NON-POINTED ABELIAN CATEGORIES

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ABSTRACT. We study a property (P) of pushouts of regular epimorphisms along monomorphisms in a regular context. We prove that (P) characterizes abelian categories among homological ones. In the non-pointed case, we show that (P) implies the normality (in the sense of Bourn) of all subobjects, that any protomodular category satisfying (P) is naturally Mal'tsev, and that an exact category is penessentially affine if and only if it is protomodular and satisfies (P). An example of such a category is the one whose objects are the abelian extensions over an object in a (strongly) semi-abelian category; by exploiting some observations in this context, we also provide a characterization of strongly semi-abelian categories by a variant of the axiom of normality of unions.

1. Introduction

Abelian categories have proved to be an extremely useful notion in many fields of mathematics. They can be defined by the so-called "Tierney Equation"

$$\text{Abelian} = \text{Additive} + \text{(Barr-)Exact},$$

which makes it clear how exactness in the sense of Barr [1] plays a fundamental role in the context of additive categories.

Various notions have been introduced as weakenings of abelianness, often by weakening one or both parts of Tierney's equation.

From this point of view, in [12], starting from a characterization of the categories of affine spaces as certain slices of categories of modules, Carboni introduced *modular categories* as a non-pointed version of additive categories. He showed that a category $\mathcal{C}$ is modular if and only if it satisfies the following conditions:

(1) the category of pointed objects $\mathbf{Pt}_1(\mathcal{C}) = 1\backslash\mathcal{C}$ is additive;

(2) $\mathcal{C}$ is equivalent to $\mathbf{Pt}_1(\mathcal{C})/(1 \to 1+1)$.

Note that $\mathcal{C}$ is pointed and modular if and only if it is additive.

After Carboni's result, Bourn studied the categories $\mathbf{Pt}_B(\mathcal{C}) = \mathbf{Pt}_1(\mathcal{C}/B)$, where B is any object of $\mathcal{C}$ [4]. In fact these categories may be seen as the fibers of the codomain functor $\mathbf{Pt}(\mathcal{C}) \to \mathcal{C}$, which is also known as the fibration of points when $\mathcal{C}$ has pullbacks. There has been extensive interest in properties relating to this functor, as illustrated by

2000 Mathematics Subject Classification: 18E08,18E10,18E13.
Key words and phrases: Abelian categories, Regular categories, Strongly semi-abelian categories.

[5, 10, 9]. An example of such a property is *protomodularity*, where the change-of-base functors between the fibers are required to be conservative [4].

By contrast with modularity, protomodularity, in the pointed context, is weaker than additivity, since it includes categories such as (non-abelian) groups. Stronger properties can be considered as non-pointed versions of additivity; for example, a category is:

- *naturally Mal'tsev* [18] if and only if all fibers $\mathbf{Pt}_B(\mathcal{C})$ are additive;
- *penessentially affine* [9] if all change-of-base functors of the fibration of points are fully faithful and create subobjects;
- *essentially affine* [4] if and only if all change-of-base functors of the fibration of points are equivalences.

All these notions coincide with additivity for pointed categories, and any of these is strictly stronger than the previous one in the list.

Protomodularity plays a key role in the definition of semi-abelian [17] and homological categories, introduced by Borceux and Bourn in [2]. In fact, homological categories are defined by weakening both summands of Tierney's Equation, since they are only regular, instead of exact, and pointed protomodular, instead of additive. Therefore, additive regular categories, including, for instance, the category of torsion free abelian groups, are examples of homological categories. In this context, many of the homological lemmas (such as the Five Lemma, the Snake Lemma, the Nine Lemma), as well as the Noether Isomorphisms Theorems, are still valid.

In this paper, we are going to consider a new point of view on a possible notion of non-pointed abelian category. We rely on a more classical description of an abelian category; namely, an abelian category is a pointed category with finite limits and colimits where all monomorphisms and epimorphisms are normal [14].

We will study, in a regular category, the following simple property, which may be seen as a generalization of the fact that every monomorphism is the kernel of its cokernel:

(P) For every span $Z \xleftarrow{p} X \xrightarrow{m} Y$ where p is a regular epimorphism and m is a monomorphism, their pushout exists and is also a pullback.

We investigate the consequences of (P), also in conjunction with protomodularity, and prove in Proposition 2.6 that categories satisfying these requirements share with abelian categories the property that every monomorphism is Bourn-normal in the sense of [7]. This observation shows that such categories are, in particular, naturally Mal'tsev. We then prove that, for a sequentiable category [8], (P) also implies (Barr)-exactness. Then, in Theorem 2.11, we prove that this property characterizes abelian categories among the homological ones.

We then show that exact protomodular categories satisfying (P) may be seen as a new non-pointed version of abelian categories, which turns out to be weaker than another possible one, namely that of exact essentially affine categories. In fact, we prove in

Theorem 3.5 that, for exact categories, property (P) characterizes penessentially affine categories among the protomodular ones. Furthermore, we show that a category which is regular protomodular, satisfies (P), and whose dual is also protomodular, is essentially affine.

We provide several examples and counterexamples of categories with property (P). An interesting example is the category of abelian extensions of an object in a semi-abelian category. By exploiting an observation from this example in the particular case of groups, we realized that we can provide two different characterizations of strongly semi-abelian categories (see Theorems 4.3 and 4.5). The second one is given by means of a variant of the axiom of normality of unions, which was introduced in [3] in relation to the representability of internal actions, and is also related to the existence of normalizers in a semi-abelian category [15].

2. Property (P) and its consequences

We are interested in studying regular categories with the following property, which holds in every abelian category:

(P) For every span $Z \xleftarrow{p} X \xrightarrow{m} Y$ where p is a regular epimorphism and m is a monomorphism, their pushout exists and is also a pullback.

We observe that in the category of groups this property is not satisfied, since there are subgroups which are not normal, and for m the inclusion of such a subgroup, taking $Z = 0$ we observe that (P) would imply that m is the kernel of its cokernel (see Proposition 2.6).

We will see that other examples can be obtained by taking slices, co-slices, and points in a category satisfying (P). Recall that the category of points $\mathbf{Pt}(\mathcal{C})$ [4] is the category whose

- objects are pairs of morphisms $p\colon X \to B$ and $s\colon B \to X$ such that $ps = 1_B$;
- morphisms $(p, s) \to (q, t)$ are pairs (f, g) of morphisms such that $gp = qf$ and $fs = tg$.

If $\mathcal{C}$ has pullbacks and pushouts, then the codomain functor $\mathbf{Pt}(\mathcal{C}) \to \mathcal{C}$ is both a (Grothendieck) fibration and opfibration (see for example [4]); a morphism of points

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, shift left, "p"] & Y \arrow[d, shift left, "q"] \\ B \arrow[r, "g"'] \arrow[u, shift left, "s"] & C \arrow[u, shift left, "t"] \end{tikzcd}$$

is then

- cartesian if the downward square is a pullback, in which case we say that $g^*(q, t) = (p, s)$;

- cocartesian if the upward square is a pushout, in which case we say that $g_*(p,s) = (q,t)$.

Thus, if $\mathcal{C}$ has pullbacks and pushouts, any $g\colon B \to C$ defines an adjunction

$$\begin{tikzcd} \mathbf{Pt}_B(\mathcal{C}) \arrow[r, bend left, "g_*"] \arrow[r, phantom, "\perp"] & \mathbf{Pt}_C(\mathcal{C}) \arrow[l, bend left, "g^*"] \end{tikzcd}$$

between the fibers over B and C.

2.1. Proposition. *If a regular category $\mathcal{C}$ satisfies (P), then every slice or co-slice of $\mathcal{C}$ also satisfies it. In particular, all the categories $\mathbf{Pt}_B(\mathcal{C})$ satisfy it.*

Proof. Since $\mathcal{C}$ is regular, $\mathcal{C}/B$ and $B\backslash\mathcal{C}$ are regular as well, and every monomorphism, regular epimorphism, pullback or pushout in $\mathcal{C}/B$ or $B\backslash\mathcal{C}$ is also a monomorphism, regular epimorphism, pullback or pushout in $\mathcal{C}$. ∎

A simple consequence of property (P) is that monomorphisms are stable under pushouts along regular epimorphisms, a property which is known to hold in all abelian categories:

2.2. Proposition. *Let $\mathcal{C}$ be a regular category satisfying (P), and let*

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "p"'] \arrow[dr, phantom, "(1)"] & Y \arrow[d, "u"] \\ Z \arrow[r, "v"'] & T \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

be a pushout square, where p is a regular epimorphism and m is a monomorphism. Then v is a monomorphism.

Proof. By property (P), the square (1) is a pullback. Let $v = nq$ be the factorization of v as a regular epimorphism followed by a monomorphism; since $\mathcal{C}$ is regular, this factorization is stable under pullbacks, so that we may consider the diagram

$$\begin{tikzcd} X \arrow[r, "q'"] \arrow[d, "p"'] \arrow[dr, phantom, "(1')"] & U \arrow[r, "n'"] \arrow[d, "h"] \arrow[dr, phantom, "(1'')"] & Y \arrow[d, "u"] \\ Z \arrow[r, "q"'] & V \arrow[r, "n"'] & T \end{tikzcd}$$

where both squares are pullbacks and $n'q' = m$. Then q' is a regular epimorphism and a monomorphism, thus it is an isomorphism. Since $(1')$ is a pullback of regular epimorphisms, it is also a pushout (by Theorem 5.2 in [13]), hence q is an isomorphism as well, which implies that v is a monomorphism. ∎

In a regular protomodular category, property (P) can be formulated alternatively:

2.3. Proposition. *Let $\mathcal{C}$ be a regular protomodular category. The following conditions are equivalent:*

(1) $\mathcal{C}$ satisfies (P);

(2) for every span as displayed in (P), there is a square

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "p"'] & Y \arrow[d, "u"] \\ Z \arrow[r, "v"'] & T \end{tikzcd}$$

which is a pullback.

Proof. That the first condition implies the second is clear. If the second condition holds, then we can write the morphism u as nq with n a monomorphism and q a regular epimorphism; since p is a regular epimorphism, v factors through n as $v = nt$, and we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "m"'] & Z \arrow[r, equal] \arrow[d, "t"] & Z \arrow[d, "v"] \\ Y \arrow[r, "q"'] & W \arrow[r, "n"'] & T \end{tikzcd}$$

where the whole rectangle and the right-hand square are pullbacks, and thus the left-hand square is a pullback as well. Since $\mathcal{C}$ is protomodular and regular, and q is a regular epimorphism, it is also a pushout by Proposition 14 in [4]. ■

In [13], the authors prove that a regular category is exact Mal'tsev if and only if pushouts of regular epimorphisms along regular epimorphisms exist and are regular pushouts, i.e. in the diagram

$$\begin{tikzcd} X \arrow[dr, "\langle p,f\rangle"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "p"'] & & \\ & Z \times_W Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "p'"] \\ & Z \arrow[r, "f'"'] & W \end{tikzcd} \qquad (1)$$

where p and f are regular epimorphisms and W is their pushout, the canonical morphism $\langle p, f\rangle\colon X \to Z \times_W Y$ is a regular epimorphism. In fact, when $\mathcal{C}$ is exact, (P) is closely related to that property:

2.4. Proposition. *Let $\mathcal{C}$ be a regular protomodular category. The following conditions are equivalent:*

(1) $\mathcal{C}$ is exact and satisfies (P);

(2) pushouts of arbitrary morphisms along regular epimorphisms exist and the canonical morphisms into the pullbacks are regular epimorphisms;

(3) for every span $Z \xleftarrow{p} X \xrightarrow{f} Y$ where p is a regular epimorphism and p, f are jointly monomorphic, their pushout exists and is also a pullback;

(4) for every span $Z \xleftarrow{p} X \xrightarrow{f} Y$ where p is a regular epimorphism and p, f are jointly monomorphic, there is a square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "u"] \\ Z \arrow[r, "v"'] & T \end{tikzcd}$$

which is a pullback.

Proof. (1) $\Rightarrow$ (2): If $\mathcal{C}$ is exact protomodular, it is also a Mal'tsev category, so that pushouts of regular epimorphisms along regular epimorphisms exist and are regular pushouts. Let $\mathcal{C}$ also satisfy (P). Then given a span

$$Z \xleftarrow{p} X \xrightarrow{f} Y$$

where p is a regular epimorphism, we write $f = mq$ for the regular epi-mono factorization of f. By taking pushouts, we get a diagram

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "p"'] & M \arrow[r, "m"] \arrow[d, "p'"] & Y \arrow[d, "p''"] \\ Z \arrow[r, "q'"'] & W \arrow[r, "m'"'] & T \end{tikzcd}$$

where both squares are pushouts. The left-hand square is then a pushout of regular epimorphisms, and thus a regular pushout, while the right-hand square is a pushout of a regular epimorphism and a monomorphism, so it is a pullback by (P). Hence, we have an isomorphism $Z \times_T Y \simeq Z \times_W M$, and the induced arrow $\langle p, f \rangle$ is isomorphic to $\langle p, q \rangle$, and it is a regular epimorphism as well.

(2) $\Rightarrow$ (3): Let p, f be jointly monomorphic, and let us consider their pushout. Then the canonical morphism $\langle p, f \rangle$ into the pullback is a monomorphism; since it is also a regular epimorphism, then it must be an isomorphism, which means that the square is a pullback.

(3) $\Rightarrow$ (4) is clear.

(4) ⇒ (1): Just observe that both exactness of $\mathcal{C}$ and property (P) are special cases of (4) where p, f are the projections of some equivalence relation, and where p is a regular epimorphism and f a monomorphism, respectively. Note that if p, f are the projections of an equivalence relation, they have a common section, so that $uf = vp$ implies $u = v$, hence the coequalizer of p and f can be computed as their pushout. ∎

In an abelian category every monomorphism is normal.

This property is in fact closely related to (P), as we will now see. Let us first recall the definition of Bourn-normal monomorphism:

2.5. DEFINITION. [7] *A morphism* $m\colon M \to X$ *is said to be* Bourn-normal *to an equivalence relation* R *on* X *if* m *induces a morphism of equivalence relations*

$$\begin{tikzcd} M \times M \arrow[r, "\widetilde{m}"] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & R \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] \\ M \arrow[r, "m"'] & X \end{tikzcd}$$

which is a discrete fibration, i.e. both commutative squares above are pullbacks.

For example, in a concrete category, a monomorphism is Bourn-normal to an equivalence relation R if it is (up to isomorphism) the injection of an equivalence class of R. It turns out that, in a semi-abelian category, Bourn-normal subobjects coincide with normal subobjects.

2.6. PROPOSITION. *If* $\mathcal{C}$ *is regular and satisfies (P), then every monomorphism* $m\colon M \to X$ *is Bourn-normal to some effective equivalence relation.*

PROOF. Consider the factorization of $M \to 1$ as a regular epimorphism $p\colon M \to Z$ followed by a monomorphism. Then the kernel pair of p is the indiscrete relation on M, and by (P) the pushout

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "p"'] & X \arrow[d, "p'"] \\ Z \arrow[r, "m'"'] & T \end{tikzcd}$$

is a pullback. Taking kernel pairs of p and p' then gives a discrete fibration

$$\begin{tikzcd} M \times M \arrow[r, "\widetilde{m}"] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] & Eq[p'] \arrow[d, shift right, "p'_1"'] \arrow[d, shift left, "p'_2"] \\ M \arrow[r, "m"'] & X \end{tikzcd}$$

which means that m is Bourn-normal to $Eq[p']$. ∎

In [2], Corollary 3.2.15, it is proved that any protomodular category where every subobject is Bourn-normal has an important additional property: it is naturally Mal'tsev.

2.7. Definition. [18, 5] *A category is* naturally Mal'tsev *if all fibers* $\mathbf{Pt}_B(\mathcal{C})$ *are additive.*

2.8. Corollary. *If $\mathcal{C}$ is regular protomodular and satisfies (P), then $\mathcal{C}$ is naturally Mal'tsev.*

If, furthermore, $\mathcal{C}$ is such that for every equivalence relation there exists a subobject which is Bourn-normal with respect to it, then $\mathcal{C}$ is exact.

Proof. By Proposition 2.6, we can apply Corollary 3.2.15 of [2], which proves that $\mathcal{C}$ is naturally Mal'tsev.

If R is an equivalence relation on an object X and $M \leq X$ is a subobject which is Bourn-normal with respect to R, the previous result shows that M is Bourn-normal with respect to an effective equivalence relation R'. Since $\mathcal{C}$ is protomodular, this implies that $R \cong R'$ (see [7]), so that R is effective. ∎

This corollary applies in particular to any pointed or quasi-pointed category, since in such a category every equivalence relation R admits a normalization $N_R = r_2 \ker(r_1)$, which is a Bourn-normal subobject with respect to R. More generally, it suffices that every object of $\mathcal{C}$ admits a morphism $a\colon A \to X$ where A is subterminal. We recall that a category is called

- *quasi-pointed* if it admits an initial object 0, a terminal object 1, and the unique arrow $0 \to 1$ is a monomorphism;
- *sequentiable* if it is protomodular, regular and quasi-pointed [8].

In a quasi-pointed category, we say that an object X has *trivial support* if there exists an arrow $X \to 0$; in the regular context, this is equivalent to the morphism $X \to 1$ having its image equal to $0 \to 1$.

2.9. Definition. [8] *In a quasi-pointed category $\mathcal{C}$, the kernel of a morphism f is the pullback of f along the morphism from the initial object:*

$$\begin{tikzcd} K[f] \arrow[r, "\ker(f)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y. \end{tikzcd}$$

If X has trivial support, then the cokernel of $f\colon X \to Y$ is the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X \arrow[d, "\mathrm{coker}(f)"] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & C[f]. \end{tikzcd}$$

Note that this notion of cokernel is not dual to the notion of kernel; if a quasi-pointed category has finite limits and colimits, then every arrow has a kernel, but not necessarily a cokernel. However, in [8], it is proved that in a sequentiable category every kernel has a cokernel, and every regular epimorphism is the cokernel of its kernel.

We can now give an alternative formulation of property (P) in sequentiable categories:

2.10. Proposition. *If $\mathcal{C}$ is sequentiable, then the following conditions are equivalent:*

(1) $\mathcal{C}$ satisfies (P);

(2) every monomorphism $m\colon M \to X$ whose domain has trivial support is normal;

(3) normal monomorphisms are stable under composition with monomorphisms.

Proof. (1) ⇒ (2): if $\mathcal{C}$ satisfies (P), and m is a monomorphism whose domain has trivial support, the morphism $M \to 0$ is a regular epimorphism, and thus we have a square

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d] & X \arrow[d, "q"] \\ 0 \arrow[r] & Q \end{tikzcd}$$

which is a pushout and a pullback, which means $q = \mathrm{coker}(m)$ and $m = \ker(q)$.

(2) ⇒ (3): it suffices to observe that the composition of a normal monomorphism and a monomorphism is always a monomorphism whose domain has trivial support.

(3) ⇒ (1): if we have a span $Z \xleftarrow{p} X \xrightarrow{m} Y$ where p is a regular epimorphism and m is a monomorphism, let us denote $n = m\ker(p)$, which is a normal monomorphism under our hypothesis. Then it must be the kernel of $q = \mathrm{coker}(n)$, and since $qm\ker(p) = qn$ factors through 0, and p is the cokernel of its kernel, there is a morphism t such that $tp = qm$. Then in the diagram

$$\begin{tikzcd} K \arrow[r, "\ker(p)"] \arrow[d] & X \arrow[r, "m"] \arrow[d, "p"] & Y \arrow[d, "q"] \\ 0 \arrow[r] & Z \arrow[r, "t"'] & Q, \end{tikzcd}$$

the left-hand square and the whole rectangle are both pushouts, since p and q are cokernels, so that the right-hand square is a pushout as well. They are also both pullbacks, since $\ker(p)$ and n are normal monomorphisms; then, since p is a regular epimorphism and $\mathcal{C}$ is regular protomodular, the right-hand square is also a pullback by Proposition 13 in [4]. ∎

As a corollary, we immediately get a characterization of abelian categories amongst homological ones:

2.11. Theorem. *A category $\mathcal{C}$ is abelian if and only if it is homological and satisfies (P).*

Proof. Any homological category satisfying (P) is pointed and naturally Mal'tsev, thus additive [18], and exact; hence it is abelian by Tierney's equation. ∎

2.12. Remark. If $\mathcal{C}$ satisfies (P), then all fibers $\mathbf{Pt}_B(\mathcal{C})$ do so as well. In particular, if $\mathcal{C}$ is protomodular and regular, then all these fibers are abelian, and in particular, exact.

Note that if R is a reflexive relation on the object X, then $r\colon R \to X \times X$ may be seen as a subobject in $\mathbf{Pt}_X(\mathcal{C})$, where it must be normal; in particular, it must be a regular monomorphism in $\mathcal{C}$, which is a necessary condition to be an effective equivalence relation.

It is not clear whether property (P) implies exactness in general. One thing we can say is the following:

2.13. Proposition. *If $\mathcal{C}$ is regular protomodular, satisfies (P), admits coequalizers of equivalence relations, and if, for all X, the functor $X \times \_$ preserves coequalizers of equivalence relations, then $\mathcal{C}$ is exact.*

Proof. Given an equivalence relation $r = \langle r_1, r_2\rangle\colon R \to X \times X$, let $q\colon X \to Q$ be the coequalizer of r_1 and r_2. We then have a diagram

$$\begin{tikzcd}
X \times R \arrow[r, shift left, "X\times r_1"] \arrow[r, shift right, "X\times r_2"'] \arrow[dr, shift left, "\pi_1"] & X \times X \arrow[r, "X\times q"] \arrow[d, shift left, "\pi_1"] & X \times Q \arrow[dl, shift left, "\pi_1"] \\
& X \arrow[ul, shift left, "{\langle 1,\delta_R\rangle}"] \arrow[u, shift left, "{\langle 1,1\rangle}"] \arrow[ur, shift left, "{\langle 1,q\rangle}"] &
\end{tikzcd}$$

in $\mathbf{Pt}_X(\mathcal{C})$. By hypothesis, $X \times q$ is the coequalizer of $X \times r_1$ and $X \times r_2$ in $\mathcal{C}$, and thus also in $\mathbf{Pt}_X(\mathcal{C})$. Furthermore $X \times R$ is an equivalence relation on $X \times X$ in $\mathbf{Pt}_X(\mathcal{C})$. Since $\mathbf{Pt}_X(\mathcal{C})$ is exact by Remark 2.12, $X \times R$ is then the kernel pair of its coequalizer, which means that the middle square in the commutative diagram

$$\begin{tikzcd}
R \arrow[r, "{\langle r_1,1\rangle}"] \arrow[d, "r_1"'] & X \times R \arrow[r, "X\times r_2"] \arrow[d, "X\times r_1"'] & X \times X \arrow[r, "\pi_2"] \arrow[d, "X\times q"] & X \arrow[d, "q"] \\
X \arrow[r, "{\langle 1,1\rangle}"'] & X \times X \arrow[r, "X\times q"'] & X \times Q \arrow[r, "\pi_2"'] & Q
\end{tikzcd}$$

is a pullback. Since the other two squares are pullbacks as well, this proves that the whole rectangle is a pullback, so that R is the kernel pair of q. ■

Note that any regular, non-exact, additive category with coequalizers, such as the categories of torsion-free abelian groups or topological abelian groups, satisfies all the hypotheses of the previous proposition, except property (P). The same happens for any semi-localization of an exact protomodular category, since there coequalizers of equivalence relations exist and are stable under pullbacks [19], but in general those categories are not exact.

3. Affineness

We have seen that property (P), for a homological category, is equivalent to abelianness; and that for a regular protomodular (not necessarily pointed) category it implies the property of being naturally Mal'tsev. As we've mentioned before, some other notions have been introduced, and shown to be equivalent to additivity in a pointed context. As it turns out, in the exact protomodular context, our property is in fact equivalent to the notion of *penessentially affine* category. This result yields an elementary characterization of the latter categories.

First we need to introduce a notion:

3.1. Definition. *Let $G\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that G* creates subobjects *if, for every monomorphism $n\colon N \to G(X)$, there exist a monomorphism $m\colon M \to X$ and an isomorphism $\varphi\colon N \to G(M)$ such that $n = G(m)\varphi$.*

Note that G creates subobjects if and only if, for any subobject n of $G(X)$, there exists a subobject m of X such that $G(m) = n$. Moreover, if G preserves finite limits, it restricts, for every object X of $\mathcal{C}$, to $Sub(X) \to Sub(G(X))$. In this case, G is conservative if and only if it is conservative on monomorphisms. This happens if and only if, when m is a subobject of X such that $G(m) = G(1_X)$ (as subobjects of $G(X)$), then $m = 1_X$ (as subobjects of X).

We will need the following criterion for when a right adjoint functor creates subobjects:

3.2. Proposition. *Let $F \dashv G$ be a pair of adjoint functors such that $G\colon \mathcal{C} \to \mathcal{D}$ reflects monomorphisms. Then G creates subobjects if and only if each component $\eta_Y\colon Y \to GF(Y)$ of the unit of the adjunction is an extremal epimorphism.*

Proof. If G creates subobjects, let us first show that, given a monomorphism $n\colon N \to G(X)$, we may take $\varphi = \eta_N$ and m to be the morphism $\widehat{n}\colon F(N) \to X$ corresponding to n under the bijection $\mathcal{D}(N, G(X)) \simeq \mathcal{C}(F(N), X)$; indeed, since $n = G(\widehat{n})\eta_N$, η_N is itself a monomorphism, and we may write $\eta_N = G(l)\psi$ for some monomorphism $l\colon L \to F(N)$ and isomorphism $\psi\colon N \to G(L)$. We also have $\psi = G(\widehat{\psi})\eta_N$ (where, again, $\widehat{\psi}\colon F(N) \to L$ correspond to ψ under the adjunction), and thus $\eta_N = G(l\widehat{\psi})\eta_N$, so that $l\widehat{\psi} = 1$. Since l is a monomorphism and a split epimorphism, it is then an isomorphism, and thus η_N is an isomorphism as well. In particular, for every monomorphism $n\colon N \to G(X)$, $G(\widehat{n})$ is also a monomorphism, and so $\widehat{n}$ is a monomorphism as well.

Now if Y is an object of $\mathcal{D}$ such that η_Y factors as ne for some monomorphism $n\colon N \to GF(Y)$, then we have

$$G(\widehat{n})GF(e)\eta_Y = G(\widehat{n})\eta_N e = ne = \eta_Y,$$

and thus $\widehat{n}F(e) = 1$. Since $\widehat{n}$ is a monomorphism it must then be an isomorphism, and thus $n = G(\widehat{n})\eta_N$ is an isomorphism.

Conversely, if all components of η are extremal epimorphisms, then if $n\colon N \to G(X)$ is a monomorphism, η_N is an isomorphism, and thus $G(\widehat{n})$ and $\widehat{n}$ are also monomorphisms. Since $n = G(\widehat{n})\eta_N$, G creates subobjects. ∎

3.3. Proposition. *If $\mathcal{C}$ has finite limits and $G\colon \mathcal{C} \to \mathcal{D}$ is a right adjoint functor that is conservative and creates subobjects, then G is fully faithful.*

Proof. If G preserves limits and reflects isomorphisms, it also reflects limits, and as a consequence it reflects monomorphisms. We then have the diagonal identity $G(\epsilon)\eta_G = 1$, with η_G an extremal epimorphism by the previous result. Thus η_G and $G(\epsilon)$ are isomorphisms, and since G is conservative, ϵ is an isomorphism, so that G is fully faithful. ∎

3.4. Definition. *A category with pullbacks is said to be*

(1) penessentially affine *[9] if all change-of-base functors g^* of the fibration of points are fully faithful and create subobjects;*

(2) essentially affine *[4] if all change-of-base functors of the fibration of points are equivalences of categories.*

The notion of penessentially affine category was introduced as a non-pointed variant of an additive category, in order to study the construction of Baer sums. As we now show, for an exact category it is equivalent to our axiom.

3.5. Theorem. *If $\mathcal{C}$ is exact, then $\mathcal{C}$ is penessentially affine if and only if it is protomodular and satisfies (P).*

Proof. Let us first assume that $\mathcal{C}$ is penessentially affine. Given a span

$$Z \xleftarrow{p} X \xrightarrow{m} Y$$

with p a regular epimorphism and m a monomorphism, we have a pullback

$$\begin{tikzcd}
X \times Y \arrow[r, "m\times 1"] \arrow[d, shift left, "\pi_1"] & Y \times Y \arrow[d, shift left, "\pi_1"] \\
X \arrow[r, "m"'] \arrow[u, shift left, "\langle 1,m\rangle"] & Y, \arrow[u, shift left, "\Delta"]
\end{tikzcd}$$

which means that $X \times Y = m^*(Y \times Y)$. Let $E[p] \mathrel{\substack{\xrightarrow{p_1} \\ \xleftarrow{\delta} \\ \xrightarrow[p_2]{}}} X$ be the kernel pair of p. Then we have a monomorphism

$$\begin{tikzcd}
E[p] \arrow[rr, "(1\times m)\langle p_1,p_2\rangle"] \arrow[dr, shift left, "p_1"] & & X \times Y \arrow[dl, shift left, "\pi_2"] \\
& X \arrow[ul, shift left, "\delta"] \arrow[ur, shift left, "\langle 1,m\rangle"] &
\end{tikzcd}$$

in $\mathbf{Pt}_X(\mathcal{C})$. Since m^* creates subobjects, there must be a subobject

$$\begin{tikzcd} S \arrow[rr, "\langle s_1,s_2\rangle"] \arrow[dr, shift left, "s_1"] & & Y\times Y \arrow[dl, shift left, "\pi_1"] \\ & Y \arrow[ul, shift left, "\delta_S"] \arrow[ur, shift left, "\Delta"] & \end{tikzcd}$$

in $\mathbf{Pt}_Y(\mathcal{C})$ such that $m^*\langle s_1, s_2\rangle = (1\times m)\langle p_1, p_2\rangle$. In particular, S is then a reflexive relation on Y. Since $\mathcal{C}$ is penessentially affine, and hence a Mal'tsev category [9], S is an equivalence relation. Since moreover $\mathcal{C}$ is exact, S is the kernel pair of its coequalizer q_S. Since we have a diagram

$$\begin{tikzcd} E[p] \arrow[r] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

where the commutative squares are pullbacks, taking coequalizers of the vertical pairs of arrows gives a pullback square

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "p"'] & Y \arrow[d, "q_S"] \\ Z \arrow[r, "m'"'] & Y/S \end{tikzcd}$$

which is also a pushout.

Let us now assume that $\mathcal{C}$ is protomodular and satisfies (P), and let $g\colon B\to C$ be a morphism in $\mathcal{C}$. We recall that if (p, s) is an object of $\mathbf{Pt}_B(\mathcal{C})$, then its image (p', s') under g_* is obtained by taking the pushout of g and s, and $g^*(p', s')$ is then obtained by taking the pullback of g and p'. Thus we have a diagram

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "s"'] & C \arrow[d, "s'"] \\ X \arrow[r, "g'"] \arrow[d, "p"'] & Y \arrow[d, "p'"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

where the upper square is a pushout, and since the vertical compositions are identities, the whole rectangle is a pushout, so that the lower square is a pushout as well. Then, by Proposition 2.4, the induced morphism $X\to B\times_C Y$, which coincides with the component at (p, s) of the unit of the adjunction $g_*\dashv g^*$, is a regular epimorphism. Hence g^* creates subobjects by Proposition 3.2, and is fully faithful by Proposition 3.3, which proves that $\mathcal{C}$ is penessentially affine. ∎

Any essentially affine category is penessentially affine, and thus satisfies (P) if it is exact. The converse is not true, even for an exact category, since for example $\mathbf{Gpd}_B(\mathbf{Grp})$ is penessentially affine and exact but not essentially affine, as reported in [9]. However, if $\mathcal{C}$ is essentially affine, then its dual $\mathcal{C}^{op}$ is also essentially affine, and thus protomodular. We say that $\mathcal{C}$ is co-protomodular if its dual is protomodular. If we add this assumption, we obtain the following result:

3.6. Proposition. *If $\mathcal{C}$ is regular, protomodular, co-protomodular, and satisfies (P), then $\mathcal{C}$ is essentially affine.*

Proof. Let us first prove that the pushout functor g_* is fully faithful. As explained at the end of the proof of Theorem 3.5, this is equivalent to the fact that, for any diagram

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "s"'] \arrow[dr, phantom, "(1)"] & C \arrow[d, "t"] \\
X \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "(2)"] & Y \arrow[d, "q"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

where the vertical sides are identities, if the square (1) is a pushout then the square (2) is a pullback.

Let us first prove that this holds whenever f, g are either both monomorphisms or both regular epimorphisms.

If f, g are regular epimorphisms and (1) is a pushout, then it is also a pullback by (P), and then (2) is a pullback since $\mathcal{C}$ is regular, protomodular and f is a regular epimorphism. Now if f, g are monomorphisms, and (1) is a pushout, then (2) is also a pushout, and thus it is a pullback by (P).

Now, in general, the regular epi-mono factorizations $f = f_1 f_2$ and $g = g_1 g_2$ give a diagram

$$\begin{tikzcd}
B \arrow[r, "g_2"] \arrow[d, "s"'] \arrow[dr, phantom, "(1'')"] & D \arrow[r, "g_1"] \arrow[d, "u"] \arrow[dr, phantom, "(1')"] & C \arrow[d, "t"] \\
X \arrow[r, "f_2"'] \arrow[d, "p"'] & Z \arrow[r, "f_1"'] \arrow[d, "r"] & Y \arrow[d, "q"] \\
B \arrow[r, "g_2"'] & D \arrow[r, "g_1"'] & C
\end{tikzcd}$$

where $ru = 1$; if the square (1) was a pushout, then, since g_2 is a (regular) epimorphism, the square $(1')$ is a pushout. Then, since $\mathcal{C}$ is co-protomodular and u is a split monomorphism, $(1'')$ is also a pushout. Then both bottom squares are pullbacks, and thus (2) is a pullback as well.

Now, for every g, since the pullback functor is conservative and its left adjoint is fully faithful, the unit of the adjunction is an isomorphism. The triangular identities then imply that the counit is an isomorphism as well, so the pullback functor is an equivalence. ∎

Thanks to Theorem 3.5, we can give a few new examples and counterexamples of categories satisfying (P).

3.7. Example. Let $\mathcal{C} = \mathbf{Mal}(\mathbf{Grp}/B)$ be the full subcategory of $\mathbf{Grp}/B$ of morphisms with abelian kernels. Then $\mathcal{C}$ is the category of Mal'tsev objects (i.e. objects equipped with an internal Mal'tsev operation) in $\mathbf{Grp}/B$, and is thus naturally Mal'tsev (see Example 2.4.6 in [2]); it is also quasi-pointed, protomodular and exact, but since it is not penessentially affine, as proved in [9], it does not satisfy (P).

We can show it directly by providing a monomorphism which is not Bourn-normal, which contradicts Proposition 2.6. Consider some morphism $f\colon G \to B$ with abelian kernel $k\colon K \to G$. We may consider k as a morphism in $\mathbf{Mal}(\mathbf{Grp}/B)$:

$$\begin{tikzcd}
K \arrow[rr, equal] \arrow[d, equal] & & K \arrow[d, "k"] \\
K \arrow[rr, "k"] \arrow[dr, "0"'] & & G \arrow[dl, "f"] \\
& B. &
\end{tikzcd}$$

Then any subgroup of K is normal in K, but not necessarily in G; suppose M is a subgroup of K which is not normal in G (see an explicit example below). Then the inclusion $m\colon M \to K$ may be seen as a monomorphism in $\mathbf{Mal}(\mathbf{Grp}/B)$, thus km gives a subobject of f:

$$\begin{tikzcd}
M \arrow[r, "m"] \arrow[d, equal] & K \arrow[r, equal] \arrow[d, equal] & K \arrow[d, "k"] \\
M \arrow[r, "m"] \arrow[dr, "0"'] & K \arrow[r, "k"] \arrow[d, "0"] & G \arrow[dl, "f"] \\
& B. &
\end{tikzcd}$$

The forgetful functor $\mathbf{Mal}(\mathbf{Grp}/B) \to \mathbf{Grp}$ preserves limits, and so, if km was Bourn-normal to some equivalence relation R in $\mathbf{Mal}(\mathbf{Grp}/B)$, it would also be Bourn-normal to R in $\mathbf{Grp}$; but this can't be true since km is not normal. Thus km is not Bourn-normal in $\mathbf{Mal}(\mathbf{Grp}/B)$.

For an explicit example, consider as G the alternating group A_4 on 4 elements, and as f the quotient morphism $q\colon A_4 \to A_4/K$, where K is the normal subgroup $\{id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Then the 2-element subgroups of K are all conjugates in A_4, and thus they are not normal.

3.8. Example. On the other hand, the category $\mathbf{AbExt}_B(\mathbf{Grp})$ of abelian extensions over a fixed group B (i.e. the full subcategory of $\mathbf{Grp}/B$ containing the surjective morphisms with abelian kernels) is essentially affine, and thus satisfies (P). This applies, more generally, to any category of abelian extensions in the sense of [11] over a fixed object in a semi-abelian category (see Theorem 7.3.5 in [2]). We can also show this directly for the category of groups, which will later help us to obtain a new characterization of strongly semi-abelian categories.

First we observe that this subcategory is closed under regular epimorphisms in $\mathbf{Grp}/B$, and thus it is also closed under pushouts along regular epimorphisms. As a consequence, it is enough to check that given a monomorphism m and a regular epimorphism p with the same domain in $\mathbf{AbExt}_B(\mathbf{Grp})$

$$\begin{tikzcd}
K[q_Z] \arrow[d, "\ker(q_Z)"'] & K[q_X] \arrow[l, "\widetilde{p}"'] \arrow[r, "\widetilde{m}"] \arrow[d, "\ker(q_X)"] & K[q_Y] \arrow[d, "\ker(q_Y)"] \\
Z \arrow[dr, "q_Z"'] & X \arrow[l, "p"'] \arrow[r, "m"] \arrow[d, "q_X"] & Y \arrow[dl, "q_Y"] \\
& B, &
\end{tikzcd}$$

the pushout of p and m is also a pullback in $\mathbf{Grp}$, which is equivalent to the composition $m\ker(p)$ being normal in $\mathbf{Grp}$.

Since

$$q_X \ker(p) = q_Z p \ker(p) = 0,$$

$\ker(p)$ must factor through $\ker(q_X)$, so that $m\ker(p)$ factors through $m\ker(q_X) = \ker(q_Y)\widetilde{m}$. Thus we can consider the diagram

$$\begin{tikzcd}
K[p] \arrow[r, "\ker(p)"] \arrow[d, "m'"'] & X \arrow[r, "p"] \arrow[d, "m"] & Z \arrow[d, "q_Z"] \\
K[q_Y] \arrow[r, "\ker(q_Y)"'] & Y \arrow[r, "q_Y"'] & B.
\end{tikzcd}$$

Now, $q_X = q_Y m$ is surjective, so for any $y \in Y$ there is some $x \in X$ such that $q_Y(m(x)) = q_Y(y)$. Then any $y \in Y$ may be written as $m(x)k$ for some $x \in X$ and $k \in K[q_Y]$, and in particular we have

$$yK[p]y^{-1} = m(x)kK[p]k^{-1}m(x)^{-1} = K[p],$$

since $K[p]$ is normal in X and in $K[q_Y]$ (since $K[q_Y]$ is abelian).

3.9. Example. If $\mathcal{C}$ is an exact Mal'tsev category, then all categories $\mathbf{Gpd}_B(\mathcal{C})$ are penessentially affine [9] and exact, and thus they satisfy (P). If $\mathcal{C}$ is semi-abelian, then the category $\mathbf{XMod}_B(\mathcal{C})$ of internal B-crossed modules in $\mathcal{C}$ is equivalent to $\mathbf{Gpd}_B(\mathcal{C})$ [16], and thus it also satisfies (P).

3.10. Example. Suppose $\mathcal{C}$ is a semi-abelian category with a diagram

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "f"'] & X \arrow[r, shift left, "p"] \arrow[d, "g"] & B \arrow[l, shift left, "s"] \arrow[d, equal] \\
A' \arrow[r, "j'"'] & X' \arrow[r, shift left, "p'"] & B \arrow[l, shift left, "s'"]
\end{tikzcd}$$

where j, j' are the kernels of p, p' respectively, A, A' are abelian, f, g are monomorphisms, and gj is not normal (see for example Counterexample 10.3 in [3]). Then, by forgetting the splittings, we obtain an example showing that (P) does not hold in the category of extensions over B with abelian kernels. On the other hand, by using each split epimorphism twice we may also see g as a monomorphism in the category of reflexive graphs, or alternatively, we may see f as a monomorphism in the category of B-precrossed modules. Since the kernels are abelian, f is in fact a morphism between Peiffer B-precrossed modules [20]; but since gj is not normal in X', f is not a normal monomorphism of precrossed modules. Thus the category of Peiffer precrossed modules does not always satisfy (P) when the "Smith is Huq" condition [21] does not hold.

4. A characterization of strongly semi-abelian categories

In Example 3.8, most of the proof is actually valid in any semi-abelian category. In some parts, however, we need some additional assumptions. When trying to find precisely for which categories the proof would work, we realized that the main property needed is in fact equivalent to the notion of *strongly semi-abelian category*:

4.1. Definition. *A semi-abelian category $\mathcal{C}$ is* strongly semi-abelian *when it is strongly protomodular [6], which in this case amounts to the following property: for any morphism of split short exact sequences*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & X \arrow[r, shift left, "q"] \arrow[d, "v"] & Y \arrow[l, shift left, "s"] \arrow[d, equal] \\ K' \arrow[r, "k'"'] & X' \arrow[r, shift left, "q'"] & Y', \arrow[l, shift left, "s'"] \end{tikzcd}$$

if u is normal then so is $k'u = vk$.

Note that the only place where the specificity of the category **Grp** seems important in Example 3.8 is in the proof that $K[p]$ is normal in Y. In fact, that proof relies on the observation that Y is the join (in its lattice of subobjects) of the two subgroups $K[q_Y]$ and $m(X)$, in which $K[p]$ is normal. This means that our proof would in fact work in any semi-abelian category that satisfies the following condition:

4.2. Definition. [3] *A semi-abelian category $\mathcal{C}$ is said to satisfy the* axiom of normality of unions *if, whenever a subobject N of X is normal in two subobjects A, B of X, N is normal in the join $A \vee B$.*

This property is linked to the representability of actions and to normalizers of subobjects (in fact, for varieties, it is equivalent to the existence of normalizers [15]). Any *category of interest* in the sense of [22] satisfies the axiom of normality of unions [3].

The axiom of normality of unions implies strong protomodularity [3], but the converse is false. For example, the category of non-associative rings is strongly semi-abelian (since it is a variety of distributive Ω_2-groups [20]) but it does not satisfy the axiom of normality of unions. As a counterexample, consider the non-associative ring formed by the free abelian group on 4 generators x, y, z, w, endowed with the multiplication defined by the table

$\cdot$	x	y	z	w
x	x	x	x	w
y	x	y	w	w
z	x	w	z	w
w	w	w	w	w

and bilinearly extended. Note that we have subrings $N = \langle x \rangle$, as well as $A = \langle x, y \rangle$ and $B = \langle x, z \rangle$. N is normal in A and B, since it is closed under multiplication by x, y and z. But, since $w = y \cdot z$, we have $A \vee B = X$, and N is not normal in X, because it is not closed under multiplication by w.

Note, however, that in Example 3.8 we only needed to use the axiom of normality of unions in the case where one subgroup considered in the union is normal. This special case will turn out to be relevant, since it is in fact equivalent to strong protomodularity.

Let us first begin with another characterization of strongly semi-abelian categories:

4.3. Theorem. *Let $\mathcal{C}$ be a semi-abelian category. Then $\mathcal{C}$ is strongly semi-abelian if and only if, whenever we have a morphism of short exact sequences*

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "u"'] & X \arrow[r, "q"] \arrow[d, "v"] & Y \arrow[d, "w"] \\ K' \arrow[r, "k'"'] & X' \arrow[r, "q'"'] & Y' \end{tikzcd}$$

with u a normal monomorphism and w a regular epimorphism, the composition $k'u$ is a normal monomorphism as well.

Proof. It is clear that any semi-abelian category satisfying the required condition must be strongly semi-abelian.

Assuming now that $\mathcal{C}$ is strongly semi-abelian, by taking the pullbacks of q along itself and of wq along q' and considering the restriction to the kernels, we obtain the following commutative diagram:

$$\begin{tikzcd}
K \arrow[rr, equal] \arrow[dr, "u"] \arrow[dd, "{\langle 0,k\rangle}"'] & & K \arrow[dr, "u"] \arrow[dd, "k"] & \\
& K' \arrow[rr, equal] \arrow[dd, "{\langle 0,k'\rangle}"'] & & K' \arrow[dd, "k'"] \\
X \times_Y X \arrow[rr, "p_2"] \arrow[dr, "1\times v"] \arrow[dd, shift right, "p_1"'] & & X \arrow[dr, "v"] \arrow[dd, "q"] & \\
& X \times_{Y'} X' \arrow[rr, "p_2'"] \arrow[dd, shift right, "p_1'"'] & & X' \arrow[dd, "q'"] \\
X \arrow[uu, shift right, "{\langle 1,1\rangle}"'] \arrow[rr, "q"] \arrow[dr, equal] & & Y \arrow[dr, "w"] & \\
& X \arrow[uu, shift right, "{\langle 1,v\rangle}"'] \arrow[rr, "wq"'] & & Y'.
\end{tikzcd}$$

Now $\langle 0, k\rangle$ (resp. $\langle 0, k'\rangle$) is the kernel of p_1 (resp. p_1'), so that $u, 1 \times v$ and 1_X form a morphism of split short exact sequences. Since $\mathcal{C}$ is strongly semi-abelian, the fact that u is normal implies that $\langle 0, k'\rangle u$ is normal as well. But, since wq is a regular epimorphism, so is p_2', and thus the image of $\langle 0, k'\rangle u$ along p_2' is normal. Since $k'u = p_2'\langle 0, k'\rangle u$, this means that $k'u$ is normal. ■

In order to have the desired characterization of strongly semi-abelian categories by means of a weakening of the axiom of normality of unions, we first need the following lemma:

4.4. Lemma. *Let $\mathcal{C}$ be a homological category, and consider a morphism of short exact sequences*

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] & X \arrow[r, "q"] \arrow[d, "v"] & Y \arrow[d, "w"] \\
K' \arrow[r, "k'"'] & X' \arrow[r, "q'"'] & Y'.
\end{tikzcd}$$

Then w is a regular epimorphism if and only if v and k' are jointly strongly epimorphic.

Proof. Assume first that w is a regular epimorphism, and let $m\colon M \to X'$ be a monomorphism through which k' and v factor as $k' = mk''$ and $v = mv'$. This yields a commutative diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "u"'] & X \arrow[r, "q"] \arrow[d, "v'"] & Y \arrow[d, "w'"] \\
K' \arrow[r, "k''"] \arrow[d, equal] \arrow[dr, phantom, "(1)"] & M \arrow[r, "q''"] \arrow[d, "m"] & Y'' \arrow[d, "m'"] \\
K' \arrow[r, "k'"'] & X' \arrow[r, "q'"'] & Y',
\end{tikzcd}$$

where q'' is the cokernel of k'' and m', w' are the maps induced by m and v', respectively. Notice that the square (1) is a pullback, so that k'' is the kernel of $q'm$ and m' is a monomorphism. We have

$$m'w'q = m'q''v' = q'mv' = q'v = wq,$$

hence $m'w' = w$, since q is a regular epimorphism, and thus m' is a regular epimorphism, in fact an isomorphism. By the Short Five Lemma, m is then an isomorphism.

Conversely, if v and k' are jointly strongly epimorphic, let $w = me$ be the regular epi-mono factorization of w, and let m' be the pullback of m along q':

$$\begin{tikzcd}
X'' \arrow[r, "q''"] \arrow[d, "m'"'] & Y'' \arrow[d, "m"] \\
X' \arrow[r, "q'"'] & Y'.
\end{tikzcd}$$

Then m' is a monomorphism, and since $meq = wq = q'v$, there exists $v' \colon X \to X''$ such that $m'v' = v$ and $q''v' = eq$. Being $q'k' = 0$, there is also $k'' \colon K' \to X''$ such that $q''k'' = 0$ and $m'k'' = k'$. Then m' and m are isomorphisms, hence w is a regular epimorphism. ■

From this lemma, we immediately obtain the last characterization of strongly semi-abelian categories:

4.5. Theorem. *A semi-abelian category is strongly semi-abelian if and only if, whenever a subobject N of X is normal in two subobjects A, B with A normal in X, N is normal in the join $A \vee B$.*

Acknowledgements

We thank the referee for providing useful comments and suggestions, and particularly for suggesting to us a simpler proof for Proposition 2.13.

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