

1 Compactness via monotonicity in nonsmooth critical point theory, with
2 application to Born-Infeld type equations

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17 **Abstract**

18 In this paper, we prove new existence and multiplicity results for critical points of lower semicon-
19 tinuous functionals in Banach spaces, complementing the nonsmooth critical point theory set forth by
20 Szulkin and avoiding the need of the Palais–Smale condition. We apply our abstract results to get
21 entire solutions with finite energy to Born-Infeld type autonomous equations. More precisely, under
22 almost optimal conditions on the nonlinearity, we construct a positive solution and infinitely many
23 solutions both in the classes of radially symmetric functions and nonradially symmetric ones.

24 **MSC2020:** 35A15; 58E05; 58E35; 35B38; 35J62.

25 **Keywords:** Nonsmooth critical point theory; Monotonicity trick; critical points; Born–Infeld equations;
26 Existence and nonexistence of solutions.

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4 1 Introduction and main abstract results

5 Let $(X, \|\cdot\|)$ be a Banach space and consider functionals of the type

$$I : X \rightarrow (-\infty, \infty], \quad I(u) \equiv \Psi(u) - \Phi(u),$$

6 where we assume that Ψ and Φ satisfy the following properties:

7 (Ψ_1) $\Psi : X \rightarrow [0, \infty]$ is lower semicontinuous and convex, and $\Psi(0) = 0$;

8 (Φ_1) $\Phi \in C^1(X, \mathbb{R})$.

9 We denote

$$D(\Psi) \equiv \{ u \in X \mid \Psi(u) < \infty \}.$$

10 This kind of functionals arises, for instance, in some problems involving PDEs with constraints for admis-
11 sible functions. A typical example we are interested in is the following:

$$\Psi(u) \equiv \begin{cases} \int_{\mathbb{R}^n} 1 - \sqrt{1 - |Du|^2} & \text{if } \|Du\|_\infty \leq 1, \\ \infty & \text{otherwise,} \end{cases} \quad \Phi(u) \equiv \int_{\mathbb{R}^n} F(u),$$

12 which comes from the prescribed mean curvature problem for space-like hypersurfaces in the Lorentz-
13 Minkowski space

$$\mathbb{L}^{n+1} \equiv \mathbb{R} \times \mathbb{R}^n \quad \text{with metric} \quad -dx_0^2 + \sum_{j=1}^n dx_j^2.$$

14 In Lorentzian geometry, space-like hypersurfaces with prescribed mean curvature play a major role high-
15 lighted, for instance, in Marsden & Tipler [47]. A space-like hypersurface whose mean curvature is a
16 given function $f : \mathbb{L}^{n+1} \rightarrow \mathbb{R}$ is described, at least locally, by the graph $x_0 = u(x_1, \dots, x_n)$ of a solution
17 $u : \Omega \rightarrow \mathbb{R}$ to

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) + f(x, u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n. \quad (1)$$

18 Formally, (1) is the Euler-Lagrange equation of the action

$$I(u) \equiv \int_{\Omega} 1 - \sqrt{1 - |Du|^2} - \int_{\Omega} F(x, u), \quad F(x, s) \equiv \int_0^s f(x, t) dt,$$

19 which is non-smooth where $|Du| = 1$ (in the terminology of [47], where the graph of u goes null).
20 Therefore, even assuming that critical points of I (in a non-smooth sense) exist, they may not correspond
21 to solutions to (1), a fact that makes the existence problem quite challenging.

22 Equation (1) also appears in the framework of the Born-Infeld theory for electromagnetism [18],
23 according to which the identity describes the interplay between an electrostatic potential u and the
24 charge density it generates, required, in this specific example, to be $f(x, u)$. If f is independent of u ,
25 including the case of f a mere Radon measure, in recent years a few authors investigated the existence and
26 regularity properties of solutions to (1). The resulting picture is still fragmentary, and many interesting

1 open problems are yet to be solved, see [15, 17, 22]. We stress that the Born-Infeld Lagrangian also
 2 appears in other branches of theoretical physics. For more details we recommend the survey [65] and the
 3 references therein.

4 In a pioneering paper, Bartnik & Simon [9] studied the Dirichlet problem for (1) in bounded domains
 5 with boundary data $\varphi \in C(\partial\Omega)$ satisfying a very mild space-like condition, and proved a general existence
 6 theorem when $f \in C(\bar{\Omega} \times \mathbb{R})$. A candidate solution is produced by minimizing I , in their setting a
 7 consequence of the direct method. A core part of their work is to show that the minimizer u actually solves
 8 (1) and enjoys nice regularity properties. Remarkably, this is achieved with no regularity requirement on
 9 $\partial\Omega$. Uniqueness of solutions holds if $f(x, s)$ is non-increasing in s (since I becomes strictly convex), but
 10 may fail otherwise. Indeed, under suitable conditions on f , for $\varphi = 0$ Bereanu, Jebelean & Mawhin [10]
 11 obtained a second solution of mountain pass type to (1). To reach the goal, they exploited a general
 12 nonsmooth critical point theory developed by Szulkin [63]. Here the boundedness of Ω implies the uniform
 13 boundedness in $C(\bar{\Omega})$ of any admissible function u with $\|Du\|_\infty \leq 1$ attaining the boundary value, an
 14 essential fact to guarantee the validity of the Palais-Smale condition. Szulkin's nonsmooth critical point
 15 theory is quite versatile, and applies to various problems that do not allow for a treatment via the classical
 16 theory by Ambrosetti and Rabinowitz [2].

17 Our starting point for the present work was the search for solutions to (1) in the autonomous case on
 18 the entire \mathbb{R}^n :

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) + f(u) = 0 \quad \text{in } \mathbb{R}^n. \quad (\mathcal{BI}_f)$$

19 We focus on solutions u vanishing at infinity. The problem was already considered in the literature, see
 20 below for more details. In this setting, as we shall see, Szulkin's theory is not sufficient anymore. Seeking
 21 for existence results tailored to (\mathcal{BI}_f) led us to complement Szulkin's theory in its general framework.

22 Our goal is to prove two existence theorems for minimax critical points of I , valid under fairly general
 23 conditions for Ψ and Φ that are suited to applications and might be somehow optimal. Differently from
 24 [63], we only require a form of *bounded* Palais-Smale condition. To find bounded Palais-Smale sequences
 25 (and then, critical points of I), our approach is based on the monotonicity trick due to Struwe [57],
 26 Jeanjean & Toland [38, 41], a tool widely used in the literature, see for instance [42]. We adapt the trick
 27 to the family of functionals

$$I_\lambda : X \rightarrow \mathbb{R}, \quad I_\lambda(u) \equiv \lambda\Psi(u) - \Phi(u)$$

28 for $\lambda \in \mathbb{R}^+$. However, to make it effective for the applications we are considering, the adaptation is far
 29 from trivial. See below for more explanations.

30 Following Szulkin [63], we set

31 **Definition 1.1.** *Let X be a Banach space and $I_\lambda = \lambda\Psi - \Phi$, with Ψ, Φ satisfying $(\Psi_1), (\Phi_1)$ and $\lambda \in \mathbb{R}^+$.*

32 • *An element $u \in X$ is called a critical point of I_λ if*

$$\lambda(\Psi(v) - \Psi(u)) - \Phi'(u)(v - u) \geq 0 \quad \text{for all } v \in X. \quad (2)$$

33 • *A sequence $\{u_j\}_{j=1}^\infty \subset X$ is called a Palais-Smale sequence for I_λ at level $c \in \mathbb{R}$ if*

34 (i) $I_\lambda(u_j) \rightarrow c$ as $j \rightarrow \infty$,

35 (ii) *there exists $\{\varepsilon_j\}_{j=1}^\infty \subset \mathbb{R}^+$ with $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$ such that*

$$\lambda(\Psi(v) - \Psi(u_j)) - \Phi'(u_j)(v - u_j) \geq -\varepsilon_j \|v - u_j\| \quad \text{for all } v \in X. \quad (3)$$

36 **Remark 1.2.** By choosing $v \in D(\Psi)$, notice that any critical point of I_λ belongs to $D(\Psi)$. Also, it is
 37 easy to see that if $u \in D(\Psi)$ is a critical point of I_λ and Ψ is differentiable at u , then u is a classical
 38 critical point of I_λ that is, $\lambda\Psi'(u) - \Phi'(u) = 0$. More generally, (2) can be rephrased as $\lambda^{-1}\Phi'(u) \in \partial\Psi(u)$,
 39 where $\partial\Psi(u)$ denotes the subdifferential of Ψ at u .

To state our results, we need some further conditions on Ψ and Φ .

(Ψ_2) $\Psi(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$.

(Ψ_3) If $\{u_j\}_{j=1}^\infty \subset D(\Psi)$ and $u \in D(\Psi)$ satisfy $u_j \rightharpoonup u$ weakly in X and $\Psi(u_j) \rightarrow \Psi(u)$, then $\|u_j - u\| \rightarrow 0$.

(Φ_2) Any bounded sequence $\{u_l\}_l$ in X has a weakly convergent subsequence $\{u_{l_j}\}_j$ to u in X such that

$$\liminf_{j \rightarrow \infty} \Phi'(u_{l_j})(v - u_{l_j}) \geq \Phi'(u)(v - u) \quad \text{for any } v \in X.$$

(IB) For any $0 < a \leq b < \infty$ and $c \in \mathbb{R}$,

$$\mathcal{K}_{[a,b]}^c \equiv \{u \in X \mid \text{for some } \lambda \in [a, b], u \text{ is a critical point of } I_\lambda \text{ with } I_\lambda(u) \leq c\}$$

is bounded in X .

Remark 1.3. Observe that (Φ_2) is satisfied if X is reflexive and $\Phi' : X \rightarrow X^*$ is compact; in this case, the liminf can be replaced by a limit, and equality holds. Notice however that (Φ_2) is a strictly weaker condition: for instance, (Φ_2) holds if X is a Hilbert space and $\Phi(u) = -\|u\|^2$. This improvement will be useful in our application to the Born-Infeld equation.

Remark 1.4. As we shall see in Proposition 4.1, properties (Ψ_3) and (Φ_2) are assumed so that I_λ satisfies a bounded Palais-Smale condition in a strengthened form, namely including a dependence on λ .

Remark 1.5. We assume (IB) as a replacement for the boundedness of Palais-Smale sequences, a condition which may fail for general I . For large classes of interesting functionals arising from nonlinear elliptic problems, (IB) may be proved via Pohozaev type identities.

Our main achievements are the following minimax theorems. We first suppose that I_λ has a uniform mountain pass geometry for λ close to 1:

$$\left\{ \begin{array}{l} \text{there exist } \rho_0, \alpha_0, \bar{\varepsilon} > 0 \text{ and } u_0 \in X \text{ such that} \\ \inf_{\|u\|=\rho_0} I_{1-\bar{\varepsilon}}(u) \equiv \alpha_0 > 0, \quad \|u_0\| > \rho_0, \quad \max\{I_{1+\bar{\varepsilon}}(0), I_{1+\bar{\varepsilon}}(u_0)\} \leq 0. \end{array} \right. \quad (\text{MP})$$

For $\lambda \in [1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$, we define the mountain pass value

$$\begin{aligned} c_0(\lambda) &\equiv \inf_{\gamma \in \Gamma_0} \sup_{t \in [0,1]} I_\lambda(\gamma(t)), \quad \text{where} \\ \Gamma_0 &\equiv \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = u_0\}. \end{aligned} \quad (4)$$

We shall prove later that $\Gamma_0 \neq \emptyset$ and $0 < \alpha_0 \leq c_0(\lambda) < \infty$ for each $\lambda \in [1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$, see Lemma 4.2. We also note that, since $\Psi \geq 0$, $c_0(\lambda)$ is nondecreasing in λ .

Theorem 1.6. *Assume that (Ψ_1), (Ψ_2), (Ψ_3), (Φ_1), (Φ_2), (IB) and (MP) hold. Then $I = I_1$ admits a critical point $u \in X$ with $I(u) = c_0(1)$.*

The second result concerns the multiplicity of critical points of I when I is even, so we assume

$$\Psi(-u) = \Psi(u), \quad \Phi(-u) = \Phi(u) \quad \text{for any } u \in X. \quad (\text{E})$$

For $k \in \mathbb{N}$, we let $\mathbb{D}^k(r)$ be the closed disk in \mathbb{R}^k with center 0 and radius r . The disk $\mathbb{D}^k(1)$ is simply denoted by \mathbb{D}^k . We assume that I has the following uniform symmetric mountain pass geometry:

$$\left\{ \begin{array}{l} \text{there exist } \rho_0, \alpha_0, \bar{\varepsilon} > 0 \text{ and an odd map } \pi_{0,k} \in C(\partial\mathbb{D}^k, X) \text{ for each } k \in \mathbb{N} \text{ such that} \\ \inf_{\|u\|=\rho_0} I_{1-\bar{\varepsilon}}(u) \equiv \alpha_0 > 0, \quad \min_{\zeta \in \partial\mathbb{D}^k} \|\pi_{0,k}(\zeta)\| > \rho_0, \quad \sup_{\zeta \in \partial\mathbb{D}^k} I_{1+\bar{\varepsilon}}(\pi_{0,k}(\zeta)) < 0. \end{array} \right. \quad (\text{SMP})$$

1 For $\lambda \in [1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$ and $k = 1, 2, \dots$, we define the minimax values

$$\begin{aligned}
 c_k(\lambda) &\equiv \inf_{\gamma \in \Gamma_k} \sup_{\zeta \in \mathbb{D}^k} I_\lambda(\gamma(\zeta)), \quad \text{where} \\
 \Gamma_k &\equiv \left\{ \gamma \in C(\mathbb{D}^k, X) \mid \gamma \text{ is odd, } \gamma = \pi_{0,k} \text{ on } \partial\mathbb{D}^k \right\}.
 \end{aligned}
 \tag{5}$$

2 Again, we shall prove that $\Gamma_k \neq \emptyset$ and that $\alpha_0 \leq c_k(\lambda) < \infty$ holds, see Lemma 4.2.

3 **Theorem 1.7.** *Assume that (Ψ_1) , (Ψ_2) , (Ψ_3) , (Φ_1) , (Φ_2) , (IB), (E) and (SMP) hold. Then, $I = I_1$*
 4 *admits infinitely many critical points $\{u_k\}_{k=1}^\infty \subset X$ with $I(u_k) = c_k(1) \rightarrow \infty$ as $k \rightarrow \infty$.*

5 The proofs of Theorems 1.6 and 1.7 will be given in Section 4. Both results crucially rely on the
 6 monotonicity trick in Theorem 4.4 below, for which we do not need properties (Ψ_3) , (Φ_2) , (IB). Comments
 7 on Theorems 1.6 and 1.7 and their proof, and comparison with the literature, can be found in Subsection
 8 1.1.

9 In Section 2, we apply Theorems 1.6 and 1.7 to obtain nontrivial solutions to (\mathcal{BI}_f) . Here, it is
 10 worth pointing out that the equivalence between critical points of I and (weak) solutions of (\mathcal{BI}_f) is
 11 not immediate, and requires a regularity result for critical points of I of independent interest. To keep
 12 the paper at a reasonable length, we only consider Born-Infeld type equations (\mathcal{BI}_f) , even though our
 13 main abstract achievements may be used in other settings: as an example, we mention quasilinear elliptic
 14 problems including (Euclidean) mean curvature type ones [28, 42, 51], plastoelasticity problems [20, 23],
 15 nonlinear obstacle problems [28], solutions to the Lorentz force equation [3, 4], see also the references
 16 therein.

17 1.1 Previous related works and novelties of our approach

18 Various critical point theories based on nonsmooth analysis (for which we refer to [25]) arose in the past
 19 40 years: for locally Lipschitz functionals we quote [24], for lower semicontinuous functionals in Banach
 20 spaces we stress the already mentioned [63], while for continuous functionals in metric spaces we highlight
 21 the theory developed independently by Corvellec, Degiovanni & Marzocchi [26–28] and by Katriel [37]
 22 (see also [43] for the Banach space setting). As described in [27, §4], functionals $I \equiv \Psi - \Phi$ satisfying
 23 (Ψ_1) , (Φ_1) fit within their framework. However, to get a critical point the authors need the Palais-Smale
 24 condition, so Theorems 1.6 and 1.7 do not follow from these results.

25 Regarding the monotonicity trick for nonsmooth functionals, we mention the work by Squassina [62].
 26 Especially, [62, Theorem 3.1] should be compared to our Theorem 4.4 and in this respect the discussion
 27 in [62, p.161–163] is informative. However, the functionals I_λ in [62] are assumed to be continuous on the
 28 whole Banach space where they are defined, and this makes a difference with our results in obtaining a
 29 deformation lemma and a bounded Palais-Smale sequence.

30 Next, we highlight the differences between our paper and [62, 63] from the technical point of view. It is
 31 known that the Palais-Smale condition plays a key role in obtaining existence and multiplicity of critical
 32 points; see [2, section 2] and [54, section 9]. Since the condition is weakened in this paper, the assertions
 33 in Theorem 1.7 are highly nontrivial. In fact, even though [62] considers higher minimax values, the
 34 existence of infinitely many critical points is not obtained. In this respect, a new idea is necessary to
 35 prove that the minimax values diverge. We next point out differences from [63]. First, for our purposes
 36 we need to refine a deformation lemma in [63, Lemmas 2.1, 2.2 and Proposition 2.3]; see Lemma 3.1.
 37 The second point regards Ekeland’s variational principle, exploited in [63] to ensure the existence of
 38 Palais–Smale sequences: in our setting, to obtain bounded Palais-Smale sequences we need to consider
 39 two different functionals I_λ and $I_{\lambda+\varepsilon}$, a fact that seems to prevent the use of Ekeland’s principle. We thus
 40 introduce a new iteration argument for deformations. This method is also useful to prove the divergence
 41 of minimax values without the full Palais-Smale condition.

1 We next describe the issues concerning a variational approach to (\mathcal{BI}_f) . If we consider (1) in a
 2 bounded domain Ω , suitably choosing the function space X we may prove that the domain $D(\Psi)$ itself is
 3 bounded in X for any nonlinearity $f \in C(\overline{\Omega} \times \mathbb{R})$, which leads to the Palais-Smale condition. However,
 4 for (\mathcal{BI}_f) , this is no longer true. The difficulty to obtain the boundedness of Palais-Smale sequences
 5 already occurs in the study of the scalar field equation

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^n, \quad (6)$$

6 where the nonlinearity f satisfies the so called Berestycki-Lions conditions (which are almost optimal for
 7 the solvability of (6)). In [12–14] the existence of least energy solutions and minimax solutions to (6)
 8 was proved by constrained minimization and minimax methods. The arguments there strongly depend
 9 on the difference between the homogeneity in λ of

$$\int_{\mathbb{R}^n} |Du|^2 \quad \text{and of} \quad \int_{\mathbb{R}^n} F(u) = \Phi(u)$$

10 with respect to the scaling $u(x) \rightarrow u(\lambda x)$, which is essential to normalize a Lagrange multiplier. This is
 11 not applicable to problem (\mathcal{BI}_f) since, under the natural scaling $u(x) \rightarrow \lambda^{-1}u(\lambda x)$ keeping the restriction
 12 $|Du| \leq 1$, Ψ and Φ display almost the same homogeneity:

$$\Psi(u_\lambda) = \lambda^{-n}\Psi(u) \quad \text{and} \quad \Phi(u_\lambda) = \lambda^{-n}\Phi(\lambda^{-1}u).$$

13 For this reason, in our setting, the use of a constrained problem seems more difficult. By the same reason,
 14 arguments based on the scaling developed in [33–35] are not straightforward to apply to the functional
 15 corresponding to (\mathcal{BI}_f) .

16 An approach to recover the existence theorems in [12–14] by means of unconstrained minimax methods
 17 was due to Struwe in [56], see also [59, section 11 of chapter II]. However, to verify a variant of the Palais-
 18 Smale condition, he still needed to introduce some constraint. It seems to us that even this method may
 19 hardly apply to (\mathcal{BI}_f) . From a different viewpoint, seeking to weaken the regularity of $-u + f(u)$ in (6)
 20 we mention [60] and [21]. There, solutions were constructed for nonlinearities of bounded variation and
 21 locally integrable, respectively.

22 In [5, 6, 16, 50], commented later on in more detail, radial solutions to (\mathcal{BI}_f) were obtained by either
 23 employing the shooting method or approximations of the operators. However, for the above reasons,
 24 nonsmooth critical point theories were never applied to (\mathcal{BI}_f) in the literature. To our knowledge, the
 25 present work is the first attempt to do so.

26 Finally, we mention the work by Jeanjean & Lu [42] where they employed the monotonicity trick to
 27 obtain infinitely many solutions to (6). To this end, in [42, Section 2], the monotonicity trick is proved
 28 for higher minimax levels as in Theorem 1.7 and in [62]. However, as pointed out in [42, Remark 2.2], the
 29 authors did not obtain the divergence of the minimax levels in an abstract setting, the property was later
 30 shown in [42, Lemma 5.7] by using the specific structure of (6) and a comparison functional (the same
 31 as in [33]). To the authors' knowledge, Theorem 1.7 seems to be the first abstract result in the literature
 32 (even for smooth functionals) to show the divergence of minimax values under general conditions not
 33 including the Palais-Smale's one.

34 2 Application to the Born-Infeld equation

35 We exploit Theorems 1.6 and 1.7 to investigate the existence of nontrivial solutions to

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) + f(u) = 0 \quad \text{in } \mathbb{R}^n, \quad (\mathcal{BI}_f)$$

1 in dimension $n \geq 3$. In principle, Theorems 1.6 and 1.7 could be applied to the case $n = 2$, too. However,
 2 some technical details in the functional-analytic setting need a certain care, and for the sake of simplicity
 3 the case $n = 2$ will be deferred to a future investigation.

4 **Definition 2.1.** Given a Radon measure ρ on \mathbb{R}^n , a function $u \in \text{Lip}(\mathbb{R}^n)$ is called a weak solution to

$$\text{div} \left(\frac{Du}{\sqrt{1-|Du|^2}} \right) + \rho = 0 \quad \text{in } \mathbb{R}^n$$

5 if

$$|Du| \leq 1 \quad \text{a.e.}, \quad \frac{1}{\sqrt{1-|Du|^2}} \in L^1_{\text{loc}}(\mathbb{R}^n), \quad (7)$$

6 and

$$\int_{\mathbb{R}^n} \frac{Du \cdot D\eta}{\sqrt{1-|Du|^2}} = \langle \rho, \eta \rangle \quad \text{for any } \eta \in \text{Lip}_c(\mathbb{R}^n) \equiv \text{Lip}(\mathbb{R}^n) \cap C_c(\mathbb{R}^n), \quad (8)$$

7 where $\langle \rho, \eta \rangle$ stands for the duality pairing.

8 The definition does not make any assumption at infinity on u . However, we will restrict ourselves to
 9 solutions (with $\rho = f(u)$) satisfying

$$u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (9)$$

10 and among them solutions lying in the energy space

$$\mathcal{D}^{1,2}(\mathbb{R}^n) \equiv \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|}, \quad \|\phi\| \equiv \left(\int_{\mathbb{R}^n} |D\phi|^2 \right)^{\frac{1}{2}}. \quad (10)$$

11 Formally, they are critical points of the functional

$$I(u) \equiv \int_{\mathbb{R}^n} \left(1 - \sqrt{1-|Du|^2} - F(u) \right) \quad \text{with } F(t) = \int_0^t f(s) ds. \quad (11)$$

12 However, the behavior of the Lagrangian density on $\{|Du| = 1\}$ prevents the differentiability of I and
 13 calls for care. Nonetheless, in Propositions 2.13 and 2.10 below we prove that a critical point u of I
 14 satisfying (9) is a weak solution of (\mathcal{BI}_f) , and that u belongs to $W_{\text{loc}}^{2,q}(\mathbb{R}^n)$ for each $q \in [2, \infty)$ (hence, to
 15 $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n)$ for each $\alpha \in (0, 1)$) and is *strictly spacelike*, in the sense that $|Du| < 1$ on \mathbb{R}^n . The regularity,
 16 whose proof needs some subtle arguments, also holds under much more general growth conditions on u
 17 at infinity. We expect it will be a handy tool for future works on Born-Infeld equations.

18 To the best of our knowledge, so far the existence problem for (\mathcal{BI}_f) was only considered when
 19 restricted to radially symmetric functions. In [6], Azzollini studied positive solutions in the model case
 20 $f(t) = |t|^{p-2}t$, and showed that the problem

$$\text{div} \left(\frac{Du}{\sqrt{1-|Du|^2}} \right) + |u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^n, \quad \lim_{|x| \rightarrow \infty} u(x) = 0, \quad (12)$$

21 - has a radially symmetric positive solution in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and infinitely many positive solutions not in
 22 $\mathcal{D}^{1,2}(\mathbb{R}^n)$ for $p \in (2^*, \infty)$, where $2^* = \frac{2n}{n-2}$;

23 - does not have radially symmetric positive solutions if $p \in (1, 2^*)$.

1 On the other hand, for $p \in (2^*, \infty)$ infinitely many radially symmetric solutions in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ to (12)
 2 were constructed by Bonheure, De Coster & Derlet in [16] using a constrained minimax argument. In the
 3 same paper, the authors posed the existence problem for $p = 2^*$ as an interesting question, see [16, page
 4 262].

5 Other nonlinearities have also been considered. A problem corresponding to the “positive mass case”
 6 of (12), namely,

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) - u + |u|^{p-2}u = 0, \quad \lim_{|x| \rightarrow \infty} u(x) = 0 \quad (13)$$

7 was studied in [5] and it was shown that (13) admits a radially symmetric positive solution in $H^1(\mathbb{R}^n)$ for
 8 each $p \in (1, \infty)$. Indeed, the solution therein was obtained by a shooting argument, for a class of locally
 9 Lipschitz functions much larger than $|u|^{p-2}u$. A general existence result for one nontrivial radial solution,
 10 closely related to our main Theorem 2.2 below, was recently obtained by Mederski & Pomponio [50], and
 11 will be commented in detail in Remark 2.3.

12 About the existence of solutions, we shall consider both radial and non-radial ones. To construct the
 13 latter, we restrict to n and $d \in \mathbb{N}$ satisfying

$$n \geq 4, \quad 2 \leq d \leq \frac{n}{2}, \quad \text{and} \quad n - 2d \neq 1. \quad (14)$$

14 Write a point of \mathbb{R}^n as $x = (x_1, x_2, x_3) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{n-2d}$ and consider the subspace of functions
 15 $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ with the following symmetries:

$$\begin{aligned} u(Ax_1, Bx_2, Cx_3) &= u(x_1, x_2, x_3) \quad \text{for each } A, B \in O(d), C \in O(n-2d), \\ u(x_2, x_1, x_3) &= -u(x_1, x_2, x_3). \end{aligned} \quad (15)$$

16 Since Bartsch & Willem [8], the above subspace has been extensively used (we refer to [49] and references
 17 therein). Notice that the only radial function therein is $u \equiv 0$, whence nontrivial solutions to $(\mathcal{B}\mathcal{I}_f)$ with
 18 the symmetries described in (15) must be non-radial. Given a mass parameter $\mathbf{m} \geq 0$ and $p \in [2, 2^*]$, we
 19 define

$$\mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n) \equiv \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{\mathcal{D}_{\mathbf{m},p}^{1,2}}}, \quad \|\phi\|_{\mathcal{D}_{\mathbf{m},p}^{1,2}} \equiv \sqrt{\|D\phi\|_2^2 + \mathbf{m}\|\phi\|_p^2}.$$

20 The exponent p will be related to \mathbf{m} according to assumption (f1) below. If $\mathbf{m} = 0$, then $\mathcal{D}_{0,p}^{1,2}(\mathbb{R}^n) =$
 21 $\mathcal{D}^{1,2}(\mathbb{R}^n)$. In this case, we agree to set $p = 2^*$. Likewise, by Sobolev’s embedding, for $\mathbf{m} > 0$ and $p = 2^*$
 22 the space $\mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n)$ is $\mathcal{D}^{1,2}(\mathbb{R}^n)$ with an equivalent norm.

23 We are ready to state our main existence theorem:

24 **Theorem 2.2.** *Assume that $n \geq 3$ and that $f \in C(\mathbb{R})$, $\mathbf{m} \geq 0$ satisfy*

25 (f1) *either*

$$\begin{aligned} \text{(f1}_0) \quad \mathbf{m} = 0, \quad & -\infty < \liminf_{s \rightarrow 0} \frac{f(s)}{|s|^{2^*-2}s} \leq \limsup_{s \rightarrow 0} \frac{f(s)}{|s|^{2^*-2}s} = 0, \quad \text{or} \\ \text{(f1}_{\mathbf{m},p}) \quad \mathbf{m} > 0, \quad & -\infty < \liminf_{s \rightarrow 0} \frac{f(s)}{|s|^{p-2}s} \leq \limsup_{s \rightarrow 0} \frac{f(s)}{|s|^{p-2}s} \equiv -\mathbf{m} \quad \text{for some } p \in [2, 2^*]; \end{aligned}$$

26 (f2) *there exists $t_0 > 0$ such that $F(t_0) \equiv \int_0^{t_0} f(t)dt > 0$.*

27 *Then, $(\mathcal{B}\mathcal{I}_f)$ admits a positive radial weak solution $u \in \mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n)$ with $I(u) \in (0, \infty)$, where I is defined
 28 in (11). Moreover, if f is odd, then $(\mathcal{B}\mathcal{I}_f)$ admits*

1 - infinitely many distinct radial weak solutions $\{u_k\}_k \subset \mathcal{D}_{m,p}^{1,2}(\mathbb{R}^n)$, possibly sign-changing, with $I(u_k) \in$
 2 $(0, \infty)$ and $I(u_k) \rightarrow \infty$ as $k \rightarrow \infty$;

3 - infinitely many distinct nonradial, sign-changing weak solutions $\{u_k\}_k \subset \mathcal{D}_{m,p}^{1,2}(\mathbb{R}^n)$ satisfying (15)
 4 with $I(u_k) \in (0, \infty)$ and $I(u_k) \rightarrow \infty$ as $k \rightarrow \infty$, provided that n, d satisfy (14).

5 Finally, any weak solution $v \in \mathcal{D}_{m,p}^{1,2}(\mathbb{R}^n)$ to (\mathcal{BI}_f) satisfies $v \in W_{\text{loc}}^{2,q}(\mathbb{R}^n)$ for each $q \in [2, \infty)$, and
 6 $\|Dv\|_\infty < 1$ in \mathbb{R}^n .

7 **Remark 2.3.** One main novelty of Theorem 2.2 is the first construction of non-radial solutions to (\mathcal{BI}_f) .
 8 Even in the radial case, however, the result is not contained in the previous literature, and indeed,
 9 under the only conditions (f1), (f2), to the best of our knowledge the existence of a radial solution in
 10 $\mathcal{D}^{1,2}(\mathbb{R}^n)$ was still unknown. The most general existence result we are aware of is [50, Theorem 1.1] by
 11 Mederski & Pomponio, where the authors produce a nontrivial radial solution in dimension $n \geq 3$ for
 12 odd nonlinearities $f \in C(\mathbb{R})$ satisfying (f2) and either

13 (i) assumption $(f1_{m,p})$, for $m > 0$ and some $p \in [2, \infty)$, or

14 (ii) the following assumption:

$$-\infty < \liminf_{s \rightarrow 0} \frac{f(s)}{|s|^{\gamma-1}} \leq \limsup_{s \rightarrow 0} \frac{f(s)}{|s|^{\gamma-1}} = 0 \quad \text{for some } \gamma > 2^*$$

15 and, if $2^* < \gamma \leq n$,

$$\limsup_{s \rightarrow \infty} \frac{f(s)}{|s|^{q^*-1}} = 0 \quad \text{for some } q \in \left(\frac{n\gamma}{n+\gamma}, n \right), \quad \text{where } q^* = \frac{nq}{n-q}.$$

16 Notice that (ii) and (i) for $p > 2^*$ are stronger than $(f1_0)$, while (i) with $p \in [2, 2^*]$ corresponds to $(f1_{m,p})$.

17 **Remark 2.4.** By appropriately defining Ψ, Φ in Theorem 2.2 so that weak solutions u to (\mathcal{BI}_f) corre-
 18 spond to critical points of $I = \Psi - \Phi$, the symmetry group G of u is chosen to ensure (Φ_2) once Φ is
 19 restricted to the subspace of G -invariant functions, and to guarantee that the corresponding restriction
 20 of I_λ still has a uniform (symmetric) mountain pass geometry. The method should be flexible enough to
 21 produce other types of nontrivial solutions.

22 **Remark 2.5.** In [64], Van Schaftingen described an abstract framework to find G -invariant minimax
 23 critical points without restricting, a priori, the functional I to a G -invariant subspace X_G (see also [62]).
 24 One advantage of the method is that the minimax values are taken among maps valued in X , not only in
 25 X_G ; hence, the approach in [64] may help to establish whether the positive, radially symmetric solution
 26 produced in Theorem 2.2 is a ground state, namely, a minimizer for I among all nonzero solutions. The
 27 semilinear case was studied in [40].

28 The following Pohozaev identity will be crucial to check (IB) for problem (\mathcal{BI}_f) .

29 **Proposition 2.6.** For $f \in C(\mathbb{R})$, let u be a weak solution to (\mathcal{BI}_f) satisfying

$$\int_{\mathbb{R}^n} \left(\frac{|Du|^2}{\sqrt{1-|Du|^2}} + F(u) \right) < \infty. \quad (16)$$

30 Then, $u \in W_{\text{loc}}^{2,q}(\mathbb{R}^n)$ for each $q \in [2, \infty)$, $|Du| < 1$ on \mathbb{R}^n and

$$\int_{\mathbb{R}^n} \frac{|Du|^2}{\sqrt{1-|Du|^2}} = n \int_{\mathbb{R}^n} \left(1 - \sqrt{1-|Du|^2} - F(u) \right) < \infty. \quad (17)$$

1 As an application of Proposition 2.6, we rule out the existence of finite energy solutions for f not
 2 supercritical. Hence, condition (f1) in Theorem 2.2 is necessary for the existence of a solution to (\mathcal{BI}_f) .
 3 On the other hand, condition (f2) is needed to show the mini-max structure (MP) or (SMP) of the energy
 4 functional, and we believe that it is necessary as well. In this respect, it is well known that (f2) is a
 5 necessary condition for the existence of a finite energy solution to (6).

6 **Proposition 2.7.** *For $n \geq 3$ there exist no weak solutions $u \not\equiv 0$ to*

$$\operatorname{div} \left(\frac{Du}{\sqrt{1-|Du|^2}} \right) + |u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^n, \quad p \in [2, 2^*] \quad (18)$$

7 *satisfying*

$$\int_{\mathbb{R}^n} (|Du|^2 + |u|^p) < \infty. \quad (19)$$

8 *In particular, (18) for $p = 2^*$ does not admit any weak solution $u \in \mathcal{D}^{1,2}(\mathbb{R}^n) \setminus \{0\}$.*

9 **Remark 2.8.** For $p = 2^*$, we thus answer in the negative the question raised in [16, page 262] for solutions
 10 in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. We expect that there exist no positive weak solutions to (18) for $p \in (2, 2^*)$ without any
 11 restriction, as shown in [31] for the semilinear case $\Delta u + |u|^{p-2}u = 0$. We also believe that the critical
 12 case $p = 2^*$ does not admit any positive weak solutions to (18). As a related problem, studying the
 13 behavior of mountain pass solutions to (18) as $p \searrow 2^*$ seems also a quite interesting issue.

14 2.1 Critical points of I and weak solutions to (\mathcal{BI}_f)

15 We first discuss the regularity properties of weak solutions in Definition 2.1. Given a Radon measure ρ
 16 on \mathbb{R}^n , define the action I_ρ on a bounded domain $\Omega \Subset \mathbb{R}^n$ as

$$I_\rho^\Omega(\psi) \equiv \int_{\Omega} \left(1 - \sqrt{1 - |D\psi|^2} \right) - \langle \rho, \psi \rangle_{\Omega},$$

17 with $\langle \cdot, \cdot \rangle$ the duality pairing and $\langle \rho, \psi \rangle_{\Omega} = \langle \rho|_{\Omega}, \psi \rangle$. For $\phi \in C(\partial\Omega)$, we consider the convex set

$$\mathcal{Y}_\phi(\Omega) \equiv \{ \psi \in W^{1,\infty}(\Omega) \mid |D\psi| \leq 1, \psi = \phi \text{ on } \partial\Omega \}.$$

18 **Remark 2.9.** If $\partial\Omega$ is not regular enough to achieve $\psi = \phi$ in the trace sense, the boundary condition
 19 has to be intended as in [9]. However, by [22, Proposition 3.5] this definition suffices to guarantee that
 20 functions $u \in \mathcal{Y}_\phi(\Omega)$ can be extended continuously to $\partial\Omega$ with boundary datum ϕ .

21 We say that

22 - u *minimizes* I_ρ *on* Ω *with boundary value* $\phi \in C(\partial\Omega)$ *if*

$$I_\rho^\Omega(u) \leq I_\rho^\Omega(\psi) \quad \text{for each } \psi \in \mathcal{Y}_\phi(\Omega);$$

23 - u *is a local minimizer for* I_ρ *if it minimizes* I_ρ^Ω *on each domain* $\Omega \Subset \mathbb{R}^n$ *with respect to the boundary*
 24 *value* $\phi = u|_{\partial\Omega}$.

25 The regularity result we need is encoded in the next proposition of independent interest, proved in
 26 Appendix B.

27 **Proposition 2.10.** *Let $u \in \operatorname{Lip}(\mathbb{R}^n)$ satisfy $|Du| \leq 1$ a.e. on \mathbb{R}^n , and let $\rho \in L_{\text{loc}}^\infty(\mathbb{R}^n)$.*

1 (i) Suppose that there exist $c_1 > 0$ and a function $h : \mathbb{R}^n \rightarrow (0, \infty)$ such that

$$\begin{aligned} h(x) &\rightarrow \infty \quad \text{as } |x| \rightarrow \infty, & \lim_{|x| \rightarrow \infty} \frac{h(x)}{|x|} &= 0, \\ |u(x)| &\leq c_1 + |x| - h(x) & \text{on } \mathbb{R}^n. \end{aligned} \tag{20}$$

2 Then, the following are equivalent:

3 (a) u is a weak solution to

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right) + \rho = 0 \quad \text{in } \mathbb{R}^n \tag{\mathcal{BI}_\rho}$$

4 as stated in Definition 2.1.

5 (b) u is a local minimizer for I_ρ .

6 Furthermore, if any between (a) and (b) occurs then $u \in W_{\text{loc}}^{2,q}(\mathbb{R}^n)$ for each $q \in [2, \infty)$, and $|Du| < 1$
 7 on \mathbb{R}^n . Moreover, for each $R, c > 0$ and $q \geq 2$ there exist $\hat{R} = \hat{R}(R, c_1, h)$ and $\delta = \delta(\hat{R}, c)$,
 8 $C_q = C(\hat{R}, c, q)$ with the following properties: if

$$\|\rho\|_{L^\infty(B_{\hat{R}}(0))} \leq c,$$

9 then

$$\sup_{B_{\hat{R}}(0)} |Du| \leq 1 - \delta, \quad \|u\|_{W^{2,q}(B_{\hat{R}}(0))} \leq C_q.$$

10 (ii) Assume that $\rho \in L^\infty(\mathbb{R}^n)$ and $u \in L^\infty(\mathbb{R}^n)$ is a weak solution to (\mathcal{BI}_ρ) with $\|\rho\|_\infty + \|u\|_\infty \leq c$.
 11 Then, there exists $\delta = \delta(c) \in (0, 1)$ such that $\|Du\|_\infty \leq 1 - \delta$.

12 **Remark 2.11.** A growth requirement like (20) is necessary for the equivalence (a) \Leftrightarrow (b) to hold. To see
 13 this, consider the example $\rho \equiv 0$ and u an affine function with slope 1. Then, u satisfies (b) but clearly
 14 does not satisfy (7) in the definition of weak solution, and (8) cannot be given a reasonable meaning.

15 **Remark 2.12.** Case (ii) in the above proposition improves on [15, Theorem 1.5].

16 We next specify the above proposition to our case of interest, the equivalence of critical points of I
 17 and solutions of (\mathcal{BI}_f) .

18 **Proposition 2.13.** Assume that $n \geq 3$ and $\mathfrak{m} \geq 0$, that $u \in \mathcal{D}_{\mathfrak{m},p}^{1,2}(\mathbb{R}^n)$, and that the functional

$$\eta \in C_c^\infty(\mathbb{R}^n) \mapsto \langle f(u), \eta \rangle \equiv \int_{\mathbb{R}^n} f(u)\eta \tag{21}$$

19 extends to a continuous one on $\mathcal{D}_{\mathfrak{m},p}^{1,2}(\mathbb{R}^n)$. Then, the following are equivalent:

20 (a) u is a weak solution to (\mathcal{BI}_f) ;

21 (b) u is a local minimizer for I_ρ with $\rho = f(u)$;

22 (c) The following inequality holds for each $\psi \in \mathcal{D}_{\mathfrak{m},p}^{1,2}(\mathbb{R}^n)$ with $\|D\psi\|_\infty \leq 1$:

$$\int_{\mathbb{R}^n} \left(\sqrt{1 - |Du|^2} - \sqrt{1 - |D\psi|^2} \right) - \int_{\mathbb{R}^n} f(u)(\psi - u) \geq 0. \tag{22}$$

1 **Remark 2.14.** Hereafter, we will often employ the following useful properties valid for $t \in [0, 1]$:

$$\frac{t^2}{2} \leq 1 - \sqrt{1 - t^2} \leq t^2, \quad 1 - \sqrt{1 - t^2} = \sum_{k=1}^{\infty} b_k t^{2k} \quad \text{with } b_k > 0,$$

2 and the pointwise inequality

$$\frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} \geq \sqrt{1 - |D\psi|^2} - \sqrt{1 - |Du|^2} \quad (23)$$

3 on the set of points $\{|Du| < 1\} \cap \{|D\psi| \leq 1\}$, see [22, Remark 3.15].

4 **Proof of Proposition 2.13.** (a) \Leftrightarrow (b) Since $n \geq 3$, from the Sobolev embedding $\mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$,
5 and from $u \in \mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n)$, $\|Du\|_{\infty} \leq 1$ we deduce that $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$. In particular, Proposition
6 2.10 ensures that (a) \Leftrightarrow (b) and that $u \in W_{\text{loc}}^{2,q}(\mathbb{R}^n)$ with $\|Du\|_{\infty} < 1$.

7 (a) \Rightarrow (c) Given $\psi \in \mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n)$ with $\|D\psi\|_{\infty} \leq 1$ and a sequence $\varepsilon_j \rightarrow 0$, we define

$$\psi_j(x) \equiv \max\{u(x), \psi(x) - \varepsilon_j\} + \min\{u(x), \psi(x) + \varepsilon_j\} - u(x) = \begin{cases} u(x) & \text{if } |\psi(x) - u(x)| < \varepsilon_j, \\ \psi(x) + \varepsilon_j & \text{if } u(x) \geq \psi(x) + \varepsilon_j, \\ \psi(x) - \varepsilon_j & \text{if } u(x) \leq \psi(x) - \varepsilon_j. \end{cases} \quad (24)$$

8 Since both u and ψ vanish as $|x| \rightarrow \infty$, by construction $u - \psi_j$ has compact support for each j . Notice also
9 that $|\psi_j(x)| \leq |u(x)| + |\psi(x) - u(x)|$ holds for any $x \in \mathbb{R}^n$. Thus, the dominated convergence theorem
10 shows that $\psi_j \rightarrow \psi$ strongly in $\mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n)$ as $j \rightarrow \infty$ (see [22, Lemma 3.7] for more details in the case
11 $\mathbf{m} = 0$. The case $\mathbf{m} > 0$ follows by minor modifications). Plugging $u - \psi_j \in \text{Lip}_c(\mathbb{R}^n)$ in the definition of
12 weak solution to (BI_f) and using (23) we get

$$\int_{\mathbb{R}^n} \left(\sqrt{1 - |D\psi_j|^2} - \sqrt{1 - |Du|^2} \right) \leq \int_{\mathbb{R}^n} \frac{Du \cdot (Du - D\psi_j)}{\sqrt{1 - |Du|^2}} = \int_{\mathbb{R}^n} f(u)(u - \psi_j).$$

13 Rearranging, we infer (22) for the test function ψ_j . On the other hand, from the definition of ψ_j and the
14 first in Remark 2.14 we observe that

$$\left| \sqrt{1 - |D\psi_j|^2} - \sqrt{1 - |Du|^2} \right| \leq 1 - \sqrt{1 - |D\psi|^2} + 1 - \sqrt{1 - |Du|^2} \leq |D\psi|^2 + |Du|^2.$$

15 Whence, inequality (22) follows by letting $j \rightarrow \infty$ and using the dominated convergence theorem together
16 with the assumed continuity of (21).

17 (c) \Rightarrow (b) Fix $\Omega \Subset \mathbb{R}^n$ and $\psi \in \mathcal{Y}_u(\Omega)$. Up to replacing ψ with ψ_j as in (24), we can assume that
18 $\text{supp}(\psi - u) \Subset \Omega$. Extend ψ to \mathbb{R}^n by setting $\psi = u$ away from Ω . Inequality (22) implies

$$\int_{\Omega} \left(\sqrt{1 - |Du|^2} - \sqrt{1 - |D\psi|^2} \right) - \int_{\Omega} f(u)(\psi - u) \geq 0.$$

19 Then, setting $\rho \equiv f(u)$ and rearranging we get $I_{\rho}^{\Omega}(u) \leq I_{\rho}^{\Omega}(\psi)$, thus u satisfies (b). \blacksquare

20 2.2 Proofs of Propositions 2.6 and 2.7

21 **Proof of Propositions 2.6.** By (16), $|Du| \in L^2(\mathbb{R}^n)$ and the interpolation inequality with $|Du| \leq 1$
22 implies $\|Du\|_q < \infty$ for each $q \geq 2$. Hence, by the proof of Morrey's embedding theorem (for instance,
23 see [19, Theorem 9.12]), for $q > n$ the $C^{0,1-n/q}$ Hölder seminorm of u on \mathbb{R}^n is globally bounded,

1 thus $|u(x)| = \mathcal{O}\left(|x|^{1-\frac{n}{q}}\right)$ as $|x| \rightarrow \infty$. Proposition 2.10 guarantees the stated regularity of u . Set for
 2 convenience $r(x) = |x|$. We prove that for every $R \in (0, \infty)$

$$\begin{aligned} & \int_{B_R} \left\{ \frac{|Du|^2}{\sqrt{1-|Du|^2}} - n \left(1 - \sqrt{1-|Du|^2}\right) + nF(u) \right\} \\ &= R \int_{\partial B_R} \left\{ F(u) + \frac{(Du \cdot Dr)^2}{\sqrt{1-|Du|^2}} - \left(1 - \sqrt{1-|Du|^2}\right) \right\} d\sigma, \end{aligned} \quad (25)$$

3 where $B_R = B_R(0)$. We consider the piecewise affine function

$$\tau_\varepsilon(t) = \begin{cases} 1 & \text{if } t \leq R - \varepsilon/2, \\ 1 - \varepsilon^{-1}(t - R + \varepsilon/2) & \text{if } R - \varepsilon/2 < t < R + \varepsilon/2, \\ 0 & \text{if } t \geq R + \varepsilon/2. \end{cases}$$

4 Because of the regularity of u , by density (8) also holds for $W^{1,q}$ test functions with compact support, so
 5 we can use $\eta_\varepsilon \equiv \tau_\varepsilon(r)rDr \cdot Du$ in (8) with $\rho = f(u)$ to obtain

$$\int_{\mathbb{R}^n} \frac{Du \cdot D\eta_\varepsilon}{\sqrt{1-|Du|^2}} = \int_{\mathbb{R}^n} f(u)\eta_\varepsilon.$$

6 From $\operatorname{div}(rDr) = n$ we get

$$f(u)\eta_\varepsilon = \operatorname{div}(\tau_\varepsilon F(u)rDr) - r\tau'_\varepsilon(r)F(u) - n\tau_\varepsilon(r)F(u),$$

7 hence

$$\int_{\mathbb{R}^n} f(u)\eta_\varepsilon = - \int_{\mathbb{R}^n} r\tau'_\varepsilon(r)F(u) - n \int_{\mathbb{R}^n} \tau_\varepsilon(r)F(u).$$

8 Notice also that

$$\frac{Du \cdot D\eta_\varepsilon}{\sqrt{1-|Du|^2}} = \frac{r\tau'_\varepsilon(r)(Du \cdot Dr)^2}{\sqrt{1-|Du|^2}} + \frac{\tau_\varepsilon(r)|Du|^2}{\sqrt{1-|Du|^2}} + \tau_\varepsilon(r)rDr \cdot D \left(1 - \sqrt{1-|Du|^2}\right).$$

9 Therefore, integrating by parts yields

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{Du \cdot D\eta_\varepsilon}{\sqrt{1-|Du|^2}} \\ &= \int_{\mathbb{R}^n} \left\{ \frac{r\tau'_\varepsilon(r)(Du \cdot Dr)^2}{\sqrt{1-|Du|^2}} + \frac{\tau_\varepsilon(r)|Du|^2}{\sqrt{1-|Du|^2}} - \{r\tau'_\varepsilon(r) + n\tau_\varepsilon(r)\} \left(1 - \sqrt{1-|Du|^2}\right) \right\} \end{aligned}$$

10 and we conclude

$$\begin{aligned} & \int_{\mathbb{R}^n} \tau_\varepsilon(r) \left[\frac{|Du|^2}{\sqrt{1-|Du|^2}} - n \left(1 - \sqrt{1-|Du|^2}\right) + nF(u) \right] \\ &= \int_{\mathbb{R}^n} r\tau'_\varepsilon(r) \left[-\frac{(Du \cdot Dr)^2}{\sqrt{1-|Du|^2}} + 1 - \sqrt{1-|Du|^2} - F(u) \right]. \end{aligned} \quad (26)$$

11 The coarea formula guarantees that

$$t \mapsto h(t) \equiv \int_{\partial B_t} \left[F(u) + \frac{(Du \cdot Dr)^2}{\sqrt{1-|Du|^2}} - \left(1 - \sqrt{1-|Du|^2}\right) \right] d\sigma \in L^1_{\text{loc}}((0, \infty)),$$

1 thus, letting $\varepsilon \rightarrow 0$ in (26) and using the definition of τ_ε and the fact $Du \in C(\mathbb{R}^n)$, we obtain (25) for
 2 any R . Moreover, by the triangle inequality and the observation that $1 - \sqrt{1 - t^2} \leq t^2$, see Remark 2.14,
 3 we get

$$|h(t)| \leq \int_{\partial B_t} \left[F(u) + \frac{|Du|^2}{\sqrt{1 - |Du|^2}} + |Du|^2 \right] d\sigma,$$

4 hence from (16) and the coarea formula we infer that $h \in L^1([1, \infty))$. Therefore, there exists a sequence
 5 $\{R_k\}$ such that $R_k \rightarrow \infty$ and $R_k h(R_k) \rightarrow 0$ as $k \rightarrow \infty$. Observing that (16) and the first in Remark
 6 2.14 guarantee the finiteness of the right-hand side of (17), letting $R = R_k \rightarrow \infty$ in (25) we conclude the
 7 desired identity. \blacksquare

8 With the aid of the Pohozaev identity in Propositions 2.6, we can show Proposition 2.7.

9 **Proof of Proposition 2.7.** Let u be a weak solution to (18) verifying (19). From $u \in L^p(\mathbb{R}^n)$ and
 10 $\|Du\|_\infty \leq 1$, we deduce that u vanishes at infinity, hence $u \in L^\infty(\mathbb{R}^n)$. Since $f(u(x)) = |u|^{p-2}u(x) \in$
 11 $L^\infty(\mathbb{R}^n)$, Proposition 2.10 (ii) guarantees that $\|Du\|_\infty < 1$, and (19) implies

$$\int_{\mathbb{R}^n} \frac{|Du|^2}{\sqrt{1 - |Du|^2}} \leq \frac{1}{\sqrt{1 - \|Du\|_\infty^2}} \int_{\mathbb{R}^n} |Du|^2 < \infty.$$

12 Thus, we apply Proposition 2.6 with $f(u) = |u|^{p-2}u$ to deduce that

$$\int_{\mathbb{R}^n} \frac{|Du|^2}{\sqrt{1 - |Du|^2}} = n \int_{\mathbb{R}^n} \left(1 - \sqrt{1 - |Du|^2} - \frac{1}{p}|u|^p \right) < \infty. \quad (27)$$

13 For $\varepsilon > 0$, we plug the compactly supported test function $\eta = \zeta_\varepsilon(u)$ with $\zeta_\varepsilon(t) \equiv (t - \varepsilon)_+ - (t + \varepsilon)_-$ in
 14 the weak definition of (18) to obtain

$$\int_{\{|u|>\varepsilon\}} |u|^{p-2}u \zeta_\varepsilon(u) = \int_{\{|u|>\varepsilon\}} \frac{|Du|^2}{\sqrt{1 - |Du|^2}}.$$

15 Letting $\varepsilon \rightarrow 0$, using the dominated convergence theorem, the identity $|Du| \equiv 0$ a.e. on $\{u = 0\}$, and
 16 substituting in (27) we thus get

$$0 = \int_{\mathbb{R}^n} \left\{ \frac{p+n}{p} \frac{|Du|^2}{\sqrt{1 - |Du|^2}} - n \left(1 - \sqrt{1 - |Du|^2} \right) \right\}.$$

17 Since $p \leq 2^*$ and

$$\frac{p+n}{p} \frac{t}{\sqrt{1-t}} - n(1 - \sqrt{1-t}) \geq \frac{n}{2} \left(\frac{t}{\sqrt{1-t}} - 2(1 - \sqrt{1-t}) \right) > 0 \quad \text{for } t \in (0, 1),$$

18 we conclude that $|Du| \equiv 0$, thus $u \equiv 0$, a contradiction. \blacksquare

19 2.3 Proof of Theorem 2.2

20 For convenience, having fixed $p_1 > \max\{n, p\}$, we define a subspace $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ of $\mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n)$ by

$$\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \equiv \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{\mathcal{Y}_{\mathbf{m},p}}}, \quad \|v\|_{\mathcal{Y}_{\mathbf{m},p}} \equiv \sqrt{\mathbf{m}\|v\|_p^2 + \|Dv\|_2^2 + \|Dv\|_{p_1}^2},$$

21 where we agree that $p = 2^*$ if $\mathbf{m} = 0$. Notice that, by Sobolev's embedding, $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \hookrightarrow \mathcal{Y}_{0,2^*}(\mathbb{R}^n)$.

1 **Lemma 2.15.** *Let $n \geq 3$. Then $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ is a reflexive Banach space and satisfies*

$$\begin{aligned} \text{(i)} \quad & \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \hookrightarrow W^{1,p}(\mathbb{R}^n), \\ \text{(ii)} \quad & \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \hookrightarrow C_0(\mathbb{R}^n) \equiv \left\{ u \in C(\mathbb{R}^n) \mid \lim_{|x| \rightarrow \infty} u(x) = 0 \right\}. \end{aligned} \quad (28)$$

2 **Proof.** The norm $\|\cdot\|_{\mathcal{Y}_{\mathbf{m},p}}$ is uniformly convex by the uniform convexity of L^q -spaces and [19, Exercise
3 3.29], thus $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ is reflexive. If $\mathbf{m} = 0$, then (i) and (ii) have been shown in [22, Proposition 3.3] (the
4 norm used there is equivalent to $\|\cdot\|_{\mathcal{Y}_{0,2^*}}$), hence (ii) for $\mathbf{m} > 0$ is a consequence of $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \hookrightarrow \mathcal{Y}_{0,2^*}(\mathbb{R}^n)$.
5 On the other hand, interpolation gives

$$\|Du\|_p \leq \|Du\|_2^\theta \|Du\|_{p_1}^{1-\theta} \leq \|u\|_{\mathcal{Y}_{\mathbf{m},p}}, \quad \frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_1},$$

6 thus (i) holds for $\mathbf{m} > 0$ as well. ■

7 In view of the above Lemma, it will be convenient to use $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ instead of $\mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n)$. Notice
8 however the following properties: for any constant $c > 0$,

$$\begin{aligned} \text{(a)} \quad & \text{for } u \text{ with } \|Du\|_\infty \leq c, \quad u \in \mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n) \iff u \in \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n), \quad \text{and} \\ & \|u\|_{\mathcal{D}_{\mathbf{m},p}^{1,2}} \leq \|u\|_{\mathcal{Y}_{\mathbf{m},p}} \leq \|u\|_{\mathcal{D}_{\mathbf{m},p}^{1,2}} + c^{1-\frac{2}{p_1}} \|u\|_{\mathcal{D}_{\mathbf{m},p}^{1,2}}^{\frac{2}{p_1}}; \\ \text{(b)} \quad & \text{if } \{u_j\} \subset \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n), \quad u \in \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \text{ and } \|Du_j\|_\infty \leq c, \quad \|Du\|_\infty \leq c, \text{ then} \\ & \|u_j - u\|_{\mathcal{Y}_{\mathbf{m},p}} \rightarrow 0 \iff \|u_j - u\|_{\mathcal{D}_{\mathbf{m},p}^{1,2}} \rightarrow 0. \end{aligned} \quad (29)$$

9 We define $\Phi, \Psi : \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \rightarrow (-\infty, \infty]$ as follows:

$$\begin{aligned} \Psi(u) &\equiv \begin{cases} \int_{\mathbb{R}^n} \left(1 - \sqrt{1 - |Du|^2} + \frac{\mathbf{m}|u|^p}{2p} \right) & \text{if } \|Du\|_\infty \leq 1; \\ \infty & \text{if } \|Du\|_\infty > 1, \end{cases} \\ \Phi(u) &\equiv \int_{\mathbb{R}^n} G(u), \quad \text{with } G(s) \equiv F(s) + \frac{\mathbf{m}}{2p}|s|^p. \end{aligned} \quad (30)$$

10 Because of the first in Remark 2.14 and (a) in (29), the domain of Ψ is

$$D(\Psi) \equiv \{ u \in \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \mid \Psi(u) < \infty \} = \{ u \in \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \mid \|Du\|_\infty \leq 1 \}. \quad (31)$$

11 **Remark 2.16.** By combining (28), the reflexivity of $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ and the fact that functions in $D(\Psi)$ are
12 equi-Lipschitz, the following holds: from every bounded sequence $\{u_k\}_{k=1}^\infty \subset D(\Psi)$ we can extract a
13 subsequence (still labelled the same) such that $u_k \rightarrow u$ weakly in $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ and locally uniformly in \mathbb{R}^n .

14 Define also

$$I_\lambda : \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \rightarrow (-\infty, \infty], \quad I_\lambda(u) \equiv \lambda\Psi(u) - \Phi(u).$$

15 We first check the conditions for Ψ .

16 **Lemma 2.17.** $\Psi : \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \rightarrow [0, \infty]$ satisfies (Ψ_1) , (Ψ_2) , (Ψ_3) .

17 **Proof.** Observe first that if $\{u_j\}_j \subset D(\Psi)$, $u \in D(\Psi)$ and

$$\mathbf{m}\|u\|_p \leq \liminf_{j \rightarrow \infty} \mathbf{m}\|u_j\|_p, \quad \|Du\|_{2k} \leq \liminf_{j \rightarrow \infty} \|Du_j\|_{2k} \quad \text{for every } k \geq 1, \quad (32)$$

1 then by the second in Remark 2.14 the following chain of inequalities holds:

$$\Psi(u) = \frac{\mathbf{m}}{2p} \|u\|_p^p + \sum_{k=1}^{\infty} b_k \int_{\mathbb{R}^n} |Du|^{2k} \leq \frac{\mathbf{m}}{2p} \|u\|_p^p + \sum_{k=1}^{\infty} b_k \liminf_{j \rightarrow \infty} \int_{\mathbb{R}^n} |Du_j|^{2k} \quad (33)$$

$$\leq \liminf_{j \rightarrow \infty} \left[\frac{\mathbf{m}}{2p} \|u_j\|_p^p + \sum_{k=1}^{\infty} b_k \int_{\mathbb{R}^n} |Du_j|^{2k} \right] = \liminf_{j \rightarrow \infty} \Psi(u_j). \quad (34)$$

2 We prove (Ψ_1) . Since $t \mapsto 1 - \sqrt{1 - t^2}$ is convex for $t \in (0, 1)$, Ψ is convex on $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$. To check that Ψ
3 is lower-semicontinuous, let $u_j \rightarrow u$ in $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$. The claim is obvious if $\|Du_j\|_{\infty} > 1$ for all large j . On
4 the other hand, if $\|Du_{j_i}\|_{\infty} \leq 1$ for a subsequence $j_i \rightarrow \infty$, then $\|Du_{j_i}\|_{2k} \rightarrow \|Du\|_{2k}$ for all $k \geq 1$. Using
5 also $\mathbf{m}\|u_{j_i}\|_p \rightarrow \mathbf{m}\|u\|_p$, we can apply (33) to deduce that $\Psi(u) \leq \liminf_{i \rightarrow \infty} \Psi(u_{j_i})$.

6 For property (Ψ_2) , if $\|Du\|_{\infty} \leq 1$, then $\|Du\|_q^q \leq \|Du\|_2^2$ for each $q \geq 2$. Hence, the first in Remark
7 2.14 and the definition of Ψ yield

$$\begin{aligned} \|u\|_{\mathcal{Y}_{\mathbf{m},p}}^2 &= \mathbf{m}\|u\|_p^2 + \|Du\|_2^2 + \|Du\|_{p_1}^2 \leq C_{p,\mathbf{m}}(\Psi(u))^{\frac{2}{p}} + \|Du\|_2^2 + \|Du\|_2^{\frac{4}{p_1}} \\ &\leq C_{p,\mathbf{m}}(\Psi(u))^{\frac{2}{p}} + b_1^{-1}\Psi(u) + (b_1^{-1}\Psi(u))^{\frac{2}{p_1}}. \end{aligned}$$

8 Thus, (Ψ_2) holds.

9 To show (Ψ_3) , assume that $\{u_j\}_j \subset D(\Psi)$ and $u \in D(\Psi)$ satisfy $u_j \rightharpoonup u$ in $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ and $\Psi(u_j) \rightarrow \Psi(u)$
10 as $j \rightarrow \infty$. The weak convergence and $u_j, u \in D(\Psi)$ yields $Du_j \rightharpoonup Du$ in $L^{2k}(\mathbb{R}^n; \mathbb{R}^n)$ as $j \rightarrow \infty$ for
11 each fixed $k \geq 1$, hence both of the inequalities in (32) hold. From the uniform convexity of L^q -spaces, it
12 suffices to prove that $\mathbf{m}\|u_j\|_p \rightarrow \mathbf{m}\|u\|_p$ and that $\|Du_j\|_{2k} \rightarrow \|Du\|_{2k}$ for each $k \in \mathbb{N}$. By contradiction,
13 if one of these fails, then up to passing to a subsequence one of the inequalities in (32) is strict. As a
14 consequence, one of the two inequalities in (33) is strict as well (recall: here we use the fact $b_k > 0$ for
15 each k), whence we would conclude from (33) that

$$\Psi(u) < \liminf_{j \rightarrow \infty} \Psi(u_j) = \Psi(u),$$

16 a contradiction. ■

17 **Lemma 2.18.** *If $n \geq 3$ and $f \in C(\mathbb{R})$ satisfies*

$$\limsup_{s \rightarrow 0} \frac{|f(s)|}{|s|^{p-1}} < \infty \quad \text{with} \quad \begin{cases} p = 2^* & \text{if } \mathbf{m} = 0, \\ p \in [2, 2^*] & \text{if } \mathbf{m} > 0, \end{cases} \quad (35)$$

18 *then $\Phi : \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \rightarrow \mathbb{R}$ satisfies (Φ_1) .*

19 **Proof.** By Lemma 2.15, $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \hookrightarrow C_0(\mathbb{R}^n)$. From (35), it is standard to see that $\Phi \in C^1(\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n), \mathbb{R})$. ■

21 Next, we check that critical points for I are weak solutions to (\mathcal{BI}_f) . In the positive mass case, this
22 is not entirely trivial due to the asymmetric role of Ψ and Φ in the definition of a critical point. We have

23 **Proposition 2.19.** *Let Φ, Ψ be as in (30), and assume (35). Then, for each $\lambda \in \mathbb{R}^+$ the following are*
24 *equivalent:*

25 - $u \in \mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n)$ is critical for $I_{\lambda} \equiv \lambda\Psi - \Phi$;

- $u \in \mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n)$ is a weak solution to

$$\operatorname{div} \left(\frac{Du}{\sqrt{1-|Du|^2}} \right) + \frac{\mathbf{m}(1-\lambda)}{2\lambda} |u|^{p-2}u + \lambda^{-1}f(u) = 0, \quad (36)$$

in particular, $u \in W_{\text{loc}}^{2,q}(\mathbb{R}^n)$ for $q \in [2, \infty)$, and $\|Du\|_\infty < 1$.

To prove the result, we need the following general lemma.

Lemma 2.20. *Let X be a normed space, $u \in X$ and $\hat{\Psi}, \hat{\Phi}$ satisfy $(\Psi_1), (\Phi_1)$, respectively. Then, the following are equivalent:*

(i) u is a critical point for $\hat{\Psi} - \hat{\Phi}$;

(ii) for some (or equivalently, every) convex, C^1 functional $T : X \rightarrow \mathbb{R}$, u is a critical point for $\hat{\Psi}_T - \hat{\Phi}_T$ with $\hat{\Psi}_T \equiv \hat{\Psi} + T$ and $\hat{\Phi}_T \equiv \hat{\Phi} + T$.

Proof. (i) \Rightarrow (ii) for each T .

Since T is convex, $T(v) - T(u) - T'(u)(v - u) \geq 0$. By combining this inequality with the definition of u being a critical point for $\hat{\Psi} - \hat{\Phi}$ we deduce

$$\hat{\Psi}_T(v) - \hat{\Psi}_T(u) - \hat{\Phi}'_T(u)(v - u) \geq 0 \quad \text{for all } v \in X,$$

thus u is critical for $\hat{\Psi}_T - \hat{\Phi}_T$.

(ii) for some $T \Rightarrow$ (i).

Assume that there exists a convex functional $T \in C^1(X, \mathbb{R})$ for which

$$(\hat{\Psi}(w) + T(w)) - (\hat{\Psi}(u) + T(u)) - (\hat{\Phi}'(u) + T'(u))(w - u) \geq 0 \quad \text{for all } w \in X.$$

Fix $v \in X$ and write $w = (1 - t)u + tv$, $t \in (0, 1]$. Since $\hat{\Psi}$ is convex and T is of class C^1 ,

$$\hat{\Psi}(w) - \hat{\Psi}(u) \leq t(\hat{\Psi}(v) - \hat{\Psi}(u)), \quad T(w) - T(u) = tT'(u)(v - u) + o(t),$$

which yields

$$t(\hat{\Psi}(v) - \hat{\Psi}(u)) + tT'(u)(v - u) + o(t) - t(\hat{\Phi}'(u) + T'(u))(v - u) \geq 0.$$

Dividing by t and letting $t \rightarrow 0$ we conclude

$$\hat{\Psi}(v) - \hat{\Psi}(u) - \hat{\Phi}'(u)(v - u) \geq 0,$$

as claimed. ■

Proof of Proposition 2.19. Applying Lemma 2.20 with

$$T(u) = \frac{\mathbf{m}\lambda}{2p} \|u\|_p^p, \quad \hat{\Psi}(u) = \lambda \int_{\mathbb{R}^n} \left(1 - \sqrt{1 - |Du|^2}\right), \quad \hat{\Phi}(u) = \int_{\mathbb{R}^n} \left(F(u) + \frac{\mathbf{m}(1-\lambda)}{2p} |u|^p\right)$$

we deduce that u is critical for I_λ if and only if it is critical for $\hat{\Psi} - \hat{\Phi}$. In this respect, notice that $\Phi, \hat{\Phi}$ and T are of class C^1 by Lemma 2.18. Proposition 2.13 then ensures that critical points of $\hat{\Psi} - \hat{\Phi}$ coincide with weak solutions to (36), and enjoy the claimed regularity properties. ■

We next prove that under our assumptions I_λ satisfies (IB).

Lemma 2.21. *Assume that $f \in C(\mathbb{R})$ satisfies (f1). Then, $I_\lambda : \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \rightarrow \mathbb{R}$ satisfies (IB).*

1 **Proof.** Without loss of generality, we can assume $c > 0$ in the definition of $\mathcal{K}_{[a,b]}^c$. By Proposition 2.19,
2 critical points $u \in \mathcal{K}_{[a,b]}^c$ for I_λ are weak solutions to (36). Moreover, $u \in W_{\text{loc}}^{2,q}(\mathbb{R}^n)$ for $q \in [2, \infty)$, and
3 $\|Du\|_\infty < 1$. The latter, together with $u \in \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ and (f1), imply

$$\int_{\mathbb{R}^n} \left(\frac{|Du|^2}{\sqrt{1-|Du|^2}} + \mathbf{m}|u|^p + |F(u)| \right) < \infty,$$

4 thus by combining the Pohozaev identity (17) and $I_\lambda(u) \leq c$ we deduce

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|Du|^2}{\sqrt{1-|Du|^2}} &= n \int_{\mathbb{R}^n} \left(1 - \sqrt{1-|Du|^2} \right) - \frac{n}{\lambda} \int_{\mathbb{R}^n} \left\{ \frac{\mathbf{m}(1-\lambda)}{2p} |u|^p + F(u) \right\} \\ &= \frac{n}{\lambda} I_\lambda(u) \leq \frac{nc}{\lambda} \leq \frac{nc}{a}. \end{aligned} \quad (37)$$

5 Therefore, $\mathcal{K}_{[a,b]}^c \subset D(\Psi)$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, hence in $\mathcal{Y}_{0,2^*}(\mathbb{R}^n)$ and in $C(\mathbb{R}^n)$ because of (28) and
6 (29). This settles the case $\mathbf{m} = 0$. Assume $\mathbf{m} > 0$, and fix δ such that

$$f(s)s + \frac{\mathbf{m}}{2}|s|^p \leq -\frac{\mathbf{m}}{4}|s|^p \quad \text{if } |s| \leq \delta.$$

7 This is possible by the definition of \mathbf{m} . Choose a constant κ so that $\|u\|_{2^*} + \|u\|_\infty \leq \kappa$ for each $u \in \mathcal{K}_{[a,b]}^c$.
8 Then, there exists a constant $C' = C'(\delta, \kappa)$ such that

$$f(s)s + \frac{\mathbf{m}}{2}|s|^p \leq -\frac{\mathbf{m}}{4}|s|^p + C'|s|^{2^*} \quad \text{for each } |s| \leq \kappa.$$

9 Using this with $I_\lambda(u) \leq c$ and $\Psi(u) \geq 0$, we get

$$c \geq I_\lambda(u) \geq - \int_{\mathbb{R}^n} \left\{ F(u) + \frac{\mathbf{m}}{2p} |u|^p \right\} \geq \frac{\mathbf{m}}{4p} \|u\|_p^p - \frac{C'}{2^*} \|u\|_{2^*}^{2^*}.$$

10 Therefore, $\mathcal{K}_{[a,b]}^c$ is bounded in $\mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n)$ and, by (29), in $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$. ■

11 Property (Φ_2) does not hold in $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ because of the translation invariance of Φ , and forces us
12 to restrict Ψ, Φ to suitable closed subspaces $X \hookrightarrow \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ where (Φ_2) is restored. Notice that any
13 of $(\Psi_1), (\Psi_2), (\Psi_3), (\Phi_1)$ is inherited by Ψ, Φ when restricted to any closed subspace X . Assume that a
14 topological group G acts continuously by isometries on $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ with an action $\cdot : G \times X \rightarrow X$, and
15 define the subspace of G -invariant functions

$$\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)_G \equiv \{ u \in \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \mid g \cdot u = u \text{ for all } g \in G \}. \quad (38)$$

16 **Example 2.22.** Let G be a closed subgroup of the isometry group $\text{Iso}(\mathbb{R}^n)$ of \mathbb{R}^n , and consider the
17 natural action induced on $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ by defining $(g \cdot u)(x) \equiv u(g^{-1}(x))$. The action is continuous and, for
18 each $g \in G$, $u \rightarrow g \cdot u$ is an isometry of $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$.

19 In the present paper we shall focus for simplicity on the next two examples:

20 **Example 2.23.** Choosing in Example 2.22 the orthogonal group $O(n) \leq \text{Iso}(\mathbb{R}^n)$ acting in the standard
21 way by matrix multiplication on \mathbb{R}^n , we obtain the subspace of radially symmetric functions with respect
22 to the origin.

1 **Example 2.24.** For n and $d \in \mathbb{N}$ satisfying

$$n \geq 4, \quad 2 \leq d \leq \frac{n}{2}, \quad \text{and} \quad n - 2d \neq 1, \quad (39)$$

2 write a point of \mathbb{R}^n as $x = (x_1, x_2, x_3) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{n-2d}$. Consider the subgroup

$$H \equiv \mathrm{O}(d) \times \mathrm{O}(d) \times \mathrm{O}(n - 2d) \leq \mathrm{O}(n)$$

3 sitting in $\mathrm{O}(n)$ as block-diagonal matrices, and the action induced by that of Example 2.23 on $\mathcal{Y}_{m,p}(\mathbb{R}^n)$.
 4 Consider also the discrete subgroup $\langle \tau \rangle$ generated by the involution $\tau \in \mathrm{Iso}(\mathbb{R}^n)$, $\tau(x_1, x_2, x_3) \equiv (x_2, x_1, x_3)$,
 5 and its action on $\mathcal{Y}_{m,p}(\mathbb{R}^n)$ given by

$$(\tau \circ u)(x) \equiv -u(\tau(x)).$$

6 Then, τ induces an action on $\mathcal{Y}_{m,p}(\mathbb{R}^n)_H$, and $(\mathcal{Y}_{m,p}(\mathbb{R}^n)_H)_{\langle \tau \rangle}$ is the subspace of H -invariant functions
 7 which are odd with respect to τ . To our knowledge, the idea of considering such symmetries to produce
 8 non-radial solutions was first introduced in [8].

9 Following [66, §1.5], we describe a sufficient condition on a group $G \leq \mathrm{Iso}(\mathbb{R}^n)$, acting on $\mathcal{Y}_{m,p}(\mathbb{R}^n)$ as
 10 in Example 2.22, so that Φ restricted to $\mathcal{Y}_{m,p}(\mathbb{R}^n)_G$ satisfies (Φ_2) . Let $m(y, r, G)$ be the maximal number
 11 of disjoint balls of radius r centered at points in the G -orbit of y , namely:

$$m(y, r, G) \equiv \sup \left\{ \ell \in \mathbb{N} \left| \begin{array}{l} \text{there exist } g_1, \dots, g_\ell \in G \text{ such that} \\ B(g_i y, r) \cap B(g_j y, r) = \emptyset \text{ if } i \neq j. \end{array} \right. \right\}$$

12 The following property is proved in [46, Théorème III.4] (see also [66, §1.5]). We include the argument
 13 for the sake of completeness.

14 **Proposition 2.25.** *Let $\{u_k\}_{k=1}^\infty \subset \mathcal{Y}_{m,p}(\mathbb{R}^n)_G \cap D(\Psi)$ be bounded in $\mathcal{Y}_{m,p}(\mathbb{R}^n)$, and assume that for some
 15 $r > 0$,*

$$\lim_{|y| \rightarrow \infty} m(y, r, G) = \infty. \quad (40)$$

16 *Then, there exists $u \in \mathcal{Y}_{m,p}(\mathbb{R}^n)_G$ such that, up to a subsequence, $\|u_k - u\|_q \rightarrow 0$ holds for any $q \in (p, \infty)$.*

17 **Remark 2.26.** A direct check shows that both of the groups $\mathrm{O}(n)$ in Example 2.23 and H in Example
 18 2.24 satisfy (40).

19 To prove Theorem 2.25, we need the following lemma:

20 **Lemma 2.27.** *Let $r > 0$ and $\{u_k\}_{k=1}^\infty \subset D(\Psi)$ be a bounded sequence in $\mathcal{Y}_{m,p}(\mathbb{R}^n)$. If*

$$\lim_{k \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{B(y,r)} |u_k|^p = 0,$$

21 *then $u_k \rightarrow 0$ strongly in $L^q(\mathbb{R}^n)$ for any $q \in (p, \infty)$.*

22 **Proof.** First, by (28) we know that $\{u_k\}_k$ is bounded in $L^\infty(\mathbb{R}^n)$. Hence, replacing balls with cubes, our
 23 assumption implies that for each $q \in [p, \infty)$

$$\lim_{k \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{y+Q} |u_k|^q = 0, \quad \text{where } Q \equiv [0, 1]^n. \quad (41)$$

1 Moreover, in view of the L^∞ bound and the interpolation inequality, it suffices to show that $\|u_k\|_q \rightarrow 0$
 2 for some $q \in (p, p^*)$. By Sobolev's embedding $W^{1,p}(Q) \hookrightarrow L^q(Q)$, we compute

$$\begin{aligned} \|u_k\|_q^q &= \sum_{y \in \mathbb{Z}^n} \int_{y+Q} |u_k|^q \leq \left(\sup_{y \in \mathbb{Z}^n} \int_{y+Q} |u_k|^q \right)^{1-\frac{p}{q}} \sum_{y \in \mathbb{Z}^n} \left(\int_{y+Q} |u_k|^q \right)^{\frac{p}{q}} \\ &\leq C_0 \left(\sup_{y \in \mathbb{Z}^n} \int_{y+Q} |u_k|^q \right)^{1-\frac{p}{q}} \sum_{y \in \mathbb{Z}^n} \|u_k\|_{W^{1,p}(y+Q)}^p \\ &= C_0 \left(\sup_{y \in \mathbb{Z}^n} \int_{y+Q} |u_k|^q \right)^{1-\frac{p}{q}} \|u_k\|_{W^{1,p}(\mathbb{R}^n)}^p. \end{aligned}$$

3 Because of (28) (i) and our assumptions, $\|u_k\|_{W^{1,p}(\mathbb{R}^n)}$ is bounded, so the conclusion follows from (41). \blacksquare

4 **Proof of Theorem 2.25.** Let $\{u_k\}_{k=1}^\infty \subset \mathcal{Y}_{m,p}(\mathbb{R}^n)_G \cap D(\Psi)$ be bounded in $\mathcal{Y}_{m,p}(\mathbb{R}^n)$. By Remark
 5 2.16, we can assume that $u_k \rightharpoonup u$ weakly in $\mathcal{Y}_{m,p}(\mathbb{R}^n)$ and locally uniformly in \mathbb{R}^n . The local uniform
 6 convergence guarantees that $u \in \mathcal{Y}_{m,p}(\mathbb{R}^n)_G$ and that $\|Du\|_\infty \leq 1$, whence $u \in D(\Psi)$ by (31). Setting
 7 $v_k := u_k - u$, the claim follows from Lemma 2.27 once we show that

$$\lim_{k \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{B(y,r)} |v_k|^p \rightarrow 0. \quad (42)$$

8 In fact, for $y \in \mathbb{R}^n$, choose $g_1, \dots, g_{m(y,r,G)} \in G$ so that $\{B(g_i y, r)\}_i$ are pairwise disjoint. By (28) (i), in
 9 our assumptions $\|v_k\|_p \leq C$ for some constant C . On the other hand,

$$C \geq \|v_k\|_p^p \geq \int_{\bigcup_{i=1}^{m(y,r,G)} B(g_i y, r)} |v_k|^p = m(y, r, G) \int_{B(y,r)} |v_k|^p.$$

10 Let $\varepsilon > 0$ be arbitrary. It follows from (40) that there exists $R_\varepsilon > 0$ such that

$$\sup_{k \geq 1} \int_{B(y,r)} |v_k|^p < \varepsilon \quad \text{for all } |y| \geq R_\varepsilon.$$

11 From $v_k \rightarrow 0$ locally uniformly in \mathbb{R}^n ,

$$\lim_{k \rightarrow \infty} \sup_{|y| \leq R_\varepsilon} \int_{B(y,r)} |v_k|^p = 0,$$

12 which implies

$$\limsup_{k \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{B(y,r)} |v_k|^p \leq \varepsilon.$$

13 Thus, (42) holds by the arbitrariness of ε . \blacksquare

14 **Proposition 2.28.** Let $G \leq \text{Iso}(\mathbb{R}^n)$ act on $\mathcal{Y}_{m,p}(\mathbb{R}^n)$ as in Example 2.22, and assume that

$$\lim_{|y| \rightarrow \infty} m(y, r, G) = \infty \quad \text{for some } r > 0.$$

15 Then, $\Phi : \mathcal{Y}_{m,p}(\mathbb{R}^n)_G \rightarrow \mathbb{R}$ satisfies (Φ_2) .

1 **Proof.** Assumption (f1) and Lemma 2.18 guarantee that $\Phi \in C^1(\mathcal{Y}_{m,p}(\mathbb{R}^n), \mathbb{R})$. Let

$$\{u_k\}_{k=1}^\infty \subset \mathcal{Y}_{m,p}(\mathbb{R}^n)_G \cap D(\Psi)$$

2 be a bounded sequence. By Lemma 2.15 and Remark 2.16, up to a subsequence, we may assume that
 3 $u_k \rightharpoonup u$ weakly in $\mathcal{Y}_{m,p}(\mathbb{R}^n)$ and locally uniformly in \mathbb{R}^n , and that $\|u_k\|_\infty \leq c_0$ for some $c_0 > 0$. We shall
 4 prove that

$$\liminf_{k \rightarrow \infty} \Phi'(u_k)(v - u_k) \geq \Phi'(u)(v - u). \quad (43)$$

5 Define

$$g(s) \equiv f(s) + \frac{m}{2}|s|^{p-2}s.$$

6 For any fixed $\varepsilon > 0$, choose $R > 0$ so that $\|v\|_{L^p(\mathbb{R}^n \setminus B_R)} < \varepsilon$. Since $u_k \rightarrow u$ uniformly in B_R , it holds

$$[\Phi'(u_k) - \Phi'(u)](v) = \int_{\mathbb{R}^n} [g(u_k) - g(u)]v = o_k(1) + \int_{\mathbb{R}^n \setminus B_R} [g(u_k) - g(u)]v$$

7 as $k \rightarrow \infty$. On the other hand, by (f1) there exists a constant C such that $|g(s)| \leq C|s|^{p-1}$ for $|s| \leq c_0$,
 8 hence

$$\begin{aligned} \left| \int_{\mathbb{R}^n \setminus B_R} [g(u_k) - g(u)]v \right| &\leq \left(\|g(u_k)\|_{\frac{p}{p-1}} + \|g(u)\|_{\frac{p}{p-1}} \right) \|v\|_{L^p(\mathbb{R}^n \setminus B_R)} \\ &\leq C_2 \left(\|u_k\|_p^{p-1} + \|u\|_p^{p-1} \right) \|v\|_{L^p(\mathbb{R}^n \setminus B_R)} \\ &\leq C_3 \varepsilon \left(\sup_k \|u_k\|_{\mathcal{Y}_{m,p}} \right)^{p-1}. \end{aligned}$$

9 Therefore, letting first $k \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we deduce that $\Phi'(u_k)(v) \rightarrow \Phi'(u)(v)$ as $k \rightarrow \infty$. Write
 10 $g(s)s = \tilde{g}_+(s) - \tilde{g}_-(s)$ where $\tilde{g}_\pm(s) \equiv \max\{0, \pm g(s)s\}$. Then, (43) follows provided that

$$\liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} (\tilde{g}_-(u_k) - \tilde{g}_+(u_k)) \geq \int_{\mathbb{R}^n} (\tilde{g}_-(u) - \tilde{g}_+(u)).$$

11 We will prove that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \tilde{g}_+(u_k) = \int_{\mathbb{R}^n} \tilde{g}_+(u), \quad \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \tilde{g}_-(u_k) \geq \int_{\mathbb{R}^n} \tilde{g}_-(u). \quad (44)$$

12 The second claim follows by Fatou's lemma. As for the first, fix $q > p$ and $\varepsilon > 0$. By (f1_m), there exists
 13 $C_{\varepsilon,q,c_0} > 0$ such that

$$\tilde{g}_+(u_k) \leq \varepsilon |u_k|^p + C_{\varepsilon,q,c_0} |u_k|^q, \quad \tilde{g}_+(u) \leq \varepsilon |u|^p + C_{\varepsilon,q,c_0} |u|^q.$$

14 Because of Proposition 2.25, we can assume that $\lim_{k \rightarrow \infty} \|u_k - u\|_q = 0$, up to subsequence. Therefore,
 15 we can fix R large enough so that

$$C_{\varepsilon,q,c_0} \int_{\mathbb{R}^n \setminus B_R} (|u_k|^q + |u|^q) < \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

16 Since $u_k \rightarrow u$ in $L_{\text{loc}}^\infty(\mathbb{R}^n)$, we compute

$$\begin{aligned} \left| \int_{\mathbb{R}^n} (\tilde{g}_+(u_k) - \tilde{g}_+(u)) \right| &\leq o_k(1) + \int_{\mathbb{R}^n \setminus B_R} |\tilde{g}_+(u_k) - \tilde{g}_+(u)| \\ &\leq o_k(1) + \varepsilon \int_{\mathbb{R}^n \setminus B_R} (|u_k|^p + |u|^p) + C_{\varepsilon,q,c_0} \int_{\mathbb{R}^n \setminus B_R} (|u_k|^q + |u|^q) \\ &\leq o_k(1) + \varepsilon C_3 \left(\|u_k\|_{\mathcal{Y}_{m,p}}^p + \|u\|_{\mathcal{Y}_{m,p}}^p \right) + \varepsilon \\ &\leq o_k(1) + C_4 \varepsilon. \end{aligned}$$

17 By letting $k \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ we obtain the first in (44), concluding the proof. \blacksquare

1 **Remark 2.29.** By modifying the above argument, in the case $\mathbf{m} = 0$ and if

$$f(s) = o\left(|s|^{2^*-1}\right) \quad \text{as } s \rightarrow 0,$$

2 one can actually show that

$$\Phi' : \mathcal{Y}_0(\mathbb{R}^n)_G \cap D(\Psi) \longrightarrow (\mathcal{Y}_0(\mathbb{R}^n))^*$$

3 is compact, a stronger property than (Φ_2) .

4 We are ready to prove our main existence result.

5 **Proof of Theorem 2.2.** First of all, by using Proposition 2.13 and Lemma 2.15 we deduce that any weak
6 solution $u \in \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ to (\mathcal{BI}_f) satisfies $u \in W_{\text{loc}}^{2,q}(\mathbb{R}^n)$ for each $q \in [2, \infty)$ and $\|Du\|_\infty < 1$. Referring to
7 Examples 2.23 and 2.24, we set for notational convenience

$$X_{\mathbf{m},1} \equiv \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)_{O(n)}, \quad X_{\mathbf{m},2} \equiv (\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)_H)_{\langle \tau \rangle}. \quad (45)$$

8 In view of (29), to prove the result we shall construct:

- 9 • a positive solution $u \in X_{\mathbf{m},1}$ with $I(u) \in (0, \infty)$;
- 10 • when f is odd, for each $j \in \{1, 2\}$, infinitely many distinct solutions $\{u_k\}_k \subset X_{\mathbf{m},j}$ with $I(u_k) \in$
11 $(0, \infty)$ and $I(u_k) \rightarrow \infty$ as $k \rightarrow \infty$ (provided n, d verify (14) when $j = 2$).

12 Define Ψ, Φ as in (30), and let

$$I_\lambda \equiv \lambda\Psi - \Phi : \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \rightarrow \mathbb{R}.$$

13 In our stated assumptions, Lemmas 2.17 and 2.18 ensure the validity of $(\Psi_1), (\Psi_2), (\Psi_3), (\Phi_1)$ on $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$.
14 In particular, from (Φ_1) and the inclusion $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \hookrightarrow \mathcal{D}_{\mathbf{m},p}^{1,2}(\mathbb{R}^n)$, we can apply Proposition 2.19 and (a)
15 in (29) to deduce that $u \in \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ is a weak solution to (\mathcal{BI}_f) if and only if

$$\Psi(v) - \Psi(u) - \Phi'(u)(v - u) \geq 0 \quad \text{for each } v \in D(\Psi).$$

16 By the definition of Ψ , the above inequality also holds when $\|Dv\|_\infty > 1$ and thus weak solutions to
17 (\mathcal{BI}_f) correspond to critical points of $I = I_1$ in the sense of Szulkin.

18 The groups $G = O(n), H$, respectively, in Examples 2.23 and 2.24, are compact and act continuously
19 on $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$, and $\langle \tau \rangle$ acts continuously by isometries on $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)_G$. Moreover, $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ is reflexive, thus
20 so is $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)_G$. Therefore, by the principle of nonsmooth symmetric criticality proved in [44, Theorem
21 3.16] (see Proposition A.1 in Appendix A for another proof which does not require reflexivity) a function
22 $u \in X_{\mathbf{m},j}$ is a critical point of $I_\lambda : \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \rightarrow \mathbb{R}$ if and only if it is critical for the restriction of I_λ to
23 $X_{\mathbf{m},j}$, namely, if and only if

$$\lambda(\Psi(v) - \Psi(u)) - \Phi'(u)(v - u) \geq 0 \quad \text{for any } v \in X_{\mathbf{m},j}.$$

24 Moreover, by Proposition 2.28 and Remark 2.26, Φ enjoys (Φ_2) provided that we restrict its domain
25 to either $X_{\mathbf{m},1}$ or to $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)_H$ in Example 2.24. In the second case (Φ_2) is therefore inherited by the
26 restriction of Φ to $X_{\mathbf{m},2}$. Lemma 2.21 guarantees that (IB) holds for $I_\lambda : X_{\mathbf{m},j} \rightarrow \mathbb{R}$. In summary,
27 $(\Psi_1), (\Psi_2), (\Psi_3)$ and $(\Phi_1), (\Phi_2), (\text{IB})$ are satisfied. To apply Theorem 1.6 (respectively, Theorem 1.7 if f
28 is odd), it is therefore sufficient to check the uniform mountain pass condition (MP) (respectively, (SMP))
29 for, say, $\lambda \in [1/2, 2]$.

30 In our assumptions, $I_\lambda(0) = 0$ for each λ . Set $\rho_0 > 0$ and consider u with $\|u\|_{\mathcal{Y}_{\mathbf{m},p}} = \rho_0$. By (28),
31 there exists a constant $C > 0$ such that $\|u\|_\infty \leq C\rho_0$. Hence, because of (f1), for each $\varepsilon > 0$ there exists
32 ρ_0 such that the following inequality holds:

$$F(u) + \frac{\mathbf{m}}{2p}|u|^p \leq \varepsilon|u|^p \quad \text{for any } u \in \mathcal{Y}_{\mathbf{m}}(\mathbb{R}^n) \text{ with } \|u\|_{\mathcal{Y}_{\mathbf{m},p}} = \rho_0.$$

1 Moreover, by using Remark 2.14,

$$\Psi(u) \geq \frac{\mathbf{m}}{4p} \|u\|_p^p + \frac{1}{2} \|Du\|_2^2.$$

2 We therefore have the following chain of inequalities for each $\lambda \geq 1/2$:

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2} \Psi(u) - \int_{\mathbb{R}^n} \left(F(u) + \frac{\mathbf{m}}{2p} |u|^p \right) \\ &\geq \left(\frac{\mathbf{m}}{4p} - \varepsilon \right) \|u\|_p^p + \frac{1}{4} \|Du\|_2^2 \geq \left(\frac{\mathbf{m}}{4p} - \varepsilon \right) \|u\|_p^p + c_1 \|u\|_{2^*}^2 + \frac{1}{8} \|Du\|_2^2, \end{aligned}$$

3 where in the last line Sobolev's inequality was used. In both cases $\mathbf{m} = 0$ ($p = 2^*$) and $\mathbf{m} > 0$, by choosing
4 $\varepsilon > 0$ small enough and ρ_0 accordingly there exists $C_2 > 0$ such that

$$I_\lambda(u) \geq C_2 \|u\|_{\mathcal{D}_{\mathbf{m},p}^{1,2}}^p \quad \text{for each } u \in \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n) \text{ with } \|u\|_{\mathcal{Y}_{\mathbf{m},p}} = \rho_0.$$

5 Hence, by (29), there exists $\alpha_0 > 0$ such that $I_\lambda(u) \geq \alpha_0$ on $\{u \mid \|u\|_{\mathcal{Y}_{\mathbf{m},p}} = \rho_0\}$ for each $\lambda \geq 1/2$.

6 (i) Property (MP).

7 In the radial case, we choose a radially symmetric function $u_0 \in C_c^\infty(\mathbb{R}^n)$ such that

$$u_0(x) \equiv \begin{cases} t_0 & \text{for } |x| \leq L, \\ 0 & \text{for } L + 3t_0 \leq |x| \end{cases}$$

8 and $\|Du_0\|_\infty < 1/2$, where t_0 is the value in (f2). For sufficiently large $L > 0$, a direct computation
9 gives $I_2(u_0) < 0$, so (MP) holds.

11 (ii) Property (SMP).

12 In the radial setting, by [13, Theorem 10], for each $k \in \mathbb{N}$ there exist R_k, M_k independent of $\zeta \in \partial\mathbb{D}^k$
13 and odd maps

$$\pi_k \in C(\partial\mathbb{D}^k, H_{\text{rad}}^1(\mathbb{R}^n))$$

14 valued in the set $H_{\text{rad}}^1(\mathbb{R}^n)$ of radial functions in $H^1(\mathbb{R}^n)$ and satisfying

$$\text{supp}(\pi_k(\zeta)) \subset B(0, R_k), \quad \|D\pi_k(\zeta)\|_\infty \leq M_k, \quad \int_{\mathbb{R}^n} F(\pi_k(\zeta)) \geq 1 \quad \text{for any } \zeta \in \partial\mathbb{D}^k. \quad (46)$$

15 Define $\gamma_{0,k}$ by

$$\gamma_{0,k}(\zeta)(x) \equiv \pi_k(\zeta) \left(\frac{x}{L} \right) \quad \text{for } L \gg 1. \quad (47)$$

16 Notice that for sufficiently large L , $\|D\gamma_{0,k}(\zeta)\|_\infty \leq M_k/L < 1$. Thus, by using (29) and (28),
17 $\pi_k(\partial\mathbb{D}^k)$ is bounded in $\mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$, hence in $C(\mathbb{R}^n)$. From the first in (46) and the dominated con-
18 vergence theorem, one readily sees that $\gamma_{0,k} \in C(\partial\mathbb{D}^k, X_{\mathbf{m},1})$, and by Remark 2.14

$$\begin{aligned} I_2(\gamma_{0,k}(\zeta)) &\leq 2L^n \int_{B(0, R_k)} \left(1 - \sqrt{1 - M_k^2/L^2} \right) - L^n \int_{\mathbb{R}^n} F(\pi_k(\zeta)) \\ &\leq L^n \left[2 \frac{M_k^2}{L^2} |B(0, R_k)| - 1 \right] < 0, \end{aligned}$$

19 which implies (SMP). Likewise, in dimension $n \geq 4$ and for each $2 \leq d \leq n/2$, in the argument
20 of [39, Lemma 4.3] the authors produced, for each $k \in \mathbb{N}$, an odd map $\pi_k \in C(\partial\mathbb{D}^k, (H^1(\mathbb{R}^n)_H)_{\langle \tau \rangle})$
21 such that (46) holds for suitable M_k, R_k independent of ζ . Defining again $\gamma_{0,k}$ as in (47), the same
22 computation as above yields $\pi_k \in C(\partial\mathbb{D}^k, X_{\mathbf{m},2})$ and $\sup_{\zeta \in \partial\mathbb{D}^k} I_2(\gamma_{0,k}(\zeta)) < 0$, as required.

1 By applying Theorem 1.6 (resp. Theorem 1.7 if f is odd), we get all of our conclusions apart from the
 2 claim that a positive solution in $X_{\mathbf{m},1}$ can be found. To see this, we consider the problem with continuous
 3 nonlinearity

$$\hat{f}(s) = \begin{cases} f(s) & \text{if } s \geq 0, \\ -\mathbf{m}|s|^{p-2}s & \text{if } s < 0. \end{cases}$$

4 Then, \hat{f} satisfies (f1_{m,p}) and (f2) since so does f , and therefore there exists a solution $u \in X_{\mathbf{m},1}$ to

$$\operatorname{div} \left(\frac{Du}{\sqrt{1-|Du|^2}} \right) + \hat{f}(u) = 0 \quad \text{in } \mathbb{R}^n.$$

5 Moreover, by Proposition 2.10 and Lemma 2.15, $u \in C^1(\mathbb{R}^n)$ and $\|Du\|_\infty < 1$. Assume that $\{u < 0\} \neq \emptyset$.
 6 As in the proof of Theorem 2.2, we may use $u_-(x) \equiv \max\{-u(x), 0\} \in \mathcal{Y}_{\mathbf{m},p}(\mathbb{R}^n)$ to deduce

$$\int_{\{u < 0\}} \left(\frac{|Du|^2}{\sqrt{1-|Du|^2}} \right) + \mathbf{m} \int_{\{u < 0\}} |u|^p = 0,$$

7 a contradiction. Therefore, $u \geq 0$ and thus u solves (BI_f). The positivity of u follows from the weak
 8 Harnack inequality (see [32, Theorem 8.18]) and the fact that $u \geq 0$ satisfies

$$0 = -\operatorname{div}(aDu) - f(u(x)) \leq -\operatorname{div}(aDu) + \left(\frac{f(u(x))}{u(x)} \right)_- u(x), \quad a(x) \equiv \frac{1}{\sqrt{1-|Du(x)|^2}}.$$

9 This concludes the proof. ■

10 3 Deformation lemmas

11 The aim of this section is to prove general deformation lemmas in order to establish the monotonicity
 12 method pioneered in [38, 41, 57, 58] for the family $\{I_\lambda\}_\lambda$. The results below relate to [63, Proposition
 13 2.3]. Although our conclusions are slightly weaker, differently from [63] we do not require a priori the
 14 validity of the Palais-Smale condition (in this respect, see also [1, Theorem 3.1]). Let us recall that a chief
 15 difficulty in proving a deformation lemma for lower semicontinuous functionals is that the closure of a
 16 relatively compact subset contained in a strip $I_\lambda^{-1}([a, b])$ may not lie in the same strip. This problem does
 17 not occur for continuous functionals as those treated in [27, Theorems 2.14 and 2.15] and [26, Theorem
 18 2.3], and makes the construction of an energy decreasing flow which produces a bounded Palais-Smale
 19 sequence a challenging task. Notice that in Lemmas 3.1 and 3.5 below, we allow I_λ to increase along the
 20 flow in some regions, albeit in a controlled way.

21 We introduce some notation: for each $\lambda > 0$, we denote by \mathcal{K}_λ the set of all critical points of I_λ :

$$\mathcal{K}_\lambda \equiv \{u \in X \mid u \text{ is a critical point of } I_\lambda\}.$$

22 Also, for a subset $C \subset X$, the symbol \overline{C} stands for the closure of C in X . We shall prove two results, the
 23 second one by assuming that Ψ, Φ are even.

24 3.1 The Deformation Lemma

25 **Lemma 3.1.** *Suppose that (Ψ_1) and (Φ_1) hold. Fix $\lambda \in \mathbb{R}^+$ and let $A \subset X$ be a closed and bounded set.
 26 For $c \in \mathbb{R}$ and $\sigma > 0$, define*

$$\mathcal{A}_{c,3\sigma} \equiv A \cap \{c - 3\sigma \leq I_\lambda \leq c + 3\sigma\}.$$

1 Assume that there exists $\delta_0 > 0$ such that for each $u \in \mathcal{A}_{c,3\sigma}$ we may find $v = v(u) \in X \setminus \{u\}$ with

$$\lambda(\Psi(v) - \Psi(u)) - \Phi'(u)(v - u) < -5\delta_0\|v - u\|. \quad (48)$$

2 Then for every set $K \subset \mathcal{A}_{c,3\sigma}$ which is relatively compact in X there exist small neighborhoods W and \widetilde{W}
3 of \overline{K} with

$$\overline{K} \subset W \subset \overline{W} \subset \widetilde{W},$$

4 $s_0 > 0$ and $\eta \in C([0, \infty) \times X, X)$ such that

5 (i) $\|\eta(s, w) - w\| \leq s$ for $(s, w) \in [0, \infty) \times X$ and $\eta(s, w) = w$ for $s \in [0, \infty)$ if $w \notin \widetilde{W}$;

6 (ii) $I_\lambda(\eta(s, w)) \leq I_\lambda(w) + \delta_0 s$ for all $(s, w) \in [0, s_0] \times X$;

7 (iii) $I_\lambda(\eta(s, w)) \leq I_\lambda(w) - 2\delta_0 s$ for all $(s, w) \in [0, s_0] \times W$ with $I_\lambda(w) \geq c - \sigma$.

8 **Remark 3.2.** Since I_λ is only assumed to be lower semicontinuous, the set $\mathcal{A}_{c,3\sigma}$ may not be closed and
9 its closure may possibly contain points in $\{I_\lambda < c - 3\sigma\}$. Consequently, (48) may fail at points of $\overline{K} \setminus K$,
10 a major obstacle to construct an energy decreasing deformation flow. This marks one of differences with
11 the deformation in [63, Proposition 2.3], where it is assumed that the set K in Lemma 3.1 is compact.

12 **Remark 3.3.** Observe that Lemma 3.1 holds for a fixed $\lambda \in \mathbb{R}^+$, and it seems difficult to control the
13 deformation for I_λ locally uniformly in λ . As a consequence, in the proof of Theorems 1.6 and 1.7
14 Ekeland's variational principle is hardly applicable, and we have to devise a different strategy based on
15 an iteration method.

16 **Proof of Lemma 3.1.** The following argument is adapted from the proof of [63, Lemma 2.2 and
17 Proposition 2.3]. Let $K \subset \mathcal{A}_{c,3\sigma}$ be relatively compact. Without loss of generality, we may suppose

$$\delta_0 < \sigma < 1. \quad (49)$$

18 From the lower semicontinuity of I_λ it follows that

$$I_\lambda(u) \leq c + 3\sigma \quad \text{for each } u \in \overline{\mathcal{A}_{c,3\sigma}}. \quad (50)$$

19 In particular, since $\overline{K} \subset \overline{\mathcal{A}_{c,3\sigma}}$, the estimate in (50) holds on \overline{K} .

20 We divide our arguments into several steps. In Step 1, we identify the building blocks to construct
21 a pseudogradient vector field in a neighborhood of K , namely, for each $u \in \overline{K}$ we find $v = v(u) \in X$
22 and a small radius $r = r(u)$ such that, loosely speaking, for $w \in B(u, r)$ the vector $v - w$ acts as a
23 pseudogradient vector: the value of I_λ does not increase too much with respect to $I_\lambda(w)$ in direction
24 $v - w$, and decreases at least at a fixed rate whenever $I_\lambda(w) \geq c - 2\sigma$. Notice that the choice $v(u)$ in (48)
25 is admissible (and works well) only if $u \in \mathcal{A}_{c,3\sigma}$. However, as $\mathcal{A}_{c,3\sigma}$ is not closed, this is not always the
26 case and the analysis for points $u \in \overline{K} \setminus \mathcal{A}_{c,3\sigma}$ is more involved.

27 **Step 1:** *The following hold:*

28 (a) Suppose $u \in \overline{K} \cap \mathcal{A}_{c,3\sigma}$ and let $v = v(u) \in X \setminus \{u\}$ be as in (48). Then there exists $r = r(u) > 0$ such
29 that $B(u, r) \subset \overline{\mathcal{A}_{c,3\sigma}} + B(0, 1)$, $v \notin B(u, r)$ and

$$\lambda(\Psi(v) - \Psi(w)) - \Phi'(w)(v - w) \leq -3\delta_0\|v - w\| \quad \text{for all } w \in B(u, r).$$

30 (b) Suppose $u \in (\overline{K} \setminus \mathcal{A}_{c,3\sigma}) \cap \mathcal{K}_\lambda$ and set $v \equiv u$. Then there exists $r = r(u) > 0$ such that $B(u, r) \subset$
31 $\overline{\mathcal{A}_{c,3\sigma}} + B(0, 1)$ and

$$\lambda(\Psi(v) - \Psi(w)) - \Phi'(w)(v - w) \leq \frac{\delta_0}{2}\|v - w\| \quad \text{for any } w \in B(u, r).$$

32 In addition, if $w \in B(u, r)$ satisfies $I_\lambda(w) \geq c - 2\sigma$, then

$$\lambda(\Psi(v) - \Psi(w)) - \Phi'(w)(v - w) \leq -3\delta_0\|v - w\|.$$

1 (c) For each $u \in \overline{K} \setminus (\mathcal{A}_{c,3\sigma} \cup \mathcal{K}_\lambda)$, there exist $v = v(u) \in X \setminus \{u\}$ and $r = r(u) > 0$ such that
 2 $B(u, r) \subset \overline{\mathcal{A}_{c,3\sigma}} + B(0, 1)$, $v \notin \overline{B(u, r)}$ and

$$\lambda(\Psi(v) - \Psi(w)) - \Phi'(w)(v - w) \leq \frac{\delta_0}{2} \|v - w\| \quad \text{for any } w \in B(u, r).$$

3 In addition, if $w \in B(u, r)$ satisfies $I_\lambda(w) \geq c - 2\sigma$, then

$$\lambda(\Psi(v) - \Psi(w)) - \Phi'(w)(v - w) \leq -3\delta_0 \|v - w\|.$$

4 **Proof.** To prove (a), let $u \in \overline{K} \cap \mathcal{A}_{c,3\sigma}$ and v be as in (48). By the lower semicontinuity of Ψ and the
 5 fact that $v \neq u$, there exists $r = r(u) > 0$ such that $B(u, r) \subset \overline{\mathcal{A}_{c,3\sigma}} + B(0, 1)$, $v \notin \overline{B(u, r)}$ and for all
 6 $w \in B(u, r)$

$$\lambda(\Psi(v) - \Psi(w)) - \Phi'(w)(v - w) < -4\delta_0 \|v - w\|.$$

7 Hence, (a) holds.

8 To prove (b) and (c), recall $\overline{K} \subset \overline{\mathcal{A}_{c,3\sigma}}$ and (50):

$$I_\lambda(u) \leq c + 3\sigma \quad \text{for any } u \in \overline{K}.$$

9 Therefore, if $u \in \overline{K} \setminus \mathcal{A}_{c,3\sigma}$, then the closedness of A and the fact $K \subset A$ yield $u \in A$ and

$$I_\lambda(u) < c - 3\sigma. \tag{51}$$

10 We prove (b). Let $u \in (\overline{K} \setminus \mathcal{A}_{c,3\sigma}) \cap \mathcal{K}_\lambda$ and put $v \equiv u$. By $u \in \mathcal{K}_\lambda$,

$$\lambda(\Psi(w) - \Psi(v)) - \Phi'(v)(w - v) \geq 0 \quad \text{for any } w \in X.$$

11 Thus,

$$\begin{aligned} \lambda(\Psi(v) - \Psi(w)) - \Phi'(w)(v - w) &\leq (\Phi'(w) - \Phi'(v))(w - v) \\ &\leq \|\Phi'(w) - \Phi'(v)\|_* \|w - v\|. \end{aligned}$$

12 By the continuity of Φ' , if $r = r(u) > 0$ is sufficiently small, then for any $w \in B(u, r) \subset \overline{\mathcal{A}_{c,3\sigma}} + B(0, 1)$

$$\lambda(\Psi(v) - \Psi(w)) - \Phi'(w)(v - w) \leq \frac{\delta_0}{2} \|w - v\|.$$

13 Hence, the first inequality in (b) holds.

14 When $w \in B(u, r)$ satisfies $I_\lambda(w) \geq c - 2\sigma$, by recalling (51) and writing $I_\lambda(u) = c - 3\sigma - a_u$ with
 15 some $a_u > 0$, it follows that

$$\begin{aligned} \sigma + a_u &\leq I_\lambda(w) - I_\lambda(u) \\ &= \lambda(\Psi(w) - \Psi(u)) + (\Phi(u) - \Phi(w)) \\ &= \lambda(\Psi(w) - \Psi(u)) + \Phi'(w)(u - w) + \int_0^1 \{\Phi'(w + \theta(u - w)) - \Phi'(w)\} (u - w) d\theta. \end{aligned} \tag{52}$$

16 From the continuity of Φ' at u , shrinking $r = r(u)$ if necessary, we obtain

$$\sigma \leq \lambda(\Psi(w) - \Psi(u)) + \Phi'(w)(u - w) \quad \text{for every } w \in B(u, r) \text{ with } I_\lambda(w) \geq c - 2\sigma$$

17 or equivalently,

$$\lambda(\Psi(u) - \Psi(w)) - \Phi'(w)(u - w) \leq -\sigma \quad \text{for every } w \in B(u, r) \text{ with } I_\lambda(w) \geq c - 2\sigma.$$

1 Recalling that we have chosen $v = u$, up to further shrinking $r = r(u) > 0$ if necessary we infer

$$\lambda(\Psi(v) - \Psi(w)) - \Phi'(w)(v - w) \leq -\sigma \leq -3\delta_0\|v - w\| \quad \text{for any } w \in B(u, r) \text{ with } I_\lambda(w) \geq c - 2\sigma,$$

2 which completes the proof of (b).

3 For (c), by $u \notin \mathcal{K}_\lambda$ we may find $v = v(u) \in X \setminus \{u\}$ and $b_u > 0$ such that

$$\lambda(\Psi(v) - \Psi(u)) - \Phi'(u)(v - u) = -2b_u < 0.$$

4 We claim that v can be chosen arbitrarily close to u . In fact, consider

$$v_t \equiv tv + (1 - t)u \quad t \in (0, 1).$$

5 From the convexity of Ψ , it follows that

$$\Psi(v_t) \leq t\Psi(v) + (1 - t)\Psi(u)$$

6 and

$$\lambda(\Psi(v_t) - \Psi(u)) - \Phi'(u)(v_t - u) \leq t\{\lambda(\Psi(v) - \Psi(u)) - \Phi'(u)(v - u)\} = -2b_ut < 0. \quad (53)$$

7 For each $t \in (0, 1)$, there exists $r_t > 0$ such that $B(u, r_t) \subset \overline{\mathcal{A}_{c, 3\sigma}} + B(0, 1)$, $v_t \notin \overline{B(u, r_t)}$ and

$$\lambda(\Psi(v_t) - \Psi(w)) - \Phi'(w)(v_t - w) < -b_ut \leq \frac{\delta_0}{2}\|v_t - w\| \quad \text{for every } w \in B(u, r_t).$$

8 Thus, the first inequality in (c) holds.

9 Let $w \in B(u, r_t)$ satisfy $I_\lambda(w) \geq c - 2\sigma$. Writing $I_\lambda(u) = c - 3\sigma - a_u$ with $a_u > 0$, in a similar way
10 to obtain (52), we have

$$\sigma + a_u \leq \lambda(\Psi(w) - \Psi(u)) - \Phi'(u)(w - u) + o(\|w - u\|).$$

11 By shrinking $r_t > 0$ if necessary, we get

$$\sigma \leq \lambda(\Psi(w) - \Psi(u)) - \Phi'(u)(w - u) \quad \text{for all } w \in B(u, r_t) \text{ with } I_\lambda(w) \geq c - 2\sigma. \quad (54)$$

12 Hence, (53) and (54) imply

$$\begin{aligned} & \lambda(\Psi(v_t) - \Psi(w)) - \Phi'(w)(v_t - w) \\ &= \lambda\{\Psi(v_t) - \Psi(u) + \Psi(u) - \Psi(w)\} - \Phi'(w)(v_t - w) + \Phi'(u)(w - u) - \Phi'(u)(w - u) \\ &\leq \lambda(\Psi(v_t) - \Psi(u)) - \Phi'(w)(v_t - w) - \Phi'(u)(w - u) - \sigma \\ &< \Phi'(u)(v_t - u) - \Phi'(w)(v_t - w) - \Phi'(u)(w - u) - \sigma \\ &= (\Phi'(u) - \Phi'(w))(v_t - w) - \sigma \\ &\leq \|\Phi'(u) - \Phi'(w)\|_* \|v_t - w\| - \sigma \\ &= \left\{ \|\Phi'(u) - \Phi'(w)\|_* - \frac{\sigma}{\|v_t - w\|} \right\} \|v_t - w\|. \end{aligned}$$

13 Since $t \in (0, 1)$ is arbitrary and we may take a smaller r_t , for sufficiently small t and r_t , we get

$$\lambda(\Psi(v_t) - \Psi(w)) - \Phi'(w)(v_t - w) \leq -3\delta_0\|v_t - w\| \quad \text{for each } w \in B(u, r_t) \text{ with } I_\lambda(w) \geq c - 2\sigma$$

14 and (c) holds. ■

1 To proceed, since \overline{K} is compact and $\Phi' \in C(X, X^*)$, there exists $r_1 > 0$ such that

$$\sup \{ \|\Phi'(v) - \Phi'(u)\|_* \mid u \in \overline{K}, \|v - u\| \leq r_1 \} \leq \frac{\delta_0}{4}. \quad (55)$$

2 **Step 2:** Construction of η .

3 **Proof.** For each $u \in \overline{K}$, we have $v = v(u)$ and $B(u, r)$ as in Step 1. Since $r = r(u)$ can be taken small,
4 we may suppose

$$r(u) \leq \frac{r_1}{2}, \quad \text{that is,} \quad \|u - w\| \leq \frac{r_1}{2} \quad \text{for all } w \in B(u, r(u)),$$

5 with r_1 as in (55). Using the compactness of \overline{K} , from $\overline{K} \subset \bigcup_{u \in \overline{K}} B(u, r(u))$ we infer the existence of
6 finitely many distinct points $u_1, \dots, u_k \in \overline{K}$ so that

$$\overline{K} \subset \bigcup_{i=1}^k B(u_i, R_i), \quad \text{dist}(\overline{K}, w) \leq \frac{r_1}{2} \quad \text{for all } w \in \bigcup_{i=1}^k B(u_i, R_i),$$

7 where we write $R_i \equiv r(u_i)$. We also use the symbol $v_i = v(u_i)$. Since the u_i 's are finite, we may find
8 $r_0 \in (0, r_1)$ such that

$$\begin{aligned} \overline{B(u_i, 2r_0)} &\subset B(u_i, R_i) && \text{for } 1 \leq i \leq k, \\ \overline{B(u_i, 2r_0)} \cap \overline{B(u_j, 2r_0)} &= \emptyset && \text{for each } i, j \in \{1, \dots, k\} \text{ with } i \neq j. \end{aligned}$$

9 Put

$$V_i \equiv B(u_i, R_i) \setminus \bigcup_{j \in \{1, \dots, k\}, j \neq i} \overline{B(u_j, r_0)} \quad \text{for } i = 1, \dots, k,$$

10 and define

$$\widetilde{W} \equiv \bigcup_{i=1}^k V_i.$$

11 From the definition of V_i and the choice of $r_0 \in (0, r_1)$, we observe that

$$\begin{aligned} \overline{B(u_i, 2r_0)} &\subset V_i \quad (1 \leq i \leq k), \quad \text{dist}(\overline{K}, w) \leq \frac{r_1}{2} \quad \text{for any } w \in \widetilde{W}, \\ \widetilde{K} &\equiv \overline{K} \cup \bigcup_{i=1}^k \overline{B(u_i, r_0)} \subset \widetilde{W}, \quad 0 < \text{dist}(\widetilde{K}, \partial \widetilde{W}). \end{aligned} \quad (56)$$

12 Notice that the last property in (56) follows from

$$\text{dist}(\overline{K}, \partial \widetilde{W}) > 0, \quad \text{dist}\left(\bigcup_{i=1}^k \overline{B(u_i, r_0)}, \partial \widetilde{W}\right) > 0.$$

13 Moreover, by the very definition of V_j ,

$$\overline{B(u_i, r_0)} \cap V_j = \emptyset \quad \text{for each } i, j \in \{1, \dots, k\} \text{ with } i \neq j. \quad (57)$$

14 For $\varepsilon > 0$, write

$$V_{i,\varepsilon} \equiv \{x \in V_i \mid \text{dist}(x, V_i^c) > \varepsilon\} \quad \text{for } i = 1, \dots, k.$$

1 From (56), there exists $\zeta_0 > 0$ such that

$$\overline{B(u_i, r_0)} \subset V_{i, 3\zeta_0} \quad \text{for } 1 \leq i \leq k, \quad \tilde{K} \subset \bigcup_{i=1}^k V_{i, 3\zeta_0} \equiv W_1.$$

2 We also set

$$W_2 \equiv \bigcup_{i=1}^k V_{i, 2\zeta_0}, \quad W_3 \equiv \bigcup_{i=1}^k V_{i, \zeta_0}, \quad W_4 \equiv \widetilde{W}.$$

3 Then,

$$\overline{W_j} \subset W_{j+1} \quad \text{and} \quad \text{dist}(W_j, X \setminus W_{j+1}) > 0 \quad (1 \leq j \leq 3).$$

4 Define

$$V_0 \equiv X \setminus \overline{W_1}, \quad V_{0, \zeta_0} \equiv X \setminus \overline{W_2}.$$

5 It is easily seen that

$$X = (X \setminus \overline{W_2}) \cup W_3 = \bigcup_{i=0}^k V_{i, \zeta_0}, \quad \text{dist}(V_{0, \zeta_0}, X \setminus V_0) = \text{dist}(\overline{W_1}, X \setminus \overline{W_2}) > 0.$$

6 Furthermore, from $\tilde{K} \subset W_1$, it follows that

$$V_0 \cap \overline{B(u_i, r_0)} = \emptyset \quad (1 \leq i \leq k). \quad (58)$$

7 For $i = 0, \dots, k$, pick $\tilde{\rho}_i \in C(X, \mathbb{R})$ so that

$$0 < \tilde{\rho}_i \quad \text{on } V_{i, \zeta_0}, \quad \text{supp } \tilde{\rho}_i \subset V_i, \quad \text{dist}(\text{supp } \tilde{\rho}_i, X \setminus V_i) > 0$$

8 and set

$$\rho_i(w) \equiv \frac{\tilde{\rho}_i(w)}{\sum_{j=0}^k \tilde{\rho}_j(w)}.$$

9 Then for each $i = 0, 1, \dots, k$,

$$\rho_i \in C(X, [0, 1]), \quad \sum_{i=0}^k \rho_i = 1 \quad \text{on } X, \quad \text{supp } \rho_i \subset V_i, \quad \text{dist}(\text{supp } \rho_i, X \setminus V_i) > 0. \quad (59)$$

10 In particular, $\rho_i = 0$ on $X \setminus W_4$ for every $i = 1, \dots, k$.

11 To define $\eta(s, w)$, we consider the decomposition $\{1, \dots, k\} = J_1 \cup J_2$, where

$$J_1 \equiv \{i \mid u_i \in \mathcal{K}_\lambda\}, \quad J_2 \equiv \{1, \dots, k\} \setminus J_1. \quad (60)$$

12 By the definition of v_i and by Step 1,

$$v_i = u_i \quad \text{if } i \in J_1, \quad v_i \notin \overline{V_i} \quad \text{if } i \in J_2. \quad (61)$$

13 For any $i \in J_1$, we define $\eta_i \in C([0, \infty) \times X, X)$ by

$$\eta_i(s, w) \equiv \begin{cases} \rho_i(w) \left[w + s \frac{v_i - w}{\|v_i - w\|} \right] & \text{if } 0 \leq s < \|v_i - w\|, \\ \rho_i(w) v_i & \text{if } s \geq \|v_i - w\|. \end{cases}$$

1 On the other hand, for $i \in J_2$ we put

$$\eta_i(s, w) \equiv \rho_i(w) \left[w + s \frac{v_i - w}{\|v_i - w\|} \right].$$

2 By (61), $v_i \notin \bar{V}_i \supset \text{supp } \rho_i$, so setting $\eta_i(s, w) = 0$ for $w = v_i$ yields a continuous map $\eta_i \in C([0, \infty) \times$
 3 $X, X)$. Finally, for $i = 0$, define

$$\eta_0(s, w) \equiv \rho_0(w)w \in C([0, \infty) \times X, X).$$

4 Using these η_i 's, we define $\eta(s, w)$ by

$$\eta(s, w) \equiv \sum_{i=0}^k \eta_i(s, w) \in C([0, \infty) \times X, X)$$

5 and set

$$W \equiv W_1 = \bigcup_{i=1}^k V_{i, 3\zeta_0}.$$

6 Remark that W is an open set satisfying $\bar{K} \subset W \subset \bar{W} \subset \widetilde{W}$. Then, we shall prove that η , W and \widetilde{W}
 7 satisfy (i)–(iii) in Lemma 3.1. ■

8 **Step 3:** η satisfies (i)–(iii).

9 **Proof.** We first prove (i). Notice that for each $(s, w) \in [0, \infty) \times X$ and $i \in J_2 \cup \{0\}$,

$$\|\eta_i(s, w) - \rho_i(w)w\| \leq \rho_i(w)s.$$

10 When $i \in J_1$, if $\rho_i(w) > 0$ and $0 \leq s < \|v_i - w\|$, then

$$\|\eta_i(s, w) - \rho_i(w)w\| \leq \rho_i(w)s.$$

11 On the other hand, if $i \in J_1$, $\rho_i(w) > 0$ and $s \geq \|v_i - w\|$, then

$$\|\eta_i(s, w) - \rho_i(w)w\| \leq \rho_i(w)\|v_i - w\| \leq \rho_i(w)s.$$

12 Combining these estimates, we observe that for every $(s, w) \in [0, \infty) \times X$,

$$\|\eta(s, w) - w\| \leq \sum_{i=0}^k \|\eta_i(s, w) - \rho_i(w)w\| \leq \sum_{i=0}^k s\rho_i(w) = s.$$

13 Thus, the first assertion in (i) holds.

14 Regarding the second assertion in (i), from the definition of \widetilde{W} and (59), it follows that if $w \notin \widetilde{W}$,
 15 then $\rho_j(w) = 0$ for every $j = 1, \dots, k$ and $\rho_0(w) = 1$. Therefore, $\eta(s, w) = \rho_0(w)w = w$ holds for any
 16 $(s, w) \in [0, \infty) \times (X \setminus \widetilde{W})$, as claimed.

17 It remains to prove (ii) and (iii). Without loss of generality, we may assume $w \in \widetilde{W}$. We split the
 18 proof into two cases:

19 **First case:** assume that

$$w \in \bigcup_{i \in J_1} B(u_i, r_0),$$

1 where J_1 is as in (60). Choose an index $i_0 \in J_1$ for which $w \in B(u_{i_0}, r_0)$. In this case, by $\overline{B(u_{i_0}, r_0)} \subset$
2 $V_{i_0, 3\zeta_0}$, (57), (58) and (59), we remark that $\rho_j(w) = 0$ for any $j \in \{0, \dots, k\}$ with $j \neq i_0$, hence

$$\rho_{i_0}(w) = 1, \quad \eta(s, w) = \eta_{i_0}(s, w). \quad (62)$$

3 Also, $u_{i_0} \in \mathcal{K}_\lambda$ and $v_{i_0} = u_{i_0}$ by (61).

4 If $w = v_{i_0}$, then by definition we have $\eta_{i_0}(s, w) = w$ for any $s \geq 0$. Also, we claim that $I_\lambda(w) < c - 3\sigma$,
5 which directly implies (ii) and there is nothing to prove for (iii). Indeed, otherwise, by combining (50),
6 $w = v_{i_0} \in \mathcal{K}_\lambda \cap \overline{K}$ and the closedness of A , from $\overline{K} \subset A$ we would deduce $w \in A \cap \{c - 3\sigma \leq I_\lambda \leq$
7 $c + 3\sigma\} = \mathcal{A}_{c, 3\sigma}$, which violates (48).

8 Assume therefore that $w \in B(u_{i_0}, r_0) \setminus \{u_{i_0}\}$. From Step 1 and the definition of η_{i_0} , if $0 \leq s \leq \|w - v_{i_0}\|$,
9 then

$$\eta(s, w) = \eta_{i_0}(s, w) = (1 - \tau_{i_0})w + \tau_{i_0}v_{i_0} \quad \text{where } \tau_{i_0} = \tau_{i_0}(s, w) \equiv \frac{s}{\|w - v_{i_0}\|} \leq 1$$

10 and by the convexity of Ψ ,

$$\begin{aligned} I_\lambda(\eta(s, w)) &\leq \lambda(1 - \tau_{i_0})\Psi(w) + \lambda\tau_{i_0}\Psi(v_{i_0}) - \Phi(\eta_{i_0}(s, w)) \\ &= I_\lambda(w) + \Phi(w) + \tau_{i_0}\lambda(\Psi(v_{i_0}) - \Psi(w)) - \Phi(\eta_{i_0}(s, w)). \end{aligned}$$

11 Notice that by $v_{i_0} = u_{i_0} \in \mathcal{K}_\lambda \cap \overline{K}$, $\tau_{i_0} \leq 1$ and $r_0 < r_1$, on the segment

$$[0, 1] \ni \theta \mapsto w_\theta \equiv w + (1 - \theta)\tau_{i_0}(v_{i_0} - w),$$

12 it holds

$$\|w_\theta - v_{i_0}\| = \{1 - (1 - \theta)\tau_{i_0}\} \|w - v_{i_0}\| \leq r_0 < r_1 \quad \text{for any } \theta \in [0, 1].$$

13 From (55), we get

$$\begin{aligned} \Phi(w) - \Phi(\eta_{i_0}(s, w)) &= \int_0^1 \frac{d}{d\theta} \Phi(w_\theta) d\theta = -\tau_{i_0}(v_{i_0} - w) \int_0^1 \Phi'(w_\theta) d\theta \\ &= -(\tau_{i_0}(v_{i_0} - w)) \left\{ \Phi'(w) + \int_0^1 [\Phi'(w_\theta) - \Phi'(v_{i_0}) + \Phi'(v_{i_0}) - \Phi'(w)] d\theta \right\}. \\ &\leq -\tau_{i_0}\Phi'(w)(v_{i_0} - w) + 2\tau_{i_0}\|v_{i_0} - w\| \max_{\theta \in [0, 1]} \|\Phi'(w_\theta) - \Phi'(v_{i_0})\|. \\ &\leq -\tau_{i_0}\Phi'(w)(v_{i_0} - w) + \frac{\delta_0}{2}\tau_{i_0}\|v_{i_0} - w\| \\ &= -\tau_{i_0}\Phi'(w)(v_{i_0} - w) + \frac{\delta_0}{2}s. \end{aligned}$$

14 This implies that for all $w \in B(u_{i_0}, r_0) \setminus \{u_{i_0}\}$ ($i_0 \in J_1$) and $s \in [0, \|v_{i_0} - w\|]$,

$$I_\lambda(\eta(s, w)) \leq I_\lambda(w) + \tau_{i_0} \left[\lambda \{ \Psi(v_{i_0}) - \Psi(w) \} - \Phi'(w)(v_{i_0} - w) \right] + \frac{\delta_0}{2}s. \quad (63)$$

15 Recalling $\tau_{i_0} \leq 1$ and Step 1 (b), we observe that for every $w \in B(u_{i_0}, r_0) \setminus \{u_{i_0}\} \subset V_{i_0}$ and $s \in$
16 $[0, \|v_{i_0} - w\|]$,

$$I_\lambda(\eta(s, w)) \leq I_\lambda(w) + \tau_{i_0} \frac{\delta_0}{2} \|v_{i_0} - w\| + \frac{\delta_0}{2}s = I_\lambda(w) + \delta_0 s. \quad (64)$$

17 Finally, if $s > \|v_{i_0} - w\|$, then $\eta(s, w) = \eta_{i_0}(s, w)$ and the definition of η_{i_0} yield

$$I_\lambda(\eta(s, w)) = I_\lambda(\eta(\|v_{i_0} - w\|, w)) \leq I_\lambda(w) + \delta_0 \|v_{i_0} - w\| \leq I_\lambda(w) + \delta_0 s.$$

18 In conclusion, (ii) is verified for all $(s, w) \in [0, \infty) \times B(u_{i_0}, r_0)$ with $i_0 \in J_1$.

1 To check (iii), let $w \in B(u_{i_0}, r_0) \setminus \{u_{i_0}\}$ satisfy $I_\lambda(w) \geq c - \sigma$. For any $s \in [0, \|v_{i_0} - w\|]$, and taking
2 into account the definition of τ_{i_0} , Step 1 (b) and (63) imply

$$I_\lambda(\eta(s, w)) \leq I_\lambda(w) - \tau_{i_0} 3\delta_0 \|v_{i_0} - w\| + \frac{\delta_0}{2} s \leq I_\lambda(w) - 2\delta_0 s.$$

3 On the other hand, when $\|v_{i_0} - w\| < s$, it follows from $I_\lambda(v_{i_0}) < c - 3\sigma$ that

$$I_\lambda(\eta(s, w)) = I_\mu(v_{i_0}) < c - 3\sigma \leq I_\lambda(w) - 2\sigma.$$

4 Since $\delta_0 < \sigma$ holds due to (49), if $\|v_{i_0} - w\| < s \leq 1$, then

$$I_\lambda(\eta(s, w)) < I_\lambda(w) - 2\delta_0 s.$$

5 Hence, for every $(s, w) \in [0, 1] \times B(u_{i_0}, r_0)$ with $i_0 \in J_1$ and $I_\lambda(w) \geq c - \sigma$,

$$I_\lambda(\eta(s, w)) \leq I_\lambda(w) - 2\delta_0 s.$$

6 Thus, (iii) holds on $[0, 1] \times \bigcup_{i \in J_1} B(u_i, r)$, too.

7 **Second case:** assume that

$$w \in \widetilde{W} \setminus \bigcup_{i \in J_1} B(u_i, r_0).$$

8 By the definition of η_i , for any $s \in [0, r_0/2]$ (so that $s < \|w - v_i\|$ for each $i \in J_1$) we have

$$\eta(s, w) = w + s \sum_{j=1}^k \rho_j(w) \frac{v_j - w}{\|v_j - w\|} = w + sx, \quad x \equiv \sum_{j=1}^k \rho_j(w) \frac{v_j - w}{\|v_j - w\|}, \quad \|x\| \leq 1. \quad (65)$$

9 Moreover, it follows from (59) and the second in (61) that

$$\text{dist}(v_j, \text{supp } \rho_j) > 0 \quad \text{for every } j \in J_2.$$

10 Also, since $v_j = u_j$ for all $j \in J_1$, we may choose $s_1 \in (0, r_0/2)$ so that

$$s \sum_{j=1}^k \frac{\rho_j(w)}{\|v_j - w\|} \leq 1 \quad \text{for every } s \in [0, s_1] \text{ and } w \in \widetilde{W} \setminus \bigcup_{i \in J_1} B(u_i, r_0).$$

11 The inequality can be rewritten as

$$\sum_{j=1}^k \tau_j(s, w) \leq 1, \quad \tau_j(s, w) \equiv s \frac{\rho_j(w)}{\|v_j - w\|} \quad \text{for each } 1 \leq j \leq k,$$

12 so that

$$\eta(s, w) = \left(1 - \sum_{j=1}^k \tau_j\right) w + \sum_{j=1}^k \tau_j v_j.$$

13 From the convexity of Ψ , we infer

$$\begin{aligned} I_\lambda(\eta(s, w)) &\leq \lambda \left[\left(1 - \sum_{j=1}^k \tau_j\right) \Psi(w) + \sum_{j=1}^k \tau_j \Psi(v_j) \right] - \Phi(\eta(s, w)) \\ &= I_\lambda(w) + \sum_{j=1}^k \tau_j \{\lambda (\Psi(v_j) - \Psi(w))\} + \Phi(w) - \Phi(\eta(s, w)). \end{aligned} \quad (66)$$

1 We shall estimate the term $\Phi(w) - \Phi(\eta(s, w))$. Since $w \in \widetilde{W}$, $\text{dist}(w, \overline{K}) \leq r_1/2$ holds due to (56). The
2 compactness of \overline{K} implies that there exists $y_w \in \overline{K}$ such that

$$\|w - y_w\| \leq \frac{r_1}{2}.$$

3 Therefore, if $0 \leq s \leq s_1$ and $\theta \in [0, 1]$, then by $s_1 < r_0/2 < r_1/2$,

$$\|w + (1 - \theta)sx - y_w\| \leq \frac{r_1}{2} + s < r_1.$$

4 By (65) and (55), for every $s \in [0, s_1]$ we have

$$\begin{aligned} \Phi(w) - \Phi(\eta(s, w)) &= \int_0^1 \frac{d}{d\theta} \Phi(w + (1 - \theta)sx) d\theta \\ &= -\Phi'(w)sx + \int_0^1 (\Phi'(w + (1 - \theta)sx) - \Phi'(w)) (-sx) d\theta \\ &= -\Phi'(w)sx + \int_0^1 (\Phi'(w + (1 - \theta)sx) - \Phi'(y_w)) (-sx) d\theta \\ &\quad + \int_0^1 (\Phi'(y_w) - \Phi'(w)) (-sx) d\theta \\ &\leq -\Phi'(w)sx + \frac{\delta_0}{2}s = -\sum_{j=1}^k \tau_j \Phi'(w)(v_j - w) + \frac{\delta_0}{2}s. \end{aligned}$$

5 Plugging into (66), for $s \in [0, s_1]$ we get

$$I_\lambda(\eta(s, w)) \leq I_\lambda(w) + \sum_{j=1}^k \tau_j \{ \lambda (\Psi(v_j) - \Psi(w)) - \Phi'(w)(v_j - w) \} + \frac{\delta_0}{2}s. \quad (67)$$

6 Notice that if $\tau_j > 0$ for some $j = 1, \dots, k$, then $w \in V_j$ and thus Step 1 gives

$$I_\lambda(\eta(s, w)) \leq I_\lambda(w) + \sum_{j=1}^k \tau_j \frac{\delta_0}{2} \|v_j - w\| + \frac{\delta_0}{2}s = I_\lambda(w) + \sum_{j=1}^k \rho_j(w) \frac{\delta_0}{2}s + \frac{\delta_0}{2}s \leq I_\lambda(w) + \delta_0 s.$$

7 This concludes the proof of (ii) whenever $(s, w) \in [0, s_1] \times (\widetilde{W} \setminus \bigcup_{i \in J_1} B(u_i, r_0))$.

8 To check the validity of (iii), assume that

$$w \in W \setminus \bigcup_{i \in J_1} B(u_i, r_0), \quad I_\lambda(w) \geq c - \sigma.$$

9 Since $W \equiv W_1$ and $V_0 \cap W = \emptyset$, (59) yields $\rho_0(w) = 0$. Thus, from (67) and Step 1 it follows $\sum_{j=1}^k \rho_j(w) =$
10 1 and

$$\begin{aligned} I_\lambda(\eta(s, w)) &\leq I_\lambda(w) + \sum_{j=1}^k \tau_j (-3\delta_0 \|v_j - w\|) + \frac{\delta_0}{2}s \\ &= I_\lambda(w) - 3\delta_0 \sum_{j=1}^k \rho_j(w)s + \frac{\delta_0}{2}s \leq I_\lambda(w) - 2\delta_0 s \end{aligned}$$

11 for each $s \in [0, s_1]$, and (iii) holds. We have thus shown properties (ii) and (iii) in all cases. ■

12 From Step 3, by choosing $s_0 \equiv \min\{s_1, 1\}$, all the conclusions in Lemma 3.1 hold and we complete
13 the proof. ■

1 3.2 Symmetric Deformation Lemma

2 Throughout this subsection we assume condition (E), which is defined as

$$\Psi(-u) = \Psi(u), \quad \Phi(-u) = \Phi(u) \quad \text{for any } u \in X.$$

3 For a subset $A \subset X$, we say that A is *symmetric* if $-A = A$ holds, where $-A \equiv \{x \in X \mid -x \in A\}$.

4 **Remark 3.4.** Under (E) and (Φ_1) , it is easily seen that $\Phi'(0) = 0$. Moreover, if Ψ satisfies (Ψ_1) , then
5 by $\Psi(u) \geq 0 = \Psi(0)$ for every $u \in X$, we infer that 0 is a critical point of I_λ .

6 To establish multiplicity results for even functionals, we need to construct an odd deformation which
7 is the counterpart of Lemma 3.1.

8 **Lemma 3.5** (Symmetric Deformation Lemma). *Assume that (Ψ_1) , (Φ_1) and (E) hold. Fix $\lambda \in \mathbb{R}^+$ and
9 let $A \subset X$ be a bounded, closed and symmetric set and for $c \in \mathbb{R}$ and $\sigma > 0$, define*

$$\mathcal{A}_{c,3\sigma} \equiv A \cap \{c - 3\sigma \leq I_\lambda \leq c + 3\sigma\}.$$

10 *Suppose that there exists $\delta_0 > 0$ such that for each $u \in \mathcal{A}_{c,3\sigma}$ we may find $v = v(u) \in X \setminus \{u\}$ with*

$$\lambda(\Psi(v) - \Psi(u)) - \Phi'(u)(v - u) < -5\delta_0\|v - u\|. \quad (68)$$

11 *Then for every relatively compact and symmetric set $K \subset \mathcal{A}_{c,3\sigma}$ there exist symmetric neighborhoods W
12 and \widetilde{W} of \overline{K} with*

$$\overline{K} \subset W \subset \overline{W} \subset \widetilde{W},$$

13 *$s_0 > 0$ and $\eta \in C([0, \infty) \times X, X)$ such that*

14 (i) $\|\eta(s, w) - w\| \leq s$ and $\eta(s, -w) = -\eta(s, w)$ for all $(s, w) \in [0, \infty) \times X$, and $\eta(s, w) = w$ for all
15 $s \in [0, \infty)$ if $w \notin \widetilde{W}$;

16 (ii) $I_\lambda(\eta(s, w)) \leq I_\lambda(w) + \delta_0 s$ for all $(s, w) \in [0, s_0] \times X$;

17 (iii) $I_\lambda(\eta(s, w)) \leq I_\lambda(w) - 2\delta_0 s$ for all $(s, w) \in [0, s_0] \times W$ with $I_\lambda(u) \geq c - \sigma$.

18 **Proof.** We follow the same scheme as in the proof of Lemma 3.1, just pointing out the differences needed
19 to get the oddness of the deformation flow.

20 By (E), notice that $\mathcal{A}_{c,3\sigma}$ is symmetric, namely $u \in \mathcal{A}_{c,3\sigma}$ yields $-u \in \mathcal{A}_{c,3\sigma}$. Therefore, for every
21 $u \in \mathcal{A}_{c,3\sigma}$, (68) and (E) imply

$$\lambda(\Psi(-v) - \Psi(-u)) - \Phi'(-u)(-v - (-u)) < -5\delta_0\|-v - (-u)\|.$$

22 For each $u \in \mathcal{A}_{c,3\sigma}$, by defining $\tilde{v}(u) \equiv \{v(u) - v(-u)\}/2$, we see from (E), (68) and the convexity of Ψ
23 that

$$\begin{aligned} \lambda(\Psi(\tilde{v}(u)) - \Psi(u)) - \Phi'(u)(\tilde{v}(u) - u) &\leq \frac{1}{2} [\lambda\{\Psi(v(u)) - \Psi(u)\} - \Phi'(u)(v(u) - u)] \\ &\quad + \frac{1}{2} [\lambda\{\Psi(v(-u)) - \Psi(-u)\} - \Phi'(-u)(v(-u) - (-u))] \\ &< \frac{1}{2} (-5\delta_0\|v(u) - u\|) + \frac{1}{2} (-5\delta_0\|v(-u) - (-u)\|) \\ &\leq -5\delta_0\|\tilde{v}(u) - u\|. \end{aligned}$$

24 Remark that $\tilde{v}(u) \neq u$ since the second-last inequality is strict. Thus, we may assume that v is odd with
25 respect to $u \in X$.

1 For Step 1 in the proof of Lemma 3.1, we remark that \overline{K} and \mathcal{K}_λ are symmetric. Therefore, in (a)
 2 and (b), we can take $v(u)$ and $r(u)$ so that $v(-u) = -v(u)$ and $r(-u) = r(u)$, and prove Step 1 (a) and
 3 (b). For Step 1 (c), if $v \in X \setminus \{u\}$ satisfies

$$\lambda(\Psi(v) - \Psi(u)) - \Phi'(u)(v - u) = -2b_u < 0,$$

4 then (E) gives

$$\lambda(\Psi(-v) - \Psi(-u)) - \Phi'(-u)(-v - (-u)) = -2b_u < 0.$$

5 Hence, following the proof of Step 1 (c), we can see that the claim in (c) holds with $v(-u) = -v(u)$ and
 6 $r(-u) = r(u)$.

7 Since \overline{K} is symmetric and compact, we notice that there are $u_1, \dots, u_k \in \overline{K}$ such that by setting
 8 $u_{k+1} = 0$ when $0 \in \overline{K}$,

$$\overline{K} \subset \begin{cases} \bigcup_{i=1}^k \{B(u_i, r(u_i)) \cup B(-u_i, r(u_i))\} & \text{if } 0 \notin \overline{K}, \\ B(u_{k+1}, r(0)) \cup \bigcup_{i=1}^k \{B(u_i, r(u_i)) \cup B(-u_i, r(u_i))\} & \text{if } 0 \in \overline{K}. \end{cases}$$

9 Write $u_{-j} \equiv -u_j$ for $1 \leq j \leq k+1$ and $R_i \equiv r(u_i)$ for $1 \leq |i| \leq k+1$, and when $0 \notin \overline{K}$ define

$$V_i \equiv B(u_i, R_i) \setminus \bigcup_{1 \leq |j| \leq k, j \neq i} \overline{B(u_j, r_0)} \quad \text{for } 1 \leq |i| \leq k$$

10 and when $0 \in \overline{K}$,

$$V_i \equiv B(u_i, R_i) \setminus \bigcup_{1 \leq |j| \leq k+1, j \neq i} \overline{B(u_j, r_0)} \quad \text{for } 1 \leq |i| \leq k, \quad V_{k+1} \equiv B(0, R_{k+1}) \setminus \bigcup_{1 \leq |j| \leq k} \overline{B(u_j, r_0)}.$$

11 In any case, we have

$$V_{-j} = -V_j \quad \text{for } 1 \leq j \leq k, \quad -V_{k+1} = V_{k+1}, \quad -\bigcup_{1 \leq |i| \leq k} V_i = \bigcup_{1 \leq |i| \leq k} V_i.$$

12 Arguing as in the proof of Step 2 of Lemma 3.1, we may find $(\tilde{\rho}_i)_{1 \leq |i| \leq k}$ (resp. $(\tilde{\rho}_i)_{1 \leq |i| \leq k}$ and $\tilde{\rho}_{k+1}$)
 13 such that $0 < \tilde{\rho}_i$ on V_{i, ζ_0} , $\text{supp } \tilde{\rho}_i \subset V_i$, $\tilde{\rho}_{-j}(u) = \tilde{\rho}_j(-u)$ for $1 \leq j \leq k$, $\tilde{\rho}_{k+1}(-u) = \tilde{\rho}_{k+1}(u)$ and
 14 $\tilde{\rho}_0(-u) = \tilde{\rho}_0(u)$. From this family of functions, we construct the partition of unity $(\rho_i)_{1 \leq |i| \leq k}$ (resp.
 15 $(\rho_i)_{1 \leq |i| \leq k}$ and ρ_{k+1}) as before.

16 To define $\eta(s, u)$, consider

$$J_1 \equiv \{i \mid 1 \leq |i| \leq k, u_i \in \mathcal{K}_\lambda\} \quad (\text{resp. } \{i \mid 1 \leq |i| \leq k \text{ or } i = k+1, u_i \in \mathcal{K}_\lambda\}), \\ J_2 \equiv \{-k, \dots, -1, 1, \dots, k\} \setminus J_1.$$

17 We remark that $k+1 \in J_1$ if $0 \in \overline{K}$ and see from the properties of V_i that for $1 \leq j \leq k$, $j \in J_1$ if
 18 and only if $-j \in J_1$. Defining $\eta_i(s, u)$ and $\eta(s, u)$ as in Step 2, we see that $\eta_{-j}(s, u) = -\eta_j(s, -u)$ for
 19 $1 \leq j \leq k$, $\eta_{k+1}(s, -u) = -\eta_{k+1}(s, u)$ and $\eta_0(s, -u) = -\eta_0(s, u)$, hence $\eta(s, -u) = -\eta(s, u)$.

20 The rest of the argument is unchanged compared to the proof of Lemma 3.1. ■

4 Proof of Theorem 1.6 and Theorem 1.7

In the proof of of Theorems 1.6 and 1.7 we assumed (IB), that is, the boundedness of the set of critical points $\mathcal{K}_{[a,b]}^c$ for each $[a, b] \subset (0, \infty)$ and $c \in \mathbb{R}$. As a matter of fact, under the further validity of $(\Psi_1), (\Psi_3), (\Phi_1)$ and (Φ_2) a stronger property holds: $\mathcal{K}_{[a,b]}^c$ is compact. More precisely, we prove the next

Proposition 4.1. *Suppose that $(\Psi_1), (\Psi_3), (\Phi_1), (\Phi_2)$ hold. Assume that there exist $c \in \mathbb{R}$ and sequences $\{\varepsilon_j\}, \{\lambda_j\} \subset (0, \infty)$ and $\{u_j\} \subset X$ with*

$$\varepsilon_j \rightarrow 0, \quad \lambda_j \rightarrow \lambda \in (0, \infty), \quad u_j \rightharpoonup u \text{ weakly in } X \quad \text{as } j \rightarrow \infty$$

and

$$\lambda_j(\Psi(v) - \Psi(u_j)) - \Phi'(u_j)(v - u_j) \geq -\varepsilon_j \|v - u_j\| \quad \text{for each } v \in X.$$

Then,

$$\lim_{j \rightarrow \infty} \|u_j - u\| = 0, \quad u \text{ is critical for } I_\lambda, \quad I_\lambda(u) = \lim_{j \rightarrow \infty} I(u_j) \in \mathbb{R}. \quad (69)$$

In particular, in the stated assumptions,

- for each $\lambda \in (0, \infty)$ and $c \in \mathbb{R}$, I_λ satisfies the bounded Palais-Smale condition at level c , namely any bounded Palais-Smale sequence for I_λ at level c has a strongly convergent subsequence in X ;

- property (IB) implies that the set $\mathcal{K}_{[a,b]}^c$ is compact for each $[a, b] \subset (0, \infty)$ and $c \in \mathbb{R}$.

Proof. We first claim that $u \in D(\Psi)$ and u is a critical point of I_λ . Remark that the sequence $\{u_j\}$ is bounded in X due to the weak convergence. Hence, up to passing to a subsequence, we can assume that $\{u_j\}$ itself satisfies the conclusion in (Φ_2) . Having fixed $v \in D(\Psi)$ and $\varepsilon \in (0, \lambda)$, we select j_0 large enough so that $\lambda_j > \lambda - \varepsilon$ for $j \geq j_0$. Since $\Psi \geq 0$, if $j \geq j_0$, then we get

$$\begin{aligned} \lambda_j \Psi(v) - (\lambda - \varepsilon) \Psi(u_j) - \Phi'(u_j)(v - u_j) &\geq \lambda_j(\Psi(v) - \Psi(u_j)) - \Phi'(u_j)(v - u_j) \\ &\geq -\varepsilon_j \|v - u_j\| \geq -C\varepsilon_j \end{aligned}$$

for some constant C . Taking a lim sup as $j \rightarrow \infty$ and using (Ψ_1) and (Φ_2) , we infer

$$\lambda \Psi(v) - (\lambda - \varepsilon) \Psi(u) - \Phi'(u)(v - u) \geq 0 \quad \text{for every } v \in X.$$

Hence, $\Psi(u) < \infty$, and letting $\varepsilon \rightarrow 0$ we deduce that u is critical for I_λ .

Next, taking the lim sup as $j \rightarrow \infty$ in the inequality

$$\lambda_j \Psi(u) - \Phi'(u_j)(u - u_j) \geq \varepsilon_j \|u - u_j\| + \lambda_j \Psi(u_j)$$

and using (Φ_2) we get $\limsup_{j \rightarrow \infty} \Psi(u_j) \leq \Psi(u)$. Since Ψ is weakly lower-semicontinuous, it follows that $\lim_{j \rightarrow \infty} \Psi(u_j) = \Psi(u)$ and thus, by (Ψ_3) , $\lim_{j \rightarrow \infty} \|u_j - u\| = 0$. Hence, by (Φ_1) , $I_\lambda(u) = \lim_{j \rightarrow \infty} I_{\lambda_j}(u_j)$, which completes the proof of (69).

To check the bounded Palais-Smale condition, it is enough to extract from a bounded Palais-Smale sequence $\{u_j\}$ at level c a weakly convergent subsequence $u_j \rightharpoonup u$ by using (Φ_2) . Applying the above, u_j converges strongly to u , as claimed.

Similarly, if (IB) holds, then pick any bounded sequence $\{u_j\} \subset \mathcal{K}_{[a,b]}^c$. Because of (Φ_2) , we can extract a weakly convergent subsequence and, up to further subsequences, we can assume that $\lambda_j \rightarrow \lambda \in [a, b]$ and $u_j \rightharpoonup u$ weakly in X as $j \rightarrow \infty$. The above guarantees that $u_j \rightarrow u$ strongly in X and that $u \in \mathcal{K}_{[a,b]}^c$. Hence, $\mathcal{K}_{[a,b]}^c$ is compact. \blacksquare

To proceed, we collect some basic properties of $c_k(\lambda), k = 0, 1, 2, \dots$, defined in (4) and (5). We first show

1 **Lemma 4.2.** *If (MP), (Ψ_1) and (Φ_1) hold, then $\Gamma_0 \neq \emptyset$ and $0 < \alpha_0 \leq c_0(\lambda) < \infty$ for each $\lambda \in [1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$.*
2 *Similarly, $\Gamma_k \neq \emptyset$ and $\alpha_0 \leq c_k(\lambda) < \infty$ hold for every $k \geq 1$ and $\lambda \in [1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$ under (SMP), (Ψ_1)*
3 *and (Φ_1) .*

4 **Proof.** For Γ_0 and $c_0(\lambda)$, consider a path $\gamma_0(t) \equiv tu_0$ ($t \in [0, 1]$) where u_0 appears in (MP). It is easily
5 seen that $\gamma_0 \in \Gamma_0$, and (Ψ_1) and $I_{1+\bar{\varepsilon}}(u_0) \leq 0$ yield $u_0 \in D(\Psi)$ and

$$0 \leq \Psi(\gamma_0(t)) \leq t\Psi(u_0) + (1-t)\Psi(0) = t\Psi(u_0) < \infty.$$

6 From $I_{1+\bar{\varepsilon}}(u_0) < 0$ and (Φ_1) we infer that $\sup_{0 \leq t \leq 1} I_\lambda(\gamma(t)) < \infty$ and $c_0(\lambda) < \infty$ for every $\lambda \in [1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$.
7 The assertion $\alpha_0 \leq c_0(\lambda)$ follows from the fact $\gamma([0, 1]) \cap \partial B(0, \rho_0)$ and (MP).

8 For Γ_k and $c_k(\lambda)$, consider a map $\tilde{\gamma}_k \in C(\mathbb{D}^k, X)$ defined by

$$\tilde{\gamma}_k(\zeta) \equiv \begin{cases} |\zeta| \pi_{0,k} \left(\frac{\zeta}{|\zeta|} \right) & \text{if } 0 < |\zeta| \leq 1, \\ 0 & \text{if } \zeta = 0. \end{cases}$$

9 Clearly $\tilde{\gamma}_k \in \Gamma_k$ holds. Moreover, (Ψ_1) , (Φ_1) and (SMP) imply

$$\begin{aligned} \sup_{\zeta \in \mathbb{D}^k} \Psi(\tilde{\gamma}_k(\zeta)) &\leq \sup_{\zeta \in \mathbb{D}^k} |\zeta| \Psi \left(\pi_{0,k} \left(\frac{\zeta}{|\zeta|} \right) \right) \leq \sup_{\zeta \in \partial \mathbb{D}^k} \Psi(\pi_{0,k}(\zeta)) \\ &= (1 + \bar{\varepsilon})^{-1} \sup_{\zeta \in \partial \mathbb{D}^k} \{I_{1+\bar{\varepsilon}}(\pi_{0,k}(\zeta)) + \Phi(\pi_{0,k}(\zeta))\} \\ &\leq (1 + \bar{\varepsilon})^{-1} \sup_{\zeta \in \partial \mathbb{D}^k} \Phi(\pi_{0,k}(\zeta)) < \infty. \end{aligned}$$

10 Hence, $c_k(\lambda) < \infty$ holds for all $\lambda \in [1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$. Furthermore, $\alpha_0 \leq c_k(\lambda)$ follows from the fact
11 $\partial B(0, \rho_0) \cap \gamma(\mathbb{D}^k) \neq \emptyset$ for each $\gamma \in \Gamma_k$. ■

12 **Proposition 4.3.** *For each $k = 0, 1, \dots$, the function $\lambda \mapsto c_k(\lambda)$ is nondecreasing on $[1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$, hence*
13 *differentiable at almost all $\lambda \in [1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$. Moreover, c_k is continuous from the right in $[1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$;*
14 *thus, c_k is upper semicontinuous.*

15 **Proof.** The monotonicity of c_k follows from $\Psi \geq 0$. For the right continuity of c_k , fix $\lambda \in [1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$ and
16 $\varepsilon \in (0, \infty)$ arbitrarily. Choose $\gamma_\varepsilon \in \Gamma_k$ so that $\sup_{\zeta \in \mathbb{D}^k} I_\lambda(\gamma_\varepsilon(\zeta)) < c_k(\lambda) + \varepsilon$. Since $\sup_{\zeta \in \mathbb{D}^k} \Psi(\gamma_\varepsilon(\zeta)) < \infty$
17 due to $\max_{\zeta \in \mathbb{D}^k} |\Phi(\gamma_\varepsilon(\zeta))| < \infty$ and $\sup_{\zeta \in \mathbb{D}^k} I_\lambda(\gamma_\varepsilon(\zeta)) < c_k(\lambda) + \varepsilon$, it follows that $0 \leq I_\mu(\gamma_\varepsilon(\zeta)) -$
18 $I_\lambda(\gamma_\varepsilon(\zeta)) < \varepsilon$ for every $\zeta \in \mathbb{D}^k$ if $\mu - \lambda > 0$ is sufficiently small. Then,

$$c(\mu) \leq \sup_{\zeta \in \mathbb{D}^k} I_\mu(\gamma_\varepsilon(\zeta)) \leq \sup_{\zeta \in \mathbb{D}^k} I_\lambda(\gamma_\varepsilon(\zeta)) + \varepsilon \leq c_k(\lambda) + 2\varepsilon$$

19 if $\mu - \lambda > 0$ is sufficiently small. This implies that c_k is right continuous. ■

20 Here is our main result on the existence of bounded Palais-Smale sequences.

21 **Theorem 4.4.** *Suppose that (Ψ_1) , (Ψ_2) , (Φ_1) and either (MP), or (E) and (SMP), hold. Assume also*
22 *that at $\lambda \in (1 - \bar{\varepsilon}, 1 + \bar{\varepsilon})$, $c'_k(\lambda)$ exists (with $k = 0$, under assumption (MP)). Then there exists a*
23 *Palais-Smale sequence $\{u_j\}_{j=1}^\infty$ for I_λ such that $I_\lambda(u_j) \rightarrow c_k(\lambda)$ and $\{u_j\}_{j=1}^\infty$ is bounded in X .*

24 **Proof.** For a fixed $k \in \{0, 1, \dots\}$, assume that $c'_k(\lambda)$ exists at some $\lambda \in (1 - \bar{\varepsilon}, 1 + \bar{\varepsilon})$. We put

$$M_0 \equiv c'_k(\lambda) + 3.$$

1 From now on, we just assume $k \geq 1$ since the case $k = 0$ can be handled in the same manner. From (Ψ_2) ,
 2 there exists $R_0 > 0$ such that

$$\{u \in X \mid \Psi(u) \leq 6M_0\} \subset B(0, R_0) \quad (70)$$

3 and we set

$$A \equiv \overline{B(0, R_0 + 1)}, \quad \mathcal{A}_{c_k(\lambda), 3\sigma} \equiv A \cap \{c_k(\lambda) - 3\sigma \leq I_\lambda \leq c_k(\lambda) + 3\sigma\}. \quad (71)$$

4 To complete the proof of Theorem 4.4, it suffices to show that $\mathcal{A}_{c_k(\lambda), 3\sigma}$ contains a Palais-Smale sequence
 5 of I_λ for every $\sigma \in (0, 1)$. For this purpose, we argue by contradiction and suppose that there is no
 6 Palais-Smale sequence in $\mathcal{A}_{c_k(\lambda), 3\sigma}$ for some $\sigma \in (0, 1)$. Since $\mathcal{A}_{c_k(\lambda), 3\sigma}$ shrinks as σ decreases, without
 7 loss of generality we may assume that

$$4\sigma < \alpha_0 = \inf_{\|u\|=\rho_0} I_{1-\bar{\varepsilon}_1}(u) \leq c_k(\lambda). \quad (72)$$

8 Since $\mathcal{A}_{c_k(\lambda), 3\sigma}$ does not contain any Palais-Smale sequence of I_λ , we may find $\delta_0 \in (0, \sigma)$ such that for
 9 each $u \in \mathcal{A}_{c_k(\lambda), 3\sigma}$ there exists $v = v(u) \in X \setminus \{u\}$ such that

$$\lambda(\Psi(v) - \Psi(u)) - \Phi'(u)(v - u) < -5\delta_0\|v - u\|.$$

10 To derive a contradiction, we shall find a suitable relatively compact set $K \subset \mathcal{A}_{c_k(\lambda), 3\sigma}$ and exploit
 11 Lemmas 3.1 and 3.5. Since $c'_k(\lambda)$ exists, there exists $h_1 > 0$ such that

$$\begin{aligned} c_k(\lambda + h) &< c_k(\lambda) + (c'_k(\lambda) + 1)h = c_k(\lambda) + (M_0 - 2)h \\ &< c_k(\lambda) + M_0h \quad \text{for any } h \in (0, h_1). \end{aligned} \quad (73)$$

12 In addition, we may also suppose

$$5M_0h_1 < \delta_0 < \sigma. \quad (74)$$

13 For $h \in (0, h_1)$, consider the following subclass Γ_k^h of Γ_k

$$\Gamma_k^h \equiv \left\{ \gamma \in \Gamma_k \mid \sup_{\zeta \in \mathbb{D}^k} I_{\lambda+h}(\gamma(\zeta)) \leq c_k(\lambda) + M_0h \right\}.$$

14 Notice that (73) implies $\Gamma_k^h \neq \emptyset$ for each $h \in (0, h_1)$. For each $h \in (0, h_1)$ and $\gamma \in \Gamma_k^h$, if $\zeta \in \mathbb{D}^k$ satisfies

$$I_\lambda(\gamma(\zeta)) \geq c_k(\lambda) - 5M_0h,$$

15 then from $\gamma \in \Gamma_k^h$ we deduce

$$\Psi(\gamma(\zeta)) = \frac{I_{\lambda+h}(\gamma(\zeta)) - I_\lambda(\gamma(\zeta))}{h} \leq \frac{c_k(\lambda) + M_0h - (c_k(\lambda) - 5M_0h)}{h} = 6M_0.$$

16 By the choice of R_0 in (70), this means that for any $h \in (0, h_1)$ and $\gamma \in \Gamma_k^h$, the set

$$K_\gamma \equiv \{\gamma(\zeta) \mid I_\lambda(\gamma(\zeta)) \geq c_k(\lambda) - 5M_0h\}$$

17 satisfies

$$K_\gamma \subset B(0, R_0) \subset A.$$

18 Furthermore, since $K_\gamma \subset \gamma(\mathbb{D}^k)$, K_γ is relatively compact in X and (74) yields $K_\gamma \subset \mathcal{A}_{c_k(\lambda), 3\sigma}$. Moreover,
 19 since we are considering the case $k \geq 1$ (thus, we are assuming (E) and (SMP)), K_γ is symmetric.

20 Next, we choose any $h \in (0, h_1)$ and $\gamma_1 \in \Gamma_k^h$, and write

$$C_1 \equiv \gamma_1(\mathbb{D}^k), \quad K_1 \equiv K_{\gamma_1} = \{\gamma_1(\zeta) \mid I_\lambda(\gamma_1(\zeta)) \geq c_k(\lambda) - 5M_0h\} \subset B(0, R_0) \cap \mathcal{A}_{c_k(\lambda), \sigma}.$$

21 By Lemma 3.5, there exist $\tilde{\tau} > 0$, $\eta \in C([0, \tilde{\tau}] \times X, X)$ and open (symmetric) sets W, \widetilde{W} in X such that

- 1 (a) $\overline{K_1} \subset W \subset \overline{W} \subset \widetilde{W}$;
2 (b) $\|\eta(s, w) - w\| \leq s$, $\eta(s, -w) = -\eta(s, w)$ for every $(s, w) \in [0, \tilde{\tau}] \times X$ and $\eta(s, w) = w$ for all $s \in [0, \tilde{\tau}]$
3 if $w \notin \widetilde{W}$;
4 (c) $I_\lambda(\eta(s, w)) \leq I_\lambda(w) + \delta_0 s$ for each $(s, w) \in [0, \tilde{\tau}] \times X$;
5 (d) $I_\lambda(\eta(s, w)) \leq I_\lambda(w) - 2\delta_0 s$ for any $(s, w) \in [0, \tilde{\tau}] \times W$ with $I_\lambda(w) \geq c_k(\lambda) - \sigma$.

6 We briefly explain the issues to overcome and the strategy. First, flowing γ_1 by η may not lead to a
7 map in Γ_k , as η may not preserve the boundary condition on $\partial\mathbb{D}^k$. For this reason, we have to modify the
8 map $\tilde{\gamma}_2 \equiv \eta(\tau_1, \gamma_1)$ in a neighborhood of $\partial\mathbb{D}^k$ to get an element $\gamma_2 \in \Gamma_k$ (perhaps, not in Γ_k^h) satisfying

$$\sup_{\zeta \in \mathbb{D}^k} I_\lambda(\gamma_2(\zeta)) \leq c_k(\lambda) + M_0 h - 2\delta_0 \tau_1 \quad \text{for some } \tau_1 \geq \tilde{\tau} > 0.$$

9 A second issue is that the time interval τ_1 might be too small to obtain the following contradiction:

$$\sup_{\zeta \in \mathbb{D}^k} I_\lambda(\gamma_2(\zeta)) < c_k(\lambda).$$

10 To remedy, we first choose τ_1 close to the maximal time for which a deformation with properties (a)–(d)
11 exists (for fixed K_1, δ_0). Next, we proceed inductively to construct Cauchy sequences of maps $\{\gamma_j\} \subset \Gamma_k$
12 and of almost maximal times $\{\tau_j\} \subset (0, \infty)$ with the property

$$\sup_{\zeta \in \mathbb{D}^k} I_\lambda(\gamma_j(\zeta)) \leq c_k(\lambda) + M_0 h - 2\delta_0 \sum_{i=1}^j \tau_i.$$

13 We shall prove that the relatively compact sets

$$K_j \equiv \left\{ \gamma_j(\zeta) \mid I_\lambda(\gamma_j(\zeta)) \geq c(\lambda) - 5M_0 h + \delta_0 \sum_{i=1}^{j-1} \tau_i \right\}$$

14 of points at which $I_\lambda \circ \gamma_j$ takes higher values are all contained in $\mathcal{A}_{c_k(\lambda), 3\sigma}$, and that their union $K_\infty \equiv$
15 $\bigcup_{j=1}^\infty K_j \subset \mathcal{A}_{c_k(\lambda), 3\sigma}$ is relatively compact, too. By choosing a deformation η_∞ in Lemma 3.5 with
16 $K = K_\infty$ we are then able to reach our desired contradiction. Here is the point where τ_j being “almost
17 maximal” plays a role.

18 **Step 1:** *construction of γ_2 , and its properties.*

19 For the given relatively compact symmetric set K_1 , we may consider

$$\mathcal{D}_1 \equiv \left\{ (\eta, \tau, W, \widetilde{W}) \mid \eta, W, \widetilde{W} \text{ satisfy (a)–(d) for } s \in [0, \tau] \right\}.$$

20 We define

$$\tilde{T}_1 \equiv \sup_{(\eta, \tau, W, \widetilde{W}) \in \mathcal{D}_1} \tau \in (0, \infty] \quad \text{and} \quad T_1 \equiv \min \left\{ \frac{2M_0 h}{\delta_0}, \tilde{T}_1 \right\}.$$

21 Remark that $\mathcal{D}_1 \neq \emptyset$, and that (74) with $h \in (0, h_1)$ gives $T_1 \leq 2M_0 h / \delta_0 \leq 2/5$. We choose $(\eta_1, \tau_1, W_1, \widetilde{W}_1) \in$
22 \mathcal{D}_1 so that

$$\frac{3}{4} T_1 < \tau_1 < T_1$$

23 and define $\tilde{\gamma}_2 \in C(\mathbb{D}^k, X)$ by

$$\tilde{\gamma}_2(\zeta) := \eta_1(\tau_1, \gamma_1(\zeta)).$$

1 Since $\tilde{\gamma}_2$ may not coincide with $\pi_{0,k}$ on $\partial\mathbb{D}^k$, in order to obtain a map in Γ_k we define, for $\theta \in (0, 1/4)$,
 2 the dilation

$$L_\theta : \mathbb{D}^k(1-\theta) \rightarrow \mathbb{D}^k, \quad L_\theta(\zeta) \equiv \frac{\zeta}{1-\theta}.$$

3 Notice that

$$\max_{\zeta \in \mathbb{D}^k(1-\theta)} |L_\theta(\zeta) - \zeta| = \frac{\theta}{1-\theta}.$$

4 When $k = 0$, we consider $L_\theta : [\theta, 1-\theta] \rightarrow [0, 1]$. By the uniform continuity of γ_1 , we may find $\theta_1 \in (0, 1/4)$
 5 such that

$$\max_{|\zeta| \leq 1-\theta_1} \|\gamma_1(\zeta) - \gamma_1(L_{\theta_1}(\zeta))\| + \max_{1-\theta_1 \leq |\zeta| \leq 1} \|\gamma_1(\zeta) - \gamma_1(\zeta/|\zeta|)\| \leq \frac{\tau_1}{2}. \quad (75)$$

6 Then we define $\gamma_2(\zeta)$ by

$$\gamma_2(\zeta) \equiv \begin{cases} \eta_1(\tau_1, \gamma_1(L_{\theta_1}(\zeta))) & \text{if } |\zeta| \leq 1-\theta_1, \\ \eta_1\left(\tau_1 \frac{1-|\zeta|}{\theta_1}, \gamma_1\left(\frac{\zeta}{|\zeta|}\right)\right) & \text{if } 1-\theta_1 \leq |\zeta| \leq 1. \end{cases}$$

7 It is easily seen that $\gamma_2 \in \Gamma_k$ and

$$\|\gamma_2(\zeta) - \gamma_1(\zeta)\| \leq \begin{cases} \|\eta_1(\tau_1, \gamma_1(L_{\theta_1}(\zeta))) - \gamma_1(L_{\theta_1}(\zeta))\| + \|\gamma_1(L_{\theta_1}(\zeta)) - \gamma_1(\zeta)\| & \text{if } |\zeta| \leq 1-\theta_1, \\ \left\| \eta_1\left(\tau_1 \frac{1-|\zeta|}{\theta_1}, \gamma_1\left(\frac{\zeta}{|\zeta|}\right)\right) - \gamma_1\left(\frac{\zeta}{|\zeta|}\right) \right\| + \left\| \gamma_1\left(\frac{\zeta}{|\zeta|}\right) - \gamma_1(\zeta) \right\| & \text{if } 1-\theta_1 \leq |\zeta| \leq 1. \end{cases}$$

8 Therefore, property (b) and (75) yield

$$\max_{\zeta \in \mathbb{D}^k} \|\gamma_1(\zeta) - \gamma_2(\zeta)\| \leq \frac{3}{2}\tau_1. \quad (76)$$

9 It follows from property (c), (SMP), (72) and (74) that if $\zeta \in \mathbb{D}^k \setminus \mathbb{D}^k(1-\theta_1)$, then

$$I_\lambda(\gamma_2(\zeta)) \leq \delta_0\tau_1 < \delta_0 < \sigma < c_k(\lambda) - 3\sigma < c_k(\lambda) - 5M_0h. \quad (77)$$

10 Similarly, when $\zeta \in \mathbb{D}^k(1-\theta_1) \setminus (\gamma_1 \circ L_{\theta_1})^{-1}(K_1)$, we have $I_\lambda(\gamma_1(L_{\theta_1}(\zeta))) < c_k(\lambda) - 5M_0h$ by the definition
 11 of K_1 , and

$$I_\lambda(\gamma_2(\zeta)) \leq I_\lambda(\gamma_1(L_{\theta_1}(\zeta))) + \delta_0\tau_1 < c_k(\lambda) + \delta_0\tau_1 - 5M_0h. \quad (78)$$

12 On the other hand, if $\zeta \in (\gamma_1 \circ L_{\theta_1})^{-1}(K_1)$, then $\gamma_1(L_{\theta_1}(\zeta)) \in K_1$, and property (d) with the fact $\gamma_1 \in \Gamma_k^h$
 13 implies

$$I_\lambda(\gamma_2(\zeta)) \leq I_\lambda(\gamma_1(L_{\theta_1}(\zeta))) - 2\delta_0\tau_1 \leq c_k(\lambda) + M_0h - 2\delta_0\tau_1.$$

14 By the properties $3T_1/4 < \tau_1 < T_1 \leq 2M_0h/\delta_0$, we see that

$$\delta_0\tau_1 - 5M_0h \leq M_0h - 2\delta_0\tau_1$$

15 and

$$c_k(\lambda) \leq \sup_{\zeta \in \mathbb{D}^k} I_\lambda(\gamma_2(\zeta)) \leq c_k(\lambda) + M_0h - 2\delta_0\tau_1. \quad (79)$$

16 **Step 2:** construction of the sequence $\{\gamma_j\}_{j=1}^\infty$.

1 Next, put

$$C_2 \equiv \gamma_2(\mathbb{D}^k), \quad K_2 \equiv \{\gamma_2(\zeta) \mid I_\lambda(\gamma_2(\zeta)) \geq c_k(\lambda) + \delta_0\tau_1 - 5M_0h\}.$$

2 By (76),

$$\text{dist}_H(C_1, C_2) \leq \frac{3}{2}\tau_1,$$

3 where dist_H stands for the Hausdorff distance between C_1 and C_2 defined by

$$\text{dist}_H(A, B) \equiv \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(A, y) \right\}.$$

4 Moreover, from (77) and (78) we infer that

$$\gamma_2^{-1}(K_2) \subset (\gamma_1 \circ L_{\theta_1})^{-1}(K_1) \subset \mathbb{D}^k(1 - \theta_1).$$

5 Hence, if $\zeta \in \gamma_2^{-1}(K_2)$, then property (b) yields

$$\|\gamma_2(\zeta) - \gamma_1(L_{\theta_1}(\zeta))\| = \|\eta_1(\tau_1, \gamma_1(L_{\theta_1}(\zeta))) - \gamma_1(L_{\theta_1}(\zeta))\| \leq \tau_1,$$

6 and the fact $\gamma_1(L_{\theta_1}(\zeta)) \in K_1 \subset B(0, R_0)$ gives

$$K_2 \subset B(0, R_0 + \tau_1).$$

7 Combining (79) with $\tau_1 < T_1 \leq 2/5$, we observe that $K_2 \subset \mathcal{A}_{c_k(\lambda), 3\sigma}$ is relatively compact. Thus, Lemma
8 3.5 ensures the existence of a deformation $(\eta, \tilde{\tau}, W_2, \tilde{W}_2)$ satisfying (a)–(d) with K_2 replacing K_1 . Then,
9 we consider

$$\mathcal{D}_2 \equiv \left\{ (\eta, \tau, W, \tilde{W}) \mid (\eta, \tau, W, \tilde{W}) \text{ satisfies (a)–(d) for } s \in [0, \tau] \text{ and } K_2 \text{ instead of } K_1 \right\}$$

10 and define

$$\tilde{T}_2 \equiv \sup_{(\eta, \tau, W, \tilde{W}) \in \mathcal{D}_2} \tau, \quad T_2 \equiv \min \left\{ \frac{2M_0h}{\delta_0} - \tau_1, \tilde{T}_2 \right\}.$$

11 Select $(\eta_2, \tau_2, W_2, \tilde{W}_2) \in \mathcal{D}_2$ and $\theta_2 \in (0, 1/4)$ so that

$$\frac{3}{4}T_2 < \tau_2 < T_2$$

12 and

$$\max_{|\zeta| \leq 1 - \theta_2} \|\gamma_2(\zeta) - \gamma_2(L_{\theta_2}(\zeta))\| + \max_{1 - \theta_2 \leq |\zeta| \leq 1} \|\gamma_2(\zeta) - \gamma_2(\zeta/|\zeta|)\| \leq \frac{\tau_2}{2}.$$

13 Define $\gamma_3 \in \Gamma_k$ by

$$\gamma_3(\zeta) \equiv \begin{cases} \eta_2(\tau_2, \gamma_2(L_{\theta_2}(\zeta))) & \text{if } |\zeta| \leq 1 - \theta_2, \\ \eta_2\left(\tau_2 \frac{1 - |\zeta|}{\theta_2}, \gamma_2\left(\frac{\zeta}{|\zeta|}\right)\right) & \text{if } 1 - \theta_2 \leq |\zeta| \leq 1. \end{cases}$$

14 Arguing as before, we may prove

$$\max_{\zeta \in \mathbb{D}^k} \|\gamma_2(\zeta) - \gamma_3(\zeta)\| \leq \frac{3}{2}\tau_2,$$

15 and

$$I_\lambda(\gamma_3(\zeta)) \begin{cases} \leq c_k(\lambda) + M_0h - 2\delta_0(\tau_1 + \tau_2) & \text{if } \zeta \in (\gamma_2 \circ L_{\theta_2})^{-1}(K_2), \\ < c_k(\lambda) + \delta_0(\tau_1 + \tau_2) - 5M_0h & \text{otherwise.} \end{cases}$$

1 Since $\tau_1 + \tau_2 < 2M_0h/\delta_0$, we see that

$$\begin{aligned} \delta_0(\tau_1 + \tau_2) - 5M_0h &< M_0h - 2\delta_0(\tau_1 + \tau_2), \\ c_k(\lambda) &\leq \sup_{\zeta \in \mathbb{D}^k} I_\lambda(\gamma_3(\zeta)) \leq c_k(\lambda) + M_0h - 2\delta_0(\tau_1 + \tau_2). \end{aligned}$$

2 By putting

$$C_3 \equiv \gamma_3(\mathbb{D}^k), \quad K_3 \equiv \{\gamma_3(\zeta) \mid I_\lambda(\gamma_3(\zeta)) \geq c_k(\lambda) + \delta_0(\tau_1 + \tau_2) - 5M_0h\},$$

3 we see from the same argument as before that

$$\begin{aligned} \text{dist}_H(C_2, C_3) &\leq \frac{3}{2}\tau_2, \quad \gamma_3^{-1}(K_3) \subset (\gamma_2 \circ L_{\theta_2})^{-1}(K_2) \subset \mathbb{D}^k(1 - \theta_2), \\ K_3 &\subset B(0, R_0 + \tau_1 + \tau_2). \end{aligned}$$

4 Thus, $K_3 \subset \mathcal{A}_{c_k(\lambda), 3\sigma}$ is relatively compact. We may therefore repeat the same procedure inductively to
5 obtain $\{K_j\}_{j=1}^\infty$, $\{\tau_j\}_{j=1}^\infty \subset (0, \infty)$, $(\eta_j, \tau_j, W_j, \widetilde{W}_j) \in \mathcal{D}_j$, $\{\theta_j\}_{j=1}^\infty \subset (0, 1/4)$, $\{L_{\theta_j}\}_{j=1}^\infty$, $\{\gamma_j\}_{j=1}^\infty \subset \Gamma_k$ such
6 that

$$\begin{aligned} C_j &\equiv \gamma_j(\mathbb{D}^k), \quad \text{dist}_H(C_{j-1}, C_j) \leq \frac{3\tau_{j-1}}{2}, \\ K_j &\equiv \left\{ \gamma_j(\zeta) \mid I_\lambda(\gamma_j(\zeta)) \geq c_k(\lambda) - 5M_0h + \delta_0 \sum_{i=1}^{j-1} \tau_i \right\} \subset B\left(0, R_0 + \sum_{i=1}^{j-1} \tau_i\right), \\ \widetilde{T}_j &\equiv \sup_{(\eta, \tau, W, \widetilde{W}) \in \mathcal{D}_j} \tau, \quad T_j \equiv \min \left\{ \frac{2M_0h}{\delta_0} - \sum_{i=1}^{j-1} \tau_i, \widetilde{T}_j \right\}, \quad \frac{3}{4}T_j < \tau_j < T_j, \\ \sup_{\zeta \in \mathbb{D}^k} I_\lambda(\gamma_{j+1}(\zeta)) &\leq c_k(\lambda) + M_0h - 2\delta_0 \sum_{i=1}^j \tau_i, \quad \sum_{i=1}^j \tau_i \leq \frac{M_0h}{2\delta_0} < \frac{2}{5}, \end{aligned} \tag{80}$$

7 where

$$\gamma_{j+1}(\zeta) \equiv \begin{cases} \eta_j(\tau_j, \gamma_j(L_{\theta_j}(\zeta))) & \text{if } |\zeta| \leq 1 - \theta_j, \\ \eta_j\left(\tau_j \frac{1 - |\zeta|}{\theta_j}, \gamma_j\left(\frac{\zeta}{|\zeta|}\right)\right) & \text{if } 1 - \theta_j \leq |\zeta| \leq 1, \end{cases} \quad \max_{\zeta \in \mathbb{D}^k} \|\gamma_j(\zeta) - \gamma_{j+1}(\zeta)\| \leq \frac{3\tau_j}{2}.$$

8 **Step 3:** a compactness argument, and conclusion.

9 Consider

$$\mathcal{S} \equiv \{K \subset X \mid K \neq \emptyset, K \text{ is compact}\}.$$

10 It is known that $(\mathcal{S}, \text{dist}_H)$ is a complete metric space (we refer e.g. to [36, Theorem 10.1.6]). Since
11 $\sum_{i=1}^\infty \tau_i < \infty$ due to (80), we have

$$\text{dist}_H(C_j, C_{j+\ell}) \leq \sum_{i=1}^{\ell} \text{dist}_H(C_{j+i-1}, C_{j+i}) \leq \sum_{i=1}^{\ell} \frac{3}{2}\tau_{j+i-1};$$

12 thus $\{C_j\}_{j=1}^\infty$ is a Cauchy sequence in $(\mathcal{S}, \text{dist}_H)$. Hence there exists $C_\infty \in \mathcal{S}$ such that

$$\text{dist}_H(C_j, C_\infty) \rightarrow 0 \text{ as } j \rightarrow \infty.$$

13 Consider

$$K_\infty \equiv \bigcup_{j=1}^{\infty} K_j.$$

1 We next claim that K_∞ is relatively compact. Indeed, let $\{u_p\}_{p=1}^\infty \subset K_\infty$ be a sequence and choose
2 $\{j_p\}_{p=1}^\infty$ satisfying $u_p \in K_{j_p}$. Without loss of generality, we may assume $j_p \rightarrow \infty$ as $p \rightarrow \infty$. By recalling
3 that $K_j \subset C_j$, $\text{dist}_H(C_j, C_\infty) \rightarrow 0$ and C_∞ is compact, $\lim_{p \rightarrow \infty} \|u_p - w_p\| = 0$ for some $\{w_p\}_{p=1}^\infty \subset C_\infty$.
4 From the compactness of C_∞ , up to a subsequence $w_p \rightarrow w_\infty \in C_\infty$ as $p \rightarrow \infty$, whence $u_p \rightarrow w_\infty$ as
5 $p \rightarrow \infty$. This proves that K_∞ is relatively compact.

6 Since $K_j \subset \mathcal{A}_{c_k(\lambda), 3\sigma}$ for every $j \geq 1$, $K_\infty \subset \mathcal{A}_{c_k(\lambda), 3\sigma}$ holds and Lemma 3.5 can be applied to obtain
7 $\tau_\infty > 0$, $\eta_\infty \in C([0, \tau_\infty] \times X, X)$ and open sets $W_\infty, \widetilde{W}_\infty$ such that $\overline{K_\infty} \subset W_\infty \subset \overline{W_\infty} \subset \widetilde{W}_\infty$ and (a)–(d)
8 hold with K_∞ instead of K_1 . Since $\sum_{i=1}^\infty \tau_i < \infty$, there are two alternatives:

$$9 \quad (\text{A}) \quad \sum_{i=1}^\infty \tau_i = \frac{2M_0h}{\delta_0} < \frac{2}{5};$$

$$10 \quad (\text{B}) \quad \sum_{i=1}^\infty \tau_i < \frac{2M_0h}{\delta_0}.$$

11 In case (A), choose a large j such that $M_0h/\delta_0 < \sum_{i=1}^j \tau_i$. Then

$$c_k(\lambda) \leq \sup_{\zeta \in \mathbb{D}^k} I_\lambda(\gamma_{j+1}(\zeta)) \leq c_k(\lambda) + M_0h - 2\delta_0 \sum_{i=1}^j \tau_i < c_k(\lambda) - M_0h,$$

12 which is a contradiction. In case (B), it follows from the definition of \widetilde{T}_j and the fact $K_j \subset K_\infty$ that
13 $\widetilde{T}_j \geq \tau_\infty$. Since $\sum_{i=1}^\infty \tau_i < \frac{2M_0h}{\delta_0}$, it follows that

$$T_j = \min \left\{ \frac{2M_0h}{\delta_0} - \sum_{i=1}^{j-1} \tau_i, \widetilde{T}_j \right\} \geq \min \left\{ \frac{2M_0h}{\delta_0} - \sum_{i=1}^\infty \tau_i, \tau_\infty \right\} > 0.$$

14 Since $\tau_j \geq \frac{3}{4}T_j$, this contradicts the fact that $\sum_{i=1}^\infty \tau_i \leq \frac{2}{5} < \infty$. Thus, in either case, we get a contradiction
15 and $\mathcal{A}_{c_k(\lambda), 3\sigma}$ contains a Palais-Smale sequence. This completes the proof of Theorem 4.4. \blacksquare

16 We are ready to prove Theorem 1.6 and part of Theorem 1.7, as a direct consequence of the next

17 **Theorem 4.5.** *Assume that (Ψ_1) , (Ψ_2) , (Ψ_3) , (Φ_1) , (Φ_2) and (IB) hold. Then, $c_0(1)$ is a critical value
18 of $I = I_1$ if (MP) holds, and $\{c_k(1)\}_{k=1}^\infty$ are critical values of I if (E) and (SMP) hold.*

19 **Proof.** We only prove that $\{c_k(1)\}_{k=1}^\infty$ are critical values of I , the other case being analogous. For $k \geq 1$,
20 $c_k(\lambda)$ is nondecreasing with respect to $\lambda \in (1 - \bar{\varepsilon}, 1 + \bar{\varepsilon}]$. Thus, there exists a sequence $\{\lambda_l\}_l \subset (1, 1 + \bar{\varepsilon})$
21 such that $\lim_{l \rightarrow \infty} \lambda_l = 1$ and $c'_k(\lambda_l)$ exists for each $l = 1, 2, \dots$. By Theorem 4.4, there exists a bounded
22 Palais-Smale sequence $\{u_{l,j}\}_j$ for I_{λ_l} with $\lim_{j \rightarrow \infty} I_{\lambda_l}(u_{l,j}) = c_k(\lambda_l)$. Then, from Proposition 4.1, up to
23 a subsequence $u_{l,j}$ converges strongly to some u_l in X as $j \rightarrow \infty$, and u_l is a critical point of I_{λ_l} with
24 $I_{\lambda_l}(u_l) = c_k(\lambda_l)$. By Proposition 4.3 and $\lambda_l > 1$, we have $c_k(\lambda_l) \rightarrow c_k(1)$ as $l \rightarrow \infty$. Furthermore, by
25 using (IB), $\{u_l\} \subset \mathcal{K}_{[1, 1+\bar{\varepsilon}]}^{c_k(1+\bar{\varepsilon})+1}$ is bounded in X . Therefore, applying again Proposition 4.1, we conclude
26 that $u_l \rightarrow u$ strongly in X , for some u which is critical for I and satisfies $I(u) = c_k(1)$. This completes
27 the proof. \blacksquare

28 The proof of Theorem 1.7 will be completed if we prove the following divergence property of $c_k = c_k(1)$.

29 **Theorem 4.6.** *Assume that (Ψ_1) , (Ψ_2) , (Ψ_3) , (Φ_1) , (Φ_2) , (IB), (E) and (SMP) hold. Then,*

$$\lim_{k \rightarrow \infty} c_k(1) = \infty.$$

1 **Proof.** To prove Theorem 4.6, we combine the arguments in [54, Chapter 9] (cf. [33]) and [63] with the
 2 iteration scheme as in the proof of Theorem 4.4. To this end, we introduce

$$\begin{aligned}\mathcal{E}_{\mathbb{R}^n} &\equiv \{E \subset \mathbb{R}^n \mid E \text{ is closed, symmetric and } 0 \notin E\}, \\ \mathcal{E} &\equiv \{F \subset X \mid F \text{ is closed, symmetric and } 0 \notin F\}, \\ \mathcal{G}(E) &\equiv \text{Krasnoselski's genus of } E.\end{aligned}$$

3 For $k \geq 1$ and $\lambda \in [1 - \bar{\varepsilon}_1, 1]$, we set

$$\begin{aligned}\Lambda_k &\equiv \left\{ \gamma \left(\overline{\mathbb{D}^n \setminus \Sigma} \right) \mid n \geq k, \gamma \in \Gamma_n, \Sigma \in \mathcal{E}_{\mathbb{R}^n}, \Sigma \cap \partial \mathbb{D}^n = \emptyset, \mathcal{G}(\Sigma) \leq n - k \right\}, \\ d_k(\lambda) &\equiv \inf_{B \in \Lambda_k} \sup_{u \in B} I_\lambda.\end{aligned}$$

4 By $\mathcal{G}(\emptyset) = 0$, $\gamma(\mathbb{D}^k) \in \Lambda_k$ holds for all $\gamma \in \Gamma_k$. Moreover, since $\Lambda_{k+1} \subset \Lambda_k$ and $\Psi \geq 0$, for every $k \geq 1$
 5 and $\lambda, \lambda_1, \lambda_2 \in [1 - \bar{\varepsilon}_1, 1]$,

$$d_k(\lambda) \leq c_k(\lambda), \quad d_k(\lambda) \leq d_{k+1}(\lambda) \quad \text{and} \quad d_k(\lambda_1) \leq d_k(\lambda_2) \quad \text{if } \lambda_1 \leq \lambda_2.$$

6 Therefore, it is enough to prove

$$d_k(1) \rightarrow \infty \quad \text{as } k \rightarrow \infty. \quad (81)$$

7 Following [54] and [63] (see also [33]), we prove the next result.

8 **Lemma 4.7.** *The following properties hold:*

- 9 (i) *if $C \in \Lambda_{k+s}$ and $Z \in \mathcal{E}$ satisfies $\mathcal{G}(Z) \leq s$ and $I_1|_Z > 0$, then $\overline{C \setminus Z} \in \Lambda_k$;*
 10 (ii) *for every $k \in \mathbb{N}$ and $C \in \Lambda_k$, it holds $C \cap \partial B(0, \rho_0) \neq \emptyset$, hence*

$$0 < \alpha_0 = \inf_{\|u\|=\rho_0} I_{1-\bar{\varepsilon}_1}(u) \leq d_k(1 - \bar{\varepsilon}_1) \leq d_k(\lambda) \quad \text{for each } \lambda \in [1 - \bar{\varepsilon}_1, 1].$$

11 **Proof. (i)** Write $C = \gamma(\overline{\mathbb{D}^n \setminus \Sigma}) \in \Lambda_{k+s}$ where $n \geq k + s$, $\gamma \in \Gamma_n$, $\Sigma \cap \partial \mathbb{D}^n = \emptyset$, $\Sigma \in \mathcal{E}_{\mathbb{R}^n}$ and
 12 $\mathcal{G}(\Sigma) \leq n - (k + s)$. Let $Z \in \mathcal{E}$ satisfy $\mathcal{G}(Z) \leq s$ and $I_1|_Z > 0$. Remark that $\gamma^{-1}(Z) \in \mathcal{E}_{\mathbb{R}^n}$. We claim
 13 that

$$\overline{C \setminus Z} = \gamma \left(\overline{\mathbb{D}^n \setminus (\Sigma \cup \gamma^{-1}(Z))} \right). \quad (82)$$

14 Notice that

$$\begin{aligned}\overline{C \setminus Z} &\supset C \setminus Z \supset \gamma(\overline{\mathbb{D}^n \setminus \Sigma}) \setminus Z \\ &= \gamma \left((\overline{\mathbb{D}^n \setminus \Sigma}) \setminus \gamma^{-1}(Z) \right) = \gamma \left(\overline{\mathbb{D}^n \setminus (\Sigma \cup \gamma^{-1}(Z))} \right).\end{aligned}$$

15 Hence, it holds that $\overline{C \setminus Z} \supset \gamma(\overline{\mathbb{D}^n \setminus (\Sigma \cup \gamma^{-1}(Z))})$. On the other hand, since $\gamma^{-1}(Z)$ is closed, we have

$$\overline{\mathbb{D}^n \setminus \Sigma} \setminus \gamma^{-1}(Z) \subset \overline{\mathbb{D}^n \setminus (\Sigma \cup \gamma^{-1}(Z))},$$

16 which implies

$$C \setminus Z = \gamma \left(\overline{\mathbb{D}^n \setminus \Sigma} \setminus \gamma^{-1}(Z) \right) \subset \gamma \left(\overline{\mathbb{D}^n \setminus (\Sigma \cup \gamma^{-1}(Z))} \right).$$

17 From the compactness of $\overline{\mathbb{D}^n \setminus (\Sigma \cup \gamma^{-1}(Z))}$, it follows that $\overline{C \setminus Z} \subset \gamma(\overline{\mathbb{D}^n \setminus (\Sigma \cup \gamma^{-1}(Z))})$. Thus, (82)
 18 holds.

19 The fact $I_1|_Z > 0 > I_1|_{\pi_{0,n}(\partial \mathbb{D}^n)}$ gives $\partial \mathbb{D}^n \cap (\Sigma \cup \gamma^{-1}(Z)) = \emptyset$. By (82) and the estimate

$$\mathcal{G}(\Sigma \cup \gamma^{-1}(Z)) \leq \mathcal{G}(\Sigma) + \mathcal{G}(\gamma^{-1}(Z)) \leq \mathcal{G}(\Sigma) + \mathcal{G}(Z) \leq n - k,$$

1 we conclude $\overline{C \setminus \bar{Z}} \in \Lambda_k$.

2 (ii) Write $C = \gamma(\mathbb{D}^n \setminus \Sigma) \in \Lambda_k$ where $n \geq k$, $\gamma \in \Gamma_n$, $\Sigma \cap \partial\mathbb{D}^n = \emptyset$ and $\mathcal{G}(\Sigma) \leq n - k$. Set

$$\widehat{U} \equiv \{\zeta \in \text{Int } \mathbb{D}^n \mid \gamma(\zeta) \in B(0, \rho_0)\}.$$

3 It is easily seen that $0 \in \widehat{U}$, $-\widehat{U} = \widehat{U}$ and $\widehat{U} \subset \mathbb{R}^n$ is open. Denote by U the connected component of \widehat{U}
 4 containing 0. Since U is a symmetric neighborhood of 0, $\mathcal{G}(\partial U) = n$ holds. Since $\|\gamma(\zeta)\| = \|\pi_{0,n}(\zeta)\| > \rho_0$
 5 for each $\zeta \in \partial\mathbb{D}^n$, we see that

$$\gamma(\partial U) \subset \partial B(0, \rho_0). \quad (83)$$

6 Define $W \equiv \{\zeta \in \mathbb{D}^n \mid \gamma(\zeta) \in \partial B(0, \rho_0)\} \in \mathcal{E}_{\mathbb{R}^n}$. Thanks to (83), we see that $\partial U \subset W$ and $n =$
 7 $\mathcal{G}(\partial U) \leq \mathcal{G}(W)$. On the other hand, letting ε small enough so that $\mathbb{D}^n(\varepsilon) \cap W = \emptyset$, by the properties of
 8 Krasnoselki's genus, $\mathcal{G}(W) \leq \mathcal{G}(\mathbb{D}^n \setminus \mathbb{D}^n(\varepsilon)) = \mathcal{G}(\partial\mathbb{D}^n) = n$. Summarizing,

$$\mathcal{G}(W) = n, \quad \mathcal{G}\left(\gamma\left(\overline{W \setminus \Sigma}\right)\right) \geq \mathcal{G}\left(\overline{W \setminus \Sigma}\right) \geq \mathcal{G}(W) - \mathcal{G}(\Sigma) \geq k > 0.$$

9 In particular, $\gamma(\overline{W \setminus \Sigma}) \neq \emptyset$. Since $\gamma(\overline{W \setminus \Sigma}) \subset C \cap \partial B(0, \rho_0)$, we have $\emptyset \neq C \cap \partial B(0, \rho_0)$. ■

10 To prove (81), we argue by contradiction and suppose that $\{d_k(1)\}_k$ is bounded. From the mono-
 11 tonicity of $d_k(\lambda)$ in k and λ , it follows that

$$d_\infty(\lambda) \equiv \lim_{k \rightarrow \infty} d_k(\lambda) \in [\alpha_0, \infty) \quad \text{for every } \lambda \in [1 - \bar{\varepsilon}_1, 1].$$

12 Notice that $\lambda \mapsto d_\infty(\lambda)$ is nondecreasing on $[1 - \bar{\varepsilon}_1, 1]$.

13 **Claim 1:** *There exists $\tilde{\lambda}_0 \in [1 - \bar{\varepsilon}_1, 1 - \frac{\bar{\varepsilon}_1}{2}]$ such that*

$$d'_\infty(\tilde{\lambda}_0) \text{ exists and } d'_\infty(\tilde{\lambda}_0) \leq 3\bar{\varepsilon}_1^{-1}\nu_0, \quad \text{where } \nu_0 \equiv d_\infty(1) - d_\infty(1 - \bar{\varepsilon}_1) \in [0, \infty).$$

14 **Proof.** Since d_∞ is nondecreasing, we obtain

$$\int_{1 - \bar{\varepsilon}_1}^1 d'_\infty(\lambda) \, d\lambda \leq d_\infty(1) - d_\infty(1 - \bar{\varepsilon}_1) = \nu_0.$$

15 Write

$$J_0 \equiv \{\lambda \in [1 - \bar{\varepsilon}_1, 1] \mid d'_\infty(\lambda) \text{ exists and } d'_\infty(\lambda) \leq 3\bar{\varepsilon}_1^{-1}\nu_0\}.$$

16 When $\nu_0 > 0$, since $0 \leq d'_\infty(\lambda)$ a.e. in $[1 - \bar{\varepsilon}_1, 1]$ and

$$\nu_0 \geq \int_{[1 - \bar{\varepsilon}_1, 1] \setminus J_0} d'_\infty(\lambda) \, d\lambda \geq 3\bar{\varepsilon}_1^{-1}\nu_0 |[1 - \bar{\varepsilon}_1, 1] \setminus J_0|,$$

17 we obtain the following estimation

$$|J_0| \geq \frac{2}{3}\bar{\varepsilon}_1. \quad (84)$$

18 Since (84) also holds when $\nu_0 = 0$, Claim 1 directly follows. ■

19 Choose $h_0 \in (0, \bar{\varepsilon}_1/2)$ so that for any $h \in [0, h_0]$,

$$d_\infty(\tilde{\lambda}_0 + h) \leq d_\infty(\tilde{\lambda}_0) + \left(d'_\infty(\tilde{\lambda}_0) + 1\right)h \leq d_\infty(\tilde{\lambda}_0) + (3\bar{\varepsilon}_1^{-1}\nu_0 + 1)h. \quad (85)$$

20 From Proposition 4.1 and $I_\lambda(0) = 0$, the set

$$\mathcal{K}_0 \equiv \left\{u \in X \mid u \text{ is a critical point of } I_{\tilde{\lambda}_0}, \frac{\alpha_0}{2} \leq I_{\tilde{\lambda}_0} \leq d_\infty(1) + 1\right\}$$

1 is compact and symmetric with $0 \notin \mathcal{K}_0$, hence $\mathcal{K}_0 \in \mathcal{E}$. Let

$$s_\infty \equiv \mathcal{G}(\mathcal{K}_0) \in \mathbb{N}.$$

2 Since \mathcal{K}_0 is compact and $I_{\tilde{\lambda}_0}$ is lower semicontinuous, we may find $r_\infty \in (0, 1)$ so that

$$\overline{\mathcal{K}_0 + B(0, 3r_\infty)} \in \mathcal{E}, \quad s_\infty = \mathcal{G}\left(\overline{\mathcal{K}_0 + B(0, 3r_\infty)}\right), \quad \frac{\alpha_0}{4} < I_{\tilde{\lambda}_0}|_{\overline{\mathcal{K}_0 + B(0, 3r_\infty)}} \leq I_1|_{\overline{\mathcal{K}_0 + B(0, 3r_\infty)}}. \quad (86)$$

3 According to (Ψ_2) , for each $M > 0$ there exists $R(M) > 0$ so that

$$\{u \in X \mid \Psi(u) \leq M\} \subset B(0, R(M)).$$

4 In what follows, we shall write

$$\begin{aligned} O_{r_\infty} &\equiv \mathcal{K}_0 + B(0, r_\infty), \quad O_{3r_\infty} \equiv \mathcal{K}_0 + B(0, 3r_\infty), \quad M_0 \equiv \bar{\varepsilon}_1^{-1}\nu_0 + 1, \\ A &\equiv \overline{B(0, R(12M_0) + 3r_\infty)} \setminus O_{r_\infty}, \quad \mathcal{A} \equiv A \cap \left\{ \frac{\alpha_0}{2} \leq I_{\tilde{\lambda}_0} \leq d_\infty(1) + 1 \right\}. \end{aligned}$$

5 Notice that $A \subset X$ is bounded, closed and symmetric. Furthermore, from Proposition 4.1 and the
6 definition of O_{r_∞} , there is no Palais-Smale sequence in \mathcal{A} . Therefore, there exists $\delta_0 \in (0, 1)$ such that
7 for every $u \in \mathcal{A}$ we may find $v = v(u) \in X \setminus \{u\}$ such that

$$\tilde{\lambda}_0(\Psi(v) - \Psi(u)) - \Phi'(u)(v - u) < -5\delta_0\|v - u\|. \quad (87)$$

8 We next choose $h_1 > 0$ so that

$$0 < h_1 \leq h_0 < \frac{\bar{\varepsilon}_1}{2}, \quad 12\delta_0^{-1}M_0h_1 < \min\left\{r_\infty, \frac{\alpha_0}{6}, \frac{1}{3}\right\} \equiv \sigma \quad (88)$$

9 and then, we can find $k_\infty \in \mathbb{N}$ such that

$$d_\infty(\tilde{\lambda}_0 + h_1) - d_k(\tilde{\lambda}_0 + h_1) \leq h_1, \quad d_\infty(\tilde{\lambda}_0) - d_k(\tilde{\lambda}_0) \leq h_1 \quad \text{for } k \geq k_\infty. \quad (89)$$

10 **Claim 2:** For every $\tilde{\lambda}_0 \leq \lambda_1 \leq \lambda_2 \leq \tilde{\lambda}_0 + h_1$ and $k_\infty \leq k_1, k_2$,

$$d_{k_2}(\lambda_2) - d_{k_1}(\lambda_1) \leq 3h_1M_0.$$

11 **Proof.** By (85), (88) and (89) with the monotonicity of $d_k(\lambda)$ on k and λ ,

$$\begin{aligned} d_{k_2}(\lambda_2) - d_{k_1}(\lambda_1) &\leq d_{k_2}(\tilde{\lambda}_0 + h_1) - d_{k_1}(\tilde{\lambda}_0) \leq d_\infty(\tilde{\lambda}_0 + h_1) - d_{k_\infty}(\tilde{\lambda}_0) \\ &\leq d_\infty(\tilde{\lambda}_0 + h_1) - d_\infty(\tilde{\lambda}_0) + h_1 \leq (3\bar{\varepsilon}_1^{-1}\nu_0 + 1)h_1 + h_1 \\ &\leq 3h_1M_0. \quad \blacksquare \end{aligned}$$

12 Recalling (87), we shall apply Lemma 3.5 with

$$\sigma = \min\left\{r_\infty, \frac{\alpha_0}{6}, \frac{1}{3}\right\}, \quad \mathcal{A}_{d_{k_\infty}(\tilde{\lambda}_0), 3\sigma} \equiv A \cap \left\{d_{k_\infty}(\tilde{\lambda}_0) - 3\sigma \leq I_{\tilde{\lambda}_0} \leq d_{k_\infty}(\tilde{\lambda}_0) + 3\sigma\right\} \subset \mathcal{A}.$$

13 Therefore, for every relatively compact symmetric set $K \subset \mathcal{A}_{d_{k_\infty}(\tilde{\lambda}_0), 3\sigma}$, we may construct an odd defor-
14 mation satisfying (i)–(iii) in Lemma 3.5. To find suitable sets K , in what follows we put

$$\mu \equiv \tilde{\lambda}_0 + h_1.$$

1 For the given integers k_∞ in (89) and s_∞ in (86), choose $C \in \Lambda_{k_\infty+s_\infty}$ so that

$$\sup_C I_\mu \leq d_{k_\infty+s_\infty}(\mu) + h_1.$$

2 Consequently, Claim 2, (88) and the inequalities $\delta_0 < 1$, $r_\infty < 1$, $M_0 \geq 1$ guarantee that

$$\sup_C I_\mu \leq d_{k_\infty}(\tilde{\lambda}_0) + (3M_0 + 1)h_1 \leq d_{k_\infty}(\tilde{\lambda}_0) + 4M_0h_1 < d_{k_\infty}(\tilde{\lambda}_0) + \sigma. \quad (90)$$

3 Moreover, if $u \in C$ satisfies

$$d_{k_\infty}(\tilde{\lambda}_0) - \sigma < d_{k_\infty}(\tilde{\lambda}_0) - 8M_0h_1 \leq I_{\tilde{\lambda}_0}(u),$$

4 then (90) yields

$$\Psi(u) = \frac{I_\mu(u) - I_{\tilde{\lambda}_0}(u)}{\mu - \tilde{\lambda}_0} \leq \frac{d_{k_\infty}(\tilde{\lambda}_0) + 4M_0h_1 - (d_{k_\infty}(\tilde{\lambda}_0) - 8M_0h_1)}{h_1} = 12M_0. \quad (91)$$

5 From (91) it follows that

$$C \cap \left\{ I_{\tilde{\lambda}_0} \geq d_{k_\infty}(\tilde{\lambda}_0) - 8M_0h_1 \right\} \subset \{u \in X \mid \Psi(u) \leq 12M_0\} \subset B(0, R(12M_0)). \quad (92)$$

6 Set

$$C_0 \equiv \overline{C \setminus \overline{O_{3r_\infty}}}.$$

7 We infer from (86), (90), (92) and Lemma 4.7 that

$$C_0 \in \Lambda_{k_\infty}, \quad K_0 \equiv C_0 \cap \left\{ d_{k_\infty}(\tilde{\lambda}_0) - 8M_0h_1 \leq I_{\tilde{\lambda}_0} \right\} \subset \mathcal{A}_{d_{k_\infty}(\tilde{\lambda}_0), 3\sigma}.$$

8 Furthermore,

$$d_X(C_0, O_{r_\infty}) \geq 2r_\infty.$$

9 Since $C_0 \in \Lambda_{k_\infty}$, we can write C_0 as

$$C_0 = \gamma_0 \left(\overline{\mathbb{D}^n \setminus \Sigma_0} \right), \quad n \geq k_\infty, \quad \gamma_0 \in \Gamma_n, \quad \Sigma_0 \in \mathcal{E}_{\mathbb{R}^n}, \quad \Sigma_0 \cap \partial\mathbb{D}^n = \emptyset, \quad \mathcal{G}(\Sigma_0) \leq n - k_\infty.$$

10 Then K_0 is rewritten as

$$K_0 = \left\{ \gamma_0(\zeta) \mid \zeta \in \overline{\mathbb{D}^n \setminus \Sigma_0}, \quad I_{\tilde{\lambda}_0}(\gamma_0(\zeta)) \geq d_{k_\infty}(\tilde{\lambda}_0) - 8M_0h_1 \right\}.$$

11 From now on, we will repeatedly use the following inequality:

$$-8M_0h_1 + \delta_0 T < 4M_0h_1 - 2\delta_0 T \quad \text{for all } T \in [0, 4\delta_0^{-1}M_0h_1]. \quad (93)$$

12 Since K_0 is relatively compact and symmetric, we may apply Lemma 3.5 and find $s_0 > 0$, $\eta \in C([0, s_0] \times X, X)$ and symmetric open sets W, \tilde{W} such that

14 (a) $\overline{K_0} \subset W \subset \overline{W} \subset \tilde{W}$.

15 (b) $\|\eta(s, w) - w\| \leq s$ and $\eta(s, -w) = -\eta(s, w)$ for every $(s, w) \in [0, s_0] \times X$.

16 (c) $I_{\tilde{\lambda}_0}(\eta(s, w)) \leq I_{\tilde{\lambda}_0}(w) + \delta_0 s$ for every $(s, w) \in [0, s_0] \times X$.

17 (d) $I_{\tilde{\lambda}_0}(\eta(s, w)) \leq I_{\tilde{\lambda}_0}(w) - 2\delta_0 s$ for every $(s, w) \in [0, s_0] \times W$ with $I_{\tilde{\lambda}_0}(w) \geq d_{k_\infty}(\tilde{\lambda}_0) - \sigma$.

1 As in the proof of Theorem 4.4, we consider a family of deformations of K_0 :

$$\mathcal{D}_0 \equiv \left\{ (\eta, \tau, W, \widetilde{W}) \mid \eta, W, \widetilde{W} \text{ satisfy (a)–(d) for } s \in [0, \tau] \right\}$$

2 and set

$$\widetilde{T}_0 \equiv \sup_{(\eta, \tau, W, \widetilde{W}) \in \mathcal{D}_0} \tau \geq s_0, \quad T_0 \equiv \min \left\{ \frac{3M_0 h_1}{\delta_0}, \widetilde{T}_0 \right\}.$$

3 Choose $(\eta_0, \tau_0, W_0, \widetilde{W}_0) \in \mathcal{D}_0$ so that

$$\frac{3}{4}T_0 < \tau_0 < T_0 \leq 3\delta_0^{-1}M_0 h_1 < \frac{r_\infty}{4},$$

4 where the last inequality comes from (88). Since for small $\theta_0 > 0$ the dilation map

$$L_{\theta_0} : \mathbb{D}^n(1 - \theta_0) \rightarrow \mathbb{D}^n, \quad L_{\theta_0}(\zeta) \equiv \frac{\zeta}{1 - \theta_0}$$

5 satisfies

$$\max_{|\zeta| \leq 1 - \theta_0} \|\gamma_0(\zeta) - \gamma_0(L_{\theta_0}(\zeta))\| + \max_{1 - \theta_0 \leq |\zeta| \leq 1} \|\gamma_0(\zeta) - \gamma_0(\zeta/|\zeta|)\| < \frac{\tau_0}{2},$$

6 we define

$$\gamma_1(\zeta) \equiv \begin{cases} \eta_0(\tau_0, \gamma_0(L_{\theta_0}(\zeta))) & \text{if } |\zeta| \leq 1 - \theta_0, \\ \eta_0\left(\tau_0 \frac{1 - |\zeta|}{\theta_0}, \gamma_0\left(\frac{\zeta}{|\zeta|}\right)\right) & \text{if } 1 - \theta_0 \leq |\zeta| \leq 1. \end{cases}$$

7 Then, $\gamma_1 \in \Gamma_n$. Since $\theta_0 > 0$ is small and L_{θ_0} is an odd homeomorphism,

$$\Sigma_1 \equiv L_{\theta_0}^{-1}(\Sigma_0) \in \mathcal{E}_{\mathbb{R}^n}, \quad \Sigma_1 \cap \partial\mathbb{D}^n = \emptyset, \quad \mathcal{G}(\Sigma_1) \leq n - k_\infty,$$

8 and moreover

$$\max_{\zeta \in \mathbb{D}^n} \|\gamma_1(\zeta) - \gamma_0(\zeta)\| \leq \frac{3}{2}\tau_0.$$

9 Define

$$C_1 \equiv \gamma_1\left(\overline{\mathbb{D}^n \setminus \Sigma_1}\right) \in \Lambda_{k_\infty}.$$

10 As in the proofs of Theorem 4.4, it follows from (93) and (90) that

$$d_{k_\infty}(\widetilde{\lambda}_0) \leq \sup_{C_1} I_{\widetilde{\lambda}_0}^- \leq d_{k_\infty}(\widetilde{\lambda}_0) + 4M_0 h_1 - 2\delta_0 \tau_0;$$

11 this implies that

$$\tau_0 \leq 2\delta_0^{-1}M_0 h_1 < \frac{r_\infty}{6}.$$

12 We next claim

$$\text{dist}_H(C_0, C_1) \leq \tau_0. \tag{94}$$

13 Let $C_1 \ni u_1 = \gamma_1(\zeta)$, where $\zeta \in \overline{\mathbb{D}^n \setminus \Sigma_1}$. If $1 - \theta_0 \leq |\zeta| \leq 1$, then by

$$\gamma_1(\zeta/|\zeta|) = \gamma_0(\zeta/|\zeta|) = \pi_{0,n}(\zeta/|\zeta|) \in C_0 \cap C_1,$$

14 the choice of θ_0 gives $\|u_1 - \gamma_0(\zeta/|\zeta|)\| \leq \tau_0$. When $|\zeta| \leq 1 - \theta_0$, we find

$$L_{\theta_0}(\zeta) \in \overline{\mathbb{D}^n \setminus \Sigma_0}, \quad \gamma_0(L_{\theta_0}(\zeta)) \in C_0, \quad \|u_1 - \gamma_0(L_{\theta_0}(\zeta))\| \leq \tau_0.$$

15 Thus, $d_X(C_0, u_1) \leq \tau_0$ and since $u_1 \in C_1$ is arbitrary, $\max_{u_1 \in C_1} d_X(C_0, u_1) \leq \tau_0$.

1 On the other hand, let $C_0 \ni u_0 = \gamma_0(\zeta)$ where $\zeta \in \overline{\mathbb{D}^n \setminus \Sigma_0}$. From the property

$$L_{\theta_0}^{-1}(\zeta) \in \overline{\mathbb{D}^n(1 - \theta_0) \setminus \Sigma_1},$$

2 it follows that

$$\gamma_1 \left(L_{\theta_0}^{-1}(\zeta) \right) \in C_1, \quad \left\| u_0 - \gamma_1(L_{\theta_0}^{-1}(\zeta)) \right\| = \left\| \gamma_0(\zeta) - \eta_0(\tau_0, \gamma_0(\zeta)) \right\| \leq \tau_0.$$

3 Hence, $d_X(u_0, C_1) \leq \tau_0$ and $\max_{u_0 \in C_0} d_X(u_0, C_1) \leq \tau_0$, which yields (94).

4 Furthermore, (94) and $d_X(C_0, O_{r_\infty}) \geq 2r_\infty$ imply the inequality

$$d_X(C_1, O_{r_\infty}) \geq 2r_\infty - \tau_0.$$

5 We next consider the set

$$K_1 \equiv \left\{ \gamma_1(\zeta) \mid \zeta \in \overline{\mathbb{D}^n \setminus \Sigma_1}, I_{\tilde{\lambda}_0}(\gamma_1(\zeta)) \geq d_{k_\infty}(\tilde{\lambda}_0) - 8M_0h_1 + \delta_0\tau_0 \right\}.$$

6 Remark that K_1 is symmetric and relatively compact. Moreover, by combining (c) with (ii) in Lemma
7 4.7 and with $\delta_0 \in (0, 1)$, for $1 - \theta_0 \leq |\zeta| \leq 1$ we have

$$\begin{aligned} I_{\tilde{\lambda}_0}(\gamma_1(\zeta)) &\leq I_{\tilde{\lambda}_0}(\gamma_0(\zeta/|\zeta|)) + \delta_0\tau_0 < \delta_0\tau_0 \\ &\leq \alpha_0 - 8M_0h_1 + \delta_0\tau_0 \leq d_{k_\infty}(\tilde{\lambda}_0) - 8M_0h_1 + \delta_0\tau_0. \end{aligned}$$

8 Hence, we obtain

$$\gamma_1^{-1}(K_1) \subset (\gamma_0 \circ L_{\theta_0})^{-1}(K_0), \quad \max_{\zeta \in \overline{\mathbb{D}^n(1 - \theta_0) \cap \mathbb{D}^n \setminus \Sigma_1}} \left\| \gamma_1(\zeta) - \gamma_0(L_{\theta_0}(\zeta)) \right\| \leq \tau_0.$$

9 Since $K_0 \subset B(0, R(12M_0))$ holds due to (92),

$$K_1 \subset B(0, R(12M_0) + \tau_0), \quad d_X(K_1, O_{r_\infty}) \geq 2r_\infty - \tau_0, \quad K_1 \subset \mathcal{A}_{d_{k_\infty}(\tilde{\lambda}_0), 3\sigma}.$$

10 By Lemma 3.5, there exist $s_1 > 0$, an odd map $\eta \in C([0, s_1] \times X, X)$ and symmetric open sets W, \tilde{W} of
11 K_1 satisfying (a)–(d) with K_1 instead of K_0 . Thus, define

$$\mathcal{D}_1 \equiv \left\{ \left(\eta, \tau, W, \tilde{W} \right) \mid \eta, W, \tilde{W} \text{ satisfy (a)–(d) for } s \in [0, \tau] \right\}$$

12 and

$$\tilde{T}_1 \equiv \sup_{(\eta, \tau, W, \tilde{W}) \in \mathcal{D}_1} \tau \geq s_1, \quad T_1 \equiv \min \left\{ \frac{3M_0h_1}{\delta_0} - \tau_0, \tilde{T}_1 \right\}.$$

13 Select $(\eta_1, \tau_1, W_1, \tilde{W}_1) \in \mathcal{D}_1$ and $0 < \theta_1 \ll 1$ such that

$$\frac{3}{4}T_1 < \tau_1 < T_1 \leq 3\frac{M_0h_1}{\delta_0} - \tau_0 < \frac{r_\infty}{4} - \tau_0$$

14 and

$$\max_{|\zeta| \leq 1 - \theta_1} \left\| \gamma_1(\zeta) - \gamma_1(L_{\theta_1}(\zeta)) \right\| + \max_{1 - \theta_1 \leq |\zeta| \leq 1} \left\| \gamma_1(\zeta) - \gamma_1(\zeta/|\zeta|) \right\| < \frac{\tau_1}{2},$$

15 where

$$L_{\theta_1}(\zeta) \equiv \frac{\zeta}{1 - \theta_1} : \mathbb{D}^n(1 - \theta_1) \rightarrow \mathbb{D}^n.$$

1 Define γ_2 by

$$\Gamma_n \ni \gamma_2(\zeta) \equiv \begin{cases} \eta_1(\tau_1, \gamma_1(L_{\theta_1}(\zeta))) & \text{if } |\zeta| \leq 1 - \theta_1, \\ \eta_1\left(\tau_1 \frac{1 - |\zeta|}{\theta_1}, \gamma_1(\zeta/|\zeta|)\right) & \text{if } 1 - \theta_1 \leq |\zeta| \leq 1. \end{cases}$$

2 From $\mathcal{G}(\Sigma_1) \leq n - k_\infty$ it follows that

$$\Sigma_2 \equiv L_{\theta_1}^{-1}(\Sigma_1) \in \mathcal{E}_{\mathbb{R}^n}, \quad \Sigma_2 \cap \partial\mathbb{D}^n = \emptyset, \quad \mathcal{G}(\Sigma_2) \leq n - k_\infty.$$

3 Writing

$$C_2 \equiv \gamma_2\left(\overline{\mathbb{D}^n \setminus \Sigma_2}\right) \in \Lambda_{k_\infty},$$

4 we may prove

$$d_{k_\infty}(\tilde{\lambda}_0) \leq \sup_{C_2} I_{\tilde{\lambda}_0} \leq d_{k_\infty}(\tilde{\lambda}_0) + 4M_0h_1 - 2\delta_0(\tau_0 + \tau_1),$$

5 which yields

$$\tau_0 + \tau_1 \leq \frac{2M_0h_1}{\delta_0} < \frac{r_\infty}{6}.$$

6 Furthermore, by an argument analogous to the one we just employed for C_0 and C_1 , we see that

$$\text{dist}_H(C_1, C_2) \leq \tau_1, \quad d_X(C_2, O_{r_\infty}) \geq 2r_\infty - (\tau_0 + \tau_1).$$

7 We next define

$$K_2 \equiv \left\{ \gamma_2(\zeta) \mid \zeta \in \overline{\mathbb{D}^n \setminus \Sigma_2}, I_{\tilde{\lambda}_0}(\gamma_2(\zeta)) \geq d_{k_\infty}(\tilde{\lambda}_0) - 8M_0h_1 + \delta_0(\tau_0 + \tau_1) \right\}$$

8 and notice that K_2 is symmetric, relatively compact, $\gamma_2^{-1}(K_2) \subset (\gamma_1 \circ L_{\theta_1})^{-1}(K_1)$ and

$$d_X(K_2, O_{r_\infty}) \geq 2r_\infty - \tau_0 - \tau_1, \quad K_2 \subset B(0, R(12M_0) + \tau_0 + \tau_1), \quad K_2 \subset \mathcal{A}_{d_{k_\infty}(\tilde{\lambda}_0), 3\sigma}.$$

9 Inductively, we obtain $\{(\eta_j, \tau_j, W_j, \tilde{W}_j)\}_{j=0}^\infty$, $\{C_j\}_{j=0}^\infty$, $\{\Sigma_j\}_{j=0}^\infty$, $\{\gamma_j\}_{j=0}^\infty$, $\{K_j\}_{j=0}^\infty$ and a sequence $\{\mathcal{D}_j\}_{j=0}^\infty$
10 of deformations of K_j such that

$$C_j = \gamma_j(\mathbb{D}^n \setminus \Sigma_j) \in \Lambda_{k_\infty}, \quad \mathcal{G}(\Sigma_j) \leq n - k_\infty, \quad d_{k_\infty}(\tilde{\lambda}_0) \leq \sup_{C_j} I_{\tilde{\lambda}_0} \leq d_{k_\infty}(\tilde{\lambda}_0) + 4M_0h_1 - 2\delta_0 \sum_{\ell=0}^j \tau_\ell,$$

$$\text{dist}_H(C_j, C_{j+1}) \leq \tau_j, \quad d_X(C_j, O_{r_\infty}) \geq 2r_\infty - \sum_{\ell=0}^{j-1} \tau_\ell, \quad \sum_{\ell=0}^j \tau_\ell \leq \frac{2M_0h_1}{\delta_0} < \frac{r_\infty}{6},$$

$$\tilde{T}_j \equiv \sup_{(\eta, \tau, W, \tilde{W}) \in \mathcal{D}_j} \tau, \quad T_j \equiv \min \left\{ \tilde{T}_j, \frac{3M_0h_1}{\delta_0} - \sum_{\ell=0}^{j-1} \tau_\ell \right\}, \quad \frac{3}{4}T_j \leq \tau_j < T_j,$$

$$K_j \equiv \left\{ \gamma_j(\zeta) \mid \zeta \in \overline{\mathbb{D}^n \setminus \Sigma_j}, I_{\tilde{\lambda}_0}(\gamma_j(\zeta)) \geq d_{k_\infty}(\tilde{\lambda}_0) - 8M_0h_1 + \delta_0 \sum_{\ell=0}^{j-1} \tau_\ell \right\},$$

$$K_j \subset B\left(0, R(12M_0) + \sum_{\ell=0}^{j-1} \tau_\ell\right).$$

11 In particular,

$$K_j \subset \mathcal{A}_{d_{k_\infty}(\tilde{\lambda}_0), 3\sigma} \quad \text{for each } j \geq 0.$$

1 Recalling that $(\mathcal{S}, \text{dist}_H)$ is a complete metric space and the above properties, we see that $\{C_j\}_{j=0}^\infty$ is a
 2 Cauchy sequence in $(\mathcal{S}, \text{dist}_H)$. Therefore, there exists $C_\infty \in \mathcal{S}$ such that $\text{dist}_H(C_j, C_\infty) \rightarrow 0$ as $j \rightarrow \infty$.
 3 As before, a set defined by

$$K_\infty \equiv \bigcup_{j=0}^{\infty} K_j \subset \mathcal{A}_{d_{k_\infty}(\tilde{\lambda}_0), 3\sigma}$$

4 is relatively compact and Lemma 3.5 with $K \equiv K_\infty$ guarantees the existence of deformation $(\eta_\infty, \tau_\infty, W_\infty, \widetilde{W}_\infty)$
 5 of K_∞ ; then $(\eta_\infty, \tau_\infty, W_\infty, \widetilde{W}_\infty) \in \mathcal{D}_j$ for each j large enough. Thus,

$$\tau_j \geq \frac{3}{4} \min \left\{ \tau_\infty, \frac{3M_0 h_1}{\delta_0} - \sum_{\ell=0}^{\infty} \tau_\ell \right\} \geq \frac{3}{4} \min \left\{ \tau_\infty, \frac{M_0 h_1}{\delta_0} \right\} \quad \text{for large enough } j,$$

6 which contradicts

$$\sum_{\ell=0}^{\infty} \tau_\ell \leq \frac{r_\infty}{6}.$$

7 Hence, we complete the proof. ■

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16 A The principle of symmetric criticality

17 The principle of nonsmooth symmetric criticality due to Kobayashi & Ôtani [44] considers a vast range
 18 of linear actions of a topological groups G on a Banach space X . In this appendix, we provide a simpler
 19 proof of the principle in a setting which applies to our main theorems. Compared to [44, Theorem 3.16],
 20 notice that Proposition A.1 does not require X to be reflexive.

21 Let X be a Banach space and G a compact topological group acting on X linearly. Assume that the
 22 map $G \times X \ni (g, u) \mapsto g \cdot u \in X$ is continuous, and consider the closed subspace

$$X_G \equiv \{ u \in X \mid g \cdot u = u \text{ for all } g \in G \}.$$

23 We consider two functionals Ψ and Φ satisfying

24 (i) $\Psi : X \rightarrow (-\infty, \infty]$ is convex and lower semicontinuous with $D(\Psi) \neq \emptyset$.

25 (ii) $\Phi \in C^1(X, \mathbb{R})$.

26 (iii) Ψ and Φ are G -invariant, that is $\Psi(g \cdot u) = \Psi(u)$ and $\Phi(g \cdot u) = \Phi(u)$ for every $g \in G$ and $u \in X$.

27 Finally, set $I(u) \equiv \Psi(u) - \Phi(u) : X \rightarrow (-\infty, \infty]$.

28 **Proposition A.1.** *Assume $u \in X_G$ is a critical point of $I|_{X_G} : X_G \rightarrow (-\infty, \infty]$. Then u is a critical*
 29 *point of $I : X \rightarrow (-\infty, \infty]$.*

1 **Proof.** Let $u \in X_G$ be a critical point of $I|_{X_G}$. Then

$$\Psi(v) - \Psi(u) - \Phi'(u)(v - u) \geq 0 \quad \text{for each } v \in X_G. \quad (95)$$

2 Fix any $w \in X$ and let μ be a normalized Haar measure on G ($\mu(G) = 1$). Since $(g, u) \mapsto g \cdot u$ is
3 continuous and G is compact, we set (cf. Rudin [55, Theorems 3.20 and 3.27, and Definition 3.26])

$$\hat{w} \equiv \int_G g \cdot w \, d\mu, \quad (96)$$

4 that is, \hat{w} is the unique element satisfying

$$\Lambda(\hat{w}) = \int_G \Lambda(g \cdot w) \, d\mu \quad \text{for every } \Lambda \in X^*.$$

5 We first show $\hat{w} \in X_G$. To this end, let $h \in G$ and write $T_h : X \ni u \mapsto h \cdot u \in X$. For each $\Lambda \in X^*$, it
6 follows from $\Lambda \circ T_h \in X^*$ and the left invariance of μ that

$$\begin{aligned} \Lambda(h \cdot \hat{w}) &= (\Lambda \circ T_h)(\hat{w}) = \int_G (\Lambda \circ T_h)(g \cdot w) \, d\mu \\ &= \int_G \Lambda(hg \cdot w) \, d\mu = \int_G \Lambda(g \cdot w) \, d\mu = \Lambda(\hat{w}). \end{aligned}$$

7 Since $\Lambda \in X^*$ is arbitrary, $h \cdot \hat{w} = \hat{w}$ holds for each $h \in G$.

8 We now prove that u is a critical point of $I : X \rightarrow (-\infty, \infty]$. For $w \in X$, consider \hat{w} in (96).
9 From [55, Theorem 3.27], \hat{w} belongs to the closure of the convex hull of the orbit $G \cdot w$. Therefore, we
10 can take convex combinations

$$w_j \equiv \sum_{k=1}^{L_j} c_{j,k} (g_{j,k} \cdot w), \quad c_{j,k} \geq 0, \quad \sum_{k=1}^{L_j} c_{j,k} = 1$$

11 such that $\|w_j - \hat{w}\| \rightarrow 0$. Since $\Phi \in C^1(X, \mathbb{R})$ satisfies $\Phi(g \cdot u) = \Phi(u)$ for every $g \in G$, it follows that

$$\Phi'(g \cdot u)(g \cdot v) = \Phi'(u)v \quad \text{for any } g \in G \text{ and } v \in X.$$

12 By the convexity and the lower semicontinuity of Ψ , we infer from (95) and $\hat{w}, u \in X_G$ (hence, $g_{j,k} \cdot u = u$)
13 that

$$\begin{aligned} 0 &\leq \Psi(\hat{w}) - \Psi(u) - \Phi'(u)(\hat{w} - u) \leq \liminf_{j \rightarrow \infty} \Psi(w_j) - \Psi(u) - \lim_{j \rightarrow \infty} \Phi'(u)(w_j - u) \\ &\leq \liminf_{j \rightarrow \infty} [\Psi(w_j) - \Psi(u) - \Phi'(u)(w_j - u)] \\ &\leq \liminf_{j \rightarrow \infty} \sum_{k=1}^{L_j} c_{j,k} [\Psi(g_{j,k} \cdot w) - \Psi(u) - \Phi'(u)(g_{j,k} \cdot w - u)] \\ &= \liminf_{j \rightarrow \infty} \sum_{k=1}^{L_j} c_{j,k} [\Psi(w) - \Psi(u) - \Phi'(g_{j,k} \cdot u)(g_{j,k} \cdot w) + \Phi'(u)u] \\ &= \liminf_{j \rightarrow \infty} \sum_{k=1}^{L_j} c_{j,k} [\Psi(w) - \Psi(u) - \Phi'(u)(w - u)] \\ &= \Psi(w) - \Psi(u) - \Phi'(u)(w - u). \end{aligned}$$

14 This means that u is a critical point of $I : X \rightarrow (-\infty, \infty]$ and the proof is completed. ■

1 B Regularity for the Born-Infeld solution

2 In this section, we prove Proposition 2.10.

3 **Proof.** We first address the equivalence in item (i).

4 (a) \Rightarrow (b).

5 Assume that u is a weak solution to (\mathcal{BI}_ρ) (in the sense of Remark 2.9 if $\partial\Omega$ is not smooth). Fix a
6 domain $\Omega \Subset \mathbb{R}^n$ and a function $\psi \in \mathcal{Y}_u(\Omega)$. Then, $u - \psi \in W^{1,\infty}(\Omega)$ and has zero boundary value. By
7 a zero extension of $u - \psi$ outside of Ω , we see that $u - \psi \in \text{Lip}_c(\mathbb{R}^n)$. We then test $u - \psi$ to (\mathcal{BI}_ρ) to
8 deduce

$$\int_{\Omega} \frac{Du \cdot (Du - D\psi)}{\sqrt{1 - |Du|^2}} = \int_{\Omega} \rho(u - \psi).$$

9 Here, due to the second in (7), the set of points where u is differentiable and $|Du| < 1$ has full measure
10 in Ω . Inequality (23) holds therein; thus we see that

$$\int_{\Omega} \sqrt{1 - |D\psi|^2} - \sqrt{1 - |Du|^2} \leq \int_{\Omega} \rho(u - \psi),$$

11 which implies $I_\rho^\Omega(u) \leq I_\rho^\Omega(\psi)$.

12 (b) \Rightarrow (a) and $u \in W_{\text{loc}}^{2,q}(\mathbb{R}^n)$ for each $q \in [2, \infty)$, $|Du| < 1$ on \mathbb{R}^n .

13 For a local minimizer u of I_ρ , we define a Lorentzian distance $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$ from 0 by

$$\ell(x) \equiv \sqrt{|x|^2 - (u(x) - u(0))^2}.$$

14 Since $|Du| \leq 1$, we have $\ell \geq 0$ on \mathbb{R}^n and moreover, in view of (20), the Lorentzian balls

$$L_R \equiv \{x \in \mathbb{R}^n \mid \ell(x) < R\}$$

15 are relatively compact in \mathbb{R}^n . Note that L_R contains the Euclidean ball $B_R(0)$, which for convenience we
16 denote by B_R . Fix $R > 0$, define $\Omega \equiv L_{4R}$ and let $\{\rho_j\}$ be a sequence of smooth functions converging to
17 ρ in $L^1(\Omega)$. Taking $t_j \in (0, 1)$ with $\lim_{j \rightarrow \infty} t_j = 1$, we consider the solution $u_j \in \mathcal{Y}_{t_j u}(\Omega)$ to

$$\begin{cases} -\operatorname{div} \left(\frac{Du_j}{\sqrt{1 - |Du_j|^2}} \right) = \rho_j & \text{on } \Omega, \\ u_j = t_j u & \text{on } \partial\Omega. \end{cases}$$

18 Since $t_j u$ is spacelike on $\partial\Omega$, more precisely $|t_j u(y) - t_j u(x)| < |y - x|$ for each $x, y \in \partial\Omega$ with $x \neq y$,
19 by [9] there exists a solution u_j , which is smooth and $|Du_j| < 1$ on Ω . We claim that $u_j \rightarrow u$ strongly in
20 $C(\overline{\Omega})$ and in $W^{1,q}(\Omega)$ for each $q \in [1, \infty)$. This can be shown by adapting an argument in [22, Proposition
21 3.11] in the following way. First, since $\{t_j u\}$ is relatively compact in $C(\partial\Omega)$, by [22, Proposition 3.5] $\{u_j\}$
22 is relatively compact in $C(\overline{\Omega})$. In particular, since $|Du_j| \leq 1$ for each j , $\{u_j\}$ is bounded in $W^{1,q}(\Omega)$ for
23 any $q \in [1, \infty]$. Then there exists $v \in \mathcal{Y}_u(\Omega)$ such that $u_j \rightarrow v$, up to a subsequence, in $C(\overline{\Omega})$ and weakly
24 in $W^{1,q}(\Omega)$ for any $q \in (1, \infty)$. To show that $v \equiv u$, notice that from the second in Remark 2.14 we get

$$\begin{aligned} \int_{\Omega} \left(1 - \sqrt{1 - |Dv|^2}\right) &= \sum_{k=1}^{\infty} b_k \|Dv\|_{2k,\Omega}^{2k} \leq \sum_{k=1}^{\infty} b_k \liminf_{j \rightarrow \infty} \|Du_j\|_{2k,\Omega}^{2k} \\ &\leq \lim_{n \rightarrow \infty} \liminf_{j \rightarrow \infty} \sum_{k=1}^n b_k \|Du_j\|_{2k,\Omega}^{2k} \\ &\leq \liminf_{j \rightarrow \infty} \int_{\Omega} \left(1 - \sqrt{1 - |Du_j|^2}\right). \end{aligned} \tag{97}$$

1 Moreover, since $u_j \rightarrow v$ in $C(\bar{\Omega})$ and $\rho_j \rightarrow \rho$ in $L^1(\Omega)$, it holds $\int_{\Omega} \rho_j u_j \rightarrow \int_{\Omega} \rho v$ as $j \rightarrow \infty$. Thus we get

$$I_{\rho}^{\Omega}(v) \leq \liminf_{j \rightarrow \infty} I_{\rho_j}^{\Omega}(u_j).$$

2 On the other hand, $u_j \in \mathcal{Y}_{t_j u}(\Omega)$ minimizes I_{ρ_j} ; thus $I_{\rho_j}^{\Omega}(u_j) \leq I_{\rho_j}^{\Omega}(t_j u)$. A simple computation gives

$$I_{\rho_j}^{\Omega}(t_j u) \rightarrow I_{\rho}^{\Omega}(u),$$

3 and therefore $I_{\rho}^{\Omega}(v) \leq I_{\rho}^{\Omega}(u)$. Since the reverse inequality holds by our assumption on u and the minimizer
4 for I_{ρ} is unique, we conclude that $v = u$ in Ω , as claimed. The argument to prove the strong convergence
5 $u_j \rightarrow u$ in $W^{1,q}(\Omega)$ for $q \in [1, \infty)$ then follows verbatim that in [22, Proposition 3.11], see the second half
6 of p.32 therein.

7 Next, we prove the regularity of u on B_R . Because of (20) and a direct comparison between the
8 Euclidean and Lorentzian distances, there exists \hat{R} depending on R, c_1, h such that

$$\Omega \equiv L_{4R} \Subset B_{\hat{R}}. \quad (98)$$

9 Indeed, in our assumption, $|u(0)| \leq c_1 - h(0)$ and

$$|u(x) - u(0)| \leq c_1 + |u(0)| + |x| - h(x) \leq \bar{c}_1 + |x| - h(x),$$

10 where $\bar{c} \equiv 2c_1 - h(0)$. If $x \in L_{4R}$, then by expanding

$$|x|^2 \leq 16R^2 + (u(x) - u(0))^2 \leq 16R^2 + (\bar{c}_1 + |x| - h(x))^2$$

11 and rearranging we get

$$2|x|(h(x) - \bar{c}_1) \leq 16R^2 + (\bar{c}_1 - h(x))^2$$

12 from which the bound $|x| \leq \hat{R}$ follows and recalling our growth assumptions on h . By a simple argument
13 by contradiction, we deduce from $u_j \rightarrow u$ in $C(\bar{\Omega})$ that for all sufficiently large j ,

$$L_{3R}^{\rho_j} \equiv \left\{ x \in \Omega \mid \ell^{\rho_j}(x) \equiv \sqrt{|x|^2 - (u_j(x) - u_j(0))^2} < 3R \right\} \Subset \Omega.$$

14 We apply [9, Lemma 2.1] and the reasoning in the proof of Theorem 4.1 therein, to deduce that there
15 exists a constant $\delta > 0$ depending on the L^{∞} bounds for u_j and ρ_j on $L_{3R}^{\rho_j}$, which are thus uniformly
16 bounded in terms of

$$\hat{R}, \quad c_1, \quad h \quad \text{and} \quad \|\rho\|_{L^{\infty}(B_{\hat{R}})},$$

17 such that

$$|Du_j| < 1 - \delta \quad \text{on} \quad L_{2R}^{\rho_j}.$$

18 Remark also that $\{u_j\}$ is bounded in $W^{2,2}(L_{2R})$. Since $B_{2R} \subset L_{2R}^{\rho_j}$, we get local uniform estimates for
19 $|Du_j|$ and for the L^2 norm of $|D^2 u_j|$ on B_{2R} . Thus, for any $\eta \in \text{Lip}_c(B_{2R})$, by letting $j \rightarrow \infty$ in

$$\int_{\mathbb{R}^n} \frac{Du_j \cdot D\eta}{\sqrt{1 - |Du_j|^2}} = \int_{\mathbb{R}^n} \rho_j \eta,$$

20 the dominated convergence theorem implies that (8) holds.

21 Finally, for any $q \in [2, \infty)$ we show the existence of $C_q = C_q(\hat{R}, c, q)$ such that $\|u\|_{W^{2,q}(B_R)} \leq C_q$. To
22 this goal, we first argue as in [22, Step 5 in Section 5.2] to obtain $Du \in C^{\alpha_R}(B_R(0))$ for some $\alpha_R \in (0, 1)$.

23 Next, from $u \in W_{\text{loc}}^{2,2}(\mathbb{R}^n)$ and writing

$$a_{ij}(x) \equiv \frac{\delta_{ij}}{\sqrt{1 - |Du(x)|^2}} + \frac{D_i u(x) D_j u(x)}{(1 - |Du(x)|^2)^{3/2}} \in C(\mathbb{R}^n),$$

1 we infer that u is a strong solution to

$$- \sum_{i,j=1}^n a_{ij}(x) D_{ij} u(x) = \rho(x) \quad \text{in } \mathbb{R}^n.$$

2 Since $\|\rho\|_{L^\infty(B_{2R})} \leq \|\rho\|_{L^\infty(B_{\bar{R}})} \leq c$, elliptic regularity yields the desired $W^{2,q}$ estimate in B_R , see for
3 instance [32, Chapter 9]. This completes the proof of (i).

4 (ii) Suppose $u \in L^\infty(\mathbb{R}^n)$ and $\rho \in L^\infty(\mathbb{R}^n)$ satisfy $\|\rho\|_\infty + \|u\|_\infty \leq c$. By (i), $u \in C^1(\mathbb{R}^n)$ and
5 $|Du(x)| < 1$ in \mathbb{R}^n . We prove the assertion by contradiction and suppose that there exist $\{\rho_i\}_i, \{u_i\}_i \in$
6 $L^\infty(\mathbb{R}^n)$, with u_i a solution with source ρ_i , and points $\{x_i\}_i \subset \mathbb{R}^n$ such that

$$\|\rho_i\|_\infty + \|u_i\|_\infty \leq c, \quad |Du(x_i)| \rightarrow 1 \quad \text{as } i \rightarrow \infty.$$

7 Define

$$\bar{u}_i(x) \equiv u_i(x + x_i), \quad \bar{\rho}_i(x) \equiv \rho_i(x + x_i).$$

8 Then, \bar{u}_i is a (strong) solution with source $\bar{\rho}_i$ on \mathbb{R}^n , hence a local minimizer of $I_{\bar{\rho}_i}$. Since \bar{u}_i is 1-Lipschitz,
9 we may also assume that $\bar{u}_i \rightarrow \bar{u}_\infty$ in $C_{loc}(\mathbb{R}^n)$, and \bar{u}_∞ is also 1-Lipschitz with $\|\bar{u}\|_\infty \leq c$. The inequality
10 $\|\bar{\rho}_i\| + \|\bar{u}_i\|_\infty \leq c$ and [9, Theorem 3.2] imply that \bar{u}_i has no light segments in the following quantitative
11 sense: for each $r > 0$, there exists $R = R(r) > 0$ such that for every $i \geq 1$,

$$L_r^{\bar{\rho}_i} \equiv \left\{ x \in \mathbb{R}^n \mid \ell^{\bar{\rho}_i}(x) \equiv \sqrt{|x|^2 - (\bar{u}_i(x) - \bar{u}_i(0))^2} < r \right\} \subset B_R.$$

12 According to the monotonicity formula in [9, Lemma 2.1], we may find $\alpha \in (0, 1/n)$ and $C = C(n) > 0$
13 such that

$$C \exp(r^2 c^2 + 1) (1 - |D\bar{u}_i(0)|)^{\frac{\alpha}{2}} \geq r^{-n} \int_{L_r^{\bar{\rho}_i}} \left(1 - |D\bar{u}_i|^2\right)^{\alpha+1} + r^{2-n} \int_{L_r^{\bar{\rho}_i}} |D^2\bar{u}_i|^2.$$

14 From $|D\bar{u}_i(0)| \rightarrow 1$ and the fact that $r > 0$ can be arbitrary, it follows that $|D\bar{u}_\infty| = 1$ and $D^2\bar{u}_\infty = 0$
15 a.e. in \mathbb{R}^n . Thus, $\bar{u}_\infty(x) = a \cdot x + b$ for some $a \in \mathbb{R}^n, b \in \mathbb{R}$ with $|a| = 1$, however, this contradicts
16 $\|\bar{u}_\infty\|_\infty \leq c$ and concludes the proof. ■

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