,j}} a_{i_1,k} \cdots a_{i_{h-2},k} \right) c_{i,k} c_{j,k}, \end{aligned} \tag{23}$$

for $2 \le h \le g-1$.

Set

$$\gamma_{0,k} = 1, \qquad \gamma_{h,k} = (-1)^h \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_h \le g-1 \\ \text{all} \neq i,j}} a_{i_1,k} \cdots a_{i_h,k}. \tag{24}$$

Then we have (cf. [4] (12))

$$(-1)^h \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_h \le g-1 \\ \text{all} \neq i,j}} a_{i_1,k} \cdots a_{i_h,k} = \sum_{l=0}^{h} \gamma_{l,k} q_{h-l,k;i,j} \tag{25}$$

for $h = 0, \ldots, g-1$.

So, by (25), formula (23) becomes

$$\begin{aligned} p_{h,k;i,j} &= \sum_{l=0}^{h-2} \gamma_{l,k} (q_{h-l,k;i,j} \delta_{i,k}\delta_{j,k} - q_{h-1-l,k;i,j} (\delta_{i,k} c_{j,k} + c_{i,k}\delta_{j,k}) \\ &\quad + q_{h-2-l,k;i,j} c_{i,k} c_{j,k}) + \gamma_{h-1,k} (q_{1,k;i,j} \delta_{i,k}\delta_{j,k} - q_{0,k;i,j} (\delta_{i,k} c_{j,k} + c_{i,k}\delta_{j,k})) \\ &\quad + \gamma_{h,k} \delta_{i,k}\delta_{j,k} = \sum_{l=0}^{h} \gamma_{l,k} \tilde{q}_{h-l,k;i,j}. \end{aligned} \tag{26}$$

So, we have

$$P_{n,k} = \sum_{m=0}^{g-1-n} \gamma_{m,k} Q_{n+m,k} \tag{27}$$

and one immediately checks that the linear systems $P_{n,k}(s_{ij}) = 0$ and (20) are equivalent. □

Proposition 4.2. *Let $g \ge 6$. For a general choice of $a_{i,k}$, $k = 1, 2$, $i = 1, \ldots, g-1$ and with conditions (15) on $c_{i,k}$, $\delta_{i,k}$, the linear system (20) has maximal rank g.*

Proof. Consider the matrix

$$M(a_{1,k}, \ldots, a_{g-1,k}) := (\tilde{q}_{h,k;i,j})_{0 \le h \le g-1, 1 \le i < j \le g-1}$$

of size $g \times \frac{(g-1)(g-2)}{2}$. We will show that the minor B_g determined by the columns with indexes $(i, j) = (1, 2), \ldots, (1, g-1), (2, [\frac{g}{2}] + 1), (g-2, g-1)$ is non zero.

Notice that

$$\begin{aligned}
&\tilde{q}_{0,k;1,j} = \mu^2, \qquad \tilde{q}_{1,k;1,j} = \mu^2 q_{1,k;1,j},\\
&\tilde{q}_{h,k;1,j} = \mu^2 q_{h,k;1,j}, \quad j < \left[\frac{g}{2}\right]+1,\ h \geq 2\\
&\tilde{q}_{0,k;1,j} = 0, \qquad \tilde{q}_{1,k;1,j} = -\mu c_{j,k},\\
&\tilde{q}_{h,k;1,j} = -\mu c_{j,k} q_{h-1,k;i,j}, \quad j \geq \left[\frac{g}{2}\right]+1,\ h \geq 2\\
&\tilde{q}_{0,k;2,[\frac{g}{2}]+1} = \tilde{q}_{0,k;g-2,g-1} = 0, \qquad \tilde{q}_{1,k;2,[\frac{g}{2}]+1} = -\mu c_{[\frac{g}{2}]+1,k},\\
&\tilde{q}_{1,k;g-2,g-1} = 0,\\
&\tilde{q}_{h,k;2,[\frac{g}{2}]+1} = -\mu c_{[\frac{g}{2}]+1,k} q_{h-1,k;2,[\frac{g}{2}]+1}, \quad h \geq 2\\
&\tilde{q}_{h,k;g-2,g-1} = q_{h-2,k;g-2,g-1} c_{g-2,k} c_{g-1,k}, \quad h \geq 2.
\end{aligned} \tag{28}$$

So, dividing the first $[\frac{g}{2}]$ columns by μ^2, the column indexed by $(2, [\frac{g}{2}]+1)$ by $-\mu c_{[\frac{g}{2}]+1,k}$,the last column by $c_{g-2,k}c_{g-1,k}$ and all the other columns indexed by $(1, j)$ with $[\frac{g}{2}]+1 \leq j \leq g-1$ by $-\mu c_{j,k}$, we see that B_g is a non zero multiple of the determinant d of the following matrix:

$$\begin{pmatrix}
1 & .. & 1 & 0 & .. & 0 & 0 & 0\\
q_{1,k;1,2} & .. & q_{1,k;1,[\frac{g}{2}]} & 1 & .. & 1 & 1 & 0\\
q_{2,k;1,2} & .. & q_{2,k;1,[\frac{g}{2}]} & q_{1,k;1,[\frac{g}{2}]+1} & .. & q_{1,k;1,g-1} & q_{1,k;2,[\frac{g}{2}]+1} & 1\\
q_{3,k;1,2} & .. & q_{3,k;1,[\frac{g}{2}]} & q_{2,k;1,[\frac{g}{2}]+1} & .. & q_{2,k;1,g-1} & q_{2,k;2,[\frac{g}{2}]+1} & q_{1,k;g-2,g-1}\\
. & .. & . & . & .. & . & . & .\\
. & .. & . & . & .. & . & . & .\\
. & .. & . & . & .. & . & . & .\\
q_{g-1,k;1,2} & .. & q_{g-1,k;1,[\frac{g}{2}]} & q_{g-2,k;1,[\frac{g}{2}]+1} & .. & q_{g-2,k;1,g-1} & q_{g-2,k;2,[\frac{g}{2}]+1} & q_{g-3,k;g-2,g-1}
\end{pmatrix}. \tag{29}$$

One can inductively compute the determinant d up to sign,

$$d = V(a_{3,k}, \ldots, a_{g-1,k}) \cdot \prod_{\substack{r=1\\ r\neq 2,[\frac{g}{2}]+1}}^{g-1} (a_{r,k} - a_{2,k}) \cdot \prod_{s=3}^{[\frac{g}{2}]} a_{s,k} \cdot \prod_{j=1}^{g-3} a_{j,k}, \tag{30}$$

where $V(a_{3,k}, \ldots, a_{g-1,k})$ is the Vandermonde determinant in the variables $a_{3,k}, \ldots, a_{g-1,k}$. To do this one can perform column and row operations.[2]

So we reduce to a $(g-3)\times(g-3)$ matrix whose columns except the last one are the columns of the Vandermonde matrix in the variables $a_{3,k}, \ldots, a_{[\frac{g}{2}],k}, a_{[\frac{g}{2}]+2,k}, \ldots, a_{g-1,k}$. Hence repeating recursively standard row and columns operations,[3] we obtain formula (30). □

[2] Substitute column $(2, [\frac{g}{2}])$ with $(2, [\frac{g}{2}]) - (1, [\frac{g}{2}])$ then divide it by $a_{2,k} - a_{1,k}$; substitute any row by itself minus $a_{1,k}$ times the preceding row; substitute each column from $(1, 3)$ to $(1, [\frac{g}{2}])$ by itself minus the first column, eliminate the first row and column and divide the column $(1, i)$ $(j = 3, \ldots, [\frac{g}{2}])$ by $a_{i,k} - a_{2,k}$; substitute each row by itself minus $a_{[\frac{g}{2}]+1,k}$ times the preceding row, eliminate the first row and the $(1, [\frac{g}{2}]+1)$-column and divide the column $(1, j)$ with $j = [\frac{g}{2}]+2, \ldots, g-1$ by $a_{j,k} - a_{[\frac{g}{2}]+1,k}$; substitute any row by itself minus $a_{2,k}$ times the preceding row, eliminate the first row and the $(2, [\frac{g}{2}]+1)$-column and divide the column $(1, j)$ with $j = 3, \ldots, [\frac{g}{2}]$ by$a_{j,k} - a_{[\frac{g}{2}]+1,k}$.

[3] Substitute each column except the first one and the last one by itself minus the first column, substitute the last one by itself minus the first column multiplied by the first coefficient of the last column, eliminate the first row and column and divide all the columns except the last one by $a_{j,k} - a_{3,k}$ and repeat.

In the following proposition we give an explicit description of the ideal $I_2(C)$ of quadrics containing $C = C_1 \cup C_2$ and we prove that C is quadratically normal.

Proposition 4.3. *Let $g \geq 6$. For a general choice of $a_{i,k}$, $k = 1, 2$, $i = 1, \dots, g-1$ and with conditions (15) on $c_{i,k}$, $\delta_{i,k}$, the linear system*

$$Q_{0,1}(s_{ij}) = \cdots = Q_{g-1,1}(s_{ij}) = Q_{1,2}(s_{ij}) = \cdots = Q_{g-2,2}(s_{ij}) = 0, \tag{31}$$

has maximal rank $2g-2$.

Proof. Since we want to prove the statement for generic $a_{i,j}$, it suffices to show it for the following choice of $a_{i,j}$, $j = 1, 2$, $i = 1, \dots, g-1$:

$$\begin{aligned} &a_{i,1} := i \cdot a, \quad i = 1, \dots, g-1; \\ &a_{1,2} := 1, \qquad a_{r,2} := r+1, \quad r = 1, \dots, g-1, \end{aligned} \tag{32}$$

where $a \neq 1$ is a non zero constant. Consider the matrix $Z := Z(a_{i,j})$ of size $(2g-2) \times \binom{g-1}{2}$ obtained by concatenating vertically $M(a_{1,1}, \dots, a_{g-1,1}) = (\tilde{q}_{h,1;i,j})_{0 \leq h \leq g-1, 1 \leq i < j \leq g-1}$, $N(a_{1,2}, \dots, a_{g-1,2}) = (\tilde{q}_{h,2;i,j})_{1 \leq h \leq g-2, 1 \leq i < j \leq g-1}$. Let us set $k := [\frac{g}{2}]$ and consider the submatrix Z_1 of Z formed by the columns of Z indexed by $(1,2), \dots, (1, g-1), (2, k+1), (g-2, g-1), (2,3), \dots, (2,k), (2,k+2), \dots, (2, g-1), (k, k+1), (k, g-1)$. We will prove that Z_1 has maximal rank $2g-2$. Note that the submatrix given by the first g rows and columns is the matrix (29) of Proposition 4.2 which is proved to be non singular. So doing operations on the columns we can assume that Z_1 is a matrix whose submatrix given by the last $g-2$ columns ad the first g rows is zero. Hence we just need to prove that the submatrix A given by the last $g-2$ rows and columns has maximal rank. If we denote by v_i the column indexed by $(1,i)$, $i = 1, \dots, g-1$, by w_i the column indexed by $(2,i)$, $i = 1, \dots, g-1$, by w the column indexed by $(k, k+1)$, by ζ the column indexed by $(k, g-1)$, the operations that we do on the columns of Z_1 are the following:

- for $i = 3, \dots, k$, substitute the column w_i with the vector $w_i + \frac{1-i}{i-2} v_i + \frac{1}{i-2} v_1$.
- for $i = k+2, \dots, g-1$, substitute the column w_i with the vector $w_i + \frac{k \cdot c_{i,1}}{c_{k+1,1}(i-2)} v_{k+1} - \frac{c_{i,1}(k-1)}{c_{k+1,1}(i-2)} w_{k+1} - \frac{i-1}{i-2} v_i$.
- substitute the column w with the vector $w + \frac{(k-1) \cdot c_{k+1,1}}{ka} v_1 - \frac{c_{k+1,1}(k-1)}{ka} v_k - (2-k) v_{k+1} - \frac{2(1-k)^2}{k} w_{k+1}$.
- substitute the column ζ with the vector $\zeta + \frac{(k-1) \cdot c_{g-1,1}}{ka(g-k-1)} v_1 - \frac{c_{g-1,1}(k-1)}{ka(g-k-1)} v_k + \frac{2(k-1)c_{g-1,1}}{(g-k-1)c_{k+1,1}} v_{k+1} - \frac{2(1-k)^2 c_{g-1,1}}{(g-k-1)kc_{k+1,1}} w_{k+1} - \frac{g-2}{g-k-1} v_{g-1}$.

To prove that the matrix A is of maximal rank $g-2$, we argue as follows. First of all one can easily check (with the same procedure as in Proposition 4.2) that the submatrix C of Z_1 formed by the columns indexed by $(1,2), (1,4), \dots, (1, g-1), (2, k+1)$ and by the last $g-2$ rows has rank $g-2$. Denote by $(\lambda_1, \dots, \lambda_{g-2})$ the coordinates of the vector given by the column of Z_1 indexed by $(1,3)$ and the last $g-2$ rows, with respect to the basis of $\mathbb{C}^{g-2}$ given by the columns of C. Then the coordinates of the columns of A with respect to the basis given by the columns of C are given by the following matrix which we will show to have maximal rank:

$$\begin{pmatrix}
\lambda_1-1 & -\frac{1}{2} & -\frac{1}{3} & \dots & -\frac{1}{k-2} & 0 & \dots & 0 & \alpha_1 & \beta_1 \\
\lambda_2 & \frac{1}{2} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\lambda_3 & 0 & \frac{1}{3} & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\
\cdot & \dots & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \dots & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & \cdot & \cdot & \dots & \cdot & \cdot & 0 & 0 & 0 \\
\lambda_{k-2} & 0 & 0 & \dots & \frac{1}{k-2} & 0 & \dots & 0 & -\alpha_1 & -\beta_1 \\
\lambda_{k-1} & 0 & 0 & \dots & 0 & \mu_{1,k+2} & \dots & \mu_{1,g-1} & \alpha_3 & \beta_3 \\
\lambda_k & 0 & 0 & \dots & 0 & \mu_{2,k+2} & \dots & 0 & 0 & 0 \\
 & \dots & \cdot & \cdot & \dots & 0 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \dots & \cdot & \cdot & \dots & \cdot & \cdot & 0 & \cdot & 0 \\
\lambda_{g-3} & 0 & 0 & 0 & \dots & 0 & 0 & \mu_{2,g-1} & 0 & \beta_4 \\
\lambda_{g-2} & 0 & 0 & 0 & \dots & \mu_{3,k+2} & \cdot\cdot & \mu_{3,g-1} & \alpha_4 & \beta_5
\end{pmatrix},$$

where, for $j = k+2, \dots, g-1$,

$$\mu_{1,j} = \frac{1}{j-2}\left(\frac{kc_{j,1}}{c_{k+1,1}} - \frac{(k+1)c_{j,2}}{c_{k+1,2}}\right), \qquad \mu_{2,j} = \frac{1}{j-2},$$

$$\mu_{3,j} = \left(\frac{k-1}{j-2}\right)\left(-\frac{c_{j,1}}{c_{k+1,1}} + \frac{c_{j,2}}{c_{k+1,2}}\right), \qquad \alpha_3 = \frac{k-2}{2},$$

$$\alpha_4 = \frac{(k-1)(4-k^2)}{2k(k+1)}, \qquad \beta_3 = \frac{1}{g-k-1}\left(\frac{-3kc_{g-1,2}}{2c_{k+1,2}} + \frac{2(k-1)c_{g-1,1}}{c_{k+1,1}}\right),$$

$$\beta_4 = \frac{1}{g-k-1}, \qquad \beta_5 = \frac{1}{g-k-1}\left(\frac{3k(k-1)c_{g-1,2}}{2(k+1)c_{k+1,2}} - \frac{2(k-1)^2 c_{g-1,1}}{kc_{k+1,1}}\right).$$

Subtracting from each of the last two columns a suitable multiple of the $(k-2)$'s column, we can assume that $\alpha_1 = \beta_1 = 0$, hence the submatrix formed by the last $g-k$ columns and the first $k-2$ rows is zero. The determinant of the submatrix given by the first $(k-2)$ rows and columns is $\frac{1}{(k-2)!}(\sum_{i=1}^{k-2}\lambda_i - 1)$, and the determinant of the submatrix given by the last $g-k$ rows and columns is a non zero multiple of

$$det\begin{pmatrix} \mu_{1,g-1} & \alpha_3 & \beta_3 \\ \mu_{2,g-1} & 0 & \beta_4 \\ \mu_{3,g-1} & \alpha_4 & \beta_5 \end{pmatrix} = \frac{-(k-1)(k-2)^2(g-k-2)}{2k(k+1)^2(k+2)(g-3)(g-k-1)} \neq 0.$$

So it remains to show that $\sum_{i=1}^{k-2}\lambda_i \neq 1$. To do this, it suffices to show that the matrix obtained by adding the row $(1, \dots, 1)$ to the submatrix of Z_1 formed by the columns indexed by $(1,2), (1,3), \dots, (1, g-1), (2, k+1)$ and the last $g-2$ rows has rank $g-1$. This can be easily seen with a procedure similar to the one used in Proposition 4.2. □

5. Surjectivity

In this section we will prove by induction on the genus the surjectivity of μ_A for a general Prym-canonical binary curve (C, A) of genus ≥ 20.

5.1. The second Gaussian map

Let us first of all analyze in detail the map μ_A of (10) when $C = C_1 \cup C_2$ is a Prym-canonical binary curve embedded in $\mathbb{P}^{g-2}$ by $\omega_C \otimes A$, where $A \in Pic^0(C)$ is non trivial of order 2.

Since $\omega_{C|C_i} = K_{C_i}(D_i)$ where D_i is the divisor of nodes in C_i, we have

$$H^0(S^2(\Omega^1_C) \otimes \omega_C^{\otimes 2}) \cong T \oplus \left(\bigoplus_{i=1,2} H^0(C_i, K_{C_i}^{\otimes 4}(2D_i)) \right),$$

where T is the torsion of $S^2(\Omega^1_C)$, which is supported at the nodes (see Lemma 2 of [4]). In fact we have an exact sequence

$$0 \to T \to S^2(\Omega^1_C) \to \mathcal{F}_C \to 0,$$

where $\mathcal{F}_C$ is a non-locally free, rank 1, torsion free sheaf on C.

To prove the surjectivity of μ_A we will show the surjectivity of the components of μ_A on both non torsion and torsion parts of $H^0(S^2(\Omega^1_C) \otimes \omega_C^{\otimes 2})$.

Consider first the non torsion component $\nu = \nu_1 \oplus \nu_2$, where

$$\nu_k : I_2(C) \to H^0(C_k, K_{C_k}^{\otimes 4}(2D_k)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g-6)), \quad k = 1, 2.$$

Recall that the curves C_l, $l = 1, 2$ are the images of the maps ϕ_l defined in (12), $\phi_l : \mathbb{P}^1 \to \mathbb{P}^{g-2}$, $l = 1, 2$

$$\phi_l(t, u) := (f_{1,l}(t), \ldots, f_{g-1,l}(t)), \qquad f_{i,l}(t) = M_l(t) \frac{(\delta_{i,l} t - c_{i,l})}{(t - a_{i,l})}.$$

Assume $Q \in I_2(C)$ is of the form (16) where s_{ij} are solutions of (20). Then using the local expression given in (11), we have

$$\nu_k(Q) = \sum_{1 \le i < j \le g-1} M_k^2(t) \left(\frac{\delta_{i,k} t - c_{i,k}}{t - a_{i,k}} \right)' \left(\frac{\delta_{j,k} t - c_{j,k}}{t - a_{j,k}} \right)' s_{ij} (dt)^4, \quad k = 1, 2. \tag{33}$$

As an element of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g-6))$, $\nu_k(Q)$ can be identified with the polynomial of degree $2g-6$ in t,

$$R_k(t) = \sum_{1 \le i < j \le g-1} M_k^2(t) \frac{(c_{i,k} - \delta_{i,k} a_{i,k})}{(t - a_{i,k})^2} \frac{(c_{j,k} - \delta_{j,k} a_{j,k})}{(t - a_{j,k})^2} s_{ij}. \tag{34}$$

To study the torsion component, we consider as in [4] the restriction τ of μ_A to $ker(\nu)$, which lands in the torsion part T of $H^0(S^2(\Omega^1_C) \otimes \omega_C^{\otimes 2})$. Then using Lemma 2 of [4] one sees that the composition of τ with the projection on the torsion part T_{P_h} at the nodes $P_1, \ldots, P_{g-1}, P_g$ is as follows: a quadric $Q \in ker(\nu)$ as in (16) is mapped to

$$dxdy \sum_{i \ne j} s_{ij} f'_{i,1}(a_{h,1}) f'_{j,2}(a_{h,2}) + 2x dxdy \sum_{i \ne j} s_{ij} f''_{i,1}(a_{h,1}) f'_{j,2}(a_{h,2}) \\ + 2y dxdy \sum_{i \ne j} s_{ij} f'_{i,1}(a_{h,1}) f''_{j,2}(a_{h,2}), \tag{35}$$

where $h = 1, \ldots, g$, $s_{ij} = s_{ji}$ and x, y are local coordinates around P_h such that C_1 is given locally by $x = 0$ and C_2 by $y = 0$ and since P_g is the image of $[0, 1]$, we set $a_{g,1} = a_{g,2} = 0$.

The description of the torsion at the point P_{g+1} is similar:

$$\begin{aligned} &dxdy\sum_{i\neq j} s_{ij}(\delta_{i,1}a_{i,1}-c_{i,1})(\delta_{j,2}a_{j,2}-c_{j,2}) \\ &\quad + 2xdxdy\sum_{i\neq j} s_{ij}a_{i,1}(\delta_{i,1}a_{i,1}-c_{i,1})(\delta_{j,2}a_{j,2}-c_{j,2}) \\ &\quad + 2ydxdy\sum_{i\neq j} s_{ij}a_{j,2}(\delta_{i,1}a_{i,1}-c_{i,1})(\delta_{j,2}a_{j,2}-c_{j,2}), \end{aligned} \tag{36}$$

where $s_{ij}=s_{ji}$ and x, y are local coordinates around P_{g+1} such that C_1 is given locally by $x=0$ and C_2 by $y=0$.

5.2. *Proof of surjectivity*

Let $C\subset\mathbb{P}^{g-2}$ be a Prym-canonical binary curve embedded by $\omega_C\otimes A$, with $A^{\otimes 2}\cong\mathcal{O}_C$, as in (15) and set $k:=[\frac{g}{2}]$. Denote by $\tilde{C}$ the partial normalization of C at the node P, where $P=P_k$ if $g=2k$, $P=P_{k+1}$ if $g=2k+1$ and by p_1, p_2 the preimages of P in $\tilde{C}$. Observe that for a general choice of the $a_{i,j}$'s, the projection π from P sends the curve C to the Prym-canonical model of $\tilde{C}$ in $\mathbb{P}^{g-3}$ given by the line bundle $K_{\tilde{C}}\otimes A'$ where A' corresponds to the point $(h'_1,\dots,h'_{g-1},1)\in\mathbb{C}^{*g}/\mathbb{C}^*$, with $h'_i=1$ for $i\le[\frac{g-1}{2}]$, $h'_i=-1$ for $i=[\frac{g-1}{2}]+1,\dots,g-1$, as in Section 2. In fact if $g=2k$, we have $k-1=[\frac{g-1}{2}]=:k'$ and $(\tilde{C},A')$ is as in (12), (15) with $a'_{i,j}=a_{i,j}$ for $i\le k'$, $j=1,2$, $a'_{i,j}=a_{i+1,j}$ for $i\ge k'+1$, $j=1,2$. If $g=2k+1$ we have $[\frac{g-1}{2}]=k$, so $(\tilde{C},A')$ is parametrized by $a'_{i,j}=a_{i,j}$ for $i\le k$, $j=1,2$, $a'_{i,j}=a_{i+1,j}$ for $i\ge k+1$, $j=1,2$.

Consider the following commutative diagrams with horizontal exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \oplus_{i=1,2}H^0(C_i,K_{C_i}^{\otimes 4}(2\tilde{D}_i)) & \longrightarrow & \oplus_{i=1,2}H^0(C_i,K_{C_i}^{\otimes 4}(2D_i)) & \longrightarrow & \oplus_{i=1,2}\mathcal{O}_{2p_i} \\ & & \uparrow{\scriptstyle\tilde{\nu}} & & \uparrow{\scriptstyle\nu} & \nearrow{\scriptstyle\chi} & \\ 0 & \longrightarrow & I_2(\tilde{C}) & \longrightarrow & I_2(C) & & \end{array} \tag{37}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{T} & \longrightarrow & T & \longrightarrow & T_P \\ & & \uparrow{\scriptstyle\tilde{\tau}} & & \uparrow{\scriptstyle\tau} & \nearrow{\scriptstyle\tau_P} & \\ 0 & \longrightarrow & \ker(\tilde{\nu}) & \longrightarrow & \ker(\nu) & & \end{array} \tag{38}$$

where D_i is the divisor of nodes of C on C_i and $\tilde{D}_i=D_i+p_i$ and ν, τ and $\tilde{\nu}$, $\tilde{\tau}$ are the maps defined in the previous section for C and $\tilde{C}$. Hence, if $\tilde{\nu}$ and χ ($\tilde{\tau}$ and τ_P, resp.) are surjective, then ν (τ, resp.) is also surjective.

Theorem 5.1. *If* $(C=C_1\cup C_2,\omega_C\otimes A)$ *is a Prym-canonical general binary curve of genus* $g\ge 20$*, then* μ_A *is surjective for* C.

Proof. The case $g=20$ is done by a direct computation with Maple (see Appendix A.1). We then proceed by induction on g: the commutativity of the diagrams (37), (38) shows that it is enough to prove the surjectivity of χ and τ_P, where $P=P_k$ for $g=2k$ and $P=P_{k+1}$ for $g=2k+1$, as above. Recall that the map ν is $\nu_1\oplus\nu_2$ where ν_1 and ν_2 are defined in (33), so we can write $\chi = \chi_1 \oplus \chi_2$, where χ_l is the composition of ν_l with the restriction to $\mathcal{O}_{2p_l}$, $l = 1, 2$. We want to compute $\chi(Q)$, where $Q \in I_2(C)$. Notice that if $Q \in I_2(C)$ is of the form (16), with the s_{ij}'s satisfying (31), then $\chi_l(Q)$ is the pair $(R_l(a_{r,l}), R_l'(a_{r,l}))$ (where $r = k$ for g even and $r = k + 1$ for g odd) corresponding to the evaluation of the polynomial $R_l(t)$ ($l = 1, 2$) of (34) and of its derivative at P. Recall that $R_l(t)$ is linear in the s_{ij}'s and denote by $R^l_{i,j}(t)$ the coefficient of s_{ij} in $R_l(t)$.

To prove the surjectivity of χ we have to show that the matrix Y of size $(2g + 2) \times \binom{g-1}{2}$ obtained by concatenating vertically the matrix Z in the proof of Proposition 4.3, and the matrix of size $4 \times \binom{g-1}{2}$ whose rows are the evaluations in P of $R^1_{i,j}$, $(R^1_{i,j})'$, $R^2_{i,j}$, $(R^2_{i,j})'$ is of maximal rank.

By formula (34) we have:

$$R^l_{i,j}(t) = (c_{i,l} - a_{i,l}\delta_{i,l})(c_{j,l} - a_{j,l}\delta_{j,l}) \cdot \prod_{r \neq i,j} (t - a_{r,l})^2. \tag{39}$$

Therefore, if $i, j \neq n$, $R^l_{i,j}(a_{n,l}) = 0$ and $(R^l_{i,j})'(a_{n,l}) = 0$. So it remains to compute $R^l_{i,k}(a_{k,l})$, $R^l_{k,j}(a_{k,l})$ and $(R^l_{i,k})'(a_{k,l})$, $(R^l_{k,j})'(a_{k,l})$, for $g = 2k$, and $R^l_{i,k+1}(a_{k+1,l})$, $R^l_{k+1,j}(a_{k+1,l})$ and $(R^l_{i,k+1})'(a_{k+1,l})$, $(R^l_{k+1,j})'(a_{k+1,l})$, for $g = 2k+1$. If $g = 2k$ and we denote by $D_{k,l} := \prod_{r \neq k}(a_{k,l} - a_{r,l})^2$, we have

$$R^l_{i,k}(a_{k,l}) = \frac{D_{k,l} \cdot a_{i,l}a_{k,l}}{(a_{k,l} - a_{i,l})^2}, \qquad R^l_{k,j}(a_{k,l}) = -\frac{D_{k,l} \cdot c_{j,l}a_{k,l}}{(a_{k,l} - a_{j,l})^2}$$

$$(R^l_{i,k})'(a_{k,l}) = \frac{2D_{k,l} \cdot a_{i,l}a_{k,l}}{(a_{k,l} - a_{i,l})^2} \cdot \sum_{r \neq i,k} \frac{1}{(a_{k,l} - a_{r,l})},$$

$$(R^l_{k,j})'(a_{k,l}) = -\frac{2D_{k,l} \cdot c_{j,l}a_{k,l}}{(a_{k,l} - a_{j,l})^2} \cdot \sum_{r \neq j,k} \frac{1}{(a_{k,l} - a_{r,l})}.$$

If $g = 2k + 1$, and we denote by $D_{k+1,l} := \prod_{r \neq k+1}(a_{k+1,l} - a_{r,l})^2$ we have

$$R^l_{i,k+1}(a_{k+1,l}) = -\frac{D_{k+1,l} \cdot a_{i,l}c_{k+1,l}}{(a_{k+1,l} - a_{i,l})^2}, \qquad R^l_{k+1,j}(a_{k+1,l}) = \frac{D_{k+1,l} \cdot c_{j,l}c_{k+1,l}}{(a_{k+1,l} - a_{j,l})^2}$$

$$(R^l_{i,k+1})'(a_{k+1,l}) = -\frac{2D_{k+1,l} \cdot a_{i,l}c_{k+1,l}}{(a_{k+1,l} - a_{i,l})^2} \cdot \sum_{r \neq i,k+1} \frac{1}{(a_{k+1,l} - a_{r,l})}$$

$$(R^l_{k+1,j})'(a_{k+1,l}) = \frac{2D_{k+1,l} \cdot c_{j,l}c_{k+1,l}}{(a_{k+1,l} - a_{j,l})^2} \cdot \sum_{r \neq j,k+1} \frac{1}{(a_{k+1,l} - a_{r,l})}.$$

To show that the matrix Y has maximal rank $2g + 2$ we will show that the minor $det N$ is non zero, where N is determined by the columns indexed by $(1, i)$, $(2, j)$, with $2 \leq i \leq g - 1$, $3 \leq j \leq g - 1$, $(k, k + 1), (k, g - 1), (g - 2, g - 1)$ and we choose the columns $(3, k)$, $(4, k)$, $(k + 1, g - 2)$, $(k + 1, g - 1)$, in the case $g = 2k$, and the columns $(3, k)$, $(4, k + 1)$, $(k + 1, g - 2)$, $(k + 1, g - 1)$, in the case $g = 2k + 1$. Notice that the square submatrix of N given by the first $2g - 2$ rows and columns is the submatrix Z_1 of Z introduced in Proposition 4.3, which is shown to be non singular for a general choice of the $a_{i,l}$'s. The columns of the submatrix G of N given by its last four columns and its first $2g - 2$ rows are clearly also columns of Z hence linearly dependent on the columns of Z_1. Therefore we perform operations on the last four columns of N to bring G to the zero matrix. So it suffices to prove that the submatrix A of order 4, given by the last 4 rows and columns is non singular for general $a_{i,l}$. To do this we choose the set of the $a_{i,l}$'s as in (32), we compute with Maple the determinant of A and we see that as a function of k it does not vanish for any integer $k \geq 10$ (see Appendix A.2). This proves that χ is surjective.

It remains to show that τ_P is surjective. Recall that $\ker(\nu)$ is defined in $I_2(C)$ by the vanishing of the polynomials $R_l(t)$, $l = 1, 2$. By the description of the torsion at the point P given in (35), we need to show the rank maximality of the matrix X of size $(2g + 5) \times \binom{g-1}{2}$ obtained by concatenating vertically the above matrix Y and the matrix of size $3 \times \binom{g-1}{2}$ whose rows are, for $g = 2k$ (hence $P = P_k$)

$$(T_1)_{ij} = f'_{i,1}(a_{k,1}) f'_{j,2}(a_{k,2}) + f'_{j,1}(a_{k,1}) f'_{i,2}(a_{k,2}),$$
$$(T_2)_{ij} = f''_{i,1}(a_{k,1}) f'_{j,2}(a_{k,2}) + f''_{j,1}(a_{k,1}) f'_{i,2}(a_{k,2}),$$
$$(T_3)_{ij} = f'_{i,1}(a_{k,1}) f''_{j,2}(a_{k,2}) + f'_{j,1}(a_{k,1}) f''_{i,2}(a_{k,2})$$

and for $g = 2k + 1$, hence $P = P_{k+1}$,

$$(T_1)_{ij} = f'_{i,1}(a_{k+1,1}) f'_{j,2}(a_{k+1,2}) + f'_{j,1}(a_{k+1,1}) f'_{i,2}(a_{k+1,2}),$$
$$(T_2)_{ij} = f''_{i,1}(a_{k+1,1}) f'_{j,2}(a_{k+1,2}) + f''_{j,1}(a_{k+1,1}) f'_{i,2}(a_{k+1,2}),$$
$$(T_3)_{ij} = f'_{i,1}(a_{k+1,1}) f''_{j,2}(a_{k+1,2}) + f'_{j,1}(a_{k+1,1}) f''_{i,2}(a_{k+1,2}).$$

We claim that the minor $det M$ of the submatrix M of X determined by the $2g + 5$ columns, indexed as the columns of N plus $(5, k + 1), (k, g - 4), (k + 1, g - 3)$ if $g = 2k$, and $(5, k + 1), (k + 1, g - 4), (k + 1, g - 3)$ if $g = 2k + 1$ is non zero. This will conclude the proof that τ_P is surjective, hence the proof of the theorem.

As above the square submatrix of M given by the first $2g - 2$ rows and columns is the submatrix Z_1 of Z introduced in Proposition 4.3, which is non singular for a general choice of the $a_{i,l}$'s. The columns of the submatrix H of M given by its last seven columns and its first $2g - 2$ rows are clearly also columns of Z hence linearly dependent on the columns of Z_1. Therefore we perform operations on the last seven columns of M to bring H to the zero matrix. So it suffices to prove that the submatrix B of order 7, given by the last 7 rows and columns is non singular for general $a_{i,l}$. To this purpose we choose the set of the $a_{i,l}$'s as in (32), we compute again with Maple the determinant of B and we see that for any integer $k \geq 10$ it does not vanish (see Appendix A.2). This proves that τ_P is surjective, hence by induction μ_A is surjective. □

6. The class

In the previous section we have proved by semicontinuity that the second Gaussian map $\mu_A : I_2(C) \to H^0(S^2(\Omega^1_C) \otimes K_C^{\otimes 2})$ has maximal rank for the general pair $[C, A]$ in $\mathcal{R}_{20}$. Notice that for $g = 20$, $dim(I_2(C)) = dim(H^0(S^2(\Omega^1_C) \otimes K_C^{\otimes 2})) = 133$. Consider the locus $\mathcal{D} = \{[C, A] \in \mathcal{R}_{20} \mid rk(\mu_A) < 133\}$. We have proved that $\mathcal{D} \neq \mathcal{R}_{20}$, hence, if it is not empty, it is an effective divisor in $\mathcal{R}_{20}$. Let $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ be the finite map which extends the forgetful map $\mathcal{R}_g \to \mathcal{M}_g$ (see [11] Section 1). The partial compactification $\tilde{\mathcal{R}}_g$ of $\mathcal{R}_g$ introduced in [11] Section 1 is the inverse image $\pi^{-1}(\tilde{\mathcal{M}}_g)$, where $\tilde{\mathcal{M}}_g := \mathcal{M}_g \cup \tilde{\Delta}_0$ and $\tilde{\Delta}_0$ is the locus of one-nodal irreducible curves. Denote by $f : \mathcal{X} \to \tilde{\mathbf{R}}_g$ the universal family and by $\mathcal{P} \in Pic(\mathcal{X})$ the corresponding Prym bundle as in [11] 1.1. Assume $g = 20$, then if $\tilde{\mathcal{D}}$ is the closure of $\mathcal{D}$ in $\tilde{\mathcal{R}}_{20}$, $\tilde{\mathcal{D}}$ is the degeneracy locus of the map

$$\tilde{\mu} : \tilde{\mathcal{I}}_2 \to f_*((\omega_f \otimes \mathcal{P})^{\otimes 2} \otimes S^2(\Omega^1_f)) \cong f_*(\omega_f^{\otimes 4} \otimes \mathcal{P}^{\otimes 2} \otimes \mathcal{I}_Z^{\otimes 2}),$$

of (9). Denote by $\mathcal{F}_i := f_*(\omega_f^{\otimes i} \otimes \mathcal{P}^{\otimes i})$. Using Grothendieck–Riemann–Roch and Proposition 1.6 of [11] one computes as in Proposition 1.7 of [11]

$$c_1(\mathcal{F}_i) = \frac{i(i-1)}{2}(12\lambda - \delta_0' - \delta_0'' - 2\delta_0^{ram}) + \lambda - \frac{i^2}{4}\delta_0^{ram},$$

where λ is the pullback of the Hodge class $\lambda \in \overline{\mathcal{M}}_g$ and δ_0', δ_0'', and δ_0^{ram} are the boundary classes defined in [11] Section 1. So we have

$$c_1(\mathcal{F}_1) = \lambda - \frac{\delta_0^{ram}}{4}, \qquad c_1(\mathcal{F}_2) = 13\lambda - \delta_0' - \delta_0'' - 3\delta_0^{ram},$$
$$c_1(S^2(\mathcal{F}_1)) = 20 \cdot c_1(\mathcal{F}_1) = 20\lambda - 5\delta_0^{ram},$$

therefore

$$c_1(\mathcal{I}_2) = c_1(S^2\mathcal{F}_1) - c_1(\mathcal{F}_2) = 7\lambda + \delta_0' + \delta_0'' - 2\delta_0^{ram}.$$

Notice that by Grothendieck–Riemann–Roch we have

$$\begin{aligned} &c_1(f_*(\omega_f^{\otimes 4} \otimes \mathcal{P}^{\otimes 2} \otimes \mathcal{I}_Z^{\otimes 2})) \\ &\quad = f_*\Big[(1 + c_1(\omega_f^{\otimes 4} \otimes \mathcal{P}^{\otimes 2}) \\ &\qquad + \frac{1}{2}c_1^2(\omega_f^{\otimes 4} \otimes \mathcal{P}^{\otimes 2}) - 2[Z]) \cdot \left(1 - \frac{c_1(\omega_f)}{2} + \frac{c_1(\omega_f)^2 + [Z]}{12}\right)\Big]_2 \\ &\quad = 73\lambda - 8(\delta_0' + \delta_0'') - 17\delta_0^{ram}, \end{aligned}$$

since $f_*(c_1(\omega_f) \cdot \mathcal{P}) = 0$, $f_*(c_1(\mathcal{P})^2) = -\delta_0^{ram}/2$, by Proposition 1.6 of [11] and by Mumford's formula, $f_*(c_1(\omega_f)^2) = 12\lambda + f_*([Z])$ and $f_*[Z] = \delta_0' + \delta_0'' + 2\delta_0^{ram}$ ([11], 1.1). So, finally we have

$$\begin{aligned} c_1(\tilde{\mathcal{D}}) &= c_1(f_*(\omega_f^{\otimes 4} \otimes \mathcal{P}^{\otimes 2} \otimes \mathcal{I}_Z^{\otimes 2})) \cdot rk(\mathcal{I}_2) - c_1(\mathcal{I}_2) \cdot rk(f_*(\omega_f^{\otimes 4} \otimes \mathcal{P}^{\otimes 2} \otimes \mathcal{I}_Z^{\otimes 2})) \\ &= 133(66\lambda - 9(\delta_0' + \delta_0'') - 15\delta_0^{ram}), \end{aligned}$$

and $c_1(\mathcal{D}) = 8778\lambda$, hence $\mathcal{D}$ is an effective divisor in $\mathcal{R}_{20}$, $\tilde{\mathcal{D}}$ is an effective divisor in $\tilde{\mathcal{R}}_{20}$ and if we denote by $\overline{\mathcal{D}}$ the closure of $\mathcal{D}$ in $\overline{\mathcal{R}}_{20}$, we have computed

$$c_1(\overline{\mathcal{D}}) = 133(66\lambda - 9(\delta_0' + \delta_0'') - 15\delta_0^{ram} - \cdots). \tag{40}$$

In fact, since the partial compactification $\tilde{\mathcal{R}}_g \subset \overline{\mathcal{R}}_g$ has the property that $\pi^{-1}(\mathcal{M}_g \cup \Delta_0) - \tilde{\mathcal{R}}_g$ has codimension ≥ 2, the expression (40) computes the coefficients of λ, δ_0', δ_0'', δ_0^{ram} in $c_1(\overline{\mathcal{D}})$.

Remark 6.1. • Using Proposition 1.9 of [11] one can find lower bounds on some of the other boundary coefficients of $\overline{\mathcal{D}}$.

• Pushing forward $\overline{\mathcal{D}}$, one gets

$$c_1(\pi_*(\overline{\mathcal{D}})) = 133(66(2^{40} - 1)\lambda - (33 \cdot 2^{38} - 9)\delta_0 - \cdots),$$

hence its slope is $\geq 8 + \frac{2}{3023656976381}$.

Appendix. Maple scripts for computations

A.1. Surjectivity for $g = 20$

We list here the Maple script we run. We will explain it afterwards: for this purpose, we added line numbers.

```
   a[1]:=[25,35,54,47,67,97,73,81,22,33,76,27,38,44,58,69,63,80,99]:
   a[2]:=[1,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20]:
   listsij:=[seq(seq(s[i,j],j=i+1..19),i=1..19)]:
   t[1]:=[0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1]:
5  A1:=mul(a[1][i],i=1..19):
   A2:=mul(a[2][i],i=1..19):
   d2:=1:
   d1:= -d2*A1/A2:
   delta:=[1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0]:
10 c[1]:=[seq(t[1][i]*a[1][i]*d1/A1,i=1..19)]:
   c[2]:=[seq(t[1][i]*a[2][i]*d2/A2,i=1..19)]:
   Z:=Matrix([seq([seq(seq((delta[i]*delta[j])*add
   (a[1][i]^m*a[1][j]^(h-m),m=0..h)-
   (delta[j]*c[1][i]+delta[i]*c[1][j])*add
   (a[1][i]^m*a[1][j]^(h-1-m),m=0..h-1)+
   (c[1][i]*c[1][j])*add(a[1][i]^m*a[1][j]^(h-2-m),m=0..h-2),
   j=i+1..19),i=1..19)],
   h=0..19), seq([seq(seq((delta[i]*delta[j])*add
   (a[2][i]^m*a[2][j]^(h-m),m=0..h)-
   (delta[j]*c[2][i]+delta[i]*c[2][j])*add
   (a[2][i]^m*a[2][j]^(h-1-m),m=0..h-1)+
   (c[2][i]*c[2][j])*add(a[2][i]^m*a[2][j]^(h-2-m),m=0..h-2),
   j=i+1..19),i=1..19)],
   h=0..19)]):
19 Zref:=Gausselim(Z,'r0') mod 131:
   r0;
                                  38
   M[1] := mul(t-a[1][i], i = 1 .. 19):
   M[2] := mul(t-a[2][i], i = 1 .. 19):
   for i from 1 to 19 do phi1[1, i]
   := diff(M[1]*(delta[i]*t-c[1][i])/(t-a[1][i]), t):
25 phi1[2, i] := diff(M[2]*(delta[i]
   *t-c[2][i])/(t-a[2][i]), t) end do:
   R[1] := add(add(s[i, j]*phi1[1, i]*phi1[1, j],
   j = i+1 .. 19), i = 1 .. 19):
   R[2] := add(add(s[i, j]*phi1[2, i]*phi1[2, j],
   j = i+1 .. 19), i = 1 .. 19):
   Eqskernu := [seq(seq(primpart(coeff(R[k], t, n)),
   n = 0 .. 34), k = 1 .. 2)]:
   K:= Gausselim(linalg[stackmatrix](Zref,
```

```
linalg[genmatrix](Eqskernu,listsij)),'r1')
mod 131):
r1;
            108
for i from 1 to 19 do phi2[1,i]
:= diff(phi1[1,i],t): phi2[2,i] := diff(phi1[2,i],t):
phi1e0[1,i]:= eval(phi1[1,i], t = 0): phi2e0[1,i]
:= eval(phi2[1,i], t = 0):
phi1e0[2,i] := eval(phi1[2,i], t = 0): phi2e0[2,i]
:= eval(phi2[2,i], t = 0):
for h to 19 do phi1e[1, i, h]:= eval(phi1[1,i], t = a[1][h]):
phi2e[1, i, h]:= eval(phi2[1,i], t = a[1][h]):
phi1e[2, i, h]:= eval(phi1[2,i], t = a[2][h]):
phi2e[2, i, h] := eval(phi2[2,i], t = a[2][h]) end do end do:
for h from 1 to 19 do
tors[h,1]:= add(add(s[i,j]*(phi1e[1,i,h]*phi1e[2, j, h]+
phi1e[1,j,h]*phi1e[2,i,h]), j = i+1 .. 19), i = 1 .. 19):
tors[h,2]:= add(add(s[i,j]*(phi2e[1,i,h]*phi1e[2, j, h]+
phi2e[1,j,h]*phi1e[2,i,h]), j = i+1 .. 19), i = 1 .. 19):
tors[h,3]:= add(add(s[i,j]*(phi1e[1,i,h]*phi2e[2, j, h]+
phi1e[1,j,h]*phi2e[2,i,h]), j = i+1 .. 19), i = 1 .. 19) end do:
tors[20,1]:= add(add(s[i j]*(phi1e0[1,i]*phi1e0[2, j]
+phi1e0[1, j]*phi1e0[2, i]),
j = i+1 .. 19), i = 1 .. 19):
tors[20,2]:= add(add(s[i,j]*(phi2e0[1,i]*phi1e0[2, j]
+phi2e0[1, j]*phi1e0[2, i]),
j = i+1 .. 19), i = 1 .. 19):
tors[20,3]:= add(add(s[i,j]*(phi1e0[1,i]*phi2e0[2, j]
+phi1e0[1, j]*phi2e0[2, i]),
j = i+1 .. 19), i = 1 .. 19):
tors[21,1]:= add(add(s[i,j]*((delta[i]*a[1][i]-c[1][i])*
(delta[j]*a[2][j]-c[2][j])+
(delta[j]*a[1][j]-c[1][j])*(delta[i]*a[2][i]-c[2][i])),
j = 1 .. 19), i = 1 .. 19):
tors[21,2]:= add(add(s[i, j]*((delta[i]*a[1][i]-c[1][i])*a[1][i]*
(delta[j]*a[2][j]-c[2][j])+
(delta[j]*a[1][j]-c[1][j])*(delta[i]*a[2][i]-c[2][i])*
a[1][j]), j = 1 .. 19), i = 1 .. 19):
tors[21,3]:= add(add(s[i, j]*((delta[i]*a[1][i]-c[1][i])*a[2][j]*
(delta[j]*a[2][j]-c[2][j])+
(delta[j]*a[1][j]-c[1][j])*(delta[i]*a[2][i]-c[2][i])*
a[2][i]), j = 1 .. 19), i = 1 .. 19):
Eqskertau:= [seq(seq(primpart(tors[h,l]), l = 1 .. 3), h = 1 .. 21)]:
Gausselim(linalg[stackmatrix](K, linalg[genmatrix]
(Eqskertau, listsij)),'r2')
mod 131):
```

```
r2;
     171
r2-r1;
                                        63
```

In lines 1–2, we define the $a_{i,j}$'s which will be used. We chose them randomly. In line 3, we collect the unknowns $\{s_{i,j}\}_{1\le i<j\le g}$ in the list listsij: there are $\binom{g-1}{2}$ of them. In line 4,5,6 we define t[1] which is the vector P_g as in Lemma 3.1, $A_i = \prod_{i=1,\dots,19} a_{r,i}$, $i = 1, 2$. In line 7,8 we define $d_2 = 1$, $d_1 = -\frac{d_2 A_1}{A_2}$, as in (15). In line 9 we define the vector δ whose components are $\delta_{i,1} = \delta_{i,2}$, as in (15). In lines 10,11 we collect the $c_{i,1}$, $c_{i,2} i = 1, \dots, 19$ as in (15), and we call them c[1][i], c[2][i]. These data give the curve C and the line bundle A as in Lemma 3.1. In lines 13–19, we define the matrix Z associated to the linear system (31), whose solutions give us the quadrics in $I_2(C)$, cf. Proposition 4.3. In line 20, Maple computes the rank r0 of Z via Gaussian elimination, by calculating modulo 131 to speed up computations. The resulting matrix is called Zref. As expected by Proposition 4.3, Maple finds r0 $= 38 = 2g - 2$ and it prints it in line 21. In lines 22, 23 we define M[j] as $\prod_{i=1,\dots,19}(t - a_{i,j})$, $j = 1, 2$. In lines 24, 25 we define phi1[j, i] as the i-th component of $\frac{d}{dt}(\phi_j(t, 1))$, $j = 1, 2$, where ϕ_j is defined in (12). In lines 26, 27 we define R[k] as the polynomial $R_k(t)$ of (34), $k = 1, 2$. In line 28, we collect in EqsKerNu the list of equations which determine $\ker(\nu)$, cf. the definition of ν in (33). In lines 29–31, Maple computes the rank r1 of the linear system EqsKerNu $\cap \ker($Zref$)$, again via Gaussian elimination modulo 131, and the resulting matrix is called K. Maple finds that r1 $= 108$ and it prints it in line 31. Since $rank(\nu) =$ r1 $-$ r0 $= 70 = 2h^0(\mathcal{O}_{\mathbb{P}^1}(2g - 6))$, for $g = 20$, we have shown that ν has maximal rank. In line 32, we define the second derivative phi2 of the $\phi_j(t, 1)$'s of (12). In lines 33–34 we define the evaluations phi1e0, phi2e0, of the first and the second derivatives of the $\phi_j(t, 1)$'s at $t = 0$, i.e. at the point P_{20} and in lines 35–38 we define their evaluations at the points P_i, $i = 1, \dots, 19$. Using them, in lines 39–51 we compute the torsion at P_i, $i = 1, \dots, 20$, cf. (35), and, in lines 52–62, the torsion at the point P_{21}, cf. (36). In lines 63 we collect in EqsKerTau the equations which determine $\ker(\tau)$ and Maple computes the rank r2 of EqsKerTau $\cap \ker($K$)$, via Gaussian elimination modulo 131 as before. Maple finds that r2 $= 171$, therefore the rank of τ is r2 $-$ r1 $= 171 - 108 = 63 = 3(g + 1) = dim(T)$, $(g = 20)$, hence also τ has maximal rank. So we have shown that μ_A is surjective for $g = 20$.

A.2. *Results of computations in Theorem 5.1*

Here we give the formulas of the determinants of the matrices A and B in the proof of Theorem 5.1. The Maple files of these computations are available under request to the authors.

If $g = 2k$, $det A = \frac{-4(4k^5+14k^4+15k^3+k^2-7k+1)}{p_1}$ where

$$p_1 = k^2(k-4)^3(k+1)(2k-1)(k-1)^3(k^2-4)(2k^3-9k^2 + 12k-4)(k-3)^2((2k-1)!)^4.$$

$det(B) = \frac{147456}{5}(k-5)(k^4-9k^3+16k^2+3k-8)\cdot p_2\cdot p_3/(q_1\cdot q_2)$, where

$$p_2 = 16k^9 - 14k^8 - 87k^7 + 121k^6 + 75k^5 - 138k^4 - 52k^3 + 54k^2 + 18k - 12,$$
$$p_3 = 6k^5 - k^4 - 12k^3 - 4k^2 + 12k - 4$$

$$q_1 = k^3(k-3)^3(k^2-4k+4)(-3k+k^2+2)(2k-1)(k-1)^8 \times a(k-4)^7(2k^3-9k^2+12k-4)$$

$$q_2 = (k^2-4)(2k^2-7k+6)((2k)!)^8$$

and one can check all the functions appearing in these expressions do not have any integral zero $k \geq 10$.

If $g = 2k+1$, $det A = \frac{-16}{15 \cdot p_4} \cdot (k-4)p_5$ where

$$p_4 = (-2+k)^4(2k-1)k^2(k-3)^3(k-1)^6(k+2)(k+1)((2+2k)!^2),$$

$$p_5 = (1168k^{14} + 2216k^{13} - 22360k^{12} - 41218k^{11} + 17145k^{10} + 47730k^9 + 46525k^8 + 38736k^7 - 70488k^6 - 58080k^5 + 35288k^4 + 14726k^3 - 6093k^2 + 66k - 465).$$

$det(B) = -\frac{3072}{25}(k-5) \cdot p_6 \cdot p_7/p_8$, where

$$p_6 = (270336 + 1257472k + 25884500k^4 - 5217504k^2 - 15573704k^3 - 6492143k^{17} + 68438542k^5 - 28031103k^6 - 108784825k^7 - 49730235k^8 - 30298961k^9 + 50987804k^{10} + 197670424k^{11} + 60883960k^{12} - 162484142k^{13} - 9462204k^{18} - 79976k^{19} + 945456k^{20} + 45632k^{21} - 44288k^{22} - 1216k^{23} + 768k^{24} - 92612465k^{14} + 54292657k^{15} + 44402735k^{16}),$$

$$p_7 = 2 + 4k - k^2 + (2k+1)!(k+1)(k^4 - 2k^3 - k^2 + 12k + 4),$$

$$p_8 = ak^6(k-4)^5(k-1)(k^3-4k^2+5k-2)^3(2k^2-7k+3)(k^2-3k+2)^3 \times (k+1)^3(k-3)^5(k+2)(2k-3)(4k^2-1)((2k+1)!)^7$$

and again one can check all these functions do not have any integral zero $k \geq 10$.

References

[1] D. Abramovich, A. Corti, A. Vistoli, Twisted bundles and admissible coverings, Comm. Algebra 31 (2003) 3547–3618. math.AG/0106211.

[2] E. Ballico, C. Casagrande, C. Fontanari, Moduli of Prym curves, Doc. Math. 9 (2004) 265–281.

[3] A. Beauville, Prym varieties and the Schottky problem, Invent. Math. 41 (1977) 149–196.

[4] A. Calabri, C. Ciliberto, R. Miranda, The rank of the 2nd Gaussian map for general curves, Michigan Math. J. 60 (3) (2011) 545–559.

[5] L. Caporaso, Brill Noether theory of binary curves, Math. Res. Lett. 17 (2) (2010) 243–262.

[6] E. Colombo, P. Frediani, Some results on the second Gaussian map for curves, Michigan Math. J. 58 (3) (2009) 745–758.

[7] E. Colombo, P. Frediani, Siegel metric and curvature of the moduli space of curves, Trans. Amer. Math. Soc. 362 (3) (2010) 1231–1246.

[8] E. Colombo, P. Frediani, On the second Gaussian map for curves on a K3 surface, Nagoya Math. J. 199 (2010) 123–136.

[9] E. Colombo, P. Frediani, G. Pareschi, Hyperplane sections of abelian surfaces, J. Algebraic Geom. 21 (1) (2012) 183–200.

[10] E. Colombo, G.P. Pirola, A. Tortora, Hodge-Gaussian maps, Ann. Scuola Normale Sup. Pisa Cl. Sci. (4) 30 (1) (2001) 125–146.

[11] G. Farkas, K. Ludwig, The Kodaira dimension of the moduli space of Prym varieties, J. Eur. Math. Soc. (JEMS) 12 (3) (2010) 755–795.

[12] R. Friedman, R. Smith, The generic Torelli theorem for the Prym map, Invent. Math. 67 (3) (1982) 473–490.
[13] M.L. Green, Infinitesimal methods in Hodge theory, in: Algebraic Cycles and Hodge Theory, Torino 1993, in: Lecture Notes in Mathematics, vol. 1594, Springer, Berlin, 1994, pp. 1–92.
[14] P. Griffiths, J. Harris, Principles of algebraic geometry, in: Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978, xii+813 pp.
[15] V.I. Kanev, A global Torelli theorem for Prym varieties at a general point, Izv. Akad. Nauk SSSR Ser. Mat. 46 (2) (1982) 244–268. 431 (in Russian).
[16] S. Kobayashi, Differential geometry of complex vector bundles, in: Publications of the Mathematical Society of Japan, 15, in: Kano Memorial Lectures, vol. 5, Princeton University Press, Princeton, NJ; Iwanami Shoten, Tokyo, 1987, xii+305 pp.
[17] C.T. Simpson, Higgs bundles and local systems, Inst. Hautes Études Sci. Publ. Math. No. 75 (1992) 5–95.