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RESEARCH

# On the theory of Bergman spaces on homogeneous Siegel domains

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**Abstract**
We consider mixed-norm Bergman spaces on homogeneous Siegel domains. In the literature, two different approaches have been considered and several results seem difficult to be compared. In this paper, we compare the results available in the literature and complete the existing ones in one of the two settings. The results we present are as follows: natural inclusions, density, completeness, reproducing properties, sampling, atomic decomposition, duality, continuity of Bergman projectors, boundary values, and transference.

**Keywords** Bergman spaces · Bergman projections · Homogeneous Siegel domains · Atomic decomposition · Boundary values

**Mathematics Subject Classification** 32A15 · 32A37 · 32A50 · 46E22

## 1 Introduction

This paper deals with some spaces of holomorphic functions on a homogeneous Siegel domain. In order to illustrate the kind of spaces and problems we are going to consider, we begin with the simplest case.

Let $\mathbb{C}_+ := \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$ be the upper half-plane. We can think of $\mathbb{C}_+$ as $\mathbb{R} + i(0, \infty)$, where $(0, \infty)$ is (essentially) the unique open (convex) cone in $\mathbb{R}$. In several variables, the upper half-plane can be generalized to tube domains over convex cones. Let $\Omega$ be an open convex cone in $\mathbb{R}^m$. Then, the domain $D = \mathbb{R}^m + i\Omega$ in $\mathbb{C}^m$ is called the tube domain over the cone $\Omega$. If the group of linear transformations of $\mathbb{R}^m$ that preserve $\Omega$ acts transitively on $\Omega$ itself, then $\Omega$ is a *homogeneous* cone and the domain becomes itself homogeneous, that is, the group of biholomorphic self-maps of $D$ (the *automorphisms* of $D$) acts transitively on $D$.

Another classical domain in several variables that extends the definition and some of the main features of $\mathbb{C}_+$ is the so-called Siegel upper half-space. Consider again the cone $(0, \infty) \subseteq \mathbb{R}$ and the hermitian quadratic map on $\mathbb{C}^n$ $\zeta \mapsto \zeta \cdot \overline{\zeta} = |\zeta|^2$. Then, the Siegel upper half-space is the domain $\mathcal{U}$ in $\mathbb{C}^n \times \mathbb{C}$

$$\mathcal{U} := \{(\zeta, z) \in \mathbb{C}^n \times \mathbb{C} \colon \operatorname{Im} z - |\zeta|^2 \in (0, \infty)\}.$$

The *homogeneous Siegel domains* are then introduced as follows—we refer to Sect. 2 for complete definitions. Let a homogeneous cone $\Omega \subset \mathbb{R}^m$ and a suitable hermitian quadratic map $\Phi \colon \mathbb{C}^n \to \mathbb{C}^m$ be given. Then, the homogeneous Siegel domain $D \subseteq \mathbb{C}^n \times \mathbb{C}^m$ is

$$D = \{(\zeta, z) \in \mathbb{C}^n \times \mathbb{C}^m \colon \operatorname{Im} z - \Phi(\zeta) \in \Omega\}.$$

(again, cf. Sect. 2 for definitions). Notice that if $n = 0$, then $D$ is the tube domain over the given cone $\Omega$.

On a homogeneous Siegel domain $D$ as above, various (mixed norm) weighted Bergman spaces have been considered in the literature. On the one hand, in [33] (for the upper

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half-plane) and [16] (for the general case), the following mixed norm weighted Bergman spaces are considered:

$$A^{p,q}_{\mathbf{s}} := \{ f \in \mathrm{Hol}(D) : \| h \mapsto \Delta^{\mathbf{s}}_{\Omega}(h) \| f_h \|_{L^p(\mathcal{N})} \|_{L^q(\nu_\Omega)} < \infty \},$$

where $\Delta^{\mathbf{s}}_{\Omega}$ are 'generalized power functions' on $\Omega$ ($\mathbf{s} \in \mathbb{R}^r$), $\nu_\Omega$ is 'the' invariant measure on $\Omega$, $\mathcal{N} = \mathbb{C}^n \times \mathbb{R}^m$ and $f_h \colon (\zeta, x) \mapsto f(\zeta, x + i\Phi(\zeta) + ih)$. On the other hand, e.g. in [1–4, 6, 7, 12, 23, 29], the following mixed-norm weighted Bergman spaces are considered:

$$\mathfrak{A}^{p,q}_{\mathbf{s}} := \{ f \in \mathrm{Hol}(D) \colon \| h \mapsto \| f_h \|_{L^{p,q}(\mathbb{R}^m, \mathbb{C}^n)} \|_{L^q(\Delta^{\mathbf{s}+\mathbf{b}/2}_{\Omega} \cdot \nu_\Omega)} < \infty \},$$

where $\mathbf{b}$ is a suitable element of $\mathbb{R}^r$ and $L^{p,q}(\mathbb{R}^m, \mathbb{C}^n) = \{ f \colon \| \zeta \mapsto \| f(\zeta, \cdot) \|_{L^p(\mathbb{R}^m)} \|_{L^q(\mathbb{C}^n)} < \infty \}$.

Two parallel theories then arise, and different conventions have been adopted. For example, the definition of the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$ suggests a natural comparison between the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$ for a fixed $\mathbf{s}$, which in turn highlights the role played by 'the' Bergman projector $\mathfrak{P}_{\mathbf{s}}$, namely the Bergman projector of the corresponding space $\mathfrak{A}^{2,2}_{\mathbf{s}}$. On the other hand, the comparison of the spaces $A^{p,q}_{\mathbf{s}}$ for fixed $\mathbf{s}$ appears to be less natural, so that more general Bergman projectors are naturally investigated. Besides that, in the study of various properties of the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$ (such as, for instance, the continuity of $\mathfrak{P}_{\mathbf{s}}$ on the space $\mathfrak{L}^{p,q}_{\mathbf{s}}$, which is defined the same way as $\mathfrak{A}^{p,q}_{\mathbf{s}}$ replacing holomorphic functions with equivalence classes of measurable functions), greater attention is placed on $p$ and $q$, rather than $\mathbf{s}$, whereas in the study of the spaces $A^{p,q}_{\mathbf{s}}$, greater attention is placed on $\mathbf{s}$, rather than $p$ or $q$, so that even when describing the same phenomena, the two parallel theories may appear quite different from one another and hard to compare.

This is the main reason which motivated us to write this work. Our goal is to describe various results for the spaces $A^{p,q}_{\mathbf{s}}$ and prove the direct counterparts in the case of the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$ for a direct comparison. Hence, in particular, we deepen the study of the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$ proving those results which, to the best of our knowledge, do not appear in the literature in the present generality. In order to tackle these issues, we are naturally led to introduce a new family of spaces $\mathcal{A}^{p,q}_{\mathbf{s}}$, which is defined in the spirit of the spaces $A^{p,q}_{\mathbf{s}}$ (and has, therefore, some technical advantages), but allows to treat also the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$ by means of suitable substitutions.

Function theory and analysis of function spaces on homogeneous Siegel domains are the classical areas of research in which complex analysis, harmonic analysis, geometry of convex cones, and representation theory all play a fundamental role, see e.g. [27, 28, 31, 32, 34] and also [24]. In more recent times, it was shown in [5] and then generalized in [4], that in order to prove the $L^p$-boundedness of the Bergman projector on some homogeneous Siegel domain of tube-type (see Subsection 2.6), it was necessary to exploit the cancellations of the kernel, a phenomenon that had never been observed before; in fact, these examples seem to remain the only instances of such behaviour. We mention in passing that, in order to exploit the cancellation of the Bergman kernel, the mixed-norm spaces were considered, hence showing their naturality in this context. It is also worth mentioning that the question of the $L^p$-boundedness of the Bergman projector on homogeneous Siegel domains is tightly connected to the sharp $\ell^2$-decoupling inequality of Bourgain and Demeter [13], see [6]. We refer also to the Introduction in [19] for a thorough discussion of this and related questions, and to [6–8, 16–19, 29] for some very recent works on the subject of the current paper.

The paper is organized as follows. In Sect. 2, we recall some basic facts and introduce some notation to deal with homogeneous Siegel domains. In order to help readers who are accustomed to different conventions, we introduce our notation axiomatically, allowing the reader to identify the (hopefully minimal) modifications needed. In Sect. 3, we briefly list several known results for the spaces $A^{p,q}_{\mathbf{s}}$ without proofs. In Sect. 4, we deal with the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$ for which we prove results which are analogous to the ones which are valid in the case of the spaces $A^{p,q}_{\mathbf{s}}$. Section 4 constitutes the main part of this paper. In order to more easily deal with the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$, we introduce another scale of spaces, denoted by $\mathcal{A}^{p,q}_{\mathbf{s}}$. In fact, the use of these latter spaces allows to simplify the notation and to more easily compare the scale of spaces $A^{p,q}_{\mathbf{s}}$ and $\mathfrak{A}^{p,q}_{\mathbf{s}}$. In addition to that, dealing with the spaces $\mathcal{A}^{p,q}_{\mathbf{s}}$ allows for a more comprehensive treatment of some topics (such as duality for $q \leq 1$) that would otherwise require the introduction of auxiliary spaces (cf., e.g. [1], where a different—yet related—description of the dual of $\mathfrak{A}^{p,p}_{\mathbf{s}}$ is determined for $p \in (0, 1)$). The topics we shall present include the following: natural inclusions, density, completeness, reproducing properties, sampling, atomic decomposition, duality, continuity of Bergman projectors, boundary values, and transference.

We did our best to acknowledge the previously known results before the statements of the most important results. We apologize for any omission.

## 2 Homogeneous Siegel domains

We denote by $E$ a complex Hilbert space of dimension $n$, by $F$ a real Hilbert space of dimension $m$, by $\Omega$ an open convex cone in $F$ not containing affine lines, and by $\Phi \colon E \times E \to F_{\mathbb{C}}$ a non-degenerate hermitian map such that $\Phi(\zeta) := \Phi(\zeta, \zeta) \in \overline{\Omega}$ for every $\zeta \in E$. Clearly, $E$ and $F$ may be replaced with the standard Euclidean spaces $\mathbb{C}^n$ and $\mathbb{R}^m$ with their natural inner products. We prefer the more abstract versions since $E$ and $F$ are in general better described as spaces of (formal) matrices, at least in most examples. Besides that, we believe that this abstract notation better underlines the roles played by the various objects under consideration. In any event, the reader may replace $E$ with $\mathbb{C}^n$ and $F$ with $\mathbb{R}^m$, if they find it convenient, without any loss of generality.

We define

$$\rho\colon E\times F_\mathbb{C}\ni(\zeta,z)\mapsto \operatorname{Im} z-\Phi(\zeta)\in F,$$

and denote by

$$D=\{(\zeta,z)\in E\times F_\mathbb{C}\colon \operatorname{Im} z-\Phi(\zeta)\in\Omega\}=\rho^{-1}(\Omega)$$

the Siegel domain associated with $\Omega$ and $\Phi$. We shall assume that $D$ is homogeneous, that is, that the group of its biholomorphisms acts transitively on it. It is known (cf., e.g. [14, Proposition 1]) that $D$ is homogeneous if and only if there is a triangular[1] subgroup $T_+$ of $GL(F)$ which acts simply transitively on $\Omega$, and for every $t\in T_+$ there is $g\in GL(E)$ such that $t\circ\Phi=\Phi\circ(g\times g)$. In this case, any other triangular subgroup of $GL(F)$ with the same properties is conjugate to $T_+$ by an element of $GL(F)$ which preserves $\Omega$. In addition, $T_+$ acts simply transitively on the right on the dual cone $\Omega'$, by transposition (cf. [38, Theorem 1]). We shall denote this latter action by $\lambda\cdot t$, for $\lambda\in\Omega'$ and $t\in T_+$; we shall consequently write $t\cdot h$ instead of $th$ for $t\in T_+$ and $h\in\Omega$. We shall still denote by $t\,\cdot$ and $\cdot\, t$ the actions of $t$ on $F_\mathbb{C}$ and $F'_\mathbb{C}$, respectively, for every $t\in T_+$.

## 2.1 Analysis on Ω

It is possible to describe the structure of $T_+$ and of its action on $\Omega$ using the theory of $T$-algebras, cf. [38], or the theory of (normal) $j$-algebras, cf. [32, 34]. In order to keep the exposition as simple as possible, we shall avoid a thorough description of the structure of $T_+$ and proceed axiomatically. We refer the reader to [16] for a more detailed treatment of the following considerations. We first observe that there are $r\in\mathbb{N}$ (called the rank of $\Omega$) and a surjective homomorphism of Lie groups

$$\Delta\colon T_+\to(\mathbb{R}_+^*)^r,$$

with kernel $[T_+,T_+]$, such that, if we fix base-points $e_\Omega\in\Omega$ and $e_{\Omega'}\in\Omega'$ and define

$$\Delta_\Omega^{\mathbf{s}}(t\cdot e_\Omega)=\Delta_{\Omega'}^{\mathbf{s}}(e_{\Omega'}\cdot t)=\Delta^{\mathbf{s}}(t)=\prod_{j=1}^r\Delta_j(t)^{s_j}\qquad(1)$$

for every $\mathbf{s}\in\mathbb{C}^r$ and for every $t\in T_+$, then $\Delta_\Omega^{\mathbf{s}}$ (and $\Delta_{\Omega'}^{\mathbf{s}}$) is bounded on bounded subsets if and only if $\operatorname{Re}\mathbf{s}\in\mathbb{R}_+^r$ (cf. [16, Lemma 2.34]). We shall further require that $\Delta(a\,\cdot)=(a,\dots,a)$ for every $a>0$, where $a\,\cdot$ denotes the homothety of ratio $a$ (which necessarily belongs to $T_+$). We remark explicitly that these conditions determine $\Delta$ up to a permutation of the coordinates (in $(\mathbb{R}_+^*)^r$).[2] Consequently, we may apply the results of [16, Chap. 2] without (essential) changes, even if a different choice of $T_+$ and $\Delta$ is made. Notice that $\Delta_\Omega^{\mathbf{s}}$ and $\Delta_{\Omega'}^{\mathbf{s}}$ extend to holomorphic functions on $\Omega+iF$ and $\Omega'+iF'$, respectively, for every $\mathbf{s}\in\mathbb{C}^r$ (cf. [16, Corollary 2.25]).

When $\Omega$ is symmetric, that is, self-dual with respect to the scalar product of $F$, then the functions $\Delta_{\mathbf{s}}$ considered in [24] coincide with the functions $\Delta_\Omega^{\mathbf{s}}$ defined in (1) for an appropriate choice of $\Delta$ (cf. [24, Chap. VI, § 3]); in particular, the 'determinant' polynomial coincides with $\Delta_\Omega^{\mathbf{1}_r}$. Generally speaking, the works which deal with the case in which $\Omega$ is symmetric generally adhere to the conventions of [24], possibly with slightly different notation, whereas the works which deal with general homogeneous cones generally adhere to the conventions described above, possibly with different notation (for example, $\Delta_\Omega^{\mathbf{s}}=Q^{\mathbf{s}}$ and $\Delta_{\Omega'}^{\mathbf{s}}=(Q^*)^{\mathbf{s}}$ in the notation of [6, 8, 29, 30]).

To simplify the notation, we state the following definition.

**Definition 2.1** We define two order relations on $\mathbb{R}^r$. On the one hand, we write $\mathbf{s}\leq\mathbf{s}'$ to mean $s_j\leq s'_j$ for every $j=1,\dots,r$ (equivalently, $\mathbf{s}'-\mathbf{s}\in\mathbb{R}_+^r$). On the other hand, we write $\mathbf{s}\preceq\mathbf{s}'$ to mean $\mathbf{s}=\mathbf{s}'$ or $s_j<s'_j$ for every $j=1,\dots,r$.

Thus, $\mathbf{s}\prec\mathbf{s}'$ (that is, $\mathbf{s}\preceq\mathbf{s}'$ and $\mathbf{s}\neq\mathbf{s}'$) if and only if $\mathbf{s}'-\mathbf{s}\in(\mathbb{R}_+^*)^r$, that is, $s_j<s'_j$ for every $j=1,\dots,r$.

**Definition 2.2** We denote by $\mathcal{H}^k$ the $k$-dimensional Hausdorff measure. There are $\mathbf{d}\prec\mathbf{0}$ and $\mathbf{b}\leq\mathbf{0}$ such that

$$\nu_\Omega:=\Delta_\Omega^{\mathbf{d}}\cdot\mathcal{H}^m,\qquad \nu_{\Omega'}:=\Delta_{\Omega'}^{\mathbf{d}}\cdot\mathcal{H}^m,\qquad\text{and}$$
$$\nu_D:=(\Delta_\Omega^{\mathbf{b}+2\mathbf{d}}\circ\rho)\cdot\mathcal{H}^{2n+2m}\qquad(2)$$

are the unique measures on $\Omega$, $\Omega'$, and $D$ (up to a multiplicative constant) which are invariant under all linear automorphisms of $\Omega$ and $\Omega'$, and all biholomorphisms of $D$, respectively (cf. [16, Propositions 2.19 and 2.44], and [24, Proposition I.3.1]).

[1] This means that all the eigenvalues of every element of $T_+$ are real. Equivalently, there is a basis of $F$ with respect to which every element of $T_+$ is represented by an upper triangular matrix, cf. [37].

[2] To see this fact, observe that, if $\Delta'\colon T_+\to(\mathbb{R}_+^*)^r$ is another homomorphism with the same properties, then there is $A\in GL(\mathbb{R}^n)$ such that $\log\Delta'=A\log\Delta$. In addition, given $\mathbf{s}\in\mathbb{R}^r$, both $\sum_j s_j\log\Delta'_j=\sum_j({}^tA\mathbf{s})_j\log\Delta_j$ and $\sum_j s_j\log\Delta_j$ induce functions which are bounded on the bounded subsets of $\Omega$ if and only if $\mathbf{s}\in\mathbb{R}_+^r$, so that ${}^tA\mathbb{R}_+^r=\mathbb{R}_+^r$ and therefore $A$ must be the composition of a permutation of the coordinates and a diagonal dilation $(x_1,\dots,x_r)\mapsto(\lambda_1x_1,\dots,\lambda_rx_r)$, $\lambda_1,\dots,\lambda_r>0$. Since $\Delta(a\,\cdot)=\Delta'(a\,\cdot)=(a,\dots,a)$ for every $a>0$, we then see that $A$ must induce the identity on the line $\mathbb{R}\mathbf{1}_r$, so that it must be a permutation of the coordinates.

*Remark 2.3* Notice that $\Delta^{-\mathbf{b}}(t) = |\det_{\mathbb{C}} g|^2$ for every $t \in T_+$ and for every $g \in GL(E)$ such that $t \cdot \Phi = \Phi \circ (g \times g)$. Further, $\Delta^{-\mathbf{d}}(a\,\cdot) = a^m$, and $\Delta^{-\mathbf{b}}(a\,\cdot) = a^n$ for every $a > 0$.

We observe explicitly that $\mathbf{d} = d$ and $\mathbf{b} = -q$ in the notation of [1, 9, 27], whereas $\mathbf{d} = -\tau$ and $\mathbf{b} = -b$ in the notation of [6, 8, 29, 30]. In particular, there is no general agreement on the sign of $\mathbf{d}$.

**Definition 2.4** There are $\mathbf{m}, \mathbf{m}' \geq \mathbf{0}$ such that $\Delta^{\mathbf{s}}_\Omega \cdot \nu_\Omega$ and $\Delta^{\mathbf{s}}_{\Omega'} \cdot \nu_{\Omega'}$ induce Radon measures on $F$ and $F'$, respectively, if and only if $\operatorname{Re}\mathbf{s} \succ \frac{1}{2}\mathbf{m}$ and $\operatorname{Re}\mathbf{s} \succ \frac{1}{2}\mathbf{m}'$, respectively (cf. [16, Proposition 2.19]).

*Remark 2.5* Notice that $\mathbf{d} = -(\mathbf{1}_r + \frac{1}{2}\mathbf{m} + \frac{1}{2}\mathbf{m}')$ (cf. [16, Definition 2.8] and the preceding remarks). We observe explicitly that $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{m}' = (n_1, \dots, n_r)$ in the notation of [6–9, 29, 30].

**Definition 2.6** For every $\mathbf{s}, \mathbf{s}' \in \mathbb{C}^r$ such that $\operatorname{Re}\mathbf{s} \succ \frac{1}{2}\mathbf{m}$ and $\operatorname{Re}\mathbf{s}' \succ \frac{1}{2}\mathbf{m}'$, we define $\Gamma_\Omega(\mathbf{s})$ and $\Gamma_{\Omega'}(\mathbf{s}')$ so that

$$\mathcal{L}(\Delta^{\mathbf{s}}_\Omega \cdot \nu_\Omega) = \Gamma_\Omega(\mathbf{s})\Delta^{-\mathbf{s}}_{\Omega'} \quad \text{and}$$
$$\mathcal{L}(\Delta^{\mathbf{s}'}_{\Omega'} \cdot \nu_{\Omega'}) = \Gamma_{\Omega'}(\mathbf{s}')\Delta^{-\mathbf{s}'}_{\Omega},$$

respectively, where $\mathcal{L}$ denotes the Laplace transform.

*Remark 2.7* Notice that $\Gamma_\Omega(\mathbf{s}) = \mathcal{L}(\Delta^{\mathbf{s}}_\Omega \cdot \nu_\Omega)(e_{\Omega'}) = c\prod_{j=1}^r \Gamma\big(s_j - \frac{1}{2}m_j\big)$ and $\Gamma_{\Omega'}(\mathbf{s}') = \mathcal{L}(\Delta^{\mathbf{s}'}_{\Omega'} \cdot \nu_{\Omega'})(e_\Omega) = c'\prod_{j=1}^r \Gamma\big(s'_j - \frac{1}{2}m'_j\big)$ for some constants $c, c' > 0$ which depend on the choice of $e_\Omega$ and $e_{\Omega'}$.

**Definition 2.8** There are two uniquely determined holomorphic families $(I^{\mathbf{s}}_\Omega)_{\mathbf{s}\in\mathbb{C}^r}$ and $(I^{\mathbf{s}}_{\Omega'})_{\mathbf{s}\in\mathbb{C}^r}$ of tempered distributions on $F$ and $F'$, respectively, such that $\mathcal{L}I^{\mathbf{s}}_\Omega = \Delta^{-\mathbf{s}}_{\Omega'}$ and $\mathcal{L}I^{\mathbf{s}}_{\Omega'} = \Delta^{-\mathbf{s}}_{\Omega}$ (cf. [16, Lemma 2.26 and Proposition 2.28]).

*Remark 2.9* Notice that $I^{\mathbf{s}}_\Omega = \frac{1}{\Gamma_\Omega(\mathbf{s})}\Delta^{\mathbf{s}}_\Omega \cdot \nu_\Omega$ and $I^{\mathbf{s}}_{\Omega'} = \frac{1}{\Gamma_{\Omega'}(\mathbf{s})}\Delta^{\mathbf{s}}_{\Omega'} \cdot \nu_{\Omega'}$ when $\operatorname{Re}\mathbf{s} \succ \frac{1}{2}\mathbf{m}$ and $\operatorname{Re}\mathbf{s} \succ \frac{1}{2}\mathbf{m}'$, respectively. In addition, $I^{\mathbf{s}}_\Omega$ and $I^{\mathbf{s}}_{\Omega'}$ are supported in $\overline{\Omega}$ and $\overline{\Omega'}$, respectively, for every $\mathbf{s} \in \mathbb{C}^r$ (cf. [16, Proposition 2.28]).

**Definition 2.10** We denote by $\mathbb{N}_\Omega$ and $\mathbb{N}_{\Omega'}$ the sets of $\mathbf{s} \in \mathbb{R}^r$ such that $\Delta^{\mathbf{s}}_\Omega$ and $\Delta^{\mathbf{s}}_\Omega$ extend to polynomials on $F$ and $F'$, respectively.

*Remark 2.11* Notice that $I^{\mathbf{s}}_\Omega$ and $I^{\mathbf{s}}_{\Omega'}$ are supported in $\{0\}$ if and only if $\mathbf{s} \in -\mathbb{N}_{\Omega'}$ and $\mathbf{s} \in -\mathbb{N}_\Omega$, respectively. Then, $\Phi_*(\mathcal{H}^{2n}) = cI^{-\mathbf{b}}_\Omega$ for a suitable constant $c > 0$ which depends on the choice of $e_{\Omega'}$ (cf. [16, Proposition 2.30]). We observe explicitly that, when $\Omega$ is symmetric, then $\mathbf{1}_r \in \mathbb{N}_\Omega$ and the differential operator $f \mapsto f * I^{-\mathbf{1}_r}_\Omega$ is simply the differential operator associated with the determinant polynomial $\Delta^{\mathbf{1}_r}_\Omega$ by means of the scalar product. This latter operator is often denoted by $\square$. In addition, if $\Omega$ is symmetric and irreducible, then $\mathbb{N}_\Omega = \{\mathbf{s} \in \mathbb{N}^r : s_1 \geq \cdots \geq s_r\}$, for an appropriate choice of $\Delta$. This latter condition completely determines $\Delta$ in this case.

## 2.2 Fourier analysis on the Šilov boundary

We now pass to the analysis of the Šilov boundary of $D$ (cf. [31] for a more general treatment of this topic). We endow $E \times F_{\mathbb{C}}$ with the 2-step nilpotent Lie group structure whose product is given by

$$(\zeta, z) \cdot (\zeta', z') := (\zeta + \zeta', z + z' + 2i\Phi(\zeta', \zeta)),$$

for every $(\zeta, z), (\zeta', z') \in E \times F_{\mathbb{C}}$. If we identify $\mathcal{N} := E \times F$ with the Šilov boundary $\rho^{-1}(0)$ of $D$ by means of the mapping $(\zeta, x) \mapsto (\zeta, x + i\Phi(\zeta))$, then $\mathcal{N}$ becomes a 2-step nilpotent Lie group with product

$$(\zeta, x)(\zeta', x') = (\zeta + \zeta', x + x' + 2\operatorname{Im}\Phi(\zeta, \zeta'))$$

for every $(\zeta, x), (\zeta', x') \in \mathcal{N}$.

Define $W := \{\lambda \in F' : \exists v \in E \setminus \{0\}\ \langle \lambda, \operatorname{Im}\Phi(\cdot\,, v)\rangle = 0\}$, so that $W$ is a proper algebraic variety in $F'$ since $\Phi$ is non-degenerate and $\overline{\Omega}$-positive. Then, for every $\lambda \in F' \setminus W$, the quotient of $\mathcal{N}$ modulo the central subgroup $\ker\lambda$ is isomorphic to a Heisenberg group (to $\mathbb{R}$, if $E = \{0\}$), so that the Stone–Von Neumann theorem (cf., e.g. [25, Theorem 1.50]) ensures the existence of a unique (up to unitary equivalence) irreducible continuous unitary representation $\pi_\lambda$ of $\mathcal{N}$ in some Hilbert space $H_\lambda$ such that $\pi_\lambda(0, x) = \mathrm{e}^{-i\langle\lambda, x\rangle}$ for every $x \in F$. One then has the Plancherel identity (cf. [16, Corollary 1.17 and Proposition 2.30]):

$$\|f\|^2_{L^2(\mathcal{N})} = c\int_{F'\setminus W} \|\pi_\lambda(f)\|^2_{\mathscr{L}^2(H_\lambda)} |\Delta^{-\mathbf{b}}_{\Omega'}(\lambda)|\,\mathrm{d}\lambda$$

for every $f \in L^1(\mathcal{N}) \cap L^2(\mathcal{N})$, where $c > 0$ is a suitable constant (which depends on the choice of $e_{\Omega'}$) and $\mathscr{L}^2(H_\lambda)$ denotes the space of Hilbert–Schmidt endomorphisms of $H_\lambda$. Note that $\Delta^{-\mathbf{b}}_{\Omega'}$ is positive on $\Omega'$ and extends to a polynomial on $F'$, so that the above formula is meaningful (cf. [16, Proposition 2.30]).

## 2.3 The CR structure of $\mathcal{N}$

For every $v \in E$, denote by $Z_v$ the left-invariant vector field on $\mathcal{N}$ which induces the Wirtinger derivative $\frac{1}{2}(\partial_v - i\partial_{iv})$ at $(0, 0)$. Then, the $Z_v$, for $v \in E$, induce a subbundle of the complexified tangent bundle of $\mathcal{N}$ which endows $\mathcal{N}$ with the structure of a CR manifold (cf. [11, Sect. 7.4]). In particular, a distribution $u$ on $\mathcal{N}$ is said to be CR if $\overline{Z_v}u = 0$ for every

$v \in E$ (cf. [11, Sects. 9.1 and 17.2]). Note that an element $f$ of $L^2(\mathcal{N})$ is CR if and only if

$$\pi_\lambda(f) = \chi_{\Lambda_+}(\lambda)\pi_\lambda(f)P_{\lambda,0}$$

for almost every $\lambda \in F' \setminus W$, where $\Lambda_+$ is the interior of the polar of $\Phi(E)$, that is, the set

$$\{\lambda \in F' : \forall \zeta \in E \setminus \{0\} \ \langle \lambda, \Phi(\zeta)\rangle > 0\},$$

and $P_{\lambda,0}$ is an orthoprojector of rank one in $H_\lambda$, for every $\lambda \in F' \setminus W$ (cf., e.g. [31] or [16, Proposition 1.19] and [15, Proposition 2.16]).

### 2.4 Metrics

We endow $D$ with a complete Riemannian metric which is invariant under the action of affine biholomorphisms (for example, the Bergman metric is complete and invariant under all biholomorphisms of $D$, cf. [16, Proposition 2.44]), and the associated distance $d$. Since the balls with respect to $d$ will only be used for bounded radii, it will not matter which distance is chosen, as long as it satisfies the preceding conditions.

We endow $\Omega$ with the Riemannian metric induced by that on $D$ by means of the submersion $\rho$ (interpreted as the projection of $D$ onto its quotient modulo the action of $\mathcal{N}$), and $\Omega'$ with the metric induced by the diffeomorphism $\Omega \ni t \cdot e_\Omega \mapsto e_{\Omega'} \cdot t^{-1} \in \Omega'$. We denote by $d_\Omega$ and $d_{\Omega'}$ the corresponding distances, and by $B_\Omega(h, R)$ and $B_{\Omega'}(\lambda, R)$ the corresponding balls of centre $h \in \Omega$ and $\lambda \in \Omega'$, respectively, and radius $R > 0$. Notice that also in this case one may choose general complete $T_+$-invariant Riemannian distances without (essentially) compromising the results which follow. Nonetheless, the relationships between $d$ and $d_\Omega$ will be useful in some places (such as in the definition of lattices given below).

Analogously, we endow $E \times \Omega$ with the Riemannian metric induced by the one on $D$ by means of the submersion

$$\rho' : D \ni (\zeta, z) \mapsto (\zeta, \rho(\zeta, z)) \in E \times \Omega,$$

interpreted as the projection of $D$ onto its quotient modulo the action of the centre $F$ of $\mathcal{N}$. We denote by $d_{E\times\Omega}$ the corresponding distance and by $B_{E\times\Omega}((\zeta, h), R)$ the corresponding ball of centre $(\zeta, h) \in E \times \Omega$ and radius $R > 0$.

We observe explicitly that both $\mathcal{N}$ and its centre $F$ are normal subgroups of the group $G_{\mathrm{Aff}}$ of affine automorphisms of $D$ (cf. [28, Proposition 2.1]). Hence, $d_\Omega$ and $d_{E\times\Omega}$ are $(G_{\mathrm{Aff}}/\mathcal{N})$- and $(G_{\mathrm{Aff}}/F)$-invariant, respectively. In particular, $d_\Omega$ and $d_{\Omega'}$ are $T_+$-invariant, while $d_{E\times\Omega}$ is invariant under the affine automorphisms of the form

$$(\zeta, h) \mapsto (g\zeta + \zeta', t \cdot h),$$

with $\zeta' \in E$, $t \in T_+$, and $g \in GL(E)$ such that $t \cdot \Phi = \Phi \circ (g \times g)$. We define $\nu_{E\times\Omega} := (\Delta_\Omega^{-\mathbf{b}-\mathbf{d}} \circ \mathrm{pr}_\Omega) \cdot \mathcal{H}^{2n+m}$, so that $\nu_{E\times\Omega}$ is $(G_{\mathrm{Aff}}/F)$-invariant.

### 2.5 Lattices

By a $(\delta, R)$-lattice on $\Omega$, with $\delta > 0$ and $R > 1$, we mean a family $(h_k)_{k\in K}$ of elements of $\Omega$ such that the balls $B_\Omega(h_k, \delta)$ are pairwise disjoint while the balls $B_\Omega(h_k, R\delta)$ cover $\Omega$. We define lattices on $\Omega'$ and $E \times \Omega$ analogously. Notice that every maximal family of elements of $\Omega$ whose mutual distances are $\geq 2\delta$ is necessarily a $(\delta, 2)$-lattice (and conversely), so that $(\delta, 2)$-lattices on $\Omega$, $\Omega'$, and $E \times \Omega$ always exist.

By an $\mathcal{N}$-$(\delta, R)$-lattice on $D$, with $\delta > 0$ and $R > 1$, we mean a family $(\zeta_{j,k}, z_{j,k})_{j\in J, k\in K}$ of elements of $D$ such that the balls $B((\zeta_{j,k}, z_{j,k}), \delta)$ are pairwise disjoint, the balls $B((\zeta_{j,k}, z_{j,k}), R\delta)$ cover $D$, and there is a $(\delta, R)$-lattice $(h_k)_{k\in K}$ on $\Omega$ such that $\rho(\zeta_{j,k}, z_{j,k}) = h_k$ for every $j \in J$ and for every $k \in K$.

By an $F$-$(\delta, R)$-lattice on $D$, with $\delta > 0$ and $R > 1$, we mean a family $(\zeta_k, z_{j,k})_{j\in J, k\in K}$ of elements of $D$ such that the balls $B((\zeta_k, z_{j,k}), \delta)$ are pairwise disjoint, the balls $B((\zeta_k, z_{j,k}), R\delta)$ cover $D$, and there is a $(\delta, R)$-lattice $(\zeta_k, h_k)_{k\in K}$ on $E \times \Omega$ such that $\rho(\zeta_k, z_{j,k}) = h_k$ for every $j \in J$ and for every $k \in K$.

By a modification of the previous argument, one may show that $\mathcal{N}$- and $F$-$(\delta, 4)$-lattices always exist on $D$ (cf. [16, Lemma 2.55]).

### 2.6 The associated tube domain

We denote by

$$T_\Omega = F + i\Omega$$

the tube domain associated with $\Omega$. Given a function $f$ on $D$, we define

$$f_h : \mathcal{N} \ni (\zeta, x) \mapsto f(\zeta, x + i\Phi(\zeta) + ih)$$

for every $h \in \Omega$, and

$$f^{(\zeta)} : T_\Omega \ni z \mapsto f(\zeta, z + i\Phi(\zeta))$$

for every $\zeta \in E$. Thus,

$$f_h^{(\zeta)} : F \ni x \mapsto f(\zeta, x + i\Phi(\zeta) + ih)$$

for every $\zeta \in E$ and for every $h \in \Omega$.

### 2.7 Two families of mixed-norm weighted bergman spaces

We now introduce the different definitions of mixed-norm Bergman spaces.

In [16, 33], mixed-norm weighted Bergman spaces are defined as

$$A_{\mathbf{s}}^{p,q}(D)=\{f\in \mathrm{Hol}(D)\colon \|h\mapsto \Delta_\Omega^{\mathbf{s}}(h)\|f_h\|_{L^p(\mathcal{N})}\|_{L^q(\nu_\Omega)}<\infty\}.$$

On the one hand, this definition highlights the role played by the Šilov boundary of $D$ and gives rise to the usual Hardy spaces when $q=\infty$ and $\mathbf{s}=\mathbf{0}$ (that is, $\Delta_\Omega^{\mathbf{s}}=1$). In particular, the non-commutative Fourier analysis on $\mathcal{N}$ comes into play. On the other hand, the weight $\Delta_\Omega^{\mathbf{s}}\circ\rho$ is considered as a multiplier of the function, and not of the measure, and the 'base measure' is chosen in such a way that it induces the invariant measures on $\mathcal{N}$ and $\Omega$. When $q=\infty$, this allows to treat a whole class of spaces which would not appear otherwise, and which play a relevant role in the duality theory of the spaces $A_{\mathbf{s}}^{p,q}(D)$ when $q\le 1$.

In [6, 29] (to cite only a few), mixed-norm weighted Bergman spaces are defined as

$$\mathfrak{A}_{\mathbf{s}}^{p,q}(D):=\{f\in \mathrm{Hol}(D)\colon \|\zeta\mapsto \|f^{(\zeta)}\|_{A^{p,q}_{(\mathbf{s}+\mathbf{b}/2)/q}(T_\Omega)}\|_{L^q(E)}<\infty\}.$$

On the one hand, this definition highlights the role played by the centre $F$ of the Šilov boundary of $D$, so that the usual (commutative) Fourier analysis on $F$ comes into play. In addition, this definition also allows to view $D$ as the union of the translates $(\zeta, i\Phi(\zeta))+T_\Omega$ of the tube domain $T_\Omega$ (identified with $\{0\}\times T_\Omega\subseteq D$), so that some of the analysis on $\mathfrak{A}_{\mathbf{s}}^{p,q}(D)$ may be reduced to a simpler analysis on $\mathfrak{A}_{\mathbf{s}}^{p,q}(T_\Omega)$. On the other hand, the weight $\Delta_\Omega^{\mathbf{s}}\circ\rho$ is considered as a multiplier of the 'base measure' $(\Delta_\Omega^{\mathbf{b}/2+\mathbf{d}}\circ\rho)\cdot\mathcal{H}^{2n+2m}=(\Delta_\Omega^{-\mathbf{b}/2-\mathbf{d}}\circ\rho)\cdot\nu_D$, and not of the function. In this way, the self-adjoint projector of $\mathfrak{L}_{\mathbf{s}}^{2,2}(D)$ (defined as $\mathfrak{A}_{\mathbf{s}}^{2,2}(D)$, but allowing $f$ to be a measurable function modulo negligible functions) onto $\mathfrak{A}_{\mathbf{s}}^{2,2}(D)$ is highlighted as the 'canonical choice' when looking for a projector of $\mathfrak{L}_{\mathbf{s}}^{p,q}(D)$ onto $\mathfrak{A}_{\mathbf{s}}^{p,q}(D)$ for different $p,q\in[1,\infty]$.

We mention that $\mathfrak{A}_{\mathbf{s}}^{p,\infty}(D)=\mathfrak{A}_{\mathbf{0}}^{p,\infty}(D)$ for every $\mathbf{s}\in\mathbb{R}^r$. Because of this fact, the case $q=\infty$ is somewhat pathological and seldom considered. For similar reasons, the duality theory for the space $\mathfrak{A}_{\mathbf{s}}^{p,q}(D)$, when $q\le 1$, is treated separately (cf., e.g. [1]).

We also observe that

$$\mathfrak{A}_{\mathbf{s}}^{p,q}(D)=\{f\in \mathrm{Hol}\,(D):\|h\mapsto \|f_h\|_{L^{p,q}(F,E)}\|_{L^q(\Delta_\Omega^{\mathbf{s}+\mathbf{b}/2}\cdot\nu_\Omega)}<\infty\},$$

where

$$\|g\|_{L^{p,q}(F,E)}:=\|\zeta\mapsto \|g(\zeta,\cdot)\|_{L^p(F)}\|_{L^q(E)}$$

for every measurable function $g\colon\mathcal{N}\to\mathbb{C}$.

## 3 The spaces $A_{\mathbf{s}}^{p,q}$

In this short section, we collect some of the main results concerning the spaces $A_{\mathbf{s}}^{p,q}(D)$ the analogues of which we wish to prove for the spaces $\mathfrak{A}_{\mathbf{s}}^{p,q}(D)$ in the next section. We recall that the spaces $A_{\mathbf{s}}^{p,q}(D)$, and $\mathfrak{A}_{\mathbf{s}}^{p,q}(D)$, are described in Sect. 2. We refer the reader to [16, 19] for the proofs of the statements of this section.

### 3.1 Elementary properties

The following result is a consequence of [16, Corollary 1.31 and Proposition 3.5].

**Proposition 3.1** *Take $p,q\in(0,\infty]$ and $\mathbf{s}\in\mathbb{R}^r$. Then, $A_{\mathbf{s}}^{p,q}(D)\neq\{0\}$ if and only if $\mathbf{s}\succ\frac{1}{2q}\mathbf{m}$ or $q=\infty$ and $\mathbf{s}\ge\mathbf{0}$. In addition, $A_{\mathbf{s}}^{p,q}(D)$ is a quasi-Banach space.*

Next, we deal with inclusions among the $A_{\mathbf{s}}^{p,q}(D)$ spaces. The result is a consequence of [16, Proposition 3.2]. It extends [33, Proposition 2.2], which corresponds to the case in which $D=\mathbb{C}_+$.

**Proposition 3.2** *Take $p_1,p_2,q_1,q_2\in(0,\infty]$ with $p_1\le p_2$, $q_1\le q_2$ and $\mathbf{s_1},\mathbf{s_2}\in\mathbb{R}^r$ with $\mathbf{s_2}=\mathbf{s_1}+\left(\frac{1}{p_2}-\frac{1}{p_1}\right)(\mathbf{b}+\mathbf{d})$. Then,*

$$A_{\mathbf{s_1}}^{p_1,q_1}(D)\subseteq A_{\mathbf{s_2}}^{p_2,q_2}(D)$$

*continuously.*

### 3.2 Reproducing kernels

Define the auxiliary function

$$B^{\mathbf{s}}_{(\zeta',z')}(\zeta,z):=\Delta_\Omega^{\mathbf{s}}\left(\frac{z-\overline{z'}}{2i}-\Phi(\zeta,\zeta')\right)$$

for every $(\zeta,z),(\zeta',z')\in(D\times\overline{D})\cup(\overline{D}\times D)$, where $\overline{D}$ denotes the closure of $D$ in $E\times F_{\mathbb{C}}$ (note that conjugation on $E$ is *not* defined).

Then, by [27, Theorem 5.4] and [9, Theorem II.6] (cf., also, [16, Proposition 3.11]), the following result holds.

**Proposition 3.3** *If $\mathbf{s}\succ\frac{1}{4}\mathbf{m}$, then $A_{\mathbf{s}}^{2,2}(D)$ is a reproducing kernel Hilbert space with reproducing kernel*

$$K^{\mathbf{s}}\colon((\zeta,z),(\zeta',z'))\mapsto c_{\mathbf{s}}B^{\mathbf{b}+\mathbf{d}-2\mathbf{s}}_{(\zeta',z')}(\zeta,z)$$

*for a suitable $c_{\mathbf{s}}\neq 0$.*

In [16], for notational convenience, the corresponding integral operators are based on $B^{\mathbf{s}}$ rather than $K^{\mathbf{s}}$, so that the operators

$$P_{\mathbf{s}}\colon f \mapsto c_{(\mathbf{b}+\mathbf{d}-\mathbf{s})/2} \int_D f(\zeta, z) B^{\mathbf{s}}_{(\zeta,z)} \Delta_\Omega^{-\mathbf{s}}(\rho(\zeta, z))\, \mathrm{d}\nu_D(\zeta, z)$$

are considered.

The following result is a consequence of [16, Proposition 3.13]. It extends [33, Theorem 3.1], which corresponds to the case in which $D = \mathbb{C}_+$.

**Proposition 3.4** *Take $p, q \in (0, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$. If:*

- $\mathbf{s} \succ \frac{1}{p}(\mathbf{b}+\mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$;
- $\mathbf{s}' \prec \frac{1}{p'}(\mathbf{b}+\mathbf{d}) - \frac{1}{2p'}\mathbf{m}', \mathbf{b}+\mathbf{d} - \frac{1}{2}\mathbf{m}$;
- $\mathbf{s}+\mathbf{s}' \prec \frac{1}{\min(1,p)}(\mathbf{b}+\mathbf{d}) - \frac{1}{2q'}\mathbf{m}'$ *or* $q' = \infty$ *and* $\mathbf{s}+\mathbf{s}' \le \frac{1}{\min(1,p)}(\mathbf{b}+\mathbf{d})$;

*then $P_{\mathbf{s}'} f = f$ for every $f \in A^{p,q}_{\mathbf{s}}(D)$.*

### 3.3 Sampling

The following sampling theorem is a consequence of [16, Theorem 3.22], where a more precise version of this result is proved. We denote by $\ell^{p,q}(J, K)$ the space of $\lambda \in \mathbb{C}^{J\times K}$ such that $\|\|\lambda_{j,k}\|_{\ell^p(J)}\|_{\ell^q(K)} < \infty$, with some abuse of notation.

**Theorem 3.5** *Take $p, q \in (0, \infty]$, $\mathbf{s} \in \mathbb{R}^r$ and $R_0 > 1$. Then, there is $\delta_0 > 0$ such that, for every $\mathcal{N}$-$(\delta, R)$-lattice $(\zeta_{j,k}, z_{j,k})_{j\in J, k\in K}$ on $D$, with $\delta \in (0, \delta_0]$ and $R \in (1, R_0]$, the mapping*

$$f \mapsto \Delta_\Omega^{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}(\rho(\zeta_{j,k}, z_{j,k})) f(\zeta_{j,k}, z_{j,k})$$

*induces an isomorphism of $A^{p,q}_{\mathbf{s}}(D)$ onto a closed subspace of $\ell^{p,q}(J, K)$.*

Here we mention that the transpose of the sampling map defined above is often considered an atomic decomposition map, especially when the duals of $A^{p,q}_{\mathbf{s}}(D)$ and $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)$ may be identified with $A^{p',q'}_{\mathbf{s}'}(D)$ and $\mathfrak{A}^{p',q'}_{\mathbf{s}'}(D)$, respectively, for some $\mathbf{s}'$.

### 3.4 Atomic decomposition and duality

**Definition 3.6** Take $p, q \in (0, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$. Then, we say that property $(L)^{p,q}_{\mathbf{s},\mathbf{s}'}$ holds if for every $\delta_0 > 0$ there is an $\mathcal{N}$-$(\delta, 4)$-lattice $(\zeta_{j,k}, z_{j,k})_{j\in J, k\in K}$, with $\delta \in (0, \delta_0]$, such that, defining $h_k := \rho(\zeta_{j,k}, z_{j,k})$ for every $j \in J$ and for every $k \in K$, the mapping

$$\Psi_A : \lambda \mapsto \sum_{j,k} \lambda_{j,k} B^{\mathbf{s}'}_{(\zeta_{j,k}, z_{j,k})} \Delta_\Omega^{(\mathbf{b}+\mathbf{d})/p-\mathbf{s}-\mathbf{s}'}(h_k)$$

is well defined (with locally uniform convergence of the sum) and maps $\ell^{p,q}(J, K)$ into $A^{p,q}_{\mathbf{s}}(D)$ continuously.

If we may take $(\zeta_{j,k}, z_{j,k})_{j\in J, k\in K}$, for every $\delta_0 > 0$ as above, in such a way that the corresponding mapping $\Psi_A$ is onto, then we say that property $(L')^{p,q}_{\mathbf{s},\mathbf{s}'}$ holds.

The next result is a consequence of [16, Theorems 3.33 and 3.34]. This result was first proved in [22, Theorem 2] when $D$ is symmetric, $p = q < \infty$, and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}\mathbf{1}_r$.[3] See also [33, Theorem 1.5], which corresponds to the case in which $D$ is the upper half-plane.

**Theorem 3.7** *Take $p, q \in ]0, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that the following hold:*

- $\mathbf{s} \succ \frac{1}{2q}\mathbf{m} + \left(\frac{1}{2\min(1,p)} - \frac{1}{2q}\right)_+ \mathbf{m}'$;
- $\mathbf{s}' \prec \frac{1}{\min(1,p)}(\mathbf{b}+\mathbf{d}) - \frac{1}{2\min(1,p)}\mathbf{m}'$;
- $\mathbf{s}+\mathbf{s}' \prec \frac{1}{\min(1,p)}(\mathbf{b}+\mathbf{d}) - \frac{1}{2q}\mathbf{m}' - \left(\frac{1}{2\min(1,p)} - \frac{1}{2q}\right)_+ \mathbf{m}$;

*Then, property $(L')^{p,q}_{\mathbf{s},\mathbf{s}'}$ holds. More precisely, the mapping $\Psi_A$ of Definition 3.6 has a continuous linear section for $\delta$ sufficiently small and $R$ bounded.*

By [19, Corollary 4.7], properties $(L)^{p,q}_{\mathbf{s},\mathbf{s}'}$ and $(L')^{p,q}_{\mathbf{s},\mathbf{s}'}$ are actually equivalent when $p, q \in [1, \infty]$.

The following result is essentially a consequence of [19, Corollary 4.14]. It extends [33, Theorem 8.2], which deals with the case in which $D$ is the upper half-plane.

**Proposition 3.8** *Take $p, q \in (0, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that property $(L)^{p,q}_{\mathbf{s},\mathbf{s}'}$ holds. Denote by $V$ the closed vector subspace of $A^{p,q}_{\mathbf{s}}(D)$ generated by the $B^{\mathbf{s}'}_{(\zeta,z)}$, $(\zeta, z) \in D$. Then, the sesquilinear form*

$$A^{p,q}_{\mathbf{s}}(D) \times A^{p',q'}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}(D) \ni (f, g) \mapsto \int_D f\overline{g}(\Delta^{-\mathbf{s}'} \circ \rho)\, d\nu_D \in \mathbb{C}$$

*induces an antilinear isomorphism of $A^{p',q'}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}(D)$ onto the dual of $V$.*

If property $(L')^{p,q}_{\mathbf{s},\mathbf{s}'}$ holds and $p, q < \infty$, then $V = A^{p,q}_{\mathbf{s}}(D)$.

### 3.5 Boundary values

We now consider the problem of determining the boundary values of the spaces $A^{p,q}_{\mathbf{s}}(D)$. We recall some definitions and results from [16], with some slight changes motivated by [14, 15].

[3] Notice that the statement of [22, Theorem 2] is incorrect because of an erroneous computation of the $A^{p,p}_{\mathbf{s}}(D)$ norm of $B^{\mathbf{s}'}_{(\zeta,z)}$ in [22, Lemma 2.2].

**Definition 3.9** We define $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ as the space of CR $\phi \in \mathcal{S}(\mathcal{N})$ such that $\mathcal{F}_F(\phi(\zeta, \cdot))$ is supported in $\overline{\Omega'}$ for every $\zeta \in E$, endowed with the topology induced by $\mathcal{S}(\mathcal{N})$. We define $\mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$ as the dual of the conjugate of $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$. In addition, we define $\widetilde{\mathcal{S}}_{\overline{\Omega'}}(\mathcal{N})$ as the space of $\phi \in \mathcal{S}(\mathcal{N})$ such that $\pi_\lambda(\phi) = \chi_{\Omega'}(\lambda) P_{\lambda,0}\pi_\lambda(\phi)P_{\lambda,0}$ for every $\lambda \in F'\setminus W$. We define

$$\mathcal{F}_{\mathcal{N}} \colon \widetilde{\mathcal{S}}_{\overline{\Omega'}}(\mathcal{N}) \ni \phi \mapsto [\lambda \mapsto \operatorname{Tr}(\pi_\lambda(\phi))].$$

Notice that $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ may be equivalently defined as the set of $\phi \in \mathcal{S}(\mathcal{N})$ such that $\pi_\lambda(\phi) = \chi_{\Omega'}(\lambda)\pi_\lambda(\phi)P_{\lambda,0}$ for every $\lambda \in F'\setminus W$, thanks to [15, Proposition 2.17]. In addition, $\mathcal{F}_{\mathcal{N}}$ induces an isomorphism of $\widetilde{\mathcal{S}}_{\overline{\Omega'}}(\mathcal{N})$ onto the space of Schwartz functions on $F'$ supported in $\overline{\Omega'}$ (cf. [15, Proposition 5.2]).

**Definition 3.10** Take $p, q \in (0, \infty]$ and $\mathbf{s} \in \mathbb{R}^r$. We define $B^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ as the space of $u \in \mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$ such that

$$(\Delta^{\mathbf{s}}_{\Omega'}(\lambda_k) u * \psi_k) \in \ell^q(K; L^p(\mathcal{N})),$$

where $(\lambda_k)_{k\in K}$ is a $(\delta, R)$-lattice on $\Omega'$ and $(\psi_k)$ is a family of elements of $\widetilde{\mathcal{S}}_{\overline{\Omega'}}(\mathcal{N})$ such that $((\mathcal{F}_{\mathcal{N}}\psi_k)(\cdot\, t_k))$ is a bounded family of positive elements of $C^\infty_c(\Omega')$,[4] with $\lambda_k = e_{\Omega'} \cdot t_k$, and

$$\sum_k \mathcal{F}_{\mathcal{N}}\psi_k \geq 1$$

on $\Omega'$.

The definition of $B^{p,q}_{\mathbf{s}}(\mathcal{N}, \Omega)$ does not depend on the choice of $\delta$, $R$, $(\lambda_k)$, and $(\psi_k)$ (cf. [16, Lemma 4.14]). In addition, $B^{p,q}_{\mathbf{s}}(\mathcal{N}, \Omega)$ is a quasi-Banach space (cf. [16, Proposition 4.16] and [14, Proposition 7.12]).

Observe that, by [16, the remarks following the statement of Lemma 5.1], there is a constant $c > 0$ such that

$$f(\zeta, z) = c \int_{\mathcal{N}} f_0(\zeta, x) B^{\mathbf{b}+\mathbf{d}}_{(\zeta, x+i\Phi(\zeta))}(\zeta, z)\, \mathrm{d}(\zeta, x)$$

for every $f \in H^2(D) = A^{2,\infty}_{\mathbf{0}}(D)$ and for every $(\zeta, z) \in D$, where $f_0$ is the limit of $(f_h)$ in $L^2(\mathcal{N})$ for $h \to 0$, $h \in \Omega$. In other words,

$$S_{(\zeta,z)} := c \left(B^{\mathbf{b}+\mathbf{d}}_{(\zeta,z)}\right)_0$$

is (the boundary values of) the Cauchy–Szegő kernel.

The following result is a consequence of [16, Proposition 4.20, Theorem 4.23, and Lemma 5.1].

[4] Notice that this means that the $(\mathcal{F}_{\mathcal{N}}\psi_k)(\cdot\, t_k)$ are supported in a fixed compact subset of $\Omega'$ and are uniformly bounded with every derivative.

**Proposition 3.11** *Take $(\lambda_k)$ and $(\psi_k)$ as in Definition 3.10, in such a way that $\sum_k (\mathcal{F}_{\mathcal{N}}\psi_k)^2 = 1$ on $\Omega'$. Then, there is a continuous sesquilinear form*

$$\begin{aligned}\langle\cdot|\cdot\rangle : B^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega) \times B^{-\mathbf{s}-(1/p-1)_+(\mathbf{b}+\mathbf{d})}_{p',q'}(\mathcal{N}, \Omega) \ni (u, u')\\ \mapsto \sum_k \langle u * \psi_k | u' * \psi_k\rangle \in \mathbb{C}\end{aligned}$$

*which induces an antilinear isomorphism of $B^{-\mathbf{s}-(1/p-1)_+(\mathbf{b}+\mathbf{d})}_{p',q'}(\mathcal{N}, \Omega)$ onto the dual of the closure of $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ in $B^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$.*

*In addition, $S_{(\zeta,z)} \in B^{-\mathbf{s}-(1/p-1)_+(\mathbf{b}+\mathbf{d})}_{p',q'}(\mathcal{N}, \Omega)$ for every $(\zeta, z) \in D$ if $\mathbf{s} \succ \frac{1}{p}(\mathbf{b}+\mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$.*

**Definition 3.12** Given $\mathbf{s} \succ \frac{1}{p}(\mathbf{b}+\mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$, we define a continuous linear operator

$$\begin{aligned}\mathcal{E} : B^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega) \ni u \mapsto [(\zeta, z) \mapsto \langle u | S_{(\zeta,z)}\rangle]\\ \in A^{\infty,\infty}_{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}(D),\end{aligned}$$

and denote by $\widetilde{A}^{p,q}_{\mathbf{s}}(D)$ its image, endowed with the corresponding topology.

Notice that $(\mathcal{E}u)_h \to u$ in $\mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$ for $h \to 0$, $h \in \Omega'$, for every $u \in B^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ (cf. [16, Theorem 5.2] and [14, Proposition 7.13]), so that $\mathcal{E}$ is one-to-one and $B^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ is the space of boundary values of $\widetilde{A}^{p,q}_{\mathbf{s}}(D)$ (when defined).

The following result is a consequence of [16, Proposition 5.4 and Corollary 5.11]. Notice that the inclusion $\mathcal{E}(\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})) \subseteq A^{p,q}_{\mathbf{s}}(D)$ does not follow from [16, Proposition 5.4], but may be obtained by means of standard arguments, making use of [16, Corollaries 2.22 and 2.35].

**Theorem 3.13** *Take $p, q \in (0, \infty]$ and $\mathbf{s} \in \mathbb{R}^r$. If $\mathbf{s} \succ \frac{1}{p}(\mathbf{b}+\mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$ and either $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$ or $q = \infty$ and $\mathbf{s} \geq \mathbf{0}$, then*

$$\mathcal{E}(\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})) \subseteq A^{p,q}_{\mathbf{s}}(D) \subseteq \widetilde{A}^{p,q}_{\mathbf{s}}(D)$$

*continuously, with equality in the second inclusion if*

$$\mathbf{s} \succ \frac{1}{2q}\mathbf{m} + \left(\frac{1}{2\min(p, p')} - \frac{1}{2q}\right)_+ \mathbf{m}'.$$

We also have transference results (cf. [19, Theorems 6.1 and 6.3]).

**Proposition 3.14** *Take $p, q \in (0, \infty]$ and $\mathbf{s} \in \mathbb{R}^r$. Then the following hold:*

(1) *if $\mathbf{s} \succ \frac{1}{p}\mathbf{d} + \frac{1}{2p'}\mathbf{m}'$ and $A^{p,p}_{\mathbf{s}}(T_\Omega) = \widetilde{A}^{p,p}_{\mathbf{s}}(T_\Omega)$, then $A^{p,p}_{\mathbf{s}}(D) = \widetilde{A}^{p,p}_{\mathbf{s}}(D)$;*

(2) *if* $\mathbf{s} \succ \frac{1}{p}(\mathbf{b}+\mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$ *and* $A^{p,q}_{\mathbf{s}}(D) = \widetilde{A}^{p,q}_{\mathbf{s}}(D)$, *then* $A^{p,q}_{\mathbf{s}-\mathbf{b}/p}(T_\Omega) = \widetilde{A}^{p,q}_{\mathbf{s}-\mathbf{b}/p}(T_\Omega)$.

Notice that, in assertion (1) above, we consider only pure norm spaces; an analogue for mixed-norm spaces holds for the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$, as we shall see below (cf. Proposition 4.26).

### 3.6 Bergman Projectors

Concerning the boundedness of Bergman projectors, we have the following results (cf. [16, Proposition 5.20]).

**Proposition 3.15** *Take* $p, q \in [1,\infty]$ *and* $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$. *If* $\mathbf{s}' \prec \mathbf{b}+\mathbf{d} - \frac{1}{2}\mathbf{m}$ *and* $P_{\mathbf{s}'}$ *induces a continuous linear projector of* $L^{p,q}_{\mathbf{s}}(D)$ *onto* $A^{p,q}_{\mathbf{s}}(D)$, *then:*

- $\mathbf{s} \succ \frac{1}{p}(\mathbf{b}+\mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$, *and* $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$ *or* $q = \infty$ *and* $\mathbf{s} \geq \mathbf{0}$;
- $\mathbf{s}' \prec \frac{1}{\min(p,p')}(\mathbf{b}+\mathbf{d} - \frac{1}{2}\mathbf{m}')$;
- $\mathbf{s}+\mathbf{s}' \prec \frac{1}{p}(\mathbf{b}+\mathbf{d}) - \frac{1}{2q}\mathbf{m}'$, *and* $\mathbf{s}+\mathbf{s}' \prec \mathbf{b}+\mathbf{d} - \frac{1}{2q'}\mathbf{m}$ *or* $q = 1$ *and* $\mathbf{s}+\mathbf{s}' \leq \mathbf{b}+\mathbf{d}$.

There are also transference results (cf. [19, Corollary 4.7 and Theorems 6.1 and 6.3]).

**Proposition 3.16** *Take* $p, q \in [1,\infty]$ *and* $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ *such that* $\mathbf{s}' \prec \mathbf{b}+\mathbf{d} - \frac{1}{2}\mathbf{m}$. *Then, the following hold:*

- *if* $P_{\mathbf{s}'}$ *induces a continuous linear projector of* $L^{p,q}_{\mathbf{s}}(D)$ *onto* $A^{p,q}_{\mathbf{s}}(D)$, *then* $P_{\mathbf{s}'}$ *induces a continuous linear projector of* $L^{p,q}_{\mathbf{s}-\mathbf{b}/p}(T_\Omega)$ *onto* $A^{p,q}_{\mathbf{s}-\mathbf{b}/p}(T_\Omega)$;
- *if* $P_{\mathbf{s}'-\mathbf{b}}$ *induces a continuous linear projector of* $L^{p,p}_{\mathbf{s}}(T_\Omega)$ *onto* $A^{p,p}_{\mathbf{s}}(T_\Omega)$, *then* $P_{\mathbf{s}'}$ *induces a continuous linear projector of* $L^{p,p}_{\mathbf{s}}(D)$ *onto* $A^{p,p}_{\mathbf{s}}(D)$.

Notice that, in the second assertion of the preceding result, we consider only pure norm spaces; an analogue for mixed-norm spaces holds for the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$, as we shall see below (cf. Corollary 4.33).

The following result is a consequence of [16, Corollary 5.27] (cf. also [19, Corollary 4.7]).

**Theorem 3.17** *Take* $p, q \in [1,\infty]$ *and* $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$. *If:*

- $\mathbf{s} \succ \frac{1}{2q}\mathbf{m} + \left(\frac{1}{2\min(p,p')} - \frac{1}{2q}\right)_+ \mathbf{m}'$;
- $\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}' \succ \frac{1}{2q'}\mathbf{m} + \left(\frac{1}{2\min(p,p')} - \frac{1}{2q'}\right)_+ \mathbf{m}'$;

*then* $P_{\mathbf{s}'}$ *induces a continuous linear projector of* $L^{p,q}_{\mathbf{s}}(D)$ *onto* $A^{p,q}_{\mathbf{s}}(D)$.

## 4 The Spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$

In this section, we consider the spaces

$$\mathfrak{A}^{p,q}_{\mathbf{s}}(D) = \{f \in \mathrm{Hol}(D) \colon \|\zeta \mapsto \|f^{(\zeta)}\|_{A^{p,q}_{(\mathbf{s}+\mathbf{b}/2)/q}(T_\Omega)}\|_{L^q(E)} < \infty\},$$

and prove the appropriate analogues of the results of the previous section valid in the case of the spaces $A^{p,q}_{\mathbf{s}}(D)$.

In order to deal with the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)$, we introduce some auxiliary spaces, namely the spaces

$$\mathcal{A}^{p,q}_{\mathbf{s}}(D) := \{f \in \mathrm{Hol}(D) \colon \|\zeta \mapsto \|f^{(\zeta)}\|_{A^{p,q}_{\mathbf{s}}(T_\Omega)}\|_{L^q(E)} < \infty\}$$

and

$$\mathcal{A}^{p,q}_{\mathbf{s},0}(D) := \mathrm{Hol}\,(D) \cap \mathcal{L}^{p,q}_{\mathbf{s},0}(D),$$

where $\mathcal{L}^{p,q}_{\mathbf{s},0}(D)$ denotes the closure of $C_c(D)$ in $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$ (defined as $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ replacing $\mathrm{Hol}(D)$ with the space of measurable functions modulo negligible functions).

The reason for the introduction of these spaces lies in the fact that they are somewhat similar to the spaces $A^{p,q}_{\mathbf{s}}$ and enjoy similar technical advantages, but may still be easily related to the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$ by means of the simple equality

$$\mathfrak{A}^{p,q}_{\mathbf{s}}(D) = \mathcal{A}^{p,q}_{(\mathbf{s}+\mathbf{b}/2)/q}(D),$$

which holds for every $p$, $q$ and $\mathbf{s}$, as one may readily see from the definitions.

We also observe that the treatment of the smaller spaces $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ is necessary for a reasonably comprehensive treatment of duality, since in general only the dual of $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ may be reasonably described. Since duality cannot be comprehensively studied using only the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$, defining an analogous space $\mathfrak{A}^{p,q}_{\mathbf{s},0}$ seems superfluous (and would only be of use when $p = \infty$ and $q < \infty$, since $\mathfrak{A}^{p,\infty}_{\mathbf{s},0}(D) = \{0\}$ for every $p \in (0,\infty]$, thanks to Proposition 4.5). Notice that analogous spaces $A^{p,q}_{\mathbf{s},0}(D)$ have also been considered in [16, 19]. Since Sect. 3 is essentially a summary of [16, 19], for the sake of simplicity we avoided the introduction of the spaces $A^{p,q}_{\mathbf{s},0}(D)$. On the contrary, we believe that the proofs of this section will benefit from the parallel treatment of the spaces $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ and $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$.

### 4.1 Elementary Properties

We begin our treatment of the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$ and $\mathcal{A}^{p,q}_{\mathbf{s}}$ by a direct comparison with the spaces $A^{p,q}_{\mathbf{s}}$.

**Lemma 4.1** *Take* $p, q \in (0,\infty]$ *and* $\mathbf{s} \in \mathbb{R}^r$. *If either* $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$, *or* $q = \infty$ *and* $\mathbf{s} \geq \mathbf{0}$, *then*

$$A^{p,q}_{\mathbf{s}}(D) = \mathcal{A}^{p,q}_{\mathbf{s}}(D)$$

*if and only if* $p=q$ *or* $E=\{0\}$. *In particular, if* $\mathbf{s}\succ\frac{1}{2}(\mathbf{m}-\mathbf{b})$ *or* $q=\infty$, *then*

$$\mathfrak{A}^{p,q}_{\mathbf{s}}(D)=A^{p,q}_{(\mathbf{s}+\mathbf{b}/2)/q}(D)$$

*if and only if* $p=q$ *or* $E=\{0\}$.

Note that, as the proof (combined with Proposition 4.5) shows, if $p\neq q$ and $E\neq\{0\}$, then $\mathcal{A}^{p,q}_{\mathbf{s}}(D)\not\subseteq A^{p,q}_{\mathbf{s}}(D)$ and $\mathcal{A}^{p,q}_{\mathbf{s}}(D)\not\supseteq A^{p,q}_{\mathbf{s}}(D)$.

***Proof*** It is clear that $A^{p,q}_{\mathbf{s}}(D)=\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ if $p=q$ (by Fubini's theorem) and if $E=\{0\}$. Conversely, assume that $p\neq q$ and that $E\neq\{0\}$, so that $\mathbf{b}\neq\mathbf{0}$. Observe that, given $t\in T_+$ and $g\in GL(E)$ such that $t\cdot\Phi=\Phi\circ(g\times g)$ and $f\in\mathrm{Hol}(D)$, one has

$$\|f\circ(g\times t)\|_{A^{p,q}_{\mathbf{s}}(D)}=\Delta^{(\mathbf{b}+\mathbf{d})/p-\mathbf{s}}(t)\|f\|_{A^{p,q}_{\mathbf{s}}(D)}$$

and

$$\|f\circ(g\times t)\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)}=\Delta^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}}(t)\|f\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)}$$

so that, letting $t\to\infty$, we see that the norms on the quasi-Banach spaces $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ and $A^{p,q}_{\mathbf{s}}(D)$ cannot be comparable. The assertion follows from the open mapping and the closed graph theorems, since both $A^{p,q}_{\mathbf{s}}(D)$ and $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ are quasi-Banach spaces (cf. Remark 3.1 and Proposition 4.5). □

The second assertion of the following result extends: [3, Proposition 3.22], which corresponds to the case in which $p_1,q_1\geq 1$, $q_2=\infty$, $n=0$, $\mathbf{s}\in\mathbb{R}\mathbf{1}_r$, and $\Omega$ is symmetric; [23, Proposition 2.3], which corresponds to the case in which $p_1,q_1\geq 1$, $q_2=\infty$, $n=0$, and $\Omega$ is symmetric; [30, Lemma 5.2], which corresponds to the case in which $p_1,q_1\geq 1$, $q_2=\infty$, and $n=0$.

**Proposition 4.2** *Take* $p_1,p_2,q_1,q_2\in(0,\infty]$ *and* $\mathbf{s_1},\mathbf{s_2}\in\mathbb{R}^r$. *If*

$$p_1\leq p_2,\qquad q_1\leq q_2,\qquad \text{and}\qquad \mathbf{s_2}=\mathbf{s_1}+\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\mathbf{d}+\left(\frac{1}{q_2}-\frac{1}{q_1}\right)\mathbf{b},$$

*then* $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1}}(D)\subseteq\mathcal{A}^{p_2,q_2}_{\mathbf{s_2}}(D)$ *and* $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1},0}(D)\subseteq\mathcal{A}^{p_2,q_2}_{\mathbf{s_2},0}(D)$. *In addition, the mappings* $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1}}(D)\ni f\mapsto f^{(\zeta)}\in\mathcal{A}^{p_1,q_1}_{\mathbf{s_1}-\mathbf{b}/q_1}(T_\Omega)$, *as* $\zeta$ *runs through* $E$, *are equicontinuous and map* $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1},0}(D)$ *into* $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1}-\mathbf{b}/q_1,0}(T_\Omega)$.

*In particular,* $\mathfrak{A}^{p_1,q_1}_{\mathbf{s_1}}(D)\subseteq\mathfrak{A}^{p_2,q_2}_{\mathbf{s_2}}(D)$ *continuously, provided that* $q_2<\infty$ *and* $\mathbf{s_2}=\frac{q_2}{q_1}\mathbf{s_1}+\left(\frac{q_2}{p_2}-\frac{q_2}{p_1}\right)\mathbf{d}+\left(\frac{1}{2}-\frac{q_2}{2q_1}\right)\mathbf{b}$. *In addition, the mappings* $\mathfrak{A}^{p_1,q_1}_{\mathbf{s_1}}(D)\ni f\mapsto f^{(\zeta)}\in\mathfrak{A}^{p_1,q_1}_{\mathbf{s_1}-\mathbf{b}/2}(T_\Omega)$, *as* $\zeta$ *runs through* $E$, *are equicontinuous.*

Before we pass to the proof, we need an analogue of [16, Lemma 3.26].

**Lemma 4.3** *There are* $R'_0>0$ *and a constant* $C>0$ *such that, for every* $p,q\in(0,\infty]$, *for every* $R'\in(0,R'_0]$, *for every* $f\in\mathrm{Hol}(D)$ *and for every* $(\zeta,h)\in E\times\Omega$,

$$\|f^{(\zeta)}_h\|_{L^p(F)}\leq C^{1/\min(1,p,q)}\left(\fint_{B_{E\times\Omega}((\zeta,h),R')}\|f^{(\zeta')}_{h'}\|^q_{L^p(F)}\,\mathrm{d}\nu_{E\times\Omega}(\zeta',h')\right)^{1/q}$$

*(modification if* $q=\infty$*).*

***Proof*** Set $\ell:=\min(1,p,q)$ to simplify the notation. By [16, Lemma 3.24], there are $R_0>0$ and $C'>0$ such that

$$|f(\zeta,z)|^\ell\leq C'\fint_{B((\zeta,z),R)}|f|^\ell\,\mathrm{d}\nu_D$$

for every $f\in\mathrm{Hol}(D)$, for every $(\zeta,z)\in D$, and for every $R\in(0,R_0]$. Then, applying Minkowski's integral inequality (with exponent $\frac{p}{\ell}$) and Young's inequality,

$$\begin{aligned}\|f^{(\zeta)}_h\|^\ell_{L^p(F)}&\leq C'C'_R\fint_{B_{E\times\Omega}((\zeta,h),R)}\left\||f^{(\zeta')}_{h'}|^\ell*[(\chi_{B((\zeta,i\Phi(\zeta)+ih),R)})^{(\zeta')}_{h'}]\check{}\right\|_{L^{p/\ell}(F)}\\&\quad\times\Delta^{\mathbf{d}}_\Omega(h')\,\mathrm{d}\nu_{E\times\Omega}(\zeta',h')\\&\leq C''\fint_{B_{E\times\Omega}((\zeta,h),R)}\|f^{(\zeta')}_{h'}\|^\ell_{L^p(F)}\frac{\Delta^{\mathbf{d}}_\Omega(h')}{\Delta^{\mathbf{d}}_\Omega(h)}\,\mathrm{d}\nu_{E\times\Omega}(\zeta',h')\end{aligned}\tag{3}$$

for every $f\in\mathrm{Hol}(D)$ and for every $h\in\Omega$, where

$$C'_R:=\frac{\nu_{E\times\Omega}(B_{E\times\Omega}((0,e_\Omega),R))}{\nu_D(B((0,ie_\Omega),R))}$$

and

$$C'':=C'\sup_{0<R\leq R_0}\sup_{(\zeta',h')\in E\times\Omega}C'_R\left\|(\chi_{B((0,ie_\Omega),R)})^{(\zeta')}_{h'}\right\|_{L^1(F)}.$$

By [16, Corollary 2.49], there is a constant $C>0$ such that, for every $f\in\mathrm{Hol}(D)$ and for every $h\in\Omega$,

$$\|f^{(\zeta)}_h\|^\ell_{L^p(F)}\leq C\fint_{B_{E\times\Omega}((\zeta,h),R)}\|f^{(\zeta')}_{h'}\|^\ell_{L^p(F)}\,\mathrm{d}\nu_{E\times\Omega}(\zeta',h'),$$

if $R\in(0,R_0]$. Then, Jensen's inequality (with exponent $\frac{q}{\ell}$) leads to the first inequality. □

***First part of the proof of Proposition 4.2*** Step I. Let us first show that there are $R,C_1>0$ such that

$$(\Delta^{\mathbf{s_2}}_\Omega(h)\|f_h\|_{L^{p_2,q_2}(F,E)})^\ell\leq C_1\fint_{B_\Omega(h,R)}(\Delta^{\mathbf{s_1}}_\Omega(h')\|f_{h'}\|_{L^{p_1,q_1}(F,E)})^\ell\,\mathrm{d}\nu_\Omega(h')\tag{4}$$

for every $f \in \mathrm{Hol}(D)$ and for every $h \in \Omega$, where $\ell := \min(1, p_1, q_1)$. By homogeneity, it will suffice to prove (4) for $h = e_\Omega$. Observe that, by [16, Lemma 3.24], there are $R > 0$ and $C_2 > 0$ such that

$$|f(\zeta, z)|^\ell \leq C_2 \fint_{B((\zeta,z),R)} |f|^\ell \, d\nu_D$$

for every $f \in \mathrm{Hol}(D)$ and for every $(\zeta, z) \in D$. Then, applying Minkowski's integral inequality and Young's inequality,[5]

$$\begin{aligned} \|f_{e_\Omega}\|^\ell_{L^{p_2,q_2}(F,E)} &\leq C_2' \fint_{B_\Omega(e_\Omega,R)} \left\| |f_h|^\ell * [(\chi_{B((0,ie_\Omega),R)})_h]^\check{} \right\|_{L^{p_2/\ell,q_2/\ell}(F,E)} \\ &\quad \times \Delta_\Omega^{\mathbf{b}+\mathbf{d}}(h) d\nu_\Omega(h) \\ &\leq C_3 \fint_{B_\Omega(e_\Omega,R)} \|f_h\|^\ell_{L^{p_1,q_1}(F,E)} \, d\nu_\Omega(h) \end{aligned}$$

for every $f \in \mathrm{Hol}(D)$ and for every $h \in \Omega$, where $C_2' = \nu_\Omega(B_\Omega(e_\Omega, R))/\nu_D(B((0, ie_\Omega), R))$ and

$$C_3 := C_2' \sup_{h \in B_\Omega(e_\Omega,R)} \Delta_\Omega^{\mathbf{b}+\mathbf{d}}(h) \|(\chi_{B((0,ie_\Omega),R)})_h\|_{L^{p_3,q_3}(F,E)}$$

and $p_3, q_3 \in [1, \infty]$ are defined so that $1 + \frac{\ell}{p_2} = \frac{\ell}{p_1} + \frac{1}{p_3}$ and $1 + \frac{\ell}{q_2} = \frac{\ell}{q_1} + \frac{1}{q_3}$.

STEP II. Applying the $L^{q_2/\ell}(\nu_\Omega)$ norm to (4) and using Jensen's inequality, we see that

$$\|f\|^\ell_{\mathcal{A}^{p_2,q_2}_{\mathbf{s}_2}(D)} \leq C_1 \|f\|^\ell_{\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1}(D)}$$

for every $f \in \mathrm{Hol}(D)$, whence the inclusion $\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1}(D) \subseteq \mathcal{A}^{p_2,q_2}_{\mathbf{s}_2}(D)$.

STEP III. For what concerns the equicontinuity of the mappings $\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1}(D) \ni f \mapsto f^{(\zeta)} \in A^{p_1,q_1}_{\mathbf{s}_1-\mathbf{b}/q_1}(T_\Omega)$, as $\zeta$ runs through $E$, observe that this is obvious when $q_1 = \infty$. Then, assume that $q_1 < \infty$ and observe that, by Lemma 4.3, there are $R', C_4 > 0$ such that

$$\begin{aligned} &\|f^{(\zeta)}_h\|_{L^{p_1}(F)} \\ &\quad \leq C_4 \left( \int_{B_{E\times\Omega}((\zeta,h),R')} \|f^{(\zeta')}_{h'}\|^{q_1}_{L^{p_1}(F)} \, d\nu_{E\times\Omega}(\zeta', h') \right)^{1/q_1} \end{aligned}$$

for every $f \in \mathrm{Hol}(D)$ and for every $(\zeta, h) \in E \times \Omega$. By [16, Corollary 2.49], there is a constant $C_5 > 0$ such that

$$\begin{aligned} &\Delta_\Omega^{\mathbf{s}_1-\mathbf{b}/q_1}(h) \|f^{(\zeta)}_h\|_{L^{p_1}(F)} \\ &\quad \leq C_5 \left( \int_{B_{E\times\Omega}((\zeta,h),R')} \Delta_\Omega^{q_1\mathbf{s}_1-\mathbf{b}}(h') \|f^{(\zeta')}_{h'}\|^{q_1}_{L^{p_1}(F)} \, d\nu_{E\times\Omega}(\zeta', h') \right)^{1/q_1} \end{aligned}$$

for every $f \in \mathrm{Hol}(D)$ and for every $(\zeta, h) \in E \times \Omega$. Applying the $L^{q_1}(\nu_\Omega)$ norm, we then see that

$$\begin{aligned} &\|f^{(\zeta)}\|_{A^{p_1,q_1}_{\mathbf{s}_1-\mathbf{b}/q_1}(T_\Omega)} \\ &\quad \leq C_5 \bigg( \int_E \int_\Omega \int_\Omega \chi_{B_{E\times\Omega}((\zeta',h'),R')}(\zeta, h) \, d\nu_\Omega(h) \\ &\qquad \times (\Delta_\Omega^{\mathbf{s}_1}(h') \|f^{(\zeta')}_{h'}\|_{L^{p_1}(F)})^{q_1} \, d\nu_\Omega(h') \, d\zeta' \bigg)^{1/q_1} \\ &\quad \leq C_6 \|f\|_{\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1}(D)} \end{aligned}$$

for every $f \in \mathrm{Hol}(D)$ and for every $\zeta \in E$, where $C_6 := C_5 \nu_\Omega(B_\Omega(e_\Omega, R'))^{1/q_1}$. □

**Corollary 4.4** *Take $p, q \in (0, \infty]$, $\mathbf{s} \in \mathbb{R}^r$, and $f \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$. Then, the function $h \mapsto \|f^{(\zeta)}_h\|_{L^p(F)}$ is decreasing (for the order induced by $\overline{\Omega}$) for every $\zeta \in E$.*

***Proof*** This follows from Proposition 4.2 and [16, Corollary 3.3]. □

The second assertion of the following result extends: [9, Corollary II.3], which corresponds to the case in which $p = q$; [3, Proposition 3.8], which corresponds to the case in which $p = q \in [1, \infty)$, $n = 0$, $\mathbf{s} \in \mathbb{R}\mathbf{1}_r$, and $\Omega$ is symmetric; [23, Theorem 2.15], which corresponds to the case in which $p \in [1, \infty]$, $q \in [1, \infty)$, $n = 0$, and $\Omega$ is symmetric.

**Proposition 4.5** *Take $p, q \in (0, \infty]$ and $\mathbf{s} \in \mathbb{R}^r$. Then, $\mathcal{A}^{p,q}_{\mathbf{s},0}(D) \neq \{0\}$ (resp. $\mathcal{A}^{p,q}_{\mathbf{s}}(D) \neq \{0\}$) if and only if $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$ (resp. $\mathbf{s} \geq \mathbf{0}$ if $q = \infty$). In particular, $\mathfrak{A}^{p,q}_{\mathbf{s}}(D) \neq \{0\}$ if and only if $\mathbf{s} \succ \frac{1}{2}(\mathbf{m} - \mathbf{b})$ or $q = \infty$.*

*In addition, $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$, $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$, and $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)$ are quasi-Banach spaces.*

***Proof*** Let $(\lambda_1, \ldots, \lambda_m)$ be a basis of $F'$ with elements in $\Omega'$. Observe first that the function

$$g^{(\varepsilon)} : D \ni (\zeta, z) \mapsto \exp\left( -\varepsilon \cos(\alpha\pi) \sum_j \langle (\lambda_j)_{\mathbb{C}}, z \rangle^\alpha \right) \in \mathbb{C} \tag{5}$$

is well defined for every $\varepsilon > 0$ and $\alpha \in (0, 1/2)$ and satisfies estimates of the form

$$|g^{(\varepsilon)}(\zeta, z)| \leq e^{-C\varepsilon(|\zeta|^{2\alpha} + |\mathrm{Re}\, z|^\alpha + |\rho(\zeta,z)|^\alpha)} \tag{6}$$

for every $\varepsilon > 0$ and for every $(\zeta, z) \in D$ and for a suitable $C$ (depending only on $\alpha$), thanks to [16, Lemma 1.22] (cf., also, [31, Lemma 8.1]). It is then readily verified that, for every $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that $\|g^{(\varepsilon)}_h\|_{L^{p,q}(F,E)} \leq C_\varepsilon e^{-C\varepsilon|h|^\alpha}$ for every $h \in \Omega$, so that $g^{(\varepsilon)}$ belongs to $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ for $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$ and to $\mathcal{A}^{p,\infty}_{\mathbf{s}}(D)$ for every $\mathbf{s} \geq 0$, thanks to [16, Proposition 2.19 and Lemma 2.34].

[5] Notice that Young's inequality may be applied to the spaces $L^{p,q}(F, E)$ since $F$ is a normal subgroup of $\mathcal{N}$ and $E$ may be identified with $\mathcal{N}/F$.

Conversely, if $q < \infty$ and $f \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$, then the mapping $h \mapsto \|f_h\|_{L^{p,q}(F,E)}$ is decreasing for the ordering induced by $\overline{\Omega}$, for every $\zeta \in E$, by Corollary 4.4. Since $\Delta^{q\mathbf{s}}_{\Omega} \cdot \nu_{\Omega}$ does *not* induce a Radon measure on $\overline{\Omega}$ unless $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$ by [16, Proposition 2.19], this proves that $f = 0$ unless $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$. If, otherwise, $f \in \mathcal{A}^{p,\infty}_{\mathbf{s},0}(D)$ or $f \in \mathcal{A}^{p,\infty}_{\mathbf{s}}(D)$, then it is clear that $f^{(\zeta)} \in \mathcal{A}^{p,\infty}_{\mathbf{s},0}(T_\Omega)$ and $f \in \mathcal{A}^{p,\infty}_{\mathbf{s}}(T_\Omega)$, respectively, for every $\zeta \in E$, so that $f = 0$ unless $\mathbf{s} \succ \mathbf{0}$ and $\mathbf{s} \geq \mathbf{0}$, respectively, by [16, Proposition 3.5].

In order to show that $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ is a quasi-Banach space, it suffices to observe that there are continuous inclusions $\mathcal{A}^{p,q}_{\mathbf{s}}(D) \subseteq \mathcal{A}^{\infty,\infty}_{\mathbf{s}-\mathbf{b}/q-\mathbf{d}/p}(D) \subseteq \mathrm{Hol}(D)$, by Proposition 4.2, and that the quasi-norm of $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ extends to a lower semi-continuous function on $\mathrm{Hol}(D)$ which is finite only on $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$. □

The second assertion of the following result extends: [3, Theorem 3.23], which corresponds to the case in which $p_1, p_2, q_1, q_2 \geq 1$, $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}\mathbf{1}_r$, $n = 0$, and $\Omega$ is symmetric; [23, Proposition 2.14], which corresponds to the case in which $p_1, p_2, q_1, q_2 \geq 1$, $n = 0$, and $\Omega$ is symmetric; [6, Lemma 3.6], which corresponds to the case in which $p_1, q_1 \geq 1$ and $p_2 = q_2 = 2$; [29, Lemma 4.3], which corresponds to the case in which $p_1, p_2, q_1, q_2 \geq 1$.

**Proposition 4.6** *Take $p_1, p_2, q_1, q_2 \in (0, \infty]$, and $\mathbf{s_1}, \mathbf{s_2} \in \mathbb{R}^r$. If $\mathcal{A}^{p_2,q_2}_{\mathbf{s_2}}(D) \neq \{0\}$, then $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1},0}(D) \cap \mathcal{A}^{p_2,q_2}_{\mathbf{s_2}}(D)$ is dense in $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1},0}(D)$ (resp. $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1}}(D) \cap \mathcal{A}^{p_2,q_2}_{\mathbf{s_2}}(D)$ is dense in $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1}}(D)$ for the weak topology $\sigma(\mathcal{A}^{p_1,q_1}_{\mathbf{s_1}}(D), \mathcal{L}^{p_1',q_1'}_{-\mathbf{s_1}+(1/q_1-1)_+\mathbf{b}+(1/p_1-1)_+\mathbf{d}}(D))$). Analogous assertions hold with $\mathcal{A}^{p_2,q_2}_{\mathbf{s_2},0}(D)$ in place of $\mathcal{A}^{p_2,q_2}_{\mathbf{s_2}}(D)$.*

*In particular, if $\mathfrak{A}^{p_2,q_2}_{\mathbf{s}_2}(D) \neq \{0\}$ and $p_1, q_1 < \infty$, then $\mathfrak{A}^{p_1,q_1}_{\mathbf{s}_1}(D) \cap \mathfrak{A}^{p_2,q_2}_{\mathbf{s}_2}(D)$ is dense in $\mathfrak{A}^{p_1,q_1}_{\mathbf{s}_1}(D)$.*

***Proof*** Define $g^{(\varepsilon)}$ as in the proof of Proposition 4.5, for some $\alpha \in (0, 1/2)$. Take $f \in \mathcal{A}^{p_1,q_1}_{\mathbf{s_1},0}(D)$. Using the estimates (6), Proposition 4.2, and Corollary 4.4, one may show that $f(\cdot + ih)g^{(\varepsilon)}$ belongs to $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1},0}(D) \cap \mathcal{A}^{p_2,q_2}_{\mathbf{s_2}}(D)$ for every $h \in \Omega$ and for every $\varepsilon > 0$. By dominated convergence one then shows that $f(\cdot + ih)g^{(\varepsilon)}$ converges to $f(\cdot + ih)$ in $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1},0}(D)$ for $\varepsilon \to 0^+$, for every $h \in \Omega$. Now, observe that the mapping $\Omega \ni h \mapsto (fg^{(\varepsilon)})_h \in L^{p_1,q_1}(F,E)$ is continuous by Corollary 4.4 and dominated convergence, so that the mapping $\Omega \ni h \mapsto f_h \in L^{p_1,q_1}(F,E)$ is continuous by the arbitrariness of $\varepsilon > 0$ and the previous arguments (combined with Corollary 4.4 again). Using this fact and Corollary 4.4, we then see that $f(\cdot + ih)$ converges to $f$ in $\mathcal{A}^{p_1,q_1}_{\mathbf{s_1},0}(D)$ for $h \to 0$. In a similar way, one deals with the other cases. □

**End of the proof of Proposition 4.2** The inclusion $\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1,0}(D) \subseteq \mathcal{A}^{p_2,q_2}_{\mathbf{s}_2,0}(D)$ follows from the continuity of the inclusion $\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1}(D) \subseteq \mathcal{A}^{p_2,q_2}_{\mathbf{s}_2}(D)$ and the density of $\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1,0}(D) \cap \mathcal{A}^{p_2,q_2}_{\mathbf{s}_2,0}(D)$ in $\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1,0}(D)$ (cf. Proposition 4.6).

The fact that the mappings $f \mapsto f^{(\zeta)}$, $\zeta \in E$, map $\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1,0}(D)$ into $\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1-\mathbf{b}/q_1,0}(T_\Omega)$ is proved similarly, using the density of $\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1,0}(D) \cap \mathcal{A}^{\min(1,p_1),\min(1,q_1)}_{\mathbf{s}_1}(D)$ in $\mathcal{A}^{p_1,q_1}_{\mathbf{s}_1,0}(D)$ and observing that $\mathcal{A}^{\min(1,p_1),\min(1,q_1)}_{\mathbf{s}_1-\mathbf{b}/\min(1,q_1)}(T_\Omega) \subseteq \mathcal{A}^{p_1,q_1}_{\mathbf{s}_1-\mathbf{b}/q_1,0}(T_\Omega)$ by the previous remarks (cf. Proposition 4.6). □

## 4.2 Reproducing kernels

Notice that $\mathcal{A}^{2,2}_{\mathbf{s}}(D) = A^{2,2}_{\mathbf{s}}(D)$, so that $P_{\mathbf{b}+\mathbf{d}-2\mathbf{s}}$ is the orthogonal projector of $\mathcal{L}^{2,2}_{\mathbf{s}}(D)$ onto $\mathcal{A}^{2,2}_{\mathbf{s}}(D)$. Analogously, if $\mathbf{s} \succ \frac{1}{2}\mathbf{m}$, then $\mathfrak{A}^{2,2}_{\mathbf{s}}(D) = A^{2,2}_{(\mathbf{s}+\mathbf{b}/2)/2}(D)$ is a reproducing kernel Hilbert space with reproducing kernel (cf. Proposition 3.3)

$$\mathfrak{K}^{\mathbf{s}} \colon ((\zeta, z), (\zeta', z')) \mapsto c_{\mathbf{s}/2+\mathbf{b}/4} B^{\mathbf{b}/2+\mathbf{d}-\mathbf{s}}_{(\zeta',z')}(\zeta, z).$$

Then,

$$\mathfrak{P}_{\mathbf{s}} = P_{\mathbf{b}/2+\mathbf{d}-\mathbf{s}} : f \mapsto c_{\mathbf{s}/2+\mathbf{b}/4} \times \int_D f(\zeta, z) \mathfrak{K}^{\mathbf{s}}_{(\zeta,z)} \Delta^{\mathbf{s}+\mathbf{b}/2+\mathbf{d}}_{\Omega}(\rho(\zeta, z)) \,\mathrm{d}(\zeta, z)$$

is the orthoprojector of $\mathfrak{L}^{2,2}_{\mathbf{s}}(D)$ onto $\mathfrak{A}^{2,2}_{\mathbf{s}}(D)$.

**Proposition 4.7** *Take $p, q \in (0, \infty]$, $\mathbf{s} \in \mathbb{R}^r$, and $\mathbf{s}' \in \mathbb{C}^r$. Then, $B^{\mathbf{s}'}_{(\zeta,z)} \in \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ (resp. $B^{\mathbf{s}'}_{(\zeta,z)} \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$) for some/every $(\zeta, z) \in D$ if and only if the following conditions hold:*

- *$\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$ (resp. $\mathbf{s} \geq \mathbf{0}$ if $q = \infty$);*
- *$\operatorname{Re}\mathbf{s}' \prec \frac{1}{p}\mathbf{d} - \frac{1}{2p}\mathbf{m}'$ (resp. $\mathbf{s}' \leq \mathbf{0}$ if $p = \infty$);*
- *$\mathbf{s} + \operatorname{Re}\mathbf{s}' \prec \frac{1}{q}\mathbf{b} + \frac{1}{p}\mathbf{d} - \frac{1}{2q}\mathbf{m}'$ (resp. $\mathbf{s} + \mathbf{s}' \leq \frac{1}{p}\mathbf{d}$ if $q = \infty$).*

*In this case,*

$$\|B^{\mathbf{s}'}_{(\zeta,z)}\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)} = \|B^{\mathbf{s}'}_{(0,ie_\Omega)}\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)} \Delta^{\mathbf{s}+\operatorname{Re}\mathbf{s}'-\mathbf{b}/q-\mathbf{d}/p}_{\Omega}(\rho(\zeta, z))$$

*for every $(\zeta, z) \in D$.*

***Proof*** Set $f := B^{\mathbf{s}'}_{(\zeta,z)}$ for some $(\zeta, z) \in D$. Assume first that the conditions in the statement are satisfied. Then, for every $\zeta' \in E$, [16, Proposition 2.41] shows that $f^{(\zeta')} \in \mathcal{A}^{p,q}_{\mathbf{s},0}(T_\Omega)$ (resp. $f^{(\zeta')} \in \mathcal{A}^{p,q}_{\mathbf{s}}(T_\Omega)$), and that

$$\|f^{(\zeta')}\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(T_\Omega)} = C_1 \Delta^{\mathbf{s}+\operatorname{Re}\mathbf{s}'-\mathbf{d}/p}_{\Omega}(h + \Phi(\zeta - \zeta')),$$

where $h = \rho(\zeta, z)$ and $C_1$ is a suitable constant. In addition, Lemma [16, Lemma 2.32] shows that $f \in \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ (resp.

$f \in A^{p,q}_{\mathbf{s},0}(D)$), and that

$$\|f\|_{A^{p,q}_{\mathbf{s}}(D)} = C_2 \Delta_\Omega^{\operatorname{Re}\mathbf{s}'-\mathbf{d}/p-\mathbf{b}/q}(h),$$

where $C_2$ is a suitable constant.[6]

Conversely, assume that $f \in \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ (resp. $f \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$). Then, Proposition 4.2 shows that $f^{(\zeta)} \in \mathcal{A}^{p,q}_{\mathbf{s}-\mathbf{b}/q,0}(T_\Omega)$ (resp. $f^{(\zeta)} \in \mathcal{A}^{p,q}_{\mathbf{s}-\mathbf{b}/q}(T_\Omega)$) for every $\zeta \in E$. Since $f^{(0)} = B^{\mathbf{s}'}_z$ (as a function on $T_\Omega$), [16, Proposition 2.41] leads to the conclusion. □

The second part of the following result extends: [9, Theorem II.6], which corresponds to the case in which $p = q$ and $\mathbf{s} \succ \frac{1}{2}\mathbf{b}+\mathbf{d}+\frac{q}{2}\mathbf{m}'$ and $\mathbf{s}'-\frac{1}{q}\mathbf{s}' \succ -\left(\frac{1}{2}+\frac{1}{2q}\right)\mathbf{b}-\frac{1}{q}\mathbf{d}+\frac{1}{2}\mathbf{m}$ when $q < 1$; [4, Proposition 2.19], which corresponds to the case in which $p = q \geq 1$, $\mathbf{s}, \mathbf{s}' \in \mathbb{R}\mathbf{1}_r$, $n = 0$, and $\Omega$ is symmetric; [6, Proposition 3.10 (ii)], which corresponds to the case in which $\mathfrak{P}_{\mathbf{s}'}$ is continuous on $\mathfrak{L}^{p,q}_{\mathbf{s}}(D)$.

**Proposition 4.8** *Take $p, q \in (0,\infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that the following hold:*

- $\mathbf{s} \succ \frac{1}{q}\mathbf{b}+\frac{1}{p}\mathbf{d}+\frac{1}{2q'}\mathbf{m}'$;
- $\mathbf{s}' \prec \frac{1}{p'}\mathbf{d}-\frac{1}{2p'}\mathbf{m}', \mathbf{b}+\mathbf{d}-\frac{1}{2}\mathbf{m}$;
- $\mathbf{s}+\mathbf{s}' \prec \frac{1}{\min(1,q)}\mathbf{b}+\frac{1}{\min(1,p)}\mathbf{d}-\frac{1}{2q'}\mathbf{m}$ *or* $\mathbf{s}+\mathbf{s}' \leq \frac{1}{q}\mathbf{b}+\frac{1}{\min(1,p)}\mathbf{d}$ *if* $q' = \infty$.

*Then, $P_{\mathbf{s}'}f = f$ for every $f \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$.*

*In particular, if $q < \infty$ and:*

- $\mathbf{s} \succ \frac{1}{2}\mathbf{b}+\frac{q}{p}\mathbf{d}+\frac{q}{2q'}\mathbf{m}'$;
- $\mathbf{s}' \succ \frac{1}{2}\mathbf{b}+\frac{1}{\max(1,p)}\mathbf{d}+\frac{1}{2p'}\mathbf{m}', \frac{1}{2}(\mathbf{m}-\mathbf{b})$;
- $\mathbf{s}'-\frac{1}{q}\mathbf{s} \succ -\frac{1}{2q'}\mathbf{b}-\left(\frac{1}{p}-1\right)_+\mathbf{d}+\frac{1}{2q'}\mathbf{m}$ *or* $q' = \infty$ *and* $\mathbf{s}'-\frac{1}{q}\mathbf{s} \geq -\left(\frac{1}{2q}-\frac{1}{2}\right)\mathbf{b}-\left(\frac{1}{p}-1\right)_+\mathbf{d}$;

*then $\mathfrak{P}_{\mathbf{s}'}f = f$ for every $f \in \mathfrak{A}^{p,q}_{\mathbf{s}}(D)$.*

***Proof*** Observe that Proposition 4.8 shows that the mapping

$$\mathcal{A}^{p,q}_{\mathbf{s}}(D) \ni f \mapsto P_{\mathbf{s}'}f(\zeta,z) = c_{(\mathbf{b}+\mathbf{d}-\mathbf{s}')/2} \times \int_D f(\zeta',z')B^{\mathbf{s}'}_{(\zeta',z')}(\zeta,z)\Delta_\Omega^{-\mathbf{s}'}(\rho(\zeta',z'))\,\mathrm{d}\nu_D(\zeta',z') \in \mathbb{C}$$

is well defined and continuous for every $(\zeta,z) \in D$ and obviously induces the mapping $f \mapsto f(\zeta,z)$ on $\mathcal{A}^{p,q}_{\mathbf{s}}(D) \cap \mathcal{A}^{2,2}_{(\mathbf{b}+\mathbf{d}-\mathbf{s}')/2}(D)$. The result follows by continuity, thanks to Proposition 4.6. □

[6] Actually, the cited reference allows to deal with the case $q < \infty$. The remaining case, though, follows from the fact that $\Delta_\Omega^{\mathbf{s}+\operatorname{Re}\mathbf{s}'-\mathbf{d}/p}(h+\cdot)$ is decreasing on $\overline{\Omega}$ thanks to [16, Corollary 2.36], and vanishes at the point at infinity of $\overline{\Omega}$ when $\mathbf{s}+\operatorname{Re}\mathbf{s}'-\mathbf{d}/p \prec \mathbf{0}$, thanks to [16, Lemma 2.35].

### 4.3 Sampling

The following sampling theorem is a consequence of the more general Theorem 4.11 presented below. We single out this result for comparison with the literature. It extends: [3, Theorem 5.6], which corresponds to the case in which $p = q \geq 1$, $\mathbf{s} \in \mathbb{R}\mathbf{1}_r$, $n = 0$, and $\Omega$ is symmetric; [8, Theorem 5.2], which corresponds to the case in which $p = q > 1$; [7, Theorem 3.3], which corresponds to the case in which $p, q \geq 1$, $n = 0$ and $\Omega$ is symmetric.

Recall that we denote by $\ell^{p,q}(J,K)$ the space of $\lambda \in \mathbb{C}^{J\times K}$ such that $\|\|\lambda_{j,k}\|_{\ell^p(J)}\|_{\ell^q(K)} < \infty$, with some abuse of notation.

**Theorem 4.9** *Take $p, q \in (0,\infty]$, $\mathbf{s} \in \mathbb{R}^r$ and $R_0 > 1$. Then, there is $\delta_0 > 0$ such that, for every $F$-$(\delta,R)$-lattice $(\zeta_k, z_{j,k})_{j\in J,k\in K}$ on $D$, with $\delta \in (0,\delta_0]$ and $R \in (1,R_0]$, the mapping*

$$f \mapsto \Delta_\Omega^{\mathbf{s}/q-\mathbf{b}/(2q)-\mathbf{d}/p}(\rho(\zeta_k,z_{j,k}))f(\zeta_k,z_{j,k})$$

*induces an isomorphism of $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)$ onto a closed subspace of $\ell^{p,q}(J,K)$.*

In order to prove the more general version of this result, we need the following definition.

**Definition 4.10** For every $\mathbf{s} \in \mathbb{R}^r$, we define $\mathscr{M}_{\mathbf{s}}$ as the space of $f \in \operatorname{Hol}(D)$ such that the function $(\zeta,z) \mapsto \Delta^{\mathbf{s}}_\Omega(\rho(\zeta,z))\mathrm{e}^{-|\zeta|^{2\alpha}-|\operatorname{Re}z|^\alpha-|\rho(\zeta,z)|^\alpha}f(\zeta,z)$ is bounded on $D$ for some $\alpha \in [0,1/2)$.

Observe that $\mathscr{M}_{\mathbf{s}} \subseteq \mathscr{M}_{\mathbf{s}'}$ for $\mathbf{s} \leq \mathbf{s}'$, and that

$$\mathcal{A}^{p,q}_{\mathbf{s}}(D) \subseteq \mathcal{A}^{\infty,\infty}_{\mathbf{s}-\mathbf{b}/q-\mathbf{d}/p}(D) \subseteq \mathscr{M}_{\mathbf{s}-\mathbf{b}/q-\mathbf{d}/p}$$

for every $\mathbf{s} \in \mathbb{R}^r$, thanks to Proposition 4.2.

**Theorem 4.11** *Take $p, q \in (0,\infty]$, $\mathbf{s} \in \mathbb{R}^r$, $R_0 > 1$ and $\delta_+ > 0$, and $\mathbf{s}' \geq \mathbf{s}-\frac{1}{q}\mathbf{b}-\frac{1}{p}\mathbf{d}$. Then, there are $\delta_-, C > 0$ such that, for every $F$-$(\delta,R)$-lattice $(\zeta_k,z_{j,k})_{j\in J,k\in K}$ on $D$, with $R \in (1,R_0]$, if we define*

$$S_+\colon \operatorname{Hol}(D) \ni f \mapsto \left(\Delta_\Omega^{\mathbf{s}-\mathbf{b}/q-\mathbf{d}/p}(h_k)\max_{\overline{B}((\zeta_k,z_{j,k}),R\delta)}|f|\right) \in \mathbb{C}^{J\times K}$$

*and*

$$S_-\colon Hol(D) \ni f \mapsto \left(\Delta_\Omega^{\mathbf{s}-\mathbf{b}/q-\mathbf{d}/p}(h_k)\min_{\overline{B}((\zeta_k,z_{j,k}),R\delta)}|f|\right) \in \mathbb{C}^{J\times K},$$

*where $h_k{:=}\rho(\zeta_k,z_{j,k})$ for every $j \in J$ and for every $k \in K$, then*

$$\frac{1}{C}\|f\|_{A^{p,q}_{\mathbf{s}}(D)} \leq \delta^{m/p+(2n+m)/q}\|S_\varepsilon f\|_{\ell^{p,q}(J,K)} \leq C\|f\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)}$$

*for every $f \in \mathrm{Hol}(D)$ and for every $\delta \in (0, \delta_+]$ if $\varepsilon = +$, and for every $f \in \mathscr{M}_{\mathbf{s}'}$ and for every $\delta \in (0, \delta_-]$ if $\varepsilon = -$. In addition,*

$$\mathcal{A}^{p,q}_{\mathbf{s},0}(D) = Hol(D) \cap S_+^{-1}(\ell^{p,q}_0(J, K)) \quad and$$
$$\mathcal{A}^{p,q}_{\mathbf{s},0}(D) = \mathscr{M}_{\mathbf{s}'}(D) \cap S_-^{-1}(\ell^{p,q}_0(J, K)),$$

*where the second equality holds provided that $\delta \le \delta_-$.*

***Proof*** For the sake of simplicity, we shall generally present the computations as if $p, q < \infty$. We leave to the reader the (purely formal) modifications which are necessary when $\max(p, q) = \infty$. Throughout the proof, for every $t \in T_+$ we shall denote by $g_t$ an element of $GL(E)$ such that $t \cdot \Phi = \Phi \circ (g_t \times g_t)$.

Step I. Define, for every $R' > 0$, for every $\zeta \in E$, and for every $h \in \Omega$,

$$\begin{aligned} M_{R'}(\zeta, h) &:= \left\| \left(\chi_{B((0,ie_\Omega),R')}\right)^{(\zeta)}_h \right\|_{L^1(F)} \\ &\le \mathcal{H}^m(\mathrm{pr}_F(B((0, ie_\Omega), R'))) < \infty \end{aligned}$$

Then, for every $\ell \in (0, \infty]$, for every $t' \in T_+$, and for every $(\zeta', x') \in \mathcal{N}$,

$$\begin{aligned} &\left\| \left(\chi_{B((\zeta',x'+i\Phi(\zeta)+it'\cdot e_\Omega),R')}\right)^{(\zeta)}_h \right\|_{L^\ell(F)} \\ &= \Delta^{-\mathbf{d}/\ell}(t') M_{R'}(g_{t'}^{-1}(\zeta - \zeta'), t'^{-1} \cdot h)^{1/\ell}, \end{aligned}$$

with the convention $0^0 = 0$. In particular,

$$\begin{aligned} &\left\| \left(\chi_{B((\zeta_k,z_{j,k}),R')}\right)^{(\zeta)}_h \right\|_{L^\ell(F)} \\ &= \chi_{B_{E\times\Omega}((\zeta_k,h_k),R')}(\zeta, h) \Delta_\Omega^{-\mathbf{d}/\ell}(h_k) M_{R'}(g_{t_k}^{-1}(\zeta - \zeta_k), t_k^{-1} \cdot h)^{1/\ell} \end{aligned}$$

for every $(\zeta, h) \in E \times \Omega$ and for every $k \in K$, where $t_k \in T_+$ is such that $h_k = t_k \cdot e_\Omega$. In addition,

$$\|M_{R'}\|_{L^\infty(\nu_{E\times\Omega})} \asymp R'^m \text{ for } R' \to 0^+.$$

For every $(\zeta, h) \in E \times \Omega$, define

$$K_{\zeta,h} := \{k \in K : (\zeta, h) \in B_{E\times\Omega}((\zeta_k, h_k), R\delta)\},$$

and observe that there is $N \in \mathbb{N}$ such that $\mathrm{Card}(K_{\zeta,h}) \le N$ for every $(\zeta, h) \in E \times \Omega$, provided that $R \le R_0$ and $\delta \le \delta_+$. We may also assume that every $(\zeta, h) \in E \times \Omega$ is contained in at most $N$ balls $B_{E\times\Omega}((\zeta_k, h_k), 2R\delta)$, $k \in K$, and that every $(\zeta, z) \in D$ is contained in at most $N$ balls $B((\zeta_k, z_{j,k}), 2R\delta)$, $(j, k) \in J \times K$, provided that $R \le R_0$ and $\delta \le \delta_+$. Finally, set $\ell := \min(1, p, q)$.

Step II. Let us prove that $S_+$ maps $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ into $\ell^{p,q}(J, K)$. Take $f \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$ and define

$$\begin{aligned} &C_{D,R'} := \nu_D(B((0, ie_\Omega), R')), \\ &C_{E\times\Omega,R'} := \nu_{E\times\Omega}(B_{E\times\Omega}((0, e_\Omega), R')), \quad \text{and} \\ &C_{\Omega,R'} := \nu_\Omega(B_\Omega(e_\Omega, R')) \end{aligned}$$

for every $R' > 0$ to simplify the notation. Then, [16, Lemma 3.24] implies that there are $R'_0 \in (0, 1/2]$ and $C_1 > 0$ such that

$$\max_{\overline{B}((\zeta_k,z_{j,k}),R\delta)} |f|^p \le \frac{C_1}{C_{D,R'_0\delta}} \int_{B((\zeta_k,z_{j,k}),(R+R'_0)\delta)} |f|^p \, \mathrm{d}\nu_D$$

for every $(j, k) \in J \times K$. Therefore, [16, Corollary 2.49] implies that there is a constant $C_2 > 0$ such that

$$\begin{aligned} (S_+ f)^p_{j,k} &\le \frac{C_2}{C_{D,R'_0\delta}} \Delta_\Omega^{p\mathbf{s}-(p/q)\mathbf{b}}(h_k) \\ &\quad \times \int_{E\times\Omega} \int_F |(\chi_{B((\zeta_k,z_{j,k}),(R+R'_0)\delta)} f)^{(\zeta)}_h(x)|^p \, \mathrm{d}x \, \mathrm{d}\nu_{E\times\Omega}(\zeta, h) \end{aligned}$$

for every $(j, k) \in J \times K$. Hence,

$$\begin{aligned} &\sum_{j\in J} (S_+ f)^p_{j,k} \\ &\quad \le \frac{C_2}{C_{D,R'_0\delta}} N \Delta_\Omega^{p\mathbf{s}-(p/q)\mathbf{b}}(h_k) \\ &\quad\quad \times \int_{B_{E\times\Omega}((\zeta_k,h_k),(R+R'_0)\delta)} \|f^{(\zeta)}_h\|^p_{L^p(F)} \, \mathrm{d}\nu_{E\times\Omega}(\zeta, h) \end{aligned}$$

for every $k \in K$. Now, Lemma 4.3 shows that there is a constant $C_3 > 0$ such that

$$\begin{aligned} &\int_{B_{E\times\Omega}((\zeta_k,h_k),(R+R'_0)\delta)} \|f^{(\zeta)}_h\|^p_{L^p(F)} \, \mathrm{d}\nu_{E\times\Omega}(\zeta, h) \\ &\quad \le C_3 \int_{B_{E\times\Omega}((\zeta_k,h_k),(R+R'_0)\delta)} \\ &\quad \left( \fint_{B_{E\times\Omega}((\zeta',h'),R'_0\delta)} \|f^{(\zeta)}_h\|^q_{L^p(F)} \, \mathrm{d}\nu_{E\times\Omega}(\zeta, h) \right)^{p/q} \\ &\quad \mathrm{d}\nu_{E\times\Omega}(\zeta', h') \\ &\quad \le C_3 \frac{C_{E\times\Omega,(R+R'_0)\delta}}{C^{p/q}_{E\times\Omega,R'_0\delta}} \\ &\quad\quad \times \left( \int_{B_{E\times\Omega}((\zeta_k,h_k),(R+2R'_0)\delta)} \|f^{(\zeta)}_h\|^q_{L^p(F)} \, \mathrm{d}\nu_{E\times\Omega}(\zeta, h) \right)^{p/q} \end{aligned}$$

for every $k \in K$. Therefore, another application of [16, Corollary 2.49] shows that there is a constant $C'_2 > 0$ such that

$$\|S_+ f\|_{\ell^{p,q}(J,K)} \le \frac{C'_2 C^{1/p}_{E\times\Omega,(R+R'_0)\delta}}{C^{1/p}_{D,R'_0\delta} C^{1/q}_{E\times\Omega,R'_0\delta}} N^{1/p+1/q} \|f\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)}.$$

Next, let us prove that $S_+(\mathcal{A}^{p,q}_{\mathbf{s},0}(D)) \subseteq \ell^{p,q}_0(J,K)$. Observe that we may assume that $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$. Then, take $\tilde p \in (0,p)$ and $\tilde q \in (0,q)$ so that $\mathbf{s}'' := \mathbf{s} - \left(\frac{1}{q} - \frac{1}{\tilde q}\right)\mathbf{b} - \left(\frac{1}{p} - \frac{1}{\tilde p}\right)\mathbf{d} \succ \frac{1}{2\tilde q}\mathbf{m}$ and observe that the preceding computations show that $S_+(\mathcal{A}^{p,q}_{\mathbf{s},0}(D) \cap \mathcal{A}^{\tilde p,\tilde q}_{\mathbf{s}''}(D)) \subseteq \ell^{\tilde p,\tilde q}(J,K) \subseteq \ell^{p,q}_0(J,K)$, so that the assertion follows by means of Proposition 4.6.

Step III. Now, take $f \in \mathrm{Hol}(D)$ and assume that $S_+ f \in \ell^{p,q}_0(J,K)$ (resp. $S_+ f \in \ell^{p,q}(J,K)$), and let us prove that $f \in \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ (resp. $f \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$). Observe first that

$$|f^{(\zeta)}_h| \le \sum_{(j,k)\in J\times K_{\zeta,h}} \Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}}(h_k) \left(\chi_{B((\zeta_k,z_{j,k}),R\delta)}\right)^{(\zeta)}_h (S_+ f)_{j,k}$$

on $F$, for every $(\zeta,h) \in E\times\Omega$, so that $f^{(\zeta)}_h \in L^p_0(F)$ (resp. $f^{(\zeta)}_h \in L^p(F)$) for every $(\zeta,h) \in E\times\Omega$. In addition,

$$\begin{aligned}\|f^{(\zeta)}_h\|_{L^p(F)} &\le N^{1/p'} \left\| \left(\Delta_\Omega^{\mathbf{b}/q-\mathbf{s}}(h_k) M_{R\delta}(g^{-1}_{t_k}(\zeta-\zeta_k), t_k^{-1}\cdot h)^{1/p} \times (S_+ f)_{j,k}\right)_{j,k} \right\|_{\ell^p(J\times K_{\zeta,h})} \\ &\le N^{1/p'+(1/p-1/q)_+} \|M_{R\delta}\|_\infty^{1/p} \times \left\| \left(\Delta_\Omega^{\mathbf{b}/q-\mathbf{s}}(h_k) \|((S_+ f)_{j,k})_j\|_{\ell^p(J)}\right)_k \right\|_{\ell^q(K_{\zeta,h})}\end{aligned}$$

for every $(\zeta,h) \in E\times\Omega$, so that $(\zeta,h) \mapsto \|f^{(\zeta)}_h\|_{L^p(F)}$ belongs to $L^q_0(E\times\Omega)$ and there is a constant $C''_2 > 0$ (cf. [16, Corollary 2.49]) such that

$$\|f\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)} \le N^{1/p'+\max(1/p,1/q)} \|M_{R\delta}\|_\infty^{1/p} C^{1/q}_{E\times\Omega,R\delta} C''_2 \|S_+ f\|_{\ell^{p,q}(J,K)}.$$

Step IV. Observe that, if $(\zeta'_{k'}, z'_{j',k'})_{j'\in J', k'\in K'}$ is an $F$-$(\delta,R')$-lattice on $D$, then there are two mappings $\iota_1 \colon K' \to K$ and $\iota_2 \colon J'\times K' \to J$ such that, setting $h'_{k'} := \rho(\zeta'_{k'}, z'_{j',k'})$ for every $j' \in J'$ and for every $k' \in K'$,

$$(\zeta'_{k'}, h'_{k'}) \in B_{E\times\Omega}((\zeta_{\iota_1(k')}, h_{\iota_1(k')}), R\delta)$$

and

$$(\zeta'_{k'}, z'_{j',k'}) \in B((\zeta_{\iota_1(k')}, z_{\iota_2(j',k'),\iota_1(k')}), R\delta)$$

for every $j' \in J'$ and for every $k' \in K'$. Define

$$S'_- \colon \mathrm{Hol}(D) \ni f \mapsto \left(\Delta_\Omega^{\mathbf{s}-\mathbf{b}/q-\mathbf{d}/p}(h'_{k'}) \min_{\overline{B}((\zeta'_{k'}, z'_{j',k'}),(R+R')\delta)} |f|\right) \in \mathbb{C}^{J'\times K'},$$

and observe that, by [16, Corollary 2.49] and the preceding remarks, there is a constant $C' > 0$ such that

$$(S'_- f)_{j',k'} \le C'(S_- f)_{\iota_2(j',k'),\iota_1(k')}$$

for every $f \in \mathrm{Hol}(D)$, for every $j' \in J'$ and for every $k' \in K'$. In addition, there is $N' \in \mathbb{N}$ such that the fibres of $\iota_1$ and $(j',k') \mapsto (\iota_2(j',k'), \iota_1(k'))$ have at most $N'$ elements. Consequently,

$$\|S'_- f\|_{\ell^{p,q}(J',K')} \le C' N'^{1/p+1/q} \|S_- f\|_{\ell^{p,q}(J,K)}$$

for every $f \in \mathrm{Hol}(D)$. In addition, if $S_- f \in \ell^{p,q}_0(J,K)$, then $S'_- f \in \ell^{p,q}_0(J',K')$.

Observe, furthermore, that if $R' \ge 8$, then we may choose $(\zeta'_{k'}, z'_{j',k'})_{j'\in J',k'\in K'}$ such that $K' = K'_1 \times K'_2$, such that $h'_{(k'_1,k'_2)} = h'_{(k''_1,k'_2)}$ for every $k'_1, k''_1 \in K'_1$ and for every $k'_2 \in K'_2$, and such that $(h'_{(k'_1,k'_2)})_{k'_2\in K'_2}$ is a $(\delta,R')$-lattice on $\Omega$ for some/every $k'_1 \in K'_1$ (argue as in the proof of [16, Lemma 2.55]).

Step V. Take $f \in \mathscr{M}_{\mathbf{s}'}(D)$ such that $S_- f \in \ell^{p,q}(J,K)$ and let us prove that

$$\|f\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)} \le C\delta^{m/p+(2n+m)/q} \|S_- f\|_{\ell^{p,q}(J,K)}$$

for a suitable constant $C > 0$ (depending only on $\delta_-$ and $R_0$), provided that $\delta_-$ is sufficiently small. Observe first that, by Step IV, up to replacing $R$ with $R+8$, we may assume that $(\zeta_k, z_{j,k})$ is an $F$-$(\delta,R)$-lattice, that $K = K'\times K''$, that $h_{(k',k'')}$ only depends on $k''$ (so that we also write $h_{k''}$ instead of $h_{(k',k'')}$), and that $(h_{k''})$ is a $(\delta,R)$-lattice in $\Omega$. Observe that, for every $(j,k) \in J\times K$, we may find $(\zeta'_{j,k}, z'_{j,k}) \in \overline{B}((\zeta_k, z_{j,k}), R\delta)$ such that

$$|f(\zeta'_{j,k}, z'_{j,k})| = \min_{\overline{B}((\zeta_k,z_{j,k}),R\delta)} |f|.$$

Now, [16, Lemmas 3.24 and 3.25] imply that there are $R'_1 \in (0,R'_0]$ and $C_3 > 0$ such that, for every $j \in J$, for every $k \in K$, for every $(\zeta,x) \in \mathcal{N}$, and for every $h \in \Omega$ such that $d((\zeta_k,z_{j,k}),(\zeta, x+i\Phi(\zeta)+ih)) < R\delta$,

$$|f^{(\zeta)}_h(x)| \le |f(\zeta'_{j,k}, z'_{j,k})| + C_3 R\delta \|\chi_{B((\zeta,x+i\Phi(\zeta)+ih),2R\delta+R'_1)} f\|_{L^p(\nu_D)},$$

provided that $R\delta \le R_1'$. Then,

$$\begin{aligned}\|f_h^{(\zeta)}\|_{L^p(F)}^p \le{}& 2^{(p-1)_+}\|M_{R\delta}\|_{L^\infty(E\times\Omega)}\\ &\times \sum_{(j,k)\in J\times K_{\zeta,h}} \Delta_\Omega^{-\mathbf{d}}(h_k)|f(\zeta'_{j,k}, z'_{j,k})|^p\\ &+2^{(p-1)_+}(C_3 R\delta)^p \Theta_1(\zeta,h),\end{aligned}$$

where

$$\Theta_1(\zeta,h):=\sum_{(j,k)\in J\times K_{\zeta,h}} \int_F (\chi_{B((\zeta_k,z_{j,k}),R\delta)})_h^{(\zeta)}(x)\\ \times \int_D \chi_{B((\zeta,x+i\Phi(\zeta)+ih),2R\delta+R_1')}|f|^p \,\mathrm{d}\nu_D\,\mathrm{d}x.$$

Now, set

$$K_h'':=\{k''\in K''\colon h\in B_\Omega(h_{k''},R\delta)\}$$

for every $h\in\Omega$ and observe that we may assume that, for every $h\in\Omega$,

$$\operatorname{Card} K_h'' \le N,$$

provided that $\delta\le\delta_+$, as in STEP I. Then, [16, Corollary 2.49] implies that there is a constant $C_4>0$ such that, if $R\delta\le R_1'$,

$$\begin{aligned}&\int_E \left(\sum_{(j,k)\in J\times K_{\zeta,h}} \Delta_\Omega^{-\mathbf{d}}(h_k)|f(\zeta'_{j,k},z'_{j,k})|^p\right)^{q/p}\mathrm{d}\zeta\\ &\le N^{(q/p-1)_+}\sum_{k''\in K_h''}\Delta_\Omega^{-(q/p)\mathbf{d}}(h_k)\\ &\quad\times\int_E \sum_{k'\colon(k',k'')\in K_{\zeta,h}}\left(\sum_{j\in J}|f(\zeta'_{j,(k',k'')},z'_{j,(k',k'')})|^p\right)^{q/p}\mathrm{d}\zeta\\ &\le C_{E,R\delta}N^{\max(1,q/p)}\sum_{k''\in K_h''}\Delta_\Omega^{-q(\mathbf{b}/q+\mathbf{d}/p)}(h_{k''})\\ &\quad\times\sum_{k'\in K'}\left(\sum_{j\in J}|f(\zeta'_{j,(k',k'')},z'_{j,(k',k'')})|^p\right)^{q/p}\\ &\le C_4\delta^{2n}N^{\max(1,q/p)}\Delta_\Omega^{-q\mathbf{s}}(h)\\ &\quad\times\sum_{k''\in K_h''}\sum_{k'\in K'}\left(\sum_{j\in J}|(S_-f)_{j,(k',k'')}|^p\right)^{q/p}\end{aligned}$$

where the first inequality follows from the convexity or subadditivity of the mapping $x\mapsto x^{q/p}$ on $\mathbb{R}_+$, while the second one follows from Tonelli's theorem, setting $C_{E,R\delta}:=\mathcal{H}^{2n}(\mathrm{pr}_E(B_{E\times\Omega}((0,e_\Omega),R\delta)))$.

Now, observe that

$$\Theta_1(\zeta,h)=\int_D |f(\zeta',z')|^p\Theta_2(\zeta',z',\zeta,h)\,\mathrm{d}\nu_D(\zeta',z'),$$

where

$$\Theta_2(\zeta',z',\zeta,h):=\\ \int_F\sum_{(j,k)\in J\times K_{\zeta,h}}(\chi_{B((\zeta_k,z_{j,k}),R\delta)\cap B((\zeta',z'),2R\delta+R_1')})_h^{(\zeta)}(x)\,\mathrm{d}x.$$

In addition, for every $(\zeta',z')\in D$ and for every $(\zeta,h)\in E\times\Omega$, setting $h':=\rho(\zeta',z')$, one has

$$\begin{aligned}&\Theta_2(\zeta',z',\zeta,h)\\ &\quad\le N\|(\chi_{B((\zeta',z'),2R\delta+R_1')})_h^{(\zeta)}\|_{L^1(F)}\\ &\quad= N M_{2R\delta+R_1'}(g_{t'}^{-1}(\zeta-\zeta'),t'^{-1}\cdot h)\Delta_\Omega^{-\mathbf{d}}(h'),\end{aligned}$$

provided that $R\le R_0$ and $\delta\le\delta_+$, where $t'\in T_+$ is such that $h'=t'\cdot e_\Omega$. Therefore, by STEP I, we see that

$$\Theta_1(\zeta,h)\le N\|M_{2R\delta+R_1'}\|_{L^\infty(E\times\Omega)}\\ \times\int_{B_{E\times\Omega}((\zeta,h),2R\delta+R_1')}\|f_{h'}^{(\zeta')}\|_{L^p(F)}^p\,\mathrm{d}\nu_{E\times\Omega}(\zeta',h'),$$

provided that $R\le R_0$ and $\delta\le\delta_+$. Then, applying Lemma 4.3 as in STEP II, we see that there is a constant $C_5>0$ such that

$$\begin{aligned}&\int_E\Theta_1(\zeta,h)^{q/p}\,\mathrm{d}\zeta\\ &\quad\le C_5\int_{B_\Omega(h,2R\delta+2R_1')}\|f_{h'}\|_{L^{p,q}(F,E)}^q\,\mathrm{d}\nu_\Omega(h')\end{aligned}$$

provided that $R\le R_0$ and $\delta\le\delta_+$.

Therefore, there is a constant $C_6>0$ such that

$$\begin{aligned}&\|f_h\|_{L^{p,q}(F,E)}\\ &\quad\le C_6\Delta_\Omega^{-\mathbf{s}}(h)\delta^{m/p+2n/q}\|S_-f\|_{\ell^{p,q}(J,K'\times K_h'')}\\ &\quad\quad+\delta C_6\left(\int_{B_\Omega(h,2R\delta+2R_1')}\|f_{h'}\|_{L^{p,q}(F,E)}^q\,\mathrm{d}\nu_\Omega(h')\right)^{1/q},\end{aligned}$$

provided that $R\le R_0$ and $\delta\le\min(R_1'/R_0,\delta_+)$.

Now, by assumption, there is $\alpha'\in(0,1/2)$ such that

$$\sup_{(\zeta,z)\in D}\Delta_\Omega^{\mathbf{s}'}(\rho(\zeta,z))\mathrm{e}^{-|\zeta|^{2\alpha'}-|\operatorname{Re} z|^{\alpha'}-|\rho(\zeta,z)|^{\alpha'}}|f(\zeta,z)|<\infty.$$

Then, take $(g^{(\varepsilon)})_{\varepsilon>0}$ as in [16, Lemma 1.22] (cf. the proof of Proposition 4.5 and formulas (5) and (6)) for some $\alpha\in(\alpha',1/2)$, so that $G(\varepsilon):=fg^{(\varepsilon)}\in\mathcal{A}^{p,\infty}_{\mathbf{s}'}(D)$ and

$S_- G(\varepsilon) \le S_- f$ for every $\varepsilon > 0$. In particular, the mapping $h \mapsto \|G(\varepsilon)_h\|_{L^{p,q}(F,E)}$ is (finite and) decreasing on $\Omega$, thanks to Corollary 4.4, for every $\varepsilon > 0$. In addition, observe that we may take $\delta_1 \in (0, \min(R'_1/R_0, \delta_+)]$ and $R'_1$ so small that $B_\Omega(e_\Omega, 2R_0\delta_1 + 2R'_1) \subseteq e_\Omega/2 + \Omega$. Then, by homogeneity,

$$B_\Omega(h, 2R_0\delta_1 + 2R'_1) \subseteq h/2 + \Omega$$

for every $h \in \Omega$. Then, the preceding estimates (applied to $G(\varepsilon)$) show that there is a constant $C'_6 > 0$ such that

$$\begin{aligned}\left\|G(\varepsilon)_h\right\|_{L^{p,q}(F,E)} &\le C'_6 \Delta_\Omega^{-\mathbf{s}}(h)\delta^{m/p+2n/q}\|S_- f\|_{\ell^{p,q}(J,K'\times K''_h)}\\ &\quad + \delta C'_6 \left\|G(\varepsilon)_{h/2}\right\|_{L^{p,q}(F,E)},\end{aligned}$$

for every $\varepsilon > 0$, provided that $\delta \le \delta_1$ and $R \le R_0$. If we define

$$\chi_\ell \colon D \ni (\zeta, z) \mapsto \chi_{e_\Omega/2^\ell+\Omega}(\rho(\zeta, z)) \in \mathbb{R}_+$$

for every $\ell \in \mathbb{N}$, then there is a constant $C''_6 > 0$ such that

$$\begin{aligned}\left\|\chi_\ell G(\varepsilon)\right\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)} &\le C''_6 \delta^{m/p+(2n+m)/q}\|S_- f\|_{\ell^{p,q}(J,K)}\\ &\quad + \delta C''_6 \left\|\chi_{\ell+1} G(\varepsilon)\right\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)}\end{aligned}$$

for every $\varepsilon > 0$ and for every $\ell \in \mathbb{N}$, provided that $\delta \le \delta_1$ and $R \le R_0$. Now, define

$$e_{\alpha'} \colon D \ni (\zeta, z) \mapsto \mathrm{e}^{|\zeta|^{2\alpha'} + |\operatorname{Re} z|^{\alpha'} + |\operatorname{Im} z - \Phi(\zeta)|^{\alpha'}} \in \mathbb{R}_+,$$

and observe that

$$\begin{aligned}\left\|G(\varepsilon)\chi_\ell\right\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)} &\le \|e_{\alpha'}^{-1} f\|_{\mathcal{L}^{\infty,\infty}_{\mathbf{s}'}(D)} \left\|e_{\alpha'} g^{(\varepsilon)}\chi_\ell\right\|_{\mathcal{L}^{p,q}_{\mathbf{s}-\mathbf{s}'}(D)}\\ &\le C_{7,\varepsilon}\|e_{\alpha'}^{-1} f\|_{\mathcal{L}^{\infty,\infty}_{\mathbf{s}'}(D)} \left\|\chi_{e_\Omega/2^\ell+\Omega}\Delta_\Omega^{\mathbf{s}-\mathbf{s}'}\mathrm{e}^{-C_{7,\varepsilon}|\cdot|^\alpha}\right\|_{L^q(\nu_\Omega)}\\ &\le C'_{7,\varepsilon}\|e_{\alpha'}^{-1} f\|_{\mathcal{L}^{\infty,\infty}_{\mathbf{s}'}(D)}\|\chi_{e_\Omega/2^\ell+\Omega}\Delta_\Omega^{\mathbf{s}-\mathbf{s}'+\mathbf{d}/q}\|_{L^\infty(\Omega)}\\ &\quad\times \|\mathrm{e}^{-C_{7,\varepsilon}|\cdot|^\alpha}\|_{L^q(\Omega)}\\ &\le C''_{7,\varepsilon} 2^{(\mathbf{s}'-\mathbf{s}-\mathbf{d}/q)\ell}\|e_{\alpha'}^{-1} f\|_{\mathcal{L}^{\infty,\infty}_{\mathbf{s}'}(D)}\end{aligned}$$

for suitable constants $C_{7,\varepsilon}, C'_{7,\varepsilon}, C''_{7,\varepsilon} > 0$, since $\mathbf{s} - \mathbf{s}' + \mathbf{d}/q \le \mathbf{0}$ (cf. [16, Corollary 2.36]). Then, fix $N' > \sum_{j=1}^r (s'_j - s_j) + m/q$ and choose $\delta_- \in (0, \delta_1]$ so that $C''_6\delta_- \le 2^{-N'}$. Observe that the preceding computations show that, if $\delta \in (0, \delta_-]$, then

$$\begin{aligned}\left\|G(\varepsilon)\chi_\ell\right\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)} &= \sum_{\ell'\in\mathbb{N}} 2^{-\ell' N'}\Bigg(\left\|G(\varepsilon)\chi_{\ell+\ell'}\right\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)}\\ &\quad - \frac{1}{2^{N'}}\left\|G(\varepsilon)\chi_{\ell+\ell'+1}\right\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)}\Bigg)\\ &\le \frac{C''_6}{1-2^{-N'}}\delta^{m/p+(2n+m)/q}\|S_- f\|_{\ell^{p,q}(J,K)}\end{aligned}$$

for every $\varepsilon > 0$ and for every $\ell \in \mathbb{N}$. Passing to the limit for $\ell \to \infty$, we then infer that $G(\varepsilon) \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$ and that

$$\left\|G(\varepsilon)\right\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)} \le \frac{C''_6}{1-2^{-N'}}\delta^{m/p+(2n+m)/q}\|S_- f\|_{\ell^{p,q}(J,K)}$$

for every $\varepsilon > 0$. Then, passing to the limit for $\varepsilon \to 0^+$, we infer that $f \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$ and that

$$\|f\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)} \le \frac{C''_6}{1-2^{-N'}}\delta^{m/p+(2n+m)/q}\|S_- f\|_{\ell^{p,q}(J,K)}.$$

Step VI. It only remains to prove that $f \in \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ for every $f \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$ such that $S_- f \in \ell^{p,q}_0(J,K)$, provided that $\delta_-$ is sufficiently small. Observe first that the preceding computations show that

$$\begin{aligned}&\|\chi_\ell(f - G(\varepsilon))\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)}\\ &\quad\le \frac{C''_6}{1-2^{-N'}}\delta^{m/p+(2n+m)/p}\|S_-(f - G(\varepsilon))\|_{\ell^{p,q}(J,K)}\end{aligned}$$

for every $\ell \in \mathbb{N}$ and for every $\varepsilon > 0$. Since $S_-(f - G(\varepsilon)) \le (S_- f)\widetilde{S}_+(1 - g^{(\varepsilon)})$, where

$$\left[\widetilde{S}_+(1 - g^{(\varepsilon)})\right]_{j,k} := \max_{B((\zeta_k, z_{j,k}), R\delta)}\left|1 - g^{(\varepsilon)}\right|$$

for every $(j,k) \in J \times K$, and since $1 - g^{(\varepsilon)} \to 0$ locally uniformly, it is readily seen that $\chi_\ell(f - G(\varepsilon)) \to 0$ in $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$ for $\varepsilon \to 0^+$, for every $\ell \in \mathbb{N}$. In particular, $f_h \in L^{p,q}_0(F,E)$ for every $h \in \Omega$, and the mapping $h \mapsto \chi_{e_\Omega/2^\ell+\Omega}(h)\Delta_\Omega^{\mathbf{s}}(h)\|f_h\|_{L^{p,q}(F,E)}$ belongs to $L^q_0(\nu_\Omega)$ for every $\ell \in \mathbb{N}$. To conclude, it will essentially suffice to show that, if $q = \infty$, then $\Delta_\Omega^{\mathbf{s}}(h)\|f_h\|_{L^{p,\infty}(F,E)} \to 0$ as $h$ approaches the boundary of $\Omega$. Observe that, by the preceding computations, there is a constant $C_8 > 0$ (namely, $C'_6\max(1, \delta_-^{m/p})$) such that

$$\begin{aligned}\|f_h\|_{L^{p,\infty}(F,E)} &\le C_8\Delta_\Omega^{-\mathbf{s}}(h)\|S_- f\|_{\ell^{p,\infty}(J,K'\times K''_h)}\\ &\quad + \delta C_8\|f_{h/2}\|_{L^{p,\infty}(F,E)}\end{aligned}$$

for every $h \in \Omega$. Observe that

$$\begin{aligned}\Delta_\Omega^{\mathbf{s}}(h)\|f_{h/2}\|_{L^{p,\infty}(F,E)} &= 2^{\mathbf{s}}\Delta_\Omega^{\mathbf{s}}(h/2)\|f_{h/2}\|_{L^{p,\infty}(F,E)}\\ &\le 2^{\mathbf{s}}\|f\|_{\mathcal{A}^{p,\infty}_{\mathbf{s}}(D)}\end{aligned}$$

for every $h \in \Omega$. Therefore, assuming that $\delta_-$ is so small that $\delta_- C_8 2^{\mathbf{s}} \le 1/2$,

$$\begin{aligned}&\Delta_\Omega^{\mathbf{s}}(h)\|f_h\|_{L^{p,\infty}(F,E)}\\ &\quad= \sum_{\ell\in\mathbb{N}} 2^{-\ell}\bigg(\Delta_\Omega^{\mathbf{s}}(h/2^\ell)\|f_{h/2^\ell}\|_{L^{p,\infty}(F,E)}\\ &\qquad - \frac{1}{2\cdot 2^{\mathbf{s}}}\Delta_\Omega^{\mathbf{s}}(h/2^\ell)\|f_{h/2^{\ell+1}}\|_{L^{p,\infty}(F,E)}\bigg)\\ &\quad\le C_8\sum_{\ell\in\mathbb{N}} 2^{-\ell}\|S_- f\|_{\ell^{p,\infty}(J,K'\times K''_{h/2^\ell})}\end{aligned}$$

for every $h \in \Omega$. Now, observe that $\eta := \min_{\ell\in\mathbb{N}} d_\Omega(e_\Omega, e_\Omega/2^{\ell+1}) > 0$, and that $\eta = \min_{\ell\in\mathbb{N}} d_\Omega(h, h/2^{\ell+1})$ for every $h \in \Omega$, by homogeneity. Therefore, if $\delta_-$ is so small that $2R_0\delta_- < \eta$, then the sets $K''_{h/2^\ell}$, as $\ell$ runs through $\mathbb{N}$, are pairwise disjoint for every $h \in \Omega$. Hence,

$$\Delta_\Omega^{\mathbf{s}}(h)\|f_h\|_{L^{p,\infty}(F,E)} \le 2C_8\|S_- f\|_{\ell^{p,\infty}(J,K'\times K'''_h)}$$

for every $h \in \Omega$, where $K'''_h := \bigcup_{\ell\in\mathbb{N}} K''_{h/2^\ell}$. Since $K'''_h$ is contained in the complement of every *fixed* finite subset of $K$ if $h \in \Omega\setminus(e_\Omega/2^\ell + \Omega)$ and $\ell$ is sufficiently large, this and the preceding remarks prove that $\Delta_\Omega^{\mathbf{s}}(h)\|f_h\|_{L^p(\mathcal{N})} \to 0$ as $h \to \infty$ in $\Omega$, provided that $\delta_-$ is sufficiently small (independently of $f$). The proof is complete. □

### 4.4 Atomic decomposition and duality

**Definition 4.12** Take $p, q \in (0,\infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$. Then, we say that property $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}',0}$ (resp. $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}'}$) holds if for every $\delta_0 > 0$ there is an $F$-$(\delta,4)$-lattice $(\zeta_k, z_{j,k})_{j\in J,k\in K}$, with $\delta \in (0,\delta_0]$, such that, defining $h_k := \rho(\zeta_k, z_{j,k})$ for every $j \in J$ and for every $k \in K$, the mapping

$$\Psi : \lambda \mapsto \sum_{j,k} \lambda_{j,k} B^{\mathbf{s}'}_{(\zeta_k,z_{j,k})}\Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}-\mathbf{s}'}(h_k)$$

is well defined (with locally uniform convergence of the sum) and maps $\ell^{p,q}_0(J,K)$ into $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ continuously (resp. maps $\ell^{p,q}(J,K)$ into $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ continuously).

If we may take $(\zeta_k, z_{j,k})_{j\in J,k\in K}$, for every $\delta_0 > 0$ as above, in such a way that the corresponding mapping $\Psi$ is onto, then we say that property $(\mathcal{L}')^{p,q}_{\mathbf{s},\mathbf{s}',0}$ (resp. $(\mathcal{L}')^{p,q}_{\mathbf{s},\mathbf{s}'}$) holds.

When $\mathbf{s}' \succ \frac{1}{2}(\mathbf{m}-\mathbf{b})$, we define also properties $(\mathfrak{L})^{p,q}_{\mathbf{s},\mathbf{s}'}$ and $(\mathfrak{L}')^{p,q}_{\mathbf{s},\mathbf{s}'}$ as properties $(\mathcal{L})^{p,q}_{(\mathbf{s}+\mathbf{b}/2)/q,\mathbf{b}/2+\mathbf{d}-\mathbf{s}'}$ and $(\mathcal{L}')^{p,q}_{(\mathbf{s}+\mathbf{b}/2)/q,\mathbf{b}/2+\mathbf{d}-\mathbf{s}'}$, respectively. These properties are therefore related to the continuity (and surjectivity) of the mapping

$$\begin{aligned}\Psi' : \lambda \mapsto &\sum_{j,k}\mathfrak{K}^{\mathbf{s}'}(\,\cdot\,,(\zeta_k,z_{j,k}))\\ &\times\Delta_\Omega^{\mathbf{b}(1/(2q)-1/2)+\mathbf{d}(1/p-1)-\mathbf{s}/q+\mathbf{s}'}(h_k)\end{aligned}$$

from $\ell^{p,q}(J,K)$ into $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)$, where $\mathfrak{K}^{\mathbf{s}'}$ denotes the reproducing kernel of $\mathfrak{A}^{2,2}_{\mathbf{s}'}(D)$.

As we shall see in Theorem 4.31 below, properties $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}',0}$ (resp. $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}'}$) and $(\mathcal{L}')^{p,q}_{\mathbf{s},\mathbf{s}',0}$ (resp. $(\mathcal{L}')^{p,q}_{\mathbf{s},\mathbf{s}'}$) are actually equivalent when $p, q \in [1,\infty]$. In addition, arguing as in the proof of Theorem 4.31, one may show that properties $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}',0}$ and $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}'}$ are equivalent when $\mathbf{s} \succ \mathbf{0}$ (which is a necessary condition for property $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}',0}$ to hold).

**Lemma 4.13** *Take $p, q \in (0,\infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that property $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}',0}$ (resp. $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}'}$) holds. Then, the following hold:*

- $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$ *(resp. $\mathbf{s} \ge \mathbf{0}$ if $q=\infty$) and* $\mathbf{s} \succ \frac{1}{q}\mathbf{b} + \frac{1}{p}\mathbf{d} + \frac{1}{2q'}\mathbf{m}'$;
- $\mathbf{s}' \in \frac{1}{\min(p,p')}\mathbf{d} - \frac{1}{2\min(p,p')}\mathbf{m}'$;
- $\mathbf{s}+\mathbf{s}' \prec \frac{1}{\min(1,q)}\mathbf{b} + \frac{1}{\min(1,p)}\mathbf{d} - \frac{1}{2q'}\mathbf{m}$ *or* $\mathbf{s}+\mathbf{s}' \le \frac{1}{q}\mathbf{b} + \frac{1}{\min(1,p)}\mathbf{d}$ *if $q'=\infty$, and* $\mathbf{s}+\mathbf{s}' \prec \frac{1}{q}\mathbf{b} + \frac{1}{p}\mathbf{d} - \frac{1}{2q}\mathbf{m}'$.

*In particular, if $q<\infty$, $\mathbf{s}' \succ \frac{1}{2}(\mathbf{m}-\mathbf{b})$ and property $(\mathfrak{L})^{p,q}_{\mathbf{s},\mathbf{s}'}$ holds, then:*

- $\mathbf{s} \succ \frac{1}{2}(\mathbf{m}-\mathbf{b}), \frac{1}{2}\mathbf{b} + \frac{q}{p}\mathbf{d} + \frac{q}{2q'}\mathbf{m}'$;
- $\mathbf{s}' \succ \frac{1}{2}\mathbf{b} + \left(1 - \frac{1}{\min(p,p')}\right)\mathbf{d} + \frac{1}{2\min(p,p')}\mathbf{m}'$;
- $\mathbf{s}' - \frac{1}{q}\mathbf{s} \succ -\left|\frac{1}{2} - \frac{1}{2q}\right|\mathbf{b} - \left(\frac{1}{p}-1\right)_+\mathbf{d} + \frac{1}{2q'}\mathbf{m}$ *or* $\mathbf{s}' - \frac{1}{q}\mathbf{s} \ge -\left|\frac{1}{2} - \frac{1}{2q}\right|\mathbf{b} - \left(\frac{1}{p}-1\right)_+\mathbf{d}$ *if $q'=\infty$, and* $\mathbf{s}' - \frac{1}{q}\mathbf{s} \succ \left(\frac{1}{2} - \frac{1}{2q}\right)\mathbf{b} + \left(1-\frac{1}{p}\right)\mathbf{d} + \frac{1}{2q}\mathbf{m}'$.

***Proof*** By Proposition 4.7, it will suffice to observe that $B^{\mathbf{s}'}_{(0,ie_\Omega)} \in \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ (resp. $B^{\mathbf{s}'}_{(0,ie_\Omega)} \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$), and to show that

$$B^{\mathbf{s}'}_{(0,ie_\Omega)} \in \mathcal{A}^{p',q'}_{\mathbf{b}/\min(1,q)+\mathbf{d}/\min(1,p)-\mathbf{s}-\mathbf{s}'}(D).$$

Take $\delta_0 > 0$. Then, there is an $F$-$(\delta,4)$-lattice $(\zeta_k, z_{j,k})_{j\in J,k\in K}$, with $\delta \le \delta_0$, such that the mapping

$$\begin{aligned}\ell^{p,q}_0(J,K) \ni \lambda \mapsto &\sum_{j,k}\lambda_{j,k}B^{\mathbf{s}'}_{(\zeta_k,z_{j,k})}\Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}-\mathbf{s}'}(h_k)\\ &\in \mathcal{A}^{p,q}_{\mathbf{s}}(D)\end{aligned}$$

is well defined and continuous, where $h_k:=\rho(\zeta_k, z_{j,k})$ for every $j \in J$ and for every $k \in K$. Observe that the continuity of the mapping $f \mapsto f(0, ie_\Omega)$ on $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ implies that there is a constant $C_1 > 0$ such that

$$\Big|\sum_{j,k} \lambda_{j,k} B^{\mathbf{s}'}_{(\zeta_k, z_{j,k})}(0, ie_\Omega)\Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}-\mathbf{s}'}(h_k)\Big| \leq C_1 \|\lambda\|_{\ell^{p,q}_0(J,K)}$$

for every $\lambda \in \ell^{p,q}_0(J, K)$. Therefore,

$$\Big(B^{\mathbf{s}'}_{(0,ie_\Omega)}(\zeta_{j,k}, z_{j,k})\Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}-\mathbf{s}'}(h_k)\Big) = \overline{\Big(B^{\mathbf{s}'}_{(\zeta_{j,k},z_{j,k})}(0, ie_\Omega)\Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}-\mathbf{s}'}(h_k)\Big)} \in \ell^{p',q'}(J, K),$$

so that the conclusion follows from Theorem 4.11 and [16, Theorem 2.47]. □

The second part of the following result extends: [10, Theorem 2.2], which corresponds to the case in which $p = q > \frac{2(n+m)}{2(n+m)+1}$, $\mathbf{s}' - \frac{1}{q}\mathbf{s} \succ -\left(\frac{1}{2} + \frac{1}{2q}\right)\mathbf{b} - \frac{1}{q}\mathbf{d} + \frac{1}{2}\mathbf{m}$, $-\left(\frac{1}{2} - \frac{1}{2q}\right)\mathbf{b} - \left(1 - \frac{1}{q}\right)\mathbf{d} + \frac{1}{2q}\mathbf{m}'$, and $\mathbf{s}' - \mathbf{s} \succ -\mathbf{d} + \frac{1}{2}\mathbf{m}'$; [8, Theorem 4.2], which deals with the extension of [10, Theorem 2.2] to the case of general homogeneous domains.[7]

**Theorem 4.14** *Take $p, q \in (0, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that the following hold:*

- $\mathbf{s} \succ \frac{1}{2q}\mathbf{m} + \left(\frac{1}{2\min(1,p)} - \frac{1}{2q}\right)_+ \mathbf{m}'$;
- $\mathbf{s} + \mathbf{s}' \prec \frac{1}{\min(1,p,q)}\mathbf{b} + \frac{1}{\min(1,p)}\mathbf{d} - \frac{1}{2q}\mathbf{m}' - \left(\frac{1}{2\min(1,p)} - \frac{1}{2q}\right)_+ \mathbf{m}$.

*Then, properties $(\mathcal{L}')^{p,q}_{\mathbf{s},\mathbf{s}',0}$ and $(\mathcal{L}')^{p,q}_{\mathbf{s},\mathbf{s}'}$ hold.*

*In particular, if $q < \infty$ and:*

- $\mathbf{s} \succ \frac{1}{2}(\mathbf{m} - \mathbf{b}) + \left(\frac{q}{2\min(1,p)} - \frac{1}{2}\right)_+ \mathbf{m}'$;
- $\mathbf{s}' - \frac{1}{q}\mathbf{s} \succ \left(\frac{1}{2} + \frac{1}{2q} - \frac{1}{\min(1,p,q)}\right)\mathbf{b} - \left(\frac{1}{p} - 1\right)_+ \mathbf{d} + \frac{1}{2q}\mathbf{m}' + \left(\frac{1}{2\min(1,p)} - \frac{1}{2q}\right)_+ \mathbf{m}$;

*then property $(\mathfrak{L}')^{p,q}_{\mathbf{s},\mathbf{s}'}$ holds.*

More precisely, the proof shows that the mapping $\Psi$ of Definition 4.12 has a continuous linear section for $\delta$ sufficiently small and $R$ bounded.

***Proof*** Take an $F$-$(\delta, R)$-lattice $(\zeta_k, z_{j,k})_{j\in J,k\in K}$ on $D$ for some $\delta > 0$ and some $R > 1$. We shall further assume, as in the proof of Theorem 4.11, that $K = K' \times K''$, and that there is a $(\delta, R)$-lattice $(h_{k''})_{k''\in K''}$ on $\Omega$ such that $h_{k''} = \rho(\zeta_{(k',k'')}, z_{j,(k',k'')})$ for every $j \in J$ and for every $(k', k'') \in K' \times K''$. We shall also write $h_{(k',k'')}$ instead of $h_{k''}$ when it simplifies the notation.

In addition, define

$$B^{\mathbf{s}''}_{j,k}:=B^{\mathbf{s}''}_{(\zeta_k, z_{j,k})}$$

for every $\mathbf{s}'' \in \mathbb{R}^r$ and for every $(j, k) \in J \times K$, in order to simplify the notation. Further, for every $\lambda \in \mathbb{C}^{J\times K}$ define

$$\Psi_+(\lambda):=\sum_{j,k} |\lambda_{j,k} B^{\mathbf{s}'}_{j,k}|\Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}-\mathbf{s}'}(h_k) \in [0, \infty]^D,$$

and, for every $\lambda \in \mathbb{C}^{(J\times K)}$,

$$\Psi(\lambda):=\sum_{j,k} \lambda_{j,k} B^{\mathbf{s}'}_{j,k}\Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}-\mathbf{s}'}(h_k) \in \mathrm{Hol}(D).$$

We shall first prove that $\lambda \mapsto \|\Psi_+(\lambda)\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)}$ is a continuous quasi-norm on $\ell^{p,q}(J, K)$. Arguing as in [16, Proposition 3.32] and using Proposition 4.7, this will prove that $\Psi$ induces continuous linear mappings $\ell^{p,q}_0(J, K) \to \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ and $\ell^{p,q}(J, K) \to \mathcal{A}^{p,q}_{\mathbf{s}}(D)$. We shall then prove that these mappings are onto and have continuous linear sections.

Step I. Assume first that $q \leq p \leq 1$. Then, for every $\zeta \in E$ and for every $h \in \Omega$,

$$\|\Psi_+(\lambda)^{(\zeta)}_h\|^p_{L^p(F)} \leq \sum_{j,k} |\lambda_{j,k}|^p \Delta_\Omega^{(p/q)\mathbf{b}+\mathbf{d}-p(\mathbf{s}+\mathbf{s}')}(h_k)\|(B^{\mathbf{s}'}_{j,k})^{(\zeta)}_h\|^p_{L^p(F)}.$$

In addition, [16, Lemma 2.39] shows that there is a constant $C_1 > 0$ such that

$$\|(B^{\mathbf{s}'}_{j,k})^{(\zeta)}_h\|^p_{L^p(F)} = C_1\Delta_\Omega^{p\mathbf{s}'-\mathbf{d}}(h + h_k + \Phi(\zeta - \zeta_k))$$

for every $(j, k) \in J \times K$, for every $\zeta \in E$, and for every $h \in \Omega$. Therefore, using the subadditivity of the mapping

[7] We note that the statement of [8, Theorem 4.2] requires far more restrictive (or even uncomparable) conditions than those of [10, Theorem 2.2], despite the fact that its proof is omitted and stated to be formally equal to that of [10, Theorem 2.2]. Concerning this fact,we simply observe that the notation of [10] and [8] are quite different, and that even the statement of [8, Proposition 4.3] does not match the one in the cited reference [10, Theorem B].

$x \mapsto x^{q/p}$ on $\mathbb{R}_+$,

$$
\begin{aligned}
&\|\Psi_+(\lambda)\|^q_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)}\\
&\quad\le C_1^{q/p}\sum_k\left(\sum_j|\lambda_{j,k}|^p\right)^{q/p}\Delta_\Omega^{\mathbf{b}+(q/p)\mathbf{d}-q(\mathbf{s}+\mathbf{s}')}(h_k)\\
&\quad\times\int_\Omega\int_E\Delta_\Omega^{q\mathbf{s}'-(q/p)\mathbf{d}}(h+h_k+\Phi(\zeta-\zeta_k))\mathrm{d}\zeta\Delta_\Omega^{q\mathbf{s}}(h)\mathrm{d}\nu_\Omega(h).
\end{aligned}
$$

Now, [16, Corollary 2.22 and Lemma 2.32] imply that there is a constant $C_2>0$ such that

$$
\begin{aligned}
&\int_\Omega\int_E\Delta_\Omega^{q\mathbf{s}'-(q/p)\mathbf{d}}(h+h_k+\Phi(\zeta-\zeta_k))\,\mathrm{d}\zeta\,\Delta_\Omega^{q\mathbf{s}}(h)\,\mathrm{d}\nu_\Omega(h)\\
&\quad=C_2\Delta_\Omega^{q(\mathbf{s}+\mathbf{s}'-\mathbf{b}/q-\mathbf{d}/p)}(h_k)
\end{aligned}
$$

for every $k\in K$. Hence,

$$
\|\Psi_+(\lambda)\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)}\le C_1^{1/p}C_2^{1/q}\|\lambda\|_{\ell^{p,q}(J,K)},
$$

whence our claim in this case.

STEP II. Assume, now, that $q\ge p\le 1$. For every $k\in K$, choose $\tau_k\in C_c(E\times\Omega)$ such that

$$
\chi_{B_{E\times\Omega}((\zeta_k,h_k),\delta/2)}\le\tau_k\le\chi_{B_{E\times\Omega}((\zeta_k,h_k),\delta)}.
$$

Observe that by the computations of STEP I and [16, Corollary 2.49], there is a constant $C_1'>0$ such that

$$
\begin{aligned}
&\|\Psi_+(\lambda)_h^{(\zeta)}\|^p_{L^p(F)}\\
&\quad\le C_1'\int_{E\times\Omega}\sum_k\lambda'_k\tau_k(\zeta',h')\Delta_\Omega^{(p/q)\mathbf{b}+\mathbf{d}-p(\mathbf{s}+\mathbf{s}')}(h')\\
&\quad\times\Delta_\Omega^{p\mathbf{s}'-\mathbf{d}}(h+h'+\Phi(\zeta-\zeta'))\,\mathrm{d}\nu_{E\times\Omega}(\zeta',h'),
\end{aligned}
$$

where $\lambda'=\left(\sum_j|\lambda_{j,k}|^p\right)_k$. Hence, by Minkowski's integral inequality and Young's inequality, and by [16, Lemma 2.32],

$$
\begin{aligned}
&\|\Psi_+(\lambda)_h\|^p_{L^{p,q}(F,E)}\\
&\quad\le C_1'\left\|\int_{E\times\Omega}\sum_k\lambda'_k\tau_k(\zeta',h')\Delta_\Omega^{(p/q)\mathbf{b}+\mathbf{d}-p(\mathbf{s}+\mathbf{s}')}(h')\right.\\
&\quad\left.\times\Delta_\Omega^{p\mathbf{s}'-\mathbf{d}}(h+h'+\Phi(\,\cdot\,-\zeta'))\,\mathrm{d}\nu_{E\times\Omega}(\zeta',h')\right\|_{L^{q/p}(E)}\\
&\quad\le C_1'\int_\Omega\left\|\left(\sum_k\lambda'_k\tau_k(\,\cdot\,,h')\right)*\Delta_\Omega^{p\mathbf{s}'-\mathbf{d}}(h+h'\right.\\
&\quad\left.+\Phi(\,\cdot\,))\right\|_{L^{q/p}(E)}\Delta_\Omega^{[(p/q)+1]\mathbf{b}+\mathbf{d}-p(\mathbf{s}+\mathbf{s}')}(h')\,\mathrm{d}\nu_\Omega(h')\\
&\quad\le C_1'\int_\Omega\sum_{k''}\left\|\sum_{k'}\lambda'_{(k',k'')}\tau_{(k',k'')}(\,\cdot\,,h')\right\|_{L^{q/p}(E)}\\
&\quad\times\|\Delta_\Omega^{p\mathbf{s}'-\mathbf{d}}(h+h'+\Phi(\,\cdot\,))\|_{L^1(E)}\\
&\quad\times\Delta_\Omega^{[(p/q)+1]\mathbf{b}+\mathbf{d}-p(\mathbf{s}+\mathbf{s}')}(h')\,\mathrm{d}\nu_\Omega(h')\\
&\quad\le C_1''\int_\Omega\sum_{k''}\chi_{B_\Omega(h_{k''},\delta)}(h')\left\|\left(\lambda'_{(k',k'')}\right)_{k'}\right\|_{L^{q/p}(K')}\\
&\quad\times\Delta_\Omega^{p\mathbf{s}'-\mathbf{b}-\mathbf{d}}(h+h')\Delta_\Omega^{\mathbf{b}+\mathbf{d}-p(\mathbf{s}+\mathbf{s}')}(h')\,\mathrm{d}\nu_\Omega(h').
\end{aligned}
$$

If we define $f:=\sum_{k''}\left\|\left(\lambda'_{(k',k'')}\right)_{k'}\right\|_{L^{q/p}(K')}\chi_{B_\Omega(h_{k''},\delta)}$, then it is clear that

$$
\begin{aligned}
\|f\|_{L^{q/p}(\nu_\Omega)}&\le\nu_\Omega(B_\Omega(e_\Omega,\delta))^{p/q}\|\lambda'\|_{L^{q/p}(K)}\\
&=\nu_\Omega(B_\Omega(e_\Omega,\delta))^{p/q}\|\lambda\|^p_{L^{p,q}(J,K)}.
\end{aligned}
$$

Then, [16, Lemma 3.35] implies that there is a constant $C_3>0$ such that

$$
\|\Psi_+(\lambda)\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)}\le C_3\|\lambda\|_{L^{p,q}(J,K)},
$$

which completes the proof of our claim in this case.

STEP III. Assume, now, that $p,q\ge 1$. For every $(j,k)\in J\times K$, choose $\tau_{j,k}\in C_c(\Omega)$ so that

$$
\chi_{B((\zeta_k,z_{j,k}),\delta/2)}\le\tau_{j,k}\le\chi_{B((\zeta_k,z_{j,k}),\delta)},
$$

and define

$$
C_4:=\sup_{(\zeta,h)\in E\times\Omega}\int_F\left(\chi_{B((0,ie_\Omega),\delta)}\right)_h^{(\zeta)}\,\mathrm{d}\mathcal{H}^m.
$$

Define

$$
\Psi'\colon\ell^{p,q}(J,K)\ni\lambda\mapsto\sum_{j,k}\lambda_{j,k}\tau_{j,k}\Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}}\circ\rho\in C(D),
$$

and let us prove that $\Psi'$ maps continuously $\ell^{p,q}(J,K)$ into $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$. Indeed, take $\lambda\in\ell^{p,q}(J,K)$ and observe that

$$
\begin{aligned}
&\|(\Psi'(\lambda))_h^{(\zeta)}\|_{L^p(F)}\\
&\quad\le\Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}}(h)\sum_{k\in K}\left\|\sum_{j\in J}|\lambda_{j,k}|\left(\chi_{B((\zeta_k,z_{j,k}),\delta)}\right)_h^{(\zeta)}\right\|_{L^p(F)}\\
&\quad\le C_4^{1/p}\sum_{k\in K}\frac{\Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}}(h)}{\Delta_\Omega^{\mathbf{d}/p}(h_k)}\chi_{B((\zeta_k,h_k),\delta)}(\zeta,h)\|(\lambda_{j,k})_j\|_{\ell^p(J)}
\end{aligned}
$$

for every $h\in\Omega$, so that by [16, Corollary 2.49], there is a constant $C_4'>0$ such that

$$
\begin{aligned}
&\|\Psi'(\lambda)\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)}\\
&\quad\le C_4'\left\|\sum_{k\in K}\chi_{B((\zeta_k,h_k),\delta)}\|(\lambda_{j,k})_j\|_{\ell^p(J)}\right\|_{L^q(\nu_{E\times\Omega})}\\
&\quad=C_4'\nu_{E\times\Omega}(B_{E\times\Omega}((0,e_\Omega),\delta))^{1/q}\|\lambda\|_{\ell^{p,q}(J,K)}.
\end{aligned}
$$

Thus, $\Psi'$ induces a continuous linear mapping $\ell^{p,q}(J,K)\to\mathcal{L}^{p,q}_{\mathbf{s}}(D)$.

Now, observe that [16, Theorem 2.47 and Corollary 2.49] imply that there is a constant $C''_4 > 0$ such that

$$|\Psi_+(\lambda)_h(\zeta,x)| \le C''_4 \int_\Omega \int_{\mathcal{N}} \Psi'(|\lambda|)_{h'}(\zeta',x') \times \left| \left( B^{\mathbf{s}'}_{(\zeta',x'+i\Phi(\zeta')+ih')} \right)_h (\zeta,x) \right| \mathrm{d}(\zeta',x') \Delta_\Omega^{\mathbf{b}+\mathbf{d}-\mathbf{s}'}(h') \,\mathrm{d}\nu_\Omega(h') = C''_4 \int_\Omega \left( \Psi'(|\lambda|)_{h'} * \left| \left( B^{\mathbf{s}'}_{(0,ih')} \right)_h \right| \right)(\zeta,x) \Delta_\Omega^{\mathbf{b}+\mathbf{d}-\mathbf{s}'}(h') \,\mathrm{d}\nu_\Omega(h')$$

for every $\lambda \in \ell^{p,q}(J,K)$, for every $h \in \Omega$, and for every $(\zeta,x) \in \mathcal{N}$. Therefore, Minkowski's integral inequality, Young's inequality (applied twice), and [16, Lemma 2.39] show that there is a constant $C_5 > 0$ such that

$$\|\Psi_+(\lambda)_h\|_{L^{p,q}(F,E)} \le C_5 \int_\Omega \|\Psi'(|\lambda|)_{h'}\|_{L^{p,q}(F,E)} \times \Delta_\Omega^{\mathbf{s}'-(\mathbf{b}+\mathbf{d})}(h+h') \Delta_\Omega^{\mathbf{b}+\mathbf{d}-\mathbf{s}'}(h') \,\mathrm{d}\nu_\Omega(h')$$

for every $\lambda \in \ell^{p,q}(J,K)$, and for every $h \in \Omega$.

Define

$$T' : f \mapsto \Delta_\Omega^{\mathbf{s}} \int_\Omega f(h') \Delta_\Omega^{\mathbf{s}'-(\mathbf{b}+\mathbf{d})}(\cdot + h') \times \Delta_\Omega^{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(h') \,\mathrm{d}\nu_\Omega(h').$$

so that $T'$ induces an endomorphism of $L^q(\nu_\Omega)$ by [16, Lemma 3.35]. Then,

$$\|\Psi_+(\lambda)\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)} \le C_5 \|T'\|_{\mathscr{L}(L^q(\nu_\Omega))} \|\Psi'(|\lambda|)\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)} \le C_5 \|T'\|_{\mathscr{L}(L^q(\nu_\Omega))} C'_4 \nu_{E\times\Omega}(B_{E\times\Omega}((0,e_\Omega),\delta))^{1/q} \times \|\lambda\|_{\ell^{p,q}(J,K)}.$$

Our claim then follows also in this case.

Step IV. Finally, assume that $p \ge 1 \ge q$. For every $(j,k) \in J \times K$, choose $\tau'_{j,k} \in C_c(F)$ so that

$$(\chi_{B((\zeta_k,z_{j,k}),\delta/2)})^{(\zeta_k)}_{h_k} \le \tau'_{j,k} \le (\chi_{B((\zeta_k,z_{j,k}),\delta)})^{(\zeta_k)}_{h_k},$$

and define

$$\Psi'' : \ell^{p,q}(J,K) \ni \lambda \mapsto \left( \sum_j \lambda_{j,k} \tau'_{j,k} \Delta_\Omega^{\mathbf{d}/p}(h_k) \right)_k \in C_c(F)^K.$$

As in Step III, one may show that $\Psi''$ induces a continuous linear mapping of $\ell^{p,q}(J,K)$ into $\ell^q(K; L^p(F))$. In addition, by means of [16, Theorem 2.47] we see that there is a constant $C_6 > 0$ such that

$$|\Psi_+(\lambda)^{(\zeta)}_h(x)| \le C_6 \sum_{k\in K} \Delta_\Omega^{\mathbf{b}/q+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(h_k) \times \int_F \Psi''(|\lambda|)_k(x') \left| \left( B^{\mathbf{s}'}_{(\zeta_k,x'+i\Phi(\zeta_k)+ih_k)} \right)^{(\zeta)}_h (x) \right| \mathrm{d}x' = C_6 \sum_{k\in K} \Delta_\Omega^{\mathbf{b}/q+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(h_k) \times \left( \Psi''(|\lambda|)_k * \left| \left( B^{\mathbf{s}'}_{(\zeta_k,i\Phi(\zeta_k)+ih_k)} \right)^{(\zeta)}_h \right| \right)(x)$$

for every $\lambda \in \ell^{p,q}(J,K)$, for every $(\zeta,h) \in E \times \Omega$, and for every $x \in F$. Then, by Minkowski's inequality, Young's inequality, and [16, Lemma 2.39], there is a constant $C'_6 > 0$ such that

$$\|\Psi_+(\lambda)^{(\zeta)}_h\|_{L^p(F)} \le C'_6 \sum_{k\in K} \|\Psi''(|\lambda|)_k\|_{L^p(F)} \Delta_\Omega^{\mathbf{b}/q+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(h_k) \times \Delta_\Omega^{\mathbf{s}'-\mathbf{d}}(h+h_k+\Phi(\zeta-\zeta_k))$$

for every $\lambda \in \ell^{p,q}(J,K)$ and for every $(\zeta,h) \in E \times \Omega$. Then, by the subadditivity of the mapping $x \mapsto x^q$ on $\mathbb{R}_+$,

$$\|\Psi_+(\lambda)\|^q_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)} \le C'^q_6 \sum_k \|\Psi''(|\lambda|)_k\|^q_{L^p(F)} \times \Delta_\Omega^{\mathbf{b}+q(\mathbf{d}-\mathbf{s}-\mathbf{s}')}(h_k) \int_{E\times\Omega} \Delta_\Omega^{q\mathbf{s}-\mathbf{b}}(h) \times \Delta_\Omega^{q(\mathbf{s}'-\mathbf{d})}(h+h_k+\Phi(\zeta-\zeta_k)) \,\mathrm{d}\nu_{E\times\Omega}(\zeta,h).$$

Now, by homogeneity,

$$\Delta_\Omega^{\mathbf{b}+q(\mathbf{d}-\mathbf{s}-\mathbf{s}')}(h_k) \int_{E\times\Omega} \Delta_\Omega^{q(\mathbf{s}'-\mathbf{d})}(h+h_k+\Phi(\zeta-\zeta_k)) \times \Delta_\Omega^{q\mathbf{s}-\mathbf{b}}(h) \,\mathrm{d}\nu_{E\times\Omega}(\zeta,h) = \int_{E\times\Omega} \Delta_\Omega^{q\mathbf{s}-\mathbf{b}}(h) \Delta_\Omega^{q(\mathbf{s}'-\mathbf{d})}(h+e_\Omega+\Phi(\zeta)) \,\mathrm{d}\nu_{E\times\Omega}(\zeta,h)$$

for every $k \in K$, and the last integral is finite by [16, Corollary 2.22 and Lemma 2.32]. Therefore, there is a constant $C_7 > 0$ such that

$$\|\Psi_+(\lambda)\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)} \le C_7 \|\Psi''(|\lambda|)\|_{\ell^q(K;L^p(F))},$$

whence our claim also in this case.

Step V. Put a well-ordering on $J \times K$ and define

$$U_{j,k} := B((\zeta_k, z_{j,k}), R\delta) \setminus \left( \bigcup_{(j',k')<(j,k)} B((\zeta_{k'}, z_{j',k'}), R\delta) \right)$$

for every $(j,k) \in J \times K$, so that $(U_{j,k})_{(j,k)\in J\times K}$ is a Borel measurable partition of $D$ (since $J$ and $K$ are countable). In addition, define $c_{j,k} := c\nu_D(U_{j,k})$ for every

$(j,k)\in J\times K$, where $c>0$ is defined so that $P_{\mathbf{s}'}f=c\int_D f(\zeta,z)B^{\mathbf{s}}_{(\zeta,z)}\Delta_\Omega^{-\mathbf{s}'}(\rho(\zeta,z))\,\mathrm{d}\nu_D(\zeta,z)$ for every $f\in C_c(D)$. Then,

$$c\,\nu_D(B((0,ie_\Omega),\delta))\leq c_{j,k}\leq c\,\nu_D(B((0,ie_\Omega),R\delta))$$

for every $(j,k)\in J\times K$. Then, define

$$S:\mathcal{A}^{p,q}_{\mathbf{s}}(D)\ni f\mapsto\left(c_{j,k}\Delta_\Omega^{\mathbf{s}-\mathbf{b}/q+\mathbf{d}/p}(h_k)f(\zeta_k,z_{j,k})\right)\in\ell^{p,q}(J,K),$$

so that Theorem 4.11 shows that $S$ is well defined and continuous, and maps $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ into $\ell^{p,q}_0(J,K)$. Define $S':=\Psi S$. Then, Proposition 4.8 implies that, for every $f\in\mathcal{A}^{p,q}_{\mathbf{s}}(D)$,

$$f-S'f=c\sum_{j,k}\int_{U_{j,k}}\Big(f(\zeta',z')B^{\mathbf{s}'}_{(\zeta',z')}\Delta_\Omega^{-\mathbf{s}'}(\rho(\zeta',z'))-f(\zeta_k,z_{j,k})B^{\mathbf{s}'}_{(\zeta_k,z_{j,k})}\Delta_\Omega^{-\mathbf{s}'}(h_k)\Big)\,\mathrm{d}\nu_D(\zeta',z').$$

Hence, [16, Theorem 2.47, Corollary 2.49, and Lemma 3.25] imply that there are $R_0'>0$ and $C_8>0$ such that

$$\begin{aligned}|(f-S'f)(\zeta,z)|&\leq C_8R\delta\sum_{j,k}c_{j,k}\sup_{(\zeta',z')\in B((\zeta_k,z_{j,k}),R\delta+R_0')}|f(\zeta',z')|\\&\quad\times|B^{\mathbf{s}'}_{(\zeta_k,z_{j,k})}(\zeta,z)|\Delta_\Omega^{-\mathbf{s}'}(h_k)\end{aligned}$$

for every $(\zeta,z)\in D$. Fix an $F$-$(1,4)$-lattice $(\zeta'_{k'},z'_{j',k'})_{j'\in J',k'\in K'}$ on $D$ and observe that the proof of [16, Proposition 2.56], together with [16, Theorem 2.47 and Corollary 2.49] again, implies that there is a constant $C_9>0$ such that

$$\begin{aligned}|(f-S'f)(\zeta,z)|&\leq C_9R\delta\sum_{j',k'}\sup_{(\zeta',z')\in B\left((\zeta'_{k'},z'_{j',k'}),R\delta+R_0'+4\right)}|f(\zeta',z')|\\&\quad\times|B^{\mathbf{s}'}_{(\zeta'_{k'},z'_{j',k'})}(\zeta,z)|\Delta_\Omega^{-\mathbf{s}'}(h'_{k'})\end{aligned}$$

for every $(\zeta,z)\in D$, where $h'_{k'}=\rho(\zeta'_{k'},z'_{j',k'})$ for every $j'\in J'$ and for every $k'\in K'$. Hence, Theorem 4.11 and the preceding steps show that there is a constant $C_{10}>0$ such that, if $R\leq R_0$ and $\delta\leq 1$, then

$$\|f-S'f\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)}\leq C_{10}\delta\|f\|_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)}.$$

Take $\delta_0>0$ so that $C_{10}\delta_0\leq\frac{1}{2}$, and assume that $\delta\leq\delta_0$. Then,

$$\Big\|\sum_{j\geq k}(I-S')^jf\Big\|^{\min(1,p,q)}_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)}\leq\sum_{j\geq k}2^{-\min(1,p,q)j}\|f\|^{\min(1,p,q)}_{\mathcal{A}^{p,q}_{\mathbf{s}}(D)}$$

for every $k\in\mathbb{N}$, so that $\sum_{j\in\mathbb{N}}(I-S')^j$ induces well defined endomorphisms of $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ and $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$, which are inverses of $S'$. Hence,

$$\Psi''':=S\sum_{j\in\mathbb{N}}(I-S')^j$$

induces well-defined and continuous linear mappings from $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ into $\ell^{p,q}_0(J,K)$ and from $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ into $\ell^{p,q}(J,K)$, and $\Psi\Psi'''=S'\sum_{j\in\mathbb{N}}(I-S')^j=I$. The proof is complete. □

**Proposition 4.15** *Take $p,q\in(0,\infty]$ and $\mathbf{s},\mathbf{s}'\in\mathbb{R}^r$ such that property $(\mathcal{L}')^{p,q}_{\mathbf{s},\mathbf{s}',0}$ holds. Then, the sesquilinear mapping*

$$(f,g)\mapsto\int_D f\overline{g}(\Delta_\Omega^{-\mathbf{s}'}\circ\rho)\,d\nu_D$$

*induces an antilinear isomorphism of $\mathcal{A}^{p',q'}_{\mathbf{b}/\min(1,q)+\mathbf{d}/\min(1,p)-\mathbf{s}-\mathbf{s}'}(D)$ onto $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)'$.*

*In particular, if $q\in(1,\infty)$, $p<\infty$, $\mathbf{s}'\succ\frac{1}{2}(\mathbf{m}-\mathbf{d})$ and property $(\mathfrak{L}')^{p,q}_{\mathbf{s},\mathbf{s}'}$ holds, then the sesquilinear mapping*

$$(f,g)\mapsto\int_D f\overline{g}(\Delta_\Omega^{\mathbf{s}'-\mathbf{b}/2-\mathbf{d}}\circ\rho)\,d\nu_D$$

*induces an antilinear isomorphism of $\mathfrak{A}^{p',q'}_{q'\mathbf{s}'+(q'/q)\mathbf{s}+q'(1-1/p)_+\mathbf{d}}(D)$ onto $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)'$.*

***Proof*** By [16, Theorem 2.47], there is a constant $C>0$ such that

$$|B^{\mathbf{s}'}_{(\zeta,z)}-B^{\mathbf{s}'}_{(\zeta',z')}|\leq C|B^{\mathbf{s}'}_{(\zeta,z)}|d((\zeta,z),(\zeta',z'))$$

for every $(\zeta,z),(\zeta',z')\in D$ such that $d((\zeta,z),(\zeta',z'))\leq 1$. Using Proposition 4.7, one then verifies that the mapping $(\zeta,z)\mapsto B^{\mathbf{s}'}_{(\zeta,z)}\in\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ is antiholomorphic, so that the mapping

$$G\colon\mathcal{A}^{p,q}_{\mathbf{s},0}(D)'\ni L\mapsto[(\zeta,z)\mapsto\overline{\langle L,B^{\mathbf{s}'}_{(\zeta,z)}\rangle}]\in\mathrm{Hol}(D)$$

is well defined. In addition, Proposition 4.7 shows that $G(L)\in A^{\infty,\infty}_{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}-\mathbf{s}'}(D)$ for every $L\in\mathcal{A}^{p,q}_{\mathbf{s},0}(D)'$. In order to prove that $G(L)\in\mathcal{A}^{p',q'}_{\mathbf{s}''}(D)$, where $\mathbf{s}'':=\mathbf{b}/\min(1,q)+\mathbf{d}/\min(1,p)-\mathbf{s}-\mathbf{s}'$, by Theorem 4.11 it will then suffice to show that $S(G(L)):=(\Delta_\Omega^{\mathbf{s}''-\mathbf{b}/q'-\mathbf{d}/p'}(h_k)G(L)(\zeta_k,z_{j,k}))\in\ell^{p',q'}(J,K)$ for some $F$-$(\delta,4)$-lattice $(\zeta_k,z_{j,k})_{j\in J,k\in K}$ on $D$ for some sufficiently small $\delta>0$. We may then choose $(\zeta_k,z_{j,k})_{j\in J,k\in K}$ so that the corresponding map

$$\Psi:\ell^{p,q}(J,K)\ni\lambda\mapsto\sum_{j,k}\lambda_{j,k}B^{\mathbf{s}'}_{(\zeta_k,z_{j,k})}\Delta_\Omega^{\mathbf{b}/q+\mathbf{d}/p-\mathbf{s}-\mathbf{s}'}(h_k)\in\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$$

is well defined, continuous, and onto. Since

$$\sum_{j,k} \lambda_{j,k} S(G(L))_{j,k} = L(\Psi(\lambda))$$

for every $\lambda \in \ell^{p,q}(J,K)$, our claim follows.

Further, observe that Proposition 4.8 shows that

$$\int_D B^{\mathbf{s}'}_{(\zeta,z)} \overline{G(L)}(\Delta_\Omega^{-\mathbf{s}'} \circ \rho)\,\mathrm{d}\nu_D = \overline{G(L)(\zeta,z)} = \langle L, B^{\mathbf{s}'}_{(\zeta,z)} \rangle$$

for every $(\zeta,z) \in D$. Since the $B^{\mathbf{s}'}_{(\zeta,z)}$, as $(\zeta,z)$ run through $D$, form a total subset of $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ (for $\Psi$ is onto), by continuity, we see that

$$\int_D f\overline{G(L)}(\Delta_\Omega^{-\mathbf{s}'} \circ \rho)\,\mathrm{d}\nu_D = \langle L, f \rangle$$

for every $f \in \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$. To conclude, it suffices to show that if $g \in \mathcal{A}^{p',q'}_{\mathbf{s}''}(D)$ and

$$\int_D f\overline{g}(\Delta_\Omega^{-\mathbf{s}'} \circ \rho)\,\mathrm{d}\nu_D = 0$$

for every $f \in \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$, then $g = 0$. However, choosing $f = B^{\mathbf{s}'}_{(\zeta,z)}$ for every $(\zeta,z) \in D$ and using Proposition 4.8 again, it is readily seen that this is the case, whence the conclusion. □

### 4.5 Boundary values

In order to define the Besov spaces of analytic type which are necessary to describe the boundary values of the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$ and $\mathcal{A}^{p,q}_{\mathbf{s}}$, we shall need some preliminary results.

**Lemma 4.16** *The continuous linear mappings $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N}) \ni u \mapsto u(\zeta,\cdot) \in \mathcal{S}_{\overline{\Omega'}}(F)$, $\zeta \in E$, induce uniquely determined continuous linear mappings $\mathcal{S}'_{\overline{\Omega'}}(\mathcal{N}) \ni u \mapsto u(\zeta,\cdot) \in \mathcal{S}'_{\overline{\Omega'}}(F)$ such that the following hold:*

(1) *for every $u \in \mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$, for every $\zeta \in E$, and for every $\phi \in \mathcal{S}(F')$ supported in $\overline{\Omega'}$,*

$$(u * \mathcal{F}_{\mathcal{N}}^{-1}(\phi))(\zeta,\cdot) = u(\zeta,\cdot) * \mathcal{F}_F^{-1}(\phi);$$

(2) *for every $u \in \mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$, the mapping $E \ni \zeta \mapsto u(\zeta,\cdot) \in \mathcal{S}'_{\overline{\Omega'}}(F)$ is of class $C^\infty$.*

As a consequence, we shall identify each $u \in \mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$ with a map $E \ni \zeta \mapsto u(\zeta,\cdot) \in \mathcal{S}'_{\overline{\Omega'}}(F)$ of class $C^\infty$.

***Proof*** Fix $\phi \in \mathcal{S}(F')$ supported in $\overline{\Omega'}$ and define $\psi := \mathcal{F}_{\mathcal{N}}^{-1}(\phi)$ and $\psi' := \mathcal{F}_F^{-1}(\phi)$. Then, $\pi_\lambda(\psi) = \phi(\lambda) P_{\lambda,0}$ and $\pi_\lambda(\delta_0 \otimes \psi') = \phi(\lambda) I$ for every $\lambda \in \Lambda_+$, so that

$$u * \psi = u * (\delta_0 \otimes \psi')$$

for every $u \in \mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$, that is,

$$(u * \psi)(\zeta,\cdot) = u(\zeta,\cdot) * \psi'$$

for every $\zeta \in E$. In particular,

$$\begin{aligned}\langle u(\zeta,\cdot) | \psi' \rangle &= (u(\zeta,\cdot) * \psi'^*)(0) = (u * \psi^*)(\zeta,0)\\ &= \langle u | \psi((\zeta,0)^{-1}\cdot) \rangle,\end{aligned}$$

so that, by the arbitrariness of $\phi$, the mapping $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N}) \ni u \mapsto u(\zeta,\cdot) \ni \mathcal{S}_{\overline{\Omega'}}(F)$ is continuous for the topologies induced by $\mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$ and $\mathcal{S}'_{\overline{\Omega'}}(F)$ on $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ and $\mathcal{S}_{\overline{\Omega'}}(F)$, respectively. Since $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ is dense in $\mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$ (because the conjugate of $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ is reflexive and the polar of $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ in the conjugate of $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ is $\{0\}$), and since $\mathcal{S}'_{\overline{\Omega'}}(F)$ is complete, the first assertion follows, as well as (1). Assertion (2) is a consequence of the fact that the mapping $\zeta \mapsto \langle u(\zeta,\cdot)|\psi'\rangle = (u * \psi^*)(\zeta,0)$ is of class $C^\infty$ on $E$ for every $\phi$ (cf. [35, p. 59, Lemma II]). □

**Lemma 4.17** *Take $p,q,p_2,q_2 \in (0,\infty]$ with $p \leq p_2$ and $q \leq q_2$, and a bounded subset $B$ of $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ such that $\mathcal{F}_{\mathcal{N}} B$ is bounded in $C^\infty_c(\Omega')$. For every $\psi \in B$ and for every $t \in T_+$, define $\psi_t := \mathcal{F}_{\mathcal{N}}^{-1}((\mathcal{F}_{\mathcal{N}}\psi)(\cdot\, t^{-1}))$. Then, there is a constant $C > 0$ such that*

$$\begin{aligned}&\|u * \psi_t\|_{L^{p_2,q_2}(F,E)}\\ &\quad \leq C \Delta^{(1/q_2 - 1/q)\mathbf{b} + (1/p_2 - 1/p)\mathbf{d}}(t) \|u * \psi_t\|_{L^{p,q}(F,E)}\end{aligned}$$

*and*

$$\|u * \psi_t * \psi'_{t'}\|_{L^{p,q}(F,E)} \leq C \|u * \psi_t\|_{L^{p,q}(F,E)}$$

*for every $u \in \mathcal{S}'(\mathcal{N})$, for every $\phi, \phi' \in B$, and for every $t, t' \in T_+$.*

***Proof*** Step I. It will suffice to prove the first assertion when $p_2 = q_2 = \infty$ and $t$ is the identity of $T_+$, by Hölder's interpolation and homogeneity. Arguing by approximation as in the proof of [16, Corollary 4.7], we may further assume that $u \in \mathcal{S}(\mathcal{N})$. Then, set $\ell := \min(1,p,q)$ and take $\tau \in \mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ so that $\psi * \tau = \psi$ for every $\psi \in B$ and observe that

$$\begin{aligned}|(u*\psi)(\zeta,x)| &\leq \|u * \psi\|^{1-\ell}_{L^\infty(\mathcal{N})} \int_{\mathcal{N}} |(u*\psi)(\zeta',x')|^\ell\\ &\quad \times |\tau((\zeta',x')^{-1}(\zeta,x))|\,\mathrm{d}(\zeta',x')\\ &\leq \|u*\psi\|^{1-\ell}_{L^\infty(\mathcal{N})} \|u*\psi\|^\ell_{L^{p,q}(F,E)} \|\tau\|_{L^{(p/\ell)',(q/\ell)'}(F,E)}\end{aligned}$$

for every $(\zeta, x) \in \mathcal{N}$, whence the first assertion.

STEP II. The second assertion follows from [16, Corollary 4.10] and Lemma 4.16. □

**Definition 4.18** Take $p, q \in (0, \infty]$ and $\mathbf{s} \in \mathbb{R}^r$. Take a $(\delta, R)$-lattice $(\lambda_k)_{k\in K}$ in $\Omega'$ for some $\delta > 0$ and some $R > 1$ and fix a bounded family $(\phi_k)_{k\in K}$ of positive elements of $C_c^\infty(\Omega')$ such that $\sum_k \phi_k(\cdot\, t_k^{-1}) \geq 1$ on $\Omega'$, where $t_k \in T_+$ and $\lambda_k = e_{\Omega'} \cdot t_k$ for every $k \in K$. Define $\psi_k := \mathcal{F}_{\mathcal{N}}^{-1}(\phi_k(\cdot\, t_k^{-1}))$. Then, we define $\mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ (resp. $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$) as the space of the $u \in \mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$ such that

$$
\begin{aligned}
&(\Delta^{\mathbf{s}}_{\Omega'}(\lambda_k)\, u * \psi_k) \in \ell^q_0(K; L^{p,q}_0(F, E))\\
&(\text{resp. } (\Delta^{\mathbf{s}}_{\Omega'}(\lambda_k)\, u * \psi_k) \in \ell^q(K; L^{p,q}(F, E))),
\end{aligned}
$$

endowed with the corresponding topology.[8] We also define $\mathfrak{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega) := \mathcal{B}^{(\mathbf{s}-\mathbf{b}/2)/q}_{p,q}(\mathcal{N}, \Omega)$.

In particular, $\mathcal{B}^{\mathbf{s}}_{p,p}(\mathcal{N}, \Omega) = B^{\mathbf{s}}_{p,p}(\mathcal{N}, \Omega)$ and $\mathring{\mathcal{B}}^{\mathbf{s}}_{p,p}(\mathcal{N}, \Omega) = \mathring{B}^{\mathbf{s}}_{p,p}(\mathcal{N}, \Omega)$ for every $p \in (0, \infty]$ and for every $\mathbf{s} \in \mathbb{R}$. We now propose a different interpretation of $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ which is particularly useful in certain situations.

***Remark 4.19*** By Lemma 4.16, $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ may be equivalently defined as the space of $u \in \mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$ such that

$$
\left\| \zeta \mapsto \|u(\zeta, \cdot)\|_{B^{\mathbf{s}}_{p,q}(F,\Omega)} \right\|_{L^q(E)} < \infty,
$$

where $\|\cdot\|_{B^{\mathbf{s}}_{p,q}(F,\Omega)}$ denotes a fixed quasi-norm on $B^{\mathbf{s}}_{p,q}(F, \Omega)$. A similar description holds also for the space $\mathfrak{B}^{p,q}_{\mathbf{s}}(\mathcal{N}, \Omega)$.

**Proposition 4.20** *Take $p_1, p_2, q_1, q_2 \in (0, \infty]$ and $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^r$ so that*

$$
\begin{aligned}
&p_1 \leq p_2, \qquad q_1 \leq q_2, \qquad \text{and}\\
&\mathbf{s}_2 = \mathbf{s}_1 + \left(\frac{1}{q_1} - \frac{1}{q_2}\right)\mathbf{b} + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)\mathbf{d}.
\end{aligned}
$$

*Then, there are continuous inclusions $\mathring{\mathcal{B}}^{\mathbf{s}_1}_{p_1,q_1}(\mathcal{N}, \Omega) \subseteq \mathring{\mathcal{B}}^{\mathbf{s}_2}_{p_2,q_2}(\mathcal{N}, \Omega)$ and $\mathcal{B}^{\mathbf{s}_1}_{p_1,q_1}(\mathcal{N}, \Omega) \subseteq \mathcal{B}^{\mathbf{s}_2}_{p_2,q_2}(\mathcal{N}, \Omega)$. In addition, the mappings $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega) \ni u \mapsto u(\zeta, \cdot) \in B^{\mathbf{s}+\mathbf{b}/q}_{p,q}(F, \Omega)$, $\zeta \in E$, are equicontinuous for every $p, q \in (0, \infty]$ and for every $\mathbf{s} \in \mathbb{R}^r$.*

*In particular, if $q_2 < \infty$ and $\mathbf{s}_3 = \frac{q_2}{q_1}\mathbf{s}_1 + \left(\frac{q_2}{2q_1} - \frac{1}{2}\right)\mathbf{b} + \left(\frac{q_2}{p_1} - \frac{q_2}{p_2}\right)\mathbf{d}$, then $\mathfrak{B}^{\mathbf{s}_1}_{p_1,q_1}(\mathcal{N}, \Omega) \subseteq \mathfrak{B}^{\mathbf{s}_3}_{p_2,q_2}(\mathcal{N}, \Omega)$. In addition, the mappings $\mathfrak{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega) \ni u \mapsto u(\zeta, \cdot) \in \mathfrak{B}^{\mathbf{s}+\mathbf{b}/2}_{p,q}(F, \Omega)$, $\zeta \in E$, are equicontinuous for every $p, q \in (0, \infty]$ and for every $\mathbf{s} \in \mathbb{R}^r$.*

[8] One may prove directly that this definition does not depend on the choice of $(\lambda_k)$ and $(\phi_k)$, arguing as in the proof of [16, Lemma 4.14] and using Lemma 4.17. Nonetheless, this follows from Remark 4.19, at least for $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$.

***Proof*** This is a consequence of Lemmas 4.16 and 4.17, and of the continuous inclusion $\ell^{q_1}(K) \subseteq \ell^{q_2}(K)$, which holds for every set $K$. □

**Proposition 4.21** *Take $p, q \in (0, \infty]$ and $\mathbf{s} \in \mathbb{R}^r$. Then, $\mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$, $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$, and $\mathfrak{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ are complete and embed continuously into $\mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$. In addition, $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ is dense in $\mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$.*

***Proof*** Since $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega) \subseteq \mathcal{B}^{\mathbf{s}+\mathbf{b}/q+\mathbf{d}/p}_{\infty,\infty}(\mathcal{N}, \Omega) = B^{\mathbf{s}+\mathbf{b}/q+\mathbf{d}/p}_{\infty,\infty}(\mathcal{N}, \Omega)$ continuously by Proposition 4.20, by [14, Proposition 7.12], we see that $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ embeds continuously into $\mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$. Completeness follows from the facts that $\mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$ is complete, that $\mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ is closed in $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$, and that the norm of $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ extends to a lower semi-continuous function on $\mathcal{S}'_{\overline{\Omega'}}(\mathcal{N})$ which is finite only on $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$.

Next, observe that $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N}) \subseteq \mathcal{B}^{\mathbf{s}'}_{l,l}(\mathcal{N}, \Omega) \subseteq \mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ thanks to Theorem 3.13 and Proposition 4.20, where $l := \min(1, p, q)$ and $\mathbf{s}' = \mathbf{s} + \left(\frac{1}{q} - \frac{1}{l}\right)\mathbf{b} + \left(\frac{1}{q} - \frac{1}{l}\right)\mathbf{d}$. For what concerns the density of $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ in $\mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$, take $u \in \mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ and $(\psi_k)$ as in Definition 4.18. It is not difficult to prove that, if we assume that $\sum_k \mathcal{F}_{\mathcal{N}}\psi_k = 1$ on $\Omega'$, then $\sum_k u * \psi_k$ converges to $u$ in $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$. In addition, clearly $u * \psi_k \in L^p_0(\mathcal{N})$ for every $k \in K$. Then, take $\psi'_k \in \mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ so that $\mathcal{F}_{\mathcal{N}}\psi'_k$ is compactly supported in $\Omega'$ and equals 1 on the support of $\mathcal{F}_{\mathcal{N}}\psi_k$. If $(f_j)$ is a sequence of elements of $C_c^\infty(\mathcal{N})$ which converges locally uniformly to $u * \psi_k$ and satisfies $|f_j| \leq |u * \psi_k|$ for every $j \in \mathbb{N}$, then $f_j * \psi'_k$ belongs to $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ and converges to $u * \psi_k$ in $L^p_0(\mathcal{N})$ by dominated convergence, since $|u * \psi_k| * |\psi'_k| \in L^p_0(\mathcal{N})$ by [14, Corollary 3.6]. It is then readily verified that $f_j * \psi'_k$ converges to $u * \psi_k$ in $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ for every $k \in \mathbb{N}$, whence the conclusion. □

**Definition 4.22** Take $(\psi_k)$ as in Definition 4.18, and assume further that $\sum_k (\mathcal{F}_{\mathcal{N}}\psi_k)^2 = 1$ on $\Omega'$. Then, the mapping

$$
(u, u') \mapsto \sum_k \langle u * \psi_k | u * \psi_k \rangle = \int_E \langle u(\zeta, \cdot) | u'(\zeta, \cdot) \rangle \, \mathrm{d}\zeta
$$

induces a well-defined continuous sesquilinear form on $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega) \times \mathcal{B}^{-\mathbf{s}}_{p',q'}(\mathcal{N}, \Omega)$, for every $p, q \in [1, \infty]$ and for every $\mathbf{s} \in \mathbb{R}^r$, which does not depend on the choice of $(\psi_k)$ (cf. [16, the proof of Proposition 4.20]).

In particular, if $q > 1$, then the same expression defines a continuous sesquilinear form on $\mathfrak{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega) \times \mathfrak{B}^{-(q'/q)\mathbf{s}+(q'/2)\mathbf{b}}_{p',q'}(\mathcal{N}, \Omega)$.

In particular, by Proposition 4.20, there is a canonical sesquilinear form on

$$
\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega) \times \mathcal{B}^{-\mathbf{s}-(1/q-1)_+\mathbf{b}-(1/p-1)_+\mathbf{d}}_{p',q'}(\mathcal{N}, \Omega)
$$

for every $p, q \in (0, \infty]$ and for every $\mathbf{s} \in \mathbb{R}^r$. We denote by $\sigma^{\mathbf{s}}_{p,q}$ the weak topology

$$\sigma(\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega), \mathring{\mathcal{B}}^{-\mathbf{s}-(1/q-1)_+\mathbf{b}-(1/p-1)_+\mathbf{d}}_{p',q'}(\mathcal{N}, \Omega)).$$

**Lemma 4.23** *Take $p, q \in [1, \infty]$ and $\mathbf{s} \succ \frac{1}{q'}\mathbf{b} + \frac{1}{p'}\mathbf{d} + \frac{1}{2q}\mathbf{m}'$. For every $(\zeta, z) \in D$, define $S_{(\zeta,z)} := c\left(B^{\mathbf{b}+\mathbf{d}}_{(\zeta,z)}\right)_0$, where $c \neq 0$ is chosen so that $f(\zeta, z) = \langle f_0 | S_{(\zeta,z)} \rangle$ for every $f \in A^{2,\infty}_{\mathbf{0}}(D)$. Then, the following hold:*

(1) *the $\Delta^{\mathbf{s}-\mathbf{b}/q'-\mathbf{d}/p'}_{\Omega}(\rho(\zeta, z))S_{(\zeta,z)}$, as $(\zeta, z)$ runs through $D$, stay in a bounded subset of $\mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$;*
(2) *for every $u \in \mathcal{B}^{-\mathbf{s}}_{p',q'}(\mathcal{N}, \Omega)$, the mapping $(\zeta, z) \mapsto \langle u | S_{(\zeta,z)} \rangle$ is holomorphic on $D$.*

***Proof*** By homogeneity, in order to prove (1), it suffices to show that $S_{(0,ie_\Omega)} \in \mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$. To this aim, take $(\lambda_k)$ and $(\psi_k)$ as in Definition 4.18. By [16, The proof of Lemma 5.1 (2)], there is a constant $C > 0$ such that the family $(\mathrm{e}^{\langle \lambda_k, e_\Omega \rangle / C} \Delta^{\mathbf{b}/q'+\mathbf{d}/p'}_{\Omega'}(\lambda_k) S_{(0,ie_\Omega)} * \psi_k)$ is bounded in $L^{p,q}_0(F, E)$. Then, [16, Proposition 2.19, Lemmas 2.34 and 2.50, and Corollary 2.49] show that $S_{(0,ie_\Omega)} \in \mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$.

For what concerns (2), take $u \in \mathcal{B}^{-\mathbf{s}}_{p',q'}(\mathcal{N}, \Omega)$ and assume further that $(\psi_k)$ satisfies the conditions of Definition 4.22, so that

$$\langle u | S_{(\zeta,z)} \rangle = \sum_k \langle u * \psi_k | S_{(\zeta,z)} * \psi_k \rangle$$

for every $(\zeta, z) \in D$. Since (1) shows that the $S_{(\zeta,z)}$ are uniformly bounded in $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ as long as $(\zeta, z)$ stays in a bounded subset of $D$, by dominated convergence, it will suffice to prove that the function $(\zeta, z) \mapsto \langle u * \psi_k | S_{(\zeta,z)} * \psi_k \rangle$ is holomorphic on $D$ for every $k \in K$. Since $u * \psi_k \in L^\infty(\mathcal{N})$ by Lemma 4.17, and since the mapping $D \ni (\zeta, z) \mapsto S_{(\zeta,z)} * \psi_k \in \mathcal{S}(\mathcal{N})$ is holomorphic by [16, (1) of Proposition 4.2 and (1) of Lemma 5.1], the assertion follows. □

**Proposition 4.24** *Take $p, q \in (0, \infty]$ and $\mathbf{s} \succ \frac{1}{q}\mathbf{b} + \frac{1}{p}\mathbf{d} + \frac{1}{2q'}\mathbf{m}'$. Define a continuous linear mapping (cf. Lemma 4.23)*

$$\mathcal{E} : \mathcal{B}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega) \ni u \mapsto [(\zeta, z) \mapsto \langle u | S_{(\zeta,z)} \rangle] \in A^{\infty,\infty}_{\mathbf{s}-\mathbf{b}/q-\mathbf{d}/p}(D),$$

*Then, the following hold:*

(1) *for every $u \in \mathcal{B}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ and for every $(\zeta, z) \in D$ (so that $u(\zeta, \cdot) \in \mathcal{B}^{-\mathbf{s}+\mathbf{b}/q}_{p,q}(F, \Omega)$ with $\mathbf{s} - \frac{1}{q}\mathbf{b} \succ \frac{1}{p}\mathbf{d} + \frac{1}{2q'}\mathbf{m}'$ by Proposition 4.20)*

$$(\mathcal{E}u)(\zeta, z) = [\mathcal{E}(u(\zeta, \cdot))](z - i\Phi(\zeta));$$

(2) *the linear mappings $u \mapsto (\mathcal{E}u)_h$, as $h$ runs through $\Omega$, induce equicontinuous endomorphisms of $\mathcal{B}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ and $\mathring{\mathcal{B}}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$;*
(3) *the mapping*

$$\Omega \cup \{0\} \ni h \mapsto (\mathcal{E}u)_h \in \mathcal{B}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$$

*is continuous if $u \in \mathring{\mathcal{B}}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$, and is continuous in the weak topology $\sigma^{-\mathbf{s}}_{p,q}$ if $u \in \mathcal{B}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$.*

The preceding statement may be also translated in terms of the spaces $\mathfrak{B}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ (except for the statements concerning the weak topology $\sigma^{-\mathbf{s}}_{p,q}$, which may not be properly stated in terms of the spaces $\mathfrak{B}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ alone). This latter result extends: [2, Proposition 3.43], which corresponds to the case in which $p = q \geq 1$, $\mathbf{s}, \mathbf{s}' \in \mathbb{R}\mathbf{1}_r$, $n = 0$, and $\Omega$ is symmetric; [23, Proposition 4.1], which corresponds to the case in which $p = q \geq 1$, $n = 0$, and $\Omega$ is symmetric.

***Proof*** (1) It will suffice to prove the assertion when $u$ is replaced by $u * \psi$ for some $\psi \in \mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ with $\mathcal{F}_{\mathcal{N}}\psi \in C^\infty_c(\Omega')$. Then,

$$[\mathcal{E}(u * \psi)]_h = (u * \psi) * (S_{(0,ih)} * \psi'),$$

where $\psi' \in \mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ and $\psi = \psi * \psi'$. It then suffices to apply Lemma 4.16.

(2) The equicontinuity of the endomorphisms $u \mapsto (\mathcal{E}u)_h$, $h \in \Omega$, of $\mathcal{B}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ follows from (1), Remark 4.19 and [16, (1) of Theorem 5.2]. The fact that these endomorphisms preserve $\mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ follows from the fact that they preserve $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$, thanks to Proposition 4.21.

(3) If $u \in \mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$, then the mapping $\Omega \cup \{0\} \ni h \mapsto (\mathcal{E}u)_h \in \mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ is continuous by [16, (1) of Lemma 5.1] and [15, Proposition 5.2], so that (2) leads to the conclusion for $u \in \mathring{\mathcal{B}}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$. The conclusion for $u \in \mathcal{B}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ follows by transposition, thanks to the formula

$$\langle (\mathcal{E}u)_h | u' \rangle = \langle u | (\mathcal{E}u')_h \rangle$$

which holds for every $u' \in \mathring{\mathcal{B}}^{\mathbf{s}-(1/q-1)_+\mathbf{b}-(1/p-1)_+\mathbf{d}}_{p',q'}(\mathcal{N}, \Omega)$ and for every $h \in \Omega$ and may be proved reducing to the case $E = \{0\}$ by means of (1), in which case it follows from the proof of [16, (3) of Theorem 5.2].

□

**Definition 4.25** Take $\mathbf{s} \succ \frac{1}{q}\mathbf{b} + \frac{1}{p}\mathbf{d} + \frac{1}{2q'}\mathbf{m}'$. Then, we define $\widetilde{\mathcal{A}}^{p,q}_{\mathbf{s},0}(D)$ and $\widetilde{\mathcal{A}}^{p,q}_{\mathbf{s}}(D)$ as the images of the spaces $\mathring{\mathcal{B}}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ and $\mathcal{B}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ under $\mathcal{E}$, endowed with the corresponding topology.

If $q < \infty$ and $\mathbf{s}' \succ \frac{1}{2}\mathbf{b} + \frac{q}{p}\mathbf{d} + \frac{q}{2q'}\mathbf{m}'$, then we shall define $\widetilde{\mathfrak{A}}^{p,q}_{\mathbf{s}}(D) := \widetilde{\mathcal{A}}^{p,q}_{(\mathbf{s}+\mathbf{b}/2)/q}(D)$.

Assertions (1), (2), and (5) of the following result extend: [2, Proposition 3.31, Theorem 1.7, and Theorem 1.8], which corresponds to the case in which $p, q \geq 1$, $\mathbf{s} \in \mathbb{R}\mathbf{1}_r$, $n = 0$, and $\Omega$ is symmetric; [23, Remark 3.21, Theorem 1.2, and Theorem 4.8], which corresponds to the case in which $p, q \geq 1$, $n = 0$, and $\Omega$ is symmetric.

**Proposition 4.26** *Take $p, q \in (0, \infty]$ and $\mathbf{s} \in \mathbb{R}^r$. Then, the following hold:*

(1) *for every $\mathbf{s}' \in \mathbb{C}^r$, the mapping $\mathcal{S}_{\overline{\Omega'}}(\mathcal{N}) \ni \phi \mapsto \phi * I_\Omega^{\mathbf{s}'} \in \mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$ induces isomorphisms of $\mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ and $\mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ onto $\mathring{\mathcal{B}}^{\mathbf{s}+\mathrm{Re}\,\mathbf{s}'}_{p,q}(\mathcal{N}, \Omega)$ and $\mathcal{B}^{\mathbf{s}+\mathrm{Re}\,\mathbf{s}'}_{p,q}(\mathcal{N}, \Omega)$, respectively; in particular, if $q < \infty$, then it induces an isomorphism of $\mathfrak{B}^{\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$ onto $\mathfrak{B}^{\mathbf{s}+q\mathrm{Re}\,\mathbf{s}'}_{p,q}(\mathcal{N}, \Omega)$;*

(2) *if $\mathbf{s} \succ \frac{1}{q}\mathbf{b} + \frac{1}{p}\mathbf{d} + \frac{1}{2q'}\mathbf{m}'$ and $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$ (resp. $\mathbf{s} \geq \mathbf{0}$ if $q = \infty$), then there are continuous inclusions*

$$\mathcal{E}(\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})) \subseteq A^{p,q}_{\mathbf{s},0}(D) \subseteq \widetilde{A}^{p,q}_{\mathbf{s},0}(D)$$
$$(\text{resp. } \mathcal{E}(\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})) \subseteq A^{p,q}_{\mathbf{s}}(D) \subseteq \widetilde{A}^{p,q}_{\mathbf{s}}(D));$$

*in particular, if $q < \infty$ and $\mathbf{s} \succ \frac{1}{2}(\mathbf{m} - \mathbf{b}), \frac{1}{2}\mathbf{b} + \frac{q}{p}\mathbf{d} + \frac{q}{2q'}\mathbf{m}'$, then there are continuous inclusions $\mathcal{E}(\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})) \subseteq \mathfrak{A}^{p,q}_{\mathbf{s}}(D) \subseteq \widetilde{\mathfrak{A}}^{p,q}_{\mathbf{s}}(D)$;*

(3) *if $\mathbf{s} \succ \frac{1}{p}\mathbf{d} + \frac{1}{2q'}\mathbf{m}'$ and $A^{p,q}_{\mathbf{s},0}(T_\Omega) = \widetilde{A}^{p,q}_{\mathbf{s},0}(T_\Omega)$ (resp. $A^{p,q}_{\mathbf{s}}(T_\Omega) = \widetilde{A}^{p,q}_{\mathbf{s}}(T_\Omega)$), then $A^{p,q}_{\mathbf{s},0}(D) = \widetilde{A}^{p,q}_{\mathbf{s},0}(D)$ (resp. $A^{p,q}_{\mathbf{s}}(D) = \widetilde{A}^{p,q}_{\mathbf{s}}(D)$); in particular, if $q < \infty$, $\mathbf{s} \succ \frac{q}{p}\mathbf{d} + \frac{q}{2q'}\mathbf{m}'$ and $\mathfrak{A}^{p,q}_{\mathbf{s}}(T_\Omega) = \widetilde{\mathfrak{A}}^{p,q}_{\mathbf{s}}(T_\Omega)$, then $\mathfrak{A}^{p,q}_{\mathbf{s}-\mathbf{b}/2}(D) = \widetilde{\mathfrak{A}}^{p,q}_{\mathbf{s}-\mathbf{b}/2,0}(D)$;*

(4) *if $\mathbf{s} \succ \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2p'}\mathbf{m}'$ and $A^{p,p}_{\mathbf{s},0}(D) = \widetilde{A}^{p,p}_{\mathbf{s},0}(D)$ (resp. $A^{p,p}_{\mathbf{s}}(D) = \widetilde{A}^{p,p}_{\mathbf{s}}(D)$), then $A^{p,p}_{\mathbf{s}-\mathbf{b}/p,0}(T_\Omega) = \widetilde{A}^{p,p}_{\mathbf{s}-\mathbf{b}/p,0}(T_\Omega)$ (resp. $A^{p,p}_{\mathbf{s}-\mathbf{b}/p}(T_\Omega) = \widetilde{A}^{p,p}_{\mathbf{s}-\mathbf{b}/p}(T_\Omega)$); in particular, if $p < \infty$, $\mathbf{s} \succ \frac{1}{2}\mathbf{b} + \mathbf{d} + \frac{p}{2p'}\mathbf{m}'$ and $\mathfrak{A}^{p,p}_{\mathbf{s}}(D) = \widetilde{\mathfrak{A}}^{p,p}_{\mathbf{s}}(D)$, then $\mathfrak{A}^{p,p}_{\mathbf{s}-\mathbf{b}/2}(T_\Omega) = \widetilde{\mathfrak{A}}^{p,p}_{\mathbf{s}-\mathbf{b}/2}(T_\Omega)$;*

(5) *if $\mathbf{s} \succ \frac{1}{2q}\mathbf{m} + \left(\frac{1}{2\min(p,p')} - \frac{1}{2q}\right)_+ \mathbf{m}'$, then $A^{p,q}_{\mathbf{s}}(D) = \widetilde{A}^{p,q}_{\mathbf{s}}(D)$ and $A^{p,q}_{\mathbf{s},0}(D) = \widetilde{A}^{p,q}_{\mathbf{s},0}(D)$. In particular, if $q < \infty$ and $\mathbf{s} \succ \frac{1}{2}(\mathbf{m} - \mathbf{b}) + \left(\frac{q}{2\min(p,p')} - \frac{1}{2}\right)_+ \mathbf{m}'$, then $\mathfrak{A}^{p,q}_{\mathbf{s}}(D) = \widetilde{\mathfrak{A}}^{p,q}_{\mathbf{s}}(D)$.*

We observe explicitly that in [16, Corollary 5.11] the assumption $\mathbf{s} \succ \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$ is redundant (as the assumption $\mathbf{s} \succ \frac{1}{q}\mathbf{b} + \frac{1}{p}\mathbf{d} + \frac{1}{2q'}\mathbf{m}'$ would be redundant in (5) above), as it is implied by the condition $\mathbf{s} \succ \frac{1}{2q}\mathbf{m} + \left(\frac{1}{2\min(p,p')} - \frac{1}{2q}\right)_+ \mathbf{m}'$. Indeed, $\left(\frac{1}{\min(p,p')} - \frac{1}{q}\right)_+ \mathbf{m}' \geq \left(\frac{1}{q'} - \frac{1}{p}\right)\mathbf{m}' \geq \frac{2}{p}\mathbf{d} + \frac{1}{q'}\mathbf{m}'$ since $2\mathbf{d} \prec -\mathbf{m} - \mathbf{m}'$.

We also mention that, if $r = 2$ (so that $\Omega$ is isomorphic to either a quadrant or a Lorentz cone), then combining [6, Theorems 6.6 and 6.8] (the latter being a consequence of [13, Theorem 1.2]) with [16, Theorem 5.10] (cf. also [19, remarks following Remark 2.6]), we see that $A^{p,q}_{\mathbf{s}}(T_\Omega) = \widetilde{A}^{p,q}_{\mathbf{s}}(T_\Omega)$ and $A^{p,q}_{\mathbf{s},0}(T_\Omega) = \widetilde{A}^{p,q}_{\mathbf{s},0}(T_\Omega)$ if and only if

$$\mathbf{s} \succ \frac{1}{2q}\mathbf{m} + \frac{1}{2}\left(\frac{1}{\min(2,p)} - \frac{1}{q}\right)_+ \mathbf{m}', \frac{1}{p}\mathbf{d} + \frac{1}{2q'}\mathbf{m}'.$$

By transference, under the same condition, we also have $A^{p,q}_{\mathbf{s}}(D) = \widetilde{A}^{p,q}_{\mathbf{s}}(D)$ and $A^{p,q}_{\mathbf{s},0}(D) = \widetilde{A}^{p,q}_{\mathbf{s},0}(D)$.

***Proof*** (1) This follows from [16, Theorem 4.26] and Remark 4.19.

(2) By [16, Proposition 5.4], there are continuous linear mappings $\mathcal{B}\colon A^{q,q}_{\mathbf{s},0}(D) \to \mathring{\mathcal{B}}^{-\mathbf{s}}_{q,q}(\mathcal{N}, \Omega)$ and $\mathcal{B}'\colon A^{p,q}_{\mathbf{s},0}(T_\Omega) \to \mathring{\mathcal{B}}^{-\mathbf{s}}_{p,q}(F, \Omega)$ such that $\mathcal{E}\mathcal{B} = I$ and $\mathcal{E}\mathcal{B}' = I$. If $u \in A^{p,q}_{\mathbf{s},0}(D) \cap A^{q,q}_{\mathbf{s},0}(D)$, then Lemma 4.16, Remark 4.19, (1) of Proposition 4.24, and Proposition 4.6 imply that $(\mathcal{B}u)(\zeta, \cdot) = \mathcal{B}'(u(\zeta, \cdot))$ for every $\zeta \in E$, and that $\mathcal{B}$ induces a continuous linear mapping of $A^{p,q}_{\mathbf{s},0}(D)$ into $\mathring{\mathcal{B}}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$. Thus, $A^{p,q}_{\mathbf{s},0}(D) \subseteq \widetilde{A}^{p,q}_{\mathbf{s},0}(D)$ continuously. The inclusion $A^{p,q}_{\mathbf{s}}(D) \subseteq \widetilde{A}^{p,q}_{\mathbf{s}}(D)$ is proved similarly (cf. the proof of Theorem 4.31). The remaining inclusions are consequences of Lemma 4.16 and Theorem 3.13.

(3) This follows from Remark 4.19 and (1) of Proposition 4.24.

(4) This follows from [19, Theorem 6.3].

(5) This follows from (3) and [16, Corollary 5.11].

□

***Remark 4.27*** We observe explicitly that $\mathbf{s} \succ \frac{1}{2}(\mathbf{m} - \mathbf{b}) + \left(\frac{q}{2\min(p,p')} - \frac{1}{2}\right)_+ \mathbf{m}'$ if and only if $\mathbf{s} \succ \frac{1}{2}(\mathbf{m} - \mathbf{b})$ and

$$q < q_{\mathbf{s}}(p) := \min(p, p') \min_{j=1,\dots,r} \frac{2s_j + b_j + m'_j - m_j}{m'_j}.$$

**Proposition 4.28** *Take $p, q \in (0, \infty]$, and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that the following hold:*

- $\mathbf{s} \succ \frac{1}{q}\mathbf{b} + \frac{1}{p}\mathbf{d} + \frac{1}{2q'}\mathbf{m}'$;
- $\mathbf{s}' \prec \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$;
- $\mathbf{s} + \mathbf{s}' \prec \frac{1}{q}\mathbf{b} + \frac{1}{p}\mathbf{d} - \frac{1}{2\max(1,q)}\mathbf{m}'$.

*Then, there is a constant $c \neq 0$ such that*

$$\langle u | u' \rangle = c \int_D (\mathcal{E}u) \overline{\mathcal{E}u' * I_\Omega^{\mathbf{s}'-\mathbf{b}-\mathbf{d}}} (\Delta_\Omega^{-\mathbf{s}'} \circ \rho) \, \mathrm{d}\nu_D \tag{7}$$

*for every $u, u' \in \mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$. In particular, the sesquilinear form*

$$(f, g) \mapsto \int_D f \overline{g} (\Delta^{-\mathbf{s}'} \circ \rho) \, \mathrm{d}\nu_D$$

*induces a continuous sesquilinear form on* $\widetilde{\mathcal{A}}^{p,q}_{\mathbf{s}}(D) \times \widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}/\min(1,q)+\mathbf{d}/\min(1,p)-\mathbf{s}-\mathbf{s}'}(D)$.

As a consequence of Corollary 4.29, the above sesquilinear form induces an antilinear isomorphism of $\widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}/\min(1,q)+\mathbf{d}/\min(1,p)-\mathbf{s}-\mathbf{s}'}(D)$ onto $\widetilde{\mathcal{A}}^{p,q}_{\mathbf{s},0}(D)'$.

***Proof*** Formula (7) follows from [16, Proposition 5.12].[9] The assertion then follows easily. □

**Corollary 4.29** *Take* $p,q \in (0,\infty]$ *and* $\mathbf{s} \in \mathbb{R}^r$. *Then, the continuous sesquilinear form* $\langle\cdot|\cdot\rangle\colon \mathcal{B}^{\mathbf{s}}_{p,q}(\mathcal{N},\Omega) \times \mathcal{B}^{-\mathbf{s}-(1/q-1)_+\mathbf{b}-(1/p-1)_+\mathbf{d}}_{p',q'}(\mathcal{N},\Omega) \to \mathbb{C}$ *induces an antilinear isomorphism of* $\mathcal{B}^{-\mathbf{s}-(1/q-1)_+\mathbf{b}-(1/p-1)_+\mathbf{d}}_{p',q'}(\mathcal{N},\Omega)$ *onto* $\mathring{\mathcal{B}}^{\mathbf{s}}_{p,q}(\mathcal{N},\Omega)'$.

One may also state an analogous result for the spaces $\mathfrak{B}^{\mathbf{s}}_{p,q}$ (necessarily restricting to $p \in (0,\infty)$ and $q \in (1,\infty)$). This latter result extends: [2, Theorem 1.4 (3)], which corresponds to the case in which $p,q \geq 1$, $\mathbf{s} \in \mathbb{R}\mathbf{1}_r$, $n=0$, and $\Omega$ is symmetric; [23, Theorem 1.1 (3)], which corresponds to the case in which $p,q \geq 1$, $n=0$, and $\Omega$ is symmetric.

***Proof*** By Proposition 4.26, we may assume that $-\mathbf{s}$ is as large as we please, so that the assertion follows from Proposition 4.28, combined with (5) of Proposition 4.26, Theorem 4.14, and Proposition 4.15. □

### 4.6 Bergman Projectors

The second part of the following result extends: [3, Theorem 4.24], which corresponds to the case in which $\mathbf{s} = \mathbf{s}' \in \mathbb{R}\mathbf{1}_r$, $n=0$, and $\Omega$ is symmetric; [23, Corollary 1.4], which deals with the case in which $\mathbf{s}=\mathbf{s}'$, $n=0$ and $\Omega$ is symmetric; [30, Theorem 3.2 (i)], which corresponds to the case in which $p=q$, $\mathbf{s}=\mathbf{s}'$, and $n=0$.

**Proposition 4.30** *Take* $p,q \in [1,\infty]$, $\mathbf{s} \in \mathbb{R}^r$, *and* $\mathbf{s}' \prec \mathbf{b}+\mathbf{d}-\frac{1}{2}\mathbf{m}$. *If* $P_{\mathbf{s}'}$ *induces an endomorphism of* $\mathcal{L}^{p,q}_{\mathbf{s},0}(D)$ *(resp. a continuous linear mapping of* $\mathcal{L}^{p,q}_{\mathbf{s},0}(D)$ *into* $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$*), then the following hold:*

(1) $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$ *(resp.* $\mathbf{s} \geq \mathbf{0}$ *if* $q=\infty$*) and* $\mathbf{s} \succ \frac{1}{q}\mathbf{b}+\frac{1}{p}\mathbf{d}+\frac{1}{2q'}\mathbf{m}'$;

(2) $\mathbf{s}' \prec \frac{1}{\min(p,p')}\mathbf{d} - \frac{1}{2\min(p,p')}\mathbf{m}'$;

(3) $\mathbf{s}+\mathbf{s}' \prec \mathbf{b}+\mathbf{d}-\frac{1}{2q'}\mathbf{m}$ *or* $\mathbf{s}+\mathbf{s}' \leq \mathbf{b}+\mathbf{d}$ *if* $q'=\infty$, *and* $\mathbf{s}+\mathbf{s}' \prec \frac{1}{q}\mathbf{b}+\frac{1}{p}\mathbf{d}-\frac{1}{2q}\mathbf{m}'$;

(4) $P_{\mathbf{s}'}$ *induces continuous linear projectors of* $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$ *and* $\mathcal{L}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ *onto* $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ *and* $\mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$, *respectively, such that*

$$\int_D f\,\overline{P_{\mathbf{s}'}g}(\Delta^{\mathbf{s}'}_\Omega\circ\rho)\,\mathrm{d}\nu_D = \int_D (P_{\mathbf{s}'}f)\overline{g}(\Delta^{\mathbf{s}'}_\Omega\circ\rho)\,\mathrm{d}\nu_D \quad (8)$$

*for every* $f \in \mathcal{L}^{p,q}_{\mathbf{s}}(D)$ *and for every* $g \in \mathcal{L}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$.

*In particular, if* $q<\infty$, $\mathbf{s}' \succ \frac{1}{2}(\mathbf{m}-\mathbf{b})$, *and* $\mathfrak{P}_{\mathbf{s}'}$ *induces a continuous linear projector of* $\mathfrak{L}^{p,q}_{\mathbf{s}}(D)$ *onto* $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)$, *then:*

- $\mathbf{s} \succ \frac{1}{2}(\mathbf{m}-\mathbf{b}), \frac{1}{2}\mathbf{b}+\frac{q}{p}\mathbf{d}+\frac{q-1}{2}\mathbf{m}'$;
- $\mathbf{s}' \succ \frac{1}{2}\mathbf{b}+\frac{1}{\max(p,p')}\mathbf{d}+\frac{1}{2\min(p,p')}\mathbf{m}'$;
- $\mathbf{s}'-\frac{1}{q}\mathbf{s} \succ \frac{1}{2q'}\mathbf{b}+\frac{1}{p'}\mathbf{d}+\frac{1}{2q}\mathbf{m}'$ *and* $\mathbf{s}'-\frac{1}{q}\mathbf{s} \succ -\frac{1}{2q'}\mathbf{b}+\frac{1}{2q'}\mathbf{m}$ *or* $q=1$ *and* $\mathbf{s}'-\mathbf{s} \geq \mathbf{0}$.

Notice that (4) uniquely determines $P_{\mathbf{s}'}$ on $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$ and $\mathcal{L}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$, since conditions (1)–(3) ensure that $B^{-\mathbf{s}'}_{(\zeta,z)} \in \mathcal{A}^{p,q}_{\mathbf{s}}(D) \cap \mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ for every $(\zeta,z) \in D$ (cf. Proposition 4.8).

In addition, if $\mathbf{s} \succ \frac{1}{2}(\mathbf{m}-\mathbf{b})$, $p,q \in (1,\infty)$, and $\mathfrak{P}_{\mathbf{s}}$ induces a continuous linear projector of $\mathfrak{L}^{p,q}_{\mathbf{s}}(D)$ onto $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)$, then:

$$p'_{\mathbf{s}} < p < p_{\mathbf{s}} \quad \text{and} \quad Q'_{\mathbf{s}}(p') < q < Q_{\mathbf{s}}(p),$$

where

$$p_{\mathbf{s}} := \min_{j=1,\dots,r} \frac{m'_j - 2d_j}{(b_j + m'_j - 2s_j)_+} \quad \text{and}$$

$$Q_{\mathbf{s}}(p) := \min_{j=1,\dots,r} \frac{s_j - \frac{1}{2}b_j + \frac{1}{2}m'_j}{\left(\frac{1}{p}d_j + \frac{1}{2}m'_j\right)_+}.$$

***Proof*** In order to prove (1)–(3), observe first that $P_{\mathbf{s}'}(C_c(D)) \subseteq \mathrm{Hol}(D)$, so that $P_{\mathbf{s}'}$ maps $\mathcal{L}^{p,q}_{\mathbf{s},0}(D)$ into $A^{p,q}_{\mathbf{s},0}(D)$ (resp. $A^{p,q}_{\mathbf{s}}(D)$). Therefore, the linear mapping

$$\begin{aligned}&\mathcal{L}^{p,q}_{\mathbf{s},0}(D) \ni f \mapsto (P_{\mathbf{s}'}f)(\zeta,z)\\ &= c_{(\mathbf{b}+\mathbf{d}-\mathbf{s})/2}\int_D f\,\overline{B^{\mathbf{s}'}_{(\zeta,z)}}(\Delta^{-\mathbf{s}'}_\Omega\circ\rho)\,\mathrm{d}\nu_D \in \mathbb{C}\end{aligned}$$

is continuous for every $(\zeta,z) \in D$, so that $B^{\mathbf{s}'}_{(\zeta,z)} \in \mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$. By Proposition 4.7, in order to complete the proof of (1)–(3), it will suffice to prove that $B^{\mathbf{s}'}_{(0,ie_\Omega)} \in \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ (resp. $B^{\mathbf{s}'}_{(0,ie_\Omega)} \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$). This is a consequence of the fact that there is $f \in C_c(D)$ such that $P_{\mathbf{s}'}(f) = B^{\mathbf{s}'}_{(0,ie_\Omega)}$ (cf. the proof of [16, Proposition 5.20]).

[9] We mention explicitly that the statement of the cited result contains a typo: instead of $\Delta^{\mathbf{s}''}_\Omega$, there should be $\Delta^{\mathbf{s}''-(\mathbf{b}+\mathbf{d})}_\Omega$ in the first formula in display, as well as in the first line of the formula in display in the proof.

Concerning (4), one first shows that formula (8) holds for every $f \in \mathcal{L}^{p,q}_{\mathbf{s},0}(D)$ and for every $g \in C_c(D)$. This, in turn, allows to show that $P_{\mathbf{s}'}$ induces a continuous linear mapping of $\mathcal{L}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',0}(D)$ into $\mathcal{L}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ and to show that formula (8) holds for every $f \in \mathcal{L}^{p,q}_{\mathbf{s},0}(D)$ and for every $g \in \mathcal{L}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',0}(D)$. One then defines

$$(P_{\mathbf{s}'} f)(\zeta, z) = c_{(\mathbf{b}+\mathbf{d}-\mathbf{s}')/2} \int_D f \overline{B^{\mathbf{s}'}_{(\zeta,z)}} (\Delta_\Omega^{-\mathbf{s}'} \circ \rho) \, \mathrm{d}\nu_D$$

for every $f \in \mathcal{L}^{p,q}_{\mathbf{s}}(D)$ (resp. for every $f \in \mathcal{L}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$), and for every $(\zeta, z) \in D$ (as a consequence of the proof of (1)–(3)). The arguments of the proof of Proposition 4.15 then allow to show that the so-defined $P_{\mathbf{s}'} f$ is holomorphic on $D$. In addition, if $(\tau_j)$ is an increasing sequence of positive elements of $C_c(D)$ which converges locally uniformly to 1, then $P_{\mathbf{s}'}(\tau_j f)$ converges locally uniformly to $P_{\mathbf{s}'} f$ and has uniformly bounded $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$ (resp. $\mathcal{L}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$) norm. This, combined with Proposition 4.8, allows to show that $P_{\mathbf{s}'}$ has been correctly extended to continuous linear mappings from $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$ and $\mathcal{L}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ onto $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ and $\mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$, respectively. Formula (8) is then proved by approximation in the general case. □

**Theorem 4.31** *Take $p, q \in [1, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$. Assume that the following hold:*

- $\mathbf{s} \succ \frac{1}{q}\mathbf{b} + \frac{1}{p}\mathbf{d} + \frac{1}{2q'}\mathbf{m}'$ *and* $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$ *(resp.* $\mathbf{s} \geq \mathbf{0}$ *if* $q = \infty$*);*
- $\mathbf{s}' \prec \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$*;*
- $\mathbf{s} + \mathbf{s}' \prec \frac{1}{q}\mathbf{b} + \frac{1}{p}\mathbf{d} - \frac{1}{2q}\mathbf{m}'$.

*Then, the following conditions are equivalent:*

(1) $\mathcal{A}^{p,q}_{\mathbf{s},0}(D) = \widetilde{\mathcal{A}}^{p,q}_{\mathbf{s},0}(D)$ *(resp.* $\mathcal{A}^{p,q}_{\mathbf{s}}(D) = \widetilde{\mathcal{A}}^{p,q}_{\mathbf{s}}(D)$*) and* $\mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D) = \widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$*;*
(2) $P_{\mathbf{s}'}$ *induces a continuous linear mapping of* $\mathcal{L}^{p,q}_{\mathbf{s},0}(D)$ *into* $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$ *and* $\mathbf{s} \succ \mathbf{0}$ *(resp.* $\mathbf{s} \geq \mathbf{0}$*);*
(3) $P_{\mathbf{s}'}$ *induces a continuous linear projector of* $\mathcal{L}^{p,q}_{\mathbf{s},0}(D)$ *onto* $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ *(resp. of* $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$ *onto* $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$*) and of* $\mathcal{L}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ *onto* $\mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$*;*
(4) $\mathbf{s} \succ \frac{1}{2q}\mathbf{m}$ *(resp.* $\mathbf{s} \geq \mathbf{0}$ *if* $q = \infty$*) and the sesquilinear mapping*

$$(f, g) \mapsto \int_D f \overline{g} (\Delta_\Omega^{-\mathbf{s}'} \circ \rho) \, \mathrm{d}\nu_D \tag{9}$$

*induces an antilinear isomorphism of* $\mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ *onto* $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)'$ *(resp. onto the dual of the closed vector subspace of* $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ *generated by the* $B^{\mathbf{s}'}_{(\zeta,z)}$, $(\zeta, z) \in D$*);*
(5) *properties* $(\mathcal{L}')^{p,q}_{\mathbf{s},\mathbf{s}',0}$ *(resp.* $(\mathcal{L}')^{p,q}_{\mathbf{s},\mathbf{s}'}$*) and* $(\mathcal{L}')^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',\mathbf{s}'}$ *hold;*
(6) *property* $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}',0}$ *(resp.* $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}'}$*) holds.*

For analogous equivalences in the context of the spaces $A^{p,q}_{\mathbf{s}}(D)$, see [19, Sect. 4].

***Proof*** (1) $\implies$ (2). Take $f \in \mathcal{L}^{p,q}_{\mathbf{s},0}(D) \cap \mathcal{L}^{2,2}_{(\mathbf{b}+\mathbf{d}-\mathbf{s}')/2}(D)$ and $\phi \in \mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$. Then, $P_{\mathbf{s}'} f \in \mathcal{A}^{2,2}_{(\mathbf{b}+\mathbf{d}-\mathbf{s}')/2}(D) = \widetilde{\mathcal{A}}^{2,2}_{(\mathbf{b}+\mathbf{d}-\mathbf{s}')/2}(D)$, so that $P_{\mathbf{s}'} f = \mathcal{E} u$ for some $u \in \mathcal{B}^{(\mathbf{s}'-\mathbf{b}-\mathbf{d})/2}(\mathcal{N}, \Omega)$ by (5) of Proposition 4.26. Then, Proposition 4.28 shows that there is $c \neq 0$ such that

$$\begin{aligned}
|\langle u|\phi\rangle| &= \left| c \int_D \mathcal{E} u \overline{\mathcal{E}(\phi * I_\Omega^{\mathbf{s}'-\mathbf{b}-\mathbf{d}})} (\Delta_\Omega^{-\mathbf{s}'} \circ \rho) \, \mathrm{d}\nu_D \right| \\
&= \left| c \int_D f \overline{\mathcal{E}(\phi * I_\Omega^{\mathbf{s}'-\mathbf{b}-\mathbf{d}})} (\Delta_\Omega^{-\mathbf{s}'} \circ \rho) \, \mathrm{d}\nu_D \right| \\
&\leq |c| \|f\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)} \|\mathcal{E}(\phi * I_\Omega^{\mathbf{s}'-\mathbf{b}-\mathbf{d}})\|_{\mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)} \\
&\leq \|f\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)} \|\phi\|_{\mathcal{B}^{\mathbf{s}}_{p',q'}(\mathcal{N},\Omega)}
\end{aligned}$$

for a suitable choice of a norm on $\mathcal{B}^{\mathbf{s}}_{p',q'}(\mathcal{N}, \Omega)$, thanks to (1) of Proposition 4.26. By Corollary 4.29, this shows that $u \in \mathcal{B}^{-\mathbf{s}}_{p,q}(\mathcal{N}, \Omega)$, that is, $P_{\mathbf{s}'} f \in \widetilde{\mathcal{A}}^{p,q}_{\mathbf{s}}(D) = \mathcal{A}^{p,q}_{\mathbf{s}}(D)$. The preceding arguments then show that $P_{\mathbf{s}'}$ induces a continuous linear mapping of $\mathcal{L}^{p,q}_{\mathbf{s},0}(D)$ into $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$.

(2) $\implies$ (3). The only assertion which is not contained in (4) of Proposition 4.30 is the following one: if $P_{\mathbf{s}'}$ induces a continuous linear mapping of $\mathcal{L}^{p,q}_{\mathbf{s},0}(D)$ into $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$ and $\mathbf{s} \succ \mathbf{0}$, then it induces an endomorphism of $\mathcal{L}^{p,q}_{\mathbf{s},0}(D)$. However, using (1)–(3) of Proposition 4.30 and Proposition 4.7 (and the assumption $\mathbf{s} \succ \mathbf{0}$ when $q = \infty$), it is clear that $B^{\mathbf{s}'}_{(\zeta,z)} \in \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ for every $(\zeta, z) \in D$, so that $P_{\mathbf{s}'}(C_c(D)) \subseteq \mathcal{A}^{p,q}_{\mathbf{s}',0}(D)$ and the assertion follows.

(3) $\implies$ (1). Take $f \in C_c(D)$ and $\phi \in \mathcal{S}_{\overline{\Omega'}}(\mathcal{N})$. By Propositions 4.26 and 4.28,

$$\begin{aligned}
\left| \int_D f \overline{\mathcal{E}\phi} (\Delta_\Omega^{-\mathbf{s}'} \circ \rho) \, \mathrm{d}\nu_D \right| &= \left| \int_D (P_{\mathbf{s}'} f) \overline{\mathcal{E}\phi} (\Delta_\Omega^{-\mathbf{s}'} \circ \rho) \, \mathrm{d}\nu_D \right| \\
&\leq \|P_{\mathbf{s}'}\| \|f\|_{\mathcal{L}^{p,q}_{\mathbf{s}}(D)} \|\mathcal{E}\phi\|_{\widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)}
\end{aligned}$$

for a suitable choice of a norm on $\widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$. By the arbitrariness of $f$, this implies that $\widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}',0}(D) \subseteq \mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ continuously. Observe that, in order to show that $\widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D) = \mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$, it will suffice to prove that, for every $f \in \widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$, there is a bounded sequence $(f_j)$ of elements of $\mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',0}(D)$ which converges pointwise to $f$ (so that $f \in \mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ with controlled norm). Then, take $u \in \mathcal{B}^{\mathbf{s}+\mathbf{s}'-\mathbf{b}-\mathbf{d}}_{p',q'}(\mathcal{N}, \Omega)$ such that $\mathcal{E}u = f$, and take $(\lambda_k)$, $(t_k)$, $(\phi_k)$, and $(\psi_k)$ as in Definition 4.18. Notice that we may choose $(\psi_k)$ so that $\sum_k \psi_k = 1$ on $\Omega'$. In addition, take $\phi' \in C_c^\infty(\Omega')$ so that $\phi' = 1$ on the support of each $\phi_k$, and set $\psi_k' := \mathcal{F}_{\mathcal{N}}^{-1}(\phi'(\cdot\, t_k^{-1}))$ for every $k \in K$, so that $\psi_k = \psi_k * \psi_k'$ and $u = \sum_k u * \psi_k$ with convergence in the weak topology $\sigma^{\mathbf{s}+\mathbf{s}'-\mathbf{b}-\mathbf{d}}_{p',q'}$. Then, fix $\tau \in C_c^\infty(\mathcal{N})$ with $\tau(0) = 1$ and set $\tau_j(\zeta, x) := \tau(2^{-j}\zeta, 2^{-2j}x)$ for every $(\zeta, x) \in \mathcal{N}$ and for every $j \in \mathbb{N}$. As in the proof of Proposition 4.21, we then see that $[(u*\psi_k)\tau_j]*\psi_k'$ converges to $u * \psi_k$ in $\mathcal{B}^{\mathbf{s}+\mathbf{s}'-\mathbf{b}-\mathbf{d}}_{p',q'}(\mathcal{N}, \Omega)$ for every $k \in K$. If $(K_j)$ is an increasing sequence of finite subsets of $K$ whose union is $K$, then one may set $f_j = \sum_{k\in K_j} \mathcal{E}([(u * \psi_k)\tau_j] * \psi_k')$ and show that the $f_j$ are uniformly bounded in $\widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ (using [14, Corollary 3.5]) and converge pointwise to $f$.

Analogously, one shows that $\widetilde{\mathcal{A}}^{p,q}_{\mathbf{s},0}(D) \subseteq \mathcal{A}^{p,q}_{\mathbf{s}}(D)$. Since $\mathcal{E}(\mathcal{S}_{\overline{\Omega'}}(\mathcal{N})) \subseteq \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ by (2) of Proposition 4.26 and (1) of Proposition 4.30, this is sufficient to prove that $\widetilde{\mathcal{A}}^{p,q}_{\mathbf{s},0}(D) = \mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ (resp. as above one shows that $\widetilde{\mathcal{A}}^{p,q}_{\mathbf{s}}(D) = \mathcal{A}^{p,q}_{\mathbf{s}}(D)$).

(1) $\implies$ (4). This follows from Proposition 4.28, once one shows that the closed vector subspace $V$ of $\mathcal{A}^{p,q}_{\mathbf{s}}(D) = \widetilde{\mathcal{A}}^{p,q}_{\mathbf{s}}(D)$ generated by the $B^{\mathbf{s}'}_{(\zeta,z)}$, $(\zeta, z) \in D$, is precisely $\widetilde{\mathcal{A}}^{p,q}_{\mathbf{s},0}(D)$. To see this latter fact, observe first that $B^{\mathbf{s}'}_{(\zeta,z)} \in \widetilde{\mathcal{A}}^{p,q}_{\mathbf{s},0}(D)$ for every $(\zeta, z) \in D$, since $B^{-\mathbf{s}'}_{(\zeta,z)} \in \mathcal{A}^{p,q}_{\mathbf{s}}(D)$ by Proposition 4.7 and since $B^{\mathbf{s}'}_{(\zeta,z)} * I^{-\mathbf{s}''}_{\Omega} = cB^{\mathbf{s}'-\mathbf{s}''}_{(\zeta,z)} \in \mathcal{A}^{p,q}_{\mathbf{s}+\mathbf{s}'',0}(D)$ for some $c \neq 0$ and some sufficiently large $\mathbf{s}'' \in \mathbb{N}_{\Omega'}$, thanks to [16, Proposition 2.29] and (1), (5) of Proposition 4.26. Since the polar of $V$ in $\mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ with respect to the sesquilinear form (9) is $\{0\}$ by Proposition 4.8, this shows that $V = \widetilde{\mathcal{A}}^{p,q}_{\mathbf{s},0}(D)$ by Proposition 4.28.

(4) $\implies$ (5). It suffices to observe that the adjoints of the sampling maps on $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$ and $\mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ (which are isomorphisms onto their images for sufficiently fine lattices, thanks to Theorem 4.11) with respect to the sesquilinear form (9) are the atomic decomposition mappings for the spaces $\mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ and $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ (resp. $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$) as one verifies without difficulty.

(5) $\implies$ (6). Obvious.

(6) $\implies$ (1). The proof is similar to that of the implication (4) $\implies$ (5). Indeed, one first observes that the adjoint of the atomic decomposition mapping $\Psi$ corresponding to property $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}',0}$ (resp. $(\mathcal{L})^{p,q}_{\mathbf{s},\mathbf{s}'}$ [10]), with respect to the sesquilinear form of Proposition 4.28, is precisely the sampling mapping on $\widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$. Since $\widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D) \subseteq \mathcal{A}^{\infty,\infty}_{\mathbf{b}/q'+\mathbf{d}/p'-\mathbf{s}-\mathbf{s}'}(D)$ by Proposition 4.24, and since we may consider lattices as fine as we please, Theorem 4.11 shows that $\widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D) \subseteq \mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$ continuously, that is, $\widetilde{\mathcal{A}}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D) = \mathcal{A}^{p',q'}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}(D)$. Reversing the argument, we then see that, for sufficiently fine lattices, the atomic decomposition mapping $\Psi$ maps $\ell^{p,q}_0(J, K)$ *onto* $\widetilde{\mathcal{A}}^{p,q}_{\mathbf{s},0}(D)$ (since its adjoint is an isomorphism onto its image). Hence, $\widetilde{\mathcal{A}}^{p,q}_{\mathbf{s},0}(D) \subseteq \mathcal{A}^{p,q}_{\mathbf{s}}(D)$. Arguing as in the proof of the implication (3) $\implies$ (1), this leads to the conclusion. □

The following result, which is a simple consequence of Theorem 4.31, extends several results known in the literature: the equivalence of (1) and (2) is [2, Theorem 1.9] when $n = 0$, $\mathbf{s} = \mathbf{s}' \in \mathbb{R}\mathbf{1}_r$, and $\Omega$ is symmetric, and [23, Theorem 1.3] when $n = 0$, $\mathbf{s} = \mathbf{s}'$, and $\Omega$ is symmetric; the implication (2) $\implies$ (4), in this context, is a simple consequence of the duality between $\mathfrak{L}^{p,q}_{\mathbf{s}}(D)$ and $\mathfrak{L}^{p',q'}_{q'(\mathbf{s}'-\mathbf{s}/q)}(D)$, cf. [3, Theorem 5.2] for the case in which $\mathbf{s} = \mathbf{s}' \in \mathbb{R}\mathbf{1}_r$, $n = 0$, and $\Omega$ is symmetric; the implication (4) $\implies$ (5) (first half) is [8, Theorem 5.1] when $p > 1$ and $\mathbf{s} = \mathbf{s}'$; the implication (2) $\implies$ (5) (first half) is [3, Theorem 5.7] when $p = q$, $\mathbf{s} = \mathbf{s}' \in \mathbb{R}\mathbf{1}_r$, $n = 0$, and $\Omega$ is symmetric, and [7, Theorem 3.4] when $\mathbf{s} = \mathbf{s}'$, $n = 0$, and $\Omega$ is symmetric; the implication (2) $\implies$ (4) is [4, Theorem 1.6] when $p = q$ and $\mathbf{s} = \mathbf{s}' \in \mathbb{R}\mathbf{1}_r$.

We also mention that [12] was devoted to a somewhat informal description of this kind of equivalences.

**Corollary 4.32** *Take $p \in [1, \infty)$, $q \in (1, \infty)$, and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that the following hold:*

- $\mathbf{s} \succ \frac{1}{2}(\mathbf{m} - \mathbf{b}), \frac{1}{2}\mathbf{b} + \frac{q}{p}\mathbf{d} + \frac{q}{2q'}\mathbf{m}'$;
- $\mathbf{s}' \succ \frac{1}{2}(\mathbf{m} - \mathbf{b})$;
- $\mathbf{s}' - \frac{1}{q}\mathbf{s} \succ \frac{1}{2q'}\mathbf{b} + \frac{1}{p'}\mathbf{d} + \frac{1}{2q}\mathbf{m}'$.

*Then, the following conditions are equivalent:*

(1) $\mathfrak{A}^{p,q}_{\mathbf{s}}(D) = \widetilde{\mathfrak{A}}^{p,q}_{\mathbf{s}}(D)$ *and* $\mathfrak{A}^{p',q'}_{q'(\mathbf{s}'-\mathbf{s}/q)}(D) = \widetilde{\mathfrak{A}}^{p',q'}_{q'(\mathbf{s}'-\mathbf{s}/q)}(D)$;
(2) $\mathfrak{P}_{\mathbf{s}'}$ *induces an endomorphism of* $\mathfrak{L}^{p,q}_{\mathbf{s}}(D)$;
(3) $\mathfrak{P}_{\mathbf{s}'}$ *induces a continuous linear mapping of* $\mathfrak{L}^{p,q}_{\mathbf{s}}(D)$ *onto* $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)$ *and of* $\mathfrak{L}^{p',q'}_{q'(\mathbf{s}'-\mathbf{s}/q)}(D)$ *onto* $\mathfrak{A}^{p',q'}_{q'(\mathbf{s}'-\mathbf{s}/q)}(D)$;
(4) $\mathbf{s} \succ \frac{1}{2}(\mathbf{m} - \mathbf{b})$ *and the sesquilinear mapping*

$$(f, g) \mapsto \int_D f\overline{g}(\Delta_\Omega^{\mathbf{s}'-\mathbf{b}/2-\mathbf{d}} \circ \rho)\, \mathrm{d}\nu_D$$

*induces an antilinear isomorphism of* $\mathfrak{A}^{p',q'}_{q'(\mathbf{s}'-\mathbf{s}/q)}(D)$ *onto* $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)'$;
(5) *properties* $(\mathfrak{L}')^{p,q}_{\mathbf{s},\mathbf{s}'}$ *and* $(\mathfrak{L}')^{p,q}_{q'(\mathbf{s}'-\mathbf{s}/q),\mathbf{s}'}$ *hold;*
(6) *property* $(\mathfrak{L})^{p,q}_{\mathbf{s},\mathbf{s}'}$ *holds.*

Combining (3) and (4) of Proposition 4.26 with Theorem 4.31, we get the following transference result.

[10] In this case, one has to preliminarily observe that the range of this latter mapping is contained in $\widetilde{\mathcal{A}}^{p,q}_{\mathbf{s},0}(D)$ as in the proof of the implication (1) $\implies$ (4).

**Corollary 4.33** *Take $p, q \in [1, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$. Then, the following hold:*

- *if $\mathbf{s}' \prec \mathbf{d} - \frac{1}{2}\mathbf{m}$ and $P_{\mathbf{s}'}$ induces a continuous linear projector of $\mathcal{L}^{p,q}_{\mathbf{s},0}(T_\Omega)$ onto $\mathcal{A}^{p,q}_{\mathbf{s},0}(T_\Omega)$ (resp. of $\mathcal{L}^{p,q}_{\mathbf{s}}(T_\Omega)$ onto $\mathcal{A}^{p,q}_{\mathbf{s}}(T_\Omega)$), then $P_{\mathbf{s}'+\mathbf{b}}$ induces a continuous linear projector of $\mathcal{L}^{p,q}_{\mathbf{s},0}(D)$ onto $\mathcal{A}^{p,q}_{\mathbf{s},0}(D)$ (resp. of $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$ onto $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$);*
- *in particular, if $\mathbf{s}' \succ \frac{1}{2}\mathbf{m}$ and $\mathfrak{P}_{\mathbf{s}'}$ induces a continuous linear projector of $\mathfrak{L}^{p,q}_{\mathbf{s}}(T_\Omega)$ onto $\mathfrak{A}^{p,q}_{\mathbf{s}}(T_\Omega)$, then $\mathfrak{P}_{\mathbf{s}'-\mathbf{b}/2}$ induces a continuous linear projector of $\mathfrak{L}^{p,q}_{\mathbf{s}-\mathbf{b}/2}(D)$ onto $\mathfrak{A}^{p,q}_{\mathbf{s}-\mathbf{b}/2}(D)$;*
- *if $\mathbf{s}' \prec \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$ and $P_{\mathbf{s}'}$ induces a continuous linear projector of $\mathcal{L}^{p,p}_{\mathbf{s},0}(D)$ onto $\mathcal{A}^{p,p}_{\mathbf{s},0}(D)$ (resp. of $\mathcal{L}^{p,p}_{\mathbf{s}}(D)$ onto $\mathcal{A}^{p,p}_{\mathbf{s}}(D)$), then $P_{\mathbf{s}'}$ induces a continuous linear projector of $\mathcal{L}^{p,p}_{\mathbf{s}-\mathbf{b}/p,0}(T_\Omega)$ onto $\mathcal{A}^{p,p}_{\mathbf{s}-\mathbf{b}/p,0}(T_\Omega)$ (resp. of $\mathcal{L}^{p,p}_{\mathbf{s}-\mathbf{b}/p}(T_\Omega)$ onto $\mathcal{A}^{p,p}_{\mathbf{s}-\mathbf{b}/p}(T_\Omega)$);*
- *in particular, if $\mathbf{s}' \prec \frac{1}{2}(\mathbf{m} - \mathbf{b})$ and $\mathfrak{P}_{\mathbf{s}'}$ induces a continuous linear projector of $\mathfrak{L}^{p,p}_{\mathbf{s}}(D)$ onto $\mathfrak{A}^{p,p}_{\mathbf{s}}(D)$, then $\mathfrak{P}_{\mathbf{s}'-\mathbf{b}/2}$ induces a continuous linear projector of $\mathfrak{L}^{p,p}_{\mathbf{s}-\mathbf{b}/2}(T_\Omega)$ onto $\mathfrak{A}^{p,p}_{\mathbf{s}-\mathbf{b}/2}(T_\Omega)$.*

***Remark 4.34*** We observe that the second assertion of the preceding result extends [6, Theorem 2.1], which corresponds to the case in which $\mathbf{s} = \mathbf{s}' \succ \frac{1}{2}(\mathbf{m} + \mathbf{m}' - \mathbf{b})$. In particular, this solves in the affirmative the first of the final remarks of [6].

Combining Theorem 4.31 with (5) of Proposition 4.26, we obtain the following result. It extends: [2, Theorem 1.9], which corresponds to the case in which $\mathbf{s} = \mathbf{s}'$, $\Omega$ is symmetric, $n = 0$, and $\mathbf{s} \in \mathbb{R}\mathbf{1}_r$; [23, Corollary 1.4], which corresponds to the case in which $\mathbf{s} = \mathbf{s}'$, $n = 0$, and $\Omega$ is symmetric; [30, Theorem 3.2 (ii)], which corresponds to the case in which $p = q$, $\mathbf{s} = \mathbf{s}'$, and $n = 0$; [29, Theorem 2.3], which corresponds to the case in which $\mathbf{s} = \mathbf{s}' \succ \frac{1}{2}(\mathbf{m} + \mathbf{m}' - \mathbf{b})$.[11]

**Corollary 4.35** *Take $p, q \in [1, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$. If:*

- $\mathbf{s} \succ \frac{1}{2q}\mathbf{m} + \left(\frac{1}{2\min(p,p')} - \frac{1}{2q}\right)_+ \mathbf{m}'$;
- $\mathbf{b} + \mathbf{d} - \mathbf{s} - \mathbf{s}' \succ \frac{1}{2q'}\mathbf{m} + \left(\frac{1}{2\min(p,p')} - \frac{1}{2q'}\right)_+ \mathbf{m}'$;

*then $P_{\mathbf{s}'}$ induces a continuous linear projector of $\mathcal{L}^{p,q}_{\mathbf{s}}(D)$ onto $\mathcal{A}^{p,q}_{\mathbf{s}}(D)$.*

*In particular, if $q < \infty$ and:*

- $\mathbf{s} \succ \frac{1}{2}(\mathbf{m} - \mathbf{b}) + \left(\frac{q}{2\min(p,p')} - \frac{1}{2}\right)_+ \mathbf{m}'$;
- $\mathbf{s}' - \frac{1}{q}\mathbf{s} \succ \frac{1}{2q'}(\mathbf{m} - \mathbf{b}) + \left(\frac{1}{2\min(p,p')} - \frac{1}{2q'}\right)_+ \mathbf{m}'$;

*then $\mathfrak{P}_{\mathbf{s}'}$ induces a continuous linear projector of $\mathfrak{L}^{p,q}_{\mathbf{s}}(D)$ onto $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)$.*

In particular, if $\mathbf{s} = \mathbf{s}' \succ \frac{1}{2}(\mathbf{m} - \mathbf{b})$ and

$$q'_{\mathbf{s}}(p) < q < q_{\mathbf{s}}(p),$$

then $\mathfrak{P}_{\mathbf{s}'}$ induces a continuous linear projector of $\mathfrak{L}^{p,q}_{\mathbf{s}}(D)$ onto $\mathfrak{A}^{p,q}_{\mathbf{s}}(D)$ (cf. Remark 4.27).

Notice that this result is more precise than the one which may be proved by means of Schur's lemma, considering the integral operator $P_{\mathbf{s}',+}$ whose kernel is the absolute value of the kernel of $P_{\mathbf{s}'}$. Cf. [26] for a general treatment of the boundedness of the operator $P_{\mathbf{s}',+}$ when $n = 0$; see also [36] for the case in which $\mathbf{s}, \mathbf{s}' \in \mathbb{R}\mathbf{1}_r$ and $\Omega$ is symmetric.

Let us also observe that, combining the characterization of the equality $\mathcal{A}^{p,q}_{\mathbf{s}} = \widetilde{\mathcal{A}}^{p,q}_{\mathbf{s}}(D)$ when $r = 2$ (cf. the remarks following the statement of Proposition 4.26) with Theorem 4.31 and Corollary 4.32, one may characterize the continuity of the Bergman projectors for tube domains over Lorentz cones (that is, for $r = 2$ and $n = 0$). The characterization of the continuity of $\mathfrak{P}_{\mathbf{s}}$ on $\mathfrak{L}^{p,q}_{\mathbf{s}}(D)$ in this case was previously obtained in [6, Theorem 2.3].

## Concluding remarks

On the one hand, we have compared two parallel theories of mixed norm Bergman spaces on homogeneous Siegel domains. On the other hand, we have extended part of the theory for the spaces $A^{p,q}_{\mathbf{s}}$ to the spaces $\mathfrak{A}^{p,q}_{\mathbf{s}}$. In doing this, we hope we have shed some light on the technically demanding subject of function theory on such domains. We believe that this is a lively area of research that in recent times has drawn the interest of many scholars. We mention that the Šilov boundary of $D$ naturally appears in the extension of the Paley–Wiener and Bernstein spaces of entire functions to higher complex dimensions, see in particular [15, 20, 21].

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[11] We mention here that in [16, p. vii] we erroneously identified the spaces described in [29] with the spaces $A^{p,q}_{\mathbf{s}}(D)$. We apologize for this lack of precision.

**Data Availability** There are no data associated with this article.

## Declarations

**Competing interests** The authors have no competing interests to declare that are relevant to the content of this article.



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