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CALABI-YAU MANIFOLDS OF TYPE A AND THE CONE CONJECTURE FOR HYPERELLIPTIC VARIETIES

Settore MAT/03

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Alla mia famiglia,
che sempre crede in me!

Un momento, stammi un po' a sentire Jim Hopkins,
tu hai la stoffa per compiere grandi imprese,
ma devi prendere in mano il timone e tracciare la tua rotta,
e devi seguirla, anche in caso di burrasca...
E quando verrà il momento in cui potrai mettere alla prova
la qualità delle tue vele e mostrare di che pasta sei fatto,
beh, spero di essere lì, a godermi lo splendore della luce
che emanerai quel giorno.

Contents

Abstract

A Calabi-Yau n-fold Y is a compact, complex, Kähler n-fold with trivial canonical bundle and $h^{i,0}(Y) = 0$ for $1 \leq i \leq n-1$. Calabi-Yau manifolds represent a fascinating objects of study due to the richness of their geometry. There exists a multitude of examples of these manifolds, but still nowadays their classification is an open problem in dimension higher than two. Among the different constructions of Calabi-Yau manifolds, we consider those given as free quotients of abelian varieties by actions of a finite group that contains no translations. These latter manifolds are called *Calabi-Yau manifolds of type A*. More in general, free quotients A/G of an abelian variety A by the action of a finite group G that does no contain any translation are called *hyperelliptic varieties*. The goal of this thesis is to investigate problems on Calabi-Yau manifolds of type A and hyperelliptic varieties.

The first part is dedicated to the study of Calabi-Yau n-folds A/G of type A. These manifolds exist only in odd dimension $n > 1$ and Oguiso and Sakurai classified them in dimension 3. In this thesis we provide higher dimensional examples that ensure the existence of Calabi-Yau manifolds of type A in all odd dimension. After that, we study the geometry of Calabi-Yau 3-folds of type A. Thanks to Oguiso and Sakurai, we know that there exist only two families $\mathcal{F}_G^A$ of Calabi-Yau 3-folds A/G of type A and each of them corresponds uniquely to a group G which acts freely on A and does not contain any translations. More in details, the family $\mathcal{F}_{\mathfrak{D}_4}^A$ is constructed by Catanese and Demleitner and the family $\mathcal{F}_{(\mathbb{Z}/2\mathbb{Z})^2}^A$ is constructed here. In particular, these families are irreducible and each $X \in \mathcal{F}_G^A$ admits a finite étale cover A which splits into the product of three elliptic curves. Our main results include the full classification of the automorphisms group and the possible quotients of manifolds in $\mathcal{F}_G^A$ for both choices of G. We prove that $\mathrm{Aut}(X)$ is finite for $X \in \mathcal{F}_G^A$. Furthermore, if $X \in \mathcal{F}_{\mathfrak{D}_4}^A$ then X/Υ is birational to a Calabi-Yau 3-folds for every $\Upsilon \leq \mathrm{Aut}(X)$, while if $X \in \mathcal{F}_{(\mathbb{Z}/2\mathbb{Z})^2}^A$ then X/Υ is birational either to a Calabi-Yau 3-fold or to a 3-fold with negative Kodaira dimension or to a 3-fold Y with trivial Kodaira dimension and $K_Y \neq 0$. In particular, we compute the Hodge numbers and the fundamental groups of Calabi-Yau 3-folds Y birational to the quotient X/Υ where $X \in \mathcal{F}_G^A$ and $\Upsilon \leq \mathrm{Aut}(X)$.

In the second part, we establish the Morrison-Kawamata cone conjecture for hyperelliptic varieties. The cone conjecture, an open problem in birational geometry, predicts that the nef and movable cones, of projective varieties Y with $K_Y \equiv 0$, admit a rational polyhedral structure under the action of automorphism and birational automorphism groups, respectively. The conjecture provides insights into both the geometry and birational geometry of varieties, specially from the point of view of Minimal Model Program. To prove the conjecture for hyperelliptic varieties, we generalize the techniques established by Prendergast-Smith for abelian varieties. Additionally, we investigate whether hyperelliptic varieties A/G have a rational polyhedral nef cones. We obtain a result in relation to the representation of G, in particular we deduce that all the Calabi-Yau manifolds of type A studied in this thesis have rational polyhedral nef cones. Finally, we study the extremal rays of the nef cones of these latter manifolds. We prove that the extremal rays are semi-ample divisors, by generalizing a proof of Oguiso and Sakurai. Thereby, we deduce that all nef divisors are semi-ample divisors.

Keywords: Fibrations (mathematics), Calabi-Yau manifolds, Abelian varieties, Automorphisms groups, Cone

Résumé

Une variété de Calabi-Yau Y de dimension n est une variété compacte, complexe, kählérienne et lisse avec un fibré canonique trivial et $h^{i,0}(Y) = 0$ pour $1 \leq i \leq n-1$. Les variétés de Calabi-Yau représentent des objets d'étude fascinants en raison de la richesse de leur géométrie. La classification est encore un problème ouvert en dimension supérieure à deux. Parmi les différentes constructions de variétés de Calabi-Yau, nous considérons celles obtenues comme quotients libres de variétés abéliennes par l'action d'un groupe fini qui ne contient pas de translations. Ces dernières variétés sont appelées *variétés de Calabi-Yau de type A*. Plus généralement, les quotients libres de variétés abéliennes A par l'action d'un groupe fini G, qui ne contient pas de translations, sont appelés *variétés hyperelliptiques*. L'objectif de cette thèse est d'étudier la géométrie et les problèmes des variétés de Calabi-Yau de type A et des variétés hyperelliptiques.

La première partie est dédiée à l'étude des variétés de Calabi-Yau de type A. Elles existent dans toutes les dimensions impaires $n > 1$ et ont été classifiées en dimension 3 par Oguiso et Sakurai. Dans cette thèse, nous donnons des exemples en dimension supérieure qui garantissent l'existence de variétés de Calabi-Yau de type A dans toutes les dimensions impaires. Ensuite, nous étudions la géométrie des 3-folds de Calabi-Yau de type A. Oguiso et Sakurai ont montré qu'il existe deux familles $\mathcal{F}_G^A$ des 3-folds de Calabi-Yau A/G de type A et que chacune correspond de manière unique à un groupe fini G qui agit librement sur A et ne contient pas de translations. La famille $\mathcal{F}_{\mathfrak{D}_4}^A$ est construite par Catanese et Demleitner, et la famille $\mathcal{F}_{(\mathbb{Z}/2\mathbb{Z})^2}^A$ est construite dans cette thèse. En particulier, ces familles sont irréductibles et chaque $X \in \mathcal{F}_G^A$ admet un revêtement étale fini A qui se décompose en un produit de trois courbes elliptiques. Nos principaux résultats incluent la classification complète des groupes des automorphismes et des quotients possibles des variétés dans $\mathcal{F}_G^A$ pour les deux choix de G. Nous prouvons que $\mathrm{Aut}(X)$ est fini pour $X \in \mathcal{F}_G^A$. De plus, si $X \in \mathcal{F}_{\mathfrak{D}_4}^A$ alors X/Υ est birationnel à une 3-fold de Calabi-Yau pour tout $\Upsilon \leq \mathrm{Aut}(X)$; si $X \in \mathcal{F}_{(\mathbb{Z}/2\mathbb{Z})^2}^A$, alors X/Υ est birationnel soit à un 3-fold de Calabi-Yau, soit à un 3-fold de dimension de Kodaira négative, soit à un 3-fold Y avec dimension de Kodaira triviale et $K_Y \neq 0$. En particulier, nous calculons les nombres de Hodge et les groupes fondamentaux pour les 3-folds de Calabi-Yau Y birationelles à X/Υ avec $X \in \mathcal{F}_G^A$ et $\Upsilon \leq \mathrm{Aut}(X)$.

Le but de la deuxième partie est d'établir la conjecture du cône de Morrison-Kawamata pour les variétés hyperelliptiques. La conjecture du cône, un problème ouvert en géométrie birationnelle, prédit que les cônes nef et mobile, des variétés projectives Y avec $K_Y \equiv 0$, admettent une structure polyédrale rationnelle sous l'action des groupes d'automorphismes et d'automorphismes birationnels, respectivement. Cette conjecture fournit des informations sur la géométrie et la géométrie birationnelle des variétés, en particulier en relation avec le Minimal Model Program. Pour prouver la conjecture pour les variétés hyperelliptiques, nous généralisons les techniques établies par Prendergast-Smith pour les variétés abéliennes. De plus, nous étudions quand les variétés hyperelliptiques A/G possèdent des cônes nef polyédriques rationnels. Nous obtenons un résultat relatif à la représentation de G, en particulier nous déduisons que toutes les variétés de Calabi-Yau de type A étudiées dans cette thèse ont des cônes nef polyédriques rationnels. Enfin, nous étudions les rayons extrémaux des cônes nef de ces dernières variétés. Nous prouvons que les rayons extrémaux sont des diviseurs semi-amples. Ainsi, nous déduisons que tous les diviseurs nef sont semi-amples, en généralisant une preuve de Oguiso et Sakurai.

Mots-clés: Fibrations (mathématiques), Calabi-Yau, Variétés de, Variétés abéliennes, Groupes d'automorphismes, Cône

Introduction

Complex algebraic manifolds with trivial canonical divisor are of great importance in algebraic geometry. For instance, they don't have a canonical model due to the triviality of their canonical divisors. Therefore one needs to investigate and find other projective models to describe their geometry. Furthermore, they appear as possible minimal models in Minimal Model Program, an algorithm whose ultimate goal is to classify algebraic varieties up to birational morphisms.

The famous Beauville-Bogomolov decomposition theorem (see Theorem 1.5.6) characterizes the manifolds with numerically trivial canonical bundle. It asserts that these manifolds decompose, up to a finite étale covering, into the product of three building blocks: complex tori, simply-connected Calabi-Yau manifolds and irreducible holomorphic symplectic manifolds.

A Calabi-Yau manifold Y is a compact, complex, Kähler manifold with trivial canonical bundle and no $(i,0)$-forms for $1 \leq i \leq \dim(Y) - 1$. Calabi-Yau manifolds represent fascinating objects of study in algebraic geometry due to the richness of their geometry. Moreover, they have various applications in other branches of science, such as physics (e.g., Mirror Symmetry, F-theory, M-theory). In dimension 1, the only topological class of such manifolds consists of genus one curves and they have a 1-dimensional moduli space [56, Chapter 8]. In dimension 2, there is again a single topological class of Calabi-Yau surfaces and all the Calabi-Yau surfaces are called K3 surfaces. However, the moduli space in this case has dimension 20 [5, Chapters VI and VIII]. In dimension 3, the classification of Calabi-Yau threefolds remains an open problem. There is no known method for constructing an infinite number of topologically distinct Calabi-Yau manifolds: there is a vast but finite list of known families of Calabi-Yau threefolds.

In this thesis we focus in the relation between complex tori and (non necessarily simply-connected) Calabi-Yau manifolds. One way to relate these manifolds is by consider quotient maps. Specifically, we consider the Calabi-Yau manifolds which admit a complex torus as étale Galois cover. In dimension 1, complex tori and Calabi-Yau curves coincide. It is well-known that the only free action are given by translations and they produce quotients that are again Calabi-Yau curves. In higher dimension the situation is more interesting, since complex tori and Calabi-Yau manifolds are topologically different. In dimension 2, there exist finite groups G acting on complex tori T such that T/G, up to desingularization, is a Calabi-Yau surface. It is worth noting that since Calabi-Yau surfaces are simply-connected then the previous construction produces always singular quotients, *i.e.* to relate complex tori and Calabi-Yau surfaces via the action of finite groups one needs to consider singular surfaces. In dimension 3, the situation is more curious. There still exist finite groups G acting on complex tori T such that T/G, up to desingularization, is a Calabi-Yau 3-folds; but moreover we can find free actions of G on T such that T/G is a Calabi-Yau 3-folds. We are particularly interested in this last situation and its generalization in higher dimension.

The quotients T/G of complex tori T by free actions of finite group G, such that T/G is not again a complex torus, are called *(generalized) hyperelliptic manifolds*. In particular, we can assume that G does not contain any translations, see Remark 4.1.2. In fact, a hyperelliptic manifold T/G is topological determined by the group G which does not contain any translation, see Remark 4.1.9, and so we called it *hyperelliptic manifold with the group* G. In the last years they have gained significant attention as they are natural generalizations of the bi-elliptic surfaces. The hyperelliptic manifolds were first introduced by H. Lange in [55] and if they are projective, *i.e.* the complex torus is an abelian variety, they are called *(generalized) hyperelliptic varieties*. Later, several mathematicians have made contributions to the study of such manifolds, for instance see [23], [25], [73], [44], [42], [29], [22]. By definition hyperelliptic manifolds exist only in dimension $n > 1$ and among them one can find Calabi-Yau manifolds. Hyperelliptic manifolds that are also Calabi-Yau manifolds exist only in odd dimension and they are always projective, see Lemma 5.6.1. They are commonly called *Calabi-Yau manifolds of type* A and they were first introduced by Oguiso and Sakurai in [73].

The goals of this thesis are to investigate problems concerning both Calabi-Yau manifolds of type A and hyperelliptic manifolds. Chapters 1 and 2 provide common preliminaries to the parts of the thesis, given the different nature of the problems investigated. In Chapter 1, we collect some basic notions and results in algebraic geometry that we will need throughout the thesis. In Chapter 2 we recall the main results and techniques on abelian varieties, which will be useful later. Then, we divide the thesis in three parts. Part I is devoted to the study of the geometry of Calabi-Yau manifolds of type A, with a focus on the three-dimensional case. Part of this study is enclosed in the article [66]. In Part II, we establish the Morrison-Kawamata cone conjecture for hyperelliptic varieties. This second work is a joint project with Ana Quedo and it is available as *preprint* in [65]. Finally, part III is devoted to further remarks and questions. Let us explain and motivate more in details the results of Part I and part II.

Part I: The Calabi-Yau manifolds of type A.

Calabi-Yau manifolds of type A exist only in odd dimension $n > 1$ (see Lemma 5.6.6) and in [73] the authors have completely classified these manifolds in dimension 3, providing also explicit examples. It is worth noting that, to the best of the author's knowledge, there are not yet any higher-dimensional examples of Calabi-Yau manifolds of type A. Therefore, a first question we aim to answer is the following:

Question 1. Do Calabi-Yau manifolds of type A exist in all dimensions?

Calabi-Yau manifolds of type A are in particular hyperelliptic manifolds. These latter are guaranteed to exist in all dimension through explicit examples, see [55, Section 4] and [1]. Looking at these examples enclosed in [1], we prove that they yield to Calabi-Yau quotients only in dimension 3, finding the same example enclosed in [73, Theorem 0.1]. Therefore, we construct, properly, free actions G on abelian varieties A to obtain, by quotient, Calabi-Yau manifolds of type A in higher dimension. More precisely, we obtain the following result.

Theorem A (see Theorem 5.6.6)**.** *Calabi-Yau $(2n+1)$-folds of type A exist for every $n \in \mathbb{N}_{\geq}1$. In particular,*

(i) *For every n, there exists a Calabi-Yau manifold $Y = A/G$ with $G \simeq (\mathbb{Z}/2\mathbb{Z})^{2n}$ and $A = E_1 \times \ldots \times E_{2n+1}$ is the product of $2n+1$ elliptic curves (non necessarily isomorphic to each other).*

(ii) *For $n = 1$, there exists a Calabi-Yau threefolds $Y = A/G$ with $G \simeq \mathfrak{D}_4$ the dihedral group of order 8 and $A = E \times E \times E'$ with E, E' elliptic curves.*

Furthermore, for every odd n there exists a free quotient $Y = A/G$ with $G \simeq \mathfrak{D}_{4n}$ with $\mathcal{K}_Y \simeq \mathcal{O}_Y$ and $h^{1,0}(Y) = h^{2,0}(Y) = 0$.

After ensuring the existence of Calabi-Yau manifolds of type A in all dimensions, we aim to study their geometry. As observed by Oguiso and Sakurai, the first interest in the Calabi-Yau manifolds of type A is motivated by the fact that they don't contain rational curves, while most of the Calabi-Yau manifolds contains rational curves. To study the geometry of these manifolds, we mainly consider the first dimension case in which these manifolds appear, *i.e.* the Calabi-Yau threefolds of type A. In [73, Theorem 0.1] the authors have proven that there exist only two families of Calabi-Yau 3-folds A/G of type A and that each of them uniquely corresponds to a finite group G which acts freely and does not contain any translation. The possible groups are: the abelian group $(\mathbb{Z}/2\mathbb{Z})^2$ and the dihedral group $\mathfrak{D}_4$ of order 8. Independently from [73], Catanese and Demleitner have constructed the whole family $\mathcal{F}^A_{\mathfrak{D}_4}$ of hyperelliptic threefolds with the group $\mathfrak{D}_4$: this family is irreducible and 2 dimensional. In particular, these threefolds admit a 16-étale cover isomorphic to $A' = E \times E \times E'$ which splits into the product of three elliptic curves (two isomorphic). Furthermore, it coincides with the family of Calabi-Yau threefolds of type A with the group $\mathfrak{D}_4$, see Lemma 6.1.2. By using similar techniques, we construct the family $\mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$ of Calabi-Yau threefolds of type A with the group $(\mathbb{Z}/2\mathbb{Z})^2$: it is irreducible and 3 dimensional; every $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$ admits a 4-étale cover $A = E_1 \times E_2 \times E_3$ which splits into the product of three (non-isomorphic) elliptic curves, see Theorem 7.1.2. We consider these two families, mainly $\mathcal{F}^A_{\mathfrak{D}_4}$, and we aim to describe the geometry of $X \in \mathcal{F}^A_G$ for both the groups G. In Chapter 6 we study $\mathcal{F}^A_{\mathfrak{D}_4}$ and in Chapter 7 the family $\mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$. The main results concern the full classification of the automorphisms group and of the quotients of such X. More in details.

Theorem B (see Theorem 6.4.1 and Theorem 7.3.1)**.** *Let $X \in \mathcal{F}^A_G$ with $X = A/G$.*

(i) *Let $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and so $A = E_1 \times E_2 \times E_3$ product of elliptic curves. Let us assume that E_i's are not isogenous to each other. Then the automorphism group of $X = A/G$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^7$. Specifically, the automorphisms on X are induced by those on A whose linear part belong to $\langle \mathrm{diag}(-1, 1, 1)\rangle$ and the translation part is given by any point of order 2.*

(ii) *Let $G \simeq \mathfrak{D}_4$ and so A is isogenous to $A' = E \times E \times E'$ product of elliptic curves. Let us assume that $\mathrm{End}_{\mathbb{Q}}(E') \not\simeq \mathbb{Q}(\zeta_6)$ then the automorphism group of X is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. Specifically, the automorphisms on X are induced by order two translations by the points (t_1, t_2, t_3) on A' such that $t_1 + t_2 \in \{0, \frac{1}{2}\}$, $t_1 \in E[2]$ and $t_3 \in E'[2]$.*

We immediately see a difference between automorphisms of $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$ and those of $X \in \mathcal{F}^A_{\mathfrak{D}_4}$. Indeed in the first case there exist automorphisms that do not preserve the volume form of X, while in the second case all automorphisms preserve the volume form of X. The different properties of automorphisms of $X \in \mathcal{F}^A_G$ for the case $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and $G \simeq \mathfrak{D}_4$, yield to different situations for the quotients of X. More precisely,

Theorem C (see Theorem 6.5.1 and Theorem 7.4.1)**.** *Let $X \in \mathcal{F}^A_G$ and $\Upsilon \leq \mathrm{Aut}(X)$. Let $\beta\colon Y \to X/\Upsilon$ be a resolution of singularities that blows up once each irreducible component in $\mathrm{Sing}(X/\Upsilon)$.*

(i) *If $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$, the followings hold.*

1. *If Υ preserves the volume form of X then β is a crepant resolution and Y is a Calabi-Yau 3-fold. In particular, there are exactly $3^3 - 1$ automorphisms $(\alpha_j)_X$ which act freely on X and the quotients $X/(\alpha_j)_X$ belong to $\mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$. There are 37 involutions with non-trivial fixed locus which preserve the volume forms of X.*

2. *If Υ does not preserve the volume form of X, we have the following cases.*

   a. *If there exists at least one $\alpha_X \in \Upsilon$ which fixes surfaces on X then Y has negative Kodaira dimension. In particular, there are 2^4 of those involutions α_X, see Remark 7.3.2.*

   b. *Otherwise, Y has trivial Kodaira dimension and K_Y is not trivial. In particular, there are 48 involutions α_X that do no fix surfaces, see Remark 7.3.2.*

(ii) *If $G \simeq \mathfrak{D}_4$. Then for each Υ we have that β is a crepant resolution and Y is a Calabi-Yau 3-fold. Moreover, there exist exactly 2 automorphisms $(\alpha_1)_X$ and $(\alpha_2)_X$ acting freely on X. In particular, $\dfrac{X}{(\alpha_j)_X}$'s belong to $\mathcal{F}^A_{\mathfrak{D}_4}$.*

Furthermore, we completely classify these quotients by computing their Hodge number and their fundamental groups: for $G \simeq \mathfrak{D}_4$ see Tables 6.2, 6.3, 6.4 and 6.5; and for $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$ see Proposition 7.5.1 and [32]. Additionally, to these studies we investigate more on the geometry of $X \in \mathcal{F}^A_G$, mainly for $G \simeq \mathfrak{D}_4$. For instance, we completely the describe the Picard group of $X \in \mathcal{F}^A_G$: for $G \simeq \mathfrak{D}_4$ see Section 6.2 and for $G = (\mathbb{Z}/2\mathbb{Z})^2$ see Section 7.2. In particular, we prove that one can choose as generators of $\mathrm{Pic}_{\mathbb{Q}}(X)$ divisors that define fibrations on X that are related with the natural projections on the cover of X. As final remark, we recall that Calabi-Yau manifolds of type A by construction have infinite fundamental group. In dimension 3 there exist other Calabi-Yau threefolds with infinite fundamental group. These latter threefolds are called *Calabi-Yau threefolds of type K* since they are covered by the product of an elliptic curve and a K3 surface. They are first introduced in [73] and later they appeared in different contexts also related with the Mirror Symmetry, see [44] and [45]. In Section 6.9, we highlight other relations between Calabi-Yau threefolds of type A and K. More in details, we prove that there exist quotients of $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ which are Calabi-Yau threefolds of type K and we present each $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ as finite cover of a certain Calabi-Yau threefolds of type K.

Part II: The Morrison-Kawamata cone conjecture for hyperelliptic varieties

Starting from the 1980s with the work of Mori, the Minimal Model Program (for short MMP) has become a powerful tool for understanding and classifying the (birational) geometry of projective varieties. The MMP can be considered a generalization of the Enriques-Kodaira classification for surfaces to higher dimensional varieties. More precisely, it is an algorithm that starting from a projective variety Y it (eventually) ends up with either a variety Y' birational to Y and with $K_{Y'}$ nef, namely *minimal model of* Y, or a Mori fiber space, *i.e.* a variety Y' birational to Y together with a fibration $\phi\colon Y' \longrightarrow Z$ whose fibers are Fano varieties. One of the core insights of the Minimal Model Program is that much information about the maps from a projective variety Y to projective space is contained in the *nef cone* $\mathrm{Nef}(Y)$ which lies in a finite dimensional vector space $N^1(Y)_{\mathbb{R}} = \mathrm{NS}(Y) \otimes \mathbb{R}$. Indeed, the study of this cone can give insight into the study of the minimal model of Y. Moreover, the nef cone contains all base-point free divisors (semi-ample divisors), thus understanding the structure of this cone allows us to understand the morphisms from Y to projective spaces. Additionally, in dimension higher than 2 the minimal models are not necessarily unique and it is not even known if the number of such minimal models is finite. Therefore, it is important to study maps between different minimal models: such maps are in fact associated with movable divisors. Therefore, the study of *the movable cone* $\overline{\mathrm{Mov}}(Y)$ is important in relation to this problem.

The important relation between the nef cone of Y and the MMP is captured by the Cone Theorem and the Contraction Theorem, see Theorems 9.1.6 and 9.1.8. They assert that the K_Y-negative part of Mori cone (dual to the nef cone) is rational polyhedral away from the K_Y-trivial hyperplane and that the extremal rays of the polyhedral part correspond to some morphisms on Y, involved in MMP.

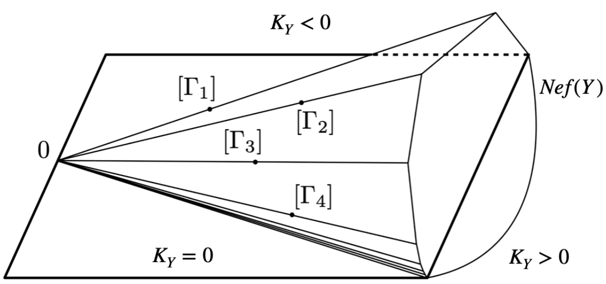

Figure 0.1: The Mori cone of a variety Y

Fano varieties represent a simplicity in that since their nef cone is rational polyhedral (finitely generated) and its extremal rays are generated by semi-ample classes. However, in general, it is difficult to describe the whole cone. For instance, the nef cone of varieties with ample canonical divisor can have infinitely many extremal rays and very little can be said in general about its structure. It remains to study the intermediate case: the nef cone of projective K-trivial manifolds. For such varieties, the nef cone ranges from polyhedral structures to round cones. The Morrison-Kawamata cone conjecture aims to explain and unify the different behaviours of the nef cone for K-trivial manifolds. Indeed it predicts the existence of a fundamental domain for the action of automorphisms group of the nef cone. In addition to study projective models of Y, we would also like to understand the birational geometry of Y by understanding the birational maps from Y and by studying the movable cone $\overline{\mathrm{Mov}}(Y)$. More in details, let $\mathrm{Nef}(Y)^+$ and $\mathrm{Mov}(Y)^+$ be the convex hulls of rational points (see Definition 9.2.2) of the nef and movable cone, respectively, of projective K-trivial manifold Y. Morrison proposed the following conjecture:

Conjecture 0.0.1 (Morrison's version, [69])**.** Let Y be a smooth projective K-trivial variety. Then

(i) There exists a rational polyhedral cone Π which is a fundamental domain for the action of the automorphism group $\mathrm{Aut}(Y)$ on the cone $\mathrm{Nef}(Y)^+$ in the following sense:

a. $\mathrm{Nef}(Y)^+ = \mathrm{Aut}(Y)\cdot\Pi$, *i.e.* $\mathrm{Nef}(Y)^+ = \bigcup_{\varphi\in\mathrm{Aut}(Y)} \varphi^*\Pi$,

b. It holds $(\mathrm{Int}\Pi)\cap\varphi^*(\mathrm{Int}\Pi) = \emptyset$ for every $\varphi^* \neq id$ in $\mathrm{GL}(N^1(Y))$.

(ii) There exists a rational polyhedral cone Π' which is a fundamental domain (in the sense above) for the action of the birational automorphism group $\mathrm{Bir}(Y)$ on the cone $\mathrm{Mov}(Y)^+$.

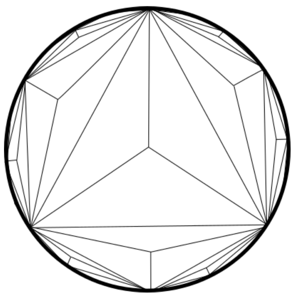

Figure 0.2: Slice of a round cone with a fundamental domain

Historically, Morrison formulated the conjecture in [69] taking inspiration from the Mirror Symmetry. Then, Kawamata [50] refined the conjecture by replacing the convex hulls in the Morrison's version with $\mathrm{Nef}(Y)^e := \mathrm{Nef}(Y) \cap \mathrm{Eff}(Y)$ and $\overline{\mathrm{Mov}}(Y)^e := \overline{\mathrm{Mov}}(Y) \cap \mathrm{Eff}(Y)$ which are more natural objects in MMP, see Conjecture 9.4.3. It is worth to observe that the two versions are not in general equivalent and the only known relation is that $\mathrm{Nef}(Y)^e \subseteq \mathrm{Nef}(Y)^+$, see Lemma 9.4.4. We also mention that there exists an extended version of the cone conjecture in the singular setting due to Totaro [85].

The cone conjecture, in particular, has twofolds consequences. It allows to study these cones by considering just their "polyhedral part" whose study is strongly related with linear algebra. It gives insight into the geometry and birational geometry of the variety. For instance: the cone conjecture for the nef cone implies that there are finitely many contractions or fiber spaces up to automorphisms, see [85, Section 1]; while the one for the movable cone would imply, modulo standard conjectures of the Minimal Model Program, the finiteness of minimal models, up to birational automorphisms, see [20, Theorem 2.14].

The cone conjecture is one of the open and challenging problems in algebraic geometry. Over the years, the conjecture has spurred a flurry of research activity, leading to significant advancements and conjectural extensions. It has been verified for numerous cases: surfaces ([83], [71],[50] and [85]), abelian varieties ([77]), irreducible holomorphic symplectic manifolds ([3], [2]). For Calabi-Yau manifolds the landscape is more complicated, indeed the conjecture is known for very specific cases ([61], [72], [75], and [58]). Since the main thread of this thesis is to study K-trivial manifolds given as free quotients of other K-trivial manifolds, in relation to the cone conjecture we aim to investigate the following problem.

Question 2. Let $\pi\colon X \longrightarrow Y = X/G$ be a finite étale cover where X is a K-trivial manifold. Assume that the cone conjecture holds for X, does it hold also for Y?

Pacienza and Sarti in [76] have given a positive answer whenever X is of IHS type and G is of prime order, the resulting manifolds are called Enriques manifolds. We explore the scenario where the cover X is an abelian variety, *i.e.* Y is a hyperelliptic variety. Specifically, our main result is the following.

Theorem D. *Let $Y = X/G$ be a hyperelliptic variety. Then, the Morrison-Kawamata cone conjecture holds in both its formulations and both for the nef and movable cone. In particular,* $\overline{\mathrm{Mov}}(Y) = \mathrm{Mov}(Y)^+ = \overline{\mathrm{Mov}}(Y)^e = \mathrm{Nef}(Y)^e = \mathrm{Nef}(Y)^+ = \mathrm{Nef}(Y)$.

The proof of the Theorem D is enclosed in Chapter 11. A core idea underlying its proof lies in the possibility to describe the nef cone of Y in terms of the G-invariant nef cone of X and to establish a connection between the automorphisms of Y and the ones in the normalizer $N_{\mathrm{Aut}(X)}(G)$. Specifically, we prove that if the cone conjecture holds for a variety X, it suffices to provide a rational polyhedral cone $\Pi \subset \left(\mathrm{Nef}(X)^G\right)^+$ such that $\mathrm{Amp}(X)^G \subset \bigcup_{h\in H} h(\Pi)$ for some $H \leq N_{\mathrm{Aut}(X)}(G)$ to obtain a rational fundamental domain for $\mathrm{Nef}(Y)$ under the action of H/G, see Proposition 11.1.1. In our setting: to reach this goal we translate our problem into a well-known problem of convex geometry about the existence of a rational fundamental domain for the action of arithmetic groups on homogeneous self-dual cones, as Prendergast-Smith did for abelian varieties [77]. It is worth noting that for hyperelliptic varieties, as for abelian varieties, the nef cone coincides with the closure of the movable cone; therefore the cone conjecture for the nef cone is equivalent to the one for movable cone.

We conclude this part of the thesis with a further investigation, which is connected to the first part. In part I we have studied the Calabi-Yau threefolds $X \in \mathcal{F}_G^A$ and since they are hyperelliptic varieties, they satisfy the cone conjecture. In fact, this result was already proven in [73, Theorem 01. (IV)] where, specifically, the authors proved that the nef cone for these threefolds is rational polyhedral. It is, therefore, natural to ask the following question:

Question 3. Given a Calabi-Yau manifold of type A, under which conditions the nef cone is rational polyhedral? Or, more generally, under which conditions the nef cone of a hyperelliptic variety $Y = A/G$ is rational polyhedral?

In Section 12.1, we provide an answer in relation to the representation of the group G. More precisely,

Theorem E (see Theorem 12.1.2 and Corollary 12.1.4)**.** *Let $Y = A/G$ be a hyperelliptic variety and η be the representation of G. We assume that A is not of CM-type. If G contains a normal abelian group H such that $\eta_{|H}$ does not contain two equals irreducible sub-representations, then the nef cone of Y is a polyhedral cone. In addition, if Y has $h^{1,0}(Y) = 0$ then $\mathrm{Aut}(Y)$ is finite.*

Furthermore, we deduce that all Calabi-Yau manifolds of type A in Theorem A have a rational polyhedral nef cone. We also observed that Oguiso and Sakurai, by proving that the Calabi-Yau threefolds of type A have rational polyhedral nef cones, have described the extremal rays of these cones. They showed that the extremal rays are the divisors generating $\mathrm{Pic}_{\mathbb{Q}}(X)$ which define fibrations on X; in particular they deduced that all nef divisors are semi-ample. Thus, we investigate on the extremal rays of the nef cone of the higher dimensional examples of Calabi-Yau manifolds that we have constructed in Theorem A and we obtain the following result.

Theorem F. *Let Y be the Calabi-Yau manifold of type A as in Theorem A. Then extremal rays of the nef cone are given by divisors which define fibrations on X induced by natural projections on A. In particular all nef divisors are semi-ample.*

We emphasize in Part III that Theorem F is related with one of the main open conjecture in MMP. As we have briefly mentioned, one of the possible outcome of the MMP is a variety Y' with $K_{Y'}$ nef. The Abundance conjecture predicts that actually $K_{Y'}$ is semi-ample. For K-trivial varieties with no $(1, 0)$-forms the abundance conjecture predicts that if D is a nef divisor then D is semi-ample. Therefore Theorem F together with [73, Theorem 0.1 (IV)] imply that all the Calabi-Yau manifolds of type A in Theorem A satisfy the Abundance conjecture (for K-trivial varieties)

1 Preliminaries from Algebraic Geometry

We assume that all algebraic varieties are irreducible, reduced and defined over $\mathbb{C}$. We mainly consider smooth and projective varieties.

1.1 Divisors and line bundles

Let Y be a smooth, projective variety.

Definition 1.1.1. A **prime divisor** is a codimension 1 subvariety of Y. A **Weil divisor** on Y is a formal finite linear combination $D = \sum_i n_i Y_i$ with Y_i's prime divisors and $n_i \in \mathbb{Z}$. A Weil divisor is called **effective** if all the coefficients n_i are non-negative.

Let $(\mathcal{M}_Y)^\times$ and $(\mathcal{O}_Y)^\times$ be the sheaves of invertible meromorphic and holomorphic functions, respectively.

Definition 1.1.2. A **Cartier divisor** is a global section of the sheaf $(\mathcal{M}_Y)^\times/(\mathcal{O}_Y)^\times$, *i.e.* a collection $\{(U_i, f_i)\}$ such that $\{U_i\}$ is a cover of Y, f_i is a section of $(\mathcal{M}_Y)^\times$ on U_i and $f_i = f_j$ on $U_i \cap U_j$ up to a multiplication by a section of $(\mathcal{O}_Y)^\times$.

We recall that since Y is smooth, there is a one-to-one correspondence between Weil and Cartier divisors, see [49, Proposition 2.3.9]. Thus we simply use the term divisor. We denote by $\mathbf{Div}(Y)$ the group of (Cartier) divisors endowed with the natural group structure.

Let $f\colon X \longrightarrow Y$ be a dominant morphism of smooth algebraic varieties, *i.e.* $f(X)$ is dense in Y. The **degree of f** is defined by

$$\deg(f) = \begin{cases} [\mathbb{C}(X) : \mathbb{C}(Y)] & \dim(X) = \dim(Y) \\ 0 & \dim(X) > \dim(Y). \end{cases}$$

We recall the relation between divisors on X and Y under the morphism f.

Definition 1.1.3. Let $D \in \mathrm{Div}(Y)$ be a prime divisor. The **the pull-back divisor** of D is defined by $f^*D = f^{-1}(D) \in \mathrm{Div}(X)$.

We extend by linearity the definition above and obtain the following morphism of groups:

$$f^*\colon \mathrm{Div}(Y) \longrightarrow \mathrm{Div}(X)$$

$$D = \sum_i n_i Y_i \longmapsto f^*D = \sum_i n_i f^{-1}(Y_i)$$

Definition 1.1.4. Let $f\colon X \longrightarrow Y$ be generically of finite degree and $D \in \mathrm{Div}(X)$ be a prime divisor. The **the push-down divisor** of D is defined by $f_*D = \deg(f)f(D)$.

We extend by linearity the definition above and obtain the following morphism of groups:

$$f_*\colon \mathrm{Div}(X) \longrightarrow \mathrm{Div}(Y)$$

$$D = \sum n_i Y_i \longmapsto f_*D = \sum n_i f_* Y_i$$

If $f\colon X \longrightarrow Y$ is generically of finite degree, the following relation holds:

$$f_* f^* D = \deg(f) D \text{ for every } D \in \mathrm{Div}(Y). \tag{1.1.1}$$

Any meromorphic function h on Y defines a divisor $D = \mathrm{div}(h) = \sum_Z \mathrm{ord}_Z(h) Z$ where Z is a prime divisor on Y and $\mathrm{ord}_Z(h)$ is the order of h along Z.

Definition 1.1.5. A divisors D is called **principal divisor** if $D = \mathrm{div}(h)$ for some meromorphic function h.

Definition 1.1.6. Two divisors D_1 and D_2 are **linearly equivalent** if $D_1 - D_2$ is principal. In this case we write $D_1 \sim_{lin} D_2$.

The relation of linear equivalence on the divisor group defines an equivalence relation on $\mathrm{Div}(Y)$.

Definition 1.1.7. We define the **Picard group Pic(Y)** the group of line bundle up to isomorphism under the tensor product $\otimes$.

Given $f\colon X \longrightarrow Y$ dominant morphism and $\mathcal{L} \in \mathrm{Pic}(Y)$ defined by $\{(U_j, \psi_{ij})\}$, we define the **pull back line bundle** $f^*\mathcal{L}$ to be the line bundle determined by $\{(f^{-1}(U_j), \psi_{ij} \circ f)\}$.

It turns out that, under smoothness and projective assumption, divisors modulo linear equivalence correspond to line bundles. More precisely, we have the following homomorphism of groups:

$$\begin{aligned} \vartheta\colon \mathrm{Div}(Y) &\hookrightarrow \mathrm{Pic}(Y) \\ D &\longmapsto \mathcal{O}_Y(D) \end{aligned} \tag{1.1.2}$$

where if $D = \{(U_i, f_i)\}$ is a (Cartier) divisor we define $\mathcal{O}_Y(D)$ to be the line bundle whose transiction functions are $f_j f_i^{-1}$ on $U_j \cap U_i$ which satisfy the cocycle conditions, see [49, Section 2.3]. The kernel of ϑ is given by the group of principal divisors [49, Lemma 2.3.14], *i.e.* ϑ factorizes through $\mathrm{Div}(Y)/\sim_{lin}$. Furthermore, whenever Y is a projective manifolds then ϑ is surjective, [49, corollary 5.3.7]. This provides the isomorphism $\mathrm{Div}(Y)/\sim_{lin} \simeq \mathrm{Pic}(Y)$ for any smooth projective variety Y. We recall that ϑ is compatible with pull-back maps, [49, Corollary 2.3.13]. Moreover, it is easy to see that the push down map is well-defined on linear equivalent divisors classes thus it induces the pushforward map $f_*\colon \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$ which maps $\mathcal{L} = \mathcal{O}_X(D)$ to $f_*\mathcal{L} = \mathcal{O}_X(f_*D)$.

An important machinery that allows us to study the Picard group is given by the exponential sequence:

$$0 \to \mathbb{Z} \to \mathcal{O}_Y \xrightarrow{exp} (\mathcal{O}_Y)^\times \to 0 \tag{1.1.3}$$

where $\mathbb{Z}$ is the sheaf of locally constant functions, $\mathcal{O}_Y$ is the sheaf of holomorphic functions. According to [49, Corollary 2.2.10], we have the following natural isomorphism $H^1(Y, (\mathcal{O}_Y)^\times) \simeq \mathrm{Pic}(Y)$. By considering the long exact sequence in cohomology induced by (1.1.3) we obtain the following map, namely the **first Chern map**

$$\begin{aligned} c_1\colon \quad \mathrm{Pic}(Y) &\longrightarrow H^2(Y, \mathbb{Z}) \\ \mathcal{L} &\longmapsto c_1(\mathcal{L}). \end{aligned}$$

We denote by $\mathrm{Pic}^0(Y) = \ker(c_1)$.

Definition 1.1.8. We denote $\mathrm{NS}(Y) = \mathrm{Im}(c_1) \simeq \dfrac{\mathrm{Pic}(Y)}{\mathrm{Pic}^0(Y)}$ and we call it **the Néron-Severi group of Y**.

Definition 1.1.9. Let $D \in \text{Div}(Y)$. The **homological class of D** is defined by $[D]_{hom} := c_1(\mathcal{O}_X(D))$.
Let $D_1, D_2 \in \text{Div}(Y)$. We say that D_1 is **homogical equivalent** to D_2, denoted by $D_1 \sim_{hom} D_2$, if their homological class are the same.

Remark 1.1.10. The homological equivalence is an equivalence relation on $\text{Div}(Y)$.

Remark 1.1.11. When Y is projective and smooth we have that $\text{Pic}^0(Y)$ can be defined as the group of divisors homological equivalent to zero. Hence $\text{NS}(Y) = \text{Div}(Y)/\sim_{hom}$.

By the Néron-Severi's Theorem, $\text{NS}(Y)$ is a finitely generated abelian group: we denote by $\rho(Y) = \text{rk}(\text{Pic}(Y)) = \text{rk}(\text{NS}(Y))$ its rank, namely the **Picard rank** of Y.

Assume that Y is projective (or Kähler), it holds the Hodge decomposition:

$$H^n(Y, \mathbb{C}) = \bigoplus_{p+q=n} H^q(Y, \Omega_Y^p) := \bigoplus H^{p,q}(Y) \tag{1.1.4}$$

where Ω_Y^p is the sheaf of p-forms. We denote by $h^{p,q} = \dim(H^{p,q}(Y))$, namely the **Hodge numbers**. We also recall the following symmetries:

$$H^{p,q}(Y) = \overline{H^{q,p}(Y)} \text{ (Hodge symmetry)} \qquad H^{p,q}(Y) = H^{n-p,n-q}(Y) \text{ (Serre duality)}.$$

Theorem 1.1.12 (Lefschetz theorem on $(1,1)$ classes)**.** [49, Section 3.3] *Let Y be a compact complex Kähler manifold. then any element in $H^{1,1}(Y) \cap H^2(Y, \mathbb{Z})$ is the cohomology class of a divisor on Y.*

Theorem 1.1.12 says that $c_1 \colon \text{Pic}(Y) \to H^{1,1}(Y, \mathbb{Z}) := H^{1,1}(Y) \cap H^2(Y, \mathbb{Z})$ is surjective, and so that the Néron-Severi group lies in the $H^{1,1}(Y)$.

Definition 1.1.13. The **holomorphic characteristic** of Y by $\chi(Y) = \sum_i (-)^i h^{j,0}(Y)$.

Definition 1.1.14. The **Euler characteristic** of Y by $e(Y) = \sum_i (-)^i b_i(Y)$ where $b_i = \dim(H^i(Y, \mathbb{Z}))$ are called **Betti numbers**.

Remark 1.1.15. By (1.1.4), $b_k(Y) = \sum_{i+j=k} h^{i,j}(Y)$.

Lemma 1.1.16. [57, Proposition 1.1.28] *Let $f \colon X \longrightarrow Y$ be a finite étale covering of Kähler manifolds. Then $\chi(X) = \deg(f)\chi(Y)$ and $e(Y) = \deg(f)e(Y)$.*

1.2 | Intersection of divisors

Let Y be a projective smooth variety over $\mathbb{C}$. There are different way to introduce intersection theory on Y. Since we are interested only in intersection between divisors we use a more simply approach, refer to [57, Chapter 1]. For more general one, we refer for example to [43, Section A].

Let $D_1, \dots, D_k$ be Cartier divisors on Y: since Y is smooth they are in bijection with the line bundles $\mathcal{O}_Y(D_i) \in \mathrm{Pic}(Y)$. We consider their cohomology class given by $c_1(\mathcal{O}_Y(D_i)) \in H^2(Y, \mathbb{Z})$. By using the cup product $\smile$ on $H^2(Y, \mathbb{Z})$ we can define the intersection of divisors as follows:

$$c_1(\mathcal{O}_Y(D_1)) \smile \dots \smile c_1(\mathcal{O}_Y(D_k)) \in H^{2k}(Y, \mathbb{Z}).$$

We generally use the following notation $D_1 \cdot \ldots \cdot D_k$ to indicate the product above.
By the Poincaré duality we have $H^{2n}(Y, \mathbb{Z}) \simeq H_0(Y, \mathbb{Z}) \simeq \mathbb{Z}$ where $\dim(Y) = n$, hence we obtain an intersection number by considering the product of exactly n divisors:

$$\begin{aligned} \mathrm{Div}(Y) \times \ldots \times \mathrm{Div}(Y) &\longrightarrow \mathbb{Z} \\ (D_1, \dots, D_n) &\longmapsto D_1 \cdot \ldots \cdot D_n \end{aligned} \tag{1.2.1}$$

The important properties of this product are the following, see [57, Remark 1.1.13]:

1. It is symmetric and multi-linear,
2. The integer $D_1 \cdot \ldots \cdot D_n$ depends only on the linear equivalence class of D_i.
3. If D_i's are effective divisors meeting transversely then $D_1 \cdot \ldots \cdot D_n = \sharp(D_1 \cap \ldots \cap D_n)$.
4. **The projection formula**: let $f\colon X \longrightarrow Y$ be a dominant morphism of varieties, then for every $D_1, D_2 \in \mathrm{Div}(Y)$
$$f_*(D_1 \cdot f^*D_2) = f_*(D_1) \cdot D_2 \tag{1.2.2}$$

If $D_1 = \ldots = D_n$ we write the product by D^n, called **self-intersection**.

Lemma 1.2.1. *Let $f\colon X \longrightarrow Y$ be a generically finite surjective morphism of smooth varieties and $\dim(X) = \dim(Y) = n$, then*

$$f^*D_1 \cdot \ldots \cdot f^*D_n = deg(f)(D_1 \cdot \ldots \cdot D_n) \tag{1.2.3}$$

for every $D_i \in \mathrm{Div}(Y)$.

Proof. Since the product of n divisors is an integer number, we have the following equality

$$f^*D_1 \cdot \ldots \cdot f^*D_n = f_*(f^*D_1 \cdot \ldots \cdot f^*D_n).$$

We apply, subsequently, property 4. (1.2.2):

$$\begin{aligned} f^*D_1 \cdot \ldots \cdot f^*D_n &= f_*(f^*D_1 \cdot \ldots \cdot f^*D_n) = f_*(f^*D_1 \cdot \ldots \cdot D_{n-1}) \cdot D_n \\ &= \ldots = f_*f^*D_1 \cdot D_2 \cdot \ldots \cdot D_n \overset{(1.1.1)}{=} \deg(f)(D_1 \cdot \ldots \cdot D_n). \end{aligned}$$

□

The intersection defined in 1.2.1, allows also to define intersection of divisors and 1-cycle.

Definition 1.2.2. A **k-cycle** on Y is a finite (formal) linear combination $\sum_i n_i V_i$ with V_i's subvarieties of dimension k and $n_i \in \mathbb{N}$. We denote by $Z_k(Y)$ the group generated by k-cycles.

Let $D \in \mathrm{Div}(Y)$ and V a subvariety of dimension 1. We have:

$$D \cdot V = c_1(\mathcal{O}_Y(D_1)) \cap [V] \in H_0(Y, \mathbb{Z}) \simeq \mathbb{Z}$$

where $[V]$ is the class of V in $H_{2(n-1)}(Y, \mathbb{Z})$. The intersection above can be linearly extended to an intersection of divisors and 1-cycle. Hence we obtain the so-called **intersection pairing**

$$\begin{aligned} \Phi_1 \colon \mathrm{Div}(Y) \times Z_1(Y) &\longrightarrow \mathbb{Z} \\ (D, V) &\longmapsto D \cdot V \end{aligned} \tag{1.2.4}$$

The intersection pairing above allows us to define an equivalence relation, namely **numerical equivalence**, on both $\mathrm{Div}(Y)$ and $Z_1(Y)$.

Definition 1.2.3. Let D_1 and D_2 in $\mathrm{Div}(Y)$. They are say to be **numerically equivalent divisor** if $D_1 \cdot C = D_2 \cdot C$ for every $C \in Z_1(Y)$. In this case we denote $D_1 \equiv D_2$. We denote $N^1(Y) = (\mathrm{Div}(Y)/\equiv)$.

Definition 1.2.4. We denote by $N^1(Y)_{\mathbb{R}} = N^1(Y) \otimes \mathbb{R}$, namely the **real Néron-Severi vector space**.

We have $N^1(Y)_{\mathbb{R}}$ a vector space of dimension $\rho(Y)$ with its standard Euclidean topology. The group of divisors of Y defines a structure of a lattice in $N^1(Y)_{\mathbb{R}}$. When an element of $N^1(Y)_{\mathbb{R}}$ is in this lattice we refer to it simply as a divisor, otherwise as $\mathbb{R}$-divisor.

Remark 1.2.5. We observe that the name Néron Severi vector space is actually related to the Néron Severi group. By Remark 1.1.11:

$$\mathrm{NS}(Y) = \mathrm{Pic}(Y)/\mathrm{Pic}^0(Y) = \mathrm{Div}(Y)/\sim_{hom}.$$

According to [57, Remark 1.1.20] we have for any $D \in \mathrm{Div}(Y)$:

$$D \equiv 0 \Leftrightarrow \exists m \in \mathbb{N}_{>0} \text{ such that } c_1(mD) \in \mathrm{Pic}^0(Y) \Leftrightarrow [mD]_{hom} = 0.$$

In other words, homological and numerical equivalence coincide up to torsion. Consequently, $N^1(Y)_{\mathbb{R}} \simeq \mathrm{NS}(Y) \otimes \mathbb{R}$ and so it has dimension $\rho(Y)$.

Definition 1.2.6. Let C_1 and C_2 in $Z_1(Y)$. They are said to be **numerically equivalent 1-cycle** if $C_1 \cdot D = C_2 \cdot D$ for every $C \in \mathrm{Div}(Y)$. In this case we denote $C_1 \equiv C_2$. We denote $N_1(Y) = (Z_1(Y)/\equiv)$.

Definition 1.2.7. We denote by $N_1(Y)_{\mathbb{R}} = N_1(Y) \otimes \mathbb{R}$.

The intersection pairing 1.2.4 extends to a non-degenerate pairing between the vector spaces $N^1(Y)_{\mathbb{R}}$ and $N_1(Y)_{\mathbb{R}}$:

$$\begin{aligned} N^1(Y)_{\mathbb{R}} \times N_1(Y)_{\mathbb{R}} &\longrightarrow \mathbb{R} \\ (D, C) &\longmapsto D \cdot C \end{aligned}$$

In particular, by construction it is a perfect pairing hence we obtain that $N^1(Y)_{\mathbb{R}}$ and $N_1(Y)_{\mathbb{R}}$ are dual vector spaces. Therefore $N_1(Y)_{\mathbb{R}}$ is a finite dimensional vector space with standard Euclidean topology.

1.3 | Maps to projective space

Let Y be a smooth algebraic variety. Divisors play a central role in the understanding of the geometry of varieties. Indeed, one way of understanding the geometry of Y is to understand the maps on Y and much information about these maps is captured by the divisors on Y. We recall that using ϑ in (1.1.2) we can associated to $D \in \mathrm{Div}(Y)$ a line bundle $\mathcal{O}_Y(D)$. Assume Y to be compact, then $H^0(Y, \mathcal{O}_Y(D))$ is either trivial or finitely generated by $\langle s_1, \ldots, s_N \rangle$. We assume to be in the last case, hence it is a finite dimensional vector space and we can use its global sections to define a rational map on Y as follows:

$$\varphi_{|D|} \colon Y \dashrightarrow \mathbb{P}(H^0(Y, \mathcal{O}_Y(D))^\vee)$$

$$y \longmapsto (s_1(y) : \ldots : s_N(y)).$$

The above map is well-defined outside the set **Bs**(D):= $\{y \in Y \mid s_i(y) = 0 \ \forall i = 1, \ldots, n\}$ which is called **base locus of** D. The properties of the map $\varphi_{|D|}$ are clearly related with properties of D. Here we recall some of them.

Definition 1.3.1. Let $D \in \mathrm{Div}(Y)$. Then

- D is **movable** if there exists $m > 0$ such that mD is effective and $\mathrm{Bs}(|mD|)$ has no component of codimension 1;
- D is **nef** (or *numerically effective*) if $D \cdot C \geq 0$ for every irreducible curve C on Y;
- D is **ample** if $\varphi_{|mD|}$ defines an embedding of Y in $\mathbb{P}^N$ for some $m >> 0$.

We make some easy observations:

1. Any ample divisor is nef. Indeed by the Nakai-Moshezan criterior [57, Theorem 1.2.23] a divisor D on Y is ample if and only if $D^{\dim(Y)} \cdot Y > 0$ for every subvariety $Y \subset X$. Thus for a curve Y we have $D \cdot Y > 0$, hence D is nef.
2. Any ample divisor is movable. Indeed if D is ample there exists $m > 0$ such that $|mD|$ defines and embedding, in particular mD is effective and $\mathrm{Bs}(|mD|) = \emptyset$. Thus D is movable.

We also remark that D is effective if $H^0(X, \mathcal{O}_X(D))$ is not trivial since there exists a bijection between effective divisors D and non-trivial global sections of $\mathcal{O}_X(D)$, see [49, Proposition 2.3.18].
It is useful to know which morphisms preserve the properties mentioned above.

Lemma 1.3.2. *Let $f \colon X \longrightarrow Y$ be a morphism of projective manifolds. Let $D \in \mathrm{Div}(Y)$.*

(i) *Then D is effective if and only if f^*D is effective.*

(ii) *If f is finite and D ample then f^*D is ample on X. If moreover f is surjective we have the viceversa.* [57, Proposition 1.2.13 and Corollary 1.2.28].

(iii) *If f is proper and D is a nef on Y then f^*D is a nef divisor on X. If moreover f is surjective it holds the viceversa.* [57, Example 1.4.4 (i)-(ii)].

Definition 1.3.3. A rational map f is said **birational map** if f is a rational map with and it has an inverse which is a rational map. A **birational morphism** $f\colon X \longrightarrow Y$ is a morphism of varieties which is also a birational map.

Lemma 1.3.4. *Let $f\colon X \longrightarrow Y$ be a birational morphism of projective varieties. If D is movable then f^*D is movable.*

Proof. It follows since according to [51, Theorem 1] any birational morphism is an isomorphism in codimension 1. □

1.4 | The canonical divisor

Let Y be a compact, projective, smooth variety. We can associate a canonical sheaf to Y as follows. Let Ω_Y be the cotangent sheaf on Y.

Definition 1.4.1. We define the **canonical sheaf** of Y by $\mathcal{K}_Y = \bigwedge^n \Omega_Y$.

Since Y is non singular, the Ω_Y is locally free of rank n, so $\mathcal{K}_Y$ is a line bundle, hence isomorphic to $\mathcal{O}_X(K_Y)$ for some (Cartier) divisor K_Y called **canonical divisor**.

Definition 1.4.2. The **Kodaira dimension** of Y is

$$k(Y) = \begin{cases} -\infty & H^0(Y, \mathcal{K}_Y^{\otimes m}) = 0 \text{ for every } m \in \mathbb{N} \\ \max\limits_{m\in\mathbb{N}} \dim\, \varphi_{|mK_Y|}(Y) & \text{otherwise.} \end{cases}$$

Proposition 1.4.3 (Adjunction formula for hypersurfaces,)**.** [49, Proposition 2.2.17] *Let Y be a smooth variety and $X \subset Y$ be an hypersurface. Then $K_X = (K_Y + X)_{|X}$.*

Proposition 1.4.4 (Riemann-Hurwitz Formula)**.** [5, Chapter I.Section 16, equation (20)] *Let us consider $f\colon X \to Y$ a generically finite morphism between smooth projective varieties. Then $K_X = f^*K_Y + R$ where R is a divisor supported on the ramification locus of f.*

1.5 | Beauville-Bogomolov decomposition theorem

Definition 1.5.1. We define the **first Chern class of Y** by $c_1(Y) = c_1(\mathcal{K}_Y)$.

Definition 1.5.2. A **K-trivial manifold** is a manifold whose canonical divisor is numerically trivial.

As we have recalled in the previous section the map associated to (multiple of) the canonical divisor can produce projective models. This is not the case of K-trivial manifolds. Let Y be a K-trivial n-fold. We recall that having trivial canonical bundle is equivalent to have a holomorphic nowhere vanishing volume form ω_Y. Whenever Y is compact the volume form defines the section (up to scalar) of the sheaf Ω_Y^n of n-form, *i.e.* $H^0(Y, \Omega_Y^n) = \langle \omega_Y \rangle \simeq \mathbb{C}$, see [49, Chapter 1]. It is clear that for K-trivial manifolds the map associated to K_Y is just a map onto a point.
The K-trivial manifolds are characterized by a decomposition theorem.

Definition 1.5.3. A **Calabi-Yau manifold** Y is compact, complex, Kähler n-fold with trivial canonical bundle and $h^{j,0}(Y) = 0$ for $1 \leq j \leq n-1$.

Definition 1.5.4. A **Irreducible Holomorphic Symplectic (IHS)** n-fold is simply connected, complex, Kähler n-fold with a non-degenerate holomorphic symplectic 2-form σ_Y such that $H^0(Y, \Omega_Y^2) = \mathbb{C}\sigma_Y$.

Remark 1.5.5. Let Y be a IHS n-fold: it follows by the existence of σ_Y that

$$H^{2j,0}(Y) = \langle (\sigma_Y)^{\wedge j} \rangle \quad j = 1, \ldots, \frac{n}{2}.$$

In particular, n is even and Y is compact.

Theorem 1.5.6 (Beauville-Bogomolov decomposition)**.** [10, Theorem 2] *Let Y be a Kähler compact K-trivial manifold. There exists a finite étale cover Y' is Y isomorphic to the product $T \times \prod_i V_i \times \prod_j X_j$ where T is a complex torus, V_i's are simply connected Calabi-Yau manifolds and X_j's are IHS manifolds. Moreover the covering is unique up to isomorphisms.*

Definition 1.5.7. Let V be a n-dimensional vector space over k. We denote the group of isometries of V by **Iso**$(V) = O(n) \ltimes k^n$ where $O(n)$ is the orthogonal group of dimension n.

Remark 1.5.8. As remarked in [9], the finite covering in Theorem 1.5.6 can be assumed to be of Galois. In particular, $\pi_1(Y)$ is a finite extension of $\mathbb{Z}^{2k} \simeq \pi_1(T)$ by a finite group G. In fact $G = G_1 \times G_2 \leq \mathrm{Iso}(\mathbb{C}^k) \times \mathrm{Aut}(M)$ with $M = \prod_i V_i \times \prod_j X_j$.

Corollary 1.5.9. *Let $Y = (T \times S)/G$ be a free finite quotient with T is a complex torus and S is compact Kähler manifold with $b_1(S) = 0$. Assume that G does not contain any element of type (t, id_S) where t is a translation on T.*

(i) *The automorphism group* $\mathrm{Aut}(Y)$ *can be identified with the normalizer of G in* $\mathrm{Aut}(T) \times \mathrm{Aut}(S)$, [9, pag. 10 (a)].

(ii) *For* $S = \prod_i S_i$ *with* S_i*'s non-isomorphic irreducible manifolds. Then* $\mathrm{Aut}(S) = \prod_i \mathrm{Aut}(S_i)$, [9, pag. 10 (b)].

1.6 | Action of finite groups of varieties

Let Y be an algebraic variety of dimension n with an action of a finite group G. Then Y/G is an algebraic variety. We are mainly interested in the case when Y is a manifold. Despite this assumption, Y/G is not necessarily smooth: this heavily depends on properties of the action of G.

Definition 1.6.1. The **fixed locus of** G **on** Y is $\mathbf{Fix}(G) = \{y \in Y \mid g(y) = y \text{ for every } g \in G\}$. The **stabilizer of** G **at** $y \in Y$ is $\mathbf{Stab}_y(G) = \{g \in G \mid g(y) = y\}$.

Definition 1.6.2. Let $\Gamma \leq \mathrm{GL}(n, \mathbb{C})$ be a finite group. An element $g \in \Gamma$ is called **quasi-reflection** or **pseudo-reflection** if $\mathrm{rank}(g - I) = 1$.

Proposition 1.6.3 (Chevalley-Shephard-Todd theorem)**.** [82, Theorem 5.1 and Section 8] *Let* G *be a finite group acting on* $V = \mathbb{C}^n$*. Then ring of invariant of* V^G *is a polynomial ring if and only if is generate by pseudo-reflections.*

In other word the Chevalley-Shephard-Todd theorem states that given a linear finite group G acting on $\mathbb{C}^n$, $\mathbb{C}^n/G$ is smooth at the origin if and only if G is generated by pseudo-reflections. This can be generalized to arbitrary complex varieties Y saying that given $G \leq \mathrm{Aut}(Y)$ finite group, Y/G is smooth if and only if for every $y \in Y$ the stabilizer of G at y is generated by pseudo-reflections. It is easy to observe that if g stabilizes only codimension 1 submanifolds then it is a pseudo-reflection (use the diagonal action of g near a fixed point given by [19, lemma 1]). Hence we summarize the result as follow.

Corollary 1.6.4. *Let* Y *be a complex manifolds and* G *be a finite group on* Y *which stabilizes only codimension* 1 *submanifolds. Then* Y/G *is smooth.*

When the fixed locus of G on Y contains an irreducible component which has codimension more than 1 then Y/G is no longer smooth. Therefore, we can consider a resolution of singularities.

Definition 1.6.5. Let Y be a normal variety such that a K_Y is $\mathbb{Q}$-Cartier, *i.e.* there exists $m \in \mathbb{Q}$ such that mK_Y is a Cartier divisor. A **resolution of singularities** is a proper birational morphism $Y' \longrightarrow Y$ with Y' smooth algebraic variety. The manifold Y' is called **desingularization** of Y.
A subvariety $Z \subset Y$ is called **exceptional divisor** if $\dim(Z) = 1$ and $\mathrm{codim} f(Z) \geq 2$.

According to the following result, any variety admits a resolution of singularities.

Theorem 1.6.6 (Hironaka [47]). *Every variety over a field of characteristic zero admits a resolution of singularities.*

Let Y be a complex algebraic variety and $f\colon Y' \longrightarrow Y$ it resolution. The following formula holds:

$$f^*K_Y + \sum_i a_i E_i = K_{Y'} \tag{1.6.1}$$

where $a_i \in \mathbb{Q}$ and E_i's exceptional prime divisors introduce by f. The rational numbers a_i's are called **discrepancy of f with respect E_i** and $\Delta := \sum_i a_i E_i$ is a $\mathbb{Q}$-divisors called **discrepancy of f**. The numbers a_i's depend only on E_i's, but not on the choice of f, see [54, Reamrk 2.23]. Depending on properties of the rational numbers a_i we can classified the type of singularities of Y and so deduce properties on their resolution. Here we are interested in the so-called canonical and terminal singularities introduced M. Reid in [78].

Definition 1.6.7. Let Y be a normal variety and $f\colon Y' \longrightarrow Y$ be a resolution of singularities, so it holds (1.6.1). Let $p \in \mathrm{Sing}(Y)$, then:

- p is said to be a **canonical singularities** if $a_i \geq 0$ in (1.6.1),
- p is said to be a **terminal singularities** if $a_i > 0$ in (1.6.1).

Definition 1.6.8. Let $f\colon Y' \longrightarrow Y$ be a resolution of singularities. It is called **crepant resolution** if $\mathcal{K}_{Y'} = f^*\mathcal{K}_Y$, *i.e.* f preserves the canonical class of Y.

Remark 1.6.9. Let Y be a normal variety and $f\colon Y' \longrightarrow Y$ be a resolution of singularities, so it holds (1.6.1). Then f is a crepant resolution if and only if in (1.6.1) $\Delta = 0$ if and only if $a_i = 0$ for all i, *i.e.* Y has only canonical singularities.

Let X be a K-trivial manifold and $G \leq \mathrm{Aut}(X)$ be a finite group that preserves the volume form ω_X on X. The smooth part of X/G admits a nowhere vanishing n-holomorphic form induced by ω_X. If X/G admits a crepant resolution we can construct a manifold Y, birational to X/G, which has trivial canonical bundle. Here we collect some of the known result concerning the 2 and 3-dimensional cases. We first recall that resolutions of singularities are local transformations, thus the study of existence of a crepant resolution of X/G is related to the one of $\mathbb{C}^n/G$ with $G \leq \mathrm{SL}(n,\mathbb{C})$.

In dimension 2, quotient singularities of $\mathbb{C}^2/G$ with $G \leq \mathrm{SL}(2,\mathbb{C})$ were first classified by Klein in 1884 [52], which are also known as **Du Val singularities** (sometimes we refer as A-D-E type singularities). In our setting they play a central role since they admit a crepant resolution, see [5, Chapter III].

Proposition 1.6.10. [5, Chapter III] *The quotients* $\mathbb{C}^2/G$ *with* $G \leq \mathrm{SL}(2,\mathbb{C})$ *a finite group admits a crepant resolution.*

Definition 1.6.11. A quotient singularity on a surface is called of type A_{n-1} if locally it is given by $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$.

When $n = 3$, the quotient singularities are classified by Blichfeldt in 1917 [14] and only in the early 1990 Roan proved the existence of crepant resolution with arguments case by case.

Proposition 1.6.12. [79, Theorem 1] *The quotients* $\mathbb{C}^3/G$ *with* $G \leq \mathrm{SL}(3,\mathbb{C})$ *a finite group admits a crepant resolution.*

Therefore one has the following result.

Proposition 1.6.13. [6] and [90] *Let* Y *be a* n*-fold with trivial canonical bundle and* $G \leq \mathrm{SL}(n,\mathbb{C})$ *be a finite group with* $n = 2, 3$*. Then* Y/G *admits a crepant resolution. Moreover, the Hodge numbers of any such a resolution do not depend on the crepant resolution we are considering.*

Remark 1.6.14. The last statement is crucial specifically in dimension 3. While in dimension 2 minimal models are unique, in dimension 3 this is not true: in fact Kawamata showed that any two birational minimal models of 3-folds can be connected by a sequence of flops, see [51, Theorem 1].

We give another characterization of canonical and terminal singularities.

Definition 1.6.15. A finite group $\Gamma \leq \mathrm{GL}(n,\mathbb{C})$ is said to be **small** if the subgroup of Γ generated by pseudo-reflection is trivial.

The condition above is equivalent to ask the fixed locus of Γ has codimension ≥ 2. Let g be an order m automorphism of $\mathbb{C}^n$. According to [19, Lemma 1], near a fixed point $y \in \mathbb{C}^n$, g can be diagonalized as $\mathrm{diag}(\zeta^{a_1}, \dots, \zeta^{a_n})$ where ζ is a primitive m-th root of unity and $0 \leq a_i \leq m-1$.

Definition 1.6.16. We define the **age of g at y** as $\mathrm{age}_y(g,\zeta) = \dfrac{\sum_{i=1}^{n} a_i}{m}$

Theorem 1.6.17. [68, Theorem 2.3] *Let* $\Gamma \leq \mathrm{GL}(n,\mathbb{C})$ *be a small group. Then:*

(i) $\mathbb{C}^n/\Gamma$ *has canonical singularities if and only if* $\mathrm{age}_y(g,\zeta) \geq 1$ *for all primitive root* ζ*, for all* $g \neq id$ *and for all singular point* y*.*

(ii) $\mathbb{C}^n/\Gamma$ *has terminal singularities if and only if* $\operatorname{age}_y(g,\zeta) > 1$ *for all primitive root* ζ, *for all* $g \neq id$ *and for all singular point* y.

In particular, $\mathbb{C}^n/\Gamma$ *admits a crepant resolution if and only if* $\operatorname{age}_y(g,\zeta) = 1$ *for all primitive root* ζ, *for all* $g \neq id$ *and for all singular point* y.

When $n \geq 4$ quotients singularities are not only canonical, they can be terminal. Thus, it fails to generalized the Proposition 1.6.13 in higher dimension, without other assumption.

Example 1. Let $\mathbb{C}^4$ together with the action of $g = \operatorname{diag}(-z_1, -z_2, -z_3, -z_4)$. An easy computation show that the age at 0 is equal to 2. Therefore $\mathbb{C}^4/g$ has terminal singularities and so does not admit a crepant resolution.

1.7 | Blowing up of a submanifolds

Let X be a smooth variety and $Y \subset X$. The blowing-up of X along Y is a geometric construction that replace Y with the projectivization of Y in X. More in details, $X \simeq \mathbb{C}^n$ with local coordinates $z_1, \ldots, z_n$. Let Y be the locus of the equation $x_1 = \cdots = x_k = 0$. The blow up of X along Y is a birational map:

$$\beta \colon \widetilde{X} \longrightarrow X$$

such that:

- $\widetilde{X} = V(x_i y_j - x_j y_i \mid i,j = 1, \ldots, k) \subseteq Y \times \mathbb{P}^{k-1}_{(y_1:\ldots:y_k)}$ is a smooth variety;
- β is proper and surjective;
- the inverse image of Y is a divisor $E = \beta^{-1}(Y) \simeq Y \times \mathbb{P}^k$;
- β is an isomorphism outside E and $\beta_{|E} \colon E \longrightarrow Y$ is the projectivization of the normal bundle of Y in X.

Let $Z, Y \subset X$ be two submanifolds and assume that Z that intersects Y. We consider $\beta \colon \widetilde{X} \longrightarrow Y$ be the blow-up of X along Y. Then the closure of $\beta^{-1}(Z \setminus Y)$ is called **strict transformation of Z under β**.

Proposition 1.7.1. [43, Exercise II.8.5] *Let* X *be a non singular variety and* Y *a subvariety of codimension* $r \geq 2$. *Let* $\beta \colon \widetilde{X} \longrightarrow X$ *be the blowing up of* X *along* Y *and let* $E = \beta^{-1}(Y)$. *Then*

$$K_{\widetilde{X}} = \beta^* K_X + (r-1)E.$$

The following result, tell us how cohomology groups change under blowing up.

Proposition 1.7.2. [89, Theorem 7.31] *Let X be a non singular variety and Y a subvariety of codimension r. We denote $\beta\colon \widetilde{X} \longrightarrow X$ be the blowing up of X along Y, $E = \beta^{-1}(Y)$ and $j : E \hookrightarrow \widetilde{Y}$. Let $h = c_1(\mathcal{O}_E(1)) \in H^2(E, \mathbb{Z})$, $\forall k = 0, \dots \dim(Y)$ we have an isomorphism of Hodge structures:*

$$H^k(X, \mathbb{Z}) \oplus \bigoplus_{i=0}^{r-2} H^{k-2i-2}(Y, \mathbb{Z}) \overset{\beta^* + \sum_i \mathrm{j}_* \circ \mathrm{h}^i \circ \beta_{\mathrm{E}}^*}{\longrightarrow} H^k(\widetilde{X}, \mathbb{Z}) \tag{1.7.1}$$

where h^i is the morphism of Hodge structures given by the cup-product by $h^i \in H^{2i}(E, \mathbb{Z})$.

We also note that on the summands $H^{k-2i-2}(Y, \mathbb{Z})$ the Hodge structure of Y is shifted by $(i+1, i+1)$ in bidegree, so $\forall k = p+q$:

$$H^{p,q}(X) \oplus \bigoplus_{i=0}^{r-2} H^{p-i-1, q-i-1}(Y) \simeq H^{p,q}(\widetilde{X}). \tag{1.7.2}$$

1.8 | A crash into representation theory

In this section we recall briefly some results of the representation theory of finite groups which will be useful trough out the thesis. For a complete discussion we refer to [34].

1.8.1 | Representation of finite groups

In the following, G is a finite group and V is a n-dimensional complex vector space.

Definition 1.8.1. A group homomorphism $\rho\colon G \longrightarrow \mathrm{GL}(V) \simeq \mathrm{GL}(n, \mathbb{C})$ is called **representation of G over V**. We denote it by (ρ, V). The dimension of V is called **the degree** of ρ.

The **trivial representation** of G is $1_G\colon G \longrightarrow \mathbb{C}^*$ such that $g \mapsto 1$ for every $g \in G$.

Definition 1.8.2. A **subrepresentation** of (ρ, V) is a vector subspace $W \subseteq V$ which is G-invariant, *i.e.* $\forall w \in W, g \in G$ $\rho(g)w \in W$.

A trivial subrepresentation of ρ is the vector space $V^G \subseteq V$, namely **G-fixed point set**, is defined by:

$$V^G = \{v \in V \mid \rho(g)v = v \ \forall g \in G\}.$$

Definition 1.8.3. A representation (ρ, V) is called **irreducible** if there is no proper non-zero invariant subspace $W \subseteq V$. Otherwise it's called **reducible**.

Remark 1.8.4. We remark that every one-dimensional representation of G is irreducible.

Definition 1.8.5. A **morphism of G-representations** (ρ_V, V) and (ρ_W, W), also called **G-equivariant map**, is a linear map $\varphi\colon V \longrightarrow W$ that is compatible with the group action, *i.e.* for every $g \in G$ the following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle \rho_V(g)}\downarrow & & \downarrow{\scriptstyle \rho_W(g)} \\ V & \xrightarrow[\varphi]{} & W. \end{array}$$

Given two G-representations (ρ, V) and (ρ', W), we can define the following G-representations:

1. **Direct sum** $(\rho \oplus \rho', V \oplus W)$ such that $\rho \oplus \rho'(g)(v \oplus w) = \rho(g)(v) \oplus \rho'(g)(w)$ for every $g \in G$, $v \in V$, $w \in W$.

2. **Tensor product** $(\rho_1 \otimes \rho_2, V \otimes W)$ such that $\rho \oplus \rho'(g)(v \otimes w) = \rho(g)(v) \otimes \rho'(g)(w)$ for every $g \in G$, $v \in V$, $w \in W$

3. **n-th tensor product** $(\rho^{\otimes n}, V^{\otimes n})$

4. **Wedge product** $(\wedge^n \rho, \bigwedge^n V)$

Definition 1.8.6. A representation which can be expressed as direct sum of other representations is called **decomposable**, otherwise **indecomposable**.

There is a connection between irreducible and indecomposable representations: if ρ is irreducible then it's indecomposable, but the converse may fail. Under the hypothesis that G is finite and V is a vector space over a field k with $\mathrm{Char}(k) = 0$, the following results guarantees the converse.

Theorem 1.8.7 (of Maschke)**.** [34, Proposition 1.5] *Let W be a subrepresentation of (ρ, V) of the finite group G. Then there exists a subrepresentation W' of V, which is called* complementary invariant*, such that $V = W \oplus W'$.*

Corollary 1.8.8. [34, Corollary 1.6] *Any representation of a finite group is the direct sum of irreducible representations.*

This property is called **complete reducibility**. The uniqueness of the decomposition is guaranteed by the following result.

Lemma 1.8.9 (Schur's Lemma)**.** [34, Lemma 1.7] *If (ρ_V, V) and (ρ_W, W) are irreducible representations of G and $\varphi\colon V \to W$ is a G-equivariant map.*

(i) *Either φ is an isomorphism or $\varphi = 0$.*

(ii) *If $V = W$ then $\varphi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}^*$ and I the identity.*

We can summarize the results above as follow.

Proposition 1.8.10. [34, Proposition 1.8] *Every representation (ρ, V) of a finite group G can be decomposed as:*

$$V \simeq V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$$

where V_i are irreducible representations such that $V_i \not\simeq V_j$ for $i \neq j$ and a_i's are the multiplicities of the factors V_i. The decomposition is unique up to isomorphisms.

Lemma 1.8.11. *Let G be a finite abelian group. Then every irreducible representation is 1-dimensional.*

Proof. Let (ρ, V) be an irreducible G-representation. For every $g \in G$, $\rho(g)\colon V \longrightarrow V$ is a G-equivariant map, since G is abelian. According to Lemma 1.8.9, $\rho(g) = \lambda_g I$ for some $\lambda_g \in \mathbb{C}^*$. Thus every subspace of V is G-invariant and since V is irreducible we must have that V is one dimensional. □

We also recall the following well-known isomorphisms of vector space:

$$\bigwedge^k V \oplus W = \bigoplus_{i+j=k} \bigwedge^i V \otimes \bigwedge^j W. \tag{1.8.1}$$

1.8.2 | Character theory

The character theory is an effective tool to study the representations of a finite group.

Definition 1.8.12. Let (ρ, V) be a representation of G. We define the **character of ρ**, denoted by χ_ρ, as the complex value function:

$$\chi_\rho : \quad G \longrightarrow \mathbb{C}$$
$$g \longmapsto \mathrm{Tr}(\rho(g))$$

which associates to g the trace of $\rho(g)$ on V. We define the **degree of χ_ρ** as the degree of ρ.

In particular we observe that χ_ρ is constant on the conjugacy classes of G, namely the **class functions** of G, and that $\chi_\rho(1_G) = \dim(V)$.

Lemma 1.8.13. [34, Proposition 2.1] *Let (ρ, V) and (ρ', W) be representations of G, the followings formulas hold:*

1. $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$,
2. $\chi_{\rho\otimes\rho'} = \chi_\rho\chi_{\rho'}$,
3. $\chi_{\bigwedge^n\rho}(g) = \frac{1}{n}\sum_{m=1}^{j} \chi_\rho(g^m)\chi_{\bigwedge^{j-m}}(g)$.

The set of the class functions of G, denoted by $\mathbb{C}_{classes}(G)$, is a $\mathbb{C}$-vector space over whose dimension is the number of the conjugacy classes of G. Let $\mathrm{Irr}(G)$ be the set of irreducible characters of G, *i.e.* characters associated to an irreducible representation; they are class functions.

Lemma 1.8.14. [34, Propsition 2.30] *The set of* $\mathrm{Irr}(G)$ *defines a basis of* $\mathbb{C}_{classes}(G)$.

For every $\alpha, \beta \in \mathbb{C}_{classes}(G)$, we define the following hermitian inner product:

$$\langle \alpha, \beta \rangle = \frac{1}{|G|}\sum_{g\in G} \overline{\alpha(g)}\beta(g) \tag{1.8.2}$$

where $\overline{\alpha(g)}$ is the conjugation in $\mathbb{C}$. The irreducible characters are orthonormal with respect to (1.8.2), see [34, Thoerem 2.12].

Lemma 1.8.15. *Let G be a finite group and C_i for $i = 1, \dots, k$ be the conjugacy classes of G and $h_i = |C_i|$, $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}$. Let us choose $g_i \in C_i$ for every i, the following relations hold:*

$$\sum_{i=1}^{k} h_i\chi_m(g_i)\overline{\chi_n(g_i)} = \delta_{mn}|G|,$$
$$\sum_{i=1}^{r} \chi_i(g_m)\overline{\chi_i(g_n)} = \delta_{mn}|C_m|.$$

It follows that any representation is determined by its characters, see [34, Corollary 2.14]. Indeed if we decompose $V \simeq V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$ with V_i as in Proposition 1.8.10, then $\chi_\rho = \sum_i a_i\chi_{\rho_i}$ where χ_{ρ_i} are linearly independent. Moreover, by [34, Corollary 2.16], the datum a_i can be computed by the following equality:

$$a_i = \langle \chi_V, \chi_{V_i} \rangle, \tag{1.8.3}$$

The inner product (1.8.2) is useful to compute the dimension of the space of invariants under the action of a group. More in details, given a finite group G and a complex representation (ρ, V), then the dimension of V^G coincides with the multiplicity of the trivial representation, namely (ρ_1, V_1), in the decomposition of V given by Proposition 1.8.10, *i.e.*

$$\dim(V^G) = \langle \chi_\rho, \chi_{\rho_1} \rangle = a_1.$$

Example 2. Let $\mathfrak{D}_m = \langle a, b \mid a^m = b^s = (ab)^2 = id \rangle$ for $m \equiv_2 0$. We also recall that $(ab^k)^2 = id$ for every $k = 0, \ldots, m-1$.

- There are $j = 1, \ldots, 4$ irreducible 1-dimensional representations given by ρ_j described by the following characters:

for $k = 0, \ldots, m-1$	a^k	ba^k
χ_1	1	1
χ_2	1	1
χ_3	$(-1)^k$	$(-1)^k$
χ_4	$(-1)^k$	$(-1)^{k+1}$

Table 1.1: 1-dimensional irreducible characters of $\mathfrak{D}_m$

- There are $0 < h < \frac{m}{2} - 1$ irreducible 2-dimensional representations given by $\widetilde{\rho_h}$ described by the following characters:

for $k = 0, \ldots, m-1$	a^k	ba^k
$\chi_{\widetilde{\rho_h}}$	$2\cos(\frac{2\pi hk}{m})$	0

Table 1.2: 2-dimensional irreducible characters of $\mathfrak{D}_m$

2 Preliminaries on abelian varieties

In this chapter we recall the main proprieties of abelian varieties that will be useful throughout this thesis. Good references are [56] and [70].

2.1 | Homomorphisms of complex tori

Let $\pi\colon V \longrightarrow T = V/\Lambda$ be a complex torus with $V \simeq \mathbb{C}^n$ and $\Lambda \simeq \mathbb{Z}^{2n}$ be a lattice. The addition on V induces a structure of a group on T as follows. Given $t_i = \pi(v_i) \in T$ for $i = 1, 2$, we let

$$t_1 + t_2 := \pi(v_1 + v_2). \tag{2.1.1}$$

Let $T' = V'/\Lambda'$ be another complex torus.

Definition 2.1.1. A **homomorphism of complex tori** $f\colon T \longrightarrow T'$ is a holomorphic map which is compatible with the group structure. A **translation by** $x_0 \in T$ is the holomorphic map defined by $t_{x_0}\colon T \longrightarrow T$ such that $t_{x_0}(t) = t + x_0$ for every $t \in T$.

According to [56, Proposition 1.2.1] every holomorphic map $f\colon T \longrightarrow T'$ between two complex tori is a composition of a homomorphism and a translation, *i.e.* it is an affine transformation. Moreover, there exist an unique $\mathbb{C}$-linear map $F\colon V \longrightarrow V'$ such that $F(\Lambda) \subseteq \Lambda'$ and $y \in V'$ such that for every $z \in T$

$$f(\underline{z}) = F(z) + t_y \tag{2.1.2}$$

where t_y denoted the translation by y. We call F the **linear part of** f and t_y **the translation part of** f.

We denote by **Hom(T, T')** the group of homomorphisms from T to T'. We get the following injective homomorphism of abelian groups:

$$\begin{aligned} \rho_a \colon \operatorname{Hom}(T, T') &\hookrightarrow \operatorname{Hom}_{\mathbb{C}}(V, V') \\ f &\longmapsto F \end{aligned} \tag{2.1.3}$$

which is called the **analytic representation**. The restriction F_Λ of F to Λ is $\mathbb{Z}$-linear and determines completely F and f. We get an injective homomorphism:

$$\begin{aligned} \rho_r \colon \operatorname{Hom}(T, T') &\hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \Lambda') \\ f &\longmapsto F_\Lambda \end{aligned} \tag{2.1.4}$$

which is called the **rational representation**.

Remark 2.1.2. There is a one-to-one correspondence

$$\{f \in \operatorname{Hom}(T, T')\} \longleftrightarrow \{\alpha \in \operatorname{Hom}_{\mathbb{C}}(V, V') \mid \alpha(\Lambda) \subset \Lambda'\}.$$

Let $T = V/\Lambda$ be a complex torus. The Hodge decomposition on $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = V \oplus \overline{V}$ implies that ρ_a and ρ_r are related via the equation $\rho_r \otimes \mathbb{C} = \rho_a \oplus \overline{\rho_a}$, see [56, Proposition 1.2.3].

Definition 2.1.3. An **isogeny** $\varphi \colon T \to T'$ of complex tori is a surjective homomorphism with finite kernel. If such φ exists then T and T' are said to be **isogenous**.

The group $\operatorname{Hom}(T, T')$ is a subgroup of $\operatorname{Hom}_{\mathbb{Q}}(T, T') = \operatorname{Hom}(T, T') \otimes \mathbb{Q}$, in particular it defines a lattice in $\operatorname{Hom}_{\mathbb{Q}}(T, T')$. We observe that $f \in \operatorname{Hom}(T, T')$ is an isogeny if and only if f is surjective and $\dim(T) = \dim(T')$. We also recall that an isogeny $id \neq \varphi \in \operatorname{Hom}(T, T')$ is invertible only in $\operatorname{Hom}_{\mathbb{Q}}(T, T')$, see [56, Proposition 1.2.6].

Definition 2.1.4. An **endomorphism** of a complex torus T is a homomorphism of T. An **automorphism** of a complex torus T is a biholomorphic map of T.

We denote by **Aut(T)** the group of automorphisms of T. Let us denote by **End(T)** the ring of endomorphisms of T and by **End$_{\mathbb{Q}}$(T)**$= \operatorname{End}(T) \otimes \mathbb{Q}$ its extension on $\mathbb{Q}$. By [56, Proposition 1.2.2]: $\operatorname{End}(T)$ is a free abelian group of finite rank and so $\operatorname{End}_{\mathbb{Q}}(T)$ is a finite dimension $\mathbb{Q}$-algebra.

Lemma 2.1.5. *The $\mathbb{Q}$-algebra $\operatorname{End}_{\mathbb{Q}}(T)$ depends only on the isogeny class of T.*

Proof. Let $\varphi\colon T \longrightarrow T'$ be an isogeny. According to [56, Propositoin 1.2.6] there exists an unique isogeny, up to isomorphisms, $\psi\colon T' \longrightarrow T$ such that

$$\deg(\varphi) = \deg(\psi) = n \quad \varphi \circ \psi = [n]_{T'} \quad \psi \circ \varphi = [n]_T$$

where $[n]_T$ and $[n]_{T'}$ are the multiplication maps respectively on T and T', *i.e.* $[n]_T(t) = nt$ for every $t \in T$ and similar for $[n]_{T'}$. We define the following $\mathbb{Q}$-algebras homomorphism

$$\mathrm{End}_{\mathbb{Q}}(T) \longrightarrow \mathrm{End}_{\mathbb{Q}}(T')$$

$$f \longmapsto \tfrac{1}{n}\varphi \circ f \circ \psi$$

which is in fact injective. It has an inverse given by $g \mapsto \frac{1}{n}\psi \circ g \circ \varphi$ for every $g \in \mathrm{End}_{\mathbb{Q}}(T')$. Thus we have an isomorphism of $\mathbb{Q}$-algebras. □

2.2 | Cohomology of complex tori

Let $T = \mathbb{C}^n/\Lambda$ be a complex torus with universal covering $\pi\colon \mathbb{C}^n \longrightarrow T$ whose kernel is Λ. Via the monodrony action, we can identify the fundamental group $\pi_1(T)$ with Λ. In particular since Λ is abelian, by the Hurewicz theorem [46, Theorem 2A.1], $\pi_1(T)$ is canonically isomorphic to $H_1(T, \mathbb{Z})$. We recall that T, as real manifold, is the product of $2n$ copies of the circle S^1. By the Künneth formula we deduce that the groups $H_j(T, \mathbb{Z})$ and $H^j(T, \mathbb{Z})$ are free abelian groups of finite rank for $j = 1, \ldots, 2n$. The main result about cohomology of complex tori is that the higher cohomology groups can be computed out of $H^1(T, \mathbb{Z})$. More precisely:

Lemma 2.2.1. [56, Lemma 1.3.1] *Let T be a complex torus. The cup product induces the following isomorphism of abelian groups $\bigwedge^n H^1(T, \mathbb{Z}) \xrightarrow{\simeq} H^n(T, \mathbb{Z})$ for every $n \geq 1$.*

Let $z = (z_1, \ldots, z_n)$ be the local coordinates on T, $|I| = i$ and $|J| = j$ be two multi-index with $i, j = 0, \ldots n$ such that $i + j \leq n$. We recall the following description:

$$H^{i,j}(T) = \langle dz_I \wedge d\overline{z_J} \rangle_{I,J}. \tag{2.2.1}$$

In particular, we deduce:

$$h^{i,j}(T) = \binom{n}{i}\binom{n}{j} \tag{2.2.2}$$

and

$$e(T) = 0 \qquad \chi(T) = 0. \tag{2.2.3}$$

2.3 | The dual complex torus

Let $T = \mathbb{C}^n/\Lambda$ be a complex torus. We denote by $\overline{\Omega} := \mathrm{Hom}_{\overline{\mathbb{C}}}(\mathbb{C}^n, \mathbb{C})$ the complex vector space of $\mathbb{C}$-antilinear forms $l\colon \mathbb{C}^n \longrightarrow \mathbb{C}$. The underlying real vector space of $\overline{\Omega}$ is isomorphic to $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{R})$ via the following isomorphism

$$\begin{aligned} \overline{\Omega} &\xrightarrow{\quad\simeq\quad} \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{R}) \\ l &\longmapsto \mathrm{Im}(l) \\ -k(iv) + ik(v) &\longleftarrow\!\shortmid\; k. \end{aligned}$$

Therefore, we obtain a $\mathbb{R}$-bilinear form:

$$\begin{aligned} \langle\ \ ,\ \ \rangle\colon \overline{\Omega} \times \mathbb{C}^n &\longrightarrow \mathbb{R} \\ (l, v) &\longmapsto \langle l, v\rangle = \mathrm{Im}(l)(v) \end{aligned}$$

Definition 2.3.1. We define the **dual lattice** $\hat{\Lambda} = \{l \in \overline{\Omega}\colon \langle l, \Lambda\rangle \subseteq \mathbb{Z}\}$. The quotient $\hat{T} = \dfrac{\overline{\Omega}}{\hat{\Lambda}}$ is a complex torus, called **dual complex torus of T**.

Lemma 2.3.2. [56, Proposition 2.4.1] *Let T be a complex torus. Then $\hat{T}$ is isomorphic to $\mathrm{Pic}^0(T)$.*

Given an homomorphism $\varphi\colon X \to X$ with analytic representation $\widetilde{\varphi} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, the (anti)-dual map $\widetilde{\varphi}^\vee : \overline{\Omega} \longrightarrow \overline{\Omega}$ induces a homomorphism $\hat{\varphi} : \hat{X} \longrightarrow \hat{X}$ called the **dual map** of φ, see [56, Section 2.4].

Let $D \in \mathrm{Pic}(T)$, for any point $x \in T$ the line bundle $t_x^*D \otimes D^{-1}$ has zero first Chern class, where t_x is the translation by x. According to the Theorem of the Square [56, Theorem 2.3.3], we get a group homomorphism for any $D \in \mathrm{Pic}(T)$:

$$\begin{aligned} \phi_D\colon T &\longrightarrow \hat{T} \simeq \mathrm{Pic}^0(T) \\ x &\longmapsto t_x^*D \otimes D^{-1} \end{aligned} \tag{2.3.1}$$

The map ϕ_D depends on the first class $c_1(D)$, see [56, Corollary 2.4.6 (a)].

Lemma 2.3.3. [56, Corollary 2.4.6. (d)] *Given $f \in \mathrm{Hom}(T, T')$ and $D \in \mathrm{Pic}(T')$ then*

$$\phi_{f^*D} = \hat{f} \circ \phi_D \circ f. \tag{2.3.2}$$

2.4 | Endomorphism algebra of abelian variety

Definition 2.4.1. An **abelian variety** is a projective complex torus.

Remark 2.4.2. The reader need to pay attention that while in the previous chapter we use the term “variety” to emphasize the singular nature of our object of study, here the same term emphasizes the projectivity of a manifolds.

Definition 2.4.3. An abelian variety is called **simple abelian variety** if it does not admit any nontrivial abelian subvariety.

Let X be an abelian variety. By using the existence of an ample line bundle on X, we obtain a decomposition of X, up to isogeny, into the product of abelian subvarieties. More precisely:

Theorem 2.4.4. [56, Poincarè Completely Reducibility Theorem, Theorem 5.3.7.] *Let X be an abelian variety. Then X is isogenous to a product $X_1^{n_1} \times \ldots \times X_k^{n_k}$ where the X_i's are simple abelian varieties, not isogeneous for $i \neq j$. The isogeny type of the X_i and the natural numbers n_i are uniquely determined by X.*

An important consequence of the Poincarè Completely Reducibility Theorem is that $\mathrm{End}_{\mathbb{Q}}(X)$ is a semisimple algebra.

Definition 2.4.5. A **division algebra** (also called *division ring* or *skew field*) is a ring in which every nonzero element has a multiplicative inverse.

Definition 2.4.6. An algebra $\mathcal{A}$ is called **simple algebra** if $\mathcal{A} \neq 0$ and it has no proper two-sided ideals.
A finite-dimensional algebra $\mathcal{A}$ is said to be **semisimple algebra** if it can be expressed as a Cartesian product of simple sub-algebras.

We denote by $\mathrm{Mat}_r(D)$ the space of square matrices $r \times r$ over the ring D.

Lemma 2.4.7. [39, IX. Proposition 1.4] *Let D be a ring. Every two-sided ideal of $\mathrm{Mat}_r(D)$ is of the form $\mathrm{Mat}_r(I)$ for an unique two-sided ideal I of D.*

Corollary 2.4.8. *If D is a division algebra, then $\mathrm{Mat}_r(D)$ is simple.*

Proof. Assume the contrary. Then by Lemma 2.4.7 any ideal of $\mathrm{Mat}_r(D)$ is of the form $\mathrm{Mat}_r(I)$ for an unique ideal I of D. But division ring has no proper ideal. Thus $\mathrm{Mat}_r(D)$ is simple. □

Corollary 2.4.9. [56, Corollary 5.3.8] *Let X be an abelian variety and $X \to X_1^{n_1} \times \ldots \times X_k^{n_k}$ be the isogeny defined by the Poincarè Complete Reducibility Theorem. Then there is an isomorphism of $\mathbb{Q}$-algebras:*

$$\mathrm{End}_{\mathbb{Q}}(X) \xrightarrow{\simeq} \mathrm{Mat}_{n_1}(D_1) \oplus \ldots \oplus \mathrm{Mat}_{n_r}(D_r)$$

where $D_j = \mathrm{End}_{\mathbb{Q}}(X_j)$ are division algebras of finite dimension over $\mathbb{Q}$. In particular, $\mathrm{End}_{\mathbb{Q}}(X)$ is a semisimple algebra.

2.5 | The Rosati involution

Let X be an abelian variety. We have seen that $\mathrm{End}_{\mathbb{Q}}(X)$ is a finite dimension semisimple $\mathbb{Q}$-algebra. We see that the presence of an ample line bundle on X gives an extra structure on $\mathrm{End}_{\mathbb{Q}}(X)$.

Let L be an ample line bundle on X, according to [56, Proposition 2.4.8] $\phi_L \colon X \longrightarrow \hat{X}$ is an isogeny and so the inverse $\phi_L^{-1} \colon \hat{X} \longrightarrow X$ is well-defined in $\mathrm{Hom}_{\mathbb{Q}}(\hat{X}, X)$. This allows to define an algebra homomorphism on $\mathrm{End}_{\mathbb{Q}}(X)$ as follows:

$$\begin{aligned} {}' \colon \mathrm{End}_{\mathbb{Q}}(X) &\longrightarrow \mathrm{End}_{\mathbb{Q}}(X) \\ \varphi &\longmapsto \varphi' := \phi_L^{-1} \circ \hat{\varphi} \circ \phi_L \end{aligned} \tag{2.5.1}$$

which is in fact an involution since $\hat{\hat{\varphi}} = \varphi$ by [56, Section 2.4 pag 35]. The involution above is called **Rosati involution** with respect L.

Definition 2.5.1. Let $\mathcal{A}_k$ be a k-algebra for a subfield $k \subset \mathbb{R}$ and τ an involution on it. We say that τ is **positive-definite with respect to the reduced trace over k** if the following holds:

$$\forall \varphi \in \mathcal{A}_k \text{ then } \mathrm{Tr}_{\mathbb{Q}}(\varphi \circ \varphi') > 0, \tag{2.5.2}$$

The Rosati involution is positive-definite with respect to the reduced trace $\mathrm{Tr}_{\mathbb{Q}}$ over $\mathbb{Q}$, see [56, Theorem 5.1.8].

Remark 2.5.2. We recall the following relations, see [56, Sectioin 5.1]

$$(f+g)' = f' + g' \qquad (fg)' = g' f'. \tag{2.5.3}$$

This extra structure on $\mathrm{End}_{\mathbb{Q}}(X)$ allows us to give a very nice description of this endomorphism algebra. We recall the following result, due to A. A. Albert, which provides a classification of $\mathbb{Q}$-division algebra with a positive-definite involution.

Theorem 2.5.3. [70, IV.21 Theorem 2, page 201] *Let $\mathcal{D}$ be a division algebra with finite rank over $\mathbb{Q}$ together with an involution τ such that $\mathrm{Tr}_{\mathbb{Q}}(x, \tau(x)) > 0$ for all $x \in \mathcal{D}$ and $x \neq 0$. Then $(\mathcal{D} \otimes \mathbb{R}, \tau)$ is isomorphic, as a $\mathbb{R}$-algebra with an involution, to one of the following:*

- $\mathbb{R} \times \ldots \times \mathbb{R}$ *and the involution is the identity.*
- $\mathbb{H} \times \ldots \times \mathbb{H}$, *where $\mathbb{H}$ is the algebra of Hamiltonian quaternions and the involution is $\tau(x) = \bar{x}$, the conjugate on each component.*
- $\mathrm{Mat}_2(\mathbb{R}) \times \ldots \times \mathrm{Mat}_2(\mathbb{R})$ *and the involution is $\tau(x) = x^t$, the transpose matrix on each component.*
- $\mathrm{Mat}_2(\mathbb{C}) \times \ldots \times \mathrm{Mat}_2(\mathbb{C})$ *and the involution is $\tau(x) = x^\dagger$, the conjugate transpose matrix on each component.*

According to Theorem 2.4.4, $\mathrm{End}_{\mathbb{Q}}(X)$ decomposes as direct sum of spaces of matrices over a division algebra with a positive definite involution. Combining this with the classification of division algebra given by Theorem 2.5.3, we obtain the following result.

Theorem 2.5.4. [77, Corollary 3.5] *Let X be an abelian variety. Then, we have the following isomorphism of $\mathbb{R}$-algebra:*

$$\psi\colon (\mathrm{End}_{\mathbb{R}}(X), ') \xrightarrow{\simeq} (\prod_i \mathrm{Mat}_{r_i}(\mathbb{R}) \times \prod_j \mathrm{Mat}_{s_j}(\mathbb{C}) \times \prod_k \mathrm{Mat}_{t_k}(\mathbb{H}), \dagger) \tag{2.5.4}$$

where the Rosati involution $'$ is sent to the positive-definite involution $\dagger$ given by the conjugate transpose on each factor.

2.6 | Polarizations on abelian varieties

The Rosati involution allows us to give an alternative description of the real Néron-Severi vector space $N^1(X)_{\mathbb{R}}$, see Definition 1.2.4, in term of $\mathbb{R}$-endomorphisms of X.

Let us consider the $\mathbb{Q}$-vector space $N^1(X)_{\mathbb{Q}} = N^1(X) \otimes \mathbb{Q}$. We define the following homomorphism of abelian groups:

$$\begin{aligned} f_L\colon N^1(X)_{\mathbb{Q}} &\longrightarrow \mathrm{End}_{\mathbb{Q}}(X) \\ D &\longmapsto \phi_L^{-1} \circ \phi_D \end{aligned} \tag{2.6.1}$$

where ϕ_L and ϕ_D are defined as in (2.3.1).

Definition 2.6.1. An endomorphism $\varphi \in \mathrm{End}(X)$ is called **symmetric endomorphism** if $\varphi' = \varphi$ where $'$ is the Rosati involution. We denote by $\mathbf{End}^s(X) \subset \mathrm{End}(X)$ the subvector space that consists of symmetric endomorphisms

Theorem 2.6.2. [56, Proposition 5.2.1] and [56, Remark 5.2.5] *Let X be an abelian variety and L be the ample line bundle which defines the Rosati involution $'$, see (2.5.1). Then $f_L \colon N^1(X)_\mathbb{Q} \longrightarrow \mathrm{End}_\mathbb{Q}(X)$, defined in (2.6.1), is an embedding. In particular, f_L defines the following isomorphism of $\mathbb{Q}$-vector space*

$$N^1(X)_\mathbb{Q} \simeq \{\varphi \in N^1(X)_\mathbb{Q} \colon \varphi = \varphi'\} = \mathrm{End}^s_\mathbb{Q}(X).$$

According to Theorem 2.5.4:

$$\psi \colon (\mathrm{End}_\mathbb{R}(X),') \xrightarrow{\ \simeq\ } (\prod_i \mathrm{Mat}_{r_i}(\mathbb{R}) \times \prod_j \mathrm{Mat}_{s_j}(\mathbb{C}) \times \prod_k \mathrm{Mat}_{t_k}(\mathbb{H}), \dagger).$$

Combing it with Theorem 2.6.2 we obtain the following result.

Proposition 2.6.3. [70, Theorem 6, pag 208] [77, Theorem 4.3] *Let X be an abelian variety. Then we have the following isomorphism of vector space:*

$$(\psi \circ f_L) \colon N^1(X)_\mathbb{R} \xrightarrow{\ \simeq\ } \bigoplus_i \mathcal{H}_{r_i}(\mathbb{R}) \oplus \bigoplus_j \mathcal{H}_{s_j}(\mathbb{C}) \oplus \bigoplus_k \mathcal{H}_{t_k}(\mathbb{H}) \subset \psi(\mathrm{End}_\mathbb{R}(X))$$

where $\mathcal{H}_n(\mathbb{F})$ is the space of hermitian matrices of dimension $n \times n$ over the field $\mathbb{F}$.

In [70, Application III pag 209], Mumford showed that $(\psi \circ f_L)$ establishes a correspondence between ample $\mathbb{R}$-divisors and positive definite matrices in $\psi(\mathrm{End}_\mathbb{R}(X))$. Here we present an alternative proof, using different techniques.

Proposition 2.6.4. [56, Theorem 2.1.2., Proposition 2.1.6 and Lemma 2.1.7.] *Let X be an abelian variety. There is a $1:1$ correspondence between $D \in \mathrm{Pic}(X)$ and hermitian form $H := c_1(D)$ on $\mathbb{C}^n$ with $\mathbb{Z}$-values on Λ.*

Lemma 2.6.5. [56, Proposition 4.5.2] *Let X be an abelian variety and $D \in \mathrm{Pic}(X)$. Then D is ample if and only if $H := c_1(D)$ is a positive-definite hermitian form.*

Lemma 2.6.6. *Let X be an abelian variety. Then the isomorphism $(\psi \circ f_L)$ establishes a bijection between ample $\mathbb{R}$-divisors and positive-definite matrices.*

Proof. For every $D \in \mathrm{N}^1(X)_\mathbb{R}$ we have

$$D \xmapsto{\ f_L\ } \phi_L^{-1}\phi_D =: \varphi \xmapsto{\ \psi\ } M \text{ hermitian matrix in } \psi(\mathrm{End}_\mathbb{R}(X)).$$

- Assume M to be positive-definite,*i.e.* there exist an invertible matrix C in $\psi(\mathrm{End}_{\mathbb{R}}(X))$ such that $M = C^{\dagger}C$. Let us denote $\gamma = \psi^{-1}(C)$. Since ψ is an isomorphism of $\mathbb{R}$-algebras and sends $'$ to $\dagger$ we have

$$\varphi = \gamma'\gamma. \tag{2.6.2}$$

To prove that D is ample, we prove that its hermitian form $c_1(D) = H$ is positive-definite, according to Lemma 2.6.5. Let $H_0 = c_1(L)$ be the positive-definite hermitian matrix associated to L, according to Lemma 2.6.5. Since $\phi_D = \phi_L\varphi$ we get $H(\cdot,\cdot) = H_0(\rho_a(\varphi)\cdot,\cdot)$. By using (2.6.2) and [56, Proposition 5.1.1] we get that for every $0 \neq v \in \mathbb{C}^n$:

$$H(v,v) = H_0(\rho_a(\varphi)\cdot,\cdot) = H_0(\rho_a(\gamma)v, \rho_a(\gamma)v) \geq 0.$$

Thus, H is positive definite and so D is ample.

- Assume D be ample, according to [56, Proposition 2.4.8.] ϕ_D is invertible in $\mathrm{End}_{\mathbb{R}}(X)$ and so M is invertible. Since M is hermitian, we prove that it is positive-definite by proving that all its eigenvalues are positive (since M is invertible it has no zero eigenvalues). Assume the contrary and let λ a negative eigenvalues of M. The matrix $-\lambda I$ is positive-definite, thus by above there exists an ample $\mathbb{R}$-divisor D_λ such that $(\psi \circ f)(D_\lambda) = -\lambda I$. We see that the matrix $M - \lambda I$ has zero as eigenvalue and so it is not invertible. On the contrary, we have:

$$(\psi \circ f)(D_\lambda + D) = M - \lambda I$$

thus $M - \lambda I$ is the image under $(\psi \circ f)$ of an ample $\mathbb{R}$-divisor $D_\lambda + D$ and so it must be invertible, as we have observe above. Therefore M has only positive eigenvalue and so it is positive definite.

□

2.7 | Elliptic curves

In this section we collect some proprieties of elliptic curves which will be useful throughout the thesis.

2.7.1 | Elliptic curves with complex multiplication

It is well-known that one dimension complex tori are projective hence they coincide with one dimensional abelian variety. In fact, they are commonly called elliptic curves. Let E be an elliptic curve. In general the endomorphism ring of E is isomorphic to $\mathbb{Z}$.

Definition 2.7.1. We define the **hyperelliptic involution** to be the involution ι_E on E such that $E/\iota_E \simeq \mathbb{P}^1$.

Definition 2.7.2. An elliptic curve E has **complex multiplication** if $\mathrm{End}(E) \supsetneq \mathbb{Z}$.

The advantages to study elliptic curves with larger endomorphism ring is that we can find endomorphisms of order grater than 2. If $\mathrm{End}(E)$ is different from $\mathbb{Z}$, then $\mathrm{End}(E)$ is an imaginary quadratic field. More precisely we have a finite list given by the following result.

Proposition 2.7.3. [56, Corollary 13.3.4] *Let E be an elliptic curve together with an automorphism f of order $d > 2$. Then (E, f) is one of the following:*

d	E	f
3	E_{ζ_3}	ζ_3
4	E_{ζ_4}	ζ_4
6	E_{ζ_3}	ζ_6

Table 2.1: Elliptic curves with complex multiplication

where $E_\nu = \mathbb{C}/(\mathbb{Z} \oplus \nu\mathbb{Z})$ and $\zeta_n = e^{\frac{2\pi i}{n}}$. In these case the $\mathrm{End}_{\mathbb{Q}}(E_\nu) \simeq \mathbb{Q}(\nu)$.

2.7.2 | Moduli space of elliptic curves

In this section we recall the description of the moduli space of the elliptic curves. For a more general description of moduli space of abelian varieties we refer to [56, Section 8].

Definition 2.7.4. We denote the upper half plane by $\mathfrak{h} = \{v \in \mathbb{C} \mid \mathrm{Im}(v) > 0\}$

Let us consider a lattice $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} \subset \mathbb{C}$, by multiply by ω_1^{-1} we obtain a lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$ with $\tau = \omega_2(\omega_1)^{-1}$: we can assume $\tau \in \mathfrak{h}$, otherwise we can simply multiply Λ by ω_2^{-1} and consider τ^{-1}. Thus we obtain a map:

$$\{\text{elliptic curves}\} \longrightarrow \mathfrak{h}$$

$$E = \mathbb{C}/(\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}) \longmapsto \omega_2(\omega_1)^{-1}$$

$$E = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}) \longleftarrow\!\shortmid \tau$$

and we see that $\mathfrak{h}$ gives a space of parameters of elliptic curves. Since we want to describe elliptic curves up to biholomorphisms, we need to consider an action of a specific group on $\mathfrak{h}$.

Definition 2.7.5. We define the **modular action** of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathfrak{h}$:

$$\mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{GL}(\mathfrak{h})$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(\tau \mapsto A \cdot \tau = \frac{a\tau + b}{c\tau + d} = \tau'\right)$$

The following result holds.

Proposition 2.7.6. [41, Proposition 1.17] *The set of biholomorphisms classes of elliptic curves is isomorphic to the quotient* $\mathcal{M}_{1,1} = \mathfrak{h}/\mathrm{SL}(2,\mathbb{Z})$.

Remark 2.7.7. We recall that sometimes authors define $\mathcal{M}_{1,1} = \mathfrak{h}/\mathrm{PSL}(2,\mathbb{Z})$ where $\mathrm{PSL}(2,\mathbb{Z})$ is the **modular group** $\mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$.

Proposition 2.7.8. [81, Chapter VII section 1.2] *The group* $\mathrm{SL}(2,\mathbb{Z})$ *is generated by* $\mathfrak{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $\mathfrak{S} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$.
The region $D = \{z \mid |z| \geq 1, |Re(z)| \leq \frac{1}{2}\}$ *is a fundamental domain for the* $\mathrm{SL}(2,\mathbb{Z})$*-action on* $\mathfrak{h}$, i.e. *all points in* $\mathfrak{h}$ *lie in the orbit of a point of* D *and the orbits of the interior of* D *are disjoint.*

Part I

Calabi-Yau manifolds of type A

L'importante non è dove tu stai,
ma dove ti stanno portando le tue scelte.

3 Summary of Part I

In the last years free quotients of complex tori are not again complex tori, called generalized hyperelliptic manifolds, have gained significant attention as they are the natural generalizations of the bi-elliptic surfaces. Among them, there exist quotients that are Calabi-Yau manifolds: in this case the cover is a projective complex torus and in fact these quotients are called Calabi-Yau manifolds of type A, see Lemma 5.6.6.

In this part of the thesis we consider the Calabi-Yau manifolds of type A investigate the following problems:

1. Do Calabi-Yau manifolds exist in all admissible dimensional cases?

2. What we can say about the geometry of Calabi-Yau manifolds of type A?

CHAPTER 4. We present the hyperelliptic manifolds and we recall the main properties about their automorphisms group, see Theorem 4.1.12, and their deformations see Section 4.2.

CHAPTER 5. We introduce the Calabi-Yau manifolds and those of type A. We recall that Calabi-Yau manifolds of type A exist only in odd dimension $n > 1$ and in [73] the authors have fully classified them in dimension 3. In this Chapter we consider the first problem and we obtain the following result.

Theorem A (see Theorem 5.6.6)**.** Calabi-Yau $(2n+1)$-folds of type A exist for every $n \in \mathbb{N}_{\geq}1$. In particular,

(i) For every n, there exists a Calabi-Yau manifold $Y = A/G$ with $G \simeq (\mathbb{Z}/2\mathbb{Z})^{2n}$ and $A = E_1 \times \ldots \times E_{2n+1}$ is the product of $2n+1$ (non necessarily isomorphic each other) elliptic curves.

(ii) For $n = 1$, there exists a Calabi-Yau threefolds $Y = A/G$ with $G \simeq \mathfrak{D}_4$ the dihedral group of order 8 and $A = E \times E \times E'$ with E, E' elliptic curves.

Furthermore, for every odd n there exists a free quotient $Y = A/G$ with $G \simeq \mathfrak{D}_{4n}$ with $\mathcal{K}_Y \simeq \mathcal{O}_Y$ and $h^{1,0}(Y) = h^{2,0}(Y) = 0$.

We observe that item (i) is a new action; while the dihedral actions where first introduced in [73] and [1].

The second problem is considered in the last chapters. More precisely, we consider the Calabi-Yau threefolds of type A classify in [73]. We recall that there are only two groups to construct these threefolds: the abelian group $(\mathbb{Z}/2\mathbb{Z})^2$ and the dihedral group $\mathfrak{D}_4$ of order 8. Our main results concern the classification of the automorphisms group and quotients of these threefolds. These results and study are also enclosed in the article [66].

Chapter 6. We study of the family $\mathcal{F}^A_{\mathfrak{D}_4}$ of Calabi-Yau threefolds of type A constructed with the group $\mathfrak{D}_4$, given in [25]. In this case each $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ is a free quotient of the abelian variety $A' = E \times E \times E'$ by a group of order 16 which contains a normal subgroup isomorphic to $\mathfrak{D}_4$. The main results are the followings.

Theorem B (see Theorem 6.4.1)**.** part (ii): Let $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ and we assume that $\mathrm{End}_{\mathbb{Q}}(E') \not\simeq \mathbb{Q}(\zeta_6)$. Then the automorphism group of X is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. Specifically, the automorphisms on X are induced by order two translations by the points $(t_1, t_2, t_3) \in A'$ satisfying certain conditions, see (6.4.2).

Theorem C (see Theorem 6.5.1 and Theorem 7.4.1)**.** part (ii): Let $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ and $\Upsilon \in \mathrm{Aut}(X)$. Let $\beta\colon Y \to X/\Upsilon$ be the blow up of the singular locus of X/Υ. Then for each Υ, β is a crepant resolution and Y a Calabi-Yau 3-fold. Moreover, there exist exactly 2 automorphisms $(\alpha_1)_X$ and $(\alpha_2)_X$ acting freely on X. In particular, $\dfrac{X}{(\alpha_j)_X}$'s belong to $\mathcal{F}^A_{\mathfrak{D}_4}$.

Chapter 7. In this chapter we consider the Calabi-Yau threefolds of type A constructed with the group $(\mathbb{Z}/2\mathbb{Z})^2$ and we apply similar studies to the one undertaken for $\mathcal{F}^A_{\mathfrak{D}_4}$. In Theorem 7.1.2 we construct the family $\mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$ of these threefolds. In particular, each $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$ is a free quotient of the abelian threefolds $A = E_1 \times E_2 \times E_3$ by the free action of $(\mathbb{Z}/2\mathbb{Z})^2$. The main results are the followings.

Theorem B (see Theorem 7.3.1)**.** part (i): Let $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$ and we assume E_i's are not isogenous each other. Then the automorphism group of $X = A/G$ is isomorphic to

$(\mathbb{Z}/2\mathbb{Z})^7$. Specifically, the automorphism on X are induced by those on A whose linear part belong to $\langle \mathrm{diag}(-1,1,1)\rangle$ and the translation part given by any point of order 2.

Theorem C (see Theorem 7.4.1)**.** part (i): Let $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$ and $\Upsilon \in \mathrm{Aut}(X)$. Let $\beta\colon Y \to X/\Upsilon$ be the blow up of the singular locus of X/Υ. The followings hold.

1. If Υ preserves the volume form of X, β is a crepant resolution and Y is a Calabi-Yau 3-fold. In particular, there are exactly $3^3 - 1$ automorphisms $(\alpha_j)_X$ which act freely on X and $X/(\alpha_j)_X$ belong to $\mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$.
2. If Υ does not preserve the volume form of X, we have the following cases.
    a. If there exists at least one $\alpha_X \in \Upsilon$ that fixes surfaces on X then Y has negative Kodaira dimension.
    b. Otherwise, Y has trivial Kodaira dimension and $K_Y \neq 0$.

4 Preliminaries on Generalized Hyperelliptic Manifolds

In this chapter we give a brief introduction on hyperelliptic manifolds recalling some of their properties focusing mainly on their automorphism group and their deformations, which will be useful throughout the thesis. Good references are [55],[25],[28].

4.1 Generalized hyperelliptic manifolds

Definition 4.1.1. A manifold not isomorphic to a complex torus but admitting a complex torus as finite étale Galois cover is called **(generalized) hyperelliptic manifolds (GHM for short)**. We called it **(generalized) hyperelliptic variety (GHV for short)** if moreover the complex torus has an ample line bundle.

Remark 4.1.2. Let Y be a GHM: since the covering is Galois there exists a complex torus T' and a finite group $G' \leq \mathrm{Aut}(T')$ acting freely such that $Y = T/G$. In particular, since Y must not be a complex torus, the group G' does not contain only translations. In fact, we assume in Definition 4.1.1 that the Galois group G' of the covering does not contain any translations. Let $\{id\} \neq G_0$ be the subgroup of translations contained in G'. Since G_0 is normal in G' then $Y = \dfrac{T}{G}$ where $T := T'/G_0$ is a complex torus and $G := G'/G_0$ is a finite group acting freely on T without containing any translation. Thus, Y is also the quotient of a complex torus T by a free action of a finite group G which does not contain any translation.

According to the classification of bi-elliptic surfaces, see [11, List VI.20, pag 84], in dimension 2 hyperelliptic manifolds are always projective and the group G is always cyclic. In higher dimension, already in dimension 3, different cases appear. For instance we can find non abelian groups, see [25].

Definition 4.1.3. Let V be a finite dimensional real vector space. A **crystallographic group** Γ is a subgroup of the isometry group of V such that it is discrete and Γ is co-compact, *i.e.* V/Γ is compact. A torsion-free crystallographic group is called **Bieberbach group.**

Remark 4.1.4. The crystallographic group Γ is torsion-free if and only if it acts freely on V, [23, Proposition 9].

Crystallographic groups are characterized by the following theorems of Bieberbach.

Theorem 4.1.5 (Bieberbach's theorems)**.** [84, Theorem 2.1]

1. *Let $\Gamma \subset \mathrm{Iso}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ be a crystallographic group. Then the group of translation $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank n in Γ, and is a maximal abelian and normal subgroup of finite index.*

2. *For each positive integer n there are only a finite number of isomorphism classes of crystallographic groups on $\mathbb{R}^n$.*

3. *Two crystallographic groups on $\mathbb{R}^n$ are isomorphic if and only if they are conjugate by an element of the affine group of $\mathbb{R}^n$.*

There is also an analogous theorem in the complex case.

Theorem 4.1.6 (Complex Bieberbach Theorems)**.** [42, Theorem 4.4]

1. *M is a flat Kähler manifold of complex dimension n if and only if there exists complex n-dimensional torus T and a finite group $G \subset \mathrm{Aut}(T)$ acting freely such that $M = T/G$.*

2. *Two flat Kähler manifolds $M = T/G$ and $M' = T'/G'$ are biholomorphic if and only if there exists a biholomorphic map $\varphi\colon T \longrightarrow T$ such that $G = \varphi^{-1}G'\varphi$*

3. *For every complex torus T there exist only finite number of flat Kähler manifolds of the form T/G, up to biholomorphism.*

Remark 4.1.7. Let $\Gamma \leq \mathrm{Iso}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ be a crystallographic group. By the first theorem of Bieberbach, Γ is characterized by the following exact sequence:

$$0 \longrightarrow \Lambda' \xrightarrow{\ i\ } \Gamma \xrightarrow{\ l\ } G' \longrightarrow 0 \tag{4.1.1}$$

where Λ' is the group of translations, $G' = \Gamma/\Lambda'$ is a finite group and $i(\lambda) = (I, \lambda)$ and $l(M, m) = M$ for every $\lambda \in \Lambda'$ and $(M, m) \in \Gamma$. Moreover Λ' has a G'-action given by the so-called **holonomy representation**:

$$L\colon G' \longrightarrow \mathrm{GL}(\Lambda')$$

$$g \in \longmapsto L(g)\colon \lambda \mapsto i^{-1}(\gamma i(l)\gamma^{-1})$$

where $l(\gamma) = g$.

Lemma 4.1.8. *Let $Y = T/G$ be a hyperelliptic manifold with $T = \mathbb{C}^n/\Lambda$. Then there exists $\Gamma \leq \mathrm{Iso}(\mathbb{C}^n)$ such that $Y = \mathbb{C}^n/\Gamma$ and $\pi_1(Y) \simeq \Gamma$. Moreover, Γ is a crystallographic group in $\mathrm{Iso}(\mathbb{C}^n)$ and fits in the following exact sequence:*

$$0 \longrightarrow \Lambda \xrightarrow{i} \Gamma \xrightarrow{l} G \longrightarrow 0. \tag{4.1.2}$$

Proof. By standard results of covering space $Y = \mathbb{C}^n/\pi_1(Y)$. By Remark 1.5.8 and Remark 4.1.7, it follows that $\pi_1(Y) \simeq \Gamma$ is a crystallographic group characterized by the exact sequence as in the statement. □

Remark 4.1.9. According to the Bieberbach's theorems: the hyperelliptic manifold T/G is topologically determined by the group G which acts freely on T and does not contain translations, that is, G characterizes the fundamental group of T/G. Hence we may say that T/G is a **hyperelliptic manifold with the group G**.

Proposition 4.1.10. *Let $Y = T/G$ be a hyperelliptic manifold with the group G. Then*

(i) *The Euler characteristic of Y is $e(Y) = 0$.*

(ii) *The manifold Y does not contain rational submanifolds.*

Proof. (i) The covering $\pi : T \to Y$ is étale hence $e(T) = \deg(\pi)e(Y)$, by Lemma 1.1.16. From the fact that $e(T) = 0$, see (2.2.3), we have the result.

(ii) Let us assume that there exists $\mathbb{P}^k \subset Y$ for some $k \in \mathbb{N}$. Since π is étale we have that $(\pi)^{-1}(\mathbb{P}^k)$ consists of $|G|$ copies of $\mathbb{P}^k$ in T, which is impossible since any complex torus does not contain rational submanifolds. Indeed if it would exist $j\colon \mathbb{P}^k \hookrightarrow T$ non-constant morphism then every k-form $w_T \in H^0(T, \Omega^k_T)$ would give rise a non-zero k-form on $\mathbb{P}^k$ which is a contradiction. This proves the statement.

□

Definition 4.1.11. Let $\pi\colon X \longrightarrow Y$ be an étale morphism of complex manifolds and $\alpha_X \in \mathrm{Aut}(X)$. Assume that there exist $\alpha_Y \in \mathrm{Aut}(Y)$ making the following diagram commutative $\begin{array}{ccc} X & \xrightarrow{\alpha_X} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[\alpha_Y]{} & Y \end{array}$. We call α_Y **automorphism induced by α_X on the quotient Y**.

In Section 1.5, see Corollary 1.5.9, we have recalled the characterization of the automorphism group of Kähler manifolds Y with $c_1(Y) = 0$ is related to the automorphism group of its finite étale cover. As special case we obtain the following.

Corollary 4.1.12. *Let $Y = T/G$ be a hyperelliptic manifold with the group G. Then the following homomorphism is surjective:*

$$\vartheta: \quad \mathrm{N}_{\mathrm{Aut}(T)}(G) \longrightarrow\!\!\!\!\rightarrow \mathrm{Aut}(Y)$$

$$\alpha_T \longmapsto \alpha_Y$$

where α_Y is the automorphism induced by α_T on the quotient Y and $\ker\vartheta = G$. In particular, we have $\mathrm{Aut}(Y) \simeq \dfrac{\mathrm{N}_{Aut(T)}(G)}{G}$.

4.2 | Deformations of GHM

We review the theory of deformations of hyperelliptic manifolds following the article [23].

Definition 4.2.1. Let Λ be a free abelian group of even rank and $G \longrightarrow \mathrm{GL}(\Lambda)$ be a faithful representation of a finite group G. A ***G*-Hodge decomposition** is a decomposition into G-invariant linear subspaces:

$$\Lambda \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1} \quad \overline{H^{1,0}} = H^{0,1}.$$

We can split $\Lambda \otimes \mathbb{C} = \bigoplus_{\chi \in \mathrm{Irr}(G)} U_\chi$ into isotypic components $U_\chi = W_\chi \otimes M_\chi$. Here, W_χ is the $\mathbb{Z}$-module corresponding to the irreducible representation χ and $M_\chi \simeq \mathbb{C}^{m_\chi}$ encodes how many times the representation with character χ appears in the decomposition.
Thus, we obtain:

$$V := H^{1,0} = \bigoplus_{\chi \in \mathrm{Irr}(G)} V_\chi$$

with $V_\chi = W_\chi \otimes M_\chi^{1,0}$.

Definition 4.2.2. The **Hodge type of a *G*-Hodge decomposition** is the collection of the dimensions $\nu(\chi) = \dim_{\mathbb{C}} M_\chi^{1,0}$. Here, χ runs over all non-real characters.

Remark 4.2.3.

1. For χ non real it holds $\nu(\chi) + \nu(\overline{\chi}) = \dim_{\mathbb{C}}(M_\chi)$.
2. All G-Hodge decompositions of a fixed Hodge type are parametrized as follows: for a real irreducible character χ, one chooses a $\frac{1}{2}\dim_{\mathbb{C}}(M_\chi)$-dimensional subspace of M_χ, and for a non-real irreducible character, one can choose a $\nu(\chi)$-dimensional subspace of M_χ.

3. The Hodge type is a invariant by deformations, see [21, Theorem 81].

Remark 4.2.4. Therefore, to give a classification of the complex tori $T = \mathbb{C}^n/\Lambda$ with a free action of G (which does not contain any translation), one needs to determine all possible complex structures on T such that the action of G is holomorphic. This corresponds to determine all possible G-Hodge decompositions on Λ.

Let Y be a compact complex manifold. We denote by $\mathrm{CS}(Y)$ the space of complex structure on Y and by $\mathrm{Diff}(Y)^+$ the group of diffeomorphisms preserving orientation on Y. The group $\mathrm{Diff}(Y)^+$ acts on $\mathrm{CS}(Y)$ We consider the subgroup $\mathrm{Diff}^0(Y) \subset \mathrm{Diff}(Y)^+$ given by the connected component of the identity.

Definition 4.2.5. We define the **Teichmüller space of** Y to be the quotient

$$\mathcal{T}(Y) = \mathrm{CS}(Y)/\mathrm{Diff}^0(Y)$$

The following result of Catanese and Corvaja describes the Teichmüller space of a hyperelliptic manifold $Y = T/G$ pointing out that it is related to the G-invariant Teichmüller space of T. More precisely:

Theorem 4.2.6. [23, Theorem 1] *Let $Y = T/G$ be a hyperelliptic manifold. The subspace of the Teichmüller space $\mathcal{T}(Y)$ corresponding to Kähler manifolds consists of a finite number of connected components, indexed by the Hodge type of the Hodge decomposition of G. In particular it is homeomorphic to $\mathcal{T}(T)^G$ the locus of fixed points for the G-action.*

5 Calabi-Yau manifolds and Calabi-Yau manifolds of type A

In this chapter we introduce the Calabi-Yau manifolds and we collect some results which will be useful through out the thesis, specifically those on actions of involutions on them. Furthermore, we introduce the Calabi-Yau manifolds of type A, *i.e.* given as free quotients of abelian varieties, and we prove Theorem A which guarantees their existence in all dimension.

5.1 | Calabi-Yau manifolds

Let Y be a Calabi-Yau n-fold. By definition (see Definition 5.3.3) $h^{j,0}(Y) = 0$ for $1 \leq j \leq n-1$, hence:

$$\chi(Y) = \begin{cases} 0 & n \text{ is odd} \\ 2 & n \text{ is even.} \end{cases} \tag{5.1.1}$$

Lemma 5.1.1. *Let Y be a Calabi-Yau n-fold. Then it is projective whenever $n \neq 2$.*

Proof. The n-fold Y is a Kähler manifold with $H^2(Y, \mathcal{O}_Y) = 0$ if $n \neq 2$. According to [48, Corollary 4.16] we have the result. □

It is worth to present some explicit constructions of Calabi-Yau manifolds. In the first construction we consider complete intersections in projective space and we use the adjuction formula, see Proposition 1.4.3.

Example 3 (Hypersurface in projective space)**.** Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of degree $n+1$. By the adjuction formula, see Proposition1.4.3, we find:

$$\mathcal{K}_Y = \mathcal{O}_Y(-n-1+\deg(Y)) = \mathcal{O}_Y.$$

Thus Y has trivial canonical bundle. By using the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and its long sequence in cohomology one can find $h^{j,0}(Y) = 0$ for every $1 \leq j \leq n-2$. Thus Y is a Calabi-Yau $(n-1)$-fold.

Another construction involves actions of finite groups on K-trivial manifold.

Example 4 (Kummer surface)**.** Let T be a 2-dimensional complex torus and ι be the involution sending $P \in T$ to its opposite $-P$ with respect to the group operation on T, (2.1.1). In local coordinates $\iota(x, y) = (-x, -y)$ for $P = (x, y) \in T$. We see that ι fixes 16 points on T which are exactly the 2-torsion points on T. Since ι preserves the volume form $\omega_T = dx \wedge dy$ on T, we obtain a volume form on the smooth part of T/ι. In particular T/ι has $16A_1$ singularities. According to Theorem 1.6.10, T/ι admits a crepant resolution. Here we recall the explicit construction. Let us consider the following diagram:

$$\begin{array}{ccc} \tilde{\iota} \circlearrowright \widetilde{T} & \xrightarrow{\quad\beta\quad} & T \\ \Big\downarrow{\scriptstyle\tilde{\pi}} & & \Big\downarrow{\scriptstyle\pi} \\ \widetilde{T}/\langle\tilde{\iota}\rangle = \widetilde{T/\langle\iota\rangle} & \xrightarrow[\gamma]{} & T/\langle\iota\rangle \end{array}$$

where γ is the blow up of the singular part of $T/\langle\iota\rangle$, β is the blow up of the fixed locus of ι and since ι preserves the blown up locus of β it lifts to an involution $\tilde{\iota}$ on $\widetilde{Y}$. One can easily check that the diagram commutes. We have: $h^{1,0}(\widetilde{T}/\langle\tilde{\iota}\rangle) = h^{1,0}(\widetilde{T})^{\tilde{\iota}} = h^{1,0}(T)^{\iota} = 0$ and $h^{2,0}(\widetilde{T}/\langle\tilde{\iota}\rangle) = h^{2,0}(T/\iota) = 1$. Moreover, according to Proposition 1.4.4 and Proposition 1.7.1:

$$\tilde{\pi}^* K_{\widetilde{T}/\langle\tilde{\iota}\rangle} + R = K_{\widetilde{T}} \qquad K_{\widetilde{T}} = \beta^* K_T + R$$

where R denotes the exceptional divisor introduced by β which, by construction, is also the ramification divisor of $\tilde{\pi}$. Since $K_T = 0$ we obtain $\pi^* K_{\widetilde{T}/\langle\tilde{\iota}\rangle} = 0$, *i.e.* $K_{\widetilde{T}/\langle\tilde{\iota}\rangle}$ is either trivial or 2-torsion. Since $h^{2,0}(\widetilde{T}/\langle\tilde{\iota}\rangle) = 1$ then $K_{\widetilde{T}/\langle\tilde{\iota}\rangle} = 0$. Thus $\widetilde{T}/\langle\tilde{\iota}\rangle = 0$ is a Calabi-Yau surface or K3 surface.

Definition 5.1.2. The Calabi-Yau surface $\widetilde{T}/\langle\tilde{\iota}\rangle$ constructed in Example 4 is called **Kummer surface** and it is denoted by $Km_2(T)$.

Remark 5.1.3. Fujiki in [33, Chapter 3] has classified all finite actions on 2-dimensional complex tori such that the quotient admits a desingularization which is a K3 surface. These surfaces are called **generalized Kummer surfaces**. Whenever these action are given by cyclic group $|G| = n \geq 2$, the resulting generalized Kummer surfaces is denoted by $Km_n(T)$.

Example 5 (Borcea-Voisin Calabi-Yau threefold)**.** Let S be a K3 surface together with an involution ι_S which does not preserve the volume form ω_S. Let E be an elliptic curve and ι be the hyperelliptic involution, see Definition 2.7.1. The automorphism $\iota_S \times \iota_E$ defines an involution on $S \times E$. Moreover, since separately ι_S and ι_E do not preserve the volume form of S and E, respectively, we obtain that $\iota_S \times \iota_E$ preserves the volume form of $S \times E$ given by $\omega_S \wedge \omega_E$. The singular quotient $(S \times E)/(\iota_S \times \iota_E)$ admits a crepant resolution, for instance by Proposition 1.6.13. Therefore we obtain a manifold Z with trivial canonical bundle and birational to $(S \times E)/(\iota_S \times \iota_E)$. Since $H^{1,0}(S \times E) = \langle \omega_E \rangle$ and $H^{2,0}(S \times E) = \langle \omega_S \rangle$ are not preserved by $\iota_S \times \iota_E$ and $h^{j,0}$ are birational invariant, we obtain $h^{1,0}(Z) = h^{2,0}(Z) = 0$. Thus Z is a Calabi-Yau threefold.

The construction above can be generalized by considering higher order of automorphisms, $n = 3, 4, 6$ and E with complex multiplication, see for instance [26]. Additionally, according to [73] and in [45] there exist Calabi-Yau threefolds admitting the product of an elliptic curve and a K3 surface as finite étale Galois cover.

5.2 | Picard group of Calabi-Yau manifolds

The Picard group plays a central role in the study of geometry of manifolds. For Calabi-Yau n-folds with $n > 2$ it has a very nice description.

We first recall an useful theorem.

Theorem 5.2.1. [46, Universal Coefficient for cohomology, Theorem 3.2] *Let Y be a topological space. Let G be a module over a principal ideal domain R, then there is an exact sequence:*

$$0 \to \mathrm{Ext}^1(H_{q-1}(Y, R), G) \to H^q(Y, G) \to \mathrm{Hom}(H_q(Y, R), G) \to 0.$$

Moreover it splits, though not naturally.

Lemma 5.2.2. *Let Y be a Calabi-Yau n-fold with $n > 2$ then:*

$$\mathrm{Pic}(Y) \simeq \mathbb{Z}^{h^{1,1}(Y)} \oplus \mathbb{A}\mathrm{b}(\pi_1(Y)) \tag{5.2.1}$$

where $\mathbb{A}\mathrm{b}(\pi_1(Y))$ denotes the abelianization of the fundamental group.

Proof. Since $h^{i,0}(Y) = 0$ for $i = 1, 2$, the first Chern map c_1 is an isomorphism

$$\mathrm{Pic}(Y) = \mathrm{NS}(Y) \simeq H^2(Y, \mathbb{Z}) \tag{5.2.2}$$

and we get $\rho(Y) = h^2(Y, \mathbb{C}) = h^{1,1}(Y)$. By Theorem 5.2.1 we obtain:

$$H^2(Y, \mathbb{Z}) \simeq \mathrm{Hom}(H_2(Y, \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}^1(H_1(Y, \mathbb{Z}), \mathbb{Z}).$$

Using the properties of the functors Hom and Ext [46, pages 195] one obtains:

$$H^2(Y, \mathbb{Z}) \simeq \mathbb{Z}^{h^2(Y,\mathbb{C})} \oplus H_1(Y, \mathbb{Z})_{tor} \tag{5.2.3}$$

where $H_1(Y, \mathbb{Z})_{tor}$ is the torsion subgroup of $H_1(Y, \mathbb{Z})$. Since $h^{1,0}(Y) = 0$ we have that $H^1(Y, \mathbb{Z}) \simeq H_1(Y, \mathbb{Z})$ is a torsion group, *i.e.* $H_1(Y, \mathbb{Z})_{tor} = H_1(Y, \mathbb{Z})$. By the Hurewicz theorem we know that $H_1(Y, \mathbb{Z}) \simeq \mathbb{A}\mathrm{b}(\pi_1(Y))$. Since $h^{2,0}(Y) = 0$ then $\rho(Y) = h^2(Y, \mathbb{C}) = h^{1,1}(Y)$. □

Remark 5.2.3. We observe that the result above holds for every compact complex Kähler manifold Y with $h^{1,0}(Y) = h^{2,0}(Y) = 0$.

5.3 | Deformations of Calabi-Yau manifolds

The key result about the space of deformation of Calabi-Yau manifolds is the Bogomolov-Tian-Todorov (unobstructed) theorem.

We have recalled the notion of Theichmüller space of a complex manifolds, see Definition 4.2.5. By studying local deformations of complex structures, in a neighbourhood of the Theichmüller space, we can deduce informations about the moduli space of a complex manifold.

Definition 5.3.1. Let Y be a complex manifold with a fixed complex structure. A **deformation** of Y consists of a smooth proper morphism $f : \mathcal{Y} \longrightarrow S$ where $\mathcal{Y}$ and S are connected spaces and $Y \simeq \mathcal{Y}_0 = f^{-1}(0)$ where 0 is a distinguished point in S.

We denote by $\mathrm{Def}(Y)$ the space of all the deformations of Y.

Theorem 5.3.2. [40, Theorem 14.10] *Let Y be a compact, Kähler, complex manifold with $H^0(Y, \mathcal{T}_Y) = 0$ and $\mathcal{K}_Y \simeq \mathcal{O}_Y$, where $\mathcal{T}_Y$ is the tangent bundle. Then $Def(Y)$ is a germ of a smooth manifold with tangent space $H^1(Y, \mathcal{T}_Y)$.*

Let Y be a manifold with $\mathcal{K}_Y \simeq \mathcal{O}_Y$ and we denote ω_Y its volume form. We have a perfect pairing induced by the wedge product:

$$\Omega^1_Y \times \Omega^{n-1}_Y \longrightarrow \Omega^n_Y \simeq \mathcal{O}_Y$$

$$(\omega_1, \omega_{n-1}) \longmapsto \omega_1 \wedge \omega_{n-1} \simeq \omega_Y.$$

In others words, by the existence of ω_Y we have a sheaves isomorphism:

$$\omega_Y : (\Omega^1_Y)^\vee \simeq \mathcal{T}_Y \xrightarrow{\sim} \Omega^{n-1}_Y$$

$$v \longmapsto \omega_Y(v, -).$$

In particular, we obtain

$$H^1(Y, \mathcal{T}_Y) \simeq H^1(Y, \Omega^{n-1}_Y) = H^{n-1,1}(Y)$$
$$H^0(Y, \mathcal{T}_Y) \simeq H^{n-1,0}(Y)$$

If Y is a Calabi-Yau n-fold with $n > 1$ then $h^0(Y, \Omega^{n-1}_Y) = h^{n-1,0}(Y) = 0$ which combining with Theorem 5.3.2 leads to the following result.

Corollary 5.3.3. *Let Y be a Calabi-Yau n-fold with $n > 1$. Then $H^{n-1,1}(Y)$ parametrizes its local deformations.*

5.4 | Automorphisms and quotients of Calabi-Yau manifolds

In this section we collect results about the automorphisms group of Calabi-Yau manifolds and their quotients.

5.4.1 | The automorphisms group

For a Calabi-Yau curve, *i.e.* elliptic curve, the automorphism group is well-known and classified. It is infinite and given by translations and complex multiplication, see Section 2.1. In higher dimension, already in dimension 2, the structure of the automorphism group of Calabi-Yau manifolds can be more complicated and its study remains an active area of research. We first present some results about the finiteness of the automorphism groups of Calabi-Yau manifolds.

Theorem 5.4.1. [72, Theorem 1.2] *Let Y be an $(2n+1)$-dimensional Calabi-Yau manifold with $\rho(Y) = 2$. Then $\mathrm{Aut}(Y)$ is finite.*

Theorem 5.4.2. [58, Theorem 1.1] *Let Y be a Calabi-Yau threefold with $\rho(Y) = 3$. Then the automorphism group* $\mathrm{Aut}(Y)$ *is either finite, or it is an almost abelian group of rank* 1, i.e. *it is isomorphic to $\mathbb{Z}$ up to finite kernel and cokernel.*

Remark 5.4.3. It is still an open problem whether or not a Calabi–Yau threefold with Picard rank equal to 3 admits infinite automorphism group. While there are example of Calabi–Yau threefolds with Picard rank equal to 3 with finite automorphism group, see [73, Theorem 0.1 IV], [45, Section 5]. We highlight that these latter examples are all given by Calabi-Yau threefolds with infinite fundamental group.

It is worth also recalling what happens for Calabi-Yau manifolds Y with $\rho(Y) \geq 4$. Borcea in [15] gave an example of Calabi-Yau threefold with Picard rank equal to 4 and infinite automorphism group. In fact it is expected a similar phenomena for any Calabi-Yau threefold Y with $\rho(Y) \geq 4$, see [37], [74].

5.4.2 | The quotients

In this section we consider actions of finite groups on Calabi-Yau manifolds, with the aim to describe their properties. We distinguish two cases depending if the resulting quotient is smooth or singular. In the second cases we mainly consider involutions.

Free actions on Calabi-Yau manifolds can be easily classified in the following way.

Proposition 5.4.4. *Let Y be a Calabi-Yau n-fold and $G \leq \mathrm{Aut}(Y)$ be a finite group acting freely on it.*

(i) *G preserves ω_Y if and only Y/G is a Calabi-Yau n-fold if and only if n is odd.*

(ii) *G does not preserves ω_Y if and only if n is even if and only if $G \simeq \mathbb{Z}/2\mathbb{Z}$. In particular, Y/G is a manifold with 2-torsion canonical bundle.*

Proof. Let us denote $\pi\colon Y \longrightarrow Z = Y/G$ the finite étale morphism, according to Proposition 1.1.16 we have:

$$\chi(Y) = |G|\,\chi(Z). \tag{5.4.1}$$

We first observe that for every $1 \leq j \leq n-1$ we have $H^{j,0}(Z) \simeq H^{j,0}(Y)^G = 0$, since Y is a Calabi-Yau manifold.

(i) The group G preserves ω_Y if and only if $\pi_*\omega_Y$ defines a volume form on Z, *i.e.* $H^{n,0}(Z) = \langle \pi_*\omega_Y \rangle$. The last condition is equivalent to say that Z is a Calabi-Yau manifold which is also equivalent to say that $\chi(Y) = \chi(Z)$. Since $|G| > 1$ and the equations (5.1.1) and (5.4.1) hold, then we have $\chi(Y) = \chi(Z)$ if and only if they are both equal to zero, and so n is odd.

(ii) The group G does not preserves ω_Y if and only if n is even, according to item (i). By (5.1.1), n is even if and only if $\chi(Y) = 2$ which by (5.4.1) is equivalent to $G \simeq \mathbb{Z}/2\mathbb{Z}$. Since G does not preserve ω_Y then Z has 2-torsion canonical bundle.

□

If the action of G on Y is not free the situation can be more complicated, due to the different nature of the fixed locus. A nice situation appear when G fixes only hypersurfaces.

Proposition 5.4.5. *Let Y be a Calabi-Yau manifold and G be a finite group. Assume that all the irreducible components of the fixed locus of G have codimension 1. Then G does not preserve ω_Y and Y/G is smooth with negative Kodaira dimension.*

Proof. By [19, Lemma 1] each $g \in G$ can be diagonalized in a neighbourhood of a fixed point and since it fixes only divisors there exists only one eigenvalues different from 1. Therefore $G \not\subset \mathrm{SL}(n, \mathbb{C})$ and so it does not preserve ω_Y. By proposition 1.6.3, Y/G is smooth. Let us consider the map $\pi\colon Y \longrightarrow Z = Y/G$. By Proposition 1.4.4 we have $\pi^* K_Z + R = K_Y = 0$ where R is the ramification divisor defined by the fixed locus of G on Y. Hence we obtain that $\pi^* K_Z = -R$ is not effective, therefore $k(Z) = -\infty$. □

If the fixed locus of G contains components of codimension grater than 1 then Y/G is singular. As we have briefly recalled in the Chapter 1 Section 1.6, we can consider a resolution of singularities whose properties heavily depends on the singularities. Here we present some results about involutions on Calabi-Yau manifolds distinguishing the case when they preserve or not the volume form of Y.

Proposition 5.4.6. *Let Y be a Calabi-Yau n-fold and $\alpha_Y \in \mathrm{Aut}(Y)$ be an involution that preserves the volume form of Y. Then*

(i) *Either $\mathrm{Fix}(\alpha_Y)$ is empty or is the finite disjoint union of even-codimensional submanifolds.*

(ii) *The fixed locus has codimension 2 if and only if $Y/\langle\alpha_Y\rangle$ admits a crepant resolution given by the blow up of Y/α_Y in its singular locus. In this case, $Y/\langle\alpha_Y\rangle$ is birational to a Calabi-Yau manifold.*

Proof. If $\mathrm{Fix}(G) \neq \emptyset$, according to [19, Lemma 1], near to a point $y \in Y$ then α_Y can be diagonalized into a matrix δ_y: since $\delta_y \in \mathrm{SL}(n, \mathbb{C})$ then the eigenvalue -1 has even multiplicity denoted by $2k_y$ with $k_y \in \mathbb{N}$. Therefore the fixed locus of α_Y is a finite

disjoint union of even-codimensional submanifolds.
Let y be a singular point. The age of α_Y at y is equal to $\frac{2k_y}{2}$, see Definition 1.6.16. According to Theorem 1.6.17: $Y/\langle\alpha_Y\rangle$ is a crepant resolution if and only if $Y/\langle\alpha_Y\rangle$ has canonical singularities if and only if $k_y = 1$ for every $y \in \mathrm{Sing}(\alpha_Y)$ if and only if the codimension of $\mathrm{Fix}(\alpha_Y)$ is equal to 2. Finally, since $h^{j,0}$ are birational invariants we get: $h^{j,0}(Z) = h^{j,0}(Y/\langle\alpha_Y\rangle) = h^{j,0}(Y)^{\alpha_Y}$. Since $H^{j,0}(Y) = 0$ for $1 \le j \le n-1$, then $h^{j,0}(Z) = 0$ for $1 \le j \le n-1$. Therefore Z is a Calabi-Yau manifold.

□

Let us consider finite groups acting on a Calabi-Yau n-fold Y such that the action does not preserve the volume form of Y.

Proposition 5.4.7. *Let Y be Calabi-Yau n-fold and $\alpha_Y \in \mathrm{Aut}(Y)$ be an involution which does not preserve ω_Y.*

(i) *If n is even then either α_Y acts freely or $\mathrm{Fix}(\alpha_Y)$ is the disjoint union of odd-codimensional submanifolds,*

(ii) *If n is odd then $\mathrm{Fix}(\alpha_Y) \neq \emptyset$ and it is the disjoint union of odd-codimensional submanifolds*

(iii) *If the fixed locus of α_Y does not contains codimension 1 submanifolds, then it admits a desingularization Z such that $k(Z) = 0$ and $h^{j,0}(Z) = 0$ for $j > 0$.*

(iv) *Otherwise $Y/\langle\alpha_Y\rangle$ has negative Kodaira dimension and $h^{j,0}(Z) = 0$ for $j > 0$.*

Proof. According to Proposition 5.4.4, α_Y can acts freely only if n is even. Assume $\mathrm{Fix}(\alpha_Y) \neq \emptyset$. Let $y \in \mathrm{Fix}(\alpha_Y)$: according to [19, Lemma 1] there exists a neighbourhood of y where α_Y can be diagonalized in δ_y. Since δ_y is an involution and $\delta_y \notin \mathrm{SL}(n, \mathbb{C})$ then the eigenvalue -1 has odd multiplicity. Therefore, the fixed locus of α_Y consists of a finite number of smooth manifolds of odd-codimension. This prove (i) and (ii).
Let F_{k_i} the k_i-dimension subset of $\mathrm{Fix}(\alpha_Y)$ with $k_i = \begin{cases} \text{even in } \{1, \dots, n-1\} & \text{if } n \text{ is odd} \\ \text{odd in } \{1, \dots, n-1\} & \text{if } n \text{ is even} \end{cases}$.
We consider the following diagram.

$$\begin{array}{ccc} \widetilde{\alpha_Y} \circlearrowright \widetilde{Y} & \xrightarrow{\quad\gamma\quad} & Y \\ {\scriptstyle 2:1}\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle 2:1} \\ Z := \widetilde{Y}/\langle\widetilde{\alpha_Y}\rangle \simeq \widetilde{Y/\langle\alpha_Y\rangle} & \xrightarrow[\beta]{} & Y/\langle\alpha_Y\rangle \end{array}$$

where γ blows up one times the irreducible components in each F_{k_i} for every $k_i \neq n-1$, since α_Y preserves the blown up locus it lifts to an involution $\widetilde{\alpha_Y}$ on $\widetilde{Y}$ and β is the blow up of the singular locus of $Y/\langle\alpha_Y\rangle$. The diagram commutes since γ blows up one times each irreducible components in F_{k_i}. The ramification divisor of π is given by $\sum_{k_i} R_{k_i}$ where $R_{k_i} = \gamma^{-1}(F_{k_i})$ for every k_i. By Proposition 1.4.4 and Proposition 1.7.1 we have:

$$\gamma^* K_Y + \sum_{k_i \neq n-1} (n-k_i-1) R_{k_i} = K_{\widetilde{Y}} \qquad K_{\widetilde{Y}} = \pi^* K_Z + \sum_{k_i} R_{k_i}.$$

Therefore we obtain:

$$\pi^* K_Z = \sum_{k_i \neq n-1} (n-k_i-2) R_{k_i} - R_{n-1} \tag{5.4.2}$$

In case (iii): there are no codimension 1 submanifolds in $\mathrm{Fix}(\alpha_Y)$ and so in formula (5.4.2) the factor R_{n-1} does not appear. Thus, we have that $\pi^* K_Z$ is effective and K_Z too. Hence $k(Z) \geq 0$. Since the Kodaira dimension cannot increase under quotient and it is a birational invariant we get $k(Y/\langle\alpha_Y\rangle) = k(Z) \leq k(Y) = 0$ and so $k(Z) = 0$. In case (iv): $\pi^* K_Z$ is not effective and so K_Z, hence $k(Y/\langle\alpha_Y\rangle) = k(Z) = -\infty$. Since $h^{j,0}$ are birational invariants and Y is a Calabi-Yau manifold, then $h^{j,0}(Z) = 0$ for $j > 0$. □

5.5 | Calabi-Yau manifolds of quotient type

Definition 5.5.1. A **Calabi-Yau manifold of quotient type** is a Calabi-Yau manifold given as free quotient of a K-trivial manifold.

According to Definition 1.5.3, Calabi-Yau manifolds can have infinite, finite or trivial fundamental group. In this thesis we mainly focus on Calabi-Yau manifolds with infinite fundamental group. We observe that these manifolds can appear only in odd dimension. Indeed let Y be a Calabi-Yau manifolds with $\pi_1(Y)$ infinite, then $\pi_1(Y)$ is a finite extension of a rank-k lattice. Thus, Y admits a finite étale cover Y' such that Y' contains a k-dimensional complex torus as a factor. Hence $e(Y') = 0$ and $e(Y) = 0$ too. According to (5.1.1) this happens if and only if Y has odd dimension.

Definition 5.5.2. A **Calabi-Yau 3-fold of type K** is a Calabi-Yau 3-fold which admits the product of a K3 surface and an elliptic curve as étale Galois cover.

Definition 5.5.3. A **Calabi-Yau manifold of type A** is a Calabi-Yau manifold which admits an abelian variety as étale Galois cover.

Remark 5.5.4. By definition, Calabi-Yau manifolds of type A are hyperelliptic varieties, hence we can assume that the Galois group does not contain any translations, see Remark 4.1.2.

Remark 5.5.5. Similar to remark 4.1.2, we can assume that given a Calabi-Yau threefolds of type K the Galois group does not contain elements of type (id, t) where t is the translation on the elliptic curve.

Let Y be a Calabi-Yau threefolds of type K. The fundamental group of Y is a finite extension of a rank-2 lattice Λ:

$$1 \longrightarrow \Lambda \longrightarrow \pi_1(Y) \longrightarrow G \longrightarrow 1. \tag{5.5.1}$$

Definition 5.5.6. We say that Y is a **Calabi-Yau threefolds of type K with the group G** if Λ in 5.5.1 is the maximal rank-2 lattice in $\pi_1(Y)$.

5.6 | Calabi-Yau manifolds of type A

Calabi-Yau manifolds of type A are completely classified in dimension 3, see [73, Theorem 0.1]; while not much is known for higher dimensional examples. In this section we produce higher dimensional examples which guarantee the existence of these manifolds in all dimension.

We observe that Calabi-Yau manifolds of type A can be defined as Calabi-Yau GHM.

Lemma 5.6.1. *Let $Y = T/G$ be a hyperelliptic n-fold with the group G. If it is a Calabi-Yau n-fold, then n is odd, $n \neq 1$ and the complex torus T has an ample line bundle.*

Proof. Let us denote $\pi\colon T \longrightarrow Y = T/G$ the finite étale Galois covering. According to the formula $\chi(T) = \deg(\pi)\chi(Y)$, see Lemma 1.1.16, and since $\chi(T) = 0$, (2.2.3), we obtain $\chi(Y) = 0$. Thus by (5.1.1), n must be odd. Moreover, by definition (Definition 4.1.1) the hyperelliptic manifolds don't exist in dimension 1 hence $n \geq 3$. Let Y be a Calabi-Yau GHM of dimension $n \geq 3$. According to Lemma 5.1.1, Y is projective, thus there exists an ample line bundle L on Y. Since $\pi : T \longrightarrow Y = T/G$ is a finite morphism, $\pi^*(L)$ defines an ample line bundle on T, by Proposition 1.3.2 (ii). Hence, T is an abelian variety. □

We observe that the Calabi-Yau manifolds of type A cannot be constructed with a cyclic group.

Lemma 5.6.2. *Let $A = \mathbb{C}^n/\Lambda$ be an abelian variety and $g \in \operatorname{Aut}(A)$ be an automorphism of finite order that acts freely. Then $h^{1,0}(A/\langle g\rangle) \neq 0$.*

Proof. According to (2.1.2), we can decompose $g(\underline{z}) = \alpha(\underline{z}) + t_g$ where α is the linear part and t the translation part. The freeness of g on A is equivalent to say that there exist no $(z, \lambda) \in \mathbb{C}^n \times \Lambda$ solution of $(\alpha - Id)(\underline{z}) = \lambda - t$. Therefore, $\alpha - Id$ is not invertible and has 1 has eigenvalue, see also [30, Remark 2.8]. Let v be the eigenvector relative to the eigenvalue 1, then g preserves the $(1,0)$-form dv. Thus dv descend to a $(1,0)$-form on $A/\langle g\rangle$. □

5.6.1 | Examples with dihedral actions

In [1], the author has constructed free actions of the dihedral group $\mathfrak{D}_{4n}$ of order $8n$ on certain abelian varieties of dimension $(2n+1)$. Here we prove that these actions lead to Calabi-Yau manifolds if and only if $n = 1$.

Let us consider $\tau, \tau' \in \mathfrak{h}$ and the elliptic curves

$$E_j = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}) \quad j = 1, \dots, 2n \qquad E_{2n+1} = \mathbb{C}/(\mathbb{Z} \oplus \tau'\mathbb{Z}).$$

We define the abelian variety

$$A' = E_1 \times \ldots \times E_{2n+1}.$$

We set, for $\underline{z} = (z_1, z_2, \dots, z_{2n}, z_{2n+1}) \in A'$:

$$r(\underline{z}) := (-z_{2n}, z_1, z_2, \dots, z_{2n-1}, z_{2n+1} + \frac{1}{4n}) = R(z) + (0, \dots, 0, \frac{1}{4n}) \tag{5.6.1}$$

$$s(\underline{z}) := (-z_{2n} + b_1, -z_{2n-1} + b_2, \dots, -z_1 + b_{2n}, -z_{2n+1}) = S(z) + (b_1, b_2, \dots, b_{2n}, 0) \tag{5.6.2}$$

where $b_{2j-1} := \dfrac{1+\tau}{2}$ and $b_{2j} := \dfrac{\tau}{2}$ for $j = 1, \dots, n$. Since s^2 is a translation, we construct the abelian variety $A := A'/s^2$ and by [1, Theorem] the group $G = \langle r, s\rangle$ induces on A a free action of the dihedral group of order $8n$ which does not contain any translations. Thus, $Y = A/G$ is a hyperelliptic variety with the group $\mathfrak{D}_{4n}$.

Lemma 5.6.3. *With the notation above. The hyperelliptic variety $Y = A/G$ with the group $\mathfrak{D}_{4n}$ is a Calabi-Yau manifold if and only if $n = 1$. For all n odd we have $\mathcal{K}_Y \simeq \mathcal{O}_Y$ and $h^{1,0}(Y) = h^{2,0}(Y) = 0$.*

Proof. We need to prove that $\mathcal{K}_Y \simeq \mathcal{O}_Y$ and $h^{j,0}(Y) = 0$ for $0 < j < 2n+1$. The first condition is equivalent to say that the representation of G goes in $\mathrm{SL}(2n+1, \mathbb{C})$. An easy computation show that:

$$\det(R) = 1 \qquad \det(S) = (-)^{n+1}.$$

Therefore, Y has trivial canonical bundle if and only if n is odd. It remains to prove that $h^{j,0}(Y) = h^{j,0}(A)^G = 0$ for $0 < j < 2n+1$ if and only if $n = 1$. Let us denote by ρ the representation of G. We decompose ρ into irreducible representations. To do this, as explained in Section 1.8, we need compute the products $\langle \chi_\rho, \chi_i \rangle$ for $i = 1,2,3,4$ and $\langle \chi_\rho, \chi_{\widetilde{\rho_h}} \rangle$ for $i = 1,2,3,4$ and $0 < h < 2n$, where χ_ρ is the characters associated to ρ and χ_i's and $\chi_{\widetilde{\rho_h}}$'s are the irreducible characters of $\mathfrak{D}_{4n}$ of dimension 1 and 2, respectively, described in Example 2.

	$\{r^k, r^{-k}\}$ for $k = 1, \dots 2n-1$	r^{2n}	sr^k for $k = 0, \dots, 4n-1$	id
χ_ρ	1	$1-2n$	-1	$2n+1$

Table 5.1: Characters of representation ρ

We have:

1. $\langle \chi_\rho, \chi_i \rangle = \begin{cases} 0 & i \neq 2 \\ 1 & i = 2 \end{cases}$

2. We have $\langle \chi_\rho, \chi_{\widetilde{\rho_h}} \rangle = \dfrac{1}{8n} \sum_{g \in G} \chi_\rho(g) \chi_{\widetilde{\rho_h}}(g)$. By a direct computation we obtain:

$$\langle \chi_\rho, \chi_{\widetilde{\rho_h}} \rangle = \frac{1}{8n} \left[4 \sum_{k=1}^{2n-1} \cos\left(\frac{\pi h k}{2n}\right) + 2(-2n+1)\cos(h\pi) + 2(2n+1) \right].$$

Claim: $\sum_{k=1}^{2n-1} \cos\left(\dfrac{\pi h k}{2n}\right) = \begin{cases} -1 & \text{h even} \\ 0 & \text{h odd} \end{cases}.$

proof of the Claim. We denote $H = \dfrac{h\pi}{2n}$ for $0 < h < 2n$ and we write:

$$\sum_{k=1}^{2n-1} \cos(Hk) = \frac{1}{2} \sum_{k=1}^{2n-1} (e^{iHk} + e^{-iHk}).$$

We recognize two finite geometric series of ratio e^{iHk} and e^{-iHk}, respectively. By using the formula $\sum_{k=0}^{N} q^k = \dfrac{1-q^N}{1-q}$, we lead to the following expression:

$$\sum_{k=1}^{2n-1} \cos(Hk) = \frac{1}{2} \left(\frac{e^{iH}(1 - e^{i(2n-1)H})}{1 - e^{iH}} + \frac{e^{-iH}(1 - e^{-i(2n-1)H})}{1 - e^{-iH}} \right).$$

Through straightforward computations and simplifications, we lead to:

$$\sum_{k=1}^{2n-1} \cos(Hk) = \frac{\cos(H) - \cos(2nH) + \cos((2n-1)H) - 1}{2 - 2\cos(H)}.$$

We make the following observations:

- $\cos(2nH) = \cos(\pi h) = \begin{cases} 1 & \text{h even} \\ -1 & \text{h odd} \end{cases}$
- $\cos((2n-1)H) = \cos(2nH)\cos(H) + \sin(2nH)\sin(H) = \begin{cases} \cos(H) & \text{h even} \\ -\cos(H) & \text{h odd} \end{cases}$

In conclusion, we obtain: $\sum_{k=1}^{2n-1} \cos(Hk) = \begin{cases} -1 & \text{h even} \\ 0 & \text{h odd} \end{cases}$ □

Therefore, by applying the Claim, we obtain:

$$\langle \chi_\rho, \chi_{\widetilde{\rho_h}} \rangle = \frac{1}{8n}\left[4\sum_{k=1}^{2n-1} \cos(Hk) + 2(-2n+1)\cos(h\pi) + 2(2n+1)\right] = \begin{cases} 0 & \text{h even} \\ 1 & \text{h odd} \end{cases}$$

Combining items 1. and 2. we obtain the following decomposition:

$$\rho = \bigoplus_{h=1 \text{ odd}}^{2n-1} \widetilde{\rho_h} \oplus \rho_2$$

where $\widetilde{\rho_h}$ and ρ_2 are the irreducible representations associated to the characters $\chi_{\widetilde{\rho_h}}$ and χ_2. This is exactly the decomposition of G on $H^{1,0}(A)$. We immediately see that $h^{1,0}(Y) = h^{1,0}(A)^G = 0$, since ρ does not contain the trivial representation ρ_1. According to Lemma 2.2.1 we have $H^{j,0}(A) = \bigwedge^j H^{1,0}(A)$ and according to (1.8.1) we have

$$\bigwedge^k \rho = \bigoplus_{i-1+\cdots_n+j=k} \bigwedge^{i_1} \widetilde{\rho_{i_1}} \otimes \ldots \otimes \bigwedge^{i_n} \widetilde{\rho_{i_n}} \otimes \bigwedge^j \rho_2.$$

To compute $h^{j,0}(Y) = h^{j,0}(A)^G$ we need to compute the multiplicity of ρ_1 in $\bigwedge^j \rho$. The following relations hold:

$$\bigwedge^2 \widetilde{\rho_{i_j}} = \rho_2 \qquad \widetilde{\rho} \otimes \rho_2 = \widetilde{\rho_{i_j}}$$

$$\bigwedge^2 \rho_2 = 0 \qquad \langle \chi_{\widetilde{\rho_{i_j}} \otimes \widetilde{\rho_{i_l}}}, \chi_1 \rangle = 0 \text{ for } i_j \neq i_l.$$

Thus, $h^{2,0}(Y) = h^{2,0}(A)^G = [\bigwedge^2 h^{1,0}(A)]^G = 0$. Similar argument lead to the following relations:

$$\bigwedge^3 \widetilde{\rho_{i_j}} = 0 \qquad \widetilde{\rho_{i_j}} \otimes \bigwedge^2 \rho_2 = 0 \qquad \bigwedge^2 \widetilde{\rho_{i_j}} \otimes \rho_2 = \rho_1$$

$$\langle \chi_{\widetilde{\rho_{i_j}} \otimes \widetilde{\rho_{i_l}}}, \chi_{\rho_2} \rangle = 0 \text{ for } i_j \neq i_l \qquad \bigwedge^3 \rho_2 = 0$$

Therefore, $h^{3,0}(Y) = h^{3,0}(A)^G = n$. In conclusion, we deduce that Y is a Calabi-Yau manifolds if and only if $n = 1$. □

In subsection 5.6.3, we insert the python code to compute all the Hodge numbers of the hyperelliptic manifold Y with the group $\mathfrak{D}_4$.

Remark 5.6.4. We observe that often in literature the definition of Calabi-Yau manifold can be "relaxed", in the following way. A Calabi-Yau manifold can be defined as a compact, Kähler, complex manifold with trivial canonical bundle and no $(1,0)$-forms. In fact, this definition is the one used by the authors of [73]. Therefore, with this definition the actions enclosed in [1], yield to Calabi-Yau $(2n+1)$-folds of type A with the group $\mathfrak{D}_{4n}$ if and only if n is odd.

5.6.2 | Examples with abelian actions

Here we construct new examples of Calabi-Yau manifolds of type A which ensure their existence in all dimension.

Let us fix $n \in \mathbb{N}$ and denote by E_j elliptic curves (not necessarily isomorphic) for $j = 1, \ldots, 2n+1$. We define the abelian variety

$$A = E_1 \times \ldots \times E_{2n+1}.$$

For $j = 1, \ldots, 2n$, we fix $u_j \in E_j[2] \setminus \{0\}$ and $v_{j+1} \in E_{j+1}[2] \setminus \{0\}$ and for $\underline{z} \in A$ we set

$$g_j(\underline{z}) = (-z_1, -z_2, \ldots, -z_{j-1}, z_j, -z_{j+1}, \ldots, -z_{2n+1}) + (0, \ldots, u_j, v_{j+1}, 0 \ldots, 0)$$

We denote $G := \langle g_1, \ldots, g_{2n} \rangle$.

Lemma 5.6.5. *With the notation above. The group G defines a free action of $(\mathbb{Z}/2\mathbb{Z})^{2n}$ on A and $Y = A/G$ is a Calabi-Yau $(2n+1)$-fold of type A with Hodge numbers:*

$$h^{i,j}(Y) = h^{i,j}(A)^G = \begin{cases} \binom{2n+1}{i} & i = j \vee i + j = 2n+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We note that $G \simeq (\mathbb{Z}/2\mathbb{Z})^{2n}$. We first prove that G defines a free action on A. We observe that for every $g \in G$ there exist at least one entry in g which is a translation, this proves that G acts freely on A. Let us fix a multi-index $i_1 < i_2 < \cdots < i_k \in \{1, \ldots, 2n\}$, we denote

$$g_{(1,\ldots,k)} := g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_k} \text{ with } g_{i_j} \in G.$$

One can easily verify that there exists at least $i_r \in \{1, \ldots, 2n\}$ such that

$$\left(g_{(1,\ldots,k)}\right)_{|E_{i_r}}(\underline{z}) = z_{i_r} + \epsilon_{i_r}$$

where $i_r = \begin{cases} i_1 & k \text{ odd} \\ i_k + 1 & k \text{ even} \end{cases}$ and $\epsilon_r = \begin{cases} u_{i_1} & k \text{ odd} \\ v_{i_k+1} & k \text{ even} \end{cases}$.
Therefore, G defines a free action on A that does not contain any translations, *i.e.* A/G is a hyperelliptic variety with the group $(\mathbb{Z}/2\mathbb{Z})^{2n}$. Moreover, since the representation of G goes into $\mathrm{SL}(2n+1, \mathbb{C})$ then $Y := A/G$ is a manifold with trivial canonical bundle. Then, we prove that in fact Y is a Calabi-Yau manifold by looking at the G-invariant cohomology of A, since $H^{i,j}(Y) = H^{i,j}(A)^G$. We recall:

$$H^{i,j}(A) = \langle dz_I \wedge d\overline{z_J} \colon I, J \in \{1, \ldots, 2n+1\} \text{ multi-index}\} \rangle.$$

We observe:

- If $|I| + |J|$ is even. The forms which are preserved by G are of the following type $\bigwedge_{i \in I} dz_i \wedge d\overline{z_i}$. So they are $(|I|, |I|)$-forms. In particular, J is determined by the choice of I. The number of such forms is the number of the choices of the multi-index I.
- If $|I| + |J|$ is odd. The forms which are preserved by G are of the following type $dz_I \wedge d\overline{z_J}$ with $I \cap J = \emptyset$ and $I \cup J = \{1, \ldots, 2n+1\}$. In particular $2n+1 = |I| + |J|$ and J is determined by the choice of I. The number of such forms is the number of the choices of the multi-index I.

Therefore we get:

$$h^{i,j}(Y) = h^{i,j}(A)^G = \begin{cases} \binom{2n+1}{i} & i = j \vee i + j = 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we obtain $H^{i,0}(Y) = H^{i,0}(A)^G \simeq \begin{cases} \mathbb{C} & i = 0, 2n+1 \\ 0 & 1 \le i \le 2n \end{cases}$, *i.e.* Y is a Calabi-Yau manifold of type A. ☐

As a consequence of this example we obtain the following result.

Theorem 5.6.6 (Theorem A). *Calabi-Yau manifolds of type A exist in all odd dimensions. In particular*

(i) *For ever n, there exists a Calabi-Yau manifold $Y = A/G$ with $G \simeq (\mathbb{Z}/2\mathbb{Z})^{2n}$ and $A = E_1 \times \ldots \times E_{2n+1}$ is the product of $2n+1$ (non necessarily isomorphic) elliptic curves.*

(ii) *For $n = 1$, there exists a Calabi-Yau threefolds $Y = A/G$ with $G \simeq \mathfrak{D}_{4n}$ the dihedral group of order $8n$ and $A = (E)^n \times E'$ with E, E' elliptic curves.*

Remark 5.6.7. It is easy to check that if Y is as in Theorem 5.6.6 then $h^{1,1}(Y) = \rho(Y)$ equals $h^{2n,1}(Y)$ the dimension of the space of local deformation of Y.

5.6.3 | Coding for Hodge numbers of GHMs with the group $\mathfrak{D}_{4n}$

The following is a Python code to compute the Hodge number of the Hyperelliptic Varieties constructed in [1].

```
import numpy as np
n = int(input("insert dimension n: "))
R =np.zeros((2*n+1,2*n+1))
S = np.zeros((2*n+1,2*n+1))
R[0,2*n-1] = -1 R[2*n,2*n] = 1
for i in range(1,2*n):
R[i,i-1] = 1
S[2*n,2*n] = -1
for i in range(0,2*n):
S[i,2*n-i-1] = -1
G = [ ]
for j in range(0,2):
for i in range(0,4*n):
G.append(np.dot(np.linalg.matrix_power(R,i), np.linalg.matrix_power(S,j)))
def chi_V(g):
return np.trace(g)
chi_values = [chi_V(g) for g in G]
def chi_wedge(g,i):
if i == 0:
return 1
chi_value= 0
```

for m in range$(1, i + 1)$:
chi_value $+ = ((-1)**(m-1)*$ chi_V(np.linalg.matrix_power$(g, m)) *$ chi_wedge$(g, i-m))$
return chi_value/i
for i in range$(2 * n + 2)$:
for j in range$(i + 1)$:
if $i + j < 2 * n + 2$:
total_sum=sum(ch_wedge(g, i)*chi_wedge(g, j) for g in G)
print(f"h^{i,j}=", total_sum$/(8 * n)$)

6 The family of Calabi-Yau threefolds of type A with the group $\mathfrak{D}_4$

This chapter is dedicated to the study of Calabi-Yau threefolds of type A with the group $\mathfrak{D}_4$. In particular, Theorem B part (ii) (as Theorem 6.4.1) and Theorem C part (ii) (as Theorem 6.5.1) are proven, see also [66].

6.1 | The Calabi-Yau 3-folds of type A with the group $\mathfrak{D}_4$

Following [25] we construct the family of Calabi-Yau 3-folds of type A with the group $\mathfrak{D}_4$.
Let us consider the abelian 3-fold $A' := E \times E \times E'$ where

$$E := E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}) \quad E' := E_{\tau'} = \mathbb{C}/(\mathbb{Z} \oplus \tau'\mathbb{Z}) \tag{6.1.1}$$

with $\tau, \tau' \in \mathfrak{h}$. For $(u_1, u_2) \in (E \times E)[2] \setminus \{(0,0)\}$ s.t. $u_1 \neq u_2$ and $u_3 \in E'[4] \setminus \{0\}$, we define

$$r(\underline{z}) := (z_2, -z_1, z_3 + u_3) \quad s(\underline{z}) := (z_2 + u_1, z_1 + u_2, -z_3) \quad \text{for } z \in A' \tag{6.1.2}$$

We denote $H := \langle r, s \rangle$ and its representation $\rho : H \longrightarrow \mathrm{GL}_3(\mathbb{C})$:

$$\rho(r) = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix} \qquad \rho(s) = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -1 \end{pmatrix} \tag{6.1.3}$$

We write $\rho = \rho_2 \oplus \rho_1$ with $\dim(\rho_j) = j$ for $j = 1, 2$ using the decomposition into irreducible representations. This is the unique faithful representation of $\mathfrak{D}_4$ on $\mathbb{C}^3$, see [73, Theorem 0.1] and [25]. We are free to choose u_1, u_2, u_3 in several ways and in the following definition we make a choice.

Definition 6.1.1. We set $u_1 := \dfrac{\tau+1}{2}, u_2 := \dfrac{\tau}{2}, u_3 = \dfrac{1}{4}$. We denote $X := \dfrac{A'}{\langle r, s\rangle}$ and $w := s^2$.

Lemma 6.1.2. *The algebraic manifold X is a Calabi-Yau 3-fold of type A with the group $\mathfrak{D}_4$ and its Hodge numbers are $(h^{1,1}(X), h^{2,1}(X)) = (2,2)$.*

Proof. We observe that X is the three dimensional case of the example enclosed in Section 5.6.1. Thus, by Lemma 5.6.3, X is a Calabi-Yau threefold. In particular, we have $X = A/G$ with $A = A'/\langle w\rangle$ and $G = H/\langle w\rangle \simeq \mathfrak{D}_4$. By using (6.1.3) and Lemma 2.2.1:

$$H^{1,1}(A)^G = \langle dz_1 \wedge d\overline{z_1} + dz_2 \wedge d\overline{z_2}, dz_3 \wedge d\overline{z_3}\rangle$$
$$H^{2,1}(A)^G = \langle dz_1 \wedge dz_2 \wedge d\overline{z_3}, dz_1 \wedge dz_3 \wedge d\overline{z_2} - dz_2 \wedge dz_3 \wedge d\overline{z_1}\rangle.$$

Therefore, $h^{1,1}(X) = h^{2,1}(X) = 2$. □

From now on $A = A'/\langle w\rangle$ and $G = H/\langle w\rangle \simeq \mathfrak{D}_4$. In this chapter, we consider X both as the quotient $\dfrac{A'}{H}$ and $\dfrac{A}{G}$. By abuse of notation we still denote by ρ the representation of G on $\mathbb{C}^3$.
Using the fact that there exists a unique faithful representation of $\mathfrak{D}_4$ on $\mathbb{C}^3$ and by Theorem 4.2.6, we lead to the following result.

Theorem 6.1.3. [24, Corollary 1.1] *The family above of Generalized Hyperelliptic 3-folds X with group $\mathfrak{D}_4$ forms an irreducible and 2-dimensional family of complex manifolds. The Kähler manifolds with the same fundamental group as X yield an open subspace of the Teichmüller space of X parametrized by the periods τ and τ' of the elliptic curves E and E'.*

Definition 6.1.4. We denote by $\mathcal{F}^A_{\mathfrak{D}_4}$ the 2-dimensional family of Calabi-Yau 3-folds of type A with the group $\mathfrak{D}_4$.

Remark 6.1.5. Roughly speaking, Theorem 6.1.3 tells us that all manifolds $Y \in \mathcal{F}^A_{\mathfrak{D}_4}$ are given as the quotient of an abelian 3-fold $E_\mu \times E_\mu \times E_{\mu'}$, isomorphic to the product of three elliptic curves, by a free action of a group H' of order 16 which contains a normal subgroup G' which is isomorphic to $\mathfrak{D}_4$ and does not contain any translation. In particular, the space of parameters of $\mathcal{F}^A_{\mathfrak{D}_4}$ is isomorphic to $\mathcal{M}_{1,1} \times \mathcal{M}_{1,1}$, where $\mathcal{M}_{1,1}$ is defined in Proposition 2.7.6.

Remark 6.1.6. We observe that $A = B \times E'$ where $w_1(\underline{z}) = (z_1 + \frac{1}{2}, z_2 + \frac{1}{2})$ is a translation on $E \times E$ and $B = (E \times E)/\langle w_1 \rangle$ is an abelian surface. Let us define:

$$\begin{aligned} r_{E\times E}(z_1, z_2) &:= (z_2, -z_1) & r_2(z_3) &:= z_3 + 1/4 \\ s_{E\times E}(z_1, z_2) &:= (z_2 + b_1, z_1 + b_2) & s_2(z_3) &:= -z_3. \end{aligned} \tag{6.1.4}$$

We denote by r_1 and s_1 the automorphisms induced on B by $r_{E\times E}$ and $s_{E\times E}$, respectively.
We set

$$G_1 := \langle r_1, s_1 \rangle \quad G_2 := \langle r_2, s_2 \rangle \tag{6.1.5}$$

which satisfy $G_2 \simeq G_1 \simeq G \simeq \mathfrak{D}_4$. In particular X is the quotient of A by the diagonal action of G.

6.2 | The Picard group of Calabi-Yau threefolds in $\mathcal{F}^A_{\mathfrak{D}_4}$

In this section we explicitly describe the Picard group of $X \in \mathcal{F}^A_{\mathfrak{D}_4}$. It is worth recalling that abstractly the Picard group of hyperelliptic manifolds is described in [22, section 2]. Although, in this situation we know that the manifolds in $\mathcal{F}^A_{\mathfrak{D}_4}$ are Calabi-Yau threefolds and so we can use the characterization of their Picard group given in Section 5.2. We are going to prove the following result.

Notation: Let W be an abelian variety, we denote by $+$ the group operation on W and by $+_{\mathrm{Div}(W)}$ the group operation on $\mathrm{Div}(W)$.

Theorem 6.2.1. *Let $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ as in Definition 6.1.1. The followings hold:*

(i) *The Picard group satisfies* $\mathrm{Pic}(X) \simeq \mathbb{Z}^2 \oplus (\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ *with*

$$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle \overline{r}, \overline{s}, \lambda_{\tau'} \rangle$$

where $\lambda_{\tau'}(z) = (z_1, z_2, z_3 + \tau') \in \mathrm{Biho}(\mathbb{C}^3)$ *and* $\overline{r}, \overline{s}$ *are respectively the lifts of* r, s *to the universal cover* $\mathbb{C}^3$ *of* A'.

(ii) *The group* $\mathrm{Pic}_{\mathbb{Q}}(X)$ *is generated by the classes of two divisors* $D_{X,1}$ *and* $D_{X,2}$ *such that*

$$(\pi_H)^* D_{X,1} = \Big(\sum_{P+P=0} (E \times P) +_{\mathrm{Div}(E\times E)} (P \times E)\Big) \times E',$$

$$(\pi_H)^* D_{X,2} = (E \times E) \times (0_{E'} +_{\mathrm{Div}(E')} Q +_{\mathrm{Div}(E')} (2Q) +_{\mathrm{Div}(E')} (3Q)) \text{ with } Q + Q = \frac{1}{2}.$$

and the followings hold

$$(D_{X,1})^3 = 0 \quad (D_{X,2})^3 = 0 \quad D_{X,1} \cdot (D_{X,2})^2 = 0 \quad (D_{X,1})^2 \cdot D_{X,2} = 4.$$

6.2.1 | The torsion part of the Picard group

According to Lemma 5.2.2 we have $\mathrm{Pic}(X) = \mathbb{Z}^{\rho(X)} \oplus \mathbb{A}\mathrm{b}(\pi_1(X))$ for any $X \in \mathcal{F}^{A}_{\mathfrak{D}_4}$. By Lemma 6.1.2 we known that $\rho(X) = 2$. It remains to compute $\mathbb{A}\mathrm{b}(\pi_1(X))$.

By construction, $\pi_1(X)$ is the finite extension of $\pi_1(A') \simeq \Lambda$ by the group $H = \langle r, s\rangle$ [1], *i.e.* we have the following exact sequence

$$0 \longrightarrow \pi_1(A') \simeq \Lambda \longrightarrow \pi_1(X) \simeq \Gamma \longrightarrow H \longrightarrow 0$$

Let us denote $\Lambda_1 = \Lambda_2 := \pi_1(E) \simeq \mathbb{Z} \oplus \tau\mathbb{Z}$ and $\Lambda_3 = \tau'\mathbb{Z} \oplus \mathbb{Z} \simeq \pi_1(E')$.
We write $\Lambda = \{\lambda \in \mathrm{Biho}(\mathbb{C}^3) \mid \lambda(\underline{z}) = (z_1 + t_1, z_2 + t_2, z_3 + t_3) \text{ and } t_i \in \Lambda_i\}$ where $\Lambda_1 = \Lambda_2 = \tau\mathbb{Z} \oplus \mathbb{Z} \simeq \pi_1(E)$ and $\Lambda_3 = \tau'\mathbb{Z} \oplus \mathbb{Z} \simeq \pi_1(E')$. We denote by $\overline{r}, \overline{s} \in \Gamma$ the lifts of $r, s \in H$ to $\mathbb{C}^3$, respectively. We define 3 elements in Λ:

$$\begin{aligned}
\lambda_1 &: (z_1, z_2, z_3) \longmapsto (z_1 + 1, z_2, z_3)\\
\lambda_2 &: (z_1, z_2, z_3) \longmapsto (z_1, z_2 + 1, z_3)\\
\lambda_3 &: (z_1, z_2, z_3) \longmapsto (z_1 + \tau, z_2 + \tau, z_3)
\end{aligned}$$

and we consider $\Sigma_1 := \langle \lambda_1, \lambda_2, \lambda_3, [\overline{r}, \overline{s}]\rangle$ a specific subgroup of $\Gamma \simeq \pi_1(X)$.

Theorem 6.2.2. *Let X be as in Definition 6.1.1, we have the following isomorphism:*

$$\mathbb{A}\mathrm{b}(\pi_1(X)) \simeq (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) = \langle \overline{r}, \overline{s}, \lambda_{\tau'}\rangle$$

where $\lambda_{\tau'}(\underline{z}) = (z_1, z_2, z_3 + \tau')$.

Proof. We observe that the group 2Λ is contained in $[\pi_1(X), \pi_1(X)]$ indeed:

$$\begin{aligned}
&(z_1 - 2t_1, z_2 - 2t_2, z_3) = [\lambda, (\overline{r})^2](\underline{z}) && \forall \lambda(\underline{z}) = (z_1 + t_1, z_2 + t_2, z_3 + t_3) \in \pi_1(A') \simeq \Lambda\\
&(z_1, z_2, z_3 - 2t_3) = [\widetilde{\lambda}, \overline{s}](\underline{z}) && \forall \widetilde{\lambda}(\underline{z}) = (z_1, z_2, z_3 + t_3) \in \pi_1(A') \simeq \Lambda
\end{aligned}$$

Let us denote by $\lambda_{1,\tau} : (z_1, z_2, z_3) \mapsto (z_1 + \tau, z_2, z_3)$. The followings hold:

$$\lambda_1 = [(\overline{r}), (\overline{s})^2](\mathrm{mod} 2\Lambda) \qquad \lambda_2 = [\overline{r}^{-1}, (\overline{s})^2](\mathrm{mod} 2\Lambda) \qquad \lambda_3 = [\lambda_{1,\tau}, \overline{r}](\mathrm{mod} 2\Lambda).$$

Thus $\langle 2\Lambda, \Sigma_1\rangle \leq [\pi_1(X), \pi_1(X)]$ and since it is normal we get the following diagram:

$$\begin{array}{ccc}
\pi_1(X) & \xrightarrow{\ \varphi\ } & \dfrac{\pi_1(X)}{\langle 2\Lambda, \Sigma_1\rangle} =: \Sigma \\
& \searrow & \Big\downarrow f \\
& & \mathbb{A}\mathrm{b}(\pi_1(X)).
\end{array} \tag{6.2.1}$$

[1] We just observe that in this case Λ is not the maximal abelian and normal subgroup of finite index.

The generators of Σ are $\{\varphi(\overline{r}), \varphi(\overline{s}), \varphi(\lambda_{\tau'}), \varphi(\lambda_{1,\tau}), \varphi(\lambda_{2,\tau})\}$ where $\varphi(\lambda_{\tau'})(\underline{z}) = (z_1, z_2, z_3 + \tau')$ and $\varphi(\lambda_{2,\tau})(\underline{z}) = (z_1, z_2 + \tau, z_3)$. The following relations hold:

$$\lambda_{2,\tau} = \lambda_3(\lambda_{1,\tau})^{-1} \Rightarrow \varphi(\lambda_{2,\tau}) = \varphi(\lambda_{1,\tau})^{-1}$$
$$(\overline{rs})^2 = \lambda_{1,\tau} \Rightarrow \varphi(\lambda_{1,\tau}) = \varphi(\overline{rs})^2 = \varphi(\overline{r})^2\varphi(\overline{s})^2.$$

Therefore:

$$\Sigma = \langle \varphi(\overline{r}), \varphi(\overline{s}), \varphi(\lambda_{\tau'}) \rangle.$$

In particular, Σ is an abelian group:

$$\varphi(\overline{r})\varphi(\overline{s}) = \varphi(\overline{s})\varphi(\overline{r}) \quad \varphi(\overline{r})\varphi(\lambda_{\tau'}) = \varphi(\lambda_{\tau'})\varphi(\overline{r}) \quad \varphi(\lambda_{\tau'})\varphi(\overline{s}) = \varphi(\lambda_{2\tau'})\varphi(\overline{s})\varphi(\lambda_{\tau'}).$$

From the fact that $\varphi(\lambda_{\tau'})$ has order two and that both $\varphi(\overline{r})$ and $\varphi(\overline{s})$ have order four, the only possibility is $\Sigma \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since the commutator subgroup $[\pi_1(X), \pi_1(X)]$ is the smallest subgroup of $\pi_1(X)$ such that the quotient group is abelian, we obtain $[\pi_1(X), \pi_1(X)] = \langle 2\Lambda, \Sigma_1 \rangle$. Thus f in diagram (6.2.1) is an isomorphism and $\mathbb{A}\mathrm{b}(\pi_1(X)) \simeq \Sigma$. □

Proof of Theorem 6.2.1 part (i). We have $\mathrm{Pic}(X) = \mathbb{Z}^{\rho(X)} \oplus \mathbb{A}b(\pi_1(X))$: since by Lemma 6.1.2 $\rho(Y) = h^{1,1}(X) = 2$ and in Theorem 6.2.2 we have described $\mathbb{A}b(\pi_1(X))$, we obtain the result. □

6.2.2 | A $\mathbb{Q}$-basis of $\mathrm{Pic}_{\mathbb{Q}}(X)$

In this section we prove Theorem 6.2.1 part (ii) providing a $\mathbb{Q}$-basis for $\mathrm{Pic}_{\mathbb{Q}}(X)$. We recall the notation $\pi_H \colon A' \longrightarrow X = A'/H$.

Proof of Theorem 6.2.1 part (ii). Let us consider the following H-invariant divisor on A':

$$D_{A',1} := \Big(\sum_{P+P=0} (E \times P) +_{\mathrm{Div}(E\times E)} (P \times E)\Big) \times E'. \tag{6.2.2}$$

We observe that the support of $D_{A',1}$ is the union of 8 surfaces all isomorphic to the surface $D_1 := \Big((0_E \times E) +_{\mathrm{Div}(E\times E)} (E \times 0_E)\Big) \times E'$ (they are the orbits of D_1 under H) and that D_1 is preserved by $\langle r^2 \rangle$. Since D_1 is a surface, we can define its image $D_{X,1} := \pi_H(D_1)$, which is in particular a divisor on X (in fact it is the reduced image of $D_{A',1}$). We obtain:

$$(\pi_H)_*(D_1) = 2D_{X,1} \quad (\pi_H)_*(D_{A',1}) = 16D_{X,1}.$$

It follows that $(\pi_H)^*(D_{X,1}) = D_{A',1}$.
Let us consider another H-invariant divisor on A':

$$D_{A',2} := (E \times E) \times (0_{E'} +_{\mathrm{Div}(E')} Q +_{\mathrm{Div}(E')} (2Q) +_{\mathrm{Div}(E')} (3Q)) \text{ with } Q + Q = \frac{1}{2}. \quad (6.2.3)$$

The support of $D_{A',2}$ consists of 4 disjoint surfaces all isomorphic to $D_2 := (E \times E) \times 0_{E'}$. In particular D_2 is preserved by $\langle s \rangle$. We define $D_{X,2} := \pi_H(D_2)$ and obtain:

$$(\pi_H)_*(D_2) = 4D_{X,2} \quad (\pi_H)_*(D_{A',2}) = 16D_{X,2}.$$

It holds $(\pi_H)^*(D_{X,2}) = D_{A',2}$.
We can compute the trilinear form on $D_{X,1}, D_{X,2}$, using the projection formula (1.2.2):

$$(D_{X,1})^3 = 0 \qquad (D_{X,2})^3 = 0 \qquad D_{X,1} \cdot (D_{X,2})^2 = 0 \quad (D_{X,1})^2 \cdot D_{X,2} = 4.$$

Hence the classes of the divisors $D_{X,1}, D_{X,2}$ are linearly independent over $\mathbb{Q}$ and define a basis of $\mathrm{Pic}_{\mathbb{Q}}(X)$. □

6.3 | Fibrations on $X \in \mathcal{F}^A_{\mathfrak{D}_4}$

In Theorem 6.2.1 we have find a $\mathbb{Q}$-basis $\mathrm{Pic}_{\mathbb{Q}}(X) = \langle D_{X,1}, D_{X,2} \rangle$. We consider these two divisors and we describe the maps associated to them.

Definition 6.3.1. A **fibration** $f : Y_1 \to Y_2$ is a proper surjective morphism of normal varieties such that $0 < \dim Y_2 < \dim Y_1$ with connected fibers.
A fibration is called **isotrivial** if there exists a open dense set $U \subseteq Y_2$ such that for every $x, y \in U$ then $f^{-1}(x) \simeq f^{-1}(y)$.
A fiber $f^{-1}(y_2)$ over a point $y_2 \in Y_2$ such that f is not smooth for every $y_1 \in f^{-1}(y_2)$ is called **multiple fiber**.

Theorem 6.3.2. *Let $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ as in Definition 6.1.1. The following hold:*

(i) *The map*

$$\varphi_{|D_{X,1}|} : X \longrightarrow \varphi_{|D_{X,1}|}(X) := Z \subset \mathbb{P}(H^0(X, \mathcal{O}_X(D_{X,1}))^\vee)$$

is an isotrivial fibration whose general fiber is isomorphic to the elliptic curve E'. The base Z is a normal Enriques surface with singularities of type $3A_1$ and $2A_3$. We have three multiple fibers with multiplicity two and two with multiplicity four.

(ii) *The map*

$$\varphi_{|D_{X,2}|} : X \longrightarrow \mathbb{P}^1$$

is an isotrivial fibration whose general fiber is isomorphic to the abelian surface B. There are four multiple fibers with multiplicity two.

Proof. (i) We recall that $(\pi_H)^*(D_{X,1}) = D_{A',1}$ where $D_{A',1}$ is defined in (6.2.2). We observe that $\varphi_{|D_{A',1}|}$ factorizes through the fibration $pr_1\colon A' \longrightarrow E \times E$, whose fibers are isomorphic to E'. Indeed by definition:

$$D_{A',1} = (\sum_{P+P=0} (E\times P) +_{\mathrm{Div}(E\times E)} (P\times E))\times E' = pr_1^*(\sum_{P+P=0} (E\times P) +_{\mathrm{Div}(E\times E)} (P\times E)) := pr_1^* D_1$$

where D_1 is an ample line bundle on $E \times E$, by [56, Proposition 4.5.2].

We split $H^0(A', \mathcal{O}_{A'}(D_{A',1})) = \bigoplus_{\chi\in\mathrm{Irr}(H)} V_\chi$ into irreducible characters decomposition: since $(\pi_H)^*(D_{X,1}) = D_{A',1}$, the pull backs under π_H of the global sections of $\mathcal{O}_X(D_{X,1})$ define global sections of $H^0(A', \mathcal{O}_{A'}(D_{A',1}))$ which, in particular, are contained in an eigenspace V_χ relative to an unique irreducible character χ of H. Thus we obtain the following commutative diagram:

$$\begin{array}{ccccc} A' & \xrightarrow{\varphi_{|D_{A',1}|}} & E\times E & \longrightarrow & \mathbb{P}(H^0(A',\mathcal{O}_{A'}(D_{A',1}))^\vee) \\ \downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle p_1} & & \vdots \\ X & \xrightarrow[\varphi_{|D_{X,1}|}]{} & (E\times E)/H_{|E\times E} & \longrightarrow & \mathbb{P}(H^0(X,\mathcal{O}_X(D_{X,1}))^\vee) \subseteq \mathbb{P}(V_\chi) \end{array}$$

Using the notation in (6.1.4) and (6.1.5) we have $Z := \dfrac{(E\times E)}{H_{|E\times E}} = \dfrac{B}{G_1}$ where the quotient $B = (E\times E)/\langle w_1\rangle$ is an abelian surface and $G_1 = \langle r_1, s_1\rangle \simeq \mathfrak{D}_4$. We can easily observe that r_1 preserves the volume form of B, $\langle r_1\rangle \simeq \mathbb{Z}/4\mathbb{Z}$ and does not contain translations. Hence by [33, Lemma 3.1] $B/\langle r_1\rangle$ is birational to a K3 surface. By [13, Proposition 2.1] the singularities of the surface $B/\langle r_1\rangle$ are $6A_1 + 4A_3$. Moreover, by studying $\forall g \in \langle w_1, r_{E\times E}\rangle$ the equations $s_{E\times E}(z) = g(z)$ we find that they don't admit solutions, hence s_1 defines a fixed point free involution on $B/\langle r_1\rangle$. Thus Z is a singular model of an Enriques surface with singularities $3A_1 + 2A_3$.

Let $z = p_1(z_1) \in Z\setminus \mathrm{Sing}(Z)$, then $(p_1\circ\varphi_{|D_{A',1}|})^{-1}(z) = \bigcup_{h\in\langle s_{E\times E}, r_{E\times E}\rangle} (h(z_1)\times E')$. These fibers are in the same orbit under the action of H and each element of H permutes them, hence π_H identifies them. Therefore, the general fiber of $\varphi_{|D_{X,1}|}$ is isomorphic to E'. Furthermore, the fibers $\varphi^{-1}_{|D_{X,1}|}(z)$ over the points in $z \in \mathrm{Sing}(Z)$ are multiple fibers. Since the singularities of Z are $3A_1 + 2A_2$, we obtain the statement.

(ii) We apply arguments similar to the ones of part (i). Let us consider $pr_2\colon A' \longrightarrow E'$ whose fibers are isomorphic to $E\times E$ and $(\pi_H)^*(D_{X,2}) = pr_2^*(0_{E'} +_{\mathrm{Div}(E')} Q +_{\mathrm{Div}(E')} (2Q) +_{\mathrm{Div}(E')} (3Q))$ with $Q+Q = \frac{1}{2}$. We obtain the following commutative diagram:

$$\begin{array}{ccccc} A' & \xrightarrow{\varphi_{|D_{A',2}|}} & E' & \longrightarrow & \mathbb{P}(H^0(A',\mathcal{O}_{A'}(D_{A',2}))^\vee) \\ \downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle p_2} & & \vdots \\ X & \xrightarrow[\varphi_{|D_{X,2}|}]{} & E'/G_2 & \longrightarrow & \mathbb{P}(H^0(X,\mathcal{O}_X(D_{X,2}))^\vee) \end{array}$$

We recall $G_2 = \langle r_2, s_2 \rangle$, see (6.1.4) and (6.1.5): since r_2 is a translation on E' and s_2 is the elliptic involution then $E'/G_2 = \mathbb{P}^1$. Let $z \in E'$ with trivial stabilizer under the action of G_2, then $(p_2 \circ \varphi_{|D_{A',2}|})^{-1}(p_2(z)) = \bigcup_{g \in G_2} (E \times E \times g(z))$ which are in the same orbit under the action of H, in particular the translation w acts on each of them and all the others $g \in G$ permute them. Thus, the quotient given by w maps them to 8 manifolds isomorphic to B and the quotient by G identifies these latter manifolds. Therefore, the general fiber of $\varphi_{|D_{X,2}|}$ is isomorphic to B. Let $z_i \in E'[2]$, they are the only points with non-trivial stabilizer under the action of G_2. Thus, the fibers over $p_2(z_i)$ are multiple fibers.

□

Remark 6.3.3. We remark that the authors of [73], using a different approach, have already proven that $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ has two fibrations induced by the natural projection on the cover A and that in fact the divisors associated to these fibrations define a $\mathbb{Q}$-basis of $\mathrm{Pic}_{\mathbb{Q}}(X)$. The main difference with our approach is that we explicitly write down the divisors generating the torsion free part of $\mathrm{Pic}(X)$.

6.4 | The automorphism group of Calabi-Yau threefolds in $\mathcal{F}^A_{\mathfrak{D}_4}$

In this section we describe the automorphism group of $X \in \mathcal{F}^A_{\mathfrak{D}_4}$. The main result is the following.

Theorem 6.4.1 (Theorem B (ii))**.** *Let $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ and assume that $\mathrm{End}_{\mathbb{Q}}(E) \neq \mathbb{Q}(\zeta_6)$. The automorphism group of X is isomorphic to $\mathrm{Aut}(X) \simeq (\mathbb{Z}/2\mathbb{Z})^4$ whose elements are induced by order two translations by the points $(t_1, t_2, t_3) \in A' = E \times E \times E'$ satisfying*

$$t_1 + t_2 \in \{0, \frac{1}{2}\} \quad t_1 \in E[2] \quad t_3 \in E'[2]. \tag{6.4.1}$$

In particular, every α_X preserves the volume form of X.

Remark 6.4.2. According to Corollary 4.1.12 we have $\mathrm{Aut}(X) \simeq \dfrac{\mathrm{N}_{\mathrm{Aut}(A)}(G)}{G}$. According to Theorem 5.4.1, the group $\mathrm{Aut}(X)$ is finite and since G is finite then $\mathrm{N}_{\mathrm{Aut}(A)}(G)$ is finite too.

Remark 6.4.3. Let $T_1 = \mathbb{C}^n/\Lambda_1$ and $T_2 = \mathbb{C}^n/\Lambda_2$ be two n-dimensional complex tori, we consider an isogeny $f : T_1 \xrightarrow{m:1} T_2$. Let $\alpha_2 \in \mathrm{Aut}(T_2)$, $\eta : \langle\alpha_2\rangle \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be its representation. Then α_2 admits at least one (and indeed m) lift to T_1 if and only if $\eta(\alpha_2)(\lambda_1) \in \Lambda_1$ for every $\lambda_1 \in \Lambda_1$ which is equivalent to $\eta(\alpha_2) \in \mathrm{End}(T_1)^{\times}$. Clearly, any lift α_1 of α_2 to T_1 belongs to $\mathrm{N}_{\mathrm{Aut}(T_1)}(\ker(f))$.

We recall that $\alpha_{A'} \in \mathrm{Aut}(A')$ is written as $\alpha_{A'} = \eta(\alpha_{A'}) + t_{\alpha_{A'}}$ where $\eta(\alpha_{A'})$ is its linear part and $t_{\alpha_{A'}}$ its translation part, see (2.1.2).

Proposition 6.4.4. *Let us assume that* $\mathrm{End}_{\mathbb{Q}}(E) \neq \mathbb{Q}(\zeta_6)$. *The following homomorphism of groups is surjective:*

$$\theta : \quad \mathrm{N}_{\mathrm{Aut}(E\times E)\times\mathrm{Aut}(E')}(H) \longrightarrow\!\!\!\!\to \mathrm{Aut}(X)$$

$$\alpha_{A'} \longmapsto \alpha_X$$

where α_X *is the automorphism induced by* $\alpha_{A'}$ *on* X. *Thus* $\mathrm{Aut}(X) \simeq \dfrac{\mathrm{N}_{\mathrm{Aut}(E\times E)\times\mathrm{Aut}(E')}(H)}{H}$.

Proof. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{N}_{\mathrm{Aut}(A')}(H) & & \\ \theta_2 \downarrow & \searrow^{\theta} & \\ \mathrm{N}_{\mathrm{Aut}(A)}(G) & \xrightarrow[\theta_1]{} & \mathrm{Aut}(X) \end{array}$$

where θ_1 is surjective by Corollary 4.1.12. In order to prove the surjectivity of θ we prove the one of θ_2.

To start, we observe that $\mathrm{N}_{\mathrm{Aut}(A)}(G) = \mathrm{N}_{\mathrm{Aut}(B)}(G_1) \times \mathrm{N}_{\mathrm{Aut}(E')}(G_2)$, where G_1 and G_2 are defined in (6.1.5). The inclusion "$\supseteq$" is trivial. For the other one, let η be a representation of $\langle\alpha_A\rangle$ on $\mathbb{C}^3$ for $\alpha_A \in \mathrm{N}_{\mathrm{Aut}(A)}(G)$. We recall that the representation ρ of G splits into $\rho = \rho_2 \oplus \rho_1$, see (6.1.3): since $\alpha_A \in \mathrm{N}_{\mathrm{Aut}(A)}(G)$, η splits into the direct sum of two representations $\eta_2 \oplus \eta_1$ such that $\dim\eta_j = j$. Hence we write $\alpha_A = (\alpha_B, \alpha_{E'})$ and we easily deduce the other inclusion. In particular, it follows that $\mathrm{N}_{\mathrm{Aut}(A')}(H) = \mathrm{N}_{\mathrm{Aut}(E\times E)}(\langle G_1, w_1\rangle) \times \mathrm{N}_{\mathrm{Aut}(E')}(G_2)$. Thus, in order to prove the surjectivity of θ_2 it's enough to prove that every α_B admits a lift to $E \times E$. In fact, since $\mathrm{N}_{\mathrm{Aut}(A)}(G)$ is finite then we are interested to prove that the maximal automorphism group of finite order on B admits a lift to $E \times E$. Moreover, B is an abelian surface which admits an action of $\mathfrak{D}_4$ which is induced by the action of $\mathfrak{D}_4$ on $E \times E$, hence we restrict our attention to the maximal automorphism group of finite order on B which contains a subgroup isomorphic to $\mathfrak{D}_4$.

We observe that B can not split into the product of two isomorphic elliptic curves. If it was true, $\gamma := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ should belong to $\mathrm{End}(B)^\times$ and since $\gamma \in \mathrm{End}(E \times E)^\times$, accordingly to Remark 6.4.3, we would have two lifts of γ to $E \times E$ given by $\widetilde{\gamma}_\epsilon(\underline{z}) = \gamma(\underline{z}) + \epsilon w_1$ for $\epsilon \in \{0, 1\}$. This leads to a contradiction since it is easy to check that $\widetilde{\gamma}_\epsilon \notin \mathrm{N}_{\mathrm{Aut}(E\times E)}(w_1)$.

Let us consider the pair (B, G') where G' is the maximal automorphisms group of finite order in $\mathrm{End}(B)^\times$ which contains a subgroup isomorphic to $\mathfrak{D}_4$. According to [33, Tables 8 and 9]: whenever $\mathrm{End}_{\mathbb{Q}}(E) \neq \mathbb{Q}(\zeta_6)$ then $G' = \langle \rho_2(r), \rho_2(s) \rangle \simeq \mathfrak{D}_4$. Hence, by Remark 6.4.3, α_B admits a lift to $E \times E$ and it belongs to $\mathrm{N}_{\mathrm{Aut}(E\times E)}(w_1)$. Therefore, θ_2 is surjective and θ too. □

Corollary 6.4.5. *With the notation and hypothesis above, every $\alpha_{A'} \in \mathrm{N}_{\mathrm{Aut}(A')}(H)$ admits a representation $\eta = \eta_2 \oplus \eta_1$ on $\mathbb{C}^3$ such that $\eta_2(\alpha_{A'}) = \rho_2(r^j s^i)$ for $j = 0, 1, 2, 3$ and $i = 0, 1$.*

Assumption: According to Proposition 6.4.4, from now on we assume $\mathrm{End}_{\mathbb{Q}}(E) \neq \mathbb{Q}(\zeta_6)$.

Lemma 6.4.6. *Let $\alpha_{A'} \in \mathrm{N}_{\mathrm{Aut}(A')}(H)$ such that $\alpha_{A'}(\underline{z}) = \eta(\alpha_{A'})\underline{z} + t_{\alpha_{A'}}$ where $t_{\alpha_{A'}}$ is the translation by the points (t_1, t_2, t_3) and η is a representation of $\alpha_{A'}$ on $\mathbb{C}^3$. The following conditions hold :*

$$\eta(\alpha_{A'}) = \rho(r^j s^i) \quad \text{for some } j = 0, 1, 2, 3 \quad i = 1, 2$$

$$t_1 + t_2 \in \{0, \tfrac{1}{2}\} \quad t_1 \in E[2] \quad 2t_3 = \begin{cases} 0 & \text{if } j = 0, 2 \\ \frac{1}{2} & \text{if } j = 1, 3. \end{cases}$$

Proof. By Corollary 6.4.5: $\eta = \eta_2 \oplus \eta_1$ and $\eta_2(\alpha_{A'}) = \rho_2(r^j s^i)$ for $j = 0, 1, 2, 3$ and $i = 0, 1$. We write $\alpha_{A'}(\underline{z}) = \eta(\alpha_{A'})\underline{z} + t_{\alpha_{A'}}$ and $\alpha_{A'}^{-1}(\underline{z}) = \eta(\alpha_{A'})^{-1}\underline{z} - \eta(\alpha_{A'})^{-1}t_{\alpha_{A'}}$. Using the expression of r, s in (6.1.2) we have:

$$\alpha_{A'}^{-1} r \alpha_{A'}(\underline{z}) = [\eta_2(\alpha_{A'})^{-1}\rho_2(r)\eta_2(\alpha_{A'}) \oplus \rho_1(r)](\underline{z}) + \eta(\alpha_{A'})^{-1}(t_2 - t_1, -t_1 - t_2, 1/4).$$

Since $\eta_2(\alpha_{A'}) = \rho_2(r^j s^i)$ for some $i = 0, 1$ and $j = 0, 1, 2, 3$, we find:

$$\eta(\alpha_{A'})^{-1}\rho(r)\eta(\alpha_{A'}) = \rho_2(s^{-i}r^{-j}rr^j s^i) \oplus \rho_1(r) = \begin{cases} \rho(r) & \text{if } i = 0, \forall j \\ \rho(r^3) & \text{if } i = 1, \forall j. \end{cases}$$

Hence $\alpha_{A'}^{-1} r \alpha_{A'} \in H$ if and only if

$$\eta(\alpha_{A'})^{-1}(t_2 - t_1, -t_1 - t_2, \frac{1}{4}) = \eta_2(\alpha_{A'})^{-1}(t_2 - t_1, -t_1 - t_2) \oplus \eta_1(\alpha_{A'})^{-1}(\frac{1}{4})$$
$$= \begin{cases} t_r \vee t_{s^2 r} & \text{if } i = 0 \\ t_{r^3} \vee t_{s^2 r^3} & \text{if } i = 1 \end{cases}$$

from which we deduce:

$$t_1 + t_2 \in \{0, \frac{1}{2}\} \quad t_1 \in E[2] \quad \text{and} \quad \begin{cases} \eta_1(\alpha_{A'}) = 1 = \rho_1(r^j s^i) & \text{if } i = 0, \forall j \\ \eta_1(\alpha_{A'}) = -1 = \rho_1(r^j s^i) & \text{if } i = 1, \forall j. \end{cases}$$

In particular we obtain $\eta(\alpha_{A'}) = \rho(r^j s^i)$. The study of $\alpha_{A'}^{-1} s \alpha_{A'} \in H$ leads to the following conditions:

$$t_1 + t_2 \in \{0, \frac{1}{2}\}, \quad t_1 \in E[2] \quad \text{and} \quad 2t_3 = \begin{cases} 0 & \text{if } j = 0, 2 \\ \frac{1}{2} & \text{if } j = 1, 3. \end{cases}$$

□

Let $\alpha_{A'} \in \mathrm{N}_{\mathrm{Aut}(A')}(H)$ be as in Lemma 6.4.6. We observe that for $j = 1, 3$ we can rewrite $t_{\alpha_{A'}}$ as the translation by the point $(0, 0, \frac{1}{4}) + t = t_r + t$ where t_r is the translation part of r (see (6.1.3)) and t is a translation by a point (t_1, t_2, t_3) which satisfies:

$$t_1 + t_2 \in \{0, \frac{1}{2}\} \quad t_1 \in E[2] \quad t_3 \in E'[2]. \tag{6.4.2}$$

Thus, we may write an automorphism $\alpha_{A'} \in \mathrm{N}_{\mathrm{Aut}(E \times E) \times \mathrm{Aut}(E')}(H)$ as follows :

$$\alpha_{A'}(z) = \rho(r^j s^i)(z) + \begin{cases} t_r + t_{\alpha_{A'}} & \text{if } j = 1, 3, \forall i \\ t_{\alpha_{A'}} & \text{if } j = 0, 2, \forall i \end{cases} \quad \text{with } t_{\alpha_{A'}} \text{ satisfying (6.4.2)} . \tag{6.4.3}$$

Theorem 6.4.7. *The automorphism group of X is isomorphic to* $\mathrm{Aut}(X) \simeq (\mathbb{Z}/2\mathbb{Z})^4$ *whose elements are induced by order two translations by the points* $(t_1, t_2, t_3) \in A'$ *satisfying* (6.4.2)*. In particular, every* α_X *preserves the volume form of* X*.*

Proof. By Lemma 6.4.6 we have $\left|\mathrm{N}_{\mathrm{Aut}(A')}(H)\right| = 2^8$, hence $|\mathrm{Aut}(X)| = \frac{2^8}{2^4} = 2^4$. Let us consider the following translations in $\mathrm{N}_{\mathrm{Aut}(A)}(H)$:

$$\begin{cases} t_r + t_{\alpha_{A'}} - t_{r^j s^i} & \text{if } j = 1, 3, \forall i \\ t_{\alpha_{A'}} + t_{r^j s^i} & \text{if } j = 0, 2, \forall i \end{cases} \tag{6.4.4}$$

with $t_{\alpha_{A'}}$ satisfying (6.4.2). It is easy to see that any $\alpha_{A'}$ as in (6.4.3) differs from a translation in (6.4.4) by an element in H. Thus they induce the same automorphism on X and consequently, every $\alpha_X \in \mathrm{Aut}(X)$ is induced by a translation on A' by a point (t_1, t_2, t_3) of order two satisfying (6.4.2): they are 2^5. Since each translation commutes with any other translation we have $\mathrm{Aut}(X)$ is abelian, hence $\mathrm{Aut}(X) \simeq (\mathbb{Z}/2\mathbb{Z})^4$. Since each translation $\alpha_{A'} \in \mathrm{N}_{\mathrm{Aut}(A')}(H)$ preserves $\omega_{A'}$ and $\omega_X = (\pi_H)_*\omega_{A'}$, each α_X in $\mathrm{Aut}(X)$ preserves ω_X. □

6.5 | Quotients of Calabi-Yau threefolds in $\mathcal{F}^A_{\mathfrak{D}_4}$

In this section we describe the quotients of X for all the possible $\Upsilon \leq \mathrm{Aut}(X)$.

Theorem 6.5.1 (Theorem C part (ii))**.** *Let $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ as in Definition 6.1.1 and $\Upsilon \leq \mathrm{Aut}(X)$. Under the same assumption of Theorem 6.4.1, each quotient X/Υ admits a crepant resolution $\beta : Y \longrightarrow X/\Upsilon$ where Y is a Calabi-Yau 3-fold. In particular, there exist exactly 2 automorphisms $(\alpha_1)_X$ and $(\alpha_2)_X$ acting freely on X. They are induced respectively by the translations $\alpha_j \in \mathrm{Aut}(A')$*

$$\alpha_1(\underline{z}) := (z_1, z_2, z_3 + \frac{\tau'}{2}) \quad \alpha_2(\underline{z}) := (z_1, z_2, z_3 + \frac{\tau'}{2} + \frac{1}{2})$$

and the $\dfrac{X}{(\alpha_j)_X}$'s belong to $\mathcal{F}^A_{\mathfrak{D}_4}$.

Let $\alpha_X \in \mathrm{Aut}(X)$ with $X \in \mathcal{F}^A_{\mathfrak{D}_4}$. According to Proposition 6.4.4, α_X is induced by $\alpha_{A'}$ in $\mathrm{N}_{\mathrm{Aut}(A')}(H)$. Thus, we have the following characterization:

$$\begin{aligned}\mathrm{Fix}(\alpha_X) &= \{\pi_H(\underline{z}) \in X \quad | \quad \alpha_X(\pi_H(\underline{z})) = \pi_H(\underline{z})\} \\ &= \pi_H(\{(\underline{z}) \in A' \quad | \quad \exists h \in H \text{ with } \alpha_{A'}(\underline{z}) = h(\underline{z})\}).\end{aligned}$$

Remark 6.5.2. Let $\alpha(\underline{z}) = (z_1 + t_1, z_2 + t_2, z_3 + t_3)$ be a translation on $\mathbb{C}^3$. Let us fix $\epsilon \in \{0, 1\}$. We denote by $\overline{H}$ a lift of H to $\mathbb{C}^3$, we consider different $\overline{h} \in \overline{H}$ and study the equations $\alpha(\underline{z}) = \overline{h}(\underline{z})$ on $\mathbb{C}^3$. In the followings, u_1 and u_2 are the ones in Definition 6.1.1.

1. If $\overline{h} \in \{id, \overline{s}^2\}$: $\alpha(\underline{z}) \in \{\underline{z}, \overline{s}^2(\underline{z})\} \Leftrightarrow \alpha \equiv id \vee \alpha \equiv s^2 \Leftrightarrow \alpha_X = id_X$.

2. If $\overline{h} \in \{\overline{s}^{2\epsilon}\overline{r}^k\}$ for $k = 1, 2, 3$:

$$\begin{aligned}\alpha(\underline{z}) = \overline{s}^{2\epsilon}\overline{r}^k(\underline{z}) \text{ admits solution} &\Leftrightarrow t_3 = \frac{1}{4} + \frac{\epsilon}{2} \text{ for } k = 1, 3 \\ \alpha(\underline{z}) = \overline{s}^{2\epsilon}\overline{r}^2(\underline{z}) \text{ admits solution} &\Leftrightarrow t_3 = \frac{1}{2}.\end{aligned}$$

3. If $\overline{h} \in \{\overline{s}^{2\epsilon+1}\}$: $\alpha(\underline{z}) \in \{\overline{s}(\underline{z}), \overline{s}^3(\underline{z})\} \Leftrightarrow t_1 + t_2 = \dfrac{1}{2} + \epsilon$.

4. If $\overline{h} \in \{\overline{r}^2\overline{s}^{2\epsilon+1}\}$: $\alpha(\underline{z}) \in \{\overline{r}^2\overline{s}(\underline{z}), \overline{r}^2\overline{s}^3(\underline{z})\} \Leftrightarrow t_1 - t_2 = \dfrac{1}{2}$.

5. If $\overline{h} \in \{\overline{r}^3\overline{s}^{2\epsilon+1}\}$: $\alpha(\underline{z}) \in \{\overline{r}^3\overline{s}(\underline{z}), \overline{r}^3\overline{s}^3(\underline{z})\} \Leftrightarrow t_2 = u_1 + \frac{\epsilon}{2}$.

6. If $\overline{h} \in \{\overline{rs}^{2\epsilon+1}\}$: $\alpha(\underline{z}) \in \{\overline{rs}(\underline{z}), \overline{rs}^3(\underline{z})\} \Leftrightarrow t_1 = u_2 + \frac{\epsilon}{2}$.

Proof of Theorem 6.5.1. By Theorem 6.4.1, $\Upsilon \simeq (\mathbb{Z}/2\mathbb{Z})^m$ for some $1 \leq m \leq 4$ and every $\upsilon \in \Upsilon$ preserves the volume form of X. According to Proposition 5.4.6 the fixed locus of Υ, if not empty, is a disjoint union of curves, *i.e.* the codimension of $\mathrm{Fix}(\Upsilon)$ is 2. Since Υ is abelian, we can split the quotient X/Υ in a subsequent quotients of order two as follows. Let $\alpha_X \in \Upsilon$: by Proposition 5.4.6 the quotient $X/\langle\alpha_X\rangle$ is birational to a Calabi-Yau 3-fold $\widetilde{X}$. Since Υ is abelian, $\Upsilon_1 := \Upsilon/\langle\alpha_X\rangle$ preserves $\mathrm{Sing}(X/\langle\alpha_X\rangle)$ and so Υ_1 lifts to an action on $\widetilde{X}$. Moreover, each element of Υ_1 preserves the volume form $\omega_{\widetilde{X}}$ of $\widetilde{X}$ since each element of Υ preserves ω_X. Thus, we have a Calabi-Yau 3-fold $\widetilde{X}$ with an action of $\Upsilon_1 \simeq (\mathbb{Z}/2\mathbb{Z})^{m-1}$ which preserves its volume form: by iterating the argument above we conclude that X/Υ admits a crepant resolution $\beta : Y \to X/\Upsilon$ with Y a Calabi-Yau 3-fold.
Let $\alpha_{A'}(\underline{z}) = (z_1 + t_1, z_2 + t_2, z_3 + t_3)$ be an automorphism of A' satisfying (6.4.2). It induces a free action on X if and only if for every $h \in H$ the equation $\alpha_{A'}(\underline{z}) = h(\underline{z})$ has no solutions. By using Remark 6.5.2, this happens if and only if $t_1 = t_2 \in \{0, \frac{1}{2}\}$ and $t_3 \in \{\frac{\tau'}{2}, \frac{\tau'+1}{2}\}$. Fixed t_3 we have $(z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3 + t_3) - (z_1, z_2, z_3 + t_3) = w(\underline{z})$, hence these two translations define the same automorphism on X. Therefore, there are only two automorphisms which act freely on X and they are induced by $\alpha_1(\underline{z}) := (z_1, z_2, z_3 + \frac{\tau'}{2})$ and $\alpha_2(\underline{z}) := (z_1, z_2, z_3 + \frac{\tau'+1}{2})$. Let us denote by $Y_j := X/\langle(\alpha_j)_X\rangle$. As we prove above, they are Calabi-Yau 3-folds. By construction $Y_j = \dfrac{A'}{\langle\alpha_j, H\rangle}$. We observe that Y_j can be obtained as free quotient of the abelian 3-fold $A_j = \dfrac{A'}{\langle\alpha_j, w\rangle}$ by the action of the finite group $\dfrac{\langle\alpha_j, H\rangle}{\langle\alpha_j, w\rangle}$. It is easy to check that this latter group is isomorphic to $\mathfrak{D}_4$ and does not contain any translation. Therefore, $Y_j \in \mathcal{F}^A_{\mathfrak{D}_4}$.

□

We finish this section by computing the fixed locus of every α_X in $\mathrm{Aut}(X)$. Let us fix $t_1 \in E[2]$, $t_1 + t_2 \in \{0, \frac{1}{2}\}$ and $t_3 \in E'[2]$. We define the elliptic curves:

$$
\begin{aligned}
C^1_{p,q} &= \{(p,q,l) \in A' \mid l \in E', 2p = t_1, 2q = t_2\}\\
C^2_{p,q} &= \{(p,q,l) \in A' \mid l \in E', 2p = t_1 + \frac{1}{2}, 2q = t_2 + \frac{1}{2}\}\\
C^3_{q,t} &= \{(p,q,l) \in A' \mid p \in E, 2q = \frac{1}{2}, 2l = \frac{1}{4} + t_3\}\\
C^4_{p,t} &= \{(p,q,l) \in A' \mid q \in E, 2p = \frac{1}{2}, 2l = \frac{3}{4} + t_3\}\\
C^{5,t_1}_{l} &= \{(p, -p + u_1 + t_1, l) \in A' \mid p \in E, 2l = \frac{1}{2} + t_3\}\\
C^{6,t_1}_{l} &= \{(p, -p + u_2 + t_1, l) \in A' \mid p \in E, 2l = \frac{1}{2} + t_3\}\\
C^{7,t_1}_{l} &= \{(p, p + u_1 + t_1, l) \in A' \mid p \in E, 2l = t_3\}\\
C^{8,t_1}_{l} &= \{(p, p + u_2 + t_1, l) \in A' \mid p \in E, 2l = t_3\}\\
C^9_{q,l} &= \{(p,q,l) \in A' \mid p \in E, 2q = 0, 2l = \frac{1}{4} + t_3\}\\
C^{10}_{p,l} &= \{(p,q,l) \in A' \mid q \in E, 2p = 0, 2l = \frac{3}{4} + t_3\}
\end{aligned}
\tag{6.5.1}
$$

Proposition 6.5.3. *Let* $\alpha_{A'}(\underline{z}) = (z_1+t_1, z_2+t_2, z_3+t_3) \in \mathrm{Aut}(A')$ *with* $t_1+t_2 \in \{0, \frac{1}{2}\}$, $t_1 \in E[2]$, *and* $t_3 \in E'[2]$ *which induces* α_X *on* X. *The fixed locus of* α_X *consists of elliptic curves and it is described in the following table:*

Table 6.1: The fixed locus of α_X on X.

t_1	t_2	t_3	$\mathrm{Fix}(\alpha_X)$	$\lvert\mathrm{Fix}(\alpha_X)\rvert$
0	0	$\frac{\tau'}{2}$	$\emptyset$	0
0	0	$\frac{\tau'+1}{2}$	$\emptyset$	0
0	0	$\frac{1}{2}$	$\pi_H(C^1_{0,0})$, $\pi_H(C^1_{\frac{\tau}{2},\frac{\tau}{2}})$, $\pi_H(C^1_{0,\frac{\tau}{2}})$, $\pi_H(C^2_{\frac{1}{4},\frac{1}{4}})$, $\pi_H(C^2_{\frac{1}{4},\frac{1}{4}+\frac{\tau}{2}})$	5
$\frac{\tau}{2}$	$\frac{\tau}{2}$	$\frac{1}{2}$	$\pi_H(C^1_{\frac{\tau}{4},\frac{\tau}{4}})$, $\pi_H(C^2_{\frac{\tau+1}{4},\frac{\tau+1}{4}})$, $\pi_H(C^3_{\frac{1}{4},\frac{3}{8}})$, $\pi_H(C^3_{\frac{1}{4},\frac{3}{8}+\frac{\tau'}{2}})$	4
$\frac{\tau}{2}$	$\frac{\tau}{2}$	$\neq \frac{1}{2}$	$\pi_H(C^3_{\frac{1}{4},\beta})$, $\pi_H(C^3_{\frac{1}{4},\beta+\frac{\tau'}{2}})$ with $2\beta = \frac{1}{4} + t_3$	2
$\frac{\tau}{2}$	$\frac{\tau+1}{2}$	$\frac{1}{2}$	$\pi_H(C^1_{\frac{\tau}{4},\frac{\tau+1}{4}})$, $\pi_H(C^1_{\frac{\tau}{4},\frac{3\tau+1}{4}})$, $\pi_H(C^9_{0,\frac{3}{8}})$, $\pi_H(C^9_{0,\frac{3}{8}+\frac{\tau'}{2}})$, $\pi_H(C^{5,\frac{\tau}{2}}_{0})$, $\pi_H(C^{5,\frac{\tau}{2}}_{\frac{\tau'}{2}})$	6
$\frac{\tau}{2}$	$\frac{\tau+1}{2}$	$\neq \frac{1}{2}$	$\pi_H(C^9_{0,\beta})$, $\pi_H(C^9_{0,\beta+\frac{\tau'}{2}})$, $\pi_H(C^{5,\frac{\tau}{2}}_{\gamma})$, $\pi_H(C^{5,\frac{\tau}{2}}_{\gamma+\frac{\tau'}{2}})$ with $2\beta = \frac{1}{4} + t_3$ and $2\gamma = \frac{1}{2} + t_3$	4
0	$\frac{1}{2}$	$\frac{1}{2}$	$\pi_H(C^1_{0,\frac{1}{4}})$, $\pi_H(C^1_{0,\frac{1}{4}+\frac{\tau}{2}})$, $\pi_H(C^{5,0}_{0})$, $\pi_H(C^{5,0}_{\frac{\tau'}{2}})$	4
0	$\frac{1}{2}$	$\neq \frac{1}{2}$	$\pi_H(C^{5,0}_{\gamma})$, $\pi_H(C^{5,0}_{\gamma+\frac{\tau'}{2}})$ with $2\gamma = \frac{1}{2} + t_3$	2

Proof. We need to study the solutions of the equations $\alpha_{A'}(\underline{z}) = h(\underline{z})$ with $h \in H$ and we use Remark 6.5.2. As evidence we explicitly show the computations in one case.
Let us consider $t_1 = t_2$ and $t_2 = \frac{1}{2}$. By Remark 6.5.2 we have just to compute the equation above for $\overline{h} \in \{\overline{s}^2\overline{r}^2, \overline{r}^2\}$. Let us denote $I^j(t_1,t_2) = \coprod_{2p=t_1, 2q=t_2} C^j_{p,q}$ with $C^j_{p,q}$ as in (6.5.1). We find $\mathrm{Fix}(\alpha_{A'}) = I^1(t_1,t_2) \coprod I^2(t_1,t_2)$. Now, we need to study the action

of H on the fixed locus. To do this we fix $t_1 = 0$. Let $C^1_{p,q} \in \mathrm{Fix}(\alpha_{A'})$, since $-p = p$ and $-q = q$ we have:

$$r \circlearrowleft C^1_{p,p} \xrightarrow{s} C^1_{p+\frac{\tau+1}{2},p+\frac{\tau}{2}} \xrightarrow{s} C^1_{p+\frac{1}{2},p+\frac{1}{2}} \xrightarrow{s} C^1_{p+\frac{\tau}{2},p+\frac{\tau+1}{2}}$$

$$\begin{array}{ccccccc} C^1_{p,p+\frac{\tau}{2}} & \xrightarrow{s} & C^1_{p+\frac{1}{2},p+\frac{\tau}{2}} & \xrightarrow{s} & C^1_{p+\frac{1}{2},p+\frac{\tau+1}{2}} & \xrightarrow{s} & C^1_{p,p+\frac{\tau+1}{2}} \\ \updownarrow r & & & & \updownarrow r & & \\ C^1_{p+\frac{\tau}{2},p} & \xrightarrow{s} & C^1_{p+\frac{\tau+1}{2},p} & \xrightarrow{s} & C^1_{p+\frac{\tau+1}{2},p+\frac{1}{2}} & \xrightarrow{s} & C^1_{p+\frac{\tau}{2},p+\frac{1}{2}}. \end{array}$$

Thus for $t_1 = 0$: $\pi_H(I^1(t_1,t_2)) = \{\pi_H(C^1_{0,0}), \pi_H(C^1_{\frac{\tau}{2},\frac{\tau}{2}}), \pi_H(C^1_{0,\frac{\tau}{2}})\}$. Similar computations for $C^2_{p,q}$ lead to $\pi_H(I^2(t_1,t_2)) = \{\pi_H(C^2_{\frac{1}{4},\frac{1}{4}}), \pi_H(C^2_{\frac{1}{4},\frac{1}{4}+\frac{\tau}{2}})\}$. By applying the same argument for the other possibility of t_1, t_2, t_3 as in the statement, we obtain the result. □

6.6 | Classification of quotients of Calabi-Yau threefolds in $\mathcal{F}^A_{\mathfrak{D}_4}$

In this section we classify the Calabi-Yau 3-folds Y obtained as crepant resolution of X/Υ with $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ and $\Upsilon \leq \mathrm{Aut}(X)$: we compute their Hodge numbers and their fundamental groups. We summarize our results in a series of tables in Section 6.10: for $\Upsilon \simeq (\mathbb{Z}/2\mathbb{Z})$ in Table 6.2, for $\Upsilon \simeq (\mathbb{Z}/2\mathbb{Z})^2$ in Table 6.3, for $\Upsilon \simeq (\mathbb{Z}/2\mathbb{Z})^3$ in Table 6.4, for $\Upsilon \simeq (\mathbb{Z}/2\mathbb{Z})^4$ in Table 6.5. We obtain the following:

Corollary 6.6.1. *There are at least* 19 *non-homeomorphic Calabi-Yau* 3*-folds Y obtained as crepant resolution of X/Υ as varying $\Upsilon \leq \mathrm{Aut}(X)$. The $h^{1,1}(Y) = h^{2,1}(Y)$ and $\pi_1(Y)$ are summarized in Section 6.10.*

6.6.1 | Hodge numbers of crepant resolution of X/Υ

In this section we recall how to compute the Hodge numbers of the Calabi-Yau 3-folds Y obtained as desingularization of the quotients X/Υ with $\Upsilon \leq \mathrm{Aut}(X)$ via the orbifold cohomology (cf. [27]).

Definition 6.6.2. An **orbifold** M is an n-dimensional variety which for every $m \in M$ there exists a neighborhood U_m of m isomorphic to $\mathbb{C}^n/G'_m$ for certain finite group G'_m.

Without going in details: there is a good way to define cohomology on orbifolds, said *orbifolds cohomology* such that in the projective case (or Kähler case) it holds the Hodge decomposition Theorem, cf. [27].

We recall the formula of orbifold cohomology in the situation which is of our interest. Let X be the Calabi-Yau 3-fold constructed in Section 6.1 and $\Upsilon \leq \mathrm{Aut(X)}$ is characterized by Theorem 6.4.1. Since Aut(X) is abelian, the set of representatives of the conjugacy classes of Υ is equal to Υ and the centralizer of each element is Υ itself. The action of α_X can be locally (near a fixed point) linearized as $\mathrm{diag}(1,-1,-1)$ hence the age of α_X (see Definition 1.6.16) in a fixed point is equal to 1 The orbifold cohomology of X/Υ is:

$$H^{p,q}_{orb}(X/\Upsilon) := H^{p,q}(X)^{\Upsilon} \oplus \bigoplus_{F \in \mathrm{Sing}(X/\Upsilon)} H^{p-1,q-1}(F). \tag{6.6.1}$$

We denote by $h^{p,q}_{orb}(X/\Upsilon)$ the dimension of $H^{p,q}_{orb}(X/\Upsilon)$.

Theorem 6.6.3. [7, Theorem 5.4 and Corollary 6.15] *Let M be a compact, Kähler, complex manifold of dimension n with trivial canonical bundle and equipped with an action of a finite group G' that preserves the volume form of M. Assume the existence of a crepant resolution $\beta : \widetilde{M} \longrightarrow M/G'$. Then*

$$h^{p,q}(\widetilde{M}) = h^{p,q}_{orb}(M/G').$$

Proposition 6.6.4. *Let $X \in \mathcal{F}^{A}_{\mathfrak{D}_4}$ and $\Upsilon \leq \mathrm{Aut}(X)$. Let $Y \longrightarrow X/\Upsilon$ be the crepant resolution constructed in Theorem 6.5.1, we have:*

$$h^{1,1}(Y) = h^{2,1}(Y) = 2 + \sum_{id \neq \alpha_X \in \Upsilon} \left| \frac{\mathrm{Fix}(\alpha_X)}{\Upsilon} \right|.$$

In particular $e(Y) = 0$.

Proof. If Υ acts freely then, by Theorem 6.5.1 part (ii), $Y = X/\Upsilon \in \mathcal{F}^{A}_{\mathfrak{D}_4}$. Thus, since $\mathcal{F}^{A}_{\mathfrak{D}_4}$ is irreducible then $h^{1,1}(Y) = h^{2,1}(Y) = h^{1,1}(X) = 2$ (see Lemma 6.1.2) and so as we have the result. Assume that Υ has non empty stabilized locus. Then X/Υ is singular and let Y be as in the statement. We use (6.6.1) to compute the Hodge numbers of X/Υ which according to Theorem 6.6.3 are equal to the ones of Y. According to Proposition 6.5.3, $\mathrm{Fix}(\Upsilon)$ consists of elliptic curves. We prove that also $\mathrm{Sing}(X/\Upsilon) = \dfrac{\mathrm{Fix}(\Upsilon)}{\Upsilon}$ consists of elliptic curves. Let $F \in \mathrm{Sing}(X/\Upsilon)$ and $p \colon X \longrightarrow X/\Upsilon$ then $F = p(F^i_{\alpha_X})$ with $F^i_{\alpha_X}$ elliptic curve fixed by $\alpha_X \in \Upsilon$. We observe that since Υ is abelian then it preserves $\mathrm{Fix}(\alpha_X)$. Hence $\dfrac{\coprod_{\alpha_X \in \Upsilon} \mathrm{Fix}(\alpha_X)}{\Upsilon} = \coprod_{\alpha_X \in \Upsilon} \dfrac{\mathrm{Fix}(\alpha_X)}{\Upsilon}$. Let $\beta_X \neq \alpha_X \in \Upsilon$, it's enough to observe that if β_X preserves $F^i_{\alpha_X}$ then it does not fix points on $F^i_{\alpha_X}$: this follows since $\beta_X(F^i_{\alpha_X}) = \pi_H(\beta_{A'}(F^i_{\alpha_X}))$, H acts freely and β_X is induced by the translation $\beta_{A'}$ on A'. Thus if Υ preserves $F^i_{\alpha_X}$, it fixes $F^i_{\alpha_X}$ or it acts as a translation on it; if Υ maps $F^i_{\alpha_X}$ to $F^j_{\alpha_X}$ then the quotient q identify these two elliptic curves. In any case $F = q(F^i_{\alpha_X})$ is an elliptic curve for every $F \in \mathrm{Sing}(X/\Upsilon)$ and so $h^{1,0}(F) = h^{0,0}(F)$. Since every element of Υ is induced by a translation, Υ acts trivially on $H^{1,1}(X)$ and $H^{2,1}(X)$, hence $h^{1,1}(X)^{\Upsilon} = h^{1,1}(X) = h^{2,1}(X) = h^{2,1}(X)^{\Upsilon}$. Therefore, using (6.6.1) we obtain:

$$\begin{aligned} h^{1,1}(Y) = h^{1,1}_{orb}(X/\Upsilon) &= h^{1,1}(X)^{\Upsilon} + \sum_{id\neq\alpha_X\in\Upsilon}\ \sum_{F\in\mathrm{Sing}(X/\Upsilon)} h^{0,0}(F) \\ &= h^{2,1}(X)^{\Upsilon} + \sum_{id\neq\alpha_X\in\Upsilon}\ \sum_{F\in\mathrm{Sing}(X/\Upsilon)} h^{1,0}(F) = h^{2,1}_{orb}(X/\Upsilon) = h^{2,1}(Y). \end{aligned}$$

Consequently $h^{1,1}(Y) = h^{2,1}(Y)$ and $e(Y) = 0$. Since $h^{0,0}(F) = 1$ and $h^{1,1}(X) = 2$ we have:

$$\begin{aligned} h^{1,1}(X/\Upsilon) &= h^{1,1}(X)^{\Upsilon} + \sum_{F\in\mathrm{Sing}(X/\Upsilon)} h^{0,0}(F) \\ &= 2 + |\mathrm{Sing}(X/\Upsilon)| \qquad (6.6.2) \\ &= 2 + \sum_{id\neq\alpha_X\in\Upsilon} \left|\frac{\mathrm{Fix}(\alpha_X)}{\Upsilon}\right|. \end{aligned}$$

□

To compute $\left|\frac{\mathrm{Fix}(\alpha_X)}{\Upsilon}\right|$ one can use the description of $\mathrm{Fix}(\alpha_X)$ in Proposition 6.5.3 and then study the action of Υ similarly to the proof on Proposition 6.5.3. We omit these computations.

6.6.2 | Fundamental group of a desingularization of X/Υ

In this section we explain how to compute the fundamental group of the Calabi-Yau 3-folds obtained as crepant resolution of X/Υ for every $\Upsilon \leq \mathrm{Aut}(X)$ via the fundamental groupoid. We only recall the results which we need, for a complete discussion we refer to [18, Chapter 11].

Definition 6.6.5. Let Γ be a group acting on a topological space. The action is called **discontinuous** if for every $y \in Y$, the stabilizer Γ_y is finite and there exists an open neighborhood V_y such that $\gamma V_y \cap V_y = \emptyset$ for every $\gamma \notin \Gamma_y$.

Theorem 6.6.6. [18, Propositions 11.2.3 and 11.5.2 (c)] *Let Y be a connected topological space and Γ be a group acting on Y. If Γ defines a discontinuous action on Y and Y is* *simply connected, the fundamental group of Y/Γ is:*

$$\pi_1\left(\frac{Y}{\Gamma}\right) \simeq \frac{\Gamma}{F_\Gamma}$$

where $F_\Gamma = \{\gamma \in \Gamma \mid \exists\ z\ \in Y \ s.t.\ \gamma \in \Gamma_z\} \trianglelefteq \Gamma$.

Lemma 6.6.7. *Every crystallographic group* $\Gamma \leq \mathrm{Iso}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ *defines a discontinuous action on* $\mathbb{R}^n$.

Proof. Since $\mathbb{R}^n$ is a Hausdorff topological space, for every $\underline{z} \in \mathbb{R}^n$ there exists $V_{\underline{z}}$ such that $\gamma V_{\underline{z}} \cap V_{\underline{z}} = \emptyset$ for every $\gamma \in \Gamma$ which is not in the stabilizer $\Gamma_{\underline{z}}$ of $\underline{z}$. We prove that the stabilizer of $\underline{z}$ is finite. By Remark 4.1.7, Γ fits in the following exact sequence

$$0 \longrightarrow \Lambda' \xrightarrow{\ i\ } \Gamma \xrightarrow{\ l\ } G' \longrightarrow 0 \tag{6.6.3}$$

where Λ' is the maximal abelian and normal subgroup of finite index in Γ. We obtain the following description:

$$\Gamma_{\underline{z}} = \{(M, \lambda') \in \Gamma \mid l(M, \lambda) = M \in G', i^{-1}(M, \lambda') = \lambda' \in \Lambda' \text{ such that } \lambda'(\underline{z}) = M^{-1}(\underline{z})\}.$$

The description above tell us that λ' is uniquely determined by M and since M varies in the finite group G', we deduce that $\Gamma_{\underline{z}}$ is finite. □

In our situation: we have X/Υ with $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ and $\Upsilon \leq \mathrm{Aut}(X)$. By Theorem 4.1.12 Υ admits a lift Υ_A to A and Υ_A is a group of translation on A. Since every holomorphic maps on A lifts to $\mathbb{C}^3$, we have a group $\Gamma \leq \mathrm{Iso}(\mathbb{C}^3)$ such that $X/\Upsilon = \mathbb{C}^3/\Gamma$. The group Γ is a crystallographic group since it is the finite extension of $\langle \pi_1(A), \Upsilon_A\rangle$ by the group G, see Remark 4.1.7. Thus Γ defines a discontinuous action on $\mathbb{C}^3$ and we can apply Theorem 6.6.6 to compute $\pi_1(X/\Upsilon)$.

Theorem 6.6.8. [53, Theorem 7.8] *Let Y_1 be a normal analytic space and let us consider a resolution of singularities $f : Y_2 \longrightarrow Y_1$. If Y_1 has quotient singularities then $\pi_1(Y_2)$ and $\pi_1(Y_1)$ are isomorphic.*

Consequently we obtain:

Corollary 6.6.9. *Let $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ and Y be a desingularization of X/Υ for $\Upsilon \leq \mathrm{Aut}(X)$. Then:*

$$\pi_1(Y) \simeq \pi_1(X/\Upsilon) \simeq \frac{\Gamma}{F_\Gamma}$$

where $Y = \mathbb{C}^3/\Gamma$ as explained before.

Remark 6.6.10. We briefly explain how to compute $\pi_1(X/\Upsilon)$. We assume $\Upsilon = \langle \alpha_X \rangle$ and denote by $\alpha_{A'}$ a lift of α_X on A'. We have X/Υ is also the quotient $\mathbb{C}^3/\Gamma_\alpha$ where the group $\Gamma_\alpha \leq \mathrm{Iso}(\mathbb{C}^3)$ is the finite extension of $\Lambda = \pi_1(A')$ by the group $\langle H, \alpha_{A'} \rangle$. We denote by $\overline{H}$ and $\overline{\alpha_{A'}}$ the lifts of H and $\alpha_{A'}$ to $\mathbb{C}^3$, respectively. By using Remark 6.5.2 one can compute the solution of $\overline{\alpha_{A'}}((\underline{z})) = (\overline{h} \circ \lambda)(\underline{z})$ with $\overline{h} \in \overline{H}$, $\lambda \in \Lambda$ and for every $\underline{z} \in \mathbb{C}^3$ and so compute F_{Γ_α} as definition. If Υ has more than one generator, said α_X and β_X: we have $F_{\Gamma_\Upsilon} = \langle \mathrm{F}_{\Gamma_{\alpha_X}}, \mathrm{F}_{\Gamma_{\beta_X}} \rangle$ and by using Remark 6.5.2, one can easily check that $\dfrac{\Gamma_\Upsilon}{\mathrm{F}_{\Gamma_\Upsilon}}$ is a subgroup of both $\dfrac{\Gamma_{\alpha_X}}{\mathrm{F}_{\Gamma_{\alpha_X}}}$ and $\dfrac{\Gamma_{\beta_X}}{\mathrm{F}_{\Gamma_{\beta_X}}}$. Thus as Υ grows then $\pi_1(Y)$ tends to the identity group, *i.e.* Y is simply-connected.

6.6.3 | The universal cover of crepant resolution of X/Υ

By tables in Section 6.10, we have that $\pi_1(Y)$ is on of the followings: a finite extension of a rank-6 lattice, a finite extension of a rank-2 lattice or a finite group. From this description, we deduce the topology of the universal cover of Y.

Corollary 6.6.11. *Let $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ be as in Definition 6.1.2, $\beta\colon Y \longrightarrow X/\Upsilon$ be a crepant resolution with Υ in* $\mathrm{Aut}(X)$ *and $\pi_1(Y)$ is described in the tables in Section 6.10. The following cases appear.*

- (i) *If $\pi_1(Y)$ is a finite extension of a rank-6 lattice, Y is a Calabi-Yau 3-fold of type A which belongs to $\mathcal{F}^A_{\mathfrak{D}_4}$.*
- (ii) *If $\pi_1(Y)$ is a finite extension of a rank-2 lattice, the universal cover of Y is isomorphic to $\mathbb{C} \times S$ where S is a K3 surface.*
- (iii) *If $\pi_1(Y)$ is finite and not trivial, the universal cover of Y is a simply-connected Calabi-Yau 3-fold. If it is trivial, Y is a simply connected Calabi-Yau threefold.*

Proof. The universal cover $\widetilde{Y}$ of Y is a 3-fold with trivial canonical bundle and by the Beauville-Bogomolov decomposition theorem 1.5.6 we deduce that:

$$\widetilde{Y} = \mathbb{C}^n \times \prod_i W_i \times \prod_j Z_j$$

where W_i's are simply connected Calabi-Yau manifolds and Z_j's are irreducible holomorphic symplectic manifolds. If $\pi_1(Y)$ is the extension of a rank-6 lattice then $\widetilde{Y} = \mathbb{C}^3$ and we have the first statement. If $\pi_1(Y)$ is the extension of a rank-2 lattice the only possibility is $\widetilde{Y} = \mathbb{C} \times W_1$ where W_1 is a K3 surface. In the last case: since $\pi_1(Y)$ is finite we cannot find a complex space as factor of $\widetilde{Y}$ and since Y has dimension 3 the only possibility is that $\widetilde{Y}$ is a simply-connected Calabi-Yau 3-fold. □

6.7 | A map of degree two defined on $\mathcal{F}^A_{\mathfrak{D}_4}$

By Theorem 6.5.1 we know the existence of quotients of $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ which belong to the same family. As consequence, we construct a map in $\mathcal{F}^A_{\mathfrak{D}_4}$ which tell us how to move in this family.

Lemma 6.7.1. *Let $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ as in Definition 6.1.2. There are two double covers of X which belong to $\mathcal{F}^A_{\mathfrak{D}_4}$ and they are isomorphic.*

Proof. From Theorem 6.5.1 we know that there exist exactly 2 quotients $X/\langle(\alpha_j)_X\rangle$ of X which belong again to $\mathcal{F}^A_{\mathfrak{D}_4}$. This result can be re-read as follows: X is the $\mathbb{Z}/2\mathbb{Z}$-free quotient of other two manifolds $Y_1, Y_2 \in \mathcal{F}^A_{\mathfrak{D}_4}$ which making the following diagram

$$\begin{array}{ccccc} A'_1 & & A' & & A'_2 \\ {\scriptstyle 16:1}\downarrow & \searrow & \downarrow & \swarrow & \downarrow{\scriptstyle 16:1} \\ Y_1 & \xrightarrow[2:1]{} & X & \xleftarrow[2:1]{} & Y_2 \end{array}$$

where $A'_j = E_{\nu_j} \times E_{\nu_j} \times E_{\nu'_j}$, with $\nu_j, \nu'_j \in \mathfrak{h}$, is the H_j-étale cover of Y_j with $|H_j| = 16$ for $j = 1, 2$, according to Remark 6.1.5. Moreover, by Theorem 6.5.1 we have $X = Y_1/\langle(\beta_1)_{Y_1}\rangle$ and $X = Y_2/\langle(\beta_2)_{Y_2}\rangle$ where β_j's are translations on A'_j respectively by the points $(0, 0, \frac{\nu'_1}{2})$ and $(0, 0, \frac{\nu'_2+1}{2})$. The commutativity of the diagram implies that $A' = A'_j/\langle\beta_j\rangle$ for $j = 1, 2$ which lead to:

$$\tau = \nu_1 = \nu_2 \qquad 2\tau' = \nu'_1 \qquad 2\tau' = \nu'_2 + 1.$$

In particular, we see that $\nu'_2 = \mathfrak{T}(\nu'_1)$ with $\mathfrak{T} \in \mathrm{SL}_2(\mathbb{Z})$ as in Proposition 2.7.8. Thus by Proposition 2.7.6 we have $E_{\nu'_1} \simeq E_{\nu'_2}$. Consequently, $Y_1 \simeq Y_2$ and this proves that the two $(\mathbb{Z}/2\mathbb{Z})$-étale covers of X which belong to $\mathcal{F}^A_{\mathfrak{D}_4}$ are isomorphic. □

Remark 6.7.2. We denote by $Y_{\mu,\mu'} \in \mathcal{F}^A_{\mathfrak{D}_4}$ the manifold whose the order 16-étale Galois cover is $E_\mu \times E_\mu \times E_{\mu'}$.

Theorem 6.7.3. *There exists a map*

$$f: \quad \mathcal{F}^A_{\mathfrak{D}_4} \xrightarrow{2:1} \mathcal{F}^A_{\mathfrak{D}_4}$$

$$Y_{\mu,\mu'} \longmapsto Y_{\mu,2\mu'}$$

where $Y_{\mu,\mu'} \xrightarrow{2:1} Y_{\mu,2\mu'}$ is a double cover. In particular f has degree two, indeed the preimage of $Y_{\mu,\mu'} \in \mathcal{F}^A_{\mathfrak{D}_4}$ are its two étale quotients $Y_{\mu,\frac{\mu'}{2}}$ and $Y_{\mu,\frac{\mu'}{2}+\frac{1}{2}}$ in $\mathcal{F}^A_{\mathfrak{D}_4}$.

Proof. Let $Y_{\mu,\mu'} \in \mathcal{F}^A_{\mathfrak{D}_4}$. By Lemma 6.7.1 the map f is well-defined and the image of $Y_{\mu,\mu'}$ is $Y_{\mu,2\mu'}$. By Theorem 6.5.1 there exist exactly two quotients of $Y_{\mu,\mu'}$ of degree 2 which belong to $\mathcal{F}^A_{\mathfrak{D}_4}$ which are $Y_{\mu,\frac{\mu'}{2}}$ and $Y_{\mu,\frac{\mu'+1}{2}}$. Thus f is a degree two map. □

Remark 6.7.4. It is worth noting that the existence of exactly two quotients of X which belong again to $\mathcal{F}^A_{\mathfrak{D}_4}$ tells us that there exist others two constructions for an abelian 3-folds with a free action of $\mathfrak{D}_4$: they are $(E_\mu \times E_\mu \times E_{\mu'})/\mathrm{T}_j$ where $\mathrm{T}_j = \langle w, \gamma_j \rangle$ and γ_j are the translation by the point $(0, 0, \frac{\mu'+j}{2})$ for $j = 0, 1$. This result was already stated in [45, Theorem 2.7], but we observe that there is an error in the computation of γ_j.

6.8 | Action of the automorphisms group on fibration of Calabi-Yau threefolds in $\mathcal{F}^A_{\mathfrak{D}_4}$

In this section we analyze the action of the automorphism group of X on the fibrations that we have described in Section 6.2.1.

Let $g\colon Y_1 \to Y_2$ be a fibration between two complex manifolds Y_i. Then $\alpha_1 \in \mathrm{Aut}(Y_1)$ preserves g if there exists an automorphism α_2 of the base Y_2 making the following diagram commutative:

$$\begin{array}{ccc} Y_1 & \xrightarrow{f} & Y_2 \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ Y_1 & \xrightarrow[f]{} & Y_2 \end{array}$$

Definition 6.8.1. If $\alpha_2 \neq id$ we say that α_1 **acts on the base**, otherwise we say that α_1 **is the identity on the base**. Let F be a fiber of f. We call F an **invariant fiber** (under the action of α_1) if $\alpha_1(F) = F$ but $F \notin \mathrm{Fix}(\alpha_1)$. We call F a **fixed fiber** if $F \in \mathrm{Fix}(\alpha_1)$.

We also recall that: if $y \in \mathrm{Fix}(\alpha_2)$, the fiber over y is preserved by α_1; so if $\alpha_2 = id$, all the fibers are preserved by α_1. Furthermore, one can easily show that $f(\mathrm{Fix}(\alpha_1)) \subseteq \mathrm{Fix}(\alpha_2)$.

It is easy to see that both $\varphi_{|D_{X,1}|}$ and $\varphi_{|D_{X,2}|}$ are preserved under the action of $\mathrm{Aut}(X)$. We recall that in Table 6.1 we have summarized the action of each α_X on X. We denote by $I^j(t_1, t_2) = \coprod_{2p=t_1, 2q=t_2} C^j_{p,q}$ where $C^j_{p,q}$ are defined in (6.5.1) for $j = 1, 2$.

We have described the fibrations $\varphi_{|D_{X,j}|}$ in Theorem 6.2.1.

The following two theorems describe the action of α_X of the fibration $\varphi_{|D_{X,1}|}$ and $\varphi_{|D_{X,2}|}$.

Theorem 6.8.2. *Let us consider the fibration $\varphi_{|D_{X,1}|} : X \longrightarrow Z$ and $\alpha_X \in \mathrm{Aut}(X)$ and by α_Z the automorphism induced by α_X on the base Z. Let us denote by $\alpha_{A'}(\underline{z}) = (z_1 + t_1, z_2 + t_2, z_3 + t_3)$ a lift of α_X to A'. Then:*

(i) *If $t_1 = t_2 = 0$, α_X is the identity on the base of $\varphi_{|D_{X,1}|}$ and α_X acts as a translation on all the fibers over $z \in Z \setminus \varphi_{|D_{X,1}|}(\mathrm{Fix}(\alpha_X))$, while fixes the ones over z in $\varphi_{|D_{X,1}|}(\mathrm{Fix}(\alpha_X))$.*

(ii) *In all the other cases, α_X acts as an involution on the base and it has non trivial fixed locus. The action of α_X on the fibers is the following:*

- *It identifies the fibers over $\{z, \alpha_Z(z)\} \in Z \setminus \mathrm{Fix}(\alpha_Z)$.*
- *It acts by translation on the fibers over $z \in \mathrm{Fix}(\alpha_Z) \setminus \varphi_{|D_{X,1}|}(\mathrm{Fix}(\alpha_X))$.*
- *It is the hyperelliptic involution (see Definition 2.7.1) on the elliptic fibers over $\varphi_{|D_{X,1}|}(\mathrm{Fix}(\alpha_X) \setminus (\pi_H(I_1(t_1,t_2) \cup I_2(t_1,t_2))))$.*
- *It fixes the fibers over $z \in \varphi_{|D_{X,1}|}(\pi_H(I_1(t_1,t_2) \cup I_2(t_1,t_2)))$.*

Proof. (i) The fibration $\varphi_{|D_{X,1}|}$ is induced by the projection of A' over the first two coordinates (see proof of Theorem 6.2.1). Since $\alpha_{A'}$ acts trivially on these coordinates, α_X is the identity on the base Z. In particular all the fibers of $\varphi_{|D_{X,1}|}$ are preserved by α_X. Since α_X is a translation on the third coordinates, it acts as a translation on the fibers over $z \in Z \setminus \varphi_{|D_{X,1}|}(\mathrm{Fix}(\alpha_X))$. If $z \in \varphi_{|D_{X,1}|}(\mathrm{Fix}(\alpha_X))$ then α_X fixes the fiber over z, since according to Table 6.1 these fibers belong to the fixed locus $\mathrm{Fix}(\alpha_X)$, if it is not empty.

(ii) In this case α_X acts as an involution on Z and we have $\emptyset \neq \varphi_{|D_{X,1}|}(\mathrm{Fix}(\alpha_X)) \subsetneq \mathrm{Fix}(\alpha_Z)$. As we have observed before, any fiber over $z \in \mathrm{Fix}(\alpha_Z)$ is preserved by α_X. In particular the action α_X on it is one of the following:

- If $z \in \mathrm{Fix}(\alpha_Z) \setminus \varphi_{|D_{X,1}|}(\mathrm{Fix}(\alpha_X))$, then $\varphi^{-1}_{|D_{X,1}|}(z) \notin \mathrm{Fix}(\alpha_X)$ is an invariant fiber and α_X acts on it by translation.
- Let $z \in \varphi_{|D_{X,1}|}(\mathrm{Fix}(\alpha_X) \setminus \pi_H(I^1(t_1,t_2) \cup I^2(t_1,t_2)))$. According to Table 6.1, the fiber $\varphi^{-1}_{|D_{X,1}|}(z)$ is not fixed by α_X. Indeed $\mathrm{Fix}(\alpha_X) \setminus \pi_H(I^1(t_1,t_2) \cup I^2(t_1,t_2))$ consists of horizontal curves for $\varphi_{|D_{X,1}|}$ and we have that $\varphi^{-1}_{|D_{X,1}|}(z)$

intersects one of these curves. Thus α_X is an involution of the elliptic curve $\varphi^{-1}_{|D_{X,1}|}(z) \simeq E'$ with non trivial fixed locus: by Riemann-Hurtwiz formula for curves, it fixes 4 points on it.

- Let $z \in \varphi_{|D_{X,1}|}(\pi_H(I_1(t_1,t_2) \cup I_2(t_1,t_2)))$. According to Table 6.1 we have $\varphi_{|D_{X,1}|}$ is fixed by α_X.

□

Theorem 6.8.3. *Let us consider the fibration* $\varphi_{|D_{X,2}|} : X \longrightarrow \mathbb{P}^1$ *and* $\alpha_X \in \mathrm{Aut}(X)$ *and by* $\alpha_{\mathbb{P}^1}$ *the automorphism induced by* α_X *on* $\mathbb{P}^1$*. We denote by* $\alpha_{A'}(\underline{z}) = (z_1+t_1, z_2+t_2, z_3+t_3)$ *a lift of* α_X *on* A'*. Then:*

(i) *If* $t_3 \in \{0, \frac{1}{2}\}$ *then* α_X *is the identity on the base of* $\varphi_{|D_{X,2}|}$ *and the action on the fibers is the following:*

- *It acts as a translation on the fibers over* $z \in \mathbb{P}^1 \setminus \varphi_{|D_{X,2}|}(\mathrm{Fix}(\alpha_X))$.
- *It is the Kummer involution on the fiber over* $z \in \varphi_{|D_{X,2}|}(\pi_H(I_1(t_1,t_2) \cup I_2(t_1,t_2)))$.
- *It fixes an elliptic curve in the fiber over* $z \in \varphi_{|D_{X,2}|}(\mathrm{Fix}(\alpha_X) \setminus \pi_H(I_1(t_1,t_2) \cup I_2(t_1,t_2)))$*. More precisely, this elliptic curve belongs to* $\mathrm{Fix}(\alpha_X) \setminus \pi_H(I_1(t_1,t_2) \cup I_2(t_1,t_2))$ *and it is mapped to* z *under* $\varphi_{|D_{X,2}|}$.

(ii) *In all the other cases* α_X *acts as an involution on the base of* $\varphi_{|D_{X,2}|}$ *and it fixes two points on it. The action of* α_X *on the fibers is the following:*

- *It identifies the fibers over* $\{z, \alpha_{\mathbb{P}^1}(z)\} \in \mathbb{P}^1 \setminus \mathrm{Fix}(\alpha_{\mathbb{P}^1})$.
- *If* $\mathrm{Fix}(\alpha_X) = \emptyset$, α_x *acts as translation on the fibers over* $z \in \mathrm{Fix}(\alpha_{\mathbb{P}^1})$.
- *If* $\mathrm{Fix}(\alpha_X) \neq \emptyset$, α_X *fixes an elliptic curve in the fiber over a point in* $\varphi_{|D_{X,2}|}(\mathrm{Fix}(\alpha_X))$*. More precisely, this elliptic curve belongs to* $\mathrm{Fix}(\alpha_X) \setminus \pi_H(I_1(t_1,t_2) \cup I_2(t_1,t_2))$ *and it is mapped to* z *under* $\varphi_{|D_{X,2}|}$.

Proof. (i) Arguments similar to the ones in the proof of Proposition 6.8.2 show that $\alpha_{\mathbb{P}^1}$ is the identity on $\mathbb{P}^1$. Thus all the fibers are preserved by α_X. The action of α_X on the fibers is the following:

- It acts by translation on the fibers over $z \in \mathbb{P}^1 \setminus \varphi_{|D_{X,2}|}(\mathrm{Fix}(\alpha_X))$.
- Let $z \in \varphi_{|D_{X,2}|}(\pi_H(I^1(t_1,t_2) \cup I^2(t_1,t_2)))$. By Table 6.1, we know that the curves in $\pi_H(I^1(t_1,t_2) \cup I^2(t_1,t_2))$ are horizontal for $\varphi_{|D_{X,2}|}$. Hence, the fiber $\varphi^{-1}_{|D_{X,2}|}(z)$ over z intersect transversely the curve on which lies the point y such

that $z = \varphi_{|D_{X,2}|}(y)$. Therefore α_X acts on this fiber fixing the intersection points: since the fiber is an abelian surface and α_X defines an involution on it which preserves its period and has non trivial fixed locus, then it fixes 16 points.

- Let $z \in \varphi_{|D_{X,2}|}(\mathrm{Fix}(\alpha_X) \setminus \pi_H(I^1(t_1,t_2) \cup I^2(t_1,t_2)))$. According to Table 6.1 there is a curve in $\mathrm{Fix}(\alpha_X) \setminus \pi_H(I^1(t_1,t_2) \cup I^2(t_1,t_2))$ mapped to z. The fiber over z is a surface and so it cannot be fixed by α_X. Thus, we obtain that α_X fixes this elliptic curve in the fiber.

(ii) In this case $\alpha_{\mathbb{P}^1}$ acts as an involution on $\mathbb{P}^1$ and, according to the Riemann-Hurtwiz formula for curves, an involution of $\mathbb{P}^1$ has two fixed points; moreover $\varphi_{|D_{X,2}|}(\mathrm{Fix}(\alpha_X))$ is contained in $\mathrm{Fix}(\alpha_{\mathbb{P}^1})$.

- If $z \in \mathbb{P}^1 \setminus \mathrm{Fix}(\alpha_Z)$, then α_X identifies the fibers over z and $\alpha_Z(z)$.
- If $\mathrm{Fix}(\alpha_X) = \emptyset$, all fibers over $z \in \mathrm{Fix}(\alpha_{\mathbb{P}^1})$ are invariants.
- If $\mathrm{Fix}(\alpha_X) \neq \emptyset$, one can check that $\varphi_{|D_{X,2}|}(\mathrm{Fix}(\alpha_X))$ contains exactly 2 points, therefore $\varphi_{|D_{X,2}|}(\mathrm{Fix}(\alpha_X)) = \mathrm{Fix}(\alpha_Z)$. Let $z \in \mathrm{Fix}(\alpha_Z)$, according to Table 6.1 and an argument similar to the one above we show that α_X fixes this elliptic curve in the fiber.

□

Since every α_X preserves the fibrations $\varphi_{|D_{X,j}|}$ for $j = 1, 2$, these fibrations induce two fibrations on the desingularization Y of the quotient X/α_X as follows:

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi_{|D_{X,1}|}} & Z \\
\downarrow & & \downarrow \\
X/\langle\alpha_X\rangle & \xrightarrow{\varphi_1} & Z/\langle\alpha_Z\rangle \\
\beta_1 \uparrow & \nearrow_{\phi_1} & \\
Y & &
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\varphi_{|D_{X,2}|}} & \mathbb{P}^1 \\
\downarrow & & \downarrow \\
X/\langle\alpha_X\rangle & \xrightarrow{\varphi_2} & \mathbb{P}^1/\langle\alpha_{\mathbb{P}^1}\rangle \\
\beta_2 \uparrow & \nearrow_{\phi_2} & \\
Y & &
\end{array}
.
$$

From Theorem 6.8.2 and 6.8.3, we deduce directly the description of the fibrations ϕ_1 and ϕ_2.

1. For the fibration $\phi_1 \colon Y \longrightarrow Z/\langle\alpha_Z\rangle$:

   (i) If α_X is the identity on Z, the base $Z/\langle\alpha_X\rangle$ is the normal Enriques surface Z.

- If $\mathrm{Fix}(\alpha_X) = \emptyset$, $Y = X/\alpha_X \in \mathcal{F}^A_{\mathfrak{D}_4}$ and we obtain the same result of Theorem 6.3.2: the fibration $\varphi_1 = \phi_1$ is over Z and the general fibers is the elliptic curve $E'/\langle\alpha_X\rangle$.
- If $\mathrm{Fix}(\alpha_X) \neq \emptyset$: the action of α_X on the fibers is either a translation or the identity. In the first case the corresponding fiber for ϕ_1 is the elliptic curve E'/α_X. In the second case the (elliptic) fibers are fixed by α_X, hence the fibration φ_1 have some fibers which are singularities of X/α_X, given by $E'/\langle\alpha_X\rangle$. The resolution β_1 blows up each of them introducing a $\mathbb{P}^1$-bundle on each of them. Therefore, the corresponding fibers of ϕ_1 are 2-dimensional and in particular are $\mathbb{P}^1$-bundles over $E'/\langle\alpha_X\rangle$.

(ii) If α_Z is not the identity, the base of ϕ_1 is the normal rational surface Z/α_Z and the general fiber is the elliptic curve E'. The action of α_X on the invariant fibers is either a translation or the hyperelliptic involution or the identity. In the first case the corresponding fiber for ϕ_1 is the elliptic curve $E'/\langle\alpha_X\rangle$, in the second case the corresponding fiber for ϕ_1 is isomorphic to an elliptic fiber of type I_0^* (it is a curve given by 5 rational curves where 4 of them intersect at exactly one point the fifth curve which has multiplicity two) and in the third case is a $\mathbb{P}^1$-bundle over E'.

We obtain an equi-dimensional fibration of $X/\langle\alpha_X\rangle$ if and only if $\mathrm{Fix}(\alpha_X) = \emptyset$; in all the other case since β_1 blows up curves which are fibers of φ_1 we have that ϕ_1 has fibers with different dimensions.

2. For the fibration $\phi_2 \colon Y \longrightarrow \mathbb{P}^1/\langle\alpha_{\mathbb{P}^1}\rangle$:

(i) If $\alpha_{\mathbb{P}^1}$ is the identity, ϕ_2 is an isotrivial fibration whose base is $\mathbb{P}^1$ and the general fiber is isomorphic to $B/\langle\alpha_X\rangle$ which is an abelian surface since by Theorem 6.8.3 α_X generically acts as a translation. We have that α_X acts on the invariant fibers either by fixing 16 points or fixing an elliptic curve on them. In the first case the corresponding fiber for ϕ_2 is a Kummer surface and in the second case the corresponding fiber of ϕ_1 is reducible and the components are given by the strict transform of $F/\langle\alpha_X\rangle$ under β_2, where F is a fiber of $\varphi_{|D_{X,2}|}$, and the $\mathbb{P}^1$-bundles introduced by β_2 over the fixed curve.

(ii) Otherwise, the fibration ϕ_2 is an isotrivial fibration whose base is the rational curve $\mathbb{P}^1/\alpha_X$ and the general fiber of ϕ_2 is isomorphic to B. If $\mathrm{Fix}(\alpha_X) = \emptyset$, $X/\langle\alpha_X\rangle \in \mathcal{F}^A_{\mathfrak{D}_4}$ and we obtain results as those of Theorem 6.2.1. Otherwise α_X acts on the invariant fibers by fixing an elliptic curve on them, as above they become reducible fibers for ϕ_2.

6.9 | Relations with Calabi-Yau threefolds of type K

As we have observed in Section 5.5 there exist two types of Calabi-Yau threefolds with infinite fundamental group: the Calabi-Yau threefolds of type A and the one of type K (see Definition 5.5.2). In this section we highlight other relations between Calabi-Yau manifolds of type A and of type K, by studying manifolds in $\mathcal{F}^A_{\mathfrak{D}_4}$. In particular we prove: there exist quotients of $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ which are Calabi-Yau threefolds of type K and we present each $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ as finite cover of certain Calabi-Yau threefolds of type K.

6.9.1 | Calabi-Yau threefolds of type K quotients of X

In Corollary 6.6.11 we have given a description of the universal cover of each Calabi-Yau 3-folds Y obtained as crepant resolution of X/Υ as varying $\Upsilon \in \mathrm{Aut}(X)$. Let us consider the Y whose universal cover is $\mathbb{C} \times S$ with S a K3 surface. In this case $\pi_1(Y)$ is characterized by the following exact sequence

$$0 \longrightarrow L_Y \longrightarrow \pi_1(Y) \longrightarrow G' \longrightarrow 0 \qquad (6.9.1)$$

where L_Y is a maximal rank-2 lattice in $\pi_1(Y)$ and G' is a finite group; so Y admits a G'-cover isomorphic to the product of a K3 surface and the elliptic curve $\mathbb{C}/L_Y$. Therefore there exist quotients of $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ which, up to a desingularization, are Calabi-Yau threefolds of type K.

Corollary 6.9.1. *Let X as in Definition 6.1.1. There are exactly two groups $\Upsilon \leq$ $\mathrm{Aut}(X)$ such that X/Υ admits as desingularization a Calabi-Yau 3-folds of type K with the group $(\mathbb{Z}/2\mathbb{Z})^2$. Moreover, the followings hold.*

(i) *If $\Upsilon = \langle \alpha_X \rangle$ is induced by $\alpha_{A'}(\underline{z}) = (z_1, z_2, z_3 + \frac{1}{2}) \in \mathrm{Aut}(A')$, the $(\mathbb{Z}/2\mathbb{Z})^2$-étale cover is isomorphic to $Km_2(B) \times \frac{E'}{(r_2)^2}$ and r_2 is defined in (6.1.4).*

(ii) *If $\Upsilon = \langle \alpha_X, \beta_X \rangle$ is induced by $\alpha_{A'}(\underline{z}) = (z_1, z_2, z_3 + \frac{1}{2})$ and $\beta_{A'}(\underline{z}) = (z_1, z_2, z_3 + \frac{\tau'}{2})$ in $\mathrm{Aut}(A')$, the $(\mathbb{Z}/2\mathbb{Z})^2$-étale cover is isomorphic to $Km_2(B) \times E'$.*

Proof. By looking at the classification of the desingularizations Y of quotients X/Υ with $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ and $\Upsilon \leq \mathrm{Aut}(X)$, see Section 6.10, there are only two groups Υ_1 and Υ_2 such that $\pi_1(Y)$ fits in the exact sequence (6.9.1) and we find $G' \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Thus there are exactly two Y's that are Calabi-Yau 3-folds of type K with the group $(\mathbb{Z}/2\mathbb{Z})^2$. In particular, these groups are $\Upsilon_1 = \langle \alpha_X \rangle$ with α_X induced by $\alpha_{A'}(\underline{z}) = (z_1, z_2, z_3 +$

$\frac{1}{2}) \in \mathrm{Aut}(A')$ and $\Upsilon_2 = \langle \alpha_X, \beta_X \rangle$ induced by $\alpha_{A'}(\underline{z}) = (z_1, z_2, z_3 + \frac{1}{2})$ and $\beta_{A'}(\underline{z}) = (z_1, z_2, z_3 + \frac{\tau'}{2})$.

(i) If $\Upsilon = \Upsilon_1$. We denote by α_A the automorphism induced by $\alpha_{A'}$ on $A = A'/w$ and we consider the group $\Gamma := \langle G, \alpha_A \rangle \leq \mathrm{Aut}(A)$ and $Y \longrightarrow A/\Gamma$ is the crepant resolution. By Remark 6.5.2 $\alpha_A r^2$ is the only element with non empty fixed locus on A. We obtain the following diagram:

$$\begin{array}{ccc}
B \times E' = A & \xrightarrow{2:1} & A/\langle \alpha_A \rangle = B \times E'' \\
{\scriptstyle 16:1}\downarrow & & \downarrow{\scriptstyle 2:1} \\
A/\Gamma & \xleftarrow[4:1]{g} & (B \times E'')/r^2 \\
\uparrow & & \uparrow \\
Y & \xleftarrow[4:1]{f} & Km_2(B) \times E''
\end{array}$$

We have: $\langle \alpha_A \rangle \trianglelefteq \Gamma$, $(\alpha_A)_{|B} = id$ and $(\alpha_A)_{|E'} = (r_2)^2$ is a translation, see (6.1.4). Therefore, we denote the elliptic curve $E'' := \dfrac{E'}{\langle (r_2)^2 \rangle}$ and we have $\dfrac{A}{\langle \alpha_A \rangle} = B \times E''$. The automorphism r^2 descends to an automorphism on $B \times E''$ which acts as $\mathrm{diag}(-1, -1)$ on the abelian surface B and as the identity on E'', hence $(B \times E'')/\langle r^2 \rangle$ is birational to $Km_2(B) \times E''$. It is easy to check that the action of the group $\Gamma/\langle \alpha_A, r^2 \rangle$ defines a free action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $(B \times E'')/r^2$. Moreover it preserves the singular locus of $(B \times E'')/r^2$, thus it lifts to a free action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $Km_2(B) \times E''$ which does not contain any element (id, t) with t a translation on E''. Therefore, since $\pi_1(E'')$ is the maximal rank-2 in $\pi_1(Y)$ then the quotient map f is the $(\mathbb{Z}/2\mathbb{Z})^2$-étale cover of Y.

(ii) If $\Upsilon = \Upsilon_2$. We denote by α_A and β_A the automorphisms induced on A by $\alpha_{A'}$ and $\beta_{A'}$, respectively. We denote the group $\Gamma := \langle G, \alpha_A, \beta_A \rangle \leq \mathrm{Aut}(A)$: by Remark 6.5.2, $\alpha_A r^2$ is the only element with non empty fixed locus and we denote by Y the Calabi-Yau 3-fold which is a desingularization of A/Γ. We have $\langle \alpha_A, \beta_A \rangle \trianglelefteq \Gamma$ we consider the quotient $A/\langle \alpha_A, \beta_A \rangle$: since α_A and β_A act only on E' and $\langle \alpha_A, \beta_A \rangle \simeq E'[2]$, we have $A/\langle \alpha_A, \beta_A \rangle \simeq A$. The automorphism r^2 descends to an automorphism on $A/\langle \alpha_A, \beta_A \rangle \simeq B \times E'$ which acts as $\mathrm{diag}(-1, -1)$ on B and as the identity on E'. We obtain the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{4:1} & A/\langle \alpha_A, \beta_A \rangle \simeq B \times E' \\ {\scriptstyle 32:1}\downarrow & & \downarrow{\scriptstyle 2:1} \\ A/\Gamma & \xleftarrow[4:1]{g} & (B \times E')/\langle r^2 \rangle \\ \uparrow & & \uparrow \\ Y & \xleftarrow[4:1]{f} & Km_2(B) \times E' \end{array}$$

As above, the action of the group $\Gamma/\langle \alpha_A, \beta_A, r^2 \rangle$ defines a free action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $Km_2(B) \times E'$ which does not contain any element (id, t) with t a translation on E'. Therefore, since $\pi_1(E')$ is the maximal rank-2 lattice in $\pi_1(Y)$ then the quotient f is the $(\mathbb{Z}/2\mathbb{Z})^2$-étale cover of Y.

□

6.9.2 | Calabi-Yau threefolds in $\mathcal{F}^{A}_{\mathfrak{D}_4}$ as cover of Calabi-Yau threefolds of type K

Let $X \in \mathcal{F}^{A}_{\mathfrak{D}_4}$ be as in Definition 6.1.1. We recall that X is given as the quotient of $B \times E'$ by the free diagonal action of $G \simeq \mathfrak{D}_4$, see Remark 6.1.6. Beside the diagonal action of G, one can look at the quotient varieties obtained by considering the actions of subgroups of G and $G_1 \times G_2$, see Section 6.1. With the notation of Section 6.1, we obtain the following diagram:

$$\begin{array}{ccc} A = (E \times E \times E')/\langle w \rangle & \xrightarrow{\simeq} & ((E \times E)/\langle w_1 \rangle) \times E' = B \times E' \simeq A \\ {\scriptstyle 4:1}\downarrow & & \downarrow{\scriptstyle 16:1} \\ A'/\langle r, w \rangle & \xrightarrow{4:1} & B/\langle r_1 \rangle \times E'/\langle r_2 \rangle \\ {\scriptstyle 2:1}\downarrow & & \downarrow{\scriptstyle 2:1} \\ X = A'/\langle r, s \rangle & \xrightarrow{4:1} & (B/\langle r_1 \rangle \times E'/\langle r_2 \rangle)/\langle s \rangle \end{array} \qquad (6.9.2)$$

On the left we are considering the action of G induced on A and on the right we are looking at the action of subgroups of G_1, G_2 and $G_1 \times G_2$ on B and E'.

Proposition 6.9.2. *With the notation above. The manifold $(B/r_1 \times E'/r_2)/s$ admits a desingularization Y which is a Calabi-Yau 3-fold of type K with the group $\mathbb{Z}/2\mathbb{Z}$. The manifold X is a degree 4 cover of Y branched along exceptional divisors introduced on Y.*

Proof. Let us look at the diagram (6.9.2). From the proof of Theorem 6.3.2 part (i) we have that $B/\langle r_1\rangle$ is birational to the K3 surface $S' = Km_4(B)$. The resolution introduces 6 rational curves C_i for $i = 1, \dots, 6$ and 4 curves F_j whose components are 3 rational curves where two of them intersect the third one in one point for $j = 1, \dots, 4$. The involution s_1 lifts to an Enriques involution on S since it preserves $\mathrm{Sing}(B/\langle r_1\rangle)$, while $E''' := E'/\langle r_2\rangle$ is an elliptic curve with the action of s_2 such that $E'''/\langle s_2\rangle \simeq \mathbb{P}^1$. In particular, we recall $s = s_1 \times s_2$. Thus, we have that $Y = \dfrac{S' \times E'''}{\langle s\rangle}$ is a Calabi-Yau 3-fold of type K. We can see that threefold $(B/\langle r_1\rangle \times E'/\langle r_2\rangle)/\langle s\rangle$ can be obtained as the quotient of $\mathbb{C}^3$ by the action $\Gamma = \langle \Lambda, r_1 \times id, id \times r_2, s\rangle$ and that Γ defines a discontinuous action on it (the proof is similar to the one of Lemma 6.6.7). Therefore we can use Theorem 6.6.6 and Theorem 6.6.8 to compute $\pi_1(Y)$. The explicit computations show that $\pi_1(E''')$ is exactly the maximal rank-2 lattice in $\pi_1(Y)$ and so, according to Definition 5.5.6, Y is a Calabi-Yau 3-fold of type K with the group $(\mathbb{Z}/2\mathbb{Z})$. We rewrite the diagram (6.9.2) using smooth varieties:

$$\begin{array}{ccccc} & & A = B \times E' & & \\ & \swarrow^{q_1} & & \searrow^{q_2} & \\ A/r & & \xrightarrow{q_3} & & S' \times E''' \\ \downarrow q_4 & & & & \downarrow q_5 \\ X = \dfrac{A}{\langle s\rangle} & & \xrightarrow{q_6} & & Y = \dfrac{S' \times E'''}{\langle s\rangle} \end{array} \qquad (6.9.3)$$

In diagram 6.9.3 the Galois coverings q_1, q_4, q_5 are étale morphisms. The morphism q_6 is finite of degree 4, since it is induced by the morphism q_3 which is finite of degree 4. By construction the exceptional divisors on $S \times E''$ define the branch locus of q_3. Since q_5 is étale then the image under q_5 of $\mathrm{Branch}(q_3)$ defines the branch locus of q_6: it consists of the exceptional divisors on $\dfrac{S' \times E''}{\langle s\rangle}$. The exceptional divisors are the surfaces $q_5(C_i \times E'')$ and $q_5(F_j \times E'')$ for $i = 1, 2, 3$ and $j = 1, 2$. □

6.10 | Tables of results

Let Y be the Calabi-Yau 3-fold desingularization of X/Υ with $X \in \mathcal{F}^A_{\mathfrak{D}_4}$ and $\Upsilon \leq \mathrm{Aut}(X)$. We denote by α_X and β_X and γ_X the generators of Υ (depending on the cardinality of Υ) and, as usual, by $\alpha_{A'}$ and $\beta_{A'}$ and $\gamma_{A'}$ their lifts on A', respectively, which are translations by a point (t_1, t_2, t_3) of order two satisfying (6.4.2). We define the following sub-lattices of $\Lambda_3 = \mathbb{Z} \oplus \tau'\mathbb{Z}$: $\Lambda_3' := \mathbb{Z} \oplus \frac{\tau'}{2}\mathbb{Z}$, $\Lambda_3'' := \mathbb{Z} \oplus \frac{\tau'+1}{2}\mathbb{Z}$ and $\Lambda_3''' := \frac{1}{2}\mathbb{Z} \oplus \tau'\mathbb{Z}$. We recall $\Lambda_1 = \Lambda_2 = \mathbb{Z} \oplus \tau\mathbb{Z}$.

$\alpha_{A'}$	$h^{1,1}(Y) = h^{2,1}(Y)$	$\pi_1(Y)$
$(0,0,\frac{\tau'}{2})$	2	$0 \to \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3' \to \pi_1(Y) \to \mathfrak{D}_4 \to 0$
$(0,0,\frac{\tau'+1}{2})$	2	$0 \to \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3'' \to \pi_1(Y) \to \mathfrak{D}_4 \to 0$
$(0,0,\frac{1}{2})$	7	$0 \to \Lambda_3''' \to \pi_1(Y) \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 0$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{1}{2})$	6	$\mathbb{Z}/2\mathbb{Z}$
$(\frac{\tau}{2},\frac{\tau}{2},\neq\frac{1}{2})$	4	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$
$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{1}{2})$	8	$\{0\}$
$(\frac{\tau}{2},\frac{\tau+1}{2},\neq\frac{1}{2})$	6	$\mathbb{Z}/2\mathbb{Z}$
$(0,\frac{1}{2},\frac{1}{2})$	6	$\mathbb{Z}/2\mathbb{Z}$
$(0,\frac{1}{2},\neq\frac{1}{2})$	4	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Table 6.2: Hodge numbers and fundamental group of crepant resolution Y of $X/(\mathbb{Z}/2\mathbb{Z})$

$\alpha_{A'}$	$\beta_{A'}$	$h^{1,1}(Y) = h^{2,1}(Y)$	$\pi_1(Y)$
$(0,0,\frac{\tau'}{2})$	$(0,0,\frac{\tau'+1}{2})$	7	$0 \to \Lambda_3 \to \pi_1(Y) \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 0$
$(0,0,\frac{\tau'}{2})$	$(\frac{\tau}{2},\frac{\tau}{2},\frac{1}{2})$	6	$\mathbb{Z}/2\mathbb{Z}$
$(0,0,\frac{\tau'}{2})$	$(\frac{\tau}{2},\frac{\tau}{2},0)$	4	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$
$(0,0,\frac{\tau'}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{1}{2})$	8	$\{0\}$
$(0,0,\frac{\tau'}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},0)$	6	$\mathbb{Z}/2\mathbb{Z}$
$(0,0,\frac{\tau'}{2})$	$(0,\frac{1}{2},\frac{1}{2})$	6	$\mathbb{Z}/2\mathbb{Z}$
$(0,0,\frac{\tau'}{2})$	$(0,\frac{1}{2},0)$	4	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$(0,0,\frac{\tau'+1}{2})$	$(\frac{\tau}{2},\frac{\tau}{2},\frac{1}{2})$	6	$\mathbb{Z}/2\mathbb{Z}$
$(0,0,\frac{\tau'+1}{2})$	$(\frac{\tau}{2},\frac{\tau}{2},0)$	4	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$
$(0,0,\frac{\tau'+1}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{1}{2})$	8	$\{0\}$
$(0,0,\frac{\tau'+1}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},0)$	6	$\mathbb{Z}/2\mathbb{Z}$
$(0,0,\frac{\tau'+1}{2})$	$(0,\frac{1}{2},\frac{1}{2})$	6	$\mathbb{Z}/2\mathbb{Z}$
$(0,0,\frac{\tau'+1}{2})$	$(0,\frac{1}{2},0)$	4	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$(0,0,\frac{1}{2})$	$(\frac{\tau}{2},\frac{\tau}{2},\frac{1}{2})$	12	$\mathbb{Z}/2\mathbb{Z}$
$(0,0,\frac{1}{2})$	$(\frac{\tau}{2},\frac{\tau}{2},\frac{\tau'}{2})$	10	$\mathbb{Z}/2\mathbb{Z}$
$(0,0,\frac{1}{2})$	$(0,\frac{1}{2},\frac{1}{2})$	12	$\mathbb{Z}/2\mathbb{Z}$
$(0,0,\frac{1}{2})$	$(0,\frac{1}{2},\frac{\tau'}{2})$	10	$\mathbb{Z}/2\mathbb{Z}$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{1}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{1}{2})$	13	$\{0\}$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{1}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{\tau'}{2})$	10	$\{0\}$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{1}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{\tau'+1}{2})$	10	$\{0\}$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{1}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},0)$	14	$\{0\}$
$(\frac{\tau}{2},\frac{\tau}{2},0)$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{1}{2})$	13	$\{0\}$
$(\frac{\tau}{2},\frac{\tau}{2},0)$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{\tau'}{2})$	8	$\mathbb{Z}/2\mathbb{Z}$
$(\frac{\tau}{2},\frac{\tau}{2},0)$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{1+\tau'}{2})$	8	$\mathbb{Z}/2\mathbb{Z}$
$(\frac{\tau}{2},\frac{\tau}{2},0)$	$(\frac{\tau}{2},\frac{\tau+1}{2},0)$	10	$\mathbb{Z}/2\mathbb{Z}$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{\tau'}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{\tau'}{2})$	8	$\mathbb{Z}/2\mathbb{Z}$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{\tau'}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{1+\tau'}{2})$	10	$\mathbb{Z}/2\mathbb{Z}$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{\tau'}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},0)$	6	$\mathbb{Z}/2\mathbb{Z}$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{\tau'+1}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{1}{2})$	7	$\{0\}$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{\tau'+1}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{\tau'}{2})$	10	$\mathbb{Z}/2\mathbb{Z}$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{\tau'+1}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},\frac{1+\tau'}{2})$	8	$\mathbb{Z}/2\mathbb{Z}$
$(\frac{\tau}{2},\frac{\tau}{2},\frac{\tau'+1}{2})$	$(\frac{\tau}{2},\frac{\tau+1}{2},0)$	6	$\mathbb{Z}/2\mathbb{Z}$

Table 6.3: Hodge numbers and fundamental group of crepant resolution Y of $X/(\mathbb{Z}/2\mathbb{Z})^2$

$\alpha_{A'}$	$\beta_{A'}$	$\gamma_{A'}$	$h^{1,1}(Y) = h^{2,1}(Y)$	$\pi_1(Y)$
$(0, 0, \frac{\tau'}{2})$	$(0, 0, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, \frac{1}{2})$	12	$\mathbb{Z}/2\mathbb{Z}$
$(0, 0, \frac{\tau'}{2})$	$(0, 0, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau+1}{2}, \frac{1}{2})$	17	$\{0\}$
$(0, 0, \frac{\tau'}{2})$	$(0, 0, \frac{1}{2})$	$(0, \frac{1}{2}, \frac{1}{2})$	12	$\mathbb{Z}/2\mathbb{Z}$
$(0, 0, \frac{\tau'}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau+1}{2}, \frac{1}{2})$	13	$\{0\}$
$(0, 0, \frac{\tau'}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau+1}{2}, 0)$	14	$\{0\}$
$(0, 0, \frac{\tau'}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, 0)$	$(\frac{\tau}{2}, \frac{\tau+1}{2}, \frac{1}{2})$	13	$\{0\}$
$(0, 0, \frac{\tau'}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, 0)$	$(\frac{\tau}{2}, \frac{\tau+1}{2}, 0)$	9	$\mathbb{Z}/2\mathbb{Z}$
$(0, 0, \frac{\tau'+1}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau+1}{2}, \frac{1}{2})$	13	$\{0\}$
$(0, 0, \frac{\tau'+1}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau+1}{2}, 0)$	14	$\{0\}$
$(0, 0, \frac{\tau'+1}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, 0)$	$(\frac{\tau}{2}, \frac{\tau+1}{2}, \frac{1}{2})$	13	$\{0\}$
$(0, 0, \frac{\tau'+1}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, 0)$	$(\frac{\tau}{2}, \frac{\tau+1}{2}, 0)$	10	$\mathbb{Z}/2\mathbb{Z}$
$(0, 0, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau+1}{2}, \frac{1}{2})$	27	$\{0\}$
$(0, 0, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau+1}{2}, \frac{\tau'}{2})$	20	$\{0\}$
$(0, 0, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, \frac{\tau'}{2})$	$(0, \frac{1}{2}, 0)$	18	$\{0\}$
$(0, 0, \frac{1}{2})$	$(\frac{\tau}{2}, \frac{\tau}{2}, \frac{\tau'}{2})$	$(0, \frac{1}{2}, \frac{\tau'}{2})$	15	$\{0\}$

Table 6.4: Hodge numbers and fundamental group of crepant resolution Y of $X/(\mathbb{Z}/2\mathbb{Z})^3$

$\Upsilon = \mathrm{Aut}(X)$	$h^{1,1}(Y) = h^{2,1}(Y)$	$\pi_1(Y)$
	27	$\{0\}$

Table 6.5: Hodge number and fundamental group of crepant resolution Y of $X/(\mathbb{Z}/2\mathbb{Z})^4$

7 The family of Calabi-Yau threefolds of type A with the group $(\mathbb{Z}/2\mathbb{Z})^2$

In this chapter we consider the Calabi-Yau threefolds of type A with the group $(\mathbb{Z}/2\mathbb{Z})^2$. Proceeding similarly to the previous one, we aim to study their geometry. In particular, Theorems B part (i) (as Theorem 7.3.1) and C part (i) (as Theorem 7.4.1) are proven, see also [66].

7.1 | The Calabi-Yau 3-folds of type A with $(\mathbb{Z}/2\mathbb{Z})^2$

We consider the abelian 3-fold $A := E_1 \times E_2 \times E_3$ where $E_j := \mathbb{C}/(\mathbb{Z} \oplus \tau_j\mathbb{Z})$ is an elliptic curve with $\tau_j \in \mathfrak{h}$, for $j = 1, 2, 3$ and the automorphisms:

$$a(\underline{z}) = (-z_1, -z_2, z_3 + u_3) \qquad b(\underline{z}) = (-z_1 + u_1, z_2 + u_2, -z_3) \tag{7.1.1}$$

where $u_j \in E_j[2] \setminus \{0\}$. One can easily prove that $G = \langle a, b\rangle$ defines a free action of $(\mathbb{Z}/2\mathbb{Z})^2$ on A, see also [73, Theorem 0.1]. According to [73, Theorem 0.1], the faithful representation of G is in fact unique. We denote $\rho\colon G \longrightarrow \mathrm{GL}(3, \mathbb{C})$ and $\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$ splits into the direct sum of three irreducible 1-dimensional representations.

Definition 7.1.1. With the notation above and fixed $u_j \in E_j[2] \setminus \{0\}$ for $j = 1, 2, 3$, we define the Calabi-Yau 3-fold $X := A/G$ of type A with the group $(\mathbb{Z}/2\mathbb{Z})^2$.

Using (7.1.1) one can prove the following relations:

$$H^{1,1}(A)^G = \langle dz_j \wedge d\overline{z_j}\rangle_{j=1,2,3} \qquad H^{2,1}(A)^G = \langle dz_j \wedge dz_k \wedge d\overline{z_j}\rangle_{i\neq k\neq j=1,2,3} \tag{7.1.2}$$

hence $h^{1,1}(X) = h^{2,1}(X) = 3$, see also Lemma 5.6.5.

Theorem 7.1.2. *There exists a* 3*-dimensional family of* $(\mathbb{Z}/2\mathbb{Z})^2$*-equivariant complex structures on a* 3*-dimensional complex torus* T*. The family of Calabi-Yau* 3*-folds of type* A *with the group* $(\mathbb{Z}/2\mathbb{Z})^2$ *is irreducible and* 3*-dimensional.*

Proof. To give a $(\mathbb{Z}/2\mathbb{Z})^2$-equivariant complex structure on $T = \mathbb{C}^3/\Lambda$ means to give a decomposition of type $\Lambda \otimes \mathbb{C} = V \oplus \overline{V}$ into $(\mathbb{Z}/2\mathbb{Z})^2$-invariant subspaces, see Remark 4.2.4. By [73, Theorem 0.1] we have a unique $(\mathbb{Z}/2\mathbb{Z})^2$-decomposition on $\mathbb{C}^3$ given by $V = V_{\chi_1} \oplus V_{\chi_2} \oplus V_{\chi_3}$ where χ_j are the irreducible characters corresponding to the irreducible representations ρ_j. Hence we obtain:

$$\Lambda \otimes \mathbb{C} = V \oplus \overline{V} = \bigoplus_{j=1}^{3} V_{\chi_j} \otimes M_{\chi_j}$$

where $V_{\overline{\chi_j}} = V_{\chi_j}$ since χ_j are real characters and $\dim_{\mathbb{C}} M_{\chi_j} = 2$. By Remark 4.2.3 the parameters of the $(\mathbb{Z}/2\mathbb{Z})^2$-Hodge decomposition are given by the choice of a 1 dimensional subspace $M^{1,0}_{\chi_j}$ of M_{χ_j} for $j = 1, 2, 3$. Hence we have a 3-dimensional family of $(\mathbb{Z}/2\mathbb{Z})^2$-equivariant complex structures on T. Let X be a Calabi-Yau 3-fold of type A with the group $(\mathbb{Z}/2\mathbb{Z})^2$: by Corollary 5.3.3, $H^{2,1}(X)$ parametrizes its local deformation. We have $H^{2,1}(X)$ is irreducible and has dimension 3, see (7.1.2). We conclude that the family of Calabi-Yau 3-folds of type A with the group $(\mathbb{Z}/2\mathbb{Z})^2$ is irreducible and 3-dimensional. □

Remark 7.1.3. We also deduce that the space that parametrizes the Calabi-Yau 3-folds of type A with the group $(\mathbb{Z}/2\mathbb{Z})^2$ is isomorphic to $(\mathcal{M}_{1,1})^3$ where $\mathcal{M}_{1,1}$ is defined in Proposition 2.7.6.

Definition 7.1.4. We denote the family constructed in Theorem 7.1.2 by $\mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$.

7.2 | The Picard group and fibrations on $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$

In this section we describe the Picard group of $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$: we produce a $\mathbb{Q}$-basis of $\mathrm{Pic}_{\mathbb{Q}}(X)$ and study the maps associated to its generators. Some of the results are enclosed in [73].

Theorem 7.2.1. *Let $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$ as above. The followings hold:*

(i) *The Picard group satisfies* $\mathrm{Pic}(X) \simeq \mathbb{Z}^2 \oplus \pi_1(X)$ *with $\pi_1(X)$ is the finite extension of $\pi_1(A)$ by the group $G = \langle a, b\rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$ defined in* (7.1.1).

(ii) *The free-torsion part* $\mathrm{Pic}_{\mathbb{Q}}(X)$ *is generated by the classes of three divisors $D_{X,1}$, $D_{X,2}$ and $D_{X,3}$ which are the fiber classes of the following fibrations*

$$\varphi_1 : X \longrightarrow \frac{E_1}{G_{|E_1}} \qquad \varphi_2 : X \longrightarrow \frac{E_2}{G_{|E_2}} \qquad \varphi_3 : X \longrightarrow \frac{E_3}{G_{|E_3}}.$$

The base $E_j/G_{|E_j}$ of each fibration φ_j is a rational curve and the general fiber is the abelian surface $E_i \times E_k$ for $i, k \neq j$.

(iii) *There are other three isotrivial fibrations on X given by:*

$$\varphi_{1,2} : X \longrightarrow \frac{E_1 \times E_2}{G_{|E_1\times E_2}} \quad \varphi_{1,3} : X \longrightarrow \frac{E_1 \times E_3}{G_{|E_1\times E_3}} \quad \varphi_{2,3} : X \longrightarrow \frac{E_2 \times E_3}{G_{|E_2\times E_3}}.$$

The general fiber of $\varphi_{i,j}$ is isomorphic to the elliptic curve E_k for $k \neq i, j = 1, 2, 3$ and the base is a normal Enriques surface whose singularities are of type $8A_1$.

Proof. (i) According to Lemma (5.2.2) we have

$$\text{Pic}(X) = \mathbb{Z}^{h^{1,1}(X)} \oplus \text{Ab}(\pi_1(X)).$$

Since $h^{1,1}(X) = 3$ and the fundamental group $\pi_1(X)$ is abelian, we obtain the result.

(ii) Let us consider the maps $\varphi_j \colon X \longrightarrow \frac{E_j}{G_{|E_j}}$. By looking at the action of G, it is easy to show that φ_j's are fibrations over a rational curve and the general fiber is the abelian surface $E_i \times E_k$ for $i \neq k \neq j = 1, 2, 3$. Let us denote by $D_{X,j}$ the fiber class of φ_j for $j = 1, 2, 3$. According to [73, Claim 2.20 pag. 61], $D_{X,j}$'s are linearly independent and, since $\rho(X) = h^{1,1}(X) = 3$ by (7.1.2), we obtain that $\{D_{X,1}, D_{X,2}, D_{X,3}\}$ defines a $\mathbb{Q}$-basis of the Picard group of X.

(iii) We see that the three projections $A \longrightarrow E_j \times E_i$ induce three fibrations on X given by $\varphi_{i,j} : X \longrightarrow \frac{E_i \times E_j}{G_{|E_i\times E_j}}$ for $i \neq j = 1, 2, 3$. A more in depth study shows that any $G_{|E_i\times E_j} \simeq (\mathbb{Z}/2\mathbb{Z})^2$ contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})$ which acts on $E_i \times E_j$ with non trivial fixed locus and it preserves the volume form of $E_i \times E_j$, hence it fixes 16 points on it. Thus we get a singular K3 surface whose singularities are of type $16A_1$. One can check that the remaining action of $G_{|E_i\times E_j}$ is a fixed point free involution on the singular $K3$ surface. Hence $\frac{E_i \times E_j}{G_{|E_i\times E_j}}$ is a singular Enriques surface with $8A_1$ singularities. Since the fibers of the projections $A \longrightarrow E_j \times E_i$ are isomorphic to E_k, the general fibers of $\varphi_{i,j}$ is still E_k with $k \neq i, j = 1, 2, 3$. We also refer to [73, Remark 2.22].

□

7.3 | Automorphism group of manifolds in $\mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$

In this section we compute the automorphism group of $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$.

Since we aim to study the general element X in the family $\mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$, we compute $\text{Aut}(X)$ under the additionally assumption that E_i's are not isogenous to each other.

Theorem 7.3.1. *The automorphism group* $\mathrm{Aut}(X) \simeq (\mathbb{Z}/2\mathbb{Z})^7$ *and its elements are induced by automorphisms on A whose linear part is in* $\langle \mathrm{diag}(-1,1,1)\rangle$ *and the translation part is any translation of order two.*

Proof. By Corollary 4.1.12 we have $\mathrm{Aut}(X) \simeq \dfrac{\mathrm{N}_{\mathrm{Aut}(A)}(G)}{G}$. Let $\alpha_A \in \mathrm{Aut}(A)$, as usual we write $\alpha_A(\underline{z}) = \eta(\alpha_A)(z) + t_{\alpha_A}$. Since $\alpha_A \in \mathrm{N}_{\mathrm{Aut}(A)}(G)$ then $\eta = \eta_1 \oplus \eta_2 \oplus \eta_3$. The condition $\alpha_A \in \mathrm{N}_{\mathrm{Aut}(A)}(G)$ leads also to:

$$\eta_j(\alpha_A) \in \langle -1 \rangle \qquad t_j \in E_j[2].$$

We obtain that $\mathrm{N}_{\mathrm{Aut}(A)}(G)$ is abelian and of cardinality $2^3 \cdot (2^2 \cdot 2^2 \cdot 2^2)$. Since each element of $\mathrm{N}_{\mathrm{Aut}(A)}(G)$ has order two, we deduce that $\mathrm{Aut}(X) \simeq (\mathbb{Z}/2\mathbb{Z})^7$. We remark that two automorphisms in $\mathrm{N}_{\mathrm{Aut}(A)}(G)$ induce the same automorphism on X if and only if they differ by an element in G. It is easy to check that every $\alpha_A \in \mathrm{N}_{\mathrm{Aut}(A)}(G)$ which preserves ω_A but it is not a translation differs from any translation of order two by an element in G. Thus every automorphism on X which preserves ω_X is induced by a translation of order two. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathrm{N}_{\mathrm{Aut}(A)}(G)$ as follows:

$$\alpha_1(\underline{z}) = \mathrm{diag}(-1,1,1) + t \qquad \alpha_2(\underline{z}) = \mathrm{diag}(1,-1,1) + t_a + t$$
$$\alpha_3(\underline{z}) = \mathrm{diag}(1,1,-1) + t_b + t \qquad \alpha_4(\underline{z}) = \mathrm{diag}(-1,-1,-1) + t_{ab} + t$$

with t any translation of order two on A and t_a,t_b, t_{ab} are the translation parts of a, b, ab respectively. One can check that α_2, α_3 and α_4 differ from α_1 by an element in G. Hence every automorphism on X which does not preserve ω_X is induced by an automorphism on A whose linear part is $\mathrm{diag}(-1,1,1)$. We conclude that an element $\alpha_X \in \mathrm{Aut}(X)$ is induced by an automorphism α_A such that the linear part is in $\langle \mathrm{diag}(-1,1,1)\rangle$ and the translation part is given by any translation of order 2. □

Remark 7.3.2. Let $\alpha_X \in \mathrm{Aut}(X)$ be induced by $\alpha_A \in \mathrm{Aut}(A)$ and $\pi\colon A \longrightarrow X$.

- Assume that α_A is a translation by the point $(t_1, t_2, t_3) \in A$ with $t_i \in E_i[2]$. An easy computation shows that the fixed locus is a finite set of elliptic curves given by $\bigcup_{id\neq g\in G} \pi(\mathrm{Fix}(\alpha_A g))$. In particular,

$$\pi(\mathrm{Fix}(\alpha_A a)) = \pi\Big(\{(p,q,l)\in A \mid 2p = t_1, 2q = t_2, l\in E_3\}\Big) \subset \mathrm{Fix}(\alpha_X) \Leftrightarrow t_3 = u_3 \tag{7.3.1}$$

$$\pi(\mathrm{Fix}(\alpha_A b)) = \pi\Big(\{(p,q,l)\in A \mid 2p = t_1+u_1, 2l = t_3, q\in E_2\}\Big) \subset \mathrm{Fix}(\alpha_X) \Leftrightarrow t_2 = u_2 \tag{7.3.2}$$

$$\pi(\mathrm{Fix}(\alpha_A ab)) = \pi\Big(\{(p,q,l)\in A \mid 2q = t_2+u_2, 2l = t_3+u_3, p\in E_1\}\Big) \subset \mathrm{Fix}(\alpha_X) \Leftrightarrow t_1 = u_1. \tag{7.3.3}$$

- Assume that $\alpha_A(\underline{z}) = (-z_1+t_1, z_2+t_2, z_3+t_3)$ with $t_i \in E_i[2]$. By explicit computations we have the followings. The 0-dimensional subset in $\mathrm{Fx}(\alpha_X)$ is given by $\pi(\mathrm{Fix}(\alpha_A ab)$ and it is never empty. In particular

$$\pi(\mathrm{Fix}(\alpha_A ab) = \pi\Big(\{(p,q,l)\in A \mid 2p = t_1+u_1, 2q = t_2+u_2, 2l = t_3+u_3\}\Big). \tag{7.3.4}$$

The 2-dimensional set in $\mathrm{Fix}(\alpha_X)$ is given by

$$\pi(\mathrm{Fix}(\alpha_A)) = \pi\Big(\{(p,q,l)\in A \mid 2p = t_1, (q,l)\in E_2\times E_3\}\Big) \neq \emptyset \Leftrightarrow t_2 = 0, t_3 = 0 \tag{7.3.5}$$

$$\pi(\mathrm{Fix}(\alpha_A a))\pi\Big(\{(p,q,l)\in A \mid 2q = t_2, (p,l)\in E_1\times E_3\}\Big) \neq \emptyset \Leftrightarrow t_1 = 0, t_3 = u_3 \tag{7.3.6}$$

$$\pi(\mathrm{Fix}(\alpha_A b))\pi\Big(\{(p,q,l)\in A \mid 2l = t_3, (p,q)\in E_1\times E_2\}\Big) \neq \emptyset \Leftrightarrow t_1 = u_1, t_2 = u_2 \tag{7.3.7}$$

and it consists of a finite numbers of abelian surfaces.

7.4 | Quotients of manifolds in $\mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$

We describe the quotient X/Υ for all possible $\Upsilon \le \mathrm{Aut}(X)$.

Theorem 7.4.1. *Let $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$ and $\Upsilon \le \mathrm{Aut}(X)$. Let $\beta : Y \to X/\Upsilon$ be the blow up of the singular locus of X/Υ. The followings hold.*

(i) *If Υ preserves the volume form of X, β is a crepant resolution and Y is a Calabi-Yau 3-fold. In particular, there are exactly $3^3 - 1$ automorphisms $(\alpha_j)_X$ which act freely on X. They are induced by the translations $\alpha_j \in \mathrm{Aut}(A)$ by the points (t_1, t_2, t_3) such that $t_i \in E_i[2]$ and $t_i \neq u_i$ for $i = 1, 2, 3$. Moreover, $X/\langle(\alpha_j)_X\rangle$ belong to $\mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$.*

(ii) *If Υ does not preserve the volume form of X, we have the following cases.*

1. *If there exists at least one element in Υ that fixes surfaces on X then Y has negative Kodaira dimension and $h^{j,0}(Y) = 0$ for $j > 0$.*
2. *Otherwise, Y has trivial Kodaira dimension, $K_Y \neq 0$ and $h^{j,0}(Y) = 0$ for $j > 0$.*

Proof. (i) Let us assume that Υ preserves the volume form of X. According to Theorem 7.3.1 $\Upsilon \simeq (\mathbb{Z}/2\mathbb{Z})^m$ with $1 \leq m \leq 6$, hence we split the quotient in a subsequent quotients of degree 2. Using similar argument to the one in the proof of Theorem 6.5.1, we conclude that $\beta : Y \longrightarrow X/\Upsilon$ is a crepant resolution and Y is a Calabi-Yau 3-fold. By Proposition 5.4.4, free actions on Calabi-Yau threefolds must preserve the volume form. Let α_A be a translation by the point (t_1, t_2, t_3) with $t_i \in E_i[2]$: the study of the equations $\alpha_A(\underline{z}) = g(\underline{z})$ with $\underline{z} \in A$ for every $g \in G$ lead to the conditions $t_i \neq u_i$ for every $i = 1, 2, 3$. Let $\alpha_A \in \mathrm{Aut}(A)$ be a translation on A of order two which induces a free action of α_X on X. Since α_X acts freely and preserves the volume form of X, by Proposition 5.4.4 $Y = \dfrac{X}{\langle\alpha_X\rangle}$ is a Calabi-Yau 3-fold. It remains to prove that $Y \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$. We consider $Y = \dfrac{A}{\langle G, \alpha_A\rangle}$: in the group $H := \langle G, \alpha_A\rangle$ there is one translation α_A, hence we construct $A/\langle\alpha_A\rangle$ which is an abelian 3-fold with the free action of $H/\langle\alpha_A\rangle$ and we have $Y = \dfrac{(A/\alpha_A)}{(H/\langle\alpha_A\rangle)}$. Since $(H/\langle\alpha_A\rangle) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and does not contain any translation it follows that $Y \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$.

(ii) In this case there exists at least one element α_X in Υ which does not preserve the volume form of X. We also observe that given two elements in Υ which does not preserve the volume form of X, their composition defines an element which preserves the volume form of X. Therefore we can split Υ in the direct product of two groups $\Upsilon_1 \times \Upsilon_2$ where $\Upsilon_1 \simeq \mathbb{Z}/2\mathbb{Z} = \langle\alpha_X\rangle$ whose generator does not preserve the volume form of X and $\Upsilon_2 \simeq (\mathbb{Z}/2\mathbb{Z})^{k-1}$ which preserves the volume form of X and $|\Upsilon| = 2^k$ for some $k = 1, \ldots, 7$.

1. Since there exist at least one elements α_X that fixes surfaces (codimension 1 submanifolds), according to Proposition 5.4.7, the quotient $X/\langle\alpha_X\rangle$ has negative Kodaira dimension and $h^{j,0}(X/\alpha_X) = 0$. Since the Kodaira dimension cannot increase under quotients and is a birational invariant then X/Υ admits a desingularization Y such that $k(Y) = -\infty$ and $h^{j,0}(Y) = 0$.

2. Since Υ does not fix surfaces we have that the automorphisms $\alpha_X\upsilon$'s fix only isolated points for every $\upsilon \in \Upsilon_2$ and Υ_2 either acts freely or fixes (elliptic) curves, see Proposition 5.4.6 and 5.4.7. We first consider the quotient by X/Υ_2 which, up to a desingularization, produces a Calabi-Yau 3-fold Z, by part (i). Since Υ is abelian and α_X preserves the fixed locus of Υ_2 then the action of $\Upsilon/\Upsilon_2 \simeq \langle\alpha_X\rangle$ on X/Υ_2 lifts to an action of an involution α_Z on Z. Moreover, α_Z does not preserve ω_Z since α_X does not preserve ω_X. By Proposition 5.4.7, α_Z fixes points or surfaces or both of them. We prove that α_Z fixes only points. Since the resolution $\gamma\colon Z \to X/\Upsilon_2$ is an isomorphism outside the blown up locus and α_X fixes only points on X/Υ_2, the only surfaces that (perhaps) α_Z could fix are the exceptional divisors. The exceptional divisors introduced by γ on Z are $\mathbb{P}^1$-bundles over the curves $C \in \mathrm{Sing}(X/\Upsilon_2)$. So if α_Z fixed an exceptional divisor E, then there should a curve C on X fixed by $\alpha_X\upsilon$ for some $\upsilon \in \Upsilon_2$. By hypothesis $\alpha_X\upsilon$ does not fix surfaces on X, therefore α_Z fixes only points on Z. Therefore, we end up with a Calabi-Yau threefold Z with an involution α_Z that does not preserve its volume forms and fixes only points. According to Proposition 5.4.7, the quotient $Z/\langle\alpha_Z\rangle$ admits a desingularization Y such that $k(Y) = 0$, K_Y is not trivial and $h^{j,0}(Y) = 0$ for $j > 0$.

□

7.5 | Hodge numbers of desingularizations of quotients of $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$

In this section we compute the Hodge numbers of Y as in Theorem 7.4.1.

Proposition 7.5.1. *Let $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$ and $\Upsilon \leq \mathrm{Aut}(X)$.*

(i) *If Υ preserves the volume form of X, then there exist $\beta\colon Y \longrightarrow X/\Upsilon$ a crepant resolution such that*

$$h^{1,1}(Y) = h^{2,1}(Y) = 3 + \sum_{id \neq \alpha_X \in \Upsilon} \left| \frac{\mathrm{Fix}(\alpha_X)}{\Upsilon} \right|.$$

In particular $e(Y) = 0$.

(ii) *If* $\Upsilon = \Upsilon_1 \times \Upsilon_2$ *(as in proof of Theorem 7.4.1) does not preserve the volume form of* X *and we denote by* α_A *a lift to* A *of the generator of* Υ_1*. There exist a resolution of singularities* $\beta\colon Y \longrightarrow X/\Upsilon$ *such that:*

$$h^{1,1}(Y) = 3 + \sum_{id \neq \upsilon \in \Upsilon_2} \left| \frac{\mathrm{Fix}(\upsilon)}{\Upsilon} \right| + |\Pi|$$
$$h^{2,1}(Y) = \sum_{F \in \mathrm{Sing}(X/\Upsilon)} g(F)$$

where g *denoted the genus of a curve and* Π *is the* 0*-dimensional subset fixed by* Υ_1 *on* X/Υ_2*. Furthermore,* β *is the blowing up of* $\mathrm{Sing}(X/\Upsilon)$ *which introduces exactly one irreducible divisor on each irreducible submanifold blown up in* X/Υ*.*

Proof. (i) If Υ preserves the volume form of X, by Theorem 7.4.1 β is a crepant resolution and so we use the orbifold cohomology formula (6.6.1) to compute the Hodge numbers of Y. We have $\mathrm{Fix}(\Upsilon) = \coprod_{\upsilon \in \Upsilon_A} \coprod_{g \in G} \pi(\mathrm{Fix}(\upsilon g))$ where Υ_A is a lift of Υ to A and $\pi : A \longrightarrow X$. Thanks to Remark 7.3.2 we kwon that $\mathrm{Fix}(\upsilon g)$ consists of a finite number of elliptic curves. Moreover, G acts on each $\mathrm{Fix}(\upsilon g)$ and since G acts freely on A it can either identify the elliptic curves in $\mathrm{Fix}(\upsilon g)$ or act on them as a translation. In any case, $\mathrm{Fix}(\Upsilon)/\Upsilon$ consists of a finite numbers of elliptic curves. We recall that each $\upsilon \in \Upsilon$ is induced by a (order two) translation on A. Applying the same proof of Proposition 6.6.4 we obtain that:

$$h^{1,1}(Y) = h^{2,1}(Y) = 3 + \sum_{id \neq \alpha_X \in \Upsilon} \left| \frac{\mathrm{Fix}(\alpha_X)}{\Upsilon} \right|.$$

In particular $e(Y) = 0$.

(ii) We have that Υ does no preserve the volume form of X. Following the proof of Theorem 7.4.1: we can write Υ as the direct product of the groups $\Upsilon_1 \times \Upsilon_2$ where $\Upsilon_1 = \langle \alpha_X \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ is cyclic of order 2 which does not preserve the volume form of X and $\Upsilon_2 \simeq (\mathbb{Z}/2\mathbb{Z})^{k-1}$ preserves the volume form of X where $|\Upsilon| = 2^k$ for $k = 1, \dots, 7$. We remark that the fixed locus of Υ_2, if not empty, consists of a finite number of elliptic curves and the one of $\alpha_X \upsilon$'s consists of finite numbers of isolated points and (possibly) smooth surfaces for every $\upsilon \in \Upsilon_2$, see Remark 7.3.2.

We consider the following commutative diagram:

$$
\begin{array}{ccccc}
 & & & & A \\
 & & & & \downarrow{\scriptstyle \pi} \\
 & & & & X \\
 & & & & \downarrow{\scriptstyle f} \\
\overline{\Upsilon_1} \circlearrowleft W & \xrightarrow{\delta} & \widetilde{\Upsilon_1} \circlearrowleft Z & \xrightarrow{\gamma} & X/\Upsilon_2 \circlearrowleft \Upsilon/\Upsilon_2 \\
{\scriptstyle 2:1}\downarrow{\scriptstyle q} & & & & {\scriptstyle p}\downarrow{\scriptstyle 2:1} \\
Y = W/\overline{\Upsilon_1} & & \xrightarrow[\beta]{\hspace{6em}} & & X/\Upsilon
\end{array}
\tag{7.5.1}
$$

Here $\gamma \colon Z \longrightarrow X/\Upsilon_2$ is the blow up of each singular curves in X/Υ_2 and Z a Calabi-Yau 3-fold, see Theorem 7.4.1 part (i). Since Υ is abelian then Υ_1 preserves the fixed locus of Υ_2. Therefore, the action of $\Upsilon_1 \simeq \Upsilon/\Upsilon_2$ on X/Υ_2 extends to an action of $\widetilde{\Upsilon_1}$ on Z. As we have observed in the proof of Theorem 7.4.1, the fixed locus of $\widetilde{\Upsilon_1}$ consists of isolated fixed points and (possibly) smooth surfaces: δ blow ups the 0-dimensional subset Π of $\mathrm{Fix}(\widetilde{\Upsilon_1})$. In particular, $\widetilde{\Upsilon_1}$ lifts to an action of $\overline{\Upsilon_1}$ on W. Finally, β is a composition of birational maps such that the diagram commutes. By construction Y is a smooth 3-fold birational to X/Υ.

We compute the Hodge numbers of Y by considering the morphisms q and $(\gamma \circ \delta)$. We have $H^{i,j}(Y) \simeq H^{i,j}(W)^{\overline{\Upsilon_1}}$ and by using the formula (1.7.2) we lead to:

$$H^{i,j}(Y) \simeq H^{i,j}(W)^{\overline{\Upsilon_1}} \simeq [H^{i,j}(Z) \oplus H^{i-1,j-1}(\Pi)]^{\widetilde{\Upsilon_1}} = H^{i,j}(Z)^{\widetilde{\Upsilon_1}} \oplus H^{i-1,j-1}(\Pi). \tag{7.5.2}$$

where the last equality follows since $\widetilde{\Upsilon_1}$ fixes Π. Since γ is a crepant resolution we use the formula (6.6.1) to describe the cohomology of Z:

$$
\begin{aligned}
H^{i,j}(Z) &\simeq H^{i,j}(X)^{\Upsilon_2} \oplus \bigoplus_{F \in \mathrm{Sing}(X/\Upsilon_2)} H^{i-1,j-1}(F) \\
&= H^{i,j}(X) \oplus \bigoplus_{F \in \mathrm{Sing}(X/\Upsilon_2)} H^{i-1,j-1}(F)
\end{aligned}
\tag{7.5.3}
$$

where according to item (i) F is an elliptic curve and the last equality follows since Υ_2 acts as the identity on the cohomology of X. By substituting (7.5.3) in (7.5.2) we obtain:

$$H^{i,j}(Y) \simeq [H^{i,j}(X) \oplus \bigoplus_{F \in \mathrm{Sing}(X/\Upsilon_2)} H^{i-1,j-1}(F)]^{\Upsilon_1} \oplus H^{i-1,j-1}(\Pi). \tag{7.5.4}$$

Using (7.1.2) and the expression of the generator of Υ_1 we obtain the followings: $H^{1,1}(X)^{\Upsilon_1} \simeq H^{1,1}(X)$ and $H^{2,1}(X)^{\Upsilon_1} = 0$. If $\Upsilon_1 \simeq \Upsilon/\Upsilon_2$ preserves the elliptic curve $F^i \in \mathrm{Sing}(X/\Upsilon_2)$, the following situations can appear: it fixes the curve, it acts by translation on it or it is the hyperelliptic involution on it. In the first two cases, $p(F^i)$ is an elliptic curve. In the last case, $p(F^i)$ is a rational curve. If $\Upsilon_1 \simeq \Upsilon/\Upsilon_2$ does not preserve on F^i, then F^i is mapped to F^j and so the quotient p identifies these two elliptic curves and $p(F^i)$ is an elliptic curve. Therefore, we obtain:

$$h^{1,1}(Y) = 3 + |\mathrm{Sing}(X/\Upsilon)| + |\Pi| = 3 + \sum_{id \neq \upsilon \in \Upsilon_2} \left| \frac{\mathrm{Fix}(\upsilon)}{\Upsilon} \right| + |\Pi|$$
$$h^{2,1}(Y) = \sum_{F \in \mathrm{Sing}(X/\Upsilon)} g(F)$$

where g denotes the genus of a curve.

Finally, we prove that Π is isomorphic, via γ, to the 0-dimensional subset fixed by Υ_1 on X/Υ_2. Since the exceptional divisors introduced by γ on Z are $\mathbb{P}^1$-bundles over the curves C in $\mathrm{Sing}(X/\Upsilon_2)$, we have to prove that whenever $\Upsilon_1 = \langle \alpha_X \rangle$ preserves a curve $C \in \mathrm{Sing}(X/\Upsilon_2)$ then $\widetilde{\Upsilon_1}$ does not fix isolated points on the fibers of the $\mathbb{P}^1$-bundle over C introduced by γ. We recall that C is an elliptic curve and so if α_X preserves C then it acts either as the identity, as a translation or as the hyperelliptic involution on it.

1. If Υ_1 fixes C, then there exists $\upsilon \in \Upsilon_2$ such that $\alpha_X \upsilon$ fixes $f^{-1}(C)$. Since $\alpha_X \upsilon$ is an involution of the Calabi-Yau 3-fold X which does not preserve ω_X, according to Proposition 5.4.7 it cannot fix curve. Therefore in this situation Υ_1 fixes a surface $S \subset X/\Upsilon_2$ that contains C. The blowing up γ introduces a $\mathbb{P}^1$-bundle E over C and $\widetilde{\Upsilon_1}$ preserves each fibers of E. Since the fibers are rational curve, $\widetilde{\Upsilon_1}$ can either act as the identity or fix two points on each of them. Then first situation cannot happen, otherwise we would find that E is fixed by $\widetilde{\Upsilon_1}$ and so we would have two surfaces E and S in $\mathrm{Fix}(\Upsilon_1)$ that intersects in C which is impossible since $\mathrm{Fix}(\Upsilon_1)$ is smooth. Thus, $\widetilde{\Upsilon_1}$ fixes two points on each fibers of E. In this case we prove that $\widetilde{\Upsilon_1}$ does not fix isolated points on the fibers of E.
2. If Υ_1 preserves C but acts as translation, then $\widetilde{\Upsilon_1}$ does not acts on the fibers of the $\mathbb{P}^1$-bundle over C introduced by γ.
3. If Υ_1 preserves C and fixes four points on it. Thus $\widetilde{\Upsilon_1}$ preserves fours fibers in the $\mathbb{P}^1$-bundle E introduced by γ over C. We prove that these fibers are fixed since they lie in a surface fixed by $\widetilde{\Upsilon_1}$ and so we deduce that $\widetilde{\Upsilon_1}$ does not fix isolated points on the fibers of E. To prove this we show that

Υ_1 fixes surfaces on X/Υ_2 that intersect C in the four points that it fixes on C. Let $\upsilon_X \in \Upsilon_2$ such that fixes the curve $C' = f^{-1}(C)$ and we write $\upsilon_A = (z_1+t_1, z_2+t_2, z_3+t_3)$ a lift of υ_X to $A = E_1 \times E_2 \times E_3$ with $t_i \in E_i[2]$. According to Remark 7.3.2, υ_X fixes a curve C' if and only if $t_i = u_i$ for at least one $i = 1, 2, 3$. We are in the case when α_X fixes four points on C'. We write $\alpha_A = (-z_1 + l_1, z_2 + l_2, z_3, l_3)$ a lift of α_X on A with $l_i \in E_i[2]$. We prove that under this hypothesis then $\alpha_X\upsilon_X$ fixes surfaces that intersect C' in the points fixed by α_X on C'. Assume that $t_1 = u_1$ then by Remark 7.3.2 we have that $C' = \pi(E^1_{q,l})$ where $E^1_{q,l}$ is an elliptic curve on A over the point (q, l) such that $2q = t_1 + u_2$ and $2l = t_3 + u_3$. By using (7.3.4), we see that the isolated fixed points by α_X lie on C' if and only

$$l_2 = t_2 \quad l_3 = t_3. \tag{7.5.5}$$

It is now an easy check to see that $\alpha_A\upsilon_A$ satisfies the condition (7.3.5) which tells us that $\alpha_X\upsilon_X$ fixes surfaces on X. In particular, by construction these surfaces intersect C' in the points which α_X fixes on C. Hence the four points fixed by Υ_1 on C, in fact lie on surfaces fixed by Υ_1. Therefore, $\widetilde{\Upsilon_1}$ fixes surfaces that contain the fibers of E that it preserves and so it fixes these fibers. Similar computations can be checked for the cases when $t_i = u_i$ for $i = 2, 3$. In the end, we prove that $\widetilde{\Upsilon_1}$ does not fix isolated points on the fibers of E.

In conclusion, we prove that Π is isomorphic, via γ, to the 0-dimensional subset fixed by Υ_1 on X/Υ_2. We deduce that implies that β is the blowing up of $\mathrm{Sing}(X/\Upsilon)$ which introduces exactly one irreducible divisor on each irreducible submanifold blown up in X/Υ.

□

We make some explicit examples.

Example 6. We consider $\Upsilon_2 = \langle \upsilon_2 \rangle$ where υ_X is induced by $\upsilon_A(\underline{z}) = (z_1 + t_1, z_2 + t_2, z_3 + u_3)$ in $\mathrm{Aut}(A)$ with $t_i \in E_i[2]$. According to Remark 7.3.2 we have

$$\mathrm{Fix}(\alpha_X) = \pi(\mathrm{Fix}(\alpha_A a)) = \pi\Big(\{(p, q, l) \in A \mid 2p = t_1, 2q = t_2, l \in E_3\}\Big)$$

which consists of 16 elliptic curves. We denote by $E^1_{p,q}$ a curve in $\mathrm{Fix}(\alpha_A a)$. An easy check shows that a acts on them as identity if and only if $t_1 = t_2 = 0$. Therefore:
$$|\mathrm{Fix}(\alpha_X)| = \begin{cases} 4 & t_1 = t_2 = 0 \\ 8 & \text{otherwise} \end{cases}$$

The argument used in the examples above can be applied to every cyclic subgroup in $\mathrm{Aut}(X)$ and we obtain the following result: let $X \in \mathcal{F}^A_{(\mathbb{Z}/2\mathbb{Z})^2}$ and $\Upsilon = \Upsilon_1 \times \Upsilon_2 \in \mathrm{Aut}(X)$ where

- Υ_1 is either id or generated by α_X induced by $\alpha_A(\underline{z}) = (-z_1 + l_2, z_2 + l_2, z_3 + l_3)$ with $l_i \in E_i[2]$;
- Υ_2 is induced by $\Upsilon_A \in \mathrm{Aut}(A)$ subgroup of translation: $\upsilon_A \in \Upsilon_A$ is a translation by the point (t_1, t_2, t_3) with $t_1 \in E_i[2]$.

Let $\beta\colon Y \longrightarrow X/\Upsilon$ as in Theorem 7.4.1.

Υ_1	Υ_2	$h^{1,1}(Y)$	$h^{2,1}(Y)$
id	$(\neq u_1, \neq u_2, \neq u_3)$	3	3
id	$(\neq \{u_1, 0\}, \neq \{u_2, 0\}, u_3)$	7	7
id	$(\neq \{u_1, 0\}, u_2, \neq \{u_3, 0\})$	7	7
id	$(u_1, \neq \{u_2, 0\}, \neq \{u_3, 0\})$	7	7
id	$(\neq u_1, u_2, u_3)$	11	11
id	$(u_1, \neq u_2, u_3)$	11	11
id	$(u_1, u_2, \neq u_3)$	11	11
id	$(0, 0, u_3)$	11	11
id	$(0, u_2, 0)$	11	11
id	$(u_1, 0, 0)$	11	11
id	(u_1, u_2, u_3)	15	15
$\forall l_i$	id	19	0

Table 7.1: Hodge numbers of Y if $\Upsilon \simeq (\mathbb{Z}/2\mathbb{Z})^2$

We compute the Hodge number of Y in two cases where Υ is non cyclic.

Example 7. We consider $\Upsilon_2 = \langle \upsilon_X \rangle$ where υ_X is induced by $\upsilon_A(\underline{z}) = (z_1 + u_1, z_2, z_3)$ and $\Upsilon_1 = \langle \alpha_X \rangle$ where α_X is induced by $\alpha_A(\underline{z}) = (-z_1, z_2 + u_2, z_3 + u_3)$. According to Remark 7.3.2:

$$\mathrm{Fix}(\Upsilon_X) = \pi(\mathrm{Fix}(\alpha_A ab)) = \pi\Big(\{(p, q, l) \in A \mid p \in E_1, 2q = u_2, 2l = u_3\}\Big).$$

Let $E^1_{q,l}$ be a curve in $\mathrm{Fix}(\alpha_A ab)$, some easy computation shows that ab acts has the identity on it. Hence $\mathrm{Fix}(\Upsilon_X)$ consists of 8 elliptic curves. Moreover α_X preserves each of these elliptic curve and it fixes 4 points on each of them given. In particular, we observe that $\alpha_X \upsilon_X$ fixes the surfaces $\pi(\{(p, q, l) \in A \mid p \in E_1, q \in E_2, 2l = u_3\}$ and one can check that the points fixes by α_X on the curves $\pi(E^1_{p,q})$ lies on the surfaces fixed by

$\alpha_X \upsilon_X$. Therefore $\dfrac{\mathrm{Fix}(\Upsilon_2)}{\Upsilon}$ consists of 8 rational curve. Finally we compute Π. Following the proof of Theorem 7.5.1 one can easily see that Π is the image under $X \longrightarrow X/\Upsilon_2$ of all isolated points fixed by $\alpha_X \upsilon_X$ and α_X that don't lie on a curve fixed by Υ_2. Indeed, as we shows in the proof of Theorem 7.5.1, if p is an isolated points fixed by α_X and lies on a curves fixed by Υ then p lies on a surface fixed by $\alpha_X \upsilon_X$ and so in the quotient X/Υ_2 the image of p is no longer an isolated fixed point. One can easily check that the isolated points fixed by $\alpha_X \upsilon_X$ and α_X don't lie on the curves fixed by Υ_2 and now it is easy to check that $|\Pi| = 16$. Thus, we obtain the following Hodge numbers:

$$h^{1,1}(Y) = 3 + 8 + 16 = 27 \qquad h^{2,1}(Y) = 0$$

Example 8. We consider $\Upsilon_2 = \langle \upsilon_X \rangle$ with υ_X induced by $\upsilon_A(\underline{z}) = (z_1 + u_1, z_2, z_3)$ and $\Upsilon_1 = \langle \alpha_X \rangle$ with α_X induced by $\alpha_A(\underline{z}) = (-z_1, z_2 + t_2, z_3 + t_3)$ with $t_i \in E_i[2] \setminus \{u_i\}$. The fixed locus $\mathrm{Fix}(\Upsilon_2)/\Upsilon_2$ is as in the previous example. We see that α_X preserve the elliptic curves in $\mathrm{Fix}(\Upsilon_2)/\Upsilon_2$ if and only if $t_2 = t_3 = 0$ and in fact it fixes them. Otherwise, α_X acts on $\mathrm{Fix}(\Upsilon_2)/\Upsilon_2$ by mapping curves to curves. As in the previous example, we compute Π. In this case we see that the all points fixed by $\alpha_X \upsilon_X$ and α_X lie on the curves fixed by Υ_2 if and only if $t_2 = t_3 = 0$. Thus

$$\pi = \begin{cases} 0 & t_2 = t_3 = 0 \\ 16 & t_2 = t_3 = 0 \end{cases} \tag{7.5.6}$$

Therefore:

$$\mathrm{Fix}(\Upsilon_2)/\Upsilon = \begin{cases} 8 & t_2 = t_3 = 0 \\ 4 & \text{otherwise} \end{cases}$$

and it consists of elliptic curves. Thus we obtain:

$$h^{1,1}(Y) = \begin{cases} 11 & t_2 = t_3 = 0 \\ 23 & \text{otherwise} \end{cases} \qquad h^{2,1}(Y) = \begin{cases} 8 & t_2 = t_3 = 0 \\ 4 & \text{otherwise.} \end{cases}$$

Part II

The Morrison-Kawamata cone conjecture for hyperelliptic varieties

A volte, dovresti provare a guardare le cose da un'altra prospettiva.
Potresti trovare un punto di vista decisamente migliore.

8 Summary of Part II

One of the way to understand the (birational) geometry of a projective variety Y is to study the nef cone $\mathrm{Nef}(Y)$ and the movable cone $\overline{\mathrm{Mov}}(Y)$, especially from the point of view of the Minimal Model Program (MMP for short). As we have briefly explain in the Introduction, it is difficult to describe these cones. For K-trivial manifolds, we expected that the nef and the movable cones are rational polyhedral up to the action of $\mathrm{Aut}(Y)$ and $\mathrm{Bir}(Y)$, respectively (see Section 9.4). This prediction is known as Morrison-Kawamata cone conjecture, see Conjecture 9.4.2 and Conjecture 9.4.3. Even if the cone conjecture is proven in numerous cases, see Section 9.4, it remains still nowadays an active area of research. In this par of the thesis, we establish the Morrison-Kawamata cone conjecture for the hyperelliptic varieties. This result is a joint work with Ana Quedo and it is available as *preperint* on Arxiv [65].

Chapter 9. We introduce the Morrison-Kawamata cone conjecture, giving also a gentle introduction to MMP.

Chapter 10. We introduce the so-called *reduction theory* which will be one of the main tool that we use to prove the cone conjecture for hyperelliptic varieties.

Chapter 11. We establish the validity of the cone conjecture for hyperelliptic varieties. More precisely:

Theorem D (Corollary 11.1.3 and Theorem 11.2.1)**.** Let $Y = X/G$ be a hyperelliptic variety. Then, the Morrison-Kawamata cone conjecture (see Conjecture 9.4.2 and 9.4.3) holds for the nef and movable cones, in both its formulations. In particular, the followings equality hold: $\overline{\mathrm{Mov}}(Y)^+ = \overline{\mathrm{Mov}}(Y)^e = \overline{\mathrm{Mov}}(Y) = \mathrm{Nef}(Y) = \mathrm{Nef}(Y)^e = \mathrm{Nef}(Y)^+$.

Chapter 12 We conclude this second part with two final investigations on the nef cone of hyperelliptic varieties, specifically of Calabi-Yau manifolds of type A. In [73, Theorem 0.1 (IV)], the author have proven that the Calabi-Yau threefolds of type A, studied din Part I, have rational polyhedral nef cones. Inspiring by this result, we investigate more on the rational polyhedral nature of hyperelliptic varieties. More precisely, we obtain the following result.

Theorem E (see Theorem 12.1.2 and Corollary 12.1.4)**.** Let $Y = A/G$ be a hyperelliptic variety and we denote by η the representation of G. We assume that A is not of CM-type. If G contains a normal abelian group H such that $\eta_{|H}$ does not contain two equals irreducible sub-representations, then the nef cone of Y is a polyhedral cone. In addition, if $h^{1,0}(Y) = 0$ then $\mathrm{Aut}(Y)$ is finite.

Furthermore, we deduce that all the Calabi-Yau manifold of type A given as in Theorem A have a rational polyhedral nef cone, see Corollary 12.1.5. Finally, we describe the extremal rays of the nef cone for these Calabi-Yau manifolds of type A, deducing in particular that all nef divisors of X are semi-ample divisors.

Theorem F (see Corollary12.2.3)**.** Let X be the Calabi-Yau manifold of type A as in Theorem D. Then extremal rays are of the nef cone are given by semi-ample divisors which define fibrations on X induced by natural projections on A. In particular all nef divisors are semi-ample divisors.

9 The Morrison-Kawamata Cone Conjecture

In this chapter we introduce the Morrison-Kawamata cone conjecture. To do this, we give a briefly introduction to the Minimal Model Program which emphasizes the importance of the study of the nef and movable cones. We also give an introduction to the formalism of convex geometry.

9.1 | A gentle introduction to Minimal Model Program

The Minimal Model Program (MMP for short) is a (non deterministic) algorithm that birationally transforms a mildly singular projective variety to one satisfying certain positivity conditions. We will not go into any details except only what will be needed for the next chapters. However a good reference for this theory is [54] and [63].

The ultimate goal of the Minimal Model Program (or the Mori Program) is to classify projective varieties up to birational morphisms. In dimension 1, two projective curves are birationally equivalent if and only if they are isomorphic. Therefore, there is an unique smooth projective model in each birational classes and we classify them via the genus $g(Y) = h^0(Y, \omega_Y)$ of a smooth projective curve Y. In dimension 2, the situation is not so simple since birational morphisms are more complicated. Indeed, birational morphisms between surfaces are composition of finite number of blow-ups and their inverses, see [11, Theorem II.11]. It turns out that any smooth projective surface can be obtained from a distinguished representative in its birational class by a sequence of blow-ups and blow downs.

Definition 9.1.1. A **(-1)-curve** on a smooth surface is a smooth rational curves with self-intersection -1.

The Castelnuovo contraction theorem [11, Theorem II.17] says that any (-1)-curve E on a smooth surface S' can be contracted and the target space is again a smooth surface. Repeating this process for any (-1)-curves, we end up with a smooth surface S with no (-1)-curves birational to S'. This output is the distinguished representative surface in the birational class of S'. Therefore, the classical MMP for surface works in the following way. We ask if a given smooth surface S have (-1)-curves: if no the algorithm stop, otherwise we contract them. The process must stop after a finite number of steps because the Picard number, which is a positive integer, drops by one every time we contract a (-1)-curve, see [11, Remark II.13.1].
When we move to the higher dimensional case, the assertion to ask "if the surface S contains (-1)-curves" does not have a generalization. Therefore, we need a slight change of perspective. We notice that (-1)-curves have negative intersection with the canonical divisor. Therefore, a natural way to generalize the MMP for surfaces in higher dimension is to ask whenever there exists curves that intersect negatively the canonical divisor and contract them. Indeed Mori proved the following result.

Theorem 9.1.2 (Mori's theorem)**.** [54, Theorem 1.13] *Let Y be a smooth projective variety and H an ample divisor on Y. Assume that there is an irreducible curve $C \subset Y$ such that $-(K_Y \cdot C) > 0$. Then there is a rational curve $E \subset X$ which, in particular, satisfies the followings:*

$$0 < -(K_Y \cdot E) \leq \dim(Y) + 1 \qquad \frac{-(E \cdot K_Y)}{(E \cdot H)} \geq \frac{-(C \cdot K_Y)}{(C \cdot H)}.$$

The theorem above suggests in fact that the presence of rational curves are related to the failure of the canonical divisor to be nef. Thus, it is should be now clear the reason of the following definition of minimal model.

Definition 9.1.3. A **minimal model of Y** is a variety Y' birational to Y such that $K_{Y'}$ is nef.

The reader need to pay attention that the definition above is not equivalent to the one classically given for surfaces. Indeed it coincides only for surfaces with positive Kodaira dimension.
According to Definition 9.1.3, it is clear that the criterior to find the minimal model is to ask "if the canonical divisors is nef". To proceed in our algorithm, it is important to study the curves that intersect negatively the canonical divisor and more generally the cone of curves.

Definition 9.1.4. Let V be a finite real vector space. A subset $C \subset V \setminus \{0\}$ is a **(convex) cone** if it is closed under addition and multiplication by positive scalar.

We recall the real vector space $N_1(Y)_{\mathbb{R}} = \Big(Z_1(Y)/\equiv\Big) \otimes \mathbb{R}$ where $Z_1(Y)$ is the group of 1-cycle, see Definition 1.2.7.

Definition 9.1.5. We define the **cone of curves** or **Mori cone** as follows

$$\mathrm{NE}(Y) = \{\sum n_i[C_i] \mid n_i \geq 0, C_i \in Z_1(Y)\} \subset N_1(Y)_{\mathbb{R}}.$$

Theorem 9.1.6 (Cone Theorem)**.** [63, Theorem 7-2-1] *Let Y be a normal projective variety. Then there are countably many rays Γ_l such that $K_Y \cdot \Gamma_l < 0$ and*

$$\overline{\mathrm{NE}}(Y) = \overline{\mathrm{NE}}(Y)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq}[\Gamma_l] \tag{9.1.1}$$

where $\overline{\mathrm{NE}}(Y)_{K_X \geq 0} = \{C \in \overline{\mathrm{NE}}(Y) \mid K_Y \cdot C \geq 0\}$. Furthermore $\Gamma_l = \mathrm{NE}(Y) \cap L^{\perp}$ for some nef line bundle L (depending on Γ_l). Moreover, for every ample $\mathbb{Q}$-divisor H on Y, there exist finitely many such rays Γ_l with

$$\overline{\mathrm{NE}}(Y) = \overline{\mathrm{NE}}(Y)_{K_X + H \geq 0} + \mathbb{R}_{\geq}[\Gamma_l]. \tag{9.1.2}$$

In particular, the rays Γ_l are discrete in the half-space $\overline{\mathrm{NE}}(Y)_{K_X<0}$.

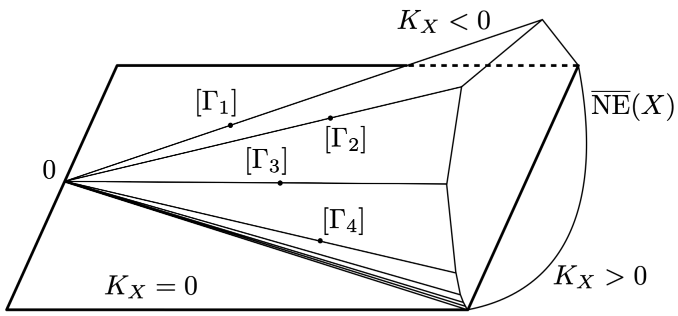

Figure 9.1: The cone of curves

The Cone Theorem gives a precise description of the cone of curves in the half part $(N_1(Y)_{\mathbb{R}})_{K_Y<0}$ saying that in this part is finitely generated. We can observe that the second equation, in Theorem 9.1.6, tell us that the rays Γ_l's can accumulate only on the hyperplane $K_Y^{\perp}$. Furthermore, there is another result that states that we can contract the extremal ray Γ_l's, known as the Contraction theorem of Kawamata and Shokurov. It was proven before the Cone's Theorem.

Definition 9.1.7. A **$\mathbb{Q}$-factorial variety** is a normal variety such that every Weil divisor D is $\mathbb{Q}$-Cartier, *i.e.* there exists $r \in \mathbb{Q}$ such that rD is a Cartier divisor.

Theorem 9.1.8 (Contraction theorem)**.** [63, Theorem 8-1-3] *Let Y be a normal projective $\mathbb{Q}$-factorial variety. With the notation of Theorem 9.1.6. For each extremal ray Γ_l of $\overline{\mathrm{NE}}(Y)$ in the half space $(N_1(Y)_{\mathbb{R}})_{K_Y<0}$ where*

$$\Gamma_l = \overline{\mathrm{NE}}(Y) \cap L^{\perp} \text{ for some nef line bundle } L$$

there exist a morphism $\phi = contr_{\Gamma_l} \colon Y \longrightarrow Z$, called ***contraction of an extremal ray*** *Γ_l with respect to K_Y such that:*

1. *ϕ is not an isomorphism;*
2. *for any curve $C \subset X$ such that $\phi(C)$ is a point, $K_Y \cdot C < 0$;*
3. *for any curve $C \subset Y$: $\phi(C)$ is a point if and only if $[C] \in \Gamma_l$;*
4. *ϕ has connected fibers with Z normal and projective.*

The above properties characterize the contractions of an extremal ray Γ_l with respect to K_Y. Furthermore,

(i) *$L = \phi^* H$ for some ample divisor H on Z;*

(ii) *$(\phi)^* \mathcal{O}_Y = \mathcal{O}_Z$;*

(iii) *for any divisors D_Y on Y:*

$$D_Y = \phi^* D_Z \text{ for some divisor } D_Z \text{ on } Z \Leftrightarrow D_Y \cdot C = 0 \text{ for any } [C] \in \Gamma_l.$$

Definition 9.1.9. Let $f \colon X \longrightarrow Y$ be a birational morphism. We define the **exceptional locus**, denoted by $\mathrm{Exc}(f) \subset X$, the locus of points where f is not an isomorphism.

An extremal contraction $\phi \colon Y \longrightarrow Z$ can be classified into one of three categories, see [54, Proposition 2.5].

1. **Divisorial contraction** if ϕ is a birational morphism and $\mathrm{Exc}(\phi)$ has codimension 1.
2. **Fiber contraction** if $\dim(Y) > \dim(Z)$: we observe that $(K_Y \cdot F) < 0$ for a general fiber F, hence $(-K_Y)_{|F}$ is ample. In this case Y is called **Mori fiber space**.
3. **Small contraction** if ϕ is birational and $\mathrm{Exc}(\phi)$ has codimension at least 2.

Together with these three types of contractions, we can describe the algorithm of the MMP in higher dimension. Let Y be a projective $\mathbb{Q}$-factorial variety. If K_Y is nef, then Y is a minimal model. Otherwise, pick $\Gamma_l \in \overline{\mathrm{NE}}(Y)_{K_Y<0}$ an extremal ray with contraction map $\phi = contr_{\Gamma_l}\colon Y \longrightarrow Z$. If ϕ is a divisorial contraction, then there are good chances that Z is still $\mathbb{Q}$-factorial and we restart the process with Z. If ϕ is a fiber contraction, we reduce the study of Y to the study of Z. When (or if) the algorithm ends the upshot is a Mori fiber space. If ϕ is a small contraction, then Z has very "bad" singularities, *i.e.* it is far away from being $\mathbb{Q}$-factorial. In order to continue the process, we introduce a new operation called **flip**, which is a topological surgery in codimension at least 2 that produce a variety Y^+ birational to Y and then we restart the algorithm with Y^+. The final outcome, if the procedure eventually finishes, is either a minimal model of Y or a Mori fiber space.

Remark 9.1.10. It seems important to point out that even if the criterior of the MMP for surface is different from the one described above, the output is the same. If classically we say that the output of the MMP for surfaces is a smooth surface with no (-1)-curves, in the modern language we say that it is either a surface S with nef canonical bundle (it is the case for $k(S) \geq 0$) or a Mori fiber space (the case with $k(S) = -\infty$).

Let us return to the MMP. Even if we have written down an algorithm, we point out that there are still several open conjecture in MMP that need to be proven in order to obtain a full birational classification of projective varieties in all dimension. For instance, whenever the MMP for a variety Y ends up with a minimal model Y' then it is conjectured that we can study Y' via the map $\varphi_{|K_{Y'}|}$, *i.e.* that $\varphi_{|mK_{Y'}|}$ is a morphism for some $m \in \mathbb{N}$.

Definition 9.1.11. Let Y be a smooth variety and $D \in \mathrm{Div}(Y)$. Then D is said to be **semi-ample** if there exists an integer $m >> 0$ such that $\mathrm{Bs}(mD) = \emptyset$.

Conjecture 9.1.12 (Abundance conjecture)**.** [54, Conjecture 3.12] Let Y be a $\mathbb{Q}$-factorial variety. If K_Y is nef then it is semi-ample.

The viceversa is also true.

Lemma 9.1.13. *Let X be a normal variety and let D be a $\mathbb{Q}$-Cartier divisor. If D is semi-ample then D is nef.*

Proof. Since D is semi-ample, by definition, there exists $n \in \mathbb{N}$ such that $|nD|$ is base-point free. We consider

$$\Phi_{|nD|} := \Phi\colon X \longrightarrow \Phi(X) \subset \mathbb{P}^M \text{ for some } M \in \mathbb{N}.$$

We denote by H the hyperplane section on $\mathbb{P}^M$ and we have $\Phi^*H = nD$. Let $C \subset X$ be a curve, by the projection formula (1.2.2) we have:

$$D \cdot C = \frac{\Phi^* H}{n} \cdot C = \frac{1}{n} H \cdot \Phi_*(C) \geq 0,$$

therefore, D is nef. □

We also remark that semi-ample divisors corresponds to faces of the nef cone that defines contractions.

Definition 9.1.14. A **face F of a cone** C is a subcone of C such that is closed under addition.

Given a contraction $f\colon Y \longrightarrow Z$ on Y then $f^*N^1(Z)_{\mathbb{R}} \cap \mathrm{Nef}(Y)$ defines a face in $\mathrm{Nef}(Y)$. Viceversa, let L be a semi-ample divisor. According to [57, Proposition 2.1.26], $\varphi_{|mL|}\colon Y \to Z$ defines a fibration on Y for some $m >> 0$. Furthermore, there exists an ample line bundle H on Z such that $f^*H = L^{\otimes m}$. We have that f^*H lies in the interior of a face of $\mathrm{Nef}(Y)$ and so we see that the faces of this cones defined by semi-ample line bundles correspond to contractions on Y.

9.2 | Preliminaries on Convex Geometry

Let V be a finite-dimensional $\mathbb{R}$-vector space.

Definition 9.2.1. We say that V has a **k-structure** for a subfield $k \subset \mathbb{R}$ if it is obtained by extension of scalars from a vector space V_k over k.

Definition 9.2.2. The **convex hull** of C is the cone of convex combination of points in C, *i.e.* $\mathrm{convhull}(C) = \{\sum a_i c_i \mid a_i \in \mathbb{R}_{\geq 0}, \sum_i a_i = 1 \text{ and } c_i \in C\}$.
The **rational hull of C**, denoted by C^+, is the convex hull of the rational points in $\overline{C}$, *i.e.* $C^+ = \mathrm{convhull}\,(\overline{C} \cap V_{\mathbb{Q}})$.

Definition 9.2.3. A cone $C \subseteq V$, with $\dim V = n$, is said to be **polyhedral** if it is finitely-generated, *i.e.* there is a set of vectors $\{v_1, \ldots, v_k\} \in V$ such that

$$C = \{a_1 v_1 + \cdots + a_k v_k \mid a_i \in \mathbb{R}_{>0}, v_i \in \mathbb{R}^n\}.$$

A polyhedral cone is said to be **rational** when it is generated by vectors with integral coordinates:

$$C = \{a_1 v_1 + \cdots + a_k v_k \mid a_i \in \mathbb{R}_{>0}, v_i \in \mathbb{Z}^n\}.$$

Definition 9.2.4. Let C be a cone in V. The **automorphism group of C** is the group **Aut**(C)= $\{\varphi \in \mathrm{GL}(V) \mid \varphi(C) = C\}$ of transformations of V preserving the cone C.

Definition 9.2.5. Let C be a cone in a real vector space V and let $\Gamma \leq \mathrm{GL}(V)$ be a subgroup that preserves the cone C. Let Π be a polyhedral subcone of C^+. We say that Π is a **fundamental domain** for the action of G on C

a. $\bigcup_{\gamma\in\Gamma} \gamma(\Pi) = C$, we write $\Gamma \cdot \Pi = C$;

b. $\gamma\big(\mathrm{Int}(\Pi)\big) \cap \mathrm{Int}(\Pi) = \emptyset$ for every $\gamma \neq id \in \Gamma$.

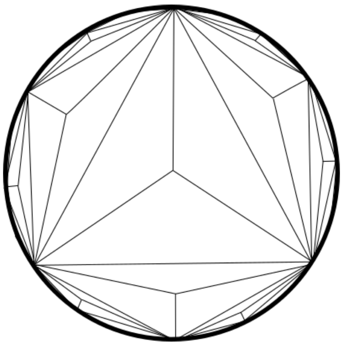

Figure 9.1: Slice of a round cone with a fundamental domain

The existence of a rational polyhedral fundamental domain is a question that has been studied deeply in convex geometry. We recall two of the main results that allow us to simplify the problem.

The first result asserts that in order to cover C with a rational polyhedral cone under the action of Γ, it is enough to cover its interior.

Proposition 9.2.6. [62, Part of Proposition-Definition 4.1] *Let C be an open convex non-degenerate cone in a finite dimensional real vectors space V with a $\mathbb{Q}$-structure $V_{\mathbb{Q}}$. Let $\Gamma \leq \mathrm{GL}(V)$ be a subgroup such that C is Γ-invariant and Γ preserves some lattice in $V_{\mathbb{Q}}$. Then the following are equivalent:*

(i) *there exists a rational polyhedral cone $\Pi \subset C^+$ with $\Gamma \cdot \Pi = C^+$;*

(ii) *there exists a rational polyhedral cone $\Pi \subset C^+$ with $\Gamma \cdot \Pi \supseteq C$.*

Moreover, in the second case we necessarily have $\Gamma \cdot \Pi = C^+$.

To get a fundamental domain we need condition (b) of Definition 9.2.5 on the interiors. The following result guarantees that, in the situation of Proposition 9.2.6 together with another reasonable condition, there exists a smaller cone $\Pi' \subseteq C^+$ which is a rational fundamental domain under the Γ-action. We underline that the existence of such Π' is guaranteed in [62, Application 4.14] where the author explicitly constructed it.

Definition 9.2.7. Let $\sigma \subset V$ be a cone in a finite dimensional real vector space V, then the dual cone $\sigma^\vee$ is defined as $\sigma^\vee = \{l \in V^\vee \mid l(s) \geq 0 \text{ for all } s \in \sigma\} \subset V^\vee$.

Lemma 9.2.8. [62, Application 4.14] *Let Λ be a finitely generated free $\mathbb{Z}$-module, and let C be a strict open cone in the $\mathbb{R}$-vector space $\Lambda_\mathbb{R} := \Lambda \otimes \mathbb{R}$. Let C^+ be the convex hull of $\overline{C} \cap \Lambda_\mathbb{Q}$. Let $(C^\vee)^\circ \subset (\Lambda_\mathbb{R})^\vee$ be the interior of the dual cone of C. Let Γ be a subgroup of $\mathrm{GL}(\Lambda)$ which preserves the cone C. Suppose that*

1. *there exists a rational polyhedral cone $\Pi \subset C^+$ such that $\Gamma \cdot \Pi \supset C$;*
2. *there exists an element $\eta \in (C^\vee)^\circ \cap (\Lambda_\mathbb{Q})^\vee$ whose stabilizer in Γ (with respect to the dual action of Γ on $(\Lambda_\mathbb{Q})^\vee$) is trivial.*

Then $\Gamma \cdot \Pi = C^+$, and in fact there exists a rational polyhedral cone $\Pi' \subset C^+$ which is a fundamental domain for the action of Γ on C^+.

9.3 | The cones of $\mathbb{R}$-divisors

Let Y be a projective, compact, smooth algebraic variety over $\mathbb{C}$ of dimension n. As explained in the previous sections, we are interested in the study cones of $\mathbb{R}$-divisors in $N^1(Y)_\mathbb{R}$, see Definition 1.2.4. Specifically, also in view of MMP, we are interested in the study of the following cones in $N^1(Y)_\mathbb{R}$:

- $\mathrm{Eff}(Y)$ is the cone spanned by effective divisors;
- $\mathrm{Mov}(Y)$ is the cone spanned by movable divisors;
- $\mathrm{Nef}(Y)$ is the cone spanned by nef divisors;
- $\mathrm{Amp}(Y)$ is the cone spanned by ample divisors.

In general $\mathrm{Mov}(Y)$ and $\mathrm{Eff}(Y)$ are neither open nor closed, thus one considers their closure. By results of Chapter 1, the following inclusions hold:

$$\mathrm{Amp}(Y) \subset \mathrm{Nef}(Y) \subset \overline{\mathrm{Mov}}(Y) \subset \overline{\mathrm{Eff}}(Y). \tag{9.3.1}$$

The fundamental results concerning nef and ample cones are the Kleiman's theorems.

Theorem 9.3.1. [57, Theorem 1.4.9] *Let X be a complete variety and D be a nef $\mathbb{R}$-divisor. Then $D^{\dim(V)} \cdot V \geq 0$ for every irreducible subvariety $V \subset X$.*

Proposition 9.3.2. [57, Theorem 1.4.23] *Let Y be a projective algebraic variety. Then*

$$\mathrm{Amp}(Y) = \mathrm{Int}(\mathrm{Nef}(Y)) \qquad \overline{\mathrm{Amp}}(Y) = \mathrm{Nef}(Y).$$

We also recall the following characterization of the **pseudoeffective cone** $\overline{\mathrm{Eff}}(Y)$.

Definition 9.3.3. Let Y be a smooth variety. Then $D \in \mathrm{Div}(Y)$ is said **big** if $|mD|$ defines a birational map onto its image for some $m >> 0$. We denote by $\mathrm{Big}(Y)$ the cone spanned by big divisors.

Lemma 9.3.4. [57, Theorem 2.2.26] *Let Y be a projective variety. Then:*

$$\mathrm{Big}(Y) = \mathrm{Int}(\overline{\mathrm{Eff}}(Y)) \qquad \overline{\mathrm{Big}}(Y) = \overline{\mathrm{Eff}}(Y).$$

We recall that the vector spaces $N^1(Y)_{\mathbb{R}}$ and $N_1(Y)_{\mathbb{R}}$ are dual vector spaces. In particular, the following result asserts that the closure of the Mori cone, which plays a central role in the contest of MMP, is dual to the nef cone.

Proposition 9.3.5. [57, Theorem 1.4.28] *Let Y be a projective variety. The cones* $\mathrm{Nef}(Y)$ *and* $\overline{\mathrm{NE}}(Y)$ *are dual.*

The Mori cones is also important as test for amplitudine of divisors. Let $D \in N^1(Y)_{\mathbb{R}}$ and we denote

$$D_{>0} = \{C \in N_1(Y)_{\mathbb{R}} \mid D \cdot C > 0\}.$$

Proposition 9.3.6 (Kleiman's Ampleness Criterior)**.** [54, Theorem 1.18] *Let Y be a projective variety and let $D \in N^1(Y)_{\mathbb{R}}$. Then D is ample if and only if*

$$\overline{\mathrm{NE}}(Y) \setminus \{0\} \subseteq D_{>0}.$$

9.4 | The Morrison-Kawamata Cone Conjecture

It should be clear at this point why the study of the nef cone is so important in the study of the geometry of a projective variety. A nice situation appear for Fano variety.

Corollary 9.4.1. *Let Y be a Fano variety,* i.e. *$-K_Y$ is ample. Then* $\mathrm{Nef}(Y)$ *and* $\overline{\mathrm{NE}}(Y)$ *are polyhedral cones.*

Proof. Since $-K_Y$ is ample, all curves C on Y intersect K_Y negatively, so $\overline{\mathrm{NE}}(Y)$ is contained in the half-space $\left(N^1(Y)_{\mathbb{R}}\right)_{K_Y<0}$. Thus, by the Cone Theorem 9.1.6 we obtain that $\overline{\mathrm{NE}}(Y) = \sum_l \Gamma_l$ is spanned by finitely many rational rays Γ_l so it is rational polyhedral. Since $\mathrm{Nef}(Y)$ is dual to $\overline{\mathrm{NE}}(Y)$, we get that it is rational polyhedral. □

The situation for K-trivial manifolds is more complicated. We start with an example, see also [85, Section 4.1].

Example 9. Let $Y = E \times E$ be the abelian surface product of two elliptic curves. The Picard group of Y has rank 3 and is spanned by the following three classes: a fiber F_1 of the first projection, a fiber F_2 of the second projection and the diagonal Δ. All three divisors intersect the other two with value 1 and have self intersection 0. We consider the cone $\mathcal{C} = \{x \in N^1(Y)_{\mathbb{R}>0} \mid x^2 \geq 0\}$. Writing $x = aF_1 + bF_2 + c\Delta$ we see that $\mathcal{C}$ is a round cone. We observe that the cone $\mathcal{C}$ contains the nef cone: indeed for any $D \in \mathrm{Nef}(Y)$, $D^2 \geq 0$. It also contains the cone $\overline{\mathrm{NE}}(Y)$: let $y \in \overline{\mathrm{NE}}(Y)$, which is in particular an effective divisor. Let α be a translation on Y, then $\alpha^* y \equiv y$ with different support. Hence $y^2 = y \cdot \alpha^* y \geq 0$. Let $\mathcal{C}^\vee = \{y \in N_1(Y)_{\mathbb{R}} \mid y \cdot x \geq 0 \text{ for every } x \in \mathcal{C}\ \}$. We immediately see that the intersection pairing defines an isomorphisms between $\mathcal{C}$ and $\mathcal{C}^\vee$. Since $\mathcal{C}$ contains both the nef cone and the cone of curves, which are dual cones, and $\mathcal{C} \simeq \mathcal{C}^\vee$ then $\mathcal{C}$ must coincide with $\mathrm{Nef}(Y)$.

The previous example shows that the nef cone for K-trivial manifolds can be not rationally polyhedral. In this case one can seek for a (rational) fundamental domain under the action of a group. Let us consider the previous example $E \times E$: we have that the nef cone is round, but as Namikawa proved in [71] by using the action of pullbacks of $\mathrm{Aut}(E \times E)$ of the nef cone one can find a rational fundamental domain. The existence of this fundamental domain can in fact be explained by the infinity of $\mathrm{Aut}(E \times E) \simeq \mathrm{PGL}(2, \mathbb{Z})$.

The cone conjecture aims to generalize this phenomena, by predicting that the nef and the movable cones of any K-trivial manifold Y have a rational fundamental domains up to the action of $\mathrm{Aut}(Y)$ and $\mathrm{Bir}(Y)$, respectively. Let us consider the action by pull backs of $\mathrm{Bir}(Y)$ on $N^1(Y)_{\mathbb{R}}$

$$\begin{aligned} \rho \colon \mathrm{Bir}(Y) &\longrightarrow \mathrm{GL}(N^1(Y)_{\mathbb{R}}) \\ f &\longmapsto f^* \end{aligned} \tag{9.4.1}$$

According to Lemma 1.3.4, $\mathrm{Bir}(Y)$ preserves the movable cone and by Proposition 1.3.2 $\mathrm{Aut}(Y)$ preserves the nef cone.

There exist two versions (non equivalent!) of the cone conjecture. Historically Morrison stated it before taking inspiration from the Mirror Symmetry, then Kawamata reformulated it in view of the MMP.

Conjecture 9.4.2 (Morrison's version)**.** [69] Let Y be a smooth projective K-trivial variety. Then

(i) There exists a rational polyhedral cone Π which is a fundamental domain (in the sense of Definition 9.2.5.) for the action of the automorphism group $\mathrm{Aut}(Y)$ on the cone $\mathrm{Nef}(Y)^+$ (see Definition 9.2.2).

(ii) There exists a rational polyhedral cone Π' which is a fundamental domain (in the sense of Definition 9.2.5) for the action of the birational automorphism group $\mathrm{Bir}(Y)$ on the cone $\mathrm{Mov}(Y)^+$.

Conjecture 9.4.3 (Kawamata's version)**.** [50] Let Y be a smooth projective K-trivial variety. Then:

(i) There exists a rational polyhedral cone Π which is a fundamental domain for the action of $\mathrm{Aut}(Y)$ on the effective nef cone $\mathrm{Nef}(Y)^e = \mathrm{Nef}(Y) \cap \mathrm{Eff}(Y)$.

(ii) There exists a rational polyhedral cone Π' which is a fundamental domain for the action of $\mathrm{Bir}(Y)$ on $\overline{\mathrm{Mov}}^e(Y) = \overline{\mathrm{Mov}}(Y) \cap \mathrm{Eff}(Y)$ the effective movable cone.

Before to explain the relation between these two versions of the cone conjecture, it is worth noting the importance of Conjecture 9.4.3 in the contest of MMP. The first item of the conjecture would imply that the faces of the nef cone corresponding to birational contractions or fiber space structures are finite up to automorphisms, see [85, Section 1], while the second one would imply, modulo standard conjectures of the Minimal Model Program, the finiteness of minimal models, birational automorphisms, see [20, Theorem 2.14].

For the nef cone, the connection between these two different versions of the cone conjecture, Conjecture 9.4.2 and Conjecture 9.4.3, can be explained by the following result, which is well-known (see for example [60, Theorem 2.15]).

Lemma 9.4.4. *Let Y be a projective variety. Then* $\mathrm{Nef}(Y)^e \subseteq \mathrm{Nef}(Y)^+$*.*

Proof. Let D in $\mathrm{Nef}(Y)^e$ so we write $D = \sum_i r_i[D_i]$ with $r_i \in \mathbb{R}_{>0}$ and D_i's are nef divisors. We can write $r_i = p_i + \delta_i$ for some $p_i \in \mathbb{Z}_{\geq 0}$ and $\delta_i \in \mathbb{R}_{\geq 0}$. We obtain that:

$$r_i[D_i] = (1-\delta_i)p_i[D_i] + \delta_i(p_i+1)[D_i]$$

is a convex combination of $p_i[D_i]$ and $(p_i+1)[D_i]$, hence $r_i[D_i]$ belongs to $\mathrm{Amp}(Y)^+$. Since $\mathrm{Amp}(Y)^+$ is a cone then $D = \sum_i r_i D_i \in \mathrm{Amp}(Y)^+ = \mathrm{Nef}(Y)^+$. □

We remark that the reverse inclusion is still wide open. The result above together with the following lemma, highlight the exact relation between the Morrison's and Kawamata's version of the cone conjecture.

Lemma 9.4.5. [35, Proposition 2.3] *Let Y be a projective variety and let $H \leq \mathrm{Aut}(Y)$ be a subgroup. Assume that there is a rational polyhedral cone $\Pi \subseteq \mathrm{Nef}^+(Y)$ such that $\mathrm{Amp}(Y) \subset H \cdot \Pi$. Then $H \cdot \Pi = \mathrm{Nef}^+(Y)$, and the H-action on $\mathrm{Nef}^+(Y)$ has a rational polyhedral fundamental domain.*

Proposition 9.4.6. *Let Y be a projective complex manifold with $c_1(Y) = 0$. Then Kawamata's Cone Conjecture holds for Y if and only if $\mathrm{Nef}(Y)^e = \mathrm{Nef}(Y)^+$ and Morrison's Cone Conjecture holds for Y.*

Proof. If we assume Kawamata' version then we have $\Pi \subset \mathrm{Nef}(Y)^e$ such that

$$\mathrm{Amp}(Y) \subset \mathrm{Nef}(Y)^e = \mathrm{Aut}(Y) \cdot \Pi.$$

Then $\Pi \subseteq \mathrm{Nef}(Y)^e \subseteq \mathrm{Nef}(Y)^+$ and so by Lemma 9.4.5, we have $\mathrm{Nef}(Y)^e = \mathrm{Nef}(Y)^+$. Therefore, Morrison's version is satisfied. The reverse implication is obvious. □

Let us denote by $\mathcal{A}(Y)$ the image of $\mathrm{Aut}(Y)$ under ρ in (9.4.1). We have the following well-known result (for people working on the cone conjecture).

Lemma 9.4.7. [36, Corollary 2.17] *Let Y be a smooth projective K-trivial variety. If $\mathrm{Nef}(Y)$ is a rational polyhedral cone, then $\mathcal{A}(Y)$ is a finite group and Morrison's Cone Conjecture holds on Y. Conversely, if $\mathcal{A}(Y)$ (resp. $\mathcal{B}(Y)$) is a finite group and we assume that Morrison's Cone Conjecture holds on Y, then $\mathrm{Nef}(Y)$ is a rational polyhedral cone.*

Proof. Suppose that $\mathrm{Nef}(Y)$ is rational polyhedral. We observe that $\mathcal{A}(Y)$ not only preserves the cone, it also acts as a permutation all the extremal rays. We can take primitive integer classes as extremal rays of the cone. Any $\varphi \in \mathcal{A}(Y)$ permutes these classes and is uniquely determined by this permutation. There are only finitely many of these permutations, so $\mathcal{A}(Y)$ is a finite group. We apply Lemma 9.4.5 to the convex cone $\mathrm{Nef}(Y)$ and the rational polyhedral subcone $\Pi = \mathrm{Nef}(Y)$, we get a rational polyhedral cone Π' which is a fundamental domain for the action of $\mathrm{Aut}(Y)$ on $\mathrm{Nef}(Y)^+$. Conversely, assume the Morrison's Cone Conjecture for Y. Then there is a rational polyhedral cone $\Pi \subseteq \mathrm{Nef}(Y)^+$ which is a fundamental domain for the action of $\mathrm{Aut}(Y)$ on $\mathrm{Nef}(Y)^+$. Since $\mathcal{A}(Y)$ is finite, $\mathrm{Nef}(Y)^+ = \mathcal{A}(Y) \cdot \Pi$ is also a rational polyhedral cone. Thus, it is closed and since $\mathrm{Amp}(Y) \subseteq \mathrm{Nef}(Y)^+ \subseteq \mathrm{Nef}(Y)$ by minimality we have $\mathrm{Nef}(Y)^+ = \mathrm{Nef}(Y)$. Therefore, $\mathrm{Nef}(Y)$ is a rational polyhedral cone. □

As final remark, we recall the following relation between the finiteness of $\mathcal{A}(Y)$ and the one of $\mathrm{Aut}(Y)$.

Lemma 9.4.8. [72, Proposition 2.4] *Let Y be a complex manifold with numerically trivial canonical bundle and $h^{1,0}(Y) = 0$. Then the kernel of*

$$\begin{aligned} \rho\colon \mathrm{Aut}(Y) &\longrightarrow \mathrm{GL}(N^1(Y)_{\mathbb{R}}) \\ f &\longmapsto f^* \end{aligned} \tag{9.4.2}$$

is a finite group. In particular, $\mathrm{Aut}(Y)$ *is finite if and only if* $\mathcal{A}(Y)$ *is.*

10 Reduction theory

The problem of computing fundamental domains for action of groups on convex cones dates back to H. Minkoswki, specifically in relation to *reduction theory.* In 1905, Minkowski made several contributions to the theory of quadratic forms, particularly to the reduction of positive-definite quadratic forms, see [64] and [31]. The term "reduced form" was first introduced by C. Hermite who in the 19th century developed the first techniques to reduce a quadratic form into an equivalent form which is easier to analyze or classify. Minkoswki expanded and refined this reduction theory with a more geometric prospective by considering some relations with lattice theory. Roughly speaking, the Minkowski's algorithm to minimize a quadratic form consists of finding a fundamental domain of a lattice, *i.e.* a region in $\mathbb{R}^n$ that contains exactly one representative of each equivalence class of points under the lattice's translations. The points in the fundamental domain of a lattice represent a unique reduced form. All the other forms can be recovered by acting on this fundamental region with $\mathbb{Z}$-valued linear transformations. Some years later, A. Weyl generalized this work and others to many other cases. In 1940, A. Borel has refined the reduction theory by introducing the concept of coarse fundamental domains, known as Siegel sets, for actions of arithmetic groups, *i.e.* linear algebraic group with integers values. The Siegel sets are better than the Minkowski's fundamental domains and in fact they are more similar to the fundamental domains introduced in the previous chapter, see Definition 9.2.5. In the 1975, A. Ash used the the notion of the Siegel sets to provide a positive answer to the following problem, see Theorem 10.3.1:

Question 4. Given a homogeneous self-dual cone C and an arithmetic group $\Gamma \leq \mathrm{Aut}(C)$, does there exist a fundamental set for the action of Γ on C?

In this chapter we give a briefly introduction on the problem above: we collect the main results around homogeneous self-dual cones and arithmetic groups and we conclude by stating the result of Ash, Theorem 10.3.1. This will be our main tool to prove the cone conjecture for hyperelliptic varieties.

10.1 | Homogeneous self-dual cones

Let C be a cone in a finite-dimensional real vector space V.

Definition 10.1.1. A cone C is said to be **homogeneous** if $\mathrm{Aut}(C)$ acts transitively on it, *i.e.* for every $x, y \in C$ there exists $\varphi \in \mathrm{Aut}(C)$ such that $\varphi(x) = y$.

Definition 10.1.2. A cone C is said to be **self-dual** if there exists a positive-definite form on V such that the resulting isomorphism between V and its dual $V^\vee$ transforms C into $C^\vee$.

The theory of a self-dual homogeneous cones is very wide, but due to results of Vinberg and Koecher, we known that homogeneous self-dual cones can be completely classified into a small number of cases, see [86].

Definition 10.1.3. Let $C_i \subset V_i$ be cones in the vector space V_i for $i = 1, 2$. We define **the direct sum of C_1 and C_2** in the vector spaces $V_1 \oplus V_2$ to be the cone $C_1 \oplus C_2 := \{v_1 + v_2 \in V_1 \oplus V_2 \mid v_i \in C_i\}$ and call a cone **indecomposable** if it cannot be written as the direct sum of two nontrivial cones.

Theorem 10.1.4. [4, Remark 1.11] *Any convex cone $C \subset V$ can be written as a direct sum $\bigoplus_i C_i$ of indecomposable cones. The product $\prod_i \mathrm{Aut}(C_i)$ is a finite-index subgroup of $\mathrm{Aut}(C)$. The cones C_i are homogeneous and self-dual if and only if C is too. Any indecomposable homogeneous self-dual cone is isomorphic to one of the following:*

1. *the cone $\mathcal{P}_r(\mathbb{R})$ of positive-definite matrices in the space $\mathcal{H}_r(\mathbb{R})$ of $r \times r$ real symmetric matrices;*
2. *the cone $\mathcal{P}_r(\mathbb{C})$ of positive-definite matrices in the space $\mathcal{H}_r(\mathbb{C})$ of $r \times r$ complex hermitian matrices;*
3. *the cone $\mathcal{P}_r(\mathbb{H})$ of positive-definite matrices in the space $\mathcal{H}_r(\mathbb{H})$ of $r \times r$ quaternionic hermitian matrices;*
4. *the spherical cone $\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \quad | \quad x_0 > \sqrt{x_1^2 + \dots x_n^2}\}$;*
5. *the 27-dimensional cone of positive-definite 3×3 octonionic hermitian matrices.*

The inner product for which the cone is self-dual is $\langle x, y\rangle = Tr(xy^*)$ *in all cases except* 4*, and we take the usual inner product on* $\mathbb{R}^{n+1}$ *in case* 4*.*

Moreover, in [87] Vinberg have computed the automorphism groups of all the cones in the list of Theorem 10.1.4. In particular, we have the following result.

Theorem 10.1.5. [87] *Let* C *be one of the cones* $\mathcal{P}_r(\mathbb{F})$ *in the previous theorem where* $\mathbb{F} = \mathbb{R}, \mathbb{C},$ *or* $\mathbb{H}$*. The identity component* $\mathrm{Aut}(C)^0$ *of the automorphism group of* C *consists of all* $\mathbb{R}$*-linear transformations of* $\mathcal{H}_r(\mathbb{F})$ *of the form* $D \longmapsto M^\dagger DM$ *for some* $M \in GL(r, \mathbb{F})$ *where* $M^\dagger$ *is the conjugate transpose.*

10.2 | Arithmetic groups

The theory of arithmetic groups is very large. For our purpose we only need the definition. For a more detailed discussion see [16] or [67].

Definition 10.2.1. An **algebraic group $\mathcal{G}$ over a field k** is an algebraic variety over k endowed with a group structure such that the following homomorphisms:

$$\mu\colon G \times G \longrightarrow G \qquad i\colon G \longrightarrow G$$

$$(g, h) \longmapsto gh \qquad g \longmapsto g^{-1}$$

are morphisms of varieties.

An algebraic group $\mathcal{G}$ is said to be **defined over a subfield $K \subset k$** if the polynomial equations defining it have coefficients in K. We denote the underlying structure of K-variety of $\mathcal{G}$ by $\mathcal{G}(K)$.

An algebraic group $\mathcal{G}$ is said to be a **linear algebraic group** if it admits a closed (with respect to the Zarisky topology) embedding $\rho : \mathcal{G} \hookrightarrow \mathrm{GL}(n, k)$ for some $n \in \mathbb{N}$, *i.e.* $\rho(\mathcal{G}) := G(k)$ is a subgroup of $\mathrm{GL}(n, k)$ defined by polynomial equations with coefficient in k.

An important result about the automorphism group of a homogeneous self-dual cone is the following.

Theorem 10.2.2. [87] *Let* $C \subset V$ *be a self-dual convex cone. Then the automorphism group* $\mathrm{Aut}(C)$ *is the group of real points of a reductive algebraic group denoted by* $\mathcal{A}ut(C)$*.*

Definition 10.2.3. [16, Sectoin 7.C] Let $\mathcal{G}$ be an algebraic linear group in $\mathrm{GL}(n, \mathbb{C})$ for some n defined over $\mathbb{Q}$. We define $\mathcal{G}(\mathbb{Z}) := \mathcal{G} \cap \mathrm{GL}(n, \mathbb{Z})$. A subgroup $\Gamma \subset \mathcal{G}(\mathbb{Q})$ is said to be **arithmetic** if it is commensurable with $\mathcal{G}(\mathbb{Z})$, *i.e.* $\mathcal{G}(\mathbb{Z}) \cap \Gamma$ is of finite index in both $\mathcal{G}(\mathbb{Z})$ and Γ.

Remark 10.2.4. It is proved in [16, Section 7.C] that the property of being arithmetic is invariant under $\mathbb{Q}$-isomorphisms.

Example 10. Let $E \times E$ the abelian surfaces. Then the group $\mathrm{End}(E \times E)^\times$ defines an arithemtic group in $\mathrm{End}_\mathbb{R}(E \times E)^\times = (\mathrm{End}(E \times E) \otimes \mathbb{R})^\times$. In fact, it is easy to observe that $\mathrm{End}_\mathbb{R}(E \times E)^\times \simeq \mathrm{GL}(2, \mathbb{R})$ is an algebraic group and $\mathrm{End}(E \times E)^\times$ is exactly the group of invertible matrices with integers coefficients.

10.3 | On the fundamental domain for homogeneous self-dual cone

Ash investigated the existence of a fundamental domain for action of arithmetic groups on homogeneous self-dual cones. One of the main tool he used was the existence of the Siegel sets introduced by Borel. We only recall the result omitting the proofs, we refer to [4, Chapter II].

Theorem 10.3.1. [4, Chapter II] *Let C be a homogeneous self-dual cone in a real vector space V with $\mathbb{Q}$-structure. Let $\mathrm{Aut}(C)$ be the automorphism group of C and let $\mathcal{A}ut(C)$ be the associated reductive algebraic group which exists in view of Theorem 10.2.2. Assume that the connected component of the identity $\mathcal{A}ut(C)^0$ is defined over $\mathbb{Q}$. Then, for any arithmetic subgroup Γ of $\mathcal{A}ut(C)^0$ there exists a rational polyhedral cone $\Pi \subset C^+$ such that $(\Gamma \cdot \Pi) \cap C = C$.*

It is worth recalling that starting from Π, it is possible to construct a rational fundamental domain for the action of Γ on C, see [4, pag. 75].

11 The cone conjecture for hyperelliptic varieties

In this chapter we prove Theorem D, the Morrison-Kawamata cone conjecture for hyperelliptic varieties, see also [65].

According to the Beauville-Bogomolov decomposition Theorem 1.5.6 every smooth projective K-trivial variety Y, up to an étale finite covering, splits into the product of three building blocks: abelian varieties, simply-connected Calabi-Yau manifolds and IHS manifolds. The Kawamata cone conjecture 9.4.3 is proven for abelian varieties ([77]), for IHS manifolds ([3] and [2]) and for certain Calabi-Yau manifolds ([61], [72], [75], and [58]). The following result guarantees that if each factor of the Beauville-Bogomolov decomposition of Y satisfies the cone conjecture then Y satisfies the cone conjecture.

Lemma 11.0.1. [43, Exercise 12.6 (b)] *Let Y_1 and Y_2 be projective varieties and assume that $H^1(Y_j, \mathcal{O}_{Y_j}) = 0$ for at least one $j = 1, 2$. Let us denote by $pr_j \colon Y_1 \times Y_2 \longrightarrow Y_j$ the projections on the j-factor for $j = 1, 2$. Then:*

$$\mathrm{Pic}(Y_1 \times Y_2) = pr_1^*\mathrm{Pic}(Y_1) \times pr_2^*\mathrm{Pic}(Y_2).$$

Corollary 11.0.2. *Let Y_1 and Y_2 two projective varieties that satisfy the Morrison cone conjecture 9.4.2 for the nef cone. Assume that $h^{1,0}(Y_j) = 0$ for at least one $j = 1, 2$. Then $Y = Y_1 \times Y_2$ satisfies the Morrison cone conjecture for the nef cone. In particular, if $Y = T \times \prod_i V_i \prod_j X_j$ is given by the Beauville-Bogomolov decomposition Theorem 1.5.6 and each factor satisfy the Morrison cone conjecture, then Y satisfies the Morrison cone conjecture.*

Proof. According to Lemma 11.0.1 we have

$$\mathrm{Pic}(Y) \otimes \mathbb{R} = pr_1^*\Big(\mathrm{Pic}(Y_1) \otimes \mathbb{R}\Big) \times pr_2^*\Big(\mathrm{Pic}(Y_2) \otimes \mathbb{R}\Big),$$

where pr_j's are the projections on the j-factor for $j = 1, 2$. Thus we obtain:

$$N^1(Y)_{\mathbb{R}} = pr_1^* N^1(Y_1)_{\mathbb{R}} \times pr_2^* N^1(Y_2)_{\mathbb{R}}.$$

By assumption there exists a rational polyhedral cones: $\Pi_{Y_j} \subset \mathrm{Nef}(Y_j)^+$ which is a fundamental domain for the action of $\mathrm{Aut}(Y_j)$ on $\mathrm{Nef}(Y_j)^+$ for $j = 1, 2$. We consider the rational polyhedral cone $\Pi_{Y_1} \times \Pi_{Y_2} \subset \mathrm{Nef}(Y_1)^+ \times \mathrm{Nef}(Y_2)^+ = \mathrm{Nef}(Y)^+$. The following inclusion holds:

$$\begin{aligned}\mathrm{Amp}(Y) \subset \mathrm{Nef}(Y)^+ &= \mathrm{Nef}(Y_1)^+ \times \mathrm{Nef}(Y_2)^+ \\ &= \bigcup_{\varphi_{Y_1} \in \mathrm{Aut}(Y_1)} \varphi_{Y_1}(\Pi_{Y_1}) \times \bigcup_{\varphi_{Y_2} \in \mathrm{Aut}(Y_2)} \varphi_{Y_2}(\Pi_{Y_2}) \\ &= \bigcup_{(\varphi_{Y_1}, \varphi_{Y_2}) \in \mathrm{Aut}(X)} (\varphi_{Y_1}, \varphi_{Y_2})(\Pi_{Y_1} \times \Pi_{Y_2}).\end{aligned}$$

Therefore, according to Lemma 9.4.5 there exists a fundamental domain for the action of $\mathrm{Aut}(Y_1) \times \mathrm{Aut}(Y_2) \subset \mathrm{Aut}(Y)$ on $\mathrm{Nef}(Y)^+$.
Let $Y = T \times \prod_i V_i \prod_j X_j$ be as in the Beauville-Bogomolov decomposition Theorem 1.5.6. Since there is at least one factor that is either an IHS manifold X_j or a Calabi-Yau manifold V_i and we have $h^{1,0}(X_j, \mathcal{O}_{X_j}) = h^{1,0}(V_i, \mathcal{O}_{V_i}) = 0$, we can apply (and generalize) the proof above. □

11.1 | The cone conjecture under étale quotients

In this section, we reformulate the cone conjecture for étale quotients and in particular for hyperelliptic varieties.

Proposition 11.1.1. *Let X be a compact projective manifold and $G \leq \mathrm{Aut}(X)$ a finite group that acts freely on it. We denote by $\pi : X \longrightarrow Y = X/G$. Assume the existence of a rational polyhedral cone $\Pi \subset (\mathrm{Nef}(X)^G)^+$ such that $\mathrm{Amp}(X)^G \subset H \cdot \Pi$ for some $H \leq \mathrm{N}_{\mathrm{Aut}(X)}(G)$. Then Y satisfies the Kawamata's cone conjecture 9.4.3.*

Proof. Let us consider $\pi_*(\Pi)$: it defines a rational polyhedral cone in $\mathrm{Nef}(Y)^+$ such that:

$$\mathrm{Amp}(Y) \subset (H/G) \cdot \pi_*(\Pi) \subset \mathrm{Aut}(Y) \cdot \pi_*(\Pi). \tag{11.1.1}$$

Thus by Lemma 9.4.5, $\mathrm{Nef}^+(Y)$ has a rational polyhedral fundamental domain. □

Proposition 11.1.2. *Let $Y = X/G$ be a hyperelliptic variety and D be a divisor on it. Then $[D] \in \mathrm{Eff}(Y)$ if and only if $[D] \in \mathrm{Nef}(Y)$. In particular, $\mathrm{Eff}(Y)$ is closed.*

Proof. The statement is true for abelian varieties, see [8, Lemma 1.1]. Let $\pi : X \longrightarrow Y$ be the finite étale covering. The statement easily follows using that D is nef/effective on Y if and only if $\pi^* D$ is nef/effective on X, see Lemma 1.3.2. □

Corollary 11.1.3. *Let Y be a hyperelliptic variety. Then*

$$\overline{\mathrm{Mov}}(Y)^+ = \overline{\mathrm{Mov}}(Y)^e = \overline{\mathrm{Mov}}(Y) = \mathrm{Nef}(Y) = \mathrm{Nef}(Y)^e = \mathrm{Nef}(Y)^+.$$

In particular, the cone conjecture for the nef cone is equivalent to the one of the movable cone. Moreover, the Morrison's Cone Conjecture 9.4.2 is equivalent to the Kawamata's Cone Conjecture 9.4.3.

Proof. According to (9.3.1) we have: $\mathrm{Nef}(Y) \subseteq \overline{\mathrm{Mov}}(Y) \subseteq \overline{\mathrm{Eff}}(Y)$. By Proposition 11.1.2 we have $\mathrm{Nef}(Y) = \mathrm{Eff}(Y) = \overline{\mathrm{Eff}}(Y)$, it follows $\mathrm{Nef}(Y) = \overline{\mathrm{Mov}}(Y) = \mathrm{Eff}(Y)$. Thus, the cone conjecture for the nef cone is equivalent to the one of the movable cone. We prove that $\mathrm{Nef}(Y)^e$ coincides with $\mathrm{Nef}(Y)^+$. It holds $\mathrm{Nef}(Y)^e \subseteq \mathrm{Nef}(Y)^+$ by Lemma 9.4.4. We observe that $\mathrm{Nef}(Y)^+ = \mathrm{Amp}(Y)^+ \subset \mathrm{Nef}(Y)$, hence since $\mathrm{Nef}(Y)^e = \mathrm{Nef}(Y)$ we obtain the reverse inclusion and so the equality. Thus, the Morrison's Cone Conjecture 9.4.2 is equivalent to the Kawamata's Cone Conjecture 9.4.3. □

11.2 | The proof of Theorem D

In the following X is an abelian variety and $G \leq \mathrm{Aut}(X)$ is a finite group which does not contain any translation.

The goal of this section is to prove the following theorem.

Theorem 11.2.1 (Theorem D)**.** *Let $Y = X/G$ be a hyperelliptic variety. Then then Conjecture 9.4.2 and Conjecture 9.4.3 hold both for the nef and the movable cones.*

We first explain the strategy. Due to the results of the preceding section, Proposition 11.1.1 and Corollary 11.1.3, to prove Theorem 11.2.1, it is enough that there exists a rational polyhedral cone $\Pi \subset (\mathrm{Nef}(X)^G)^+$ such that

$$\mathrm{Amp}(X)^G \subset H \cdot \Pi \tag{11.2.1}$$

for some $H \leq \mathrm{N}_{\mathrm{Aut}(X)}(G)$. To achieve this, we adopt the following strategy: we establish that the cone $\mathrm{Amp}(X)^G$ is a homogeneous self-dual cone and that the centralizer $H := \mathrm{C}_{\mathrm{Aut}(X)}(G)$ defines an action of an arithmetic group on it. This results allow us to invoke the main result of reduction theory, Theorem 10.3.1, which guarantees the existence of the desired Π satisfying (11.2.1) for the H-action on $\mathrm{Amp}(X)^G$.

We prove some useful facts.

Lemma 11.2.2. *Let X be an abelian variety and let us consider the following homomorphism:*

$$\rho\colon \mathrm{Aut}(X) \longrightarrow \mathrm{GL}(N^1(X)_{\mathbb{R}})$$

$$\varphi \longmapsto (\varphi^*\colon D \longmapsto \varphi^* D).$$

Then, $\rho(\mathrm{Aut}(X)) = \rho(\mathrm{End}(X)^\times)$. In other words, every translation of X acts as the identity on $N^1(X)_{\mathbb{R}}$.

Proof. Let t be a translation on X and $[D]_{num} \in N^1(Y)_{\mathbb{R}}$ denotes the numerical class of D, see 1.2.3. Then $\mathcal{O}_X(t^*D - D) \in \mathrm{Pic}^0(X)$, *i.e.* $[t^*D - D]_{hom} = 0$ in $H^2(X, \mathbb{Z})$. By Remark 1.2.5 homological and numerical equivalence coincide up to torsion, so on $N^1(Y)_{\mathbb{R}}$, we have that $[t^*D - D]_{num} = 0$ (see Definition 1.1.9) and so the result. □

The homomorphism ρ in Lemma 11.2.2 can be extended to an action of $\mathrm{End}_{\mathbb{R}}(X)^\times$ on $\mathrm{End}_{\mathbb{R}}(X)$. We define the following action:

$$\begin{aligned} \alpha\colon \mathrm{End}_{\mathbb{R}}(X)^\times &\longrightarrow \mathrm{GL}(\mathrm{End}_{\mathbb{R}}(X)) \\ \varphi &\longmapsto \alpha(\varphi)\colon l \mapsto \varphi' \circ l \circ \varphi \end{aligned} \tag{11.2.2}$$

where $'$ is the Rosati involution defined in (2.5.1). We recall that the following embedding of vector spaces, defined in Theorem 2.6.2:

$$f_L\colon \mathrm{N}^1(X)_{\mathbb{R}} \hookrightarrow \mathrm{End}_{\mathbb{R}}(X)$$

$$D \longmapsto \phi_L^{-1} \circ \phi_D$$

where L is the ample line bundle on X defining the Rosati involution $'$ and ϕ_L, ϕ_D are defined as in (2.3.1).

Lemma 11.2.3. *The action α defined in (11.2.2) extends the action of $\mathrm{Aut}(X)$ on $N^1(X)_{\mathbb{R}}$ by pull back.*

Proof. Let $\varphi \in \mathrm{Aut}(X)$. We have to prove that the following diagram

$$\begin{array}{ccc} N^1(X)_{\mathbb{R}} & \xhookrightarrow{f_L} & \mathrm{End}_{\mathbb{R}}(X) \\ \downarrow{\varphi^*} & & \downarrow{\alpha(\varphi)} \\ N^1(X)_{\mathbb{R}} & \xhookrightarrow{f_L} & \mathrm{End}_{\mathbb{R}}(X) \end{array}$$

is commutative. Let $D \in N^1(X)_{\mathbb{R}}$, we have:

$$f_L(\varphi^* D) = \phi_L^{-1}\phi_{\varphi^* D} \overset{(2.3.2)}{=} \phi_L^{-1}(\hat{\varphi} \circ \phi_D \circ \varphi) \overset{(2.5.1)}{=} \varphi' \phi_L^{-1}\phi_D \varphi = \alpha(\varphi)(f_L(D)).$$

□

11.2.1 | The G-invariant $\mathbb{R}$-algebra of an abelian variety

Definition 11.2.4. $\mathbf{End}_{\mathbb{Q}}(X)^G := \{\varphi \in \mathrm{End}_{\mathbb{Q}}(X) \mid \alpha(g)(\varphi) = \varphi \text{ for every } g \in \mathrm{Lin}(G)\}$ where $\mathrm{Lin}(G)$ is the the linear part of G.

The Rosati involution $'$ depends on the choice of the ample line bundle L. Since we are considering X with an action of a finite group G, we can choose L to be G-invariant.[1] Hence we have

$$\phi_L = \phi_{g^* L} \overset{(2.3.2)}{=} \hat{g}\phi_L g \text{ for every } g \in G. \tag{11.2.3}$$

This leads to the following relation:

$$\forall g \in G \qquad g' \overset{(2.5.1)}{=} \phi_L^{-1}\hat{g}\phi_L = \phi_L^{-1}(\hat{g}\phi_L g)g^{-1} \overset{(11.2.3)}{=} g^{-1}. \tag{11.2.4}$$

Therefore we obtain:

$$\mathrm{C}_{\mathrm{End}(X)^\times}(G) = \{\varphi \in \mathrm{End}(X)^\times \mid g'\varphi g = \varphi \text{ for every } g \in \mathrm{Lin}(G)\} = (\mathrm{End}(X)^G)^\times. \tag{11.2.5}$$

We have the following characterization of $\mathrm{End}_{\mathbb{Q}}(X)^G$.

Theorem 11.2.5. *Let X be an abelian variety and $G \leq \mathrm{Aut}(X)$ be a finite group. Then:*

(i) $\mathrm{End}_{\mathbb{Q}}(X)^G$ *is a finite dimensional $\mathbb{Q}$-algebra with an involution ι given by $\iota(x) = x'$ for every $x \in \mathrm{End}_{\mathbb{Q}}(X)^G \subseteq \mathrm{End}_{\mathbb{Q}}(X)$ which is positive-definite with respect to the trace reduced over $\mathbb{Q}$, see Definition 2.5.1.*

(ii) *We have the following isomorphism of $\mathbb{R}$-algebras:*

$$\Psi\colon (\mathrm{End}_{\mathbb{Q}}(X)^G \otimes \mathbb{R}, \iota) \xrightarrow{\simeq} (\prod_i \mathrm{Mat}_{l_i}(\mathbb{R}) \times \prod_j \mathrm{Mat}_{m_j}(\mathbb{C}) \times \prod_k \mathrm{Mat}_{n_k}(\mathbb{H}), \dagger)$$

[1] If L is not invariant, we can consider $\sum_{g \in G} g^* L$ which defines a G-invariant ample line bundle since $L \neq 0$ and ample.

where the involution ι is sent to the conjugate transpose $\dagger$ on each factor.

Before proving the theorem above we recall fundamental results about finite-dimensional algebras with positive-definite involution.

Lemma 11.2.6. [88, Lemma 8.4.5] *Let $\mathcal{A}_k$ be a k-algebra, for a subfield $k \subset \mathbb{R}$, and τ be a positive-definite involution with respect to the trace (see Definition 2.5.1). Then $\mathcal{A}_k$ is semisimple, see Definition 2.4.6.*

Remark 11.2.7. Let $\mathcal{A}_k$ as in Lemma 11.2.6 and consider the decomposition $\mathcal{A}_k = \prod_i \mathcal{A}_i$ into simple sub-algebras $\mathcal{A}_i \subset \mathcal{A}_k$. Then τ preserves this decomposition, *i.e.* $\tau(\mathcal{A}_i) = \mathcal{A}_i$ for all i. Indeed if $\tau(\mathcal{A}_i) = \mathcal{A}_j$ for $i \neq j$, then $\mathcal{A}_j$ is a simple factor and $\mathcal{A}_i\mathcal{A}_j = 0$. Therefore $\mathrm{Tr}(\mathcal{A}_i\tau(\mathcal{A}_i)) = \mathrm{Tr}(\mathcal{A}_i\mathcal{A}_j) = 0$ which is a contradiction since τ is positive definite with respect to the trace.

Lemma 11.2.8. [56, Lemma 5.5.1] *For any simple $\mathbb{R}$-algebra $\mathcal{A}_\mathbb{R}$ of finite dimension with a positive-definite involution there is an isomorphism of $\mathbb{R}$-algebras from $(\mathcal{A}_\mathbb{R}, \tau)$ to $(\mathrm{Mat}_n(\mathbb{F}), \dagger)$ for some $n \in \mathbb{N}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\dagger$ is the correspondent conjugate transpose on each field.*

We are in position to prove Theorem 11.2.5.

proof of Theorem 11.2.5. (i) We prove that $\mathrm{End}_\mathbb{Q}(X)^G$ is a well-defined sub-algebra of the finite dimensional $\mathbb{Q}$-algebra $\mathrm{End}_\mathbb{Q}(X)$, *i.e.* the algebra operations of $\mathrm{End}_\mathbb{Q}(X)$ are G-equivariants. In the following $x, y \in \mathrm{End}_\mathbb{Q}(X)$, $\lambda \in \mathbb{Q}$ and $g \in G$:

1. $\alpha(g)(x+y) = g'(x+y)g = g'xg + g'yg = \alpha(g)(x) + \alpha(g)(y)$,
2. $\alpha(g)(\lambda x) = g'\lambda xg = \lambda\alpha(g)(x)$,
3. $\alpha(g)(xy) = g'xyg = g'xgg'yg = \alpha(g)(x)\alpha(g)(y)$ we use $g'g = id$ by (11.2.4).

Clearly, the multiplicative and additive identity are in $\mathrm{End}_\mathbb{Q}(X)^G$ as well as the multiplicative and additive inverse. Thus, $\mathrm{End}_\mathbb{Q}(X)^G$ is a finite-dimensional $\mathbb{Q}$-algebra. We prove that the Rosati involution $'$ on $\mathrm{End}_\mathbb{Q}(X)$ is G-equivariant, *i.e.* ι is well-defined as involution in $\mathrm{End}_\mathbb{Q}(X)^G$. For every $g \in G$, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{End}_\mathbb{Q}(X) & \xrightarrow{\ '\ } & \mathrm{End}_\mathbb{Q}(X) \\ \downarrow{\scriptstyle \alpha(g)} & & \downarrow{\scriptstyle \alpha(g)} \\ \mathrm{End}_\mathbb{Q}(X) & \xrightarrow[\ '\ ]{} & \mathrm{End}_\mathbb{Q}(X). \end{array}$$

Indeed for every $\varphi \in \mathrm{End}_{\mathbb{Q}}(X)$ it holds

$$\alpha(g)(\varphi') = g'\varphi' g \overset{(2.5.3)}{=} (g'\varphi g)' = (\alpha(g)\varphi)'.$$

Therefore ι is well-defined on $\mathrm{End}_{\mathbb{Q}}(X)^G$. In particular, it is still an involution and positive-definite with respect to trace over $\mathbb{Q}$.

(ii) According to (i), $\mathrm{End}_{\mathbb{R}}(X)^G$ is a finite dimensional $\mathbb{Q}$-algebra with a positive-definite involution ι. Hence by Lemma 11.2.6, it is semisimple. Therefore we get a decomposition

$$\mathrm{End}_{\mathbb{R}}(X)^G = \prod_i \mathcal{A}_i$$

where $\mathcal{A}_i$'s are simple $\mathbb{R}$-algebras. Moreover, according to Remark 11.2.7 the positive-definite involution ι preserves each factor $\mathcal{A}_i$ and so it restricts to a positive-definite involution at each factor $\mathcal{A}_i$. By applying Lemma 11.2.8 at each simple $\mathbb{R}$-algebra $\mathcal{A}_i$ with a positive-definite involution, we obtain the following isomorphism of $\mathbb{R}$-algebras:

$$\Psi\colon (\mathrm{End}_{\mathbb{R}}(X)^G, \iota) \xrightarrow{\simeq} (\prod_i \mathrm{Mat}_{l_i}(\mathbb{R}) \times \prod_j \mathrm{Mat}_{m_j}(\mathbb{C}) \times \prod_k \mathrm{Mat}_{n_k}(\mathbb{H}), \dagger).$$

□

11.2.2 | The G-invariant ample cone of an abelian variety

Definition 11.2.9. We define the $\mathbb{R}$-vector space $\boldsymbol{N^1(X)^G_{\mathbb{R}}}$:= $\{D \in N^1(X)_{\mathbb{R}}$ s.t. G-invariant$\}$. We define the G-invariant ample cone $\mathbf{Amp(X)^G}$:= $\{D \in \mathrm{Amp}(X)$ s.t. G-invariant$\}$.

Theorem 11.2.10. *Let X be an abelian variety and $G \leq \mathrm{Aut}(X)$ be a finite group. Then the G-invariant ample cone is isomorphic to :*

$$(\Psi \circ F)\colon \mathrm{Amp}(X)^G \xrightarrow{\simeq} \bigoplus_i \mathcal{P}_{l_i}(\mathbb{R}) \oplus \bigoplus_j \mathcal{P}_{m_i}(\mathbb{C}) \oplus \bigoplus_k \mathcal{P}_{n_k}(\mathbb{H}) \subseteq \Psi(\mathrm{End}_{\mathbb{R}}(X)^G)$$

where $\mathcal{P}_l(\mathbb{F})$ is the cone of positive-definite hermitian matrices of dimension l over the field $\mathbb{F}$, $F = (f_L)_{|N^1(X)^G_{\mathbb{R}}}$ with f_L as in (2.6.1) and Ψ is defined in Theorem 11.2.5. In particular, it is a homogeneous self-dual cone.

Proof. By Theorem 2.6.2, we have the following isomorphism of $\mathbb{R}$-vector spaces:

$$f_L\colon N^1(X)_{\mathbb{R}} \xrightarrow{\simeq} \mathrm{End}^s_{\mathbb{R}}(X)$$

$$D \longmapsto \phi_L^{-1}\phi_D,$$

where $\mathrm{End}^s_{\mathbb{R}}(X)$ denotes the space of $\mathbb{R}$-endomorphisms on X fixed by the Rosati involution, see Definition 2.6.1. According to Lemma 11.2.3, for every $g \in G$ we have the following commutative diagram:

$$\begin{array}{ccc} N^1(X)_{\mathbb{R}} & \xrightarrow{f_L} & \mathrm{End}^s_{\mathbb{R}}(X) \\ {\scriptstyle g^*}\big\downarrow & & \big\downarrow{\scriptstyle \alpha(g)} \\ N^1(X)_{\mathbb{R}} & \xrightarrow[f_L]{} & \mathrm{End}^s_{\mathbb{R}}(X) \end{array}$$

Therefore, f_L is G-equivariant and the following morphism of $\mathbb{R}$-vector spaces is well-defined:

$$F = (f_L)_{|N^1(X)^G_{\mathbb{R}}} \colon N^1(X)^G_{\mathbb{R}} \longrightarrow \mathrm{End}_{\mathbb{R}}(X)^G$$

$$D \longmapsto \phi_L^{-1}\phi_D.$$

In particular, F is still an embedding since f_L is an embedding. Furthermore, since f_L establishes a bijection between $N^1(X)_{\mathbb{R}}$ and $'$-invariant element in $\mathrm{End}_{\mathbb{R}}(X)$, see Theorem 2.6.1, we get that F establishes a bijection between $N^1(X)^G_{\mathbb{R}}$ and ι-invariant elements in $\mathrm{End}_{\mathbb{R}}(X)^G$. This bijection is actually an isomorphism of vector spaces. Since by Theorem 11.2.5:

$$\Psi \colon (\mathrm{End}_{\mathbb{R}}(X)^G, \iota) \xrightarrow{\simeq} (\prod_i \mathrm{Mat}_{l_i}(\mathbb{R}) \times \prod_j \mathrm{Mat}_{m_j}(\mathbb{C}) \times \prod_k \mathrm{Mat}_{n_k}(\mathbb{H}), \dagger)$$

we have that $(\Psi \circ F)$ maps $\mathbb{R}$-divisors in $N^1(X)^G_{\mathbb{R}}$ to matrices in $\Psi(\mathrm{End}_{\mathbb{R}}(X)^G)$ fixed by $\dagger$, *i.e.* the space of hermitian matrices. More precisely, we obtain the following isomorphisms of vector spaces:

$$(\Psi \circ F) \colon N^1(X)^G_{\mathbb{R}} \xrightarrow{\simeq} \bigoplus_i \mathcal{H}_{l_i}(\mathbb{R}) \oplus \bigoplus_j \mathcal{H}_{m_i}(\mathbb{C}) \oplus \bigoplus_k \mathcal{H}_{n_k}(\mathbb{H}) \subset \Psi(\mathrm{End}_{\mathbb{R}}(X)^G)$$

$$D \longmapsto (\prod_i \phi_i, \prod_j \phi_j, \prod_k \phi_k)$$

As we have recalled in Lemma 2.6.6 the isomorphism $(\psi \circ f_L)$ maps ample $\mathbb{R}$-divisors in $N^1(X)_{\mathbb{R}}$ to positive definite matrices in $\psi(\mathrm{End}_{\mathbb{R}}(X))$. One can easily see that the same proof holds for $(\Psi \circ F)$. Therefore, we obtain:

$$(\Psi \circ F) \colon \mathrm{Amp}(X)^G \xrightarrow{\simeq} \bigoplus_i \mathcal{P}_{l_i}(\mathbb{R}) \oplus \bigoplus_j \mathcal{P}_{m_i}(\mathbb{C}) \oplus \bigoplus_k \mathcal{P}_{n_k}(\mathbb{H}) \subset \Psi(\mathrm{End}_{\mathbb{R}}(X)^G).$$

By Theorem 10.1.4 we know that a cone $C = \bigoplus_i C_i$ is homogeneous self-dual if and only if each indecomposable cone C_i's is too. By the classification of the homogeneous self-dual indecomposable cones, see Theorem 10.1.4, we know that each factor $\mathcal{P}_l(\mathbb{F})$ of $\mathrm{Amp}(X)^G$ is a homogeneous self-dual cone, hence we have that $\mathrm{Amp}(X)^G$ is a homogeneous self-dual cone.

□

11.2.3 | The action of the centralizer on the G-invariant ample cone

We prove that the centralizer $\mathrm{C}_{\mathrm{Aut}(X)}(G)$ defines an arithmetic subgroup in $\mathrm{Aut}(\mathrm{Amp}(X)^G)^0$.

Lemma 11.2.11. *Let X be an abelian variety and $G \leq \mathrm{Aut}(X)$ be a finite subgroup. The group of units $(\mathrm{End}_{\mathbb{R}}(X)^G)^{\times}$ is an affine algebraic group defined over $\mathbb{Q}$ and $\mathrm{C}_{\mathrm{End}(X)^{\times}}(G)$ is an arithmetic subgroup.*

Proof. By Theorem 11.2.5 part (i) $\mathrm{End}_{\mathbb{Q}}(X)^G$, as finite dimensional $\mathbb{Q}$-algebra, is a finite dimensional $\mathbb{Q}$-vector space and, since $\mathrm{End}_{\mathbb{R}}(X)^G = \mathrm{End}_{\mathbb{Q}}(X)^G \otimes \mathbb{R}$, it defines a $\mathbb{Q}$-structure on $\mathrm{End}_{\mathbb{R}}(X)^G$. We set the following isomorphisms of affine spaces:

$$\mathrm{End}_{\mathbb{Q}}(X)^G \simeq \mathbb{Q}^d \qquad \mathrm{End}_{\mathbb{R}}(X)^G \simeq \mathbb{Q}^d \otimes \mathbb{R} \simeq \mathbb{R}^d.$$

We denote by $\mathcal{A}_{\mathbb{Q}}$ the d-dimensional $\mathbb{Q}$-algebra $\mathrm{End}_{\mathbb{Q}}(X)^G$ and by $\mathcal{A}_{\mathbb{R}}$ its extension over $\mathbb{R}$, that we call $\mathrm{End}_{\mathbb{R}}(X)^G$. We consider the following injective map:

$$j\colon (\mathcal{A}_{\mathbb{R}})^{\times} \hookrightarrow \mathcal{A}_{\mathbb{R}} \times \mathcal{A}_{\mathbb{R}}$$

$$x \longmapsto (x, x^{-1}).$$

This map yields to the following description of $(\mathcal{A}_{\mathbb{R}})^{\times}$ as a Zariski closed subset in $\mathcal{A}_{\mathbb{R}} \times \mathcal{A}_{\mathbb{R}} \simeq \mathbb{R}^{2d}$:

$$(\mathcal{A}_{\mathbb{R}})^{\times} \overset{j}{\simeq} \{(x,y) \in \mathcal{A}_{\mathbb{R}} \times \mathcal{A}_{\mathbb{R}} \mid xy - 1 = 0\} = V(xy-1) \subset \mathcal{A}_{\mathbb{R}} \times \mathcal{A}_{\mathbb{R}} \simeq \mathbb{R}^{2d}.$$

Therefore, $(\mathcal{A}_{\mathbb{R}})^{\times}$ is an affine algebraic subgroup $\mathcal{G}$ of the affine space $\mathbb{R}^{2d}$. Since the equation defining $(\mathcal{A}_{\mathbb{R}})^{\times}$ is in fact defined over $\mathbb{Q}$, we have that $(\mathcal{A}_{\mathbb{R}})^{\times}$ is an affine variety defined over $\mathbb{Q}$. Moreover, we observe that the group of $\mathbb{Q}$-points of $(\mathcal{A}_{\mathbb{R}})^{\times}$ is $(\mathcal{A}_{\mathbb{Q}})^{\times}$.
We now prove the arithmetic part. Given an abelian variety $X \simeq \mathbb{C}^n/\Lambda$ there is the following faithful representation, see (2.1.4):

$$\rho_r\colon \mathrm{End}(X) \hookrightarrow \mathrm{End}_{\mathbb{Z}}(\Lambda) \simeq \mathrm{Mat}_{2n}(\mathbb{Z})$$

$$\varphi \longmapsto \widetilde{\varphi}$$

where $\widetilde{\varphi}$ is the unique $\mathbb{C}$-linear map such that $\widetilde{\varphi}(\Lambda) \subseteq \Lambda$ inducing φ. We extend it $\mathbb{R}$-linearly and then we restrict $(\rho_r)_{\mathbb{R}}$ to the group of units of G-invariant $\mathbb{R}$-endomorphisms, leading to the following embedding:

$$(\rho_r)_{\mathbb{R}} \colon (\mathrm{End}_{\mathbb{R}}(X)^G)^{\times} \hookrightarrow \mathrm{GL}_{\mathbb{R}}(\Lambda \otimes \mathbb{R}) \simeq \mathrm{GL}_{2n}(\mathbb{R})$$

which, in particular, is a morphism of $\mathbb{Q}$-algebraic groups, *i.e.* it is a closed embedding. Since $(\rho_r)_{\mathbb{R}}\Big((\mathrm{End}(X)^G)^{\times}\Big) \subset \mathrm{GL}_{2n}(\mathbb{Z})$, we obtain:

$$(\mathrm{End}(X)^G)^{\times} \simeq \mathrm{Im}((\rho_r)_{\mathbb{R}}) \cap \mathrm{GL}_{2n}(\mathbb{Z}) = (\rho_r)_{\mathbb{R}}\Big((\mathrm{End}_{\mathbb{R}}(X)^G)^{\times}\Big) \cap \mathrm{GL}_{2n}(\mathbb{Z}).$$

Therefore, denoting the algebraic group $\mathcal{G} = (\rho_r)_{\mathbb{R}}\Big((\mathrm{End}_{\mathbb{R}}(X)^G)^{\times}\Big)$ we have

$$\mathcal{G}(\mathbb{Z}) \overset{(\rho_r)_{\mathbb{R}}}{\simeq} (\mathrm{End}(X)^G)^{\times},$$

see by Definition 10.2.3. Together with (11.2.5), we prove that that $\mathrm{C}_{\mathrm{End}(X)^{\times}}(G) = (\mathrm{End}(X)^G)^{\times}$ is an arithmetic group in $(\mathrm{End}_{\mathbb{R}}(X)^G)^{\times}$.

□

Lemma 11.2.12. *Let X be an abelian variety and $G \leq \mathrm{Aut}(X)$ finite group. The following $\mathbb{Q}$-morphism of algebraic groups*

$$\beta \colon (\mathrm{End}_{\mathbb{R}}(X)^G)^{\times} \longrightarrow \mathrm{Aut}(\mathrm{Amp}(X)^G)^0 \subset \mathrm{GL}(N^1(X)^G_{\mathbb{R}})$$

$$\varphi \longmapsto \beta(\varphi) \colon x \mapsto \iota(\varphi) \circ x \circ \varphi$$

is surjective.

Proof. It is clear that β is a morphism of algebraic groups. Moreover it is also well-defined as morphism of $\mathbb{Q}$-varieties $(\mathrm{End}_{\mathbb{Q}}(X)^G)^{\times} \longrightarrow \mathrm{GL}(N^1(X)^G_{\mathbb{Q}})$, where $N^1(X)^G_{\mathbb{Q}} = (N^1(X) \otimes \mathbb{Q})^G$.

Assume, for simplicity, that $\mathrm{End}_{\mathbb{R}}(X)^G$ has a single direct factor. Using the notation of Theorem 11.2.5:

$$\Psi \colon (\mathrm{End}_{\mathbb{R}}(X)^G, \iota) \xrightarrow{\simeq} (\mathrm{Mat}_l(\mathbb{F}), \dagger)$$
$$\Psi \colon (\mathrm{End}_{\mathbb{R}}(X)^G)^{\times} \xrightarrow{\simeq} \mathrm{GL}_l(\mathbb{F})$$

and by Theorem 11.2.10 we have

$$(\Psi \circ F) \colon \mathrm{Amp}(X)^G \xrightarrow{\simeq} \mathcal{P}_l(\mathbb{F})$$

where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Thus we have:

$$\begin{array}{ccc} (\mathrm{End}_{\mathbb{R}}(X)^G)^\times & \xrightarrow{\beta} & \mathrm{Aut}(\mathrm{Amp}(X)^G)^0 \\ \simeq\downarrow\Psi & & \Psi\downarrow\simeq \\ GL_l(\mathbb{F}) & \xrightarrow{\beta} & \mathrm{Aut}(\mathcal{P}_l(\mathbb{F}))^0 \end{array} \qquad \begin{array}{ccc} \varphi & \longmapsto & \beta(\varphi)\colon x \mapsto \iota(\varphi)\circ x \circ \varphi \\ \Psi\downarrow & & \downarrow\Psi \\ M & \longmapsto & \beta(M)\colon D \mapsto M^\dagger D M \end{array}$$

Theorem 10.1.5 guarantees the surjectivity of β since every automorphism in $\mathrm{Aut}(\mathcal{P}_l(\mathbb{F}))^0$ is of the form $D \mapsto M^\dagger D M$ with $M \in GL_l(\mathbb{F})$.

If $\mathrm{End}_{\mathbb{R}}(X)^G \simeq \prod_i \mathrm{Mat}_{l_i}(\mathbb{R}) \times \prod_j \mathrm{Mat}_{m_j}(\mathbb{C}) \times \prod_k \mathrm{Mat}_{n_k}(\mathbb{H})$ we can generalize the proof above since for a $C = \bigoplus_i C_i$ the identity component $\mathrm{Aut}(C)^0$ is isomorphic to $\prod_i \mathrm{Aut}(C_i)^0$ by Theorem 10.1.4. Thus, the surjectivity of β follows by applying the previous proof at each factor. □

Remark 11.2.13. We note that by definition β of Lemma 11.2.12 is nothing else than the restriction of α defined in (11.2.2) to $(\mathrm{End}_{\mathbb{R}}(X)^G)^\times$. In particular, since by Theorem 11.2.3 α is the extension of the action of $\mathrm{Aut}(X)$ by pull-backs on $N^1(X)_{\mathbb{R}}$, we deduce that β is the extension of the action of $\mathrm{C}_{\mathrm{Aut}(X)}(G)$ by pull-backs on $N^1(X)_{\mathbb{R}}^G$. Therefore, by the Lemma 11.2.2 we have:

$$\beta(\mathrm{C}_{\mathrm{Aut}(X)}(G)) = \beta(\mathrm{C}_{\mathrm{End}(X)^\times}(G)).$$

Proposition 11.2.14. *Let X be an abelian variety and $G \leq \mathrm{Aut}(X)$ be a finite group. Then the centralizer $\mathrm{C}_{\mathrm{Aut}(X)}(G)$ defines an arithmetic subgroup in $\mathrm{Aut}(\mathrm{Amp}(X)^G)^0$.*

Proof. Let us consider the $\mathbb{Q}$-morphism of algebraic groups:

$$\beta\colon (\mathrm{End}_{\mathbb{R}}(X)^G)^\times \longrightarrow \mathrm{GL}(N^1(X)_{\mathbb{R}}^G).$$

By Lemma 11.2.12 we have $\mathrm{Aut}(\mathrm{Amp}(X)^G)^0$ is the image of β, hence it is an algebraic group defined over $\mathbb{Q}$. By Remark 11.2.13 we have $\beta(\mathrm{C}_{\mathrm{Aut}(X)}(G)) = \beta(\mathrm{C}_{\mathrm{End}(X)^\times}(G))$ and by Lemma 11.2.11 we have that $\mathrm{C}_{\mathrm{End}^\times(X)}(G)$ is an arithmetic subgroup of $(\mathrm{End}_{\mathbb{R}}(X)^G)^\times$. According to [16, Remark 8.22] the property to be arithmetic is preserved under $\mathbb{Q}$-epimorphism. Therefore, we obtain that $\beta(\mathrm{C}_{\mathrm{Aut}(X)}(G))$ is an arithmetic subgroup in $\mathrm{Aut}(\mathrm{Amp}(X)^G)^0$. □

Now, we are in position to prove Theorem 11.2.1

Proof of Theorem 11.2.1. According to Proposition 11.1.1 and Corollary 11.1.3, it is enough to prove the existence of a rational polyhedral cone $\Pi \subset \mathrm{Nef}(X)^G = (\mathrm{Nef}(X)^G)^+$ such that $\mathrm{Amp}(X)^G \subset H \cdot \Pi$ for some $H \leq \mathrm{N}_{\mathrm{Aut}(X)}(G)$. By Theorem 11.2.10, we have that $\mathrm{Amp}(X)^G$ is a homogeneous self-dual cone. By Proposition 11.2.14, the centralizer $H := \mathrm{C}_{\mathrm{Aut}(X)}(G)$ defines an action of an arithmetic group on it. By Theorem 10.3.1.

There exists a rational polyhedral cone $\Pi \subset \mathrm{Nef}(X)^G$ such that $\mathrm{Amp}(X)^G \subset \mathrm{C}_{\mathrm{Aut}(X)}(G) \cdot \Pi$. □

12 On the nef cone of Calabi-Yau manifolds of type A

In this chapter we prove Theorem E about the rational polyhedral nature of the nef cone of hyperelliptic varieties deducing also that all Calabi-Yau manifolds Y of type A constructed in Theorem A have a rational polyhedral cone. Then in Theorem F we describe the extremal rays of the nef cone for these Calabi-Yau manifolds, deducing in particular that all nef divisors of X are semi-ample divisors

12.1 | Polyhedral nature of nef cone of hyperelliptic varieties

In part I, we have studied the Calabi-Yau threefolds of type A and since they are hyperelliptic varieties, they satisfy the cone conjecture according to Theorem 11.2.1. In fact this latter observation was already proven in [73, Theorem 0.1 (IV)] and, specifically, it is proven that for these threefolds the nef cone is rational polyhedral. This result agrees with the finiteness of the automorphisms group for these threefolds, see Lemma 9.4.7. It is natural to ask the following question:

Question 5. Given a hyperelliptic variety $Y = X/G$, under which conditions its nef cone of is rational polyhedral? Or equivalent, under which conditions its automorphism group of is finite?

We observe that this question is related to the one asked in [42, Section 6].

Definition 12.1.1. Let K be a totally complex quadratic extension of a totally number field of degree g over $\mathbb{Q}$. A **CM-type of K** is a set $\Phi = \{\sigma_1, \dots, \sigma_n\}$ of non complex conjugate embeddings $K \hookrightarrow \mathbb{C}$.
An abelian variety $X = \mathbb{C}^n/\Lambda$ is said to be **of CM-type** (K, Φ) if there exists an embedding $\eta \colon K \hookrightarrow \mathbb{C}$ where $\rho_a \circ \eta = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$.

Theorem 12.1.2. *Let $Y = X/G$ be a hyperelliptic variety and we denote by η the representation of G. We assume that X is not of CM-type. If G contains a normal abelian group H such that $\eta_{|H}$ does not contains two equals irreducible sub-representations, then the nef cone of Y is a rational polyhedral cone.*

Proposition 12.1.3. *Let $Y = X/G$ be a projective hyperelliptic n-fold with G an abelian group. Assume that X is not of CM-type and that the representation η of G does not contains two equal irreducible representations. Then $\mathrm{C}_{\mathrm{End}(X)^\times}(G)$ is finite. In particular, every $\alpha \in \mathrm{C}_{\mathrm{End}(X)^\times}(G)$ has order two.*

Proof. By Proposition 1.8.10 the representation $\eta\colon G \longrightarrow \mathrm{GL}(\mathbb{C}^n)$ decomposes into the direct sum $\bigoplus_{i=1}^{n} \eta_i$ of irreducible representations η_i and since G is abelian $\dim(\eta_i) = 1$ for every i, by Lemma 1.8.11. In other words, $\eta(g)$ has a diagonal form for every $g \in G$. Let $\alpha \in \mathrm{End}(X)^\times$: it can be though as a linear transformation of $V = \mathbb{C}^n$. We have that $\alpha \in \mathrm{C}_{\mathrm{End}(X)^\times}(G)$ if and only if for every $g \in G$ the following diagram is commutative

$$\begin{array}{ccc} V & \xrightarrow{\eta(g)} & V \\ \alpha\downarrow & & \downarrow\alpha \\ V & \xrightarrow[\eta(g)]{} & V \end{array}$$

which can be rephrased as follows: α is a G-linear transformation of V with respect to η. By hypothesis $\eta = \bigoplus_{i=1}^{n} \eta_i$ with $\eta_i \neq \eta_j$ for every $j \neq i = 1, \dots, n$. By the Schur's Lemma 1.8.9, we have that $\alpha = \mathrm{diag}(\mu_1, \dots, \mu_n)$. Indeed since $\eta_i \neq \eta_j$ for any $i \neq j$ we have that

$$\alpha\colon V_i \longrightarrow V_j$$

is zero whenever $i \neq j$ where $\eta_k\colon G \longrightarrow \mathrm{GL}(V_i)$. Since $\alpha \in \mathrm{End}(X)^\times = \mathrm{GL}_n(\mathbb{Z})$ we get that $\mu_i \in \{\pm 1\}$, equivalently, α has finite order equals to two. Therefore, $\mathrm{C}_{\mathrm{End}(X)^\times}(G)$ is finite. ☐

Now we are in position to prove Theorem 12.1.2.

proof of Theorem 12.1.2. By assumption $\eta_{|H}$ does not contain two equal irreducible sub-representations. Thus $\mathrm{C}_{\mathrm{End}(X)^\times}(H)$ is finite and since $\mathrm{C}_{\mathrm{End}(X)^\times}(G) \leq \mathrm{C}_{\mathrm{End}(X)^\times}(H)$ then $\mathrm{C}_{\mathrm{End}(X)^\times}(G)$ is finite. According to the Theorem 11.2.1, Y satisfies the Morrison's cone conjecture, specifically there exists a rational polyhedral cone $\Pi \subset \mathrm{Nef}(Y)$ such that

1. $\mathrm{Nef}(Y) = \dfrac{\mathrm{C}_{\mathrm{End}(X)^\times}(G)}{G} \cdot \Pi$,
2. Π is a fundamental domain.

Since $\dfrac{\mathrm{C}_{\mathrm{End}(X)^\times}(G)}{G}$ is finite, following the proof of Lemma 9.4.7 we deduce that $\mathrm{Nef}(Y)$ is rational polyhedral. □

As we have already observed in Section 9.4, having a rational polyhedral cone does not imply that the automorphism group is finite. We recall an easy example.

Example 11. Let $S = E \times F/G$ be a bi-elliptic surface. According to [80] its nef cones is rational polyhedral. But its automorphism group is not finite since the elliptic curve on which G acts as translation is contained in $\mathrm{C}_{\mathrm{Aut}(E\times F)}(G))$, see [12]. Thus $\mathrm{Aut}(S)$ is infinite since it contains translations.

The example above shows that given $Y = X/G$ a hyperelliptic variety even if we can control the cardinality of the centralizer, we can have some infinity part in $\mathrm{Aut}(Y)$ coming from translations on X. A different situation appear under the hypothesis that the hyperelliptic manifolds has trivial irregularity.

Corollary 12.1.4. *Let $Y = X/G$ be a hyperelliptic variety. In the same hypothesis of Theorem 12.1.2. Assume moreover that $h^{1,0}(Y) = 0$. Then the nef cone is rational polyhedral and the automorphism group is finite.*

Proof. The first assertion is Theorem 12.1.2. For the second one: we have that $\mathrm{C}_{\mathrm{End}(X)^\times}(G)$ is finite. Let us consider the following homomorphism of groups:

$$\varphi\colon \mathrm{N}_{\mathrm{Aut}(X)}(G) \longrightarrow \mathrm{Aut}(G)$$

$$f \longmapsto \varphi(f)\colon (g \mapsto f^{-1}gf)$$

whose kernel is $\mathrm{C}_{\mathrm{Aut}(X)}(G)$. Since G is finite we have $\mathrm{Aut}(G)$ is finite. Therefore we obtain:

$$\mathrm{C}_{\mathrm{Aut}(X)}(G) \text{ is finite if and only if } \mathrm{N}_{\mathrm{Aut}(X)}(G) \text{ is finite.} \tag{12.1.1}$$

Let us consider:

$$\rho\colon \mathrm{Aut}(X) \longrightarrow \mathrm{GL}(N^1(X)_{\mathbb{R}})$$

$$f \longmapsto f^*$$

which according to Lemma 9.4.8 has finite kernel, since $h^{1,0}(Y) = 0$. According to Lemma 11.2.2 we have that $\rho\Big(\mathrm{C}_{\mathrm{Aut}(X)}(G)\Big) = \rho\Big(\mathrm{C}_{\mathrm{End}(X)^\times}(G)\Big)$. Since $\mathrm{C}_{\mathrm{End}(X)^\times}(G)$ is finite and ρ has finite kernel then $\mathrm{C}_{\mathrm{Aut}(X)}(G)$ must be finite. Thus by (12.1.1), $\mathrm{N}_{\mathrm{Aut}(X)}(G)$ is finite and since G is finite we have that $\mathrm{Aut}(Y) = \dfrac{\mathrm{N}_{\mathrm{Aut}(X)}(G)}{G}$ is finite. □

Corollary 12.1.5. *All the Calabi-Yau manifolds of type A in Theorem A have rational polyhedral nef cone and finite automorphism group.*

Proof. It's enough to check that for the known examples of Calabi-Yau manifolds of type A enclosed in Theorem 5.6.6, we are in the hypothesis of Corollary 12.1.4. □

12.2 | The extremal rays of the nef cone

In [73, Theorem 0.1], the authors proved that the nef cone of the Calabi-Yau threefolds Y of type A is rational polyhedral and they also described explicitly the extremal rays of the nef cone. More in details, they proved that the extremal rays are the divisors which generate $\mathrm{Pic}_{\mathbb{Q}}(Y)$ which define fibrations on Y. Thus, in particular, the extremal rays are semi-ample divisors. As a consequence, they deduce that each rational nef divisor is semi-ample.

Since in Corollary 12.1.5, we prove that all Calabi-Yau manifolds of type A in Theorem A have rational polyhedral nef cone, we investigate on the extremal rays of this cone generalizing the result of Oguiso and Sakurai. More precisely, we obtain the following.

Theorem 12.2.1. *Let Y be the Calabi-Yau manifold of type A with the group $(\mathbb{Z}/2\mathbb{Z})^{2n}$. Then every nef divisor is semi-ample.*

Proof. We recall:

$$A = E_1 \times \cdots \times E_{2n+1}$$

where E_j's are elliptic curves. By Lemma 5.6.5 we have that $h^{1,1}(Y) = 2n+1$ and since the $h^{2,0}(Y) = 0$ it holds $\rho(Y) = h^{1,1}(Y) = 2n+1$, where $\rho(Y)$ denotes the Picard rank. We consider the following fibrations on A:

$$p_j \colon A \longrightarrow E_j \quad j = 1, \dots, 2n+1.$$

Each p_j is G-equivariant, *i.e.* the following diagram is commutative:

$$\begin{array}{ccc} p_j \colon A & \longrightarrow & E_j \\ \downarrow & & \downarrow \\ f_j \colon Y = A/G & \longrightarrow & E_j/G_{|E_j} \simeq \mathbb{P}^1 \end{array}$$

In particular, f_j's are still fibrations. We denote by $[F_j]$ the class of the fiber of f_j for $j = 1, \dots, 2n+1$, which in particular are nef divisors.

Claim 1: $\{[F_1], \dots, [F_{2n+1}]\}$ defines a $\mathbb{Q}$-basis of $\mathrm{Pic}_{\mathbb{Q}}(Y)$. We have already observed that $\mathrm{rk}(\mathrm{Pic}_{\mathbb{Q}}(Y)) = 2n+1$, hence it is enough to prove that the F_j's are linearly independent. We note that these divisors have no trivial intersection with each other. Let H be an ample line bundle on Y, the following relations hold:

$$(F_j)^l \cdot H^{2n+1-l} = 0 \quad 1 < l \leq 2n+1 \tag{12.2.1}$$
$$F_{i_1} \cdot F_{i_2} \dots F_{i_k} \cdot H^{2n+1-k} \neq 0 \quad i_1 \leq \cdots \leq i_k \in 1, \dots, 2n+1. \tag{12.2.2}$$

Let us consider F_1 and F_2: if F_2 was linearly dependent from F_1, by (12.2.1) we would have $F_1 \cdot F_2 \dots F_{2n} \cdot H^{2n+1-2} = 0$ which contradicts (12.2.2). Thus, $[F_1]$ and $[F_2]$ are linearly independent. Iterating the argument above to $\mathcal{B} := \{[F_1], \dots, [F_{2n+1}]\}$, we prove that $\mathcal{B}$ defines a $\mathbb{Q}$-basis of $\mathrm{Pic}_{\mathbb{Q}}(Y)$.
Claim 2: $\{[F_1], \dots, [F_{2n+1}]\}$ span the nef cone of Y, *i.e.*:

$$\mathrm{Nef}(Y) = \mathbb{R}_{\geq 0}[F_1] \oplus \cdots \oplus \mathbb{R}_{\geq 0}[F_{2n+1}] := C. \tag{12.2.3}$$

Since F_j's are nef divisors, the inclusion "$\supseteq$" follows. For the reverse inclusion: let us consider H an ample divisor on Y. Since $H \in \mathrm{Amp}(Y) \subset \mathrm{Pic}_{\mathbb{R}}(Y)$, by Claim 1 there exist $c_i \in \mathbb{R}$ such that:

$$H = \sum_{i=1}^{2n+1} c_i [F_i].$$

The followings hold:

$$0 < H \cdot \prod_{i \neq j=1}^{2n+1} F_j = c_i \text{ for all } i = 1, \dots, 2n+1.$$

Thus, $\mathrm{Amp}(Y) \subseteq C \subseteq \mathrm{Nef}(Y)$. Since C is closed and contains $\mathrm{Amp}(Y)$, by minimality $C \supseteq \mathrm{Nef}(Y)$ and so the equality holds. In conclusion, the nef cone of Y it is a rational polyhedral cone generated by $[\overline{F_j}]$ for $j = 1, \dots, 2n+1$.
Finally, we observe that F_j's are semi-ample divisors and so $\mathrm{Nef}(Y)$ is generated by semi-ample divisors. Let us take $[D] \in \mathrm{Nef}(Y) \cap N^1_{\mathbb{Q}}(Y) \subset \mathrm{Pic}_{\mathbb{Q}}(Y)$, we have:

$$D \equiv \sum_{i=1}^{2n+1} q_i F_i \qquad q_i \in \mathbb{Q}.$$

Since $h^{1,0}(Y) = 0$ then $\mathrm{Pic}^0(X) = 0$. Therefore there exist $m \in \mathbb{Z}$ such that

$$mD \sim_{lin} m \sum_{i=1}^{2n+1} q_i F_i$$

where $\sim_{lin}$ denotes the linear equivalence. Thus, since F_i's are semi-ample divisors we deduce that mD is a semi-ample divisor and so D is. $\square$

Remark 12.2.2. The proof above can be applied whenever we have a hyperelliptic variety $Y = A/G$ with $h^{1,0}(Y) = 0$ with exactly $h^{1,1}(Y)$ linearly independent fibrations.

In conclusion, combining Theorem 12.2.1, [73, Theorem 0.1], Theorem 7.2.1 and Theorem 6.3.2 we obtain the following.

Corollary 12.2.3 (Theorem F)**.** *Let Y be the Calabi-Yau manifold of type A as in Theorem D. Then extremal rays of the nef cone are given by semi-ample divisors which define fibrations on X induced by natural projections on A. In particular all nef divisors are semi-ample divisors.*

Part III

Further questions and investigations

Se credi abbastanza in te stesso
e con abbastanza coraggio,
tutto è possibile.

* Problem 1 *

In all examples of the Calabi-Yau manifolds of type A, presented in this thesis, they admit an étale cover that splits into the product of elliptic curves. We observe that even if this is the easiest construction, it is not expected in general to be the only one. In fact, more generally, this is not expected to be true also for hyperelliptic manifolds. For instance in [28, Chapter 3] the author showed that there exists hyperelliptic manifold whose cover splits into the product of lower dimensional abelian varieties but not all of them have dimension 1. Thus we can expect a similar phenomena for Calabi-Yau manifolds of type A. Indeed one of the naif way to construct Calabi-Yau manifolds of type A is to look at the actions producing hyperelliptic manifolds and select the one that give Calabi-Yau quotients. Thus we ask the following question.

Problem 1. *Are there Calabi-Yau manifolds of type A whose cover is not isogenous to the product of elliptic curves?*

* Problem 2 *

Another investigation is to consider more in details the "up to étale cases" in the decomposition theorem 1.5.6, with a focus on Calabi-Yau quotients. For instance, non-simply connected Calabi-Yau threefolds play a central role in the study of string compactifications. As we observe in Section 5.5, Calabi-Yau threefolds of quotient type can be obtained in different ways. In dimension three we can distinguish three main situations: the one given as free quotient of a simply connected Calabi-Yau threefolds, the Calabi-Yau threefolds of type A and the one of type K. We highlight that in the first case we obtain a Calabi-Yau quotient with finite fundamental group, while in the last two cases the fundamental group is infinite. The first situation is undertaken in [17] and [38] and in this situation the simply-connected Calabi-Yau threefolds are mainly given as complete intersections in projective spaces. The last two situations are studied in [73], [32], [44], [45] and here. In higher dimension more situations appear. For instance, the generalization of Calabi-Yau threefolds of type K in higher dimension seems an interesting problem. We consider the following example.

Example 12. Let S and S' be two K3 surfaces. Let ι_S be an Enriques involution on S and $\iota_{S'}$ be a non symplectic involution on S', *i.e.* it does not preserve $\omega_{S'}$. Then

$\iota = \iota_S \times \iota_{S'}$ defines a free action on $S \times S'$ which preserves the volume form. Thus the quotient $Y = (S \times S')/\iota$ is a fourfold with trivial canonical bundle. Moreover, $H^{1,0}(Y) = H^{3,0}(Y) = 0$ since $H^{1,0}(S) = H^{1,0}(S') = 0$. Since ι_S and $\iota_{S'}$ are non-symplectic involutions then $H^{2,0}(Y) = 0$. Hence Y is a Calabi-Yau fourfold.

Example 13. Let S be a K3 surface and E, E' two elliptic curves. Let ι_S be an Enriques involution on S, ι_E and $\iota_{E'}$ be the hyperelliptic involution on E and E', respectively, and t_E and $t_{E'}$ be translations of order two on E and E', respectively. Let $\alpha = \iota_S \times \iota_E \times t_{E'}$ and $\beta = \iota_S \times t_E \times \iota_{E'}$, then $G\langle\alpha, \beta\rangle$ define a free action on $S \times E \times E'$ which preserves the volume form. One can easily check that G does not preserve any $(i, 0)$-form for $i = 1, 2, 3$. Thus $Y = (S \times E \times E')G$ is a Calabi-Yau fourfold.

The examples above shows that when one tries to generalize in higher dimension the Calabi-Yau threefolds of type K as Calabi-Yau manifolds whose universal cover contains a K3 surface different situations appear. We also recall as showed in [45] that Calabi-Yau threefolds of type K are interesting also from the point of view of the Mirror Symmetry. Therefore, it seems natural investigate the following problem.

Problem 2. *Define and study Calabi-Yau manifolds whose étale cover contains at least one $K3$ surface.*

* Problem 3 *

According to Theorem F, we know that all nef divisors of all the Calabi-Yau manifolds given in Theorem A are semi-ample. We observe that this result is in fact related with one of the main conjecture in MMP. In Chapter 9.1 we have briefly introduced the Abundance conjecture 9.1.12. We recall that it predicts that given a $\mathbb{Q}$-factorial variety Y with $K_{Y'}$ nef then $K_{Y'}$ is semi-ample. For the state of ars on this conjecture we refer to [59]. For K-trivial varieties, the Abundance conjecture has an alternative statement.

Conjecture 12.2.4. [59, Conjecture 4.8] Let Y be a projective manifold with $H^1(Y, \mathcal{O}_Y) = 0$ such that $c_1(Y) = 0$. If D is a nef divisor on Y, then Y is semi-ample.

The condition $H^1(X, \mathcal{O}_X) = 0$ is needed to exclude the case of abelian varieties. We deduce that Theorem F, guarantees the validity of Conjecture 12.2.4 for all Calabi-Yau manifolds of type A in Theorem A (which for the best of the author's knowledge are the only ones known). Therefore, it is natural to investigate following problem:

Problem 3. *Under which conditions Calabi-Yau manifolds of type A satisfy the Conjecture 12.2.4? Or, more in general, under which conditions hyperelliptic varieties with no* $(1,0)$*-forms satisfy the Conjecture 12.2.4?*

We hope to return to the problems mentioned above in the future.

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