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A comparison between geometric quasi-functors and Fourier-Mukai functors

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Introduction

One of the best contribution that algebraic geometry gave to mathematics was the possibility to study geometric objects with the powerful language of category theory. In fact, one can attach to any scheme X some natural abelian categories and therefore all the formalisms of category theory can in some sense be embedded into the geometry. One of the most interesting problems became then to understand "how much information" one needs to remember about the geometric object X, in order to describe it in a satisfactory way. Derived categories turned out to be the best object for this purpose and hence the study of them became a key research theme in algebraic geometry. It was discovered that they do not carry anymore the structure of an abelian category, instead they can be endowed with the weaker structure of a triangulated category. Such a structure, unluckyily, has some problematic features - as was know to mathematicians since the very beginning.

On the other hand, the study of functors between derived categories led to the discovery that an incredible number of them has the particular shape of a Fourier-Mukai functor - a kind of functor that is also very simple and very geometric in nature. Starting from the seminal works of Mukai in the 80's, an increasing class of functors between derived categories has been proved to be of Fourier-Mukai type. Conversely, the examples of non-Fourier-Mukai functors that were discovered since the second decade of this century has been understood to be related to the pathologic structure of triangulated categories.

For these reasons mathematicians have moved to higher categorical structures in order to have a deeper understanding of this phenomenom. Working in the greater generality of differential graded categories has permitted to solve a lot of problems and to have a neater and simpler description of the theory.

The work of this PhD thesis fits in this framework and aims to add a brick in the comprehension of the relationship between Fourier-Mukai functors and the differential graded world.

Let us now be a bit more detailed. Let X and Y be two smooth proper schemes and denote by $X \overset{p}{\leftarrow} X \times Y \overset{q}{\rightarrow} Y$ the two projections. For any object $\mathcal{E}$ in the bounded derived category of coherent sheaves $D^b(\mathrm{Coh}(X \times Y))$ the *Fourier-Mukai functor of kernel* $\mathcal{E}$ is a triangulated functor

$$\Phi_{\mathcal{E}} : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(Y))$$

defined by sending an object $A \in D^b(\mathrm{Coh}(X))$ to $Rq_*(\mathcal{E} \otimes^L p^*A)$. Actually, it turns out that all the triangulated functors one deals with in ordinary life are of this form; moreover Orlov proved in [36] that, if X and Y are smooth projective varieties over a field, then any fully faithful triangulated functor between $D^b(\mathrm{Coh}(X))$ and $D^b(\mathrm{Coh}(Y))$ is of Fourier-Mukai type for some kernel $\mathcal{E}$ (unique up to isomorphism).

But the growing hope for all the triangulated functors to be of Fourier-Mukai type (removing the fully faithfulness hypotesis in Orlov Theorem) was definitely stopped in 2014, when the first example of a non-Fourier-Mukai functor was exhibited [39].

In the meantime mathematicians have started to investigate the realm of differential graded (dg) categories and its connection with Fourier-Mukai functors. In particular, in [4] the authors proved - in the case of X and Y being smooth algebraic varieties - that any triangulated functor $\mathsf{F} : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(Y))$ which is liftable at the dg level must be of Fourier-Mukai type. A natural question that one may ask is:

(♣) if such a triangulated functor F is given, how can one find (the isomorphism class of) the element $\mathcal{E}_{\mathsf{F}} \in D^b(\mathrm{Coh}(X \times Y))$ such that $\Phi_{\mathcal{E}_{\mathsf{F}}}$ is isomorphic to F?

In the article [45] Toën studied the properties of **Hqe**, the localization of the category of (small) dg categories with respect to *quasi-equivalences*. He came out with a bijection between the isomorphism classes of objects in $D(\mathrm{Qcoh}(X \times Y))$, the derived category of quasicoherent sheaves on $X \times Y$, and the set of morphisms in **Hqe** between $D^{dg}(\mathrm{Qcoh}(X))$ and $D^{dg}(\mathrm{Qcoh}(Y))$, two dg enhancements of $D(\mathrm{Qcoh}(X))$ and $D(\mathrm{Qcoh}(Y))$, respectively. He claimed, moreover, that such a bijection was exactly the way of producing the kernel of question (♣). A proof of that fact was given in [31] even in a more general form.

Actually, in the case of smooth and proper schemes, Fourier-Mukai functors can be defined also between $\mathfrak{Perf}(X)$ and $\mathfrak{Perf}(Y)$, the subcategory of $D(\mathrm{Qcoh}(X))$ and $D(\mathrm{Qcoh}(Y))$ consisting of perfect complexes. Denote by $\mathfrak{Perf}^{dg}(X)$ and $\mathfrak{Perf}^{dg}(Y)$ two dg enhancements of $\mathfrak{Perf}(X)$ and $\mathfrak{Perf}(Y)$, repsectively. In this thesis we are going to prove the following

Theorem (see Theorem 4.4.1)**.** *Let X and Y be two smooth proper schemes over a field. Then there exists a bijective map*

$$\gamma : \mathrm{Iso}(\mathfrak{Perf}(X \times Y)) \xrightarrow{1:1} \mathbf{Hqe}(\mathfrak{Perf}^{dg}(X), \mathfrak{Perf}^{dg}(Y))$$

compatible with Fourier-Mukai kernels; i.e. such that, for any $\mathcal{E} \in \mathfrak{Perf}(X \times Y)$, we have $\mathrm{H}^0(\gamma(\mathcal{E})) \simeq \Phi_{\mathcal{E}}$.

Our proof heavily relies on the explicit computation of a dg enhancement of the Fourier-Mukai functor $\Phi_{\mathcal{E}}$, that will take a consistent part of our work. This fact allows our bijection γ to be very explicit as well; therefore it gives an easy way of exhibiting the kernel $\mathcal{E}_{\mathsf{F}} \in \mathfrak{Perf}(X \times Y)$ of (the analogous version of) question (♣).

During the proof of this result we will highlight how a lot of intermediate steps can be proved in a wider generality (without the assumption of being over a field) and how we believe our strategy could lead to a full proof of Theorem 4.4.1 under these weaker hypoteses.

The present thesis is divided in four chapters. The first two chapters are on the theoretical background, the third is devoted to the construction of an explicit dg enhancement of $\Phi_{\mathcal{E}}$ while the last one deals with all the steps for the proof of Theorem 4.4.1.

The first chapter aims to recall some (perhaps well-known) key concepts; anyway, some knowledge about the basics of category theory is assumed. We start by surveying the general theory of triangulated categories, derived categories and derived functors. After that, we move to the geometric context in order to describe the incarnations of those concepts that will be used throughout this work. We end with a discussion on Fourier-Mukai functors and their features.

In the second chapter we introduce the world of differential graded categories and we study its most important concepts like dg modules and pretriangulated dg categories. We spend some time on the *extensions of dg functors* and on the properties of the localized category **Hqe**. We define the way of lifting triangulated categories and exact functors at the dg level and we spend some words on the problems concerning *uniqueness* of those lifts.

The third chapter starts with a discussion on the hypoteses under which our results will be proved. We then move to the study of a particular type of Čech enhancement; it will be used for producing explicit lifts of the three (derived) functors whose composition yields Fourier-Mukai functors: pullback, pushforward and tensor product.

With those dg lifts, in the fourth chapter we will produce a proof of Theorem 4.4.1. In order to do that, we need to properly define the kernel $\mathcal{E}_{\mathsf{F}}$ and to deal with several intermediate steps. We then conclude with some comments and by proving some results towards a further generalisation of what we have done.

Chapter 1

Derived categories of geometric flavour

In this first chapter we are going to introduce the main characters of the present work: Fourier-Mukai functors. They are a particular class of derived functors between the derived categories of sheaves of modules on a scheme (and many of their "relatives"). For this reason we need to introduce the concepts of derived category of an abelian category and of exact functor between them. We will focus, anyway, on the geometric incarnations of those concepts and we will study particular examples of derived functors that turns out to be very important for our purposes. We begin by presenting the axioms of triangulated categories, since they are the theoretical structure that derived categories naturally possess. Most of the basic facts about category theory are assumed to be well-known to the reader.

1.1 Triangulated and derived categories

We start this Section by briefly recalling the notion of triangulated category and we continue by highlighting the main steps in the construction of the derived category of an abelian category, which gives the main (and the most important) example of triangulated category. Essentially, one can think about triangulated categories as the correct framework for the study of derived categories.

Definition 1.1.1. Let $\mathcal{D}$ be an additive category endowed with an additive equivalence $[1]_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$. We will denote it just by $[1]$ when no confusion can occur and we will write, for any object A of $\mathcal{D}$ and for any arrow f of $\mathcal{D}$, $A[1]$ instead of $[1](A)$ and $f[1]$ instead of $[1](f)$. A *triangle* in $\mathcal{D}$ is the datum of three objects A, B and C of $\mathcal{D}$ and three arrows:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1].$$

A *morphism of triangles* in $\mathcal{D}$ is a commutative diagram of the form

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

where the rows are triangles in $\mathcal{D}$.

Definition 1.1.2. An additive category $\mathcal{D}$ is called *triangulated* if it is endowed with an additive equivalence $[1] : \mathcal{D} \to \mathcal{D}$ and a set of *distinguished triangles* which satisfy the following axioms:

TR1 Any triangle of the form

$$A \xrightarrow{\mathrm{id}_A} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished;
any triangle isomorphic to a distinguished triangle is distinguished;
any morphism $f : A \to B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1].$$

C is called a *cone* of the morphism f.

TR2 The triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is a distinguished triangle.

TR3 Suppose there exists a commutative diagram of distinguished triangles with vertical arrows f and g:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

then the diagram can be completed to a morphism of triangles by a (not necessarily unique) morphism $h : C \to C'$.

TR4 This fourth and last axiom is a bit technical. It roughly states that given three morphisms $f : A \to B$, $g : B \to C$ and their composition $g \circ f : A \to C$ then the three mapping cones of each of these morphisms (of whose we know the existence - altough they are not unique - by [TR1]) can be composed into a distinguished triangle such that "everything commutes". To be precise we say that given three distinguished triangles

$$A \xrightarrow{f} B \xrightarrow{h} C' \longrightarrow A[1],$$

$$B \xrightarrow{g} C \xrightarrow{k} A' \longrightarrow B[1],$$

$$A \xrightarrow{g\circ f} C \xrightarrow{\ell} B' \longrightarrow A[1],$$

there exists a distinguished triangle

$$C' \xrightarrow{u} B' \xrightarrow{v} A' \xrightarrow{w} C'[1]$$

such that the following diagram commutes:

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{h} & C' & \longrightarrow & A[1] \\
{\scriptstyle \mathrm{id}_A}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle u}\vdots & & \downarrow{\scriptstyle \mathrm{id}_{A[1]}} \\
A & \xrightarrow{g\circ f} & C & \xrightarrow{\ell} & B' & \longrightarrow & A[1] \\
{\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}_C}\downarrow & & {\scriptstyle v}\vdots & & \downarrow{\scriptstyle f[1]} \\
B & \xrightarrow{g} & C & \xrightarrow{k} & A' & \longrightarrow & B[1] \\
{\scriptstyle h}\downarrow & & {\scriptstyle \ell}\downarrow & & {\scriptstyle \mathrm{id}_{A'}}\downarrow & & \downarrow{\scriptstyle h[1]} \\
C' & \xrightarrow{u} & B' & \xrightarrow{v} & A' & \xrightarrow{w} & C'[1]
\end{array}$$

This last axiom is generally called *octahedron axiom* because, in one and probably the most famous of its different reformulations, it can be written using the vertices of an octahedron (see for example [48]). In our exposition we have followed [23] and [34], but the interested reader can find other presentations in [17].

Remark 1.1.3. An easy consequence of the axioms is the fact that the composition of two consecutive arrows of a distinguished triangle is zero. In fact, the following commutative diagram of distinguished triangles

$$\begin{array}{ccccccc}
A & \xrightarrow{\mathrm{id}_A} & A & \longrightarrow & 0 & \longrightarrow & A[1] \\
{\scriptstyle \mathrm{id}_A}\downarrow & & {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id}_{A[1]}} \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1]
\end{array}$$

can be completed, by axiom TR3, to a morphism of (distinguished) triangles yielding $g \circ f = 0$. By making use of TR2 we can then conclude that the composition of any two consecutive arrows of a distinguished triangle is the zero morphism.

From the above remark it is easy to deduce the following

Proposition 1.1.4. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ *be a distinguished triangle in a triangulated category* $\mathcal{D}$*. Then, for any object* G *of* $\mathcal{D}$ *we have the following induced exact sequences:*

$$\mathrm{Hom}_{\mathcal{D}}(G, A) \longrightarrow \mathrm{Hom}_{D}(G, B) \longrightarrow \mathrm{Hom}_{D}(G, C)$$

$$\mathrm{Hom}_{\mathcal{D}}(C, G) \longrightarrow \mathrm{Hom}_{D}(B, G) \longrightarrow \mathrm{Hom}_{D}(A, G)$$

Furthermore, since from TR2 *we get that also the sequence*

$$\mathrm{Hom}_{D}(G, B) \longrightarrow \mathrm{Hom}_{D}(G, C) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(G, A[1])$$

is exact (and the same happens when we apply $\mathrm{Hom}(-, G)$*), what we obtain are actually long exact sequences.*

Remark 1.1.5. The axiom TR1 tells us that any morphism $f : A \longrightarrow B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1].$$

Notice that this can be done essentially in a unique way. In fact if

$$A \xrightarrow{f} B \xrightarrow{g'} C' \xrightarrow{h'} A[1]$$

is another distinguished triangle we have, by TR3, a morphism of triangles

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\ \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \mathrm{id}_B} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}_{A[1]}} \\ A & \xrightarrow{f} & B & \xrightarrow{g'} & C' & \xrightarrow{h'} & A[1] \end{array}$$

and by Proposition 1.1.4 and the Five Lemma we can conclude that α must be an isomorphism. Observe that the non-uniqueness of the filling map α yields the fact that the cone of f is unique up to a non-unique isomorphism.

Remark 1.1.6. Let us now consider any triangulated category and any non-zero object R in it. We can always write the following commutative diagram where the two rows are distinguished triangles and the non-labelled maps are the zero maps:

$$\begin{array}{ccccccc} R & \longrightarrow & 0 & \longrightarrow & R[1] & \xrightarrow{\mathrm{id}_{R[1]}} & R[1] \\ \downarrow & & \downarrow & & & & \downarrow \\ 0 & \longrightarrow & R[1] & \xrightarrow{\mathrm{id}_{R[1]}} & R[1] & \longrightarrow & 0. \end{array}$$

Actually both the identity $\mathrm{id}_{R[1]} : R[1] \longrightarrow R[1]$ and the zero map $0 : R[1] \longrightarrow R[1]$ can be used to complete the above diagram to a morphism of distinguished triagles. This is an example of the *non-functoriality of the cone* that is one of the main problematic features of triangulated categories.

Definition 1.1.7. An additive functor $\mathsf{F} : \mathcal{D} \longrightarrow \mathcal{D}'$ between two triangulated categories is called *exact* (or *triangulated*) if it satisfies the following conditions:

i) There exists a natural isomorphism

$$\mathsf{F} \circ [1]_{\mathcal{D}} \xrightarrow{\sim} [1]_{\mathcal{D}'} \circ \mathsf{F}.$$

ii) Any distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

in $\mathcal{D}$ is mapped to a distinguished triangle

$$\mathsf{F}(A) \longrightarrow \mathsf{F}(B) \longrightarrow \mathsf{F}(C) \longrightarrow \mathsf{F}(A)[1]$$

in $\mathcal{D}'$, where $\mathsf{F}(A[1])$ is identified with $\mathsf{F}(A)[1]$ via the functor isomorphism in i).

Definition 1.1.8. A *triangulated subcategory* of the triangulated category $\mathcal{D}$ is an additive subcategory $\mathcal{D}' \subseteq \mathcal{D}$ admitting a triangulated structure such that the inclusion $\mathcal{D}' \hookrightarrow \mathcal{D}$ is an exact functor.

Let us now just spend a couple of words in order to introduce the important notion of compact object of a triangulated category and its connection with the different notions of generation.

Definition 1.1.9. An object C of a triangulated category $\mathcal{D}$ with arbitrary coproducts is called *compact* if, for any family $\{E_i\}$ of object of $\mathcal{D}$ the canonical map

$$\oplus_i \mathrm{Hom}_{\mathcal{D}}(C, E_i) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(C, \oplus_i E_i)$$

is an isomorphism. We denote by $\mathcal{D}^c$ the set of compact objects of $\mathcal{D}$.

From now until the end of this section let us denote by $\beta = \{B_i\}_{i \in I}$ a set of objects of a triangulated category $\mathcal{D}$.

Definition 1.1.10. We say that β *classically generates* $\mathcal{D}$ if the smallest full triangulated subcategory of $\mathcal{D}$ containing β and closed under isomorphisms and direct summands is equal to $\mathcal{D}$ itself.

By the right orthogonal $\beta^{\perp}$ in $\mathcal{D}$ we denote the full subcategory of $\mathcal{D}$ whose objects A have the property that $\mathrm{Hom}(B_i[n], A) = 0$ for all $i \in I$ and for all $n \in \mathbb{Z}$. $\beta^{\perp}$ is closed under isomorphisms and direct summands.

Definition 1.1.11. We say that β *generates* $\mathcal{D}$ if $\beta^{\perp} = 0$. Suppose moreover that $\mathcal{D}$ has arbitrary coproducts: we say that $\mathcal{D}$ is *compactly generated* if it is generated by its compact objects.

Clearly if β classically generates $\mathcal{D}$ then it generates $\mathcal{D}$, but the converse is false. Moreover we have the following

Theorem 1.1.12 (see Theorem 2.1.2 in [5])**.** *Assume that the triangulated category $\mathcal{D}$ is compactly generated. Then a set of objects $\beta \subset \mathcal{D}^c$ classically generates $\mathcal{D}^c$ if and only if it generates $\mathcal{D}$.*

Triangulated categories can be seen as a kind of generalisation of abelian categories, where distinguished triangles play the role of short exact sequences. However, it has been clear from the very beginning that triangulated categries are in some sense "badly behaved": in particular the non-uniqueness of the filling in TR3 is the cause of many problems, like the so called *non-functoriality of the cone* of a morphism as we have seen in Remark 1.1.6. Moreover, if $\mathcal{D}$ and $\mathcal{D}'$ are two triangulated categories there is no natural way of endowing the category of exact functors between them with a triangulated structure. We will see in the next Chapter how this and other kind of problems can be overcome by employing differential graded categories.

It is now the time to introduce derived categories and see how they naturally fit into the framework of triangulated categories.

Starting from a preadditive category $\mathcal{A}$ one can construct the category $\mathrm{Kom}(\mathcal{A})$ whose objects are complexes of objects of $\mathcal{A}$ and whose morphisms are complex morphisms: degreewise morphisms commuting with the differential. It is still a preadditive category (it is abelian if we start from an abelian category). The idea of derived categories is that we want an object in which we can identify complexes with the same cohomology (i.e. that are *quasi-isomorphic*). On the other hand, we want such an object to contain "homotopical information": for this purpose we need to remember more of a complex than just its cohomology. For a more detailed discussion, the reader may look at [44].

One can define the derived category of the abelian category $\mathcal{A}$ in terms of universal properties:

Definition 1.1.13. For any abelian category $\mathcal{A}$ the *derived category of* $\mathcal{A}$ is a category $\mathrm{D}(\mathcal{A})$ such that there exists a functor

$$\mathsf{Q} : \mathrm{Kom}(\mathcal{A}) \longrightarrow \mathrm{D}(\mathcal{A})$$

sending any quasi-isomorphism of $\mathrm{Kom}(\mathcal{A})$ to an isomorphism of $\mathrm{D}(\mathcal{A})$. Moreover Q has to be such that for any functor $\mathsf{F} : \mathrm{Kom}(\mathcal{A}) \longrightarrow \mathcal{D}$ sending quasi-isomorphisms to isomorphisms there exists a unique functor $\mathsf{F}' : \mathrm{D}(\mathcal{A}) \longrightarrow \mathcal{D}$ making the following diagram commute

$$\begin{array}{ccc} \mathrm{Kom}(\mathcal{A}) & \xrightarrow{\quad \mathsf{Q} \quad} & \mathrm{D}(\mathcal{A}) \\ & {\scriptstyle \mathsf{F}}\searrow \qquad \swarrow {\scriptstyle \mathsf{F}'} & \\ & \mathcal{D} & \end{array}$$

This definition is a purely categorical one. In order to explicitly define $\mathrm{D}(\mathcal{A})$, several steps have to be made. Our interest here is just to spell out how objects and arrows of $\mathrm{D}(\mathcal{A})$ look like.

For what concerns the objects of $\mathrm{D}(\mathcal{A})$ that's the easy part: they are just the same as $\mathrm{Kom}(\mathcal{A})$. The definition of morphisms is more subtle. First of all, one can define $\mathcal{K}(\mathcal{A})$, the homotopy category of $\mathcal{A}$ taking as objects the same as $\mathrm{Kom}(\mathcal{A})$ and as morphisms *morphisms of complexes modulo homotopy*. Actually the definition of $\mathrm{Kom}(\mathcal{A})$ makes sense whenever $\mathcal{A}$ is a preadditive category. Since homotopy equivalent morphisms have the same cohomology the notion of quasi-isomorphism is still well defined for morphisms in $\mathcal{K}(\mathcal{A})$. If $A^\bullet$ and $B^\bullet$ are two objects of $\mathrm{D}(\mathcal{A})$ the set of morphisms between them in $\mathrm{D}(\mathcal{A})$ is the set of all equivalence classes of "roofs" of the form

$$\begin{array}{ccc} & C^\bullet & \\ \swarrow & & \searrow \\ A^\bullet & & B^\bullet \end{array}$$

where the dotted arrow is a quasi-isomorphism. Two of such roofs are equivalent if the are dominated by a third roof in the homotopy category $\mathcal{K}(\mathcal{A})$. A precise statement of this, together with a careful discussion on composition of morphism and in general on the construction of $\mathrm{D}(\mathcal{A})$ can be found in Chapter 2 of [21].

Remark 1.1.14. Actually the procedure of formally inverting a class of arrows in a category $\mathcal{C}$ is a standard one called *localization*. It is well known that localizing a category can produce something that is not a category anymore in the sense that we may loose control on the arrows between two objects: they may not be collected into a set. One can avoid all those troubles if $\mathcal{C}$ possesses the structure of a *model category*.

Roughly speaking, model categories are categories with three classes of arrows (called *fibrations*, *cofibrations* and *weak equivalences*) satisfying certain properties. We will not write down all the axioms of a model category (the interested reader can find all the details in [20]); we will content ourselves by saying that if $\mathcal{C}$ has the structure of a model category with quasi equivalences W, then the localization $\mathcal{C}[W]^{-1}$ of $\mathcal{C}$ with repect to W is still a category and can be described in a nice way.
What happens is that (at least if $\mathcal{A}$ is a Groethendieck abelian category) one can endow $\mathrm{Kom}(\mathcal{A})$ with a model structure whose set of weak equivalences is the set W of quasi-isomorphisms.

The categories $\mathcal{K}(\mathcal{A})$ and $\mathrm{D}(\mathcal{A})$ are not abelian anymore; indeed they possess a triangulated structure.

Proposition 1.1.15. *Let $\mathcal{A}$ be an abelian category. Both the categories $\mathcal{K}(\mathcal{A})$ and $\mathrm{D}(\mathcal{A})$ are triangulated categories whose distinguished triangles are the ones isomorphic in $\mathcal{K}(\mathcal{A})$ (respectively in $\mathrm{D}(\mathcal{A})$) to triangles of the form*

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow C(f) \longrightarrow A^\bullet[1]$$

where f is a morphism of complexes and $C(f)$ is its mapping cone and $[1]$ is the usual shift of complexes.

Besides the notion of unbounded derived category $\mathrm{D}(\mathcal{A})$ we will denote by $\mathrm{D}^?(\mathcal{A})$ with $? = b, +, -$ the bounded (resp. bounded below, bounded above) derived category that can be defined by suitably substitute $\mathrm{Kom}^?(\mathcal{A})$ and $\mathcal{K}^?(\mathcal{A})$ to $\mathrm{Kom}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ (see Proposition 2.30 of [21]).

1.2 Derived functors

We start now a discussion on derived functors. Let $\mathsf{F} : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. It gives rise to a well defined functor $\mathrm{Kom}(\mathsf{F}) : \mathrm{Kom}(\mathcal{A}) \longrightarrow \mathrm{Kom}(\mathcal{B})$ as well as $\mathcal{K}(\mathsf{F}) : \mathcal{K}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{B})$. But it is easy to see that $\mathcal{K}(\mathsf{F})$ induces a functor between derived categories if and only if it maps acyclic complexes to acyclic complexes; in other words, when F is an *exact* functor.

But most of the times, the functors that one encounters are only left or right exact. In those cases, in order to end up with a functor between the derived categories, a more sophisticated procedure has to be adopted.

Before introducing it, we need to recall some important definition. Let $\mathcal{A}$ and be an abelian category.

Definition 1.2.1. A complex $I^\bullet$ of objects of $\mathcal{A}$ is called *h-injective* if for any acyclic complex $M^\bullet$ of objects of $\mathcal{A}$ we have that $\mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(M^\bullet, I^\bullet)$ is acyclic.

Definition 1.2.2. An *h-injective resolution* of an object $C^\bullet \in \mathrm{Kom}(\mathcal{A})$ is a quasi-isomorphism $C^\bullet \longrightarrow I^\bullet$ with $I^\bullet$ an h-injective complex of objects of $\mathcal{A}$.

Dually, we have the notion of h-projective complex and h-projective resolution.

Definition 1.2.3. A complex $P^\bullet$ of objects of $\mathcal{A}$ is called *h-projective* if for any acyclic complex $M^\bullet$ of objects of $\mathcal{A}$ we have $\mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(P^\bullet, M^\bullet)$ is acyclic.

Definition 1.2.4. An *h-projective resolution* of an object $C^\bullet \in \mathrm{Kom}(\mathcal{A})$ is a quasi-isomorphism $P^\bullet \longrightarrow C^\bullet$ with $P^\bullet$ an h-projective complex of objects of $\mathcal{A}$.

We will see later that also the notion of h-flat resolution is important for defining the derived functor we are interested in.

Definition 1.2.5. A complex $K^\bullet$ of objects of $\mathcal{A}$ is called *h-flat* if for any acyclic complex $M^\bullet$ of objects of $\mathcal{A}$ we have that $M^\bullet \otimes K^\bullet$ is acyclic.

Definition 1.2.6. An *h-flat resolution* of an object $C^\bullet \in \mathrm{Kom}(\mathcal{A})$ is a quasi-isomorphism $K^\bullet \longrightarrow C^\bullet$ with $C^\bullet$ an h-flat complex of objects of $\mathcal{A}$.

We can now come back to the original problem; suppose for example that $\mathsf{F} : \mathcal{A} \longrightarrow \mathcal{B}$ is a left exact functor. If there exists a subcategory $\mathcal{I}_\mathsf{F} \subset \mathcal{K}(\mathcal{A})$ such that:

(i) $\mathcal{K}(\mathsf{F})$ sends any acyclic complex of $\mathcal{I}_\mathsf{F}$ to an acyclic complex of $\mathcal{K}(\mathcal{B})$;

(ii) any object in $\mathcal{K}(\mathcal{A})$ is quasi-isomorphic to an object of $\mathcal{I}_\mathsf{F}$.

then the right derived functor $R\mathsf{F} : \mathrm{D}(\mathcal{A}) \longrightarrow \mathrm{D}(\mathcal{B})$ can be defined. In fact, condition (ii) gives us that the localization of $\mathcal{I}_\mathsf{F}$ by quasi-isomorphisms between complexes with objects in $\mathcal{I}_\mathsf{F}$ is equivalent to $\mathrm{D}(\mathcal{A})$ (let us call such equivalence ι). Moreover, condition (i) ensures that $\mathcal{K}(\mathsf{F})$ can be defined on this localization and hence one can define the *right derived functor of* F to be $R\mathsf{F} := \bar{\mathsf{Q}}_\mathcal{B} \circ \mathcal{K}(\mathsf{F}) \circ \iota^{-1}$, where $\bar{\mathsf{Q}}_\mathcal{B} : \mathcal{K}(\mathcal{B}) \longrightarrow \mathrm{D}(\mathcal{B})$ is the natural functor which factorizes the one of Definition 1.1.13.

Remark 1.2.7. Observe that if every complex of objects of $\mathcal{A}$ has an h-injective resolution, then the class of h-injective objects can play the role of $\mathcal{I}_\mathsf{F}$ for any left exact functor F (see [28], Corollary (2.3.2.3)). This happens for example whenever $\mathcal{A}$ is a Grothendieck category - see Section 1.3 below - by Theorem 5.4 of [1].

Clearly, if F is a right exact functor, the *left derived functor of* F can be defined in an analogous way. In this case, the class of h-projective objects can play the role of $\mathcal{I}_\mathsf{F}$. Unfortunately, it is not true that any complex of objects of a Grothendieck category admits an h-projective resolution. But often other kind of resolutions, like h-flat ones, can be employed.

1.3 Derived categories and derived functors in the geometric context

Let us now move to the geometric situation. In all this section we will assume all schemes to be quasi-compact and quasi-separated over a commutative ring $\mathbb{k}$.

Given any scheme X we can consider the category $\mathrm{Sh}(X)$ of sheaves of $\mathcal{O}_X$-modules and its subactegories $\mathrm{Qcoh}(X)$ and $\mathrm{Coh}(X)$ of quasi-coherent (respectively, coherent) sheaves. All those three categories are abelian and we can consider their (bounded or unbounded) derived categories. Moreover we have that $\mathrm{Sh}(X)$ and $\mathrm{Qcoh}(X)$ are *Grothendieck categories* meaning that they are abelian categories which are closed under small coproducts, possess a small set of generators and are such that the direct limits of short exact sequences are exact.

In this thesis we will be primarily interested in the full subcategory $\mathfrak{Perf}(X)$ of $D(\mathrm{Sh}(X))$ consisting of perfect complexes i.e. complexes that are locally quasi-isomorphic to a bounded complex of vector bundles. Under our hypoteses $\mathfrak{Perf}(X)$ coincides with the full subcategory of compact objects of $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$ (see Theorem 3.1.1 of [5]). By $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$ we mean the full subcategory of $D(\mathrm{Sh}(X))$ consisting of complexes with quasi-coherent cohomologies.

When the scheme X is also separated, the natural functor $D(\mathrm{Qcoh}(X)) \longrightarrow D(\mathrm{Sh}(X))$ induces an equivalence:

$$D(\mathrm{Qcoh}(X)) \simeq D_{\mathrm{Qcoh}}(\mathrm{Sh}(X)). \tag{1.1}$$

This equivalence restricts to an equivalence between $\mathfrak{Perf}(X)$ and the full subcategory of $D(\mathrm{Qcoh}(X))$ consisting of complexes that are locally quasi-isomorphic to a bounded complex of vector bundles.

Let us now take some time to discuss the derived functors that will be needed in the present work.

Pushforward. Let $f : X \longrightarrow Y$ be a morphism of schemes. It yields functors:

$$f^* : \mathrm{Sh}(Y) \longrightarrow \mathrm{Sh}(X) \qquad\qquad f_* : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(Y)$$

with f^* right exact and f_* left exact since we have the adjuction $f^* \dashv f_*$. According to what we have said in the above section (see Remark 1.2.7), the derived functor

$$Rf_* : \mathrm{D}(\mathrm{Sh}(X)) \longrightarrow \mathrm{D}(\mathrm{Sh}(Y))$$

exists and we have that $Rf_*(D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))) \subseteq D_{\mathrm{Qcoh}}(\mathrm{Sh}(Y))$, by Section 3.9 of [28]. Such a functor, in the case f is flat and proper and Y is Noetherian, restricts - from [42, Tag 0B6F] - to a functor

$$Rf_* : \mathfrak{Perf}(X) \longrightarrow \mathfrak{Perf}(Y).$$

Pullback. For any morphism $f : X \longrightarrow Y$ we have that, by means of *h-flat resolutions* (*k-flat* in the terminology of [41], *q-flat* in the one of [28]), we can define the left derived functor:

$$Lf^* : D(\mathrm{Sh}(Y)) \longrightarrow D(\mathrm{Sh}(X))$$

and - always by the results in Section 3.9 of [28] - it is such that $Lf^*(D_{\mathrm{Qcoh}}(\mathrm{Sh}(Y))) \subseteq D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$. Moreover, it restricts to

$$Lf^* : \mathfrak{Perf}(Y) \longrightarrow \mathfrak{Perf}(X)$$

by [42, Tag 09UA].

Tensor product. Let $\mathcal{E} \in \mathrm{Sh}(X)$. Tensorization by $\mathcal{E}$ defines a right exact functor

$$\mathcal{E} \otimes - : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(X). \tag{1.2}$$

Following Section 2.5 of [28] we can define, by means of *h-flat resolutions*, the left derived functor

$$\mathcal{E} \otimes^L - : D(\mathrm{Sh}(X)) \longrightarrow D(\mathrm{Sh}(X)).$$

When $\mathcal{E}$ is an element of $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$ the functor (1.2) restricts to a functor

$$\mathcal{E} \otimes^L - : D_{\mathrm{Qcoh}}(\mathrm{Sh}(X)) \longrightarrow D_{\mathrm{Qcoh}}(\mathrm{Sh}(X)).$$

Moreover, since the tensor product of bounded complexes of vector bundles is still a bonded complex of vector bundles we obtain a functor

$$\mathcal{E} \otimes^L - : \mathfrak{Perf}(X) \longrightarrow \mathfrak{Perf}(X).$$

every time we choose $\mathcal{E} \in \mathfrak{Perf}(X)$.

Boxtimes. Let us denote by

$$X \xleftarrow{p} X \times Y \xrightarrow{q} Y$$

the two projection. The *boxtimes* (bi)functor (called also *external tensor product*) is defined as

$$- \boxtimes \sim := p^*(-) \otimes q^*(\sim) : \mathrm{Sh}(X) \times \mathrm{Sh}(Y) \longrightarrow \mathrm{Sh}(X \times Y).$$

Thanks to the discusion above and to the fact that the pullback of h-flat complexes is a h-flat complex as well (see [42, Tag 06YC]), we can compute by means of h-flat resolutions the derived functor

$$- \boxtimes^L \sim := Lp^*(-) \otimes^L Lq^*(\sim) : D(\mathrm{Sh}(X)) \times D(\mathrm{Sh}(Y)) \longrightarrow D(\mathrm{Sh}(X \times Y))$$

that restricts to the derived category of complexes of sheaves of modules with quasi-coherent cohomology, and also to

$$- \boxtimes^L \sim : \mathfrak{Perf}(X) \times \mathfrak{Perf}(Y) \longrightarrow \mathfrak{Perf}(X \times Y).$$

Observe that if our schemes are smooth or if $\Bbbk$ is a field then the two projections are exact and hence we have $Lp^* = p^*$ and $Lq^* = q^*$. Suppose moreover that the schemes are *globally strictly perfect* i.e. such that any perfect complex on them is quasi-isomorphic to a bounded complex of vector bundles; in Chapters 3 and 4 we will always be under such hypotesis. We get then that even the tensor product is exact since bounded complexes of vector bundles are h-flat. Therefore we can write $\boxtimes^L = \boxtimes$ and we get the exact functor:

$$- \boxtimes \sim : \mathfrak{Perf}(X) \times \mathfrak{Perf}(Y) \longrightarrow \mathfrak{Perf}(X \times Y).$$

Notice that when $\Bbbk$ is a field - without any further hypoteses - the exactness of $\boxtimes$ already comes from [31] Lemma A.14.

We conclude the present section with three useful formulas, in the version that will be used in the sequel.

Proposition 1.3.1 (Projection formula, see [29] Proposition 5.2.32 and [47] Exercise 16.3.H)**.** *Let $f : X \longrightarrow Y$ be a quasi-compact and separated morphism of schemes and let $\mathcal{F} \in \mathrm{Qcoh}(X)$, $\mathcal{G} \in \mathrm{Qcoh}(Y)$. We have a canonical homomorphism*

$$(f_*\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G} \longrightarrow f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$$

that is an isomorphism if $\mathcal{G}$ is flat or f is affine.

Remark 1.3.2. Notice that, in virtue of the equivalence (1.1), the above proposition still holds for $\mathcal{F} \in D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$ and $\mathcal{G} \in D_{\mathrm{Qcoh}}(\mathrm{Sh}(Y))$.

Proposition 1.3.3 (Base change, see [31] Lemma A.1)**.** *Let $f : Y \longrightarrow X$ be a morphism of ringed spaces. Let $U \subset X$ be an open subset and $V := f^{-1}(U)$. Consider them as ringed spaces in the natural way: we have a cartesian diagram of ringed spaces*

$$\begin{array}{ccc} V & \xrightarrow{f'} & U \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \iota} \\ Y & \xrightarrow{f} & X. \end{array}$$

Then, for any $\mathcal{G} \in \mathrm{Sh}(Y)$ there is a natural isomorphism of $\mathcal{O}_U$-modules $\iota^* f_* \mathcal{G} \simeq f'_* j^* \mathcal{G}$.

Another version of base change (see [42, Tag 02KH]) is given by the following

Proposition 1.3.4 (Flat base change)**.** *Consider a cartesian diagram of schemes*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X. \end{array}$$

and suppose that f is flat and g is quasi-compact and quasi-separated. Then, for any $\mathcal{F} \in \mathrm{Qcoh}(Y)$ and for any $i \geq 0$ there is a natural isomorphism $g^* R^i f_* \mathcal{F} \simeq R^i f'_* g'^* \mathcal{F}$.

1.4 Fourier-Mukai functors

In this section we are going to introduce a really important class of exact functors between derived categories and we will see how this importance is justified.

Let X and Y be again two quasi compact and separated schemes over $\Bbbk$, which is supposed to be a commutative ring. We still denote by

$$X \xleftarrow{p} X \times Y \xrightarrow{q} Y$$

the two projections.

Definition 1.4.1. Let $\mathcal{E}$ be any object of $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X \times Y))$. The *Fourier-Mukai functor of kernel* $\mathcal{E}$ is the functor $\Phi_{\mathcal{E}} : D_{\mathrm{Qcoh}}(\mathrm{Sh}(X)) \longrightarrow D_{\mathrm{Qcoh}}(\mathrm{Sh}(Y))$ defined by the assignation

$$A \longmapsto Rq_*(\mathcal{E} \otimes^L Lp^* A)$$

Notice that under our hypoteses the definition is well-posed thanks to the discussion in the previous section. If moreover the schemes are flat, proper and Noetherian then - if $\mathcal{E}$ is taken in $\mathfrak{Perf}(X \times Y)$ - we can define the Fourier-Mukai functor

$$\Phi_{\mathcal{E}} : \mathfrak{Perf}(X) \longrightarrow \mathfrak{Perf}(Y).$$

Remark 1.4.2. It turns out that Fourier-Mukai functors have a lot of interesting properties (see for example [21], Chapter 5 for the case of smooth projective schemes over a field):

- they are always exact functors: this comes just by the definition;

- if one of the schemes is flat then the composition of two Fourier-Mukai functors is again (isomorphic to) a Fourier-Mukai functor;
- in the case of smooth and proper schemes over a field, Fourier-Mukai functors $\Phi_{\mathcal{E}} : \mathfrak{Perf}(X) \longrightarrow \mathfrak{Perf}(Y)$ always possess both left and right adjoint, which are still Fourier-Mukai functors.

Another property of Fourier-Mukai functors is the one given by the following Lemma. It follows from standard computations as highlighted, for example, in Section 4.3 of [9] but for sake of completness we present an outline of the proof. Consider the diagonal embedding $\Delta : X \longrightarrow X \times X$ and denote by $\mathcal{O}_\Delta$ the structure sheaf of its image in $X \times X$. Consider moreover the two projections

$$\begin{array}{ccc} & X \times X \times X \times Y & \\ \swarrow^{p_{12}} & & \searrow^{p_{34}} \\ X \times X & & X \times Y \end{array}$$

and define

$$- \hat{\boxtimes} \sim := p_{12}^*(-) \otimes p_{34}^*(\sim) : \mathrm{Sh}(X \times X) \times \mathrm{Sh}(X \times Y) \longrightarrow \mathrm{Sh}(X \times X \times X \times Y).$$

In complete analogy with the boxtimes functor we defined in Section 1.3, it can be derived giving rise to a functor between the derived categories. Moreover it restricts to perfect complexes and — if we are dealing with smooth and globally strictly perfect schemes — it coincides with the exact functor

$$- \hat{\boxtimes} \sim := p_{12}^*(-) \otimes p_{34}^*(\sim) : \mathfrak{Perf}(X \times X) \times \mathfrak{Perf}(X \times Y) \longrightarrow \mathfrak{Perf}(X \times X \times X \times Y).$$

Lemma 1.4.3. *In the hypoteses that all our schemes are smooth and globally strictly perfect we have, for any $\mathcal{E} \in \mathfrak{Perf}(X \times Y)$, the isomorphism $\Phi_{\mathcal{O}_\Delta \hat{\boxtimes} \mathcal{E}}(\mathcal{O}_\Delta) \cong \mathcal{E}$.*

Proof. In order to simplify the notation we will write the product of schemes instead of the product of the category of perfect complexes on them.

For any i, j and z in $\{1,2,3,4\}$ let us denote by p_{ijz} the projection from $X \times X \times X \times Y$ onto the i-th, j-th and z-th terms and by p_{ij} the projection from $X \times X \times X \times Y$ onto the i-th and the j-th terms. For any h and k in $\{1,2,3\}$, denote by $\tilde{p}_{hk}$ the projection from $X \times X \times Y$ onto the h-th and k-th terms and by $\tilde{p}_k$ the projection from $X \times X \times Y$ onto the k-th term.

Let us moreover define

$$\Delta^{12} \times \mathrm{id}_{X \times Y} : X \times X \times Y \longrightarrow X \times X \times X \times Y \qquad (x, x', y) \longmapsto (x, x, x', y)$$

and

$$\Delta^{13} \times \mathrm{id}_{X \times Y} : X \times X \times Y \longrightarrow X \times X \times X \times Y \qquad (x, x', y) \longmapsto (x, x', x, y)$$

We can finally write:

$$
\begin{aligned}
&\Phi_{\mathcal{O}_\Delta \hat{\boxtimes} \mathcal{E}}(\mathcal{O}_\Delta) = p_{24*}(\mathcal{O}_\Delta \hat{\boxtimes} \mathcal{E} \otimes p_{13}^*\mathcal{O}_\Delta) = p_{24*}(p_{12}^*\mathcal{O}_\Delta \otimes p_{34}^*\mathcal{E} \otimes p_{13}^*\mathcal{O}_\Delta) \\
&= p_{24*}(p_{12}^*\Delta_*\mathcal{O}_X \otimes p_{34}^*\mathcal{E} \otimes p_{13}^*\Delta_*\mathcal{O}_X) \\
&\overset{(a)}{\simeq} (\tilde{p}_{23} \circ p_{124})_*(p_{12}^*\Delta_*\mathcal{O}_X \otimes p_{34}^*\mathcal{E} \otimes (\Delta^{13} \times \mathrm{id}_{X \times Y})_* \tilde{p}_1^*\mathcal{O}_X) \\
&\overset{(b)}{\simeq} \tilde{p}_{23*} \circ p_{124*} \circ (\Delta^{13} \times \mathrm{id}_{X \times Y})_* \Big((\Delta^{13} \times \mathrm{id}_{X \times Y})^* ((p_{12}^*\Delta_*\mathcal{O}_X \otimes p_{34}^*\mathcal{E}) \otimes \tilde{p}_1^*\mathcal{O}_X\Big) \\
&\overset{(c)}{\simeq} \tilde{p}_{23*}(\Delta^{13} \times \mathrm{id}_{X \times Y})^* \Big(p_{12}^*\Delta_*\mathcal{O}_X \otimes p_{34}^*\mathcal{E}\Big) \\
&\overset{(d)}{\simeq} \tilde{p}_{23*}\Big((\Delta^{13} \times \mathrm{id}_{X \times Y})^* p_{12}^*\Delta_*\mathcal{O}_X \otimes (\Delta^{13} \times \mathrm{id}_{X \times Y})^* p_{34}^*\mathcal{E}\Big) \\
&\overset{(e)}{\simeq} \tilde{p}_{23*}\Big(\tilde{p}_{12}^*\Delta_*\mathcal{O}_X \otimes (\Delta^{13} \times \mathrm{id}_{X \times Y})^* p_{34}^*\mathcal{E}\Big) \\
&\overset{(f)}{\simeq} \tilde{p}_{23*}\Big((\Delta \times \mathrm{id}_Y)_* p^*\mathcal{O}_X \otimes (\Delta^{13} \times \mathrm{id}_{X \times Y})^* p_{34}^*\mathcal{E}\Big) \\
&\overset{(g)}{\simeq} \tilde{p}_{23*}(\Delta \times \mathrm{id}_Y)_* \Big(p^*\mathcal{O}_X \otimes (\Delta \times \mathrm{id}_Y)^* (\Delta^{13} \times \mathrm{id}_{X \times Y})^* p_{34}^*\mathcal{E}\Big) \overset{(h)}{\simeq} \mathcal{E}.
\end{aligned}
$$

where we have used:

- in (a), in addition to the trivial equality $p_{24} = \tilde{p}_{23} \circ p_{124}$, the flat base change (Proposition 1.3.4) for the diagram

$$
\begin{array}{ccc}
X \times X \times Y & \xrightarrow{\Delta^{13} \times \mathrm{id}_{X \times Y}} & X \times X \times X \times Y \\
\big\downarrow{\scriptstyle \tilde{p}_1} & & \big\downarrow{\scriptstyle p_{13}} \\
X & \xrightarrow{\quad \Delta \quad} & X \times X;
\end{array}
$$

- in (b) the projection formula (Proposition 1.3.1);
- in (c) the equality $p_{124} \circ (\Delta^{13} \times \mathrm{id}_{X \times Y}) = \mathrm{id}_{X \times X \times Y}$ and the isomorphism $\tilde{p}_1^*\mathcal{O}_X \simeq \mathcal{O}_{X \times X \times Y}$;
- in (d) the commutativity between tensor product and inverse image;
- in (e) the equality $p_{12} \circ (\Delta^{13} \times \mathrm{id}_{X \times Y}) = \tilde{p}_{12}$;
- in (f) the flat base change (Proposition 1.3.4) with respect to the diagram

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\Delta \times \mathrm{id}_Y} & X \times X \times Y \\
\big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \tilde{p}_{12}} \\
X & \xrightarrow{\quad \Delta \quad} & X \times Y;
\end{array}
$$

- in (g) the projection formula (Proposition 1.3.1);
- in (h) the equalities $\tilde{p}_{23} \circ (\Delta \times \mathrm{id}_Y) = \mathrm{id}_{X \times Y} = p_{34} \circ (\Delta^{13} \times \mathrm{id}_{X \times Y}) \circ (\Delta \times \mathrm{id}_Y)$ and the isomorphism $p^*(\mathcal{O}_X) \simeq \mathcal{O}_{X \times Y}$.

$\square$

But probably the most notable feature of Fourier-Mukai functors - despite it could be not so obviout at a first sight - is that they are ubiquitous. For example: by using projection formula 1.3.1 and the immediate equalities $p \circ \Delta = \mathrm{id} = q \circ \Delta$ we can write - for any object $A \in D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$:

$$\begin{aligned}\Phi_{\mathcal{O}_\Delta}(A) = Rq_*(\mathcal{O}_\Delta \otimes^L p^*A) = Rq_*(\Delta_*(\mathcal{O}_X) \otimes^L p^*A) \simeq Rq_*(\Delta_*(\mathcal{O}_X \otimes^L \Delta^*p^*A)) \\ \simeq (Rq \circ \Delta)_*(\mathcal{O}_X \otimes^L (p \circ \Delta)^*A) = \mathrm{id}_*(\mathcal{O}_X \otimes^L \mathrm{id}^*A) \simeq A\end{aligned}$$

This tells us that the Fourier-Mukai functor $\Phi_{\mathcal{O}_\Delta}$ is nothing else than the identity on $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$.

Definition 1.4.4. Given an exact functor F between $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$ and $D_{\mathrm{Qcoh}}(\mathrm{Sh}(Y))$ we say that F is *of Fourier-Mukai type* if there exists an object $\mathcal{E}$ in $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X \times Y))$ such that F is isomorphic to $\Phi_{\mathcal{E}}$ of Definition 2.6.3 (we will say in this case that $\mathcal{E}$ is a *Fourier-Mukai kernel* - or simply a *kernel* - of F).

It is not hard to prove that, if we consider for example an element A of $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$, the (derived) tensor product $- \otimes^L A$ is isomorphic to $\Phi_{\Delta_* A}$. Moreover, for any morphism $f : X \longrightarrow Y$, the derived direct image functor Rf_* is of Fourier-Mukai type with kernel $\mathcal{O}_{\Gamma_f}$, where $\Gamma_f \subset X \times Y$ is the graph of f. And also the pullback Lf^* along f is of Fourier-Mukai type with kernel the same object (but swapping p and q) of $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X \times Y))$ (actually, since it is the left adjoint of Rf_*, for smooth and proper schemes over a field, this should be clear also by Remark 1.4.2).

The list of functors of Fourier-Mukai type can obviously continue, but in order to make a fundamental step in the understanding of "how many" exact functors between derived categories are of Fourier-Mukai type one must cite the following beautiful result, due to Orlov:

Theorem 1.4.5 ([36] Theorem 2.2)**.** *Let X and Y be two smooth projective varieties and let*

$$\mathsf{F} : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(Y))$$

be an exact functor. If F is fully faithful then there exists an object $\mathcal{E}$, unique up to isomorphisms, such that F is isomorphic to $\Phi_{\mathcal{E}}$.

As one can easily believe, this Theorem is one of the milestones in the field. Actually, the original statement requires F to have both left and right adjoints but this has been proven to hold for any exact functor in our situation (it is due to Theorem 1.1 of [5]; see also [11], Proposition 3.5).

Remark 1.4.6. Orlov Theorem has been generalised in various directions: in [24] a version for smooth stacks, obtained as global quotients, is provided; Canonaco and Stellari proved in [9] a generalization to the twisted setting under milder assumptions on the functor; in [30] the case of projective (possibly singular) schemes is treated; in [8] the authors showed that - under some assumptions - the faithfulness of the exact functor F can be deduced from it fullness; in [12] can be found a version in the supported case; finally, Olander proved in [35] the version for smooth and proper varieties over a field.

In fact for some time it was believed that it could hold without the assumption of fully-faithfulness. But recently (see [39], [49] or [38]) examples of exact functors between bounded derived categories of coherent sheaves that are not of Fourier-Mukai type were discovered, even in the case of smooth projective schemes over a field.

On the other hand, also the *uniqueness* of the kernel is an interesting problem in itself: there are well-known examples (even for the derived categories of elliptic curves) of non isomorphic kernels whose Fourier-Mukai functors become isomorphic (see [10]). Anyway, it has been proved that the cohomology sheaves of the kernels of two isomorphic Fourier-Mukai functors are uniquely determined up to isomorphism (see again [10], Theorem 1.2).

Those examples has shed light on the fact that the problem of understanding which exact functors are of Fourier-Mukai type is more involved and that actually "pathologies" like the existence of non-Fourier-Mukai functors are due to the bad behaviour of triangulated categories. For this reason mathematicians have moved to higher categorical structures in order to have a deeper and more complete picture of the situation.

What we are going to do now is in fact to present one of such possible generalisations: the theory of differential graded categories. With this new tool we will see, in Section 2.6, how the situation we have presented here can change if seen from an *enhanced* perspective.

Chapter 2

Differential graded categories and (quasi-) functors

In this chapter we present the theory of differential graded categories, the language that will be used throughout the present work. They are a powerful way of enhancing triangulated categories; for this reason, after presenting the basic definitions and examples, we move to speak about pretriangulated dg categories.

We need also to understand the structure and the main properties of **Hqe**, the category obtained by formally inverting quasi-equivalences in the category of (small) dg categories. In fact it turns out to be the correct ambient where many problems of the "triangulated world" can be solved.

Morphisms in **Hqe** are strongly linked to dg (bi-)modules, therefore we will spend some words on them and on the way of extending dg functors to the categories of dg modules. The final part is devoted to present the way(s) of enhancing triangulated categories and triangulated functors to the differential graded level and to discuss some related issues.

For what concerns the notation, in this chapter — for any category $\mathcal{A}$ and for any pair of objects A, A' in $\mathcal{A}$ — we will write $\mathcal{A}(A, A')$ to denote the set of arrows in $\mathcal{A}$ from A to A'.

2.1 Dg categories and dg functors

Let $\Bbbk$ denote a commutative ring. We will assume all categories to be $\Bbbk$-*linear* i.e. the Hom-space will be a $\Bbbk$-module with a $\Bbbk$-bilinear composition. All our functors will be $\Bbbk$-*linear* as well i.e. such that the induced map between the Hom-spaces is $\Bbbk$-linear.

Let us recall here the (well-known) definition of tensor product of complexes (of $\Bbbk$-modules).

Definition 2.1.1. Let $M^\bullet$ and $N^\bullet$ be two complexes of $\Bbbk$-modules. The *tensor product* $M^\bullet \otimes_\Bbbk N^\bullet$ is a complex (of $\Bbbk$-modules) such that

$$\left(M^\bullet \otimes_\Bbbk N^\bullet\right)^i := \bigoplus_{p+q=i} M^p \otimes N^q.$$

The differential is defined as follows: let $f \in M^p$, $g \in N^q$ and $i := p + q$ then

$$d^i_{M^\bullet \otimes_\Bbbk N^\bullet}(f \otimes g) := d^p_{M^\bullet}(f) \otimes g + (-1)^p f \otimes d^q_{N^\bullet}(g).$$

Definition 2.1.2. A *differential graded (dg) category* is a category $\mathcal{A}$ such that, for any couple of objects X, Y in $\mathcal{A}$, the space of morphisms $\mathcal{A}(X,Y)$ is a $\mathbb{Z}$-graded $\mathbb{k}$-module endowed with a differential d: $\mathcal{A}(X,Y) \longrightarrow \mathcal{A}(X,Y)$ of degree 1. We require also the composition maps

$$\begin{aligned} \mathcal{A}(Y,Z)\otimes_{\mathbb{k}}\mathcal{A}(X,Y) &\longrightarrow \mathcal{A}(X,Z) \\ g \otimes f &\longmapsto g \circ f \end{aligned}$$

to be morphisms of complexes for any X, Y and Z in $\mathrm{Ob}(\mathcal{A})$.

The tensor product appearing in the above definition is the ordinary tensor product of complexes of Definition 2.1.1.

Actually, we can see a dg category $\mathcal{A}$ also as a category enriched over the (closed monoidal) category $C(\mathbb{k})$ of cochain complexes of $\mathbb{k}$-modules i.e. we are requiring all the Hom-spaces of $\mathcal{A}$ to be cochain complexes of $\mathbb{k}$-modules.

From the request on the composition we get $\deg(g \circ f) = \deg(g) + \deg(f)$ for f and g homogeneous morphisms and also $d(g \circ f) = d(g) \circ f + (-1)^{\deg(g)} g \circ d(f)$. In the special case of $f = g = 1$ we can see that the identity of each object must be a closed morphism of degree 0.

Example 2.1.3. Let us present the basic examples of dg categories:

(i) Any $\mathbb{k}$-linear category $\mathcal{A}$ is a dg category, once we set, for every couple of objects X, Y in $\mathrm{Ob}(\mathcal{A})$:

$$\mathcal{A}(X,Y)^i = \begin{cases} \mathrm{Hom}_{\mathcal{A}}(X,Y) & \text{when } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

(ii) Let A be a dg algebra over $\mathbb{k}$. We can define the dg category $\mathcal{A}$ as follows:

$$\mathrm{Ob}(\mathcal{A}) = \{*\}; \qquad \mathcal{A}(*,*) = A.$$

Hence we have that any dg algebra (over $\mathbb{k}$) gives rise to a dg category with one object. Notice that we need A to be a dg algebra (and not just a dg module) in virtue of the request on the compositon map. On the other hand, every dg category with one object defines a dg algebra: the endomorphism space of the unique object.

(iii) If $\mathcal{A}$ is an ordinary category we can define a dg category $C_{dg}(\mathcal{A})$ of (cochain) complexes of $\mathcal{A}$. Its objects are complexes in $\mathcal{A}$, while morphism are defined to be

$$C_{dg}(\mathcal{A})(X^\bullet, Y^\bullet)^n := \prod_{i\in\mathbb{Z}} \mathcal{A}(X^i, Y^{n+i})$$

where $X^\bullet$ and $Y^\bullet$ are in $\mathrm{Ob}(C_{dg}(\mathcal{A}))$. Note that, since we have assumed all categories to be $\mathbb{k}$-linear each term in $\prod_{i\in\mathbb{Z}} \mathcal{A}(X^i, Y^{n+i})$ is a $\mathbb{k}$-module.

Let $f = (f^i)_{i\in\mathbb{Z}}$ be an element of $C_{dg}(\mathcal{A})(X^\bullet, Y^\bullet)^n$. We define the differential to be

$$d^n(f) = d^n((f^i)_{i\in\mathbb{Z}}) = (d^n(f)^i)_{i\in\mathbb{Z}} := (d^{i+n}_{Y^\bullet} \circ f^i - (-1)^n f^{i+1} \circ d^i_{X^\bullet})_{i\in\mathbb{Z}}.$$

In the case we consider as $\mathcal{A}$ the category of $\mathbb{k}$-modules we obtain the dg category $C_{dg}(\mathbb{k}\text{-Mod})$.

Definition 2.1.4. Given a dg category $\mathcal{A}$ we can define the *opposite dg category* $\mathcal{A}^\circ$ to be the category with the same objects as $\mathcal{A}$ and such that, for any X, Y in $\mathrm{Ob}(\mathcal{A}^\circ)$:

$$\mathcal{A}^\circ(X,Y) := \mathcal{A}(Y,X).$$

The composition $g \circ f$ in $\mathcal{A}^\circ$ - with f and g homogeneous - is given by $(-1)^{\deg(f)\deg(g)} f \circ g$ in $\mathcal{A}$.

Definition 2.1.5. Given a dg category $\mathcal{A}$ we can define the following two categories:

- The *underlying category* $Z^0(\mathcal{A})$ with the same objects as $\mathcal{A}$ and such that, for any X and Y in $\mathrm{Ob}(\mathcal{A})$

  $$Z^0(\mathcal{A})(X,Y) := Z^0(\mathcal{A}(X,Y))$$

  where by Z^0 of a complex we mean the kernel of the differential d^0.

- The *homotopy category* $H^0(\mathcal{A})$ with the same objects as $\mathcal{A}$ and such that, for any X and Y in $\mathrm{Ob}(\mathcal{A})$

  $$H^0(\mathcal{A})(X,Y) := H^0(\mathcal{A}(X,Y))$$

  where by H^0 of a complex we mean its 0-th cohomology $\mathbb{k}$-module.

Definition 2.1.6. Two objects of a dg category $\mathcal{A}$ are *dg isomorphic* if there exists an isomorphism between them in $Z^0(\mathcal{A})$ (called a *dg isomorphism*). Moreover, they are *homotopy equivalent* if there exists a morphism between them in $Z^0(\mathcal{A})$ whose image in $H^0(\mathcal{A})$ is an isomorphism (in this case it is called a *homotopy equivalence*).

Example 2.1.7. Let us consider again Example 2.1.3 (iii). We have:

$$d^0(f) = 0 \quad \Longleftrightarrow \quad d^i_{Y^\bullet} \circ f^i = f^{i+1} \circ d^i_{X^\bullet} \quad \text{for any } i \in \mathbb{Z}.$$

i.e. $f \in \ker(d^0)$ if and only if it is a morphism of complexes. On the other hand:

$$d^{-1}(g) = f \quad \Longleftrightarrow \quad f = (d^{i-1}_{Y^\bullet} \circ g^i + g^{i+1} \circ d^i_{X^\bullet})_{i\in\mathbb{Z}}$$

i.e. $f \in \mathrm{Im}(d^{-1})$ if and only if it is null-homotopic.

From what we have just said it is now clear that $Z^0(C_{dg}(\mathcal{A}))$ is the usual category $\mathrm{Kom}(\mathcal{A})$ of complexes of $\mathcal{A}$ and that $H^0(C_{dg}(\mathcal{A}))$ is the usual homotopy category $\mathcal{K}(\mathcal{A})$ of $\mathcal{A}$. Moreover observe that two complexes in $\mathcal{A}$ are homotopy equivalent in the sense of Definition 2.1.6 if and only if they are homotopy equivalent complexes in the usual way.

Definition 2.1.8. A *dg functor* $\mathsf{F} : \mathcal{A} \longrightarrow \mathcal{B}$ between two dg categories is given by a map: $\mathrm{Ob}(\mathcal{A}) \longrightarrow \mathrm{Ob}(\mathcal{B})$ and, for every X, Y in $\mathrm{Ob}(\mathcal{A})$, by a morphism of complexes of $\mathbb{k}$-modules

$$\mathsf{F}_{X,Y} : \mathcal{A}(X,Y) \longrightarrow \mathcal{B}(\mathsf{F}(X),\mathsf{F}(Y))$$

which is compatible with compositions and units.

Clearly, a dg functor $\mathsf{F} : \mathcal{A} \longrightarrow \mathcal{B}$ induces a functor $H^0(\mathsf{F}) : H^0(\mathcal{A}) \longrightarrow H^0(\mathcal{B})$.

Definition 2.1.9. We can define the category **dgCat** (or $\mathbf{dgCat}_{\mathbb{k}}$ if the ground ring is not clear from the context) whose objects are (small) dg categories and whose morphisms are dg-functors.

The category **dgCat** has the empty dg category $\emptyset$ as initial object and the dg category with one object — and with the zero ring as the endomorphism ring — as terminal object.

Definition 2.1.10. Given two dg categories $\mathcal{A}$ and $\mathcal{B}$ we can define two new dg categories:

- The *tensor product* $\mathcal{A} \otimes \mathcal{B}$, whose objects are pairs (A,B) with $A \in \mathcal{A}$, $B \in \mathcal{B}$ and the Hom-spaces are defined as follows:

  $$(\mathcal{A} \otimes \mathcal{B})((A, B), (A', B')) := \mathcal{A}(A, A') \otimes_{\Bbbk} \mathcal{B}(B, B').$$

  where the last tensor product is the tensor product of complexes of $\Bbbk$-modules over the ground ring $\Bbbk$ (see Definition 2.1.1).

- The *dg functors category* $\underline{Hom}(\mathcal{A}, \mathcal{B})$, whose objects are dg functors between $\mathcal{A}$ and $\mathcal{B}$ and Hom-spaces are $\underline{Hom}(\mathsf{F}, \mathsf{G}) = \bigoplus_{i\in\mathbb{Z}} \underline{Hom}(\mathsf{F}, \mathsf{G})^i$ with

  $$\underline{Hom}(\mathsf{F}, \mathsf{G})^i := \{\text{dg natural transformations } \phi : \mathsf{F} \to \mathsf{G} \text{ of degree } i\}.$$

  where a dg natural transformation $\phi : \mathsf{F} \to \mathsf{G}$ of degree i is the datum, for any object of A of $\mathcal{A}$, of a degree i morphism $\phi_A : \mathsf{F}(A) \longrightarrow \mathsf{G}(A)$ such that, for any homogeneous $f \in \mathcal{A}(A, A')$, we have $\mathsf{G}(f) \circ \phi_A = (-1)^{\deg(f)i}\phi_{A'} \circ \mathsf{F}(f)$. The differential is induced by the one of $\mathcal{B}(\mathsf{F}(A), \mathsf{G}(A))$.

Remark 2.1.11. Observe that the dg category $\underline{Hom}(\mathcal{A}, \mathcal{B})$ has many objects even if $\mathcal{A}$ and $\mathcal{B}$ only have one. In fact, suppose $Ob(\mathcal{A}) = \{A\}$ and $Ob(\mathcal{B}) = \{B\}$. A dg functor F between $\mathcal{A}$ and $\mathcal{B}$ in this case is uniquely defined by a morphism of dg algebras, see Example 2.1.3 (ii),

$$\mathsf{F}_{A,B} : \mathcal{A}(A, A) \longrightarrow \mathcal{B}(B, B)$$

and hence we can have plenty of them. This is a considerable difference between the world of dg categories and the one of dg algebras.

Remark 2.1.12. Tensor product of dg categories is (up to isomorphism) associative, commutative and with $\Bbbk$ acting as the identity: in other words, it defines a symmetric monoidal structure on **dgCat**. Moreover, such a monoidal structure is *closed* i.e. for any $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in **dgCat** we have a natural isomorphism:

$$\underline{Hom}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \underline{Hom}(\mathcal{A}, \underline{Hom}(\mathcal{B}, \mathcal{C})). \tag{2.1}$$

At the end of this section we introduce a class of maps between dg categories whose importance will be crucial throughout this work. It can be seen as a mixture between categorical equivalences and quasi-isomorphisms.

Definition 2.1.13. A dg functor $\mathsf{F} : \mathcal{A} \longrightarrow \mathcal{B}$ is a *quasi-equivalence* if $H^0(\mathsf{F})$ is an equivalence and F is *quasi-fully faithfull* i.e. for any A and B in $\mathcal{A}$ the map $\mathsf{F}_{A,B}$ of Definition 2.1.8 is a quasi-isomorphism.

Observe that, if a dg functor F is quasi-fully faithfull it suffices to require that $H^0(\mathsf{F})$ is essentially surjective on objects in order to have a quasi-equivalence.

2.2 Dg modules

A really important notion in the theory of dg categories is the one of dg modules.

Definition 2.2.1. Let $\mathcal{A}$ be a dg category. A *right dg $\mathcal{A}$-module* (or a *right dg module over $\mathcal{A}$*) is a dg functor

$$\mathsf{F} : \mathcal{A}^{\circ} \longrightarrow C_{dg}(\Bbbk\text{-Mod})$$

Right dg $\mathcal{A}$-modules can therefore clearly be collected into a dg category

$$\text{dgMod}(\mathcal{A}) := \underline{Hom}(\mathcal{A}^{\circ}, C_{dg}(\Bbbk\text{-Mod})).$$

Clearly we have also the analogous notion of a *left dg $\mathcal{A}$-module* as a dg functor $\mathsf{F} : \mathcal{A} \longrightarrow C_{dg}(\Bbbk\text{-Mod})$. But from now on, if not otherwise specified, all our dg modules will be right dg modules.

Also at the dg level it is possible to define a version of the Yoneda embedding. For any dg category $\mathcal{A}$ we have a dg functor

$$Y^{dg}_{\mathcal{A}} : \mathcal{A} \longrightarrow \text{dgMod}(\mathcal{A})$$

defined by sending an object A to $\mathcal{A}(-, A)$. It is called *Yoneda dg functor* and it has all the nice properties one could expect: it is fully faithful and injective on objects. Moreover, for any dg $\mathcal{A}$-module M and for any $A \in \mathcal{A}$ we have a natural isomorphism

$$\text{dgMod}(\mathcal{A})(Y^{dg}_{\mathcal{A}}(A), \mathsf{M}) \xrightarrow{\sim} \mathsf{M}(A). \tag{2.2}$$

Definition 2.2.2. In dgMod($\mathcal{A}$) we have some interesting full dg subcategories. In particular, a dg $\mathcal{A}$-module M is said to be:

1. *acyclic* if $\mathsf{M}(A)$ is an acyclic complex for any object $A \in \mathcal{A}$. We denote by Ac($\mathcal{A}$) the subcategory of acyclic dg-modules;
2. *h-projective* if $H^0(\text{dgMod}(\mathcal{A}))(\mathsf{M}, \mathsf{N}) = 0$ for any acyclic dg-module N. We denote by h-proj($\mathcal{A}$) the subcategory of h-projective dg-modules;
3. *representable* if it lies in the image of the Yoneda dg functor. We denote by $\bar{\mathcal{A}} \subset \text{dgMod}(\mathcal{A})$ the closure of the image of $Y^{dg}_{\mathcal{A}}$ by homotopy equivalent objects;

It is clear by (2.2) that any representable dg-module is h-projective.

We stop for the moment the discussion on dg modules: we will continue it after having introduced the very important notion of pretriangulated dg category.

2.3 Pretriangulated dg categories

We have said that dg categories are one of the ways mathematicians have enhanced triangulated categories in order to overcome their "bad features". So one may wonder how dg categories contain a triangulated structure: we will answer this question providing the definition of pretriangulated dg categories. It turns out that one of the main difference between triangulated and dg categories lies in the fact that a triangulated category is an additive category plus some additional structure (and in fact we can have, for example, different triangulated structures on the same category) while being pretriangulated is a property that a dg category could or could not have.

A crucial point is the fact that in the context of dg categories it is possible to define functorially the shift of an object and the cone of a closed degree 0 morphism.

Definition 2.3.1. Let $\mathcal{A}$ be a dg-category, $A \in \mathcal{A}$ and $n \in \mathbb{Z}$. A *n-shift* of A is an object $A[n] \in \mathcal{A}$ together with two closed morphisms

$$n_A : A[n] \longrightarrow A \qquad\qquad n^A : A \longrightarrow A[n]$$

of degree n and $-n$ respectively and such that $n_A \circ n^A = \mathrm{id}_A$ and $n^A \circ n_A = \mathrm{id}_{A[n]}$.

It is easy to see that the object $A[n]$, once it exists, is unique up to a natural dg isomorphism.

Definition 2.3.2. Let $\mathcal{A}$ be a dg category and let $f : A \longrightarrow B$ be a closed morphism of degree 0 in $\mathcal{A}$; suppose moreover that $\mathcal{A}$ contains the shift $A[1]$. A *cone of* f is an object $C(f) \in \mathcal{A}$ together with degree 0 morphisms

$$A[1] \underset{p}{\overset{i}{\rightleftarrows}} C(f) \underset{j}{\overset{s}{\rightleftarrows}} B$$

such that

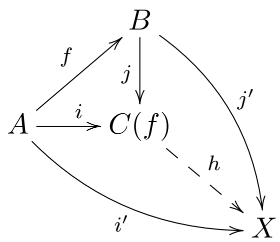
$$p \circ i = \mathrm{id}_{A[1]}, \qquad s \circ j = \mathrm{id}_B, \qquad s \circ i = 0 = p \circ j, \qquad i \circ p + j \circ s = \mathrm{id}_{C(f)} \qquad (2.3)$$

and

$$d(j) = 0 = d(p), \qquad d(i) = j \circ f \circ 1_A, \qquad d(s) = -f \circ 1_A \circ p. \qquad (2.4)$$

Notice that conditions (2.3) characterize $C(f)$ as the biproduct of $A[1]$ and B in $\mathcal{A}$, seen as a $\Bbbk$-linear category. Moreover it is also uniquely determined up to a *unique* dg isomorphism. In fact it satisfies a universal property, according to the following:

Proposition 2.3.3 ([19] Proposition 2.3.4)**.** *Let $\mathcal{A}$ be a dg category, and let $f : A \longrightarrow B$ be a closed morphism of degree* 0*. View the inclusion map $i : A[1] \longrightarrow C(f)$ as a degree -1 morphism $A \longrightarrow C(f)$. Then, $d(i) = j \circ f$, and for any closed degree* 0 *map $j' : B \longrightarrow X$ and any degree -1 map $i' : A \longrightarrow X$ such that $d(\grave{\imath}') = j' \circ f$, there exists a unique closed degree* 0 *map $h : C(f) \longrightarrow X$ such that the diagram below*



commutes, i.e. such that $h \circ j = j'$ and $h \circ i = i'$.

The notion of cone of a closed degree 0 morphism in a differential graded category can be seen as a generalization of the cone of a morphism of complexes. If $f : A \longrightarrow B$ is a closed degree 0 morphism, then a sequence of the form

$$A \xrightarrow{f} B \xrightarrow{j} C(f) \xrightarrow{p} A[1]$$

where the notation is the same as in Definition 2.3.2, is called *preexact triangle*. Preexact triangles - as well as shifts and cones of degree 0 morphisms - are preserved by any dg functor.

Definition 2.3.4. A dg category $\mathcal{A}$ is *strongly pretriangulated* if it contains the shift $A[n]$ for any object $A \in \mathcal{A}$ and for every $n \in \mathbb{Z}$ and if it moreover contains the cone of every closed degree 0 morphism of $\mathcal{A}$.

A dg category $\mathcal{A}$ is *pretriangulated* if there exists a quasi-equivalence $\mathcal{A} \longrightarrow \mathcal{A}'$ with $\mathcal{A}'$ strongly pretriangulated.

Remark 2.3.5. The dg category $C_{dg}(\mathcal{A})$ is strongly pretriangulated for any additive category $\mathcal{A}$.

Remark 2.3.6. If the dg category $\mathcal{B}$ is (strongly) pretriangulated, then so is $\underline{Hom}(\mathcal{A}, \mathcal{B})$ for any dg category $\mathcal{A}$.

If $\mathcal{A}$ is a pretriangulated dg category then its homotopy category $\mathrm{H}^0(\mathcal{A})$ is a triangulated category: the images of preexact triangles form a set of distinguished triangles for $\mathrm{H}^0(\mathcal{A})$. Notice the analogy with Proposition 1.1.15. Moreover, any dg functor $\mathsf{F} : \mathcal{A} \longrightarrow \mathcal{B}$ between pretriangulated dg categories gives rise to a triangulated functor $\mathrm{H}^0(\mathsf{F}) : \mathrm{H}^0(\mathcal{A}) \longrightarrow \mathrm{H}^0(\mathcal{B})$.

Definition 2.3.7. For every dg category $\mathcal{A}$ we will denote by $\mathrm{Pretr}(\mathcal{A})$ the smallest full dg subcategory of h-proj$(\mathcal{A})$ which contains $Y^{dg}_{\mathcal{A}}(\mathcal{A})$, and which is closed under shifts and cones of degree-0 morphisms.

For any dg category $\mathcal{A}$, $\mathrm{Pretr}(\mathcal{A})$ is a (strongly) pretriangulated dg category and clearly the Yoneda embedding restricts to it. Actually $\mathrm{Pretr}(\mathcal{A})$ is equivalent to the explicit construction that permits to formally add to $\mathcal{A}$ all shifts, all cones, cones of morphisms between cones, etc. The latter construction is usually denoted by $\mathcal{A}^{\mathrm{pretr}}$: one can look at [16] Section 2.4 or [4] Section 4 for the explicit construction.

Proposition 2.3.8. *A dg category $\mathcal{A}$ is pretriangulated (respectively, strongly pretriangulated) if and only if the restricted Yoneda embedding $\mathcal{A} \hookrightarrow \mathrm{Pretr}(\mathcal{A})$ is a quasi-equivalence (respectively, an equivalence).*

In virtue of the above result we can see that a dg category $\mathcal{A}$ is pretriangulated if and only if the essential image of $\mathrm{H}^0(Y^{dg}_{\mathcal{A}}) : \mathrm{H}^0(\mathcal{A}) \longrightarrow \mathrm{H}^0(\mathrm{dgMod}(\mathcal{A}))$ is a triangulated subcategory of $\mathrm{H}^0(\mathrm{dgMod}(\mathcal{A}))$.

2.4 Dg modules, again

We have that the dg categories $\mathrm{Ac}(\mathcal{A})$, h-proj$(\mathcal{A})$ and $\mathrm{dgMod}(\mathcal{A})$ introduced in Section 2.2 are strongly pretriangulated and both $\mathrm{H}^0(\mathrm{Ac}(\mathcal{A}))$ and $\mathrm{H}^0(\text{h-proj}(\mathcal{A}))$ are triangulated subcategories of the triangulated category $\mathrm{H}^0(\mathrm{dgMod}(\mathcal{A}))$. Moreover $\mathrm{H}^0(\mathrm{Ac}(\mathcal{A}))$, $\mathrm{H}^0(\text{h-proj}(\mathcal{A}))$ and $\mathrm{H}^0(\mathrm{dgMod}(\mathcal{A}))$ have arbitrary coproducts and there is a *semi-orthogonal decomposition* (look at Section 3 of [25]):

$$\mathrm{H}^0(\mathrm{dgMod}(\mathcal{A})) = \langle \mathrm{H}^0(\mathrm{Ac}(\mathcal{A})), \mathrm{H}^0(\text{h-proj}(\mathcal{A})) \rangle. \tag{2.5}$$

Definition 2.4.1. We can define the dg category $\mathrm{Perf}(\mathcal{A})$ of *perfect* dg-modules as the full dg subcategory of h-proj$(\mathcal{A})$ whose objects are the compact objects of the triangulated category $\mathrm{H}^0(\text{h-proj}(\mathcal{A}))$.

We can see, using the isomorphism (2.2), that the Yoneda dg functor $Y^{dg}_{\mathcal{A}}$ factors through $\mathrm{Perf}(\mathcal{A})$. Therefore we can write the following chain of inclusions:

$$Y^{dg}_{\mathcal{A}}(\mathcal{A}) \subseteq \bar{\mathcal{A}} \subseteq \mathrm{Perf}(\mathcal{A}) \subseteq \text{h-proj}(\mathcal{A}) \subseteq \mathrm{dgMod}(\mathcal{A})$$

Remark 2.4.2. Actually one can equivalently define $\mathrm{Perf}(\mathcal{A})$ to be the full dg subcategory of $\text{h-proj}(\mathcal{A})$ consisting of dg modules which are homotopy equivalent to a direct summand of an object in $\mathrm{Pretr}(\mathcal{A})$. In this way we get that $\mathrm{H}^0(\mathrm{Perf}(\mathcal{A}))$ is the *idempotent completion* (or the *Karoubi envelope*) of $\mathrm{H}^0(\mathrm{Pretr}(\mathcal{A}))$.
This also implies that $\mathrm{H}^0(\mathrm{Perf}(\mathcal{A}))$ is closed under direct summands i.e. it is a *thick* triangulated subcategory of $\mathrm{H}^0(\text{h-proj}(\mathcal{A}))$ (and of $\mathrm{H}^0(\mathrm{dgMod}(\mathcal{A}))$ as well). Actually the definition of a thick subcategory requires also the closure under isomorphisms but in the present case it is straightforward since an object isomorphic to a compact one is compact itself.

Definition 2.4.3. A dg category $\mathcal{A}$ is called *perfect* if it is pretriangulated and its homotopy category $\mathrm{H}^0(\mathcal{A})$ is Karoubian (i.e. idempotent complete).

We can see from Remark 2.4.2 that the dg category $\mathrm{Perf}(\mathcal{A})$ is always a perfect dg category. Moreover, any dg category that is quasi-equivalent to a perfect dg category is itself perfect.

Proposition 2.4.4 ([4] Proposition 4.20)**.** *If $\mathcal{A}$ is a perfect dg category then the dg Yoneda embedding $Y^{dg}_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathrm{Perf}(\mathcal{A})$ is a quasi-equivalence.*

Definition 2.4.5. Given a dg category $\mathcal{A}$ we can also define its *derived category* to be the triangulated category:

$$D(\mathcal{A}) := \frac{\mathrm{H}^0(\mathrm{dgMod}(\mathcal{A}))}{\mathrm{H}^0(\mathrm{Ac}(\mathcal{A}))}$$

where we have made use of the Verdier quotient (see [48]).

We know by [16] section 3.3 that there always exists a triangulated functor

$$\mathrm{H}^0(\mathrm{dgMod}(\mathcal{A}))/\mathrm{H}^0(\mathrm{Ac}(\mathcal{A})) \longrightarrow \mathrm{H}^0(\mathrm{dgMod}(\mathcal{A})/\mathrm{Ac}(\mathcal{A}))$$

where $\mathrm{dgMod}(\mathcal{A})/\mathrm{Ac}(\mathcal{A})$ is the Drinfeld quotient that he defined in the article cited above. Such a triangulated functor is an equivalence every time $\mathrm{dgMod}(\mathcal{A})$ is *homotopically flat*. For the definition of homotopically flat (or h-flat) dg categories, look at Remark 2.5.12.

Remark 2.4.6. There is an equivalence $\mathrm{H}^0(\text{h-proj}(\mathcal{A})) \simeq D(\mathcal{A})$. This is obvious if we remind of the semi-orthogonal decomposition (2.5).

2.5 Bimodules and quasi-functors

In this section we will introduce the machinery of *extension of dg functors* that will be intensively used throughout this thesis. We will follow the exposition in [13], Section 3.1.

Let $\mathcal{A}$ be any dg category, let $\mathsf{M} \in \mathrm{dgMod}(\mathcal{A})$ and $\mathsf{N} \in \mathrm{dgMod}(\mathcal{A}^\circ)$. We can define a map

$$\mathbb{T} : \bigoplus_{A,B\in\mathcal{A}} \mathsf{M}(B) \otimes_{\Bbbk} \mathcal{A}(A,B) \otimes_{\Bbbk} \mathsf{N}(A) \longrightarrow \bigoplus_{C\in\mathcal{A}} \mathsf{M}(C) \otimes_{\Bbbk} \mathsf{N}(C)$$

that acts as follows: given $x \in \mathsf{M}(B)$ omogeneous of degree m, $y \in \mathsf{N}(A)$ and $f : A \to B$ of degree l we have

$$\mathbb{T}(x, f, y) := \mathsf{M}(f)(x) \otimes y - (-1)^{ml} x \otimes \mathsf{N}(f)(y) \in \mathsf{M}(A) \otimes_{\Bbbk} \mathsf{N}(A) \oplus \mathsf{M}(B) \otimes_{\Bbbk} \mathsf{N}(B).$$

Definition 2.5.1. According to the notation above, given two dg modules $\mathsf{M} \in \mathrm{dgMod}(\mathcal{A})$ and $\mathsf{N} \in \mathrm{dgMod}(\mathcal{A}^\circ)$ we define *the tensor product of* M *and* N *over* $\mathcal{A}$ to be

$$\mathsf{M} \otimes_{\mathcal{A}} \mathsf{N} := \mathrm{coker}(\mathbb{T})$$

that lives in $\mathrm{dgMod}(\Bbbk) = C_{dg}(\Bbbk\text{-Mod})$.

Clearly, one can repeat the same definition taking $\mathsf{M} \in \mathrm{dgMod}(\mathcal{A} \otimes \mathcal{B})$ and $\mathsf{N} \in \mathrm{dgMod}(\mathcal{B}^\circ \otimes \mathcal{C})$. In this case, $\mathsf{M} \otimes_{\mathcal{B}} \mathsf{N}$ is an object in $\mathrm{dgMod}(\mathcal{A} \otimes \mathcal{C})$.

Given two dg categories $\mathcal{A}$ and $\mathcal{B}$, as a particular case of the natural isomorphism (2.1) we get the isomorphism of dg categories

$$\eta_{\mathcal{A},\mathcal{B}} : \mathrm{dgMod}(\mathcal{A}^\circ \otimes \mathcal{B}) \longrightarrow \underline{Hom}(\mathcal{A}, \mathrm{dgMod}(\mathcal{B})) \tag{2.6}$$

Now, given a dg functor $\mathsf{F} : \mathcal{A} \longrightarrow \mathrm{dgMod}(\mathcal{B})$ we call *extension of F* the dg functor

$$\hat{\mathsf{F}} : \mathrm{dgMod}(\mathcal{A}) \longrightarrow \mathrm{dgMod}(\mathcal{B})$$

defined as the tensorization (over $\mathcal{A}$) by the element $E_{\mathsf{F}} \in \mathrm{dgMod}(\mathcal{A}^\circ \otimes \mathcal{B})$ corresponding to F via (2.6). We also have a natural way to define a dg functor

$$\tilde{\mathsf{F}} : \mathrm{dgMod}(\mathcal{B}) \longrightarrow \mathrm{dgMod}(\mathcal{A})$$

by sending a dg $\mathcal{B}$-module M to the dg $\mathcal{A}$-module $\mathrm{dgMod}(\mathcal{B})(\mathsf{F}(-), \mathsf{M})$.

If now $\mathsf{G} : \mathcal{A} \longrightarrow \mathcal{B}$ is a dg functor we define $\mathrm{Ind}_{\mathsf{G}} := \widehat{Y^{dg}_{\mathcal{B}} \circ \mathsf{G}}$ and $\mathrm{Res}_{\mathsf{G}} := \widetilde{Y^{dg}_{\mathcal{B}} \circ \mathsf{G}}$. Observe that, by the isomorphism (2.2), we have $\mathrm{Res}_{\mathsf{G}}(\mathsf{M}) = \mathsf{M}(\mathsf{G}(-))$.

Proposition 2.5.2 ([13] Proposition 3.2, [16] Section 14.9)**.** *Let $\mathsf{G} : \mathcal{A} \longrightarrow \mathcal{B}$ be a dg functor. We have the following properties:*

(*i*) $\mathrm{Ind}_{\mathsf{G}}$ *is left adjoint to* $\mathrm{Res}_{\mathsf{G}}$*;*

(*ii*) $\mathrm{Ind}_{\mathsf{G}} \circ Y^{dg}_{\mathcal{A}}$ *is dg isomorphic to* $Y^{dg}_{\mathcal{B}} \circ \mathsf{G}$*;*

(*iii*) $\mathrm{Ind}_{\mathsf{G}}$ *restricts to h-projective dg modules and such a restriction is a quasi-equivalence if* G *is a quasi-equivalence;*

(*iv*) $\mathrm{Res}_{\mathsf{G}}$ *restricts to h-projective dg modules if and only if* $\mathrm{Res}_{\mathsf{G}}(\bar{\mathcal{B}}) \subseteq \text{h-proj}(\mathcal{A})$*; moreover* $\mathrm{Res}_{\mathsf{G}}$ *restricts to perfect dg modules if and only if* $\mathrm{Res}_{\mathsf{G}}(\bar{\mathcal{B}}) \subseteq \mathrm{Perf}(\mathcal{A})$*;*

(*v*) $\mathrm{H}^0(\mathrm{Ind}_{\mathsf{G}})$ *and* $\mathrm{H}^0(\mathrm{Res}_{\mathsf{G}})$ *commute with arbitrary direct sums.*

Proof. the only statement that is not covered by the cited results is the second part of (*iv*). To prove (the non-trivial implication of) it we just need to recall that $\mathrm{Perf}(\mathcal{B})$ is obtained from $\bar{\mathcal{B}}$ through shifts, cones of degree-0 morphisms and direct summands in homotopy and that $\mathrm{Perf}(\mathcal{A})$ is closed under those operations (see Remark 2.4.2, Definition 2.3.7 and the discussion below it). □

Proposition 2.5.3 ([4] Lemma 4.14)**.** *Let* $\mathsf{G} : \mathcal{A} \longrightarrow \mathcal{B}$ *be a dg functor. Then* $\mathrm{Ind}_{\mathsf{G}}$ *restricts also to perfect dg modules and such a restriction is a quasi-equivalence if* G *is a quasi-equivalence.*

If $\mathsf{G} : \mathcal{A} \longrightarrow \mathcal{B}$ is a dg functor we can see that $\mathrm{Ind}_{\mathsf{G}}$ restricts to a dg functor between $\mathrm{Pretr}(\mathcal{A})$ and $\mathrm{Pretr}(\mathcal{B})$. In fact, since any element of $\mathrm{Pretr}(\mathcal{A})$ is obtained from representable $\mathcal{A}$-modules by means of iterated shifts and cones and since $\mathrm{Ind}_{\mathsf{G}}$ preserves shifts and cones (being a dg functor), we can conclude by Proposition 2.5.2.

Proposition 2.5.4 (see [4], Remark 4.12)**.** *If a dg functor* $\mathsf{G} : \mathcal{A} \longrightarrow \mathcal{B}$ *is a quasi-equivalence then the induced functor*

$$\mathrm{Ind}_{\mathsf{G}} : \mathrm{Pretr}(\mathcal{A}) \longrightarrow \mathrm{Pretr}(\mathcal{B})$$

is a quasi-equivalence as well.

Moreover, for a dg functor $\mathsf{G} : \mathcal{A} \longrightarrow \mathcal{B}$, we can define also a derived version of the dg functors $\mathrm{Ind}_{\mathsf{G}}$ and $\mathrm{Res}_{\mathsf{G}}$. Namely we have

$$L\mathrm{H}^0(\mathrm{Ind}_{\mathsf{G}}) : D(\mathcal{A}) \longrightarrow D(\mathcal{B}) \qquad \text{and} \qquad R\mathrm{H}^0(\mathrm{Res}_{\mathsf{G}}) : D(\mathcal{B}) \longrightarrow D(\mathcal{A})$$

and they still are a pair of adjoint functors (see, for example, [2] Section 10).

Remark 2.5.5. Actually, since $\mathrm{Res}_{\mathsf{G}}$ preserves acyclic dg modules (it is in some sense the "dual" statement of Proposition 2.5.2 (iii)), it naturally passes to the derived categories (see Definition 2.4.5) and hence the right derived functor $R\mathrm{H}^0(\mathrm{Res}_{\mathsf{G}})$ coincides with $\mathrm{H}^0(\mathrm{Res}_{\mathsf{G}})$. If we now consider the equivalence between $D(\mathcal{A})$ and $\mathrm{H}^0(\text{h-proj}(\mathcal{A}))$ we can see that the left derived functor $L\mathrm{H}^0(\mathrm{Ind}_{\mathsf{G}})$ in this setting is exactly $\mathrm{H}^0(\mathrm{Ind}_{\mathsf{G}})$ while $R\mathrm{H}^0(\mathrm{Res}_{\mathsf{G}})$ sends an object $\mathsf{M} \in \mathrm{H}^0(\text{h-proj}(\mathcal{B}))$ to an h-projective resolution of $\mathrm{H}^0(\mathrm{Res}_{\mathsf{G}}(\mathsf{M}))$ (see [16] Section 14.12) since we have that in general $\mathrm{Res}_{\mathsf{G}}$ does not restrict to h-projective dg modules.

We now present some result that will be used in the second part of the work.

Lemma 2.5.6. *Let* $\mathcal{A}$ *be a perfect dg category and consider the Yoneda dg functor* $Y_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathrm{Perf}(\mathcal{A})$*. Then the dg functor* $\mathrm{Res}_{Y_{\mathcal{A}}}$ *restricts to perfect dg modules and provides a quasi-equivalence* $\mathrm{Res}_{Y_{\mathcal{A}}} : \mathrm{Perf}(\mathrm{Perf}(\mathcal{A})) \longrightarrow \mathrm{Perf}(\mathcal{A})$*.*

Proof. Let us start by showing that $\mathrm{Res}_{Y_{\mathcal{A}}}$ restricts to perfect dg modules. Thanks to Proposition 2.5.2 (iv) it suffices to show that $\mathrm{Res}_{Y_{\mathcal{A}}}(\mathrm{Hom}_{\mathrm{Perf}(\mathcal{A})}(-, A)) \subseteq \mathrm{Perf}(\mathcal{A})$ for any $A \in \mathrm{Perf}(\mathcal{A})$. Now, by the defnition of Res and by the dg Yoneda Lemma we can write

$$\mathrm{Res}_{Y_{\mathcal{A}}}(\mathrm{Hom}_{\mathrm{Perf}(\mathcal{A})}(-, A)) = \mathrm{Hom}_{\mathrm{Perf}(\mathcal{A})}(Y_{\mathcal{A}}(-), A) \simeq A(-) \in \mathrm{Perf}(\mathcal{A}).$$

We have therefore that $\mathrm{Res}_{Y_{\mathcal{A}}}$ restricts to perfect dg modules and hence it is a right adjoint to $\mathrm{Ind}_{Y_{\mathcal{A}}} : \mathrm{Perf}(\mathcal{A}) \longrightarrow \mathrm{Perf}(\mathrm{Perf}(\mathcal{A}))$. But the latter dg functor is a quasi-equivalence by Proposition 2.5.3, since $\mathcal{A}$ is perfect.
It follows that $\mathrm{Res}_{Y_{\mathcal{A}}} : \mathrm{Perf}(\mathrm{Perf}(\mathcal{A})) \longrightarrow \mathrm{Perf}(\mathcal{A})$ is a quasi-equivalence as well. □

Lemma 2.5.7. *If* $\mathcal{A}$ *is a perfect dg category and* $\mathcal{B}$ *is any dg category then the dg functor* $\mathrm{Res}_{Y_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{B}}}$ *restricts to perfect dg modules and provides a quasi-equivalence* $\mathrm{Res}_{Y_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{B}}} : \mathrm{Perf}(\mathrm{Perf}(\mathcal{A}) \otimes \mathcal{B}) \longrightarrow \mathrm{Perf}(\mathcal{A} \otimes \mathcal{B})$*.*

Proof. Let us start again by showing that the dg functor under investigation restricts to h-projective dg modules. Always by Proposition 2.5.2 (iv) this amounts to checking that $\mathrm{Res}_{Y_{\mathcal{A}}\otimes \mathrm{id}_{\mathcal{B}}}\Big(\mathrm{Hom}_{\mathrm{Perf}(\mathcal{A})\otimes\mathcal{B}}((-,=),(A,B))\Big) \subseteq \text{h-proj}(\mathcal{A}\otimes\mathcal{B})$ for any $A \in \mathrm{Perf}(\mathcal{A})$ and $B \in \mathcal{B}$. We can write

$$\begin{aligned}\mathrm{Res}_{Y_{\mathcal{A}}\otimes \mathrm{id}_{\mathcal{B}}}\Big(\mathrm{Hom}_{\mathrm{Perf}(\mathcal{A})\otimes\mathcal{B}}((-,=),(A,B))\Big) &= \mathrm{Hom}_{\mathrm{Perf}(\mathcal{A})\otimes\mathcal{B}}((Y_{\mathcal{A}}(-),=),(A,B)) \\ &= \mathrm{Hom}_{\mathrm{Perf}(\mathcal{A})}(Y_{\mathcal{A}}(-),A)\otimes \mathrm{Hom}_{\mathcal{B}}(=,B) \simeq A(-)\otimes \mathrm{Hom}_{\mathcal{B}}(=,B),\end{aligned}$$

again by the dg Yoneda Lemma. Now, since $\mathcal{A}$ is a perfect dg category there exists an object $\tilde{A} \in \mathcal{A}$ such that $A(-)$ is homotopy equivalent to $\mathrm{Hom}_{\mathcal{A}}(-,\tilde{A})$. Moreover, tensorization by $\mathrm{Hom}_{\mathcal{B}}(=,B) \in \mathrm{dgMod}(\mathcal{B})$ provides a dg functor from $\mathrm{dgMod}(\mathcal{A})$ to $\mathrm{dgMod}(\mathcal{A}\otimes\mathcal{B})$ (see, for example, Remark 2.5 of [13]). Therefore $A(-)\otimes\mathrm{Hom}_{\mathcal{B}}(=,B)$ is homotopy equivalent to $\mathrm{Hom}_{\mathcal{A}}(-,\tilde{A})\otimes\mathrm{Hom}_{\mathcal{B}}(=,B) = \mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}((-,=),(\tilde{A},B)) \in \mathrm{Perf}(\mathcal{A}\otimes\mathcal{B})$. It follows that $\mathrm{Res}_{Y_{\mathcal{A}}\otimes \mathrm{id}_{\mathcal{B}}}\Big(\mathrm{Hom}_{\mathrm{Perf}(\mathcal{A})\otimes\mathcal{B}}((-,=),(A,B))\Big)$ actually belongs to $\text{h-proj}(\mathcal{A}\otimes\mathcal{B})$.
For the remaining part of the proof we proceed exactly as in the proof of Lemma 2.5.6 and we can conclude. ☐

Corollary 2.5.8. *If $\mathcal{A}$ and $\mathcal{B}$ are two perfect dg categories then the dg functor* $\mathrm{Res}_{Y_{\mathcal{A}}\otimes Y_{\mathcal{B}}}$ *gives rise to a quasi-equivalence* $\mathrm{Res}_{Y_{\mathcal{A}}\otimes Y_{\mathcal{B}}} : \mathrm{Perf}(\mathrm{Perf}(\mathcal{A})\otimes\mathrm{Perf}(\mathcal{B})) \longrightarrow \mathrm{Perf}(\mathcal{A}\otimes\mathcal{B})$.

Proof. Clearly we can prove - in complete analogy with Lemma 2.5.7 - that also

$$\mathrm{Res}_{\mathrm{id}_{\mathrm{Perf}(\mathcal{A})}\otimes Y_{\mathcal{B}}} : \mathrm{Perf}(\mathrm{Perf}(\mathcal{A})\otimes\mathrm{Perf}(\mathcal{B})) \longrightarrow \mathrm{Perf}(\mathrm{Perf}(\mathcal{A})\otimes\mathcal{B})$$

is a quasi-equivalence. Therefore the claim follows from the fact that the composition of two quasi-equivalences is a quasi-equivalence, too. ☐

One of the important features of differential graded categories is that **dgCat** can be endowed with the structure of a *model category* whose *weak equivalences* are the quasi-equivalences (see [43]). This assures us that localizing **dgCat** with respect to the set W of quasi-equivalences still gives rise to a category. We set

$$\mathbf{Hqe} := \mathbf{dgCat}[W^{-1}].$$

This category has been believed to be the right place for working with dg categories from a homotopical point of view and it will be one of the main characters of the present work. So let us summarize here the main properties of **Hqe**.

We will follow [14] and call *quasi-functor* any morphism in **Hqe**. An element $\mathsf{F} \in \mathbf{Hqe}(\mathcal{A},\mathcal{B})$ can always be represented (see [6] Theorem 1) by a roof

$$\begin{array}{ccc} & \mathcal{A}' & \\ \swarrow & & \searrow \\ \mathcal{A} & & \mathcal{B} \end{array}$$

where the left-pointing arrow is a quasi-equivalence. It induces a well-defined (up to isomorphism) functor $\mathrm{H}^0(\mathsf{F}) : \mathrm{H}^0(\mathcal{A}) \longrightarrow \mathrm{H}^0(\mathcal{B})$.

Definition 2.5.9. Let F and G be two dg functors from $\mathcal{A}$ to $\mathcal{B}$. A dg natural transformation $\theta : \mathsf{F} \longrightarrow \mathsf{G}$ is a *termwise homotopy equivalence* if it is closed of degree 0 and $\theta_A : \mathsf{F}(A) \longrightarrow \mathsf{G}(A)$ is a homotopy equivalence for any $A \in \mathcal{A}$.

Observe that θ is a termwise homotopy equivalence if and only if the natural transformation $\mathrm{H}^0(\theta) : \mathrm{H}^0(\mathsf{F}) \longrightarrow \mathrm{H}^0(\mathsf{G})$ is an isomorphism.

Proposition 2.5.10 ([13] Corollary 2.12)**.** *Let F and G be two dg functors from $\mathcal{A}$ to $\mathcal{B}$. If there exists a termwise homotopy equivalence between them, then the images of F and G in **Hqe**$(\mathcal{A},\mathcal{B})$ are the same.*

We have already observed in Remark 2.1.12 that **dgCat** is a closed (symmetric) monoidal category. What about **Hqe**? Since the tensor product of dg categories does not behave well with respect to quasi-equivalences (as it should be clear since in the definition appears the tensor product between complexes of $\Bbbk$-modules) we need to suitably derive it. It can be done by means of *h-projective resolution*.

Definition 2.5.11. A dg category $\mathcal{A}$ is *h-projective* if for any A, $B \in \mathcal{A}$ the complex $\mathcal{A}(A,B)$ lies in h-proj$(C_{dg}(\Bbbk\text{-Mod}))$.

Note that, if $\Bbbk$ is a field, any dg category is h-projective. In general, starting from any dg category $\mathcal{A}$ one can construct an h-projective dg category $\mathcal{A}^{hp}$ quasi-equivalent to $\mathcal{A}$ (see Section 13.5 of [16]); hence the *derived tensor product* of two dg categories can be defined as

$$\mathcal{A} \otimes^{\mathbf{L}} \mathcal{B} := \mathcal{A}^{hp} \otimes \mathcal{B}.$$

Remark 2.5.12. Actually it suffices that one of the two dg categories is *h-flat* i.e. such that all its complex morphisms are h-flat complexes (see Definition 1.2.5): this means that tensoring them with an acyclic complex of $\Bbbk$-modules still produces an acyclic complex of $\Bbbk$-modules. The reader can have a look at Section 3.2 of [7] for an explicit construction of functorial h-flat resolutions. Clearly an h-projective dg category is also h-flat.

The derived tensor product defines a symmetric monoidal structure on **Hqe**. In 2007 Toën proved that it is also closed.

Theorem 2.5.13 ([45], Theorem 6.1 or [13], Theorem 1.1)**.** *Given three dg categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ there exists a dg category $R\underline{Hom}(\mathcal{B},\mathcal{C})$ and a functorial isomorphism in* **Hqe**

$$\mathbf{Hqe}(\mathcal{A} \otimes^{\boldsymbol{L}} \mathcal{B}, \mathcal{C}) \simeq \mathbf{Hqe}(\mathcal{A}, R\underline{Hom}(\mathcal{B},\mathcal{C}))$$

The dg category $R\underline{Hom}(\mathcal{B},\mathcal{C})$ is naturally isomorphic to the dg category

$$\text{h-proj}(\mathcal{B}^\circ \otimes^{\mathbf{L}} \mathcal{C})^{\mathrm{rqr}}$$

of right quasi-representable h-projective dg modules over $\mathcal{B}^\circ \otimes^{\mathbf{L}} \mathcal{C}$.

Definition 2.5.14. Let $\mathcal{B}$ and $\mathcal{C}$ be two dg categories. The category of *right quasi-representable* dg $\mathcal{B}^\circ \otimes \mathcal{C}$-modules is the full dg subcategory of h-proj$(\mathcal{B}^\circ \otimes \mathcal{C})$ consisting of dg modues M such that $\eta_{\mathcal{B},\mathcal{C}}(\mathsf{M})(\mathcal{B}) \subseteq \bar{\mathcal{C}}$, where η is the isomorphism of (2.6).

In addition we have, see [45] Theorem 4.2 or [13] Theorem 1.1:

Theorem 2.5.15. *For any dg categories $\mathcal{A}$ and $\mathcal{B}$ there exists a natural bijection*

$$\mathbf{Hqe}(\mathcal{A},\mathcal{B}) \longleftrightarrow \mathrm{Iso}(\mathrm{H}^0(\text{h-proj}(\mathcal{A}^\circ \otimes^{\boldsymbol{L}} \mathcal{B})^{\mathrm{rqr}})) \tag{2.7}$$

2.6 Dg enhancements and lifts

The theory of differential graded categories whose essential aspects we have diven into in the previous Sections allows us to look back at the problem of detecting representable exact functors from a higher perspective. But before doing it we need to carefully define the way of enhancing triangulated categories (and triangulated functors) to the dg level.

Definition 2.6.1. Let $\mathcal{D}$ be a triangulated category. A *dg enhancement* of $\mathcal{D}$ is a pair $(\mathcal{A}, \mathsf{E})$ where $\mathcal{A}$ is a pretriangulated dg category and E is an exact equivalence

$$\mathsf{E} : \mathrm{H}^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}$$

Remark 2.6.2. We can recall now that we have already encountered examples of dg enhancement, in the previous Sections:

(a) By Example 2.1.7 we can see that the dg category $C_{dg}(\mathcal{A})$, which is pretriangulated, is a dg enhancement of the homotopy category $\mathcal{K}(\mathcal{A})$ whenever $\mathcal{A}$ is additive;

(b) The pretriangulated dg category h-proj$(\mathcal{A})$ is a dg enhancement of $D(\mathcal{A})$, by Remark 2.4.6, for every dg category $\mathcal{A}$;

(c) In general, if $\mathcal{A}$ is a pretriangulated dg category and $\mathcal{B}$ is a pretriangulated full dg subcategory of $\mathcal{A}$ we have that the Drinfeld quotient $\mathcal{A}/\mathcal{B}$ is a dg enhancement of the Verdier quotient $\mathrm{H}^0(\mathcal{A})/\mathrm{H}^0(\mathcal{B})$. In the case of $\mathcal{A}$ being homotopically flat this is exactly Theroem 3.4 of [16]. Otherways the claim is obtained by passing through h-flat resolutions (see again [16], or [7] Section 3.2 and in particular the discussion before Remark 3.11 in it).

In the geometric case, there are many ways of enhancing a triangulated category. Let us consider, for example, the category $D^b(\mathrm{Coh}(X))$ in the case of X being a smooth projective scheme over a field.

- Consider the dg category $\mathrm{Inj}(X)$ of bounded below complexes of injective quasi-coherent sheaves on X with bounded coherent cohomologies (see, for example, [4]). It is a pretriangulated dg category and it is not hard to provide a natural exact equivalence $\mathrm{H}^0(\mathrm{Inj}(X)) \cong D^b(\mathrm{Coh}(X))$.

- In virtue of Remark 2.6.2 (a) and (c), we can see that the Drinfeld quotient of $C^b_{dg}(\mathrm{Coh}(X))$ (a version of the object defined in Example 2.1.7 where we are considering only bounded complexes) by its full pretriangulated dg subcategory $\mathrm{Ac}^b_{dg}(\mathrm{Coh}(X))$ of acyclic complexes is another dg enhancement of $D^b(\mathrm{Coh}(X))$.

- In [4] it is defined also an enhancement of $D^b(\mathrm{Coh}(X))$ constructed starting from Čech resolution of bounded complexes of vector bundles (Lemma 5.6). This kind of enhancements, in the refined version of [31] Appendix C, turns out to be the most suitable for our purposes: we will therefore describe it in details later, see Section 3.2.

- In [31] the authors construct another enhancement of $D^b(\mathrm{Coh}(X))$ (cfr. Proposition 3.15), consisting in certain non-full subcategories of the dg category $C_{dg}(\mathrm{Sh}(X))$.

It is therefore natural to wonder how are all those dg enhancements related. In the literature, there are different ways of doing it:

Definition 2.6.3. We say that the triangulated category $\mathcal{D}$ *has a unique enhancement* if given $(\mathcal{A}, \mathsf{E})$ and $(\mathcal{A}', \mathsf{E}')$ two dg enhancements of $\mathcal{D}$ there exists a pretriangulated dg category $\mathcal{A}''$ together with two quasi equivalences

$$\begin{array}{ccc} & \mathcal{A}'' & \\ \overset{\sim}{\swarrow} & & \overset{\sim}{\searrow} \\ \mathcal{A} & & \mathcal{A}'. \end{array}$$

This in particular implies that (actually, is equivalent to) $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic in **Hqe**.

The problem of uniqueness of dg enhancement has been quite intensively studied during the last decade. The first important result was obtained in 2010 by Lunts and Orlov [30] where they prove that $D^b(\mathrm{Coh}(X))$ has unique enhancement when X is a quasi-projective scheme; they proved also that $D(\mathrm{Qcoh}(X))$ has unique enhancement in the case of a quasi-compact and separated scheme X with enough locally free sheaves. These and the other seminal results in [30] have been improved in many directions. Recently, in [7] Canonaco, Neeman, and Stellari managed to prove the uniqueness of dg enhancement for all kinds of derived categories (either bounded or unbounded or bounded above/below) of any abelian category. They also proved that $\mathfrak{Perf}(X)$ has a unique enhancement for any quasi-compact and quasi-separated scheme.

The careful reader may have noticed that in Definition 2.6.3 the exact equivalences E and E', that are part of the data of a dg enhancement, do not play any role. If one wants to care about them another definition of uniqueness has to be introduced.

Definition 2.6.4. The triangulated category $\mathcal{D}$ *has a strongly unique enhancement* if given $(\mathcal{A}, \mathsf{E})$ and $(\mathcal{A}', \mathsf{E}')$ two dg enhancements of $\mathcal{D}$ there exists a quasi-functor $\mathsf{F} \in \mathbf{Hqe}(\mathcal{A}, \mathcal{A}')$ and an isomorphism of exact functors

$$\mathsf{E} \simeq \mathsf{E}' \circ \mathrm{H}^0(\mathsf{F}). \tag{2.8}$$

There is also the intermediate notion of *semi-strongly unique enhancement* requiring just $\mathsf{E}(A) \simeq \mathsf{E}' \circ \mathrm{H}^0(\mathsf{F})(A)$ for any object A of $\mathcal{A}$ instead of the isomorphism of functors (2.8).

Remark 2.6.5. While — as we have observed — uniqueness is well understood in a broad range of situations, strongly uniqueness is in some sense more obscure. Some results has been proved: for example, for a smooth projective scheme X over a field $D^b(\mathrm{Coh}(X))$ has a strongly unique enhancement (Theorem 2.14 of [30]). But still a lot of open questions resist.

Besides the notion of (dg) enhancement of a triangulated category, there is also a way of enhancing triangulated functors.

Definition 2.6.6. Let $f : \mathcal{D} \longrightarrow \mathcal{T}$ be an exact functor between triangulated categories and let $(\mathcal{A}, \mathsf{E})$ and $(\mathcal{B}, \mathsf{U})$ be two dg enhancements of $\mathcal{D}$ and $\mathcal{T}$, respectively. A *dg lift* of f is a morphism $\mathsf{F} \in \mathbf{Hqe}(\mathcal{A}, \mathcal{B})$ such that the diagram

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{A}) & \xrightarrow{\mathrm{H}^0(\mathsf{F})} & \mathrm{H}^0(\mathcal{B}) \\ {\scriptstyle \mathsf{E}}\downarrow & & \downarrow{\scriptstyle \mathsf{U}} \\ \mathcal{D} & \xrightarrow{f} & \mathcal{T} \end{array}$$

is commutative up to isomorphisms i.e. there is an isomorphism of exact functors $\mathsf{U} \circ \mathrm{H}^0(\mathsf{F}) \cong f \circ \mathsf{E}$.

This is a good point for a jump back to Section 1.4: we have now all the tools needed for understanding how the somehow obscure situation we have depicted there finds a new description in the dg realm.

Suppose now X and Y to be two smooth projective schemes. We have seen that given any object $\mathcal{E} \in D^b(\mathrm{Coh}(X \times Y))$ we can naturally attach to it an exact functor $\Phi_{\mathcal{E}} : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(Y))$. This can be expressed with the existence of a functor

$$D^b(\mathrm{Coh}(X \times Y)) \longrightarrow \mathrm{Fun}(D^b(\mathrm{Coh}(X)), D^b(\mathrm{Coh}(Y))) \tag{2.9}$$

where $\mathrm{Fun}(D^b(\mathrm{Coh}(X))\ ,\ D^b(\mathrm{Coh}(Y)))$ denotes the category of exact functors between $D^b(\mathrm{Coh}(X))$ and $D^b(\mathrm{Coh}(Y))$.

Such a functor is unfortunately neither essentially surjective nor essentially injective — in Remark 1.4.6 we have spoken about the existence of non-Fourier-Mukai functors as well as the fact that two non-isomorphic objects in $D^b(\mathrm{Coh}(X \times Y))$ can give rise to isomorphic Fourier-Mukai functors. A much more detailed dsicussion on the problematic aspects of the functor (2.9) can be found, for example, in [14] Section 6.

In [4] the authors began to investigate this problem employing the machinery of differential graded categories and they came up with the following:

Theorem 2.6.7 ([4] Theorem 6.8)**.** *Let X and Y be smooth projective varieties and let $f : D^b(\mathrm{Coh}(X)) \longrightarrow D^b(\mathrm{Coh}(Y))$ be an exact functor. If f admits a dg lift, then f is isomorphic to the Fourier-Mukai functor $\Phi_{\mathcal{E}_f}$ for some $\mathcal{E}_f \in D^b(\mathrm{Coh}(X \times Y))$.*

This suggests that one may consider the functor (2.9) by taking values in the class of morphism in **Hqe** between two dg enhancements of $D^b(\mathrm{Coh}(X))$ and $D^b(\mathrm{Coh}(Y))$. I want to recall now that we have already encountered the bijection (2.7) concerning morphisms in **Hqe** between two dg categories. Toën proved in [45], as a consequence of the existence of internal homs in **Hqe**, the following result

Proposition 2.6.8 ([45], Corollary 8.15)**.** *Let X and Y be two smooth proper schemes over a commutative ring. Then there exists an isomorphism*

$$R\underline{Hom}(\mathfrak{Perf}^{dg}(X), \mathfrak{Perf}^{dg}(Y)) \xrightarrow{\sim} \mathfrak{Perf}^{dg}(X \times Y)$$

in **Hqe***.*

In the above statement $\mathfrak{Perf}^{dg}(X)$ and $\mathfrak{Perf}^{dg}(Y)$ stand for dg enhancements of $\mathfrak{Perf}(X)$ and, respectively, of $\mathfrak{Perf}(Y)$. This proposition yields - taking the isomorphisms classes in H^0 - a bijection

$$\mathbf{Hqe}(\mathfrak{Perf}^{dg}(X), \mathfrak{Perf}^{dg}(Y)) \longleftrightarrow \mathrm{Iso}(\mathfrak{Perf}(X \times Y)). \tag{2.10}$$

Now, consider an object $\mathcal{E} \in \mathfrak{Perf}(X \times Y)$ and its associated Fourier-Mukai functor $\Phi_{\mathcal{E}} : \mathfrak{Perf}(X) \longrightarrow \mathfrak{Perf}(Y)$. A natural question arise:

What is the relationship between $\Phi_{\mathcal{E}}$ and the element $\mathsf{F}_{\mathcal{E}}$ of $\mathbf{Hqe}(\mathfrak{Perf}^{dg}(X), \mathfrak{Perf}^{dg}(Y))$ corresponding to the isomorphism class of $\mathcal{E}$ through the bijection (2.10)?

What one expects is to have $\mathrm{H}^0(\mathsf{F}_{\mathcal{E}}) \simeq \Phi_{\mathcal{E}}$. Toën claimed an analogous fact in [45], after Corollary 8.12. This would also give us a way, in the hypoteses of Theorem 2.6.7, of producing the kernel $\mathcal{E}_f$. A positive answer to a closely related question has been given in [31].

We will see in the following Chapters how it is possible to define a bijection that behaves exactly in this way.

Chapter 3

A dg lift of the Fourier-Mukai functor $\Phi_{\mathcal{E}}$

We start this Chapter with a discussion on the hypoteses we are going to assume. After that we introduce the kind of Čech enhancements that will be used in order to lift all the "pieces" of Fourier-Mukai functors in a compatible way. We move then to the explicit dg lift of pullback, pushforward and tensor product.

We finish by pointing out that, with some small modification in the construction, the dg lift of the Fourier-Mukai functor we end up with can still be produced in a bigger level of generality.

3.1 Description of the general setting

What we are going to do in this thesis is to prove, under suitable hypoteses that will be discussed later, the existence of a bijective map

$$\gamma : \mathrm{Iso}(\mathfrak{Perf}(X \times Y)) \xrightarrow{1:1} \mathbf{Hqe}(\mathfrak{Perf}^{dg}(X), \mathfrak{Perf}^{dg}(Y))$$

compatible with Fourier-Mukai kernels; this means that γ is such that, for any $\mathcal{E} \in \mathfrak{Perf}(X \times Y)$, we have $\mathrm{H}^0(\gamma(\mathcal{E})) \simeq \Phi_{\mathcal{E}}$. The dg categories $\mathfrak{Perf}^{dg}(X)$ and $\mathfrak{Perf}^{dg}(Y)$ are two fixed dg enhancements of $\mathfrak{Perf}(X)$ and $\mathfrak{Perf}(Y)$ respectively. Look at Theorem 4.4.1 for the precise statement.

Our strategy consists essentially in two steps: the first one is to produce an explicit dg lift $\Phi_{\mathcal{E}}^{dg}$ of the Fourier-Mukai functor $\Phi_{\mathcal{E}}$, where $\mathcal{E}$ is an object of $\mathfrak{Perf}(X \times Y)$; the second one is to show that we can construct the bijection γ essentially by sending the isomorphism class of $\mathcal{E}$ to $\Phi_{\mathcal{E}}^{dg}$.
This will require a discrete amount of work that will be carried along the present and the following Chapter. But before starting, we need to spend a couple of words about the hypoteses we will be working under.

We will construct the bijection γ (and prove its properties) assuming that X and Y satisfy condition $(*)$, where we say that any scheme X satisfies $(*)$ if:

$(*)$ X is a smooth and proper scheme over a field $\Bbbk$.

The construction of the dg lift $\Phi_{\mathcal{E}}$ will be done by means of Čech enhancements, in the version presented in [31] Appendix C. We will make use of some of the results there, that are proved for schemes X which satisfy the condition

(GSP+) X is a quasi-compact, separated, Nagata, locally integral scheme over a commutative ring $\Bbbk$. Moreover, any perfect complex on X is quasi-isomorphic to a bounded complex of vector bundles.

Recall that a ring A is Nagata if it is Noetherian and if for every prime ideal $\mathfrak{p}$ of A and for every finite extension L of $\mathrm{Frac}(A/\mathfrak{p})$ the integral closure of $A/\mathfrak{p}$ in L is finite over $A/\mathfrak{p}$. A scheme X is *Nagata* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subseteq X$ such that the ring $\mathcal{O}_X(U)$ is Nagata.

Remark 3.1.1. A scheme X satisfying the condition (GSP+) is Noetherian: in fact it is quasi-compact by assumption and clearly a Nagata scheme is locally Noetherian.

Remark 3.1.2. We can see that under the assumptions $(*)$ any scheme X satisfies the condition (GSP+). In fact:

(a) X is obviously quasi-compact and separated: it follows from properness;

(b) X is also Nagata since it is of finite type over a field, see the discussion at the beginning of Appendix C in [31];

(c) X is locally integral. In fact, since it is smooth over a field, [42, Tag 056S] implies that X is regular: in particular the local rings $\mathcal{O}_{X,x}$ are regular for any $x \in X$, hence $\mathcal{O}_{X,x}$ are integral domains for any $x \in X$ (by Proposition 4.2.11 of [29]).

(d) X is also regular as we observed above. Therefore - by Remark 2.4 of [31] and by the fact that a Noetherian separated regular scheme has the resolution property - any perfect complex on X is quasi-isomorphic to a bounded complex of vector bundles.

Observe that moreover any scheme X satisfying $(*)$ is of finite dimension. In fact we have that $\dim(X) = \sup_\alpha \dim(U_\alpha)$ for any open covering $\{U_\alpha\}$ of X. Since we are in the quasi-compact case the supremum is actually a maximum and hence it suffices to show that any affine open $U = \mathrm{Spec}(A)$ of X has finite dimension. Now, $\dim(U) = \dim(A)$ and A has finite dimension since it is a finitely generated $\Bbbk$-algebra.

Remark 3.1.3. We have that, if X satisfies $(*)$, Proposition 2.1 of [31] applies and hence $\mathfrak{Perf}(X) = D^b(\mathrm{Coh}(X))$.

Actually, many of the parts of the proof we are going to present hold true in greater generality: in particular we do not need to work with schemes over a field. It is sufficient for them to satisfy the condition

$(**)$ X is a smooth, proper, Nagata, locally integral scheme over a commutative ring $\Bbbk$. Moreover, any perfect complex on X is quasi-isomorphic to a bounded complex of vector bundles.

Observe that this condition clearly implies condition (GSP+): it is a consequence of point (a) in Remark 3.1.2.

We point out also that a scheme X satisfying condition $(**)$ is Noetherian, by Remark 3.1.1.

Along the way, we will highlight which results still hold in the $(**)$ case.

3.2 Choice of the enhancements and auxiliary results

In this section we set the basis for the construction of an explicit dg lift of the Fourier-Mukai functor $\Phi_{\mathcal{E}} : \mathfrak{Perf}(X) \longrightarrow \mathfrak{Perf}(Y)$. Such a lift will be computed by lifting each of the three (derived) functors $\Phi_{\mathcal{E}}$ is made of: the pullback along the first projection, the tensor product with the element $\mathcal{E}$ of $\mathfrak{Perf}(X \times Y)$ and the pushforward along the second projection.

As we have seen in Definition 2.6.6, part of the data of the dg lift of a triangulated functor $\mathsf{F} : \mathcal{A} \longrightarrow \mathcal{B}$ is the choice of a dg enhancement of $\mathcal{A}$ and $\mathcal{B}$, respectively. We are therefore looking for lifts of pullback, pushforward and tensor product that are computed using the same dg enhancements (in our case, Čech enhancements) in order to avoid troubles with quasi-equivalences. Moreover, Čech enhancements are good tools for having an explicit definition of the dg lifts and this will be very important in the next Chapter.

In the present and future sections of this Chapter — except Section 3.8 — we will working assuming all the schemes satisfy condition $(*)$.

Let us now begin with the description of those kind of enhancements. Throughout this work $\mathcal{U} := \{U_i\}_{i\in I}$ will be a finite affine open covering of X; if A is an $\mathcal{O}_X-$ module, we will denote by $\mathcal{C}_{\mathcal{U}}(A)$ the Čech resolution of A with respect to the covering $\mathcal{U}$. It is defined as

$$\mathcal{C}^n_{\mathcal{U}}(A) := \bigoplus_{i_0<\ldots<i_n \in I} (j_{i_0,\ldots,i_n})_*(j_{i_0,\ldots,i_n})^* A$$

where $U_{i_0\ldots i_n} := U_{i_0} \cap \ldots \cap U_{i_0}$ and $j_{i_0,\ldots,i_n}$ is the open inclusion $U_{i_0\ldots i_n} \hookrightarrow X$. Differentials are defined in the usual way: the map

$$d : \mathcal{C}^p_{\mathcal{U}}(A) \longrightarrow \mathcal{C}^{p+1}_{\mathcal{U}}(A)$$

is such that

$$d(s)_{i_0\ldots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0\ldots \hat{i}_j \ldots i_{p+1} | U_{i_0\ldots i_{p+1}}}$$

where by $s_{i_0\ldots i_p}$ we denote a local section of $(j_{i_0,\ldots,i_p})_*(j_{i_0,\ldots,i_p})^* A$ and the "hat" on an index means that it has been removed. When $A^\bullet$ is a complex of $\mathcal{O}_X-$ modules $\mathcal{C}_{\mathcal{U}}(A^\bullet)$ is defined to be the totalization of the obvious double complex.

Following [31] Appendix C, we set $\mathcal{P}_*(\mathcal{U})$ to be the smallest full dg subcategory of $C(\mathrm{Qcoh}(X))$ containing all the complexes $\mathcal{C}_{\mathcal{U}}(P)$ for every vector bundle P on X and closed under shifts and cones of closed morphisms of degree zero. We denote by $\mathcal{P}(\mathcal{U})$ the full dg subcategory of $C_{dg}(\mathrm{Sh}(X))$ that contains all the objects of $\mathcal{P}_*(\mathcal{U})$ and is closed under taking homotopy equivalent objects. Observe that any object of the form $\mathcal{C}_{\mathcal{U}}(R)$, with R a bounded complex of vector bundles, still belongs to $\mathcal{P}(\mathcal{U})$. It is the dg category that in [31] is denoted by $\mathrm{Perf}_{\check{C}ob*}(X)$. The dg category $\mathcal{P}(\mathcal{U})$ is strongly pretriangulated and moreover we have the following

Proposition 3.2.1 ([31], Proposition C.1 together with subsection C.1.1)**.** *The dg category $\mathcal{P}(\mathcal{U})$ is a dg enhancement of $\mathfrak{Perf}(X)$.*

Actually, it can be seen from the proof of Proposition C.1 in [31] that the exact equivalence $\omega_X : \mathrm{H}^0(\mathcal{P}(\mathcal{U})) \longrightarrow \mathfrak{Perf}(X)$ acts on objects as the identity.

Remark 3.2.2. Note that, by Remark C.3 of [31], any object of $\mathcal{P}(\mathcal{U})$ is homotopy equivalent to an object of the form $\mathcal{C}_{\mathcal{U}}(R)$, with R a bounded complex of vector bundles.

Analogously we will denote by $\mathcal{V} := \{V_j\}_{j\in J}$ a finite affine covering of Y; observe that $\mathcal{U}\times\mathcal{V} := \{U_i \times V_j\}_{(i,j)\in I\times J}$ is a finite affine covering of $X\times Y$. So we can define also the dg categories $\mathcal{P}_*(\mathcal{V})$, $\mathcal{P}(\mathcal{V})$, $\mathcal{P}_*(\mathcal{U}\times\mathcal{V})$ and $\mathcal{P}(\mathcal{U}\times\mathcal{V})$, together with the equivalences

$$\omega_Y : \mathrm{H}^0(\mathcal{P}(\mathcal{V})) \longrightarrow \mathfrak{Perf}(Y) \qquad \text{and} \qquad \omega_{X\times Y} : \mathrm{H}^0(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \longrightarrow \mathfrak{Perf}(X\times Y).$$

With this kind of enhancement we will try to lift all the three "pieces" of the Fourier-Mukai functor.

Remark 3.2.3. Observe that, by definition, the inclusion $\iota_{\mathcal{U}} : \mathcal{P}_*(\mathcal{U}) \hookrightarrow \mathcal{P}(\mathcal{U})$ is a quasi-equivalence as well as $\iota_{\mathcal{V}} : \mathcal{P}_*(\mathcal{V}) \hookrightarrow \mathcal{P}(\mathcal{V})$ and $\iota_{\mathcal{U}\times\mathcal{V}} : \mathcal{P}_*(\mathcal{U}\times\mathcal{V}) \hookrightarrow \mathcal{P}(\mathcal{U}\times\mathcal{V})$. Moreover, since we are dealing with h-projective dg categories (see the remark below), the inclusion

$$L := \iota_{\mathcal{U}} \otimes \iota_{\mathcal{V}} : \mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}) \hookrightarrow \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})$$

is a quasi-equivalence, too by Remark 2.8 of [13].

Remark 3.2.4. Observe also that, since we are working over a field, all those dg categories will be h-projective and therefore also h-flat. In particular this avoids troubles whenever we are dealing with the tensor product of dg categories in **Hqe**. Actually we will see (in Proposition 3.8.1) that all the categories we are dealing with are h-flat even under the milder hypotesis $(**)$.

In the following pages we are going to prove some importat properties of the dg version of boxtimes bifunctor we have introduced in Section 1.3. We have said that one can define a bifunctor

$$-\boxtimes \sim := p^*(-) \otimes q^*(\sim) : \mathrm{Sh}(X) \times \mathrm{Sh}(Y) \longrightarrow \mathrm{Sh}(X\times Y).$$

that - in our hypoteses - is exact whenever we fix one of the two arguments (see the discussion about the boxtimes in Section 1.3). It naturally induces a dg bifunctor

$$\boxtimes : \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}) \longrightarrow \mathcal{P}(\mathcal{U}\times\mathcal{V}).$$

The fact that the image $\boxtimes(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))$ is actually contained in $\mathcal{P}(\mathcal{U}\times\mathcal{V})$ follows from Proposition C.11, Corollary C.12 and subsection C.3.3 of [31]. Notice that those results, originally proved in the hypotesis of being over a field, actually hold also when the schemes satisfy condition $(**)$ — in this case the role of Lemma A.16 (c) in [31] is played by Lemma 3.2.8 below.

Remark 3.2.5. From those last two results it follows that - if A is a bounded complex of vector bundles on X and B is a bounded complex of vector bundles on Y - there is a homotopy equivalence between $\mathcal{C}_{\mathcal{U}}(A) \boxtimes \mathcal{C}_{\mathcal{V}}(B)$ and $\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(A \boxtimes B)$.

Proposition 3.2.6 ([31], Lemma C.14 and subsection C.3.3)**.** *The dg-functor* $\boxtimes$: $\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}) \longrightarrow \mathcal{P}(\mathcal{U}\times\mathcal{V})$ *is quasi-fully faithful.*

Actually, a stronger result than Proposition 3.2.6 holds true.

Proposition 3.2.7. *For any couple of objects (A, B) and (A', B') of $\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})$ we have that the canonical map*

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}((A, B), (A', B')) \longrightarrow \mathrm{Hom}_{\mathcal{P}(\mathcal{U}\times\mathcal{V})}(A \boxtimes B, A' \boxtimes B')$$

is a homotopy equivalence. Moreover, such a map is an isomorphism every time A, A', B and B' are Čech resolutions of bounded complexes of vector bundles (on X and Y, respectively).

Proof. Let us start by considering vector bundles P and P' on X and Q and Q' on Y. We need that the natural morphism of complexes (♡)

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{C}_{\mathcal{U}}(P),\mathcal{C}_{\mathcal{U}}(P'))\otimes\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{C}_{\mathcal{V}}(Q),\mathcal{C}_{\mathcal{V}}(Q')) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X\times Y}}(\mathcal{C}_{\mathcal{U}}(P)\boxtimes\mathcal{C}_{\mathcal{V}}(Q),\mathcal{C}_{\mathcal{U}}(P')\boxtimes\mathcal{C}_{\mathcal{V}}(Q'))$$

is an isomorphism.

The i-th degree of the right hand side is given by:

$$\bigoplus_{\omega}\mathrm{Hom}_{\mathcal{O}_{X\times Y}}\Big((\mathcal{C}_{\mathcal{U}}(P)\boxtimes\mathcal{C}_{\mathcal{V}}(Q))^{\omega},(\mathcal{C}_{\mathcal{U}}(P')\boxtimes\mathcal{C}_{\mathcal{V}}(Q'))^{\omega+i}\Big)=$$

$$\bigoplus_{\omega}\mathrm{Hom}_{\mathcal{O}_{X\times Y}}\Big(\bigoplus_{a+b=\omega}\bigoplus_{i_0<..<i_a}\bigoplus_{j_0<..<j_b}(P_{U_{i_0..i_a}}\boxtimes Q_{V_{j_0..j_b}}),\bigoplus_{c+d=\omega+i}\bigoplus_{l_0<..<l_c}\bigoplus_{t_0<..<t_d}(P'_{U_{l_0..l_c}}\boxtimes Q'_{V_{t_0..t_d}})\Big)$$

The latter, thanks to the Lemma below, is isomorphic to (♠)

$$\bigoplus_{\omega}\bigoplus_{a+b=\omega}\bigoplus_{c+d=\omega+i}\bigoplus_{i_0..i_a}\bigoplus_{j_0..j_b}\bigoplus_{l_0..l_c}\bigoplus_{t_0..t_d}\mathrm{Hom}_{\mathcal{O}_{X\times Y}}\Big((P\boxtimes Q)_{U_{i_0..i_a}\times V_{j_0..j_b}},(P'\boxtimes Q')_{U_{l_0..l_c}\times V_{t_0..t_d}}\Big)$$

On the other hand the i-th degree of the complex on the left is:

$$\bigoplus_{r+s=i}\Big(\bigoplus_{h}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{C}_{\mathcal{U}}(P)^{h},\mathcal{C}_{\mathcal{U}}(P')^{h+r})\otimes\bigoplus_{k}\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{C}_{\mathcal{V}}(Q)^{k},\mathcal{C}_{\mathcal{V}}(Q')^{k+s})\Big)=$$

$$\bigoplus_{r+s=i}\Big(\bigoplus_{h}\mathrm{Hom}_{\mathcal{O}_X}(\bigoplus_{i_0<..<i_h}P_{U_{i_0..i_h}},\bigoplus_{l_0<..<l_{h+r}}P'_{U_{l_0..l_{h+r}}})\otimes\bigoplus_{k}\mathrm{Hom}_{\mathcal{O}_Y}(\bigoplus_{j_0<..<j_k}Q_{V_{j_0..j_k}},\bigoplus_{t_0<..<t_{k+s}}Q'_{V_{t_0..t_{k+s}}})\Big)=$$

$$\bigoplus_{r+s=i}\bigoplus_{h}\bigoplus_{k}\bigoplus_{i_0..i_h}\bigoplus_{l_0..l_{h+r}}\bigoplus_{j_0..j_k}\bigoplus_{t_0..t_{k+s}}\Big(\mathrm{Hom}_{\mathcal{O}_X}(P_{U_{i_0..i_h}},P'_{U_{l_0..l_{h+r}}})\otimes\mathrm{Hom}_{\mathcal{O}_Y}(Q_{V_{j_0..j_k}},Q'_{V_{t_0..t_{k+s}}})\Big)$$

and - up to a reordering of the terms (for example with the substitution $h=a$, $k=b$, $r=c-a$, $s=d-b$) - it is the same as (♣)

$$\bigoplus_{\omega}\bigoplus_{a+b=\omega}\bigoplus_{c+d=\omega+i}\bigoplus_{i_0..i_a}\bigoplus_{j_0..j_b}\bigoplus_{l_0..l_c}\bigoplus_{t_0..t_d}\Big(\mathrm{Hom}_{\mathcal{O}_X}(P_{U_{i_0..i_a}},P'_{U_{l_0..l_c}})\otimes\mathrm{Hom}_{\mathcal{O}_Y}(Q_{V_{j_0..j_b}},Q'_{V_{t_0..t_d}})\Big)$$

But the latter, as well as (♠), is zero unless $U_{l_0..l_c}\subseteq U_{i_0..i_a}$ and $V_{t_0..t_d}\subseteq V_{j_0..j_b}$ by Lemma A.5 of [31] and, if these conditions are verified, reasoning as in the proof of Lemma C.14 of the same paper we can conclude that (♠) and (♣) are exactly the same. This yields the fact that the map (♡) is an isomorphism of complexes, as wished.

With essentially the same computations it is possible to see that such an isomorphisms holds true even in the case of P, P', Q and Q' being bounded complexes of vector bundles.

If we now consider generic objects of $\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})$, also in virtue of Remark 3.2.2, it is easy to see that the map (♡) is a homotopy equivalence in this case, considering the fact that covariant and contravariant Hom functors are defined on the whole dg category $C_{dg}(\mathrm{Sh}(X))\times C_{dg}((\mathrm{Sh}(Y))$ as well as the boxtimes is well defined on $C_{dg}((\mathrm{Sh}(X\times Y))$. □

Lemma 3.2.8. *Let X and Y be as in our hypoteses, let $\iota: U\longrightarrow X$ and $j: V\longrightarrow Y$ be the inclusions of affine open subsets. Then for any P (resp. Q) vector bundle on X (resp. Y) we have the following isomorphism of $\mathcal{O}_{X\times Y}$-modules*

$$\iota_*\iota^*P\boxtimes j_*j^*Q=p^*\iota_*\iota^*P\otimes q^*j_*j^*Q\simeq(\iota\times j)_*(\iota\times j)^*(p^*P\otimes q^*Q)$$

Proof. Let us start by considering the cartesian diagram

$$\begin{array}{ccc} U\times Y & \xrightarrow{p_{|U}} & U \\ {\scriptstyle \iota\times \mathrm{id}_Y}\downarrow & & \downarrow{\scriptstyle \iota} \\ X\times Y & \xrightarrow{p} & X \end{array}$$

By applying *base change* (see Proposition 1.3.3) we get

$$p^*\iota_*\iota^*P \simeq (\iota\times \mathrm{id}_Y)_* p^*_{|U}\iota^*P = (\iota\times \mathrm{id}_Y)_*(\iota\times \mathrm{id}_Y)^*p^*P$$

Clearly we have a completely analogous situation on Y hence we can write:

$$p^*\iota_*\iota^*P\otimes q^*j_*j^*Q \simeq (\iota\times \mathrm{id}_Y)_*(\iota\times \mathrm{id}_Y)^*p^*P\otimes(\mathrm{id}_X\times j)_*(\mathrm{id}_X\times j)^*q^*Q.$$

Now let us consider

$$(\iota\times j)_*(\iota\times j)^*(p^*P\otimes q^*Q) = (\mathrm{id}_X\times j)_*(\iota\times \mathrm{id}_V)_*\Big((\mathrm{id}_U\times j)^*(\iota\times \mathrm{id}_Y)^*p^*P\otimes(\iota\times \mathrm{id}_V)^*(\mathrm{id}_X\times j)^*q^*Q\Big)$$

The immersion $\iota : U \longrightarrow X$ is separated because U is affine [42, Tag 01KN], hence $\iota\times\mathrm{id}_V$ is separated as well since it is the base change of a separated morphism [42, Tag 01KU]. It is also quasi-compact: thus, being the module $(\mathrm{id}_X\times j)^*q^*Q$ flat (since the pullback of a vector bundle is still a vector bundle), we can apply *projection formula* (see Proposition 1.3.1) and gain that what we have written above is isomorphic to

$$\begin{aligned}(\mathrm{id}_X\times j)_*\Big((\iota\times \mathrm{id}_V)_*(\mathrm{id}_U\times j)^*(\iota\times \mathrm{id}_Y)^*p^*P\otimes(\mathrm{id}_X\times j)^*q^*Q\Big)\simeq\\ \simeq(\mathrm{id}_X\times j)_*\Big((\mathrm{id}_X\times j)^*(\iota\times \mathrm{id}_Y)_*(\iota\times \mathrm{id}_Y)^*p^*P\otimes(\mathrm{id}_X\times j)^*q^*Q\Big)\end{aligned}$$

where for the last isomorphism we have used again *base change*. Now, since the map $(\mathrm{id}_X\times j)$ is affine (as base change of an affine map), we can still apply *projection formula* and get that the latter is isomorphic to

$$(\iota\times \mathrm{id}_Y)_*(\iota\times \mathrm{id}_Y)^*p^*P\otimes(\mathrm{id}_X\times j)_*(\mathrm{id}_Y\times j)^*q^*Q$$

This concludes the proof. □

Proposition 3.2.7 yields the following

Corollary 3.2.9. *The dg functor* $\boxtimes : \mathcal{P}_*(\mathcal{U})\otimes\mathcal{P}_*(\mathcal{V}) \longrightarrow \mathcal{P}(\mathcal{U}\times\mathcal{V})$ *is fully faithful.*

Let us now denote by $\boxtimes_|$ the restriction to $\mathcal{P}_*(\mathcal{U})\otimes\mathcal{P}_*(\mathcal{V})$ of $\boxtimes$ and let us abbreviate by Y the dg Yoneda embedding $Y^{dg}_{\mathcal{P}(\mathcal{U}\times\mathcal{V})}$. The latter is a quasi-equivalence from $\mathcal{P}(\mathcal{U}\times\mathcal{V})$ to $\mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))$, since $\mathcal{P}(\mathcal{U}\times\mathcal{V})$ is a perfect dg category (see Section 3.3). We can prove the following

Proposition 3.2.10. *The dg-functor* $\mathrm{Res}_{\boxtimes_|}\circ Y : \mathcal{P}(\mathcal{U}\times\mathcal{V}) \longrightarrow \mathrm{dgMod}(\mathcal{P}_*(\mathcal{U})\otimes\mathcal{P}_*(\mathcal{V}))$ *induces a quasi-equivalence from* $\mathcal{P}(\mathcal{U}\times\mathcal{V})$ *to* $\mathrm{Perf}(\mathcal{P}_*(\mathcal{U})\otimes\mathcal{P}_*(\mathcal{V}))$.

Proof. We start by noticing that Corollary 3.2.9 gives us the commutativity of the following diagram

$$\begin{array}{ccc} \mathrm{dgMod}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})) & \xleftarrow{\ \mathrm{Res}_{\boxtimes_|}\ } & \mathrm{dgMod}(\mathcal{P}(\mathcal{U} \times \mathcal{V})) \\ \uparrow{\scriptstyle \tilde{Y}} & & \uparrow{\scriptstyle Y} \\ \mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}) & \xrightarrow{\ \boxtimes_|\ } & \mathcal{P}(\mathcal{U} \times \mathcal{V}) \end{array}$$

where we have denoted by $\tilde{Y}$ the dg Yoneda embedding $Y^{dg}_{\mathcal{P}_*(\mathcal{U})\otimes\mathcal{P}_*(\mathcal{V})}$. Let us now consider the dg category $\mathcal{B} := \boxtimes_|(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}))$ We have:

(*i*) $\mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y)(\mathrm{H}^0(\mathcal{B})) \subseteq \mathrm{H}^0(\mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})))$**:** In fact, thanks to the commutativity of the above diagram we can write

$$\begin{aligned} \mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y)(\mathrm{H}^0(\mathcal{B})) &= \mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y \circ \boxtimes_|)(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})) \\ &= \mathrm{H}^0(\tilde{Y})(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})) \subseteq \mathrm{H}^0(\mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))); \end{aligned}$$

(*ii*) $\mathrm{H}^0(\mathcal{P}(\mathcal{U} \times \mathcal{V}))$ **is classically generated by** $\mathrm{H}^0(\mathcal{B})$**:** in fact in our hypoteses we can apply the Lemma 3.4.1 of [5] together with the equivalence (1.1) and get that $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X \times Y))$ is generated by a single compact object $E \boxtimes F$, where E and F are the compact generators of $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$ and $D_{\mathrm{Qcoh}}(\mathrm{Sh}(Y))$, respectively. It follows that, see Theorem 1.1.12 and Definition 1.1.10, $\mathrm{H}^0(\mathcal{P}(\mathcal{U} \times \mathcal{V})) \simeq \mathfrak{Perf}(X \times Y) = D_{\mathrm{Qcoh}}(\mathrm{Sh}(X \times Y))^c$ is the smallest thick triangulated subcategory of itself containing $E \boxtimes F$ and therefore it is also the smallest thick triangulated subcategory of itself containing $\mathrm{H}^0(\mathcal{B})$: this precisely means that $\mathrm{H}^0(\mathcal{P}(\mathcal{U} \times \mathcal{V}))$ is classically generated by $\mathrm{H}^0(\mathcal{B})$;

(*iii*) $\mathrm{H}^0(\mathrm{Perf}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})))$ **is a thick triangulated subcategory of the triangulated category** $\mathrm{H}^0(\mathrm{dgMod}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})))$ by what we said in Remark 2.4.2.

Therefore we get that the image of $\mathrm{H}^0(\mathcal{P}(\mathcal{U} \times \mathcal{V}))$ through $\mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y)$ is contained in $\mathrm{H}^0(\mathrm{Perf}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})))$. Moreover, the commutativity of the diagram above tells us that $\mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y)_{|\mathrm{H}^0(\mathcal{B})}$ is a fully faithful functor: this fact, together with (*ii*) implies that $\mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y)$ is fully faithful as well.

Summarizing, we have a fully faithful exact functor:

$$\mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y) : \mathrm{H}^0(\mathcal{P}(\mathcal{U} \times \mathcal{V})) \longrightarrow \mathrm{H}^0(\mathrm{Perf}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}))).$$

Now, since $\mathrm{H}^0(\mathrm{Perf}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})))$ is classically generated by $\mathrm{H}^0(\tilde{Y})(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})) = \mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y)(\mathrm{H}^0(\mathcal{B}))$, in order to have that $\mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y)(\mathrm{H}^0(\mathcal{P}(\mathcal{U} \times \mathcal{V})))$ contains $\mathrm{H}^0(\mathrm{Perf}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})))$ (and therefore they are equal) we just need to show that it is closed under direct summands, cones and shifts and that contains $\mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y)(\mathrm{H}^0(\mathcal{B}))$ (but the latter fact is immediate). Finally, the fact that $\mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y)(\mathrm{H}^0(\mathcal{P}(\mathcal{U} \times \mathcal{V})))$ is closed under direct summands, cones and shifts follows since $\mathrm{H}^0(\mathcal{P}(\mathcal{U} \times \mathcal{V}))$ is idempotent complete, it is closed under cones and shifts and $\mathrm{H}^0(\mathrm{Res}_{\boxtimes_|} \circ Y)$ is fully faithful. □

Corollary 3.2.11. *The dg functor* $\mathrm{Res}_{\boxtimes_| |\mathrm{Perf}(\mathcal{P}(\mathcal{U} \times \mathcal{V}))} : \mathrm{Perf}(\mathcal{P}(\mathcal{U} \times \mathcal{V})) \longrightarrow \mathrm{Perf}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}))$ *is a quasi-equivalence.*

Proof. It follows from the Proposition and from the fact that Y is a quasi-equivalence from $\mathcal{P}(\mathcal{U}\times\mathcal{V})$ to $\mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))$. □

Lemma 3.2.12. *Let $L : \mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}) \longrightarrow \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})$ be the inclusion. Then* $\mathrm{Res}_L(\text{h-proj}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))) \subseteq \text{h-proj}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}))$.

Proof. We know that it suffices to show — see Proposition 2.5.2 — that

$$\mathrm{Res}_L(\overline{\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})}) \subseteq \text{h-proj}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})).$$

Let $\alpha := \mathrm{Hom}_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}\Big((-,=),(\mathcal{C}_{\mathcal{U}}(A),\mathcal{C}_{\mathcal{V}}(B))\Big) = Y_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}(\mathcal{C}_{\mathcal{U}}(A),\mathcal{C}_{\mathcal{V}}(B))$. We have:

$$\begin{aligned}\mathrm{Res}_L(\alpha) &= \mathrm{Hom}_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}\Big(L((-,=)),(\mathcal{C}_{\mathcal{U}}(A),\mathcal{C}_{\mathcal{V}}(B))\Big) = \\ &= \mathrm{Hom}_{\mathcal{P}_*(\mathcal{U})\otimes\mathcal{P}_*(\mathcal{V})}\Big((-,=),(\mathcal{C}_{\mathcal{U}}(A),\mathcal{C}_{\mathcal{V}}(B))\Big) \in \text{h-proj}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}))\end{aligned}$$

Now, we know that dg functors preserves shifts, cones of degree 0 morphisms and homotopy equivalences. Moreover, we have that $\text{h-proj}(\mathcal{A})$ is closed for homotopy equivalent objects for any dg category $\mathcal{A}$. Therefore we obtain that $\mathrm{Res}_L \circ Y_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}(C) \in \text{h-proj}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}))$ for any object $B \in \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})$. For the same reasons we have also $\mathrm{Res}_L(\overline{\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})}) \subseteq \text{h-proj}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}))$. □

From the above proof and from the fact that $\mathrm{Perf}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}))$ is closed under shifts, cones of degree 0 morphisms and homotopy equivalences we have

Corollary 3.2.13. $\mathrm{Res}_L(\mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))) \subseteq \mathrm{Perf}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}))$.

Now, since L is a quasi-equivalence (see Remark 3.2.3) we have by Proposition 2.5.2 (iii) (respectively, by Proposition 2.5.3) that the restrictions of Ind_L to h-projective dg modules (respectively, to perfect dg modules) is a quasi-equivalence. From the latter two results we can conclude that also the restriction of Res_L to h-projective (respectively, perfect) dg modules is a quasi-equivalence, since it is the adjoint of the version of Ind_L restricted to h-projective (respectively, perfect) dg modules.

Lemma 3.2.14. *The dg functor $\mathrm{Ind}_{\boxtimes} : \mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})) \longrightarrow \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))$ is a quasi-equivalence and its inverse in* **Hqe** *is equal to $\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|}$.*

Proof. From the equality $\boxtimes_| = \boxtimes \circ L$ we obtain $\mathrm{Ind}_{\boxtimes_|} = \mathrm{Ind}_{\boxtimes} \circ \mathrm{Ind}_L$. But now $\mathrm{Ind}_{\boxtimes_|} : \mathrm{Perf}(\mathcal{P}_*(\mathcal{U})\otimes\mathcal{P}_*(\mathcal{V})) \longrightarrow \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))$ is a quasi-equivalence since it is the left adjoint of the quasi-equivalence $\mathrm{Res}_{\boxtimes_|}$. Also $\mathrm{Ind}_L : \mathrm{Perf}(\mathcal{P}_*(\mathcal{U})\otimes\mathcal{P}_*(\mathcal{V})) \longrightarrow \mathrm{Perf}(\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V}))$ is a quasi-equivalences by the discussion above. It follows that $\mathrm{Ind}_{\boxtimes} = \mathrm{Ind}_{\boxtimes_|} \circ \mathrm{Res}_L$ in **Hqe** is a quasi-equivalence and, by taking inverses, we get our claim. □

3.3 Notation

Let us now spend some words to set the notation we will use from now on.

We will abbreviate by $Y_{\mathcal{U}}$ the dg Yoneda embedding

$$Y^{dg}_{\mathcal{P}(\mathcal{U})} : \mathcal{P}(\mathcal{U}) \longrightarrow \mathrm{dgMod}(\mathcal{P}(\mathcal{U})).$$

Note that it is a quasi-equivalence onto $\text{Perf}(\mathcal{P}(\mathcal{U}))$ since $\mathcal{P}(\mathcal{U})$ is a perfect dg category (look at Definition 2.4.3 and at Proposition 2.4.4). In fact it is pretriangulated by definition and $\mathfrak{Perf}(X)$ is a Karoubian category for any quasi-compact quasi-separated scheme X (see [5] Proposition 2.1.1).
Clearly we will adopt analogous convention for $Y^{dg}_{\mathcal{P}(\mathcal{V})}$ and $Y^{dg}_{\mathcal{P}(\mathcal{U}\times\mathcal{V})}$.

Moreover we will call

$$\pi_{\mathcal{U}} : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})$$

the dg functor sending an object A to $(A, \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y))$ and defined in the obvious way on morphisms. The fact that it is a dg functor is easy to check: by definition of tensor product of dg categories, we have $(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))(A \otimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y), A' \otimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)) := \mathcal{P}(\mathcal{U})(A, A') \otimes \mathcal{P}(\mathcal{V})(\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y), \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y))$. We need to show that it satisfies the equality:

$$d_{\mathcal{P}(\mathcal{U})}(\alpha) \otimes \text{id}_{\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)} = d_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}(\alpha \otimes \text{id}_{\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)})$$

for any $\alpha : A \longrightarrow A'$. But, by definition of tensor product of complexes, we have

$$d_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}(\alpha \otimes \text{id}_{\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)}) = d_{\mathcal{P}(\mathcal{U})}(\alpha) \otimes \text{id}_{\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)} + (-1)^{deg(\alpha)} \alpha \otimes d_{\mathcal{P}(\mathcal{V})}(\text{id}_{\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)})$$

and we are done since, by definition of dg category, the identity of an object is a closed map.

We define moreover the dg functor

$$\pi_{\mathcal{V}} : \mathcal{P}(\mathcal{V}) \longrightarrow \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})$$

sending an object B to $(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), B)$.

We will need also "restricted" versions of $\pi_{\mathcal{U}}$ and of $\pi_{\mathcal{V}}$.
Namely, $\bar{\pi}_{\mathcal{U}} : \mathcal{P}_*(\mathcal{U}) \to \mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})$ is the dg functor sending an object A to $(A, \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y))$ while $\bar{\pi}_{\mathcal{V}} : \mathcal{P}_*(\mathcal{V}) \longrightarrow \mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})$ is the dg functor sending an object B to $(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), B)$.

In the same spirit we define also the dg functors

$$\pi_1 : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U}) \qquad \text{and} \qquad \pi_2 : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U})$$

sending an object $B \in \mathcal{P}(\mathcal{U})$ respectively to $(B, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))$ and to $(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), B)$.

3.4 Pullback

Let us start by considering the *pullback* functor

$$p^* : \mathfrak{Perf}(X) \longrightarrow \mathfrak{Perf}(X \times Y),$$

where the map $p : X \times Y \longrightarrow X$ is the projection on the first factor. Note that, since p is flat, p^* does not need to be derived. We want to compute a dg lift of such a functor, via the Čech resolutions i.e. to find a dg functor $\mathsf{G} : \mathcal{P}(\mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{U}\times\mathcal{V})$ such that the following diagram

$$\begin{array}{ccc} H^0(\mathcal{P}(\mathcal{U})) & \xrightarrow{H^0(\mathsf{G})} & H^0(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \\ \downarrow{\scriptstyle \omega_X} & & \downarrow{\scriptstyle \omega_{X\times Y}} \\ \mathfrak{Perf}(X) & \xrightarrow{p^*} & \mathfrak{Perf}(X\times Y) \end{array}$$

commutes.

We define the dg functor G as the composition $\boxtimes \circ \pi_{\mathcal{U}}$ where $\pi_{\mathcal{U}}$ is the dg functor we have just defined.

Proposition 3.4.1. *The dg functor* $\boxtimes \circ \pi_{\mathcal{U}} : \mathcal{P}(\mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{U} \times \mathcal{V})$ *is a dg lift of the triangulated functor* p^**.*

Proof. We need to show that there is an isomorphism η of triangulated functors from $p^* \circ \omega_X$ to $\omega_{X \times Y} \circ H^0(\boxtimes \circ \pi_{\mathcal{U}})$.
By Remark C.3 of [31], any object of $\mathcal{P}(\mathcal{U})$ is homotopy equivalent (and hence isomorphic in $\mathrm{H}^0(\mathcal{P}(\mathcal{U}))$) to an object of the form $\mathcal{C}_{\mathcal{U}}(R)$, with R a bounded complex of vector bundles. Let therefore A be a bounded complex of vector bundles. We have:

- $p^* \circ \omega_X(\mathcal{C}_{\mathcal{U}}(A)) = p^*(\mathcal{C}_{\mathcal{U}}(A))$;
- $\omega_{X \times Y} \circ \mathrm{H}^0(\boxtimes \circ \pi_{\mathcal{U}})(\mathcal{C}_{\mathcal{U}}(A)) = \omega_{X \times Y}(\mathcal{C}_{\mathcal{U}}(A) \boxtimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)) = \mathcal{C}_{\mathcal{U}}(A) \boxtimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y) = p^*\mathcal{C}_{\mathcal{U}}(A) \otimes q^*\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y) \simeq p^*\mathcal{C}_{\mathcal{U}}(A) \otimes q^*\mathcal{O}_Y \simeq p^*\mathcal{C}_{\mathcal{U}}(A)$.

The two isomorphisms are meant in $\mathfrak{Perf}(X \times Y)$ and are due to the fact that q is a flat map and hence q^* is exact.
So we have an isomorphism $\eta_{\mathcal{C}_{\mathcal{U}}(A)} : p^* \circ \omega_X(\mathcal{C}_{\mathcal{U}}(A)) \longrightarrow \omega_{X \times Y} \circ \mathrm{H}^0(\boxtimes \circ \pi_{\mathcal{U}})(\mathcal{C}_{\mathcal{U}}(A))$ that is natural. In fact, if $f : \mathcal{C}_{\mathcal{U}}(A) \longrightarrow \mathcal{C}_{\mathcal{U}}(B)$ is a morphism in $\mathrm{H}^0(\mathcal{P}(\mathcal{U}))$, we can easily check the equality $\eta_{\mathcal{C}_{\mathcal{U}}(B)} \circ p^* \circ \omega_X(f) = \omega_{X \times Y} \circ \mathrm{H}^0(\boxtimes \circ \pi_{\mathcal{U}})(f) \circ \eta_{\mathcal{C}_{\mathcal{U}}(A)}$. □

3.5 Pushforward

For what concerns the (derived) *pushforward* functor Rp_*, we can use a more formal argument in order to get a dg lift for it with respect to Čech enhancements using the fact that Rp_* is right adjoint to $Lp^* = p^*$.

Remember that, by Proposition 2.5.3, given a dg functor $\mathsf{G} : \mathcal{A} \longrightarrow \mathcal{B}$ between two dg categories we have that the dg functor $\mathrm{Ind}_{\mathsf{G}} : \mathrm{dgMod}(\mathcal{A}) \longrightarrow \mathrm{dgMod}(\mathcal{B})$ restricts to perfect dg modules.

Therefore the dg functor

$$\mathrm{Ind}_{\boxtimes \circ \pi_{\mathcal{U}}} = \mathrm{Ind}_{\boxtimes} \circ \mathrm{Ind}_{\pi_{\mathcal{U}}} : \mathrm{Perf}(\mathcal{P}(\mathcal{U})) \longrightarrow \mathrm{Perf}(\mathcal{P}(\mathcal{U} \times \mathcal{V}))$$

is another dg lift of p^*.

Proposition 3.5.1. *The dg functor* $\mathsf{G} := \mathrm{Res}_{\pi_{\mathcal{U}}} \circ \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|}$ *is a dg lift of* Rp_**.*

Proof. We can deduce from Lemma 3.2.14 that $\mathrm{H}^0(\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|})$ is a right adjoint to $\mathrm{H}^0(\mathrm{Ind}_{\boxtimes})$; moreover, we know by Proposition 2.5.2 (i) that $\mathrm{Res}_{\pi_{\mathcal{U}}}$ is a right adjoint to $\mathrm{Ind}_{\pi_{\mathcal{U}}}$ at the level of dg modules. Therefore we have that G must be a dg lift of the right adjoint of p^* once we show that it restricts to perfect dg modules.

We have already seen that both Ind_L and $\mathrm{Res}_{\boxtimes_|}$ restricts to perfect dg modules. To what concerns $\mathrm{Res}_{\pi_{\mathcal{U}}}$ let (A,B) be an object of $\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})$:

$$\mathrm{Res}_{\pi_{\mathcal{U}}}(\mathrm{Hom}_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}(-,(A,B))) = \mathrm{Hom}_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}(\pi_{\mathcal{U}}(-),(A,B)) =$$
$$\mathrm{Hom}_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}((-,\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)),(A,B)) = \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(-,A)\otimes_{\Bbbk}\mathrm{Hom}_{\mathcal{P}(\mathcal{V})}(\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y),B).$$

Now, we claim that $\mathrm{Hom}_{\mathcal{P}(\mathcal{V})}(\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y),B)$ is quasi-isomorphic to a bounded complex of projective (actually free) $\Bbbk$ - modules of finite rank. In fact, it is bounded from the definition of Čech resolution and from the finiteness of the covering $\mathcal{V}$. It is moreover trivially made of free modules since $\Bbbk$ is a field. Finally it suffices to show that it has finite dimensional cohomologies: since we know that $\mathcal{P}(\mathcal{V})$ is a dg enhancement of $\mathfrak{Perf}(Y)$ we have

$$\mathrm{H}^i(\mathrm{Hom}_{\mathcal{P}(\mathcal{V})}(\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y),B)) = \mathrm{H}^0(\mathrm{Hom}_{\mathcal{P}(\mathcal{V})}(\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y),B[i])) =$$
$$\mathrm{Hom}_{\mathrm{H}^0(\mathcal{P}(\mathcal{V}))}(\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y),B[i]) \simeq \mathrm{Hom}_{\mathfrak{Perf}(Y)}(\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y),B[i])$$

and the latter is finite dimensional over $\Bbbk$ by a theorem of Serre (see [18] Théorème III.3.2.1 and Corollaire III.3.2.3) since Y is proper. It follows that $\mathrm{Hom}_{\mathcal{P}(\mathcal{V})}(\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y),B)$ lies in $\mathrm{Perf}(\Bbbk)$ for any $B\in\mathcal{P}(\mathcal{V})$.

But now, since $\Bbbk$ is a field, we have that $\mathrm{H}^0(\mathrm{Pretr}(\Bbbk))$ is an idempotent complete category and hence $\mathrm{Perf}(\Bbbk)$ coincides with $\mathrm{Pretr}(\Bbbk)$ (cfr. Remark 2.4.2): therefore $\mathrm{Hom}_{\mathcal{P}(\mathcal{V})}(\mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y),B)$ is obtained from $\Bbbk$ with a finite number of shifts and cones (of degree 0 morphisms). But all those shifts and cones are preserved by the tensorization with $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(-,A)\in\mathrm{Perf}(\mathcal{P}(\mathcal{U}))$. It follows that $\mathrm{Res}_{\pi_{\mathcal{U}}}(\mathrm{Hom}_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}(-,(A,B)))$ is obtained from $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(-,A)\in\mathrm{Perf}(\mathcal{P}(\mathcal{U}))$ with a finite number of shifts and cones of degree 0 morphisms and thus it lies in $\mathrm{Perf}(\mathcal{P}(\mathcal{U}))$ since the latter is closed with respect to all those cones and shifts. Finally, the claim follows by part (iv) of Proposition 2.5.2. □

Remark 3.5.2. It is obvious that $\mathrm{Res}_{\pi_{\mathcal{V}}}\circ\mathrm{Ind}_L\circ\mathrm{Res}_{\boxtimes_|}:\mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))\longrightarrow\mathrm{Perf}(\mathcal{P}(\mathcal{V}))$ will be a dg lift of $Rq_*:\mathfrak{Perf}(X\times Y)\longrightarrow\mathfrak{Perf}(Y)$.

3.6 Tensor product

It's now time to consider *tensor product.* We need to lift, for an object $\mathcal{E}\in\mathfrak{Perf}(X)$, the triangulated functor $-\otimes^L\mathcal{E}:\mathfrak{Perf}(X)\longrightarrow\mathfrak{Perf}(X)$.

Let us define $\mathcal{P}_{\otimes}(\mathcal{U})$ to be the smallest full dg subcategory of $C_{dg}(\mathrm{Sh}(X))$ containing all the objects of the form $\mathcal{C}_{\mathcal{U}}(A)$ and $\mathcal{C}_{\mathcal{U}}(A)\otimes\mathcal{C}_{\mathcal{U}}(B)$ for A and B vector bundles on X, and closed under shifts, cones and taking homotopy equivalent objects. We want to show the following

Proposition 3.6.1. *The dg category $\mathcal{P}_{\otimes}(\mathcal{U})$ is a dg enhancement of $\mathfrak{Perf}(X)$.*

Proof. We will proceed by following the idea of the proof of Proposition C.1 of [31]. Clearly the natural functor

$$\mathrm{H}^0(\mathcal{P}_{\otimes}(\mathcal{U}))\longrightarrow\mathfrak{Perf}(X)$$

is essentially surjective. It remains to prove that

$$\mathrm{Hom}_{\mathrm{H}^0(C_{dg}(\mathrm{Sh}(X)))}(P, Q[n]) \longrightarrow \mathrm{Hom}_{D(\mathrm{Sh}(X))}(P, Q[n])$$

is an isomorphism, for any $n \in \mathbb{Z}$, in the (four) cases where P and Q are of the form $\mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(B)$ or $\mathcal{C}_{\mathcal{U}}(R)$ for A, B and R vector bundles on X. Note that $\mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(B)$ is made up from objects of the form $A_U \otimes B_V \simeq (A \otimes B)_{U \cap V}$, where U and V are in $\mathcal{U}$. Hence the two cases where P is of the form $\mathcal{C}_{\mathcal{U}}(R)$ can be treated as in the proof of Proposition C.1 of [31].

For the remaining cases our claim is to show that:

$$\mathrm{Hom}_{\mathrm{H}^0(C_{dg}(\mathrm{Sh}(X)))}(\mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(B), R_V[n]) \longrightarrow \mathrm{Hom}_{D(\mathrm{Sh}(X))}(\mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(B), R_V[n])$$

is an isomorphism, for any $n \in \mathbb{Z}$, where V is the intersection of some elements of $\mathcal{U}$. Consider the composition:

$$A \otimes B \xrightarrow{\alpha \otimes \mathrm{id}_B} \mathcal{C}_{\mathcal{U}}(A) \otimes B \xrightarrow{\mathrm{id}_{\mathcal{C}_{\mathcal{U}}(A)} \otimes \beta} \mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(B)$$

where α (resp. β) is the quasi-isomorphism between A (resp. B) and its Čech resolution. Note that $\alpha \otimes \mathrm{id}_B$ is still a quasi-isomorphism since B is flat; moreover $\mathrm{id}_{\mathcal{C}_{\mathcal{U}}(A)} \otimes \beta$ is a quasi-isomorphism, too since $\mathcal{C}_{\mathcal{U}}(A)$ is h-flat (see the Lemma below).

Now let us consider the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{H}^0(C_{dg}(\mathrm{Sh}(X)))}(\mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(B), R_V[n]) & \longrightarrow & \mathrm{Hom}_{D(\mathrm{Sh}(X))}(\mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(B), R_V[n]) \\
\downarrow {\scriptstyle (\mathrm{id}_{\mathcal{C}_{\mathcal{U}}(A)} \otimes \beta)^*} & & \downarrow \\
\mathrm{Hom}_{\mathrm{H}^0(C_{dg}(\mathrm{Sh}(X)))}(\mathcal{C}_{\mathcal{U}}(A) \otimes B, R_V[n]) & \longrightarrow & \mathrm{Hom}_{D(\mathrm{Sh}(X))}(\mathcal{C}_{\mathcal{U}}(A) \otimes B, R_V[n]) \\
\downarrow {\scriptstyle (\alpha \otimes \mathrm{id}_B)^*} & & \downarrow \\
\mathrm{Hom}_{\mathrm{H}^0(C_{dg}(\mathrm{Sh}(X)))}(A \otimes B, R_V[n]) & \xrightarrow{\delta} & \mathrm{Hom}_{D(\mathrm{Sh}(X))}(A \otimes B, R_V[n])
\end{array}$$

where the two right vertical arrows are isomorphism by what we have just observed. The bottom map corresponds, via the adjuction $j^* \dashv j_*$ (where j is the inclusion of V in X), to

$$\mathrm{Hom}_{\mathrm{H}^0(C(\mathrm{Sh}(V)))}(j^*(A \otimes B), j^*(R)[n]) \longrightarrow \mathrm{Hom}_{D(\mathrm{Sh}(V))}(j^*(A \otimes B), j^*(R)[n]).$$

Now, since both $j^*(A \otimes B)$ and $j^*(R)[n]$ are quasi-coherent $\mathcal{O}_V$-modules — and taking into account equivalence (1.1) — we can conclude that the above map is an isomorphism since $j^*(A \otimes B)$ is projective in $\mathrm{Qcoh}(V)$. Therefore δ is an isomorphism as well.

In order to show that $(\alpha \otimes \mathrm{id}_B)^*$ is an isomorphism we will prove that

$$\omega : \mathrm{Hom}_{C_{dg}(\mathrm{Sh}(X))}(\mathcal{C}_{\mathcal{U}}(A) \otimes B, R_V) \longrightarrow \mathrm{Hom}_{C_{dg}(\mathrm{Sh}(X))}(A \otimes B, R_V)$$

is a quasi-isomorphism. But now we can reason exactly as in the proof of Lemma C.2 of [31]: the degree 0 component of the right hand side is

$$H := \mathrm{Hom}_{\mathcal{O}_V}((A \otimes B)_{|V}, R_{|V})$$

and all other components are trivial. The left handside, instead, is made up of direct sums of objects $\mathrm{Hom}_{\mathcal{O}_X}((A \otimes B)_W, R_V)$ where W is the intersection of a (positive) number of element of $\mathcal{U}$. These objects - thanks to [31], Lemma A.5 - are equal to H if $V \subseteq W$ and are zero otherwise. Whence we obtain that the left hand side is equal to the chain comlplex that has in degree i a number of copies of H that depends on $s_0 < \cdots < s_i$ with all these variables lying in the set of elements of $\mathcal{U}$ containing V. Therefore we can see ω as the augmentation map of the chain complex of a simplex with coefficients in H that is actually a homotopy equivalence and hence a quasi-isomorphism.
The same argument can be used to show that $(\mathrm{id}_{\mathcal{C}_{\mathcal{U}}(A)} \otimes \beta)^*$ is an isomorphism and finally we get our claim. □

Lemma 3.6.2. *Let X be a quasi-compact separated scheme and let be P a vector bundle (or a bounded complex of vector bundles) on X. Then $\mathcal{C}_{\mathcal{U}}(P)$ is an h-flat complex of $\mathcal{O}_X$ - modules.*

Proof. Since $\mathcal{C}_{\mathcal{U}}(P)$ is a bounded complex - and since the finite direct sum of flat modules is still flat - our claim follows once we have proved that j_*j^*P is a flat $\mathcal{O}_X$ - module, where $j : U \to X$ is the inclusion of an affine open set (see [42, Tag 06YD]).
Let $\gamma : N \longrightarrow N'$ be an injective morphism of $\mathcal{O}_X$ - modules. We want to prove that $\mathrm{id} \otimes \gamma : j_*j^*P \otimes N \longrightarrow j_*j^*P \otimes N'$ is still injective.
Now, in our case the morphism j is affine therefore we can make use of the projection formula (see Proposition 1.3.1) and get that $j_*j^*P \otimes M \simeq j_*(j^*P \otimes j^*M)$ for any $\mathcal{O}_X$ - module M. But this means that $\mathrm{id} \otimes \gamma$ is the pushforward along j of the map

$$j^*P \otimes j^*N \xrightarrow{\mathrm{id}\otimes j^*\gamma} j^*P \otimes j^*N'.$$

Since j is an open immersion it follows that it is flat and hence j^* is exact and $j^*\gamma$ is still injective. Moreover j^*P is still a vector bundle and in particular it is flat therefore $\mathrm{id} \otimes j^*\gamma$ is again an injective map. But finally $j_*(\mathrm{id} \otimes j^*\gamma) = \mathrm{id} \otimes \gamma$ is injective as well. □

Corollary 3.6.3. *The two dg-categories $\mathcal{P}(\mathcal{U})$ and $\mathcal{P}_{\otimes}(\mathcal{U})$ are actually equal.*

Proof. The inclusion $\mathcal{P}(\mathcal{U}) \subseteq \mathcal{P}_{\otimes}(\mathcal{U})$ holds by definition. Now, we have that $\mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(B)$ and $\mathcal{C}_{\mathcal{U}}(A \otimes B)$ are isomorphic in $\mathfrak{Perf}(X)$ i.e. are quasi-isomorphic. But this, by the above Proposition, implies that they are isomorphic in $\mathrm{H}^0(\mathcal{P}_{\otimes}(\mathcal{U}))$ and it means that they are homotopy equivalent. Whence $\mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(B)$ lies already in $\mathcal{P}(\mathcal{U})$ and we get the other inclusion. □

Observe that this last corollary tells us that $\mathcal{P}(\mathcal{U})$ contains all the elements of the form $\mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(B)$, for A and B vector bundles on X.

Thanks to the discussion above the dg functor

$$\mathsf{K} : \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{U})$$

that sends $(A, B) \in \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U})$ into $A \otimes B$ is well defined.

In addition, if we fix a bounded complex R of vector bundles of X, we can define the dg functor:

$$\mathsf{K}_{\mathcal{C}_{\mathcal{U}}(R)} : \mathcal{P}(\mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{U}) \qquad A \longmapsto A \otimes \mathcal{C}_{\mathcal{U}}(R).$$

where A is any object of $\mathcal{P}(\mathcal{U})$.

Let now $\mathcal{E}$ be an object of $\mathfrak{Perf}(X)$. In our hypoteses we know that it is quasi-isomorphic to some bounded complex of vector bundles E. We have the following

Proposition 3.6.4. *The functor* $\mathsf{K}_{\mathcal{C}_{\mathcal{U}}(E)}$ *is a dg-lift of* $-\otimes^L \mathcal{E} : \mathfrak{Perf}(X) \longrightarrow \mathfrak{Perf}(X)$.

Proof. We have already observed (Remark 3.2.2) that any element of $\mathcal{P}(\mathcal{U})$ is homotopy equivalent to one of the form $\mathcal{C}_{\mathcal{U}}(A)$ and that tensorization preserves homotopy equivalent objects in $C_{dg}(\mathrm{Sh}(X))$. Whence it suffices to prove our claim for an element of the form $\mathcal{C}_{\mathcal{U}}(A)$. We have:

- $\omega_X \circ \mathrm{H}^0(\mathsf{K}_{\mathcal{C}_{\mathcal{U}}(E)})(\mathcal{C}_{\mathcal{U}}(A)) = \omega_X(\mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(E)) = \mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(E)$;
- $(-\otimes^L \mathcal{E}) \circ \omega_X(\mathcal{C}_{\mathcal{U}}(A)) = (-\otimes^L \mathcal{E})(\mathcal{C}_{\mathcal{U}}(A)) = \mathcal{C}_{\mathcal{U}}(A) \otimes^L \mathcal{E} \simeq \mathcal{C}_{\mathcal{U}}(A) \otimes E$.

and hence we can define the natural transformation

$$\eta : (-\otimes^L \mathcal{E}) \circ \omega_X \longrightarrow \omega_X \circ \mathrm{H}^0(\mathsf{K}_{\mathcal{C}_{\mathcal{U}}(E)})$$

by taking, for any $(\mathcal{C}_{\mathcal{U}}(A) \in \mathcal{P}(\mathcal{U})$, the isomorphism in $\mathfrak{Perf}(X)$ from $\mathcal{C}_{\mathcal{U}}(A) \otimes E$ to $\mathcal{C}_{\mathcal{U}}(A) \otimes \mathcal{C}_{\mathcal{U}}(E)$ that we have since $\mathcal{C}_{\mathcal{U}}(A)$ is a bounded complex of flat modules (as the one between $\mathcal{C}_{\mathcal{U}}(A) \otimes^L \mathcal{E}$ and $\mathcal{C}_{\mathcal{U}}(A) \otimes E$).
Now, for any map $f : \mathcal{C}_{\mathcal{U}}(A) \longrightarrow \mathcal{C}_{\mathcal{U}}(B)$, we have $\omega_X \circ \mathrm{H}^0(\mathsf{K}_{\mathcal{C}_{\mathcal{U}}(E)})(f) = f \otimes \mathrm{id}_{\mathcal{C}_{\mathcal{U}}(E)}$ and $(-\otimes^L \mathcal{E}) \circ \omega_X(f) \simeq f \otimes \mathrm{id}_E$ and hence we can clearly see that $\omega_X \circ \mathrm{H}^0(\mathsf{K}_{\mathcal{C}_{\mathcal{U}}(E)})(f) \circ \eta_{\mathcal{C}_{\mathcal{U}}(A)} = \eta_{\mathcal{C}_{\mathcal{U}}(B)} \circ (-\otimes^L \mathcal{E}) \circ \omega_X(f)$.

We have therefore proved the commutativity of the diagram

$$\begin{array}{ccc} H^0(\mathcal{P}(\mathcal{U})) & \xrightarrow{H^0(\mathsf{K}_{\mathcal{C}_{\mathcal{U}}(E)})} & H^0(\mathcal{P}(\mathcal{U})) \\ \downarrow{\scriptstyle \omega_X} & & \downarrow{\scriptstyle \omega_X} \\ \mathfrak{Perf}(X) & \xrightarrow{-\otimes^L \mathcal{E}} & \mathfrak{Perf}(X), \end{array}$$

yielding that $\mathsf{K}_{\mathcal{C}_{\mathcal{U}}(E)}$ is a dg lift of the derived tensor product, as wished. □

3.7 A dg lift of $\Phi_{\mathcal{E}}$

In the previous Sections we have constructed a "piecewise" dg-lift of the Fourier-Mukai functor: it can be written down as follows

$$\Phi_{\mathcal{E}}^{dg}(-) := \mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U}\times\mathcal{V}} \circ \mathsf{K}_{\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)} \circ \boxtimes \circ \pi_{\mathcal{U}}(-).$$

where $\mathcal{E}$ is an element of $\mathfrak{Perf}(X \times Y)$ and E is a (fixed) bounded complex of vector bundles quasi-isomorphic to $\mathcal{E}$.

We will draw a diagram in order to make clear to what functors we are referring to:

$$\mathcal{P}(\mathcal{U}) \xrightarrow{\boxtimes\circ\pi_{\mathcal{U}}} \mathcal{P}(\mathcal{U}\times\mathcal{V}) \xrightarrow{\mathsf{K}_{\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)}} \mathcal{P}(\mathcal{U}\times\mathcal{V}) \xrightarrow{Y_{\mathcal{U}\times\mathcal{V}}} \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \xrightarrow{\mathsf{G}} \mathrm{Perf}(\mathcal{P}(\mathcal{V}))$$

We have abbreviated $\mathsf{G} := \mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|}$.

Remark 3.7.1. Observe that if E' is another bounded complex of vector bundles on $X \times Y$ quasi-isomorphic to $\mathcal{E} \in \mathfrak{Perf}(X \times Y)$ then if we use E' instead of E for the definition of $\Phi_{\mathcal{E}}^{dg}$ we obtain the same dg functor in **Hqe**. In fact we get that $\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)$ and $\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E')$ are quasi-isomorphic but - since they both live in the enhancement $\mathcal{P}(\mathcal{U}\times\mathcal{V})$ - they are actually homotopy equivalent. This gives us a natural transformation between the two version of $\Phi_{\mathcal{E}}^{dg}$ that is clearly a termwise homotopy equivalence. Hence we can conclude by Proposition 2.5.10.

What we have done in the present Chapter can be summarized by saying that we have constructed, for any $\mathcal{E} \in \mathfrak{Perf}(X \times Y)$, a commutative diagram of exact functors

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{P}(\mathcal{U})) & \xrightarrow{\mathrm{H}^0(\Phi_{\mathcal{E}}^{dg})} & \mathrm{H}^0(\mathrm{Perf}(\mathcal{P}(\mathcal{V}))) \\ \downarrow{\scriptstyle \omega_X} & & \downarrow{\scriptstyle \omega_Y \circ \mathrm{H}^0(Y_{\mathcal{V}})^{-1}} \\ \mathfrak{Perf}(X) & \xrightarrow{\Phi_{\mathcal{E}}} & \mathfrak{Perf}(Y) \end{array}$$

where ω_X and $\omega_Y \circ \mathrm{H}^0(Y_{\mathcal{V}})^{-1}$ are exact equivalences.

3.8 The most general case

We would like to point out here that the three dg lifts we have discussed in the previous pages can actually be computed also in the more general setting (∗∗). We have used the hypotesis of being over a field in two situations: the first one is when we said that, given two quasi-equivalences $F : \mathcal{A} \longrightarrow \mathcal{A}'$ and $G : \mathcal{B} \longrightarrow \mathcal{B}'$, the dg functor $F \otimes G : \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{A}' \otimes \mathcal{B}'$ is a quasi-equivalence as well. This happened for example for the dg functor L (see Remark 3.2.3). The second situation is in the proof of Proposition 3.5.1.

To what concerns the first problem, we can prove the following

Proposition 3.8.1. *The dg category $\mathcal{P}(\mathcal{U})$ is h-flat.*

Proof. Recall that for $\mathcal{P}(\mathcal{U})$ to be h-flat means that for any two of its objects A and B the dg functor

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A, B) \otimes -$$

preserves quasi-isomorphisms.

Let us consider the case of $A = \mathcal{C}_{\mathcal{U}}(P)$ and $B = \mathcal{C}_{\mathcal{U}}(Q)$, where P and Q are vector bundles on X: since $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(P), \mathcal{C}_{\mathcal{U}}(Q))$ is a bounded complex we have (see [42, Tag 06YD]) that it is h-flat if

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(P), \mathcal{C}_{\mathcal{U}}(Q))^n = \bigoplus_h \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(P)^h, \mathcal{C}_{\mathcal{U}}(Q)^{h+n})$$

is a flat $\mathcal{O}_X$-module for any n. But the latter is equal to

$$\bigoplus_h \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}\Big(\bigoplus_{i_0,..,i_h} P_{U_{i_0,..,i_h}}, \bigoplus_{j_0,..,j_{h+n}} Q_{U_{j_0,..,j_{h+n}}}\Big)$$
$$\simeq \bigoplus_h \bigoplus_{i_0,..,i_h} \bigoplus_{j_0,..,j_{h+n}} \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(P_{U_{i_0,..,i_h}}, Q_{U_{j_0,..,j_{h+n}}}).$$

Therefore — since direct sums of flat modules are again flat — we just need to check that the $\Bbbk$-module $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(P_U, Q_V)$ is flat where U and V are affine open sets. But, in our hypoteses, we can make use of Lemma A.5 of [31] and get that such a module vanishes if $V \not\subseteq U$ while in the case of $V \subseteq U$ it is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_V}(P_{|V}, Q_{|V})$$

and so we can reduce ourselves to the affine situation.
Suppose $V = \mathrm{Spec}(R)$: thus we need to show that $\mathrm{Hom}_R(A, B)$ is a flat $\Bbbk$-module for any finite projective R-modules A and B. We start by noticing that since, by hypoteses, R is a flat $\Bbbk$-module (as our scheme X is smooth) it is sufficient to check that $\mathrm{Hom}_R(A, B)$ is flat as an R-module. Now - by the projectivity of A - there exists an R-module A' and an $r \in \mathbb{N}$ such that $A \oplus A' = R^r$ hence we have:

$$\begin{aligned} \mathrm{Hom}_R(A, B) \oplus \mathrm{Hom}_R(A', B) &\cong \mathrm{Hom}_R(A \oplus A', B) \cong \mathrm{Hom}_R(R^r, B) \\ &\cong \oplus_r \mathrm{Hom}_R(R, B) \cong \oplus_r B \end{aligned}$$

where $\bigoplus_r$ stands for the direct sum of r copies. Now, by hypotesis B (and $\oplus_r B$, too) is a projective R-module hence $\mathrm{Hom}_R(A, B) \oplus \mathrm{Hom}_R(A', B)$ must be a projective R-module as well. It follows that $\mathrm{Hom}_R(A, B)$ is a projective, and therefore flat, R-module (we have used here that both direct sums and direct summands of projective modules are still projective).

Now suppose that A_1 and B are objects of $\mathcal{P}(\mathcal{U})$ such that $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A_1, B)$ is h-flat. If A' is a shift of A_1 then $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A', B)$ is clearly still h-flat. Let now A' be the cone of the (closed degree 0) morphism $f : A_1 \to A_2$, where A_2 is another object of $\mathcal{P}(\mathcal{U})$ such that $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A_2, B)$ is h-flat. Applying the dg functor $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(-, B)$ to the distinguished triangle $A_1 \xrightarrow{f} A_2 \to A'$ it yields a distinguished triangle

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A', B) \longrightarrow \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A_2, B) \longrightarrow \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A_1, B)$$

in $K(\mathrm{Sh}(X))$ and therefore $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A', B)$ is h-flat as well by [42, Tag 06Y2]. This shows that $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A', B)$ is h-flat for any object $A' \in \mathcal{P}(\mathcal{U})$. The case where B is a generic object of $\mathcal{P}(\mathcal{U})$ can be treated in a similar way.

Let A and B be two objects of $\mathcal{P}(\mathcal{U})$ that are homotopy equivalent to some objects (say respectively $\tilde{A}$ and $\tilde{B}$) obtained from Čech resolution of vector bundles by iteratively taking shifts or cone of degree 0 morphisms. Hence $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\tilde{A}, \tilde{B})$ is an h-flat complex for what we have just shown. Now, since for any object $C \in \mathcal{P}(\mathcal{U})$ we have that both $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(-, C)$ and $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(C, -)$ are well defined dg functors on all $C_{dg}(\mathrm{Sh}(X))$, we get the following chain of homotopy equivalences:

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A, B) \simeq \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\tilde{A}, B) \simeq \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\tilde{A}, \tilde{B}).$$

Therefore tensoring with $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A, B)$ preserves quasi-isomorphisms since tensoring with $\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\tilde{A}, \tilde{B})$ does (tensor product is a dg functor on $C_{dg}(\mathrm{Sh}(X))$ and hence preserves homotopy equivalent objects). This proves our claim. □

Corollary 3.8.2. *The dg category $\mathcal{P}_*(\mathcal{U})$ is h-flat as well.*

Proof. It follows since $\mathcal{P}_*(\mathcal{U})$ is a full dg subcategory of the h-flat dg category $\mathcal{P}(\mathcal{U})$. □

Those facts assure us that all the tensor products between our dg enhancements are well defined in **Hqe** without the need of dealing with h-projective (or h-flat) resolutions. This also implies that the product of quasi-equivalences is a again a quasi-equivalence (in fact, it is the "h-flatness" of the dg category that actually matter in Remark 2.8 of [13]).

To what concerns Proposition 3.5.1, if we now do not have that $\mathrm{Res}_{\boxtimes\circ\pi_{\mathcal{U}}}$ restricts to perfect dg modules, a more sophisticated argument can be adopted in order to end up with a dg lift of the derived pushforward along the projection.

For any dg category $\mathcal{A}$ consider the Drinfeld quotient $\mathrm{dgMod}(\mathcal{A})/\mathrm{Ac}(\mathcal{A})$: as we have seen in Section 2.4, we always have a triagulated functor to its homotopy category from the Verdier quotient $\mathrm{H}^0(\mathrm{dgMod}(\mathcal{A}))/\mathrm{H}^0(\mathrm{Ac}(\mathcal{A}))$ that is an equivalence if, for example, $\mathrm{dgMod}(\mathcal{A})$ is *homotopically flat*. Now, following Section 3.2 of [7] we have a "functorial" h-flat resolution $\mathsf{I} : \mathcal{C}^{hf} \longrightarrow \mathcal{C}$ which is the identity on objects. We have therefore the following dg functors

$$\text{h-proj}(\mathcal{A}) \xrightarrow{\ \iota\ } \mathrm{dgMod}(\mathcal{A}) \xleftarrow{\ \mathsf{I}\ } \mathrm{dgMod}(\mathcal{A})^{hf} \xrightarrow{\ \mathsf{Q}\ } \mathrm{dgMod}(\mathcal{A})^{hf}/\mathrm{Ac}(\mathcal{A})' \quad (3.1)$$

whose composition η is a quasi-equivalence. By $\mathrm{Ac}(\mathcal{A})'$ we mean the full dg subcategory of $\mathrm{dgMod}(\mathcal{A})^{hf}$ with the same object as $\mathrm{Ac}(\mathcal{A})$. We will denote by H the dg quasi-functor $\eta^{-1} \circ \mathsf{Q} \circ \mathsf{I}^{-1} : \mathrm{dgMod}(\mathcal{A}) \longrightarrow \text{h-proj}(\mathcal{A})$ that sends a dg $\mathcal{A}$- module to its h-projective resolution.

In particular we have, for any $\mathsf{M} \in \text{h-proj}(\mathcal{A})$ and for any $\mathsf{N} \in \mathrm{dgMod}(\mathcal{A})$ a quasi-isomorphism

$$\mathrm{Hom}_{\mathrm{dgMod}(\mathcal{A})}(\mathsf{M}, \mathsf{N}) \xrightarrow{\sim} \mathrm{Hom}_{\text{h-proj}(\mathcal{A})}(\mathsf{M}, \mathsf{H}(\mathsf{N})).$$

Lemma 3.8.3. *For any dg functor* $\mathsf{F} : \mathcal{A} \longrightarrow \mathcal{B}$ *we have an adjunction* $\mathrm{H}^0(\mathrm{Ind}_{\mathsf{F}}) \dashv \mathrm{H}^0(\mathsf{H} \circ \mathrm{Res}_{\mathsf{F}})$ *at the level of h-projective dg modules.*

Proof. Thanks to what we said above, for any $\mathsf{M} \in \text{h-proj}(\mathcal{A})$ and for any $\mathsf{N} \in \text{h-proj}(\mathcal{B})$ we have a quasi-isomorphism:

$$\mathrm{Hom}_{\text{h-proj}(\mathcal{B})}(\mathrm{Ind}_{\mathsf{F}}(\mathsf{M}), \mathsf{N}) = \mathrm{Hom}_{\mathrm{dgMod}(\mathcal{B})}(\mathrm{Ind}_{\mathsf{F}}(\mathsf{M}), \mathsf{N}) \simeq$$
$$\mathrm{Hom}_{\mathrm{dgMod}(\mathcal{A})}(\mathsf{M}, \mathrm{Res}_{\mathsf{F}}(\mathsf{N})) \xrightarrow{\sim} \mathrm{Hom}_{\text{h-proj}(\mathcal{A})}(\mathsf{M}, \mathsf{H} \circ \mathrm{Res}_{\mathsf{F}}(\mathsf{N}))$$

and this is enough to conclude. □

Proposition 3.8.4. *The dg quasi-functor*

$$\mathsf{H} \circ \mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_{L} \circ \mathrm{Res}_{\boxtimes_{|}} : \text{h-proj}(\mathcal{P}(\mathcal{U} \times \mathcal{V})) \longrightarrow \text{h-proj}(\mathcal{P}(\mathcal{U}))$$

restricts to perfect dg modules and such a restriction is a dg lift of Rp_*.

Proof. in virtue of the above lemma, the only thing we need to show is the fact that - having denoted $\mathsf{G} := \mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_{L} \circ \mathrm{Res}_{\boxtimes_{|}}$ - the dg functor $\mathsf{H} \circ \mathsf{G}$ restricts to perfect complexes. Let us consider the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{H}^0(\text{h-proj}(\mathcal{P}(\mathcal{U}))) & \xrightarrow{\mathrm{H}^0(\mathrm{Ind}_{\boxtimes\circ\pi_{\mathcal{U}}})} & \mathrm{H}^0(\text{h-proj}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))) \\
\uparrow & & \uparrow \\
\mathrm{H}^0(\mathrm{Perf}(\mathcal{P}(\mathcal{U}))) & \longrightarrow & \mathrm{H}^0(\mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))) \\
\sim\uparrow & & \uparrow\sim \\
\mathrm{H}^0(\mathcal{P}(\mathcal{U})) & \xrightarrow{\mathrm{H}^0(\boxtimes\circ\pi_{\mathcal{U}})} & \mathrm{H}^0(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \\
\sim\uparrow & & \uparrow\sim \\
\mathfrak{Perf}(X) & \xrightarrow{p^*} & \mathfrak{Perf}(X\times Y).
\end{array}
$$

This tells us that p^* is isomorphic to $\mathrm{H}^0(\boxtimes\circ\pi_{\mathcal{U}})$ at the level of $\mathrm{H}^0(\mathcal{P}(\mathcal{U}))$ and of $\mathrm{H}^0(\mathrm{Perf}(\mathcal{P}(\mathcal{U})))$.

Moreover we have that $\mathrm{H}^0(\text{h-proj}(\mathcal{P}(\mathcal{U})))$ is equivalent to $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$: take $\mathcal{C} = D^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$ a dg enhancement of $D_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$, $\mathcal{B} = \mathcal{P}(\mathcal{U})$ in Proposition 1.17 of [30] and notice that we have a quasi-equivalence between the dg category $\mathrm{SF}(\mathcal{A})$ of semi-free dg modules over $\mathcal{A}$ and h-proj$(\mathcal{A})$ for any dg category $\mathcal{A}$ (see, for example section 3.1 of [25]). For example, $D^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$ can be the Drinfeld quotient

$$C^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X))/Ac^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X))$$

between the dg category of complexes of sheaves of modules with quasi coherent cohomology and its full dg subcategory of acyclic complexes. It follows that we have a dg quasi-functor $\phi_X : \mathcal{D}^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X)) \longrightarrow \text{h-proj}(\mathcal{P}(\mathcal{U}))$ such that $\mathrm{H}^0(\phi_X)$ is an equivalence. Clearly we have the equivalence between $\mathrm{H}^0(\text{h-proj}(\mathcal{P}(\mathcal{U}\times\mathcal{V})))$ and $D^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X\times Y))$ as well.

Now the functor p^* defines a dg functor $C^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X)) \longrightarrow C^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X\times Y))$ that, since p^* is exact, restricts to a dg functor

$$\frac{C^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X))}{Ac^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X))} \longrightarrow \frac{C^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X\times Y))}{Ac^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X\times Y))}$$

that is a dg lift of the triangulated functor p^*. We are therefore in the following situation

$$
\begin{array}{ccc}
D^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X)) & \xrightarrow{p^*} & D^{dg}_{\mathrm{Qcoh}}(\mathrm{Sh}(X\times Y)) \\
\phi_X\downarrow & & \downarrow\phi_{X\times Y} \\
\text{h-proj}(\mathcal{P}(\mathcal{U})) & \xrightarrow{\mathrm{Ind}_{\mathsf{F}}} & \text{h-proj}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))
\end{array}
$$

Note that $\mathrm{H}^0(\phi)$ is an equivalence. We can see, as a consequence of Proposition 3.10 in [13], that $\mathrm{H}^0(\mathrm{Ind}_{\boxtimes\circ\pi_{\mathcal{U}}})$ and $\mathrm{H}^0(\phi_{X\times Y})\circ\mathrm{H}^0(p^*)\circ\mathrm{H}^0(\phi_X)^{-1}$ are isomorphic as functors from $\mathrm{H}^0(\text{h-proj}(\mathcal{P}(\mathcal{U})))$ if they are isomorphic on $\mathrm{H}^0(\mathcal{P}(\mathcal{U}))$. Since this is precisely the case, we can conclude that $\mathrm{H}^0(\mathrm{Ind}_{\boxtimes\circ\pi_{\mathcal{U}}})$ and $p^* = \mathrm{H}^0(p^*)$ are isomorphic also at the level of $D_{\mathrm{Qcoh}}(\mathrm{Sh}(-))$.

We have also that the triangulated functor $\mathrm{H}^0(\mathsf{H} \circ \mathsf{G})$ is right adjoint to $\mathrm{H}^0(\mathrm{Ind}_{\boxtimes \circ \pi_{\mathcal{U}}})$ at the level of derived categories (see Lemma 3.2.14 and Remark 2.5.5 - or the above Lemma together with Remark 2.4.6) therefore we get that $\mathrm{H}^0(\mathsf{H} \circ \mathsf{G})$ must be isomorphic to Rp_* at the level of $D_{\mathrm{Qcoh}}(\mathrm{Sh}(-))$. But now the geometry tells us that, under our hypoteses, Rp_* sends perfect complexes to perfect complexes - as we observed in Section 1.3. We thus have that $\mathrm{H}^0(\mathsf{H} \circ \mathsf{G})$ (and hence $\mathsf{H} \circ \mathsf{G}$, too) sends the compact objects of $\mathrm{H}^0(\text{h-proj}(\mathcal{P}(\mathcal{U})))$ to compact objects of $\mathrm{H}^0(\text{h-proj}(\mathcal{P}(\mathcal{U} \times \mathcal{V})))$ and our proof is concluded. □

Therefore we have computed a dg lift of the Fourier-Mukai functor $\Phi_{\mathcal{E}}$ even when our schemes are over a commutative ring, precisely in the setting $(**)$. This lift can be written as

$$\Phi_{\mathcal{E}}^{dg}(-) := \mathsf{H} \circ \mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U} \times \mathcal{V}} \circ \mathsf{K}_{\mathcal{C}_{\mathcal{U} \times \mathcal{V}}(E)} \circ \boxtimes \circ \pi_{\mathcal{U}}(-).$$

Chapter 4

An explicit bijection

In this chapter we are going to prove Theorem 4.4.1. We will define the bijection γ by estensively using the dg lift $\Phi_{\mathcal{E}}^{dg}$ we have computed in the previous Chapter. For proving surjectivity we need to define a slightly different version $\tilde{\Phi}_{\mathcal{E}}^{dg}$ of such a lift. Moreover, we need to properly define a candidate for the kernel $\mathcal{E}_{\mathsf{F}}$ of the Fourier-Mukai functor associated to a dg quasi-functor F. The injectivity of γ is due to Proposition 4.3.1, where we prove that - essentially - at the dg level different kernels produce different Fourier-Mukai functors.

Also in this Chapter, except for Section 4.5, we will work under the hypotesis of all our schemes satisfy condition $(*)$.

4.1 Construction of the kernel $\mathcal{E}_{\mathsf{F}}$

Our aim is to define a bijective map

$$\gamma : \mathrm{Iso}(\mathfrak{Perf}(X \times Y)) \xrightarrow{1:1} \mathbf{Hqe}(\mathcal{P}(\mathcal{U}), \mathcal{P}(\mathcal{V}))$$

and we want to do it by sending an object $\mathcal{E}$ to something that is essentially $\Phi_{\mathcal{E}}^{dg}$. If we want to prove its surjectivity we need, for any quasi-functor $\mathsf{F} : \mathcal{P}(\mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{V})$, to suitably find a candidate for being the kernel $\mathcal{E}_{\mathsf{F}}$ such that $\Phi_{\mathcal{E}_{\mathsf{F}}}^{dg} = \mathsf{F}$ in $\mathbf{Hqe}$.

Our idea is roughly to choose *the image via* $\mathrm{id} \otimes \mathsf{F}$ *of the diagonal bimodule.* We will give a precise definition later on, but please be aware that this idea is not new: it matches what happens in the triangulated case (see [36]). Before we can write it in a proper way we need some preliminary result:

Consider now the diagonal map $\Delta : X \longrightarrow X \times X$. The pullback Δ^* clearly gives rise to a dg functor between $C_{dg}((\mathrm{Sh}(X \times X)) \longrightarrow C_{dg}(\mathrm{Sh}(X))$. Let us define $\mathcal{P}_{\Delta^*}(\mathcal{U})$ to be the smallest full dg subcategory of $C_{dg}(\mathrm{Sh}(X))$ containing all the objects of the form $\mathcal{C}_{\mathcal{U}}(R)$ and $\Delta^* \mathcal{C}_{\mathcal{U} \times \mathcal{U}}(A)$ for R and A vector bundles on X and $X \times X$ respectively, and closed under shifts, cones and taking homotopy equivalent objects. We want to show the following

Proposition 4.1.1. *The dg category* $\mathcal{P}_{\Delta^*}(\mathcal{U})$ *is a dg enhancement of* $\mathfrak{Perf}(X)$.

Proof. The proof readily follows the one of Proposition 3.6.1.

Note that, for any vector bundle A on $X \times X$, $\Delta^* \mathcal{C}_{\mathcal{U} \times \mathcal{U}}(A)$ is made up from objects of the form $\Delta^*(A_{U \times V}) \simeq (\Delta^* A)_{U \cap V}$, where U and V are in $\mathcal{U}$.

Moreover, we have a quasi-isomorphism $\alpha : \Delta^*A \longrightarrow \Delta^*\mathcal{C}_{\mathcal{U}\times\mathcal{U}}(A)$: in fact, since both A and $\mathcal{C}_{\mathcal{U}\times\mathcal{U}}(A)$ are flat, it is precisely the image of the quasi-isomorphism $A \longrightarrow \mathcal{C}_{\mathcal{U}\times\mathcal{U}}(A)$ via the exact functor $L\Delta^*$.

Finally, the map

$$\mathrm{Hom}_{\mathrm{H}^0(C(\mathrm{Sh}(V)))}(j^*\Delta^*A, j^*R[n]) \longrightarrow \mathrm{Hom}_{D(\mathrm{Sh}(V))}(j^*\Delta^*A, j^*R[n])$$

is an isomorphism since $j^*\Delta^*A$ is a vector bundle on the affine scheme V and hence it is projective in $\mathrm{Qcoh}(V)$. Again we have to use the fact that both $j^*\Delta^*A$ and $j^*R[n]$ are quasi-coherent $\mathcal{O}_V$-modules together with equivalence (1.1). □

Corollary 4.1.2. *The two dg-categories $\mathcal{P}(\mathcal{U})$ and $\mathcal{P}_{\Delta^*}(\mathcal{U})$ are actually equal.*

Proof. The inclusion $\mathcal{P}(\mathcal{U}) \subseteq \mathcal{P}_{\Delta^*}(\mathcal{U})$ holds by definition. Now, we have that $\Delta^*\mathcal{C}_{\mathcal{U}\times\mathcal{U}}(A)$ and $\mathcal{C}_{\mathcal{U}}(\Delta^*A)$ are isomorphic in $\mathfrak{Perf}(X)$ i.e. are quasi-isomorphic. But this, by the above Proposition, implies that they are isomorphic in $\mathrm{H}^0(\mathcal{P}_{\Delta^*}(\mathcal{U}))$ and it means that they are homotopy equivalent. Whence $\Delta^*\mathcal{C}_{\mathcal{U}\times\mathcal{U}}(A)$ lies already in $\mathcal{P}(\mathcal{U})$ and we get the other inclusion. □

Observe that this last corollary tells us that $\mathcal{P}(\mathcal{U})$ contains all the elements of the form $\Delta^*\mathcal{C}_{\mathcal{U}\times\mathcal{U}}(A)$, for any vector bundle A on $X\times X$. Hence we can define a dg functor $\Delta^*_{dg} : \mathcal{P}(\mathcal{U}\times\mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{U})$ as the restriction to $\mathcal{P}(\mathcal{U}\times\mathcal{U})$ of the dg functor $\Delta^* : C_{dg}((\mathrm{Sh}(X\times X) \longrightarrow C_{dg}(\mathrm{Sh}(X))$. Therefore it is clear that we have the following:

Lemma 4.1.3. *The dg functor $\Delta^*_{dg} : \mathcal{P}(\mathcal{U}\times\mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{U})$ is a dg lift of of the triangulated functor $L\Delta^* : \mathfrak{Perf}(X\times X) \longrightarrow \mathfrak{Perf}(X)$.*

We would like now to produce a dg lift of $R\Delta_*$. The problem is that in this case $\mathrm{Res}_{\Delta^*_{dg}}$ does not restrict to perfect dg modules. So we need to proceed as in Section 3.8 and compose it with the projection to the quotient of dg modules by acyclic ones.

Remark 4.1.4. Unlike Section 3.8, we are over a field and hence the construction we have made there can be simplified: in fact all the dg categories are now homotopically flat. We have therefore that the composition η of (3.1) can be written now as

$$\text{h-proj}(\mathcal{A}) \xrightarrow{\ \iota\ } \mathrm{dgMod}(\mathcal{A}) \xrightarrow{\ \mathsf{Q}\ } \mathrm{dgMod}(\mathcal{A})/\mathrm{Ac}(\mathcal{A}).$$

We will denote again by H the dg quasi-functor $\eta^{-1} \circ \mathsf{Q} : \mathrm{dgMod}(\mathcal{A}) \longrightarrow \text{h-proj}(\mathcal{A})$ that sends a dg $\mathcal{A}$-module to its h-projective resolution.

Now observe that, since in our hypoteses $\Delta : X \longrightarrow X \times X$ is a closed immersion (we are on a separated scheme), we have $R\Delta_* = \Delta_*$.

Lemma 4.1.5. *In our hypoteses the triangulated functor $\Delta_* : \mathfrak{Perf}(X) \longrightarrow \mathfrak{Perf}(X\times X)$ sends perfect complexes to perfect complexes.*

Proof. The scheme X is smooth and hence the natural morphism $X\times X \longrightarrow X$ coming from the definition of fibered product is a smooth morphism; it follows that its section Δ is a regular immersion ([3] Proposition 1.10). Since moreover $\Delta : X \longrightarrow X\times X$ is a proper morphism of noetherian schemes we have that - according to Corollary 4.8.1 and Exercise 4.1.1 of [22] - Δ_* sends perfect complexes to perfect complexes. □

We have now all the ingredients for proving the following:

Lemma 4.1.6. *The dg quasi-functor* $\mathsf{H} \circ \mathrm{Res}_{\Delta^*_{dg}} : \mathrm{Perf}(\mathcal{P}(\mathcal{U})) \longrightarrow \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{U}))$ *is a dg lift of* $R\Delta_* = \Delta_*$.

Proof. Thanks to Lemma 4.1.3 and Lemma 4.1.5, a variant of Propostion 3.8.4 applies in this case, with Δ^*_{dg} in place of F and Δ in place of p. In fact we have that, by Remark 2.5.5, the triangulated functor $\mathrm{H}^0(\mathsf{H} \circ \mathrm{Res}_{\Delta^*_{dg}})$ is right adjoint to $\mathrm{H}^0(\mathrm{Ind}_{\Delta^*_{dg}})$ at the level of derived categories and therefore $\mathrm{H}^0(\mathsf{H} \circ \mathrm{Res}_{\Delta^*_{dg}})$ must be isomorphic to Δ_* at the level of $D_{\mathrm{Qcoh}}(\mathrm{Sh}(-))$. □

Before coming to the definition of $\mathcal{E}_F$ we need a technical result. Let us consider $\mathrm{Res}_{\boxtimes_|} : \mathrm{dgMod}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \longrightarrow \mathrm{dgMod}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}))$; observe that it restricts to h-projective dg modules, by Proposition 2.5.2 (iv) and since we already know from Corollary 3.2.11 that it restricts to perfect dg modules. It therefore makes sense to state the following

Lemma 4.1.7. *We have the commutativity in* **Hqe** *of the diagram:*

$$\begin{array}{ccc} \mathrm{dgMod}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) & \xrightarrow{\mathrm{Res}_{\boxtimes_|}} & \mathrm{dgMod}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})) \\ \downarrow{\scriptstyle \mathsf{H}} & & \downarrow{\scriptstyle \mathsf{H}} \\ \text{h-proj}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) & \xrightarrow{\mathrm{Res}_{\boxtimes_|}} & \text{h-proj}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})). \end{array}$$

Proof. Recallt that H was defined as the composition $\eta^{-1} \circ \mathsf{Q}$ so we can rewrite the rectangle above as:

$$\begin{array}{ccc} \mathrm{dgMod}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) & \xrightarrow{\mathrm{Res}_{\boxtimes_|}} & \mathrm{dgMod}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})) \\ \downarrow{\scriptstyle \mathsf{Q}} & & \downarrow{\scriptstyle \mathsf{Q}} \\ \frac{\mathrm{dgMod}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))}{\mathrm{Ac}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))} & \xrightarrow{\overline{\mathrm{Res}_{\boxtimes_|}}} & \frac{\mathrm{dgMod}(\mathcal{P}_*(\mathcal{U})\otimes\mathcal{P}_*(\mathcal{V}))}{\mathrm{Ac}(\mathcal{P}_*(\mathcal{U})\otimes\mathcal{P}_*(\mathcal{V}))} \\ {\scriptstyle \eta}\uparrow{\scriptstyle \sim} & & {\scriptstyle \sim}\uparrow{\scriptstyle \eta} \\ \text{h-proj}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) & \xrightarrow{\mathrm{Res}_{\boxtimes_|}} & \text{h-proj}(\mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V})) \end{array}$$

where $\overline{\mathrm{Res}_{\boxtimes_|}}$ is the (unique) dg functor making the upper rectangle commutes. And this happens since the dg functor $\mathrm{Res}_{\mathsf{F}}$ preserves acyclic dg modules for any dg functor F. The commutativity of the lower diagram comes from the fact that η is by definition $\mathsf{Q} \circ \iota$, where ι is the inclusion of the dg category of h-projective dg modules into the dg category of dg modules. □

Remark 4.1.8. Observe that the proof above actually shows that for any dg functor F such that $\mathrm{Res}_{\mathsf{F}}$ restricts to h-projective dg modules we have $\mathsf{H} \circ \mathrm{Res}_{\mathsf{F}} = \mathrm{Res}_{\mathsf{F}} \circ \mathsf{H}$ in **Hqe**.

Now we are ready to define the kernel $\mathcal{E}_{\mathsf{F}}$. First of all, since F is a quasi-functor between $\mathcal{P}(\mathcal{U})$ and $\mathcal{P}(\mathcal{V})$, we know that it can be represented by

$$\mathcal{P}(\mathcal{U}) \xleftarrow[\sim]{\ \mathsf{I}\ } \mathcal{P}(\mathcal{U})' \xrightarrow{\ \mathsf{F}'\ } \mathcal{P}(\mathcal{V})$$

for some dg category $\mathcal{P}(\mathcal{U})'$ and some dg functors F' and I, where I is a quasi-equivalence and hence $\mathsf{F} = \mathsf{F}' \circ \mathsf{I}^{-1}$ in **Hqe**. Similarly to what we set in Section 3.3 we will denote $Y^{dg}_{\mathcal{P}(\mathcal{U})'}$ by $Y_{\mathcal{U}'}$.

We define α to be the image of $\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)$ via the following composition of dg quasi-functors

$$\mathcal{P}(\mathcal{U}) \xrightarrow{Y_{\mathcal{U}}} \mathrm{Perf}(\mathcal{P}(\mathcal{U})) \xrightarrow{\mathsf{H}\circ\mathrm{Res}_{\Delta^*_{dg}}} \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{U})) \xrightarrow{\mathrm{Ind}_L\circ\mathrm{Res}_{\boxtimes_|}} \mathrm{Perf}(\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{U})).$$

Thanks to Lemma 4.1.7 we have

$$\begin{aligned}\alpha :&= \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ \mathsf{H} \circ \mathrm{Res}_{\Delta^*_{dg}} \circ Y_{\mathcal{U}}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)) \simeq \mathrm{Ind}_L \circ \mathsf{H} \circ \mathrm{Res}_{\boxtimes_|} \circ \mathrm{Res}_{\Delta^*_{dg}} \circ Y_{\mathcal{U}}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)) \\ &\simeq \mathrm{Ind}_L \circ \mathsf{H} \circ \mathrm{Res}_{\Delta^*_{dg}\circ\boxtimes_|} \circ Y_{\mathcal{U}}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)) \simeq \mathrm{Ind}_L \circ \mathsf{H} \circ \mathrm{Res}_{\mathsf{K}_|} \circ Y_{\mathcal{U}}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))\end{aligned}$$

in $\mathrm{H}^0(\mathrm{Perf}(\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{U})))$, where $\mathsf{K} : \mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{U})$ is the dg functor we have introduced in Section 3.6 and with $\mathsf{K}_|$ we have denoted its restriction to $\mathcal{P}_*(\mathcal{U})\otimes\mathcal{P}_*(\mathcal{U})$. The last isomorphism is a consequence of the one between $\Delta^* \circ \boxtimes$ and $\otimes$ that holds already at the level of $C_{dg}(\mathrm{Sh}(X)) \times C_{dg}(\mathrm{Sh}(X))$.

Observe that we have $\mathsf{K}_| = \mathsf{K} \circ L$ therefore $\mathrm{Res}_{\mathsf{K}_|} = \mathrm{Res}_L \circ \mathrm{Res}_{\mathsf{K}}$. From this and by Remark 4.1.8 (remember that Res_L restricts to h-projective dg modules by Lemma 3.2.12) we obtain

$$\mathrm{Ind}_L \circ \mathsf{H} \circ \mathrm{Res}_{\mathsf{K}_|} = \mathrm{Ind}_L \circ \mathsf{H} \circ \mathrm{Res}_L \circ \mathrm{Res}_{\mathsf{K}} = \mathrm{Ind}_L \circ \mathrm{Res}_L \circ \mathsf{H} \circ \mathrm{Res}_{\mathsf{K}} = \mathsf{H} \circ \mathrm{Res}_{\mathsf{K}}$$

in **Hqe** and whence we have $\alpha \simeq \mathsf{H} \circ \mathrm{Res}_{\mathsf{K}} \circ Y_{\mathcal{U}}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))$ in $\mathrm{H}^0(\mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U})))$.

Now, I is a quasi-equivalence by hypoteses and therefore, since we are dealing with h-flat dg categories, $\mathrm{id} \otimes \mathsf{I}$ is a quasi-equivalence as well (see Remark 2.8 of [13]). It follows by Proposition 2.5.2 (iii) that also

$$\mathrm{Ind}_{\mathrm{id}\otimes\mathsf{I}} : \mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U})') \longrightarrow \mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U}))$$

is a quasi-equivalence. We therefore choose α' to be an element of $\mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U})')$ such that

$$\mathrm{H}^0(\mathrm{Ind}_{\mathrm{id}\otimes\mathsf{I}})(\alpha') \simeq \alpha$$

and finally we set our candidate kernel

$$\mathcal{E}_{\mathsf{F}} := \mathrm{Ind}_{\mathrm{id}\otimes\mathsf{F}'}(\alpha') \in \mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})). \tag{4.1}$$

4.2 A new version of the dg lift

The only problem in the definition of $\mathcal{E}_{\mathsf{F}}$ is that it actually belongs to $\mathrm{Perf}(\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V}))$ instead of $\mathcal{P}(\mathcal{U}\times\mathcal{V})$, where the object $\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)$ quasi-isomorphic to $\mathcal{E}$ appearing in the definition of $\Phi^{dg}_{\mathcal{E}}$ lives. For this reason we are going to construct a slightly different version of the latter. In order to do that, we need some auxiliary lemmas.

Let us now define the map $p^*_{|\mathcal{V}} : \mathcal{P}(\mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{U}\times\mathcal{V})$ as the composition between the two dg functors $p^* : \mathcal{P}(\mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{U}\times Y)$ and $\mathcal{C}_{X\times\mathcal{V}}(-) : \mathcal{P}(\mathcal{U}\times Y) \longrightarrow \mathcal{P}(\mathcal{U}\times\mathcal{V})$.

The functor p^* is just the ordinary pullback along p: it sends an object $\mathcal{C}_{\mathcal{U}}(A)$ to $p^*\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(A) \simeq \mathcal{C}_{\mathcal{U}\times Y}(p^*A)$. The functor $\mathcal{C}_{X\times\mathcal{V}}(-)$ is the one given by taking the Čech resolution with respect to the covering $X\times\mathcal{V}$. They are both well defined as dg functors between the category of complexes of sheaves of modules (for $\mathcal{C}_{X\times\mathcal{V}}(-)$ it comes from the fact that our open covering is finite). Hence they are well defined with respect to homotopy equivalent objects.

We want to prove the following

Lemma 4.2.1. *For any bounded complex E of vector bundles of $\mathrm{Perf}(X\times Y)$ we have that the two dg functors $\mathsf{K}_{\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)} \circ \boxtimes \circ \pi_{\mathcal{U}}(-)$ and $\mathsf{K}_{p^*_{|\mathcal{V}}(-)}(\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E))$ from $\mathcal{P}(\mathcal{U})$ to $\mathcal{P}(\mathcal{U}\times\mathcal{V})$ are equal in **Hqe**.*

Proof. It is easy to see that $p^*_{|\mathcal{V}}$ sends the object $\mathcal{C}_{\mathcal{U}}(A)$ to $\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(p^*A)$. Let us now prove that we have an equality in **Hqe** between $\mathsf{K}_{\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)} \circ \boxtimes \circ \pi_{\mathcal{U}}(-)$ and $\mathsf{K}_{p^*_{|\mathcal{V}}(-)}(\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E))$. For any $\mathcal{C}_{\mathcal{U}}(A) \in \mathcal{P}(\mathcal{U})$ we have that:

$$\mathsf{K}_{\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)} \circ \boxtimes \circ \pi_{\mathcal{U}}(\mathcal{C}_{\mathcal{U}}(A)) = \mathsf{K}_{\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)}(\mathcal{C}_{\mathcal{U}}(A) \boxtimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)) =$$
$$(\mathcal{C}_{\mathcal{U}}(A) \boxtimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)) \otimes \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)$$

and

$$\mathsf{K}_{p^*_{|\mathcal{V}}(\mathcal{C}_{\mathcal{U}}(A))}(\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)) = \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(p^*A) \otimes \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E).$$

are homotopy equivalent - see Remark 3.2.5. It is then easy to see that we have a termwise homotopy equivalence and, by Proposition 2.5.10, our proof is concluded. □

Now we can see $\mathrm{Ind}_{\mathsf{K}_{p^*_{|\mathcal{V}}(-)}}$ as a dg functor

$$\mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \otimes \mathcal{P}(\mathcal{U}) \longrightarrow \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))$$

sending an object $(A, B) \in \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \otimes \mathcal{P}(\mathcal{U})$ to $\mathrm{Ind}_{\mathsf{K}_{p^*_{|\mathcal{V}}(B)}}(A)$. In the same way $\mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}$ is the dg functor

$$\mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})) \otimes \mathcal{P}(\mathcal{U}) \longrightarrow \mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))$$

sending an object $(C, D) \in \mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})) \otimes \mathcal{P}(\mathcal{U})$ to $\mathrm{Ind}_{\mathsf{K}_{(D)}\otimes\mathrm{id}}(C)$. We want to prove the following

Lemma 4.2.2. *We have the commutativity in **Hqe** of the diagram:*

$$\begin{array}{ccc}
\mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \otimes \mathcal{P}(\mathcal{U}) & \xrightarrow{\mathrm{Ind}_{\mathsf{K}_{p^*_{|\mathcal{V}}(-)}}} & \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \\
\Big\downarrow {\scriptstyle (\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|})\otimes\mathrm{id}} & & \Big\downarrow {\scriptstyle \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|}} \\
\mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})) \otimes \mathcal{P}(\mathcal{U}) & \xrightarrow{\mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}} & \mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})).
\end{array}$$

Proof. Let us start by preliminarily showing that, for any object $B \in \mathcal{P}(\mathcal{U})$ we have the equality

$$\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ \mathrm{Ind}_{\mathsf{K}_{p^*_{|\mathcal{V}}(B)}} = \mathrm{Ind}_{\mathsf{K}_B\otimes\mathrm{id}} \circ \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \tag{4.2}$$

in **Hqe**. This comes once we have that $\mathsf{K}_{p^*_{|\mathcal{V}}(B)} \circ \boxtimes = \boxtimes \circ (\mathsf{K}_B \otimes \mathrm{id})$ in **Hqe**. Actually we can reduce ourselves to the case of $B = \mathcal{C}_{\mathcal{U}}(A)$. In fact, since we are dealing with dg functors they preserve shifts and cones of morphisms of degree zero; moreover such functors are defined on $C(\mathrm{Sh}(-))$ and therefore preserve homotopy equivalences. Now, for any $(\mathcal{C}_{\mathcal{U}}(C), \mathcal{C}_{\mathcal{V}}(D)) \in \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})$ we have:

- $\mathsf{K}_{p^*_{|\mathcal{V}}(\mathcal{C}_\mathcal{U}(A))} \circ \boxtimes(\mathcal{C}_\mathcal{U}(C), \mathcal{C}_\mathcal{V}(D)) = \mathsf{K}_{\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(p^*A)}(\mathcal{C}_\mathcal{U}(C) \boxtimes \mathcal{C}_\mathcal{V}(D))$
  $= \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(p^*A) \otimes (\mathcal{C}_\mathcal{U}(C) \boxtimes \mathcal{C}_\mathcal{V}(D))$;
- $\boxtimes \circ (\mathsf{K}_{\mathcal{C}_\mathcal{U}(A)} \otimes \mathrm{id})(\mathcal{C}_\mathcal{U}(C), \mathcal{C}_\mathcal{V}(D)) = \boxtimes(\mathcal{C}_\mathcal{U}(C) \otimes \mathcal{C}_\mathcal{U}(A), \mathcal{C}_\mathcal{V}(D))$
  $= (\mathcal{C}_\mathcal{U}(C) \otimes \mathcal{C}_\mathcal{U}(A)) \boxtimes \mathcal{C}_\mathcal{V}(D)$.

and both objects are homotopy equivalent to $\mathcal{C}_{\mathcal{U}\times\mathcal{V}}((A \otimes C) \boxtimes D)$ - by Remark 3.2.5 and by the proof of Corollary 3.6.3. Therefore we have a termwise homotopy equivalence and we can apply, as in the previous Lemma, Proposition 2.5.10.
Hence we get also

$$\mathrm{Ind}_{\mathsf{K}_{p^*_{|\mathcal{V}}(B)}} \circ \mathrm{Ind}_\boxtimes = \mathrm{Ind}_\boxtimes \circ \mathrm{Ind}_{\mathsf{K}_B \otimes \mathrm{id}}.$$

Now we recall that $\mathrm{Ind}_\boxtimes$ is invertible in **Hqe** and $\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|}$ is its inverse: we can therefore deduce the equality (4.2).

For a proof of the original claim we need to show that for any element (E, F) of $\mathrm{Perf}(\mathcal{P}(\mathcal{U} \times \mathcal{V})) \otimes \mathcal{P}(\mathcal{U})$ we have a homotopy equivalence between

$$\mathrm{Ind}_{\mathsf{K}_{(-)} \otimes \mathrm{id}} \circ \big((\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|}) \otimes \mathrm{id}\big)(E, F) \qquad \text{and} \qquad \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ \mathrm{Ind}_{\mathsf{K}_{p^*_{|\mathcal{V}}(-)}}(E, F).$$

Actually it suffices to show it when (E, F) is of the form $\big(\mathrm{Hom}_{\mathcal{P}(\mathcal{U}\times\mathcal{V})}(-, \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(G)), H\big) \in \mathrm{Perf}(\mathcal{P}(\mathcal{U} \times \mathcal{V})) \otimes \mathcal{P}(\mathcal{U})$ where G is a bounded complex of vector bundles on $X \times Y$. We can write:

$$\begin{aligned}
&\mathrm{Ind}_{\mathsf{K}_{(-)} \otimes \mathrm{id}} \circ \big((\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|}) \otimes \mathrm{id}\big)\big(\mathrm{Hom}_{\mathcal{P}(\mathcal{U}\times\mathcal{V})}(-, \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(G)), H\big) = \\
&\mathrm{Ind}_{\mathsf{K}_{(-)} \otimes \mathrm{id}}\Big(\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|}\big(\mathrm{Hom}_{\mathcal{P}(\mathcal{U}\times\mathcal{V})}(-, \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(G))\big), H\Big) = \\
&\mathrm{Ind}_{\mathsf{K}_H \otimes \mathrm{id}}\Big(\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|}\big(\mathrm{Hom}_{\mathcal{P}(\mathcal{U}\times\mathcal{V})}(-, \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(G))\big)\Big)
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ \mathrm{Ind}_{\mathsf{K}_{p^*_{|\mathcal{V}}(-)}}\big(\mathrm{Hom}_{\mathcal{P}(\mathcal{U}\times\mathcal{V})}(-, \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(G)), H\big) = \\
&\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ \mathrm{Ind}_{\mathsf{K}_{p^*_{|\mathcal{V}}(H)}}\big(\mathrm{Hom}_{\mathcal{P}(\mathcal{U}\times\mathcal{V})}(-, \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(G))\big)
\end{aligned}$$

But the latter element, thanks to equation (4.2), is homotopy equivalent to

$$\mathrm{Ind}_{\mathsf{K}_H \otimes \mathrm{id}} \circ \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|}\big(\mathrm{Hom}_{\mathcal{P}(\mathcal{U}\times\mathcal{V})}(-, \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(G))\big)$$

and hence we get the desired result. □

Thanks to Lemma 4.2.1, Proposition 2.5.2 (ii) and Lemma 4.2.2 we can write:

$$\begin{aligned}
\Phi^{dg}_{\mathcal{E}}(-) :&= \mathrm{Res}_{\pi_\mathcal{V}} \circ \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U}\times\mathcal{V}} \circ \mathsf{K}_{\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)} \circ \boxtimes \circ \pi_\mathcal{U}(-) \\
&= \mathrm{Res}_{\pi_\mathcal{V}} \circ \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U}\times\mathcal{V}} \circ \mathsf{K}_{p^*_{|\mathcal{V}}(-)}(\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)) \\
&= \mathrm{Res}_{\pi_\mathcal{V}} \circ \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ \mathrm{Ind}_{\mathsf{K}_{p^*_{|\mathcal{V}}(-)}} \circ Y_{\mathcal{U}\times\mathcal{V}}(\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)) \\
&= \mathrm{Res}_{\pi_\mathcal{V}} \circ \mathrm{Ind}_{\mathsf{K}_{(-)} \otimes \mathrm{id}} \circ \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U}\times\mathcal{V}}(\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E)).
\end{aligned}$$

Now we can define the dg functor

$$\tilde{\Phi}^{dg}_\beta(-) := \mathrm{Res}_{\pi_\mathcal{V}} \circ \mathrm{Ind}_{\mathsf{K}_{(-)} \otimes \mathrm{id}}(\beta) : \mathcal{P}(\mathcal{U}) \longrightarrow \mathrm{Perf}(\mathcal{P}(\mathcal{V}))$$

for any element $\beta \in \mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))$. What we have proved in the last pages can therefore be summarised saying that we have the equality

$$\Phi^{dg}_{\mathcal{E}} = \tilde{\Phi}^{dg}_{\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U}\times\mathcal{V}}(\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E))} \quad \text{in } \mathbf{Hqe}.$$

Notice now that the kernel of $\tilde{\Phi}^{dg}$ actually belongs to $\mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))$: therefore it makes perfect sense to write $\tilde{\Phi}^{dg}_{\mathcal{E}_{\mathsf{F}}}$ and what we want to do now is to prove the following crucial result.

Proposition 4.2.3. *Let* $\mathsf{F} : \mathcal{P}(\mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{V})$ *be any dg quasi-functor and let* $\mathcal{E}_{\mathsf{F}}$ *be the element of* $\mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))$ *we have defined in (4.1). Then* $\tilde{\Phi}^{dg}_{\mathcal{E}_{\mathsf{F}}} = Y_{\mathcal{V}} \circ \mathsf{F}$ *in* ***Hqe***.

The proof of this Proposition will be carried over the remaining part of the present Section, involving many auxiliary results.

Recall that F can be represented in **Hqe** by a "roof" $\mathcal{P}(\mathcal{U}) \xleftarrow{\mathsf{I}} \mathcal{P}(\mathcal{U})' \xrightarrow{\mathsf{F}'} \mathcal{P}(\mathcal{V})$, where I is a quasi-equivalence.

Lemma 4.2.4. *Let* $\mathsf{G} : \mathcal{P}(\mathcal{U})' \longrightarrow \mathcal{P}(\mathcal{V})$ *be any dg functor, then we have an equality in* ***Hqe*** *between* $\mathrm{Ind}_{\mathsf{K}_{(-)}\otimes \mathrm{id}} \circ \mathrm{Ind}_{\mathrm{id}\otimes\mathsf{G}}$ *and* $\mathrm{Ind}_{\mathrm{id}\otimes\mathsf{G}} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes \mathrm{id}}$.

Proof. this fact comes from the obvious equality in **Hqe** between $(\mathsf{K}_{(-)} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathsf{G})$ and $(\mathrm{id} \otimes \mathsf{G}) \circ (\mathsf{K}_{(-)} \otimes \mathrm{id})$. □

Let us now define the dg functor

$$\pi'_2 : \mathcal{P}(\mathcal{U})' \longrightarrow \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U})'$$

given by $A \longmapsto (\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), A)$. It has clearly the same behavior of $\pi_{\mathcal{U}}$ and so it is not hard to believe that $\mathrm{Res}_{\pi'_2}$ restricts to perfect dg modules (see Proposition 3.5.1). We therefore can state the following

Lemma 4.2.5. *The two dg functors* $\mathrm{Ind}_{\mathsf{F}'} \circ \mathrm{Res}_{\pi'_2}$ *and* $\mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_{\mathrm{id}\otimes\mathsf{F}'}$ *from* $\mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U})')$ *to* $\mathrm{Perf}(\mathcal{P}(\mathcal{V}))$ *are equal in* ***Hqe***.

Proof. Let us consider $A \in \mathcal{P}(\mathcal{U})$, $B \in \mathcal{P}(\mathcal{U})'$ and consider the element $Y_{\mathcal{U}\times\mathcal{U}'}(A, B)$. We have:

$$\begin{aligned}
&\mathrm{Ind}_{\mathsf{F}'} \circ \mathrm{Res}_{\pi'_2} \circ Y_{\mathcal{U}\times\mathcal{U}'}(A, B) = \\
&\mathrm{Ind}_{\mathsf{F}'}\Big(\mathrm{Hom}_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{U})'}(\pi'_2(-), (A, B))\Big) = \\
&\mathrm{Ind}_{\mathsf{F}'}\Big(\mathrm{Hom}_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{U})'}((\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), -), (A, B))\Big) = \\
&\mathrm{Ind}_{\mathsf{F}'}\Big(\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), A) \otimes \mathrm{Hom}_{\mathcal{P}(\mathcal{U})'}(-, B)\Big)
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&\mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_{\mathrm{id}\otimes\mathsf{F}'} \circ Y_{\mathcal{U}\times\mathcal{U}'}(A, B) = \mathrm{Res}_{\pi_{\mathcal{V}}} \circ Y_{\mathcal{U}\times\mathcal{V}} \circ (\mathrm{id} \otimes \mathsf{F}')(A, B) = \\
&\mathrm{Res}_{\pi_{\mathcal{V}}} \circ Y_{\mathcal{U}\times\mathcal{V}}\Big(A, \mathsf{F}'(B)\Big) = \mathrm{Hom}_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}(\pi_{\mathcal{V}}(-), (A, \mathsf{F}'(B))) = \\
&\mathrm{Hom}_{\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V})}((\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), -), (A, \mathsf{F}'(B))) = \\
&\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), A) \otimes \mathrm{Hom}_{\mathcal{P}(\mathcal{V})}(-, \mathsf{F}'(B)) = \\
&\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), A) \otimes Y_{\mathcal{V}}(\mathsf{F}'(B)) = \\
&\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), A) \otimes \mathrm{Ind}_{\mathsf{F}'}(\mathrm{Hom}_{\mathcal{P}(\mathcal{U})'}(-, B)).
\end{aligned}$$

Now, we know by definition (see Section 2.5) that the dg functor $\mathrm{Ind}_{\mathsf{F}}$ acts as a tensorization with the dg bimodule $\mathsf{E}_{Y_{\mathcal{V}}\circ\mathsf{F}}$ corresponding to $Y_{\mathcal{V}} \circ \mathsf{F}$ via the isomorphism (2.6). Therefore, denoting $T := \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), A) \in \mathrm{dgMod}(\Bbbk) = C_{dg}(\Bbbk\text{-Mod})$, we have

$$\mathrm{Ind}_{\mathsf{F}'} \circ \mathrm{Res}_{\pi_2'} \circ Y_{\mathcal{U}\times\mathcal{U}'}(A, B) = \Big(T \otimes \mathrm{Hom}_{\mathcal{P}(\mathcal{U})'}(-, B)\Big) \otimes_{\mathcal{P}(\mathcal{V})} \mathsf{E}_{Y_{\mathcal{V}}\circ\mathsf{F}'}$$

and

$$\mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_{\mathrm{id}\otimes\mathsf{F}'} \circ Y_{\mathcal{U}\times\mathcal{U}'}(A, B) = T \otimes \Big(\mathrm{Hom}_{\mathcal{P}(\mathcal{U})'}(-, B) \otimes_{\mathcal{P}(\mathcal{V})} \mathsf{E}_{Y_{\mathcal{V}}\circ\mathsf{F}'}\Big).$$

The two expressions are equal thanks to the associativity of the tensor product of dg modules (see Remark 2.6 of [13]). Our claim therefore comes from part 2 of Theorem 7.2 in [45]. □

Recall that a few pages ago we defined α as $\mathsf{H} \circ \mathrm{Res}_{\mathsf{K}} \circ Y_{\mathcal{U}}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))$ and α' such that $\mathrm{H}^0(\mathrm{Ind}_{\mathrm{id}\otimes\mathsf{I}})(\alpha') \simeq \alpha$.

Proposition 4.2.6. *We have* $\mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}(\alpha') = \mathsf{H} \circ \mathrm{Res}_{\mathrm{id}\otimes\mathsf{I}} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}(\alpha)$ *in* **Hqe***.*

Proof. By definition we have that α is homotopy equivalent to $\mathrm{Ind}_{\mathrm{id}\otimes\mathsf{I}}(\alpha')$, therefore we have, in **Hqe**:

$$\begin{aligned}&\mathsf{H} \circ \mathrm{Res}_{\mathrm{id}\otimes\mathsf{I}} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}(\alpha) = \mathsf{H} \circ \mathrm{Res}_{\mathrm{id}\otimes\mathsf{I}} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}} \circ \mathrm{Ind}_{\mathrm{id}\otimes\mathsf{I}}(\alpha') = \\ &\mathsf{H} \circ \mathrm{Res}_{\mathrm{id}\otimes\mathsf{I}} \circ \mathrm{Ind}_{\mathrm{id}\otimes\mathsf{I}} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}(\alpha').\end{aligned}$$

when in the last equality we have used Lemma 4.2.4. Now, the claim follows from the fact that the quasi-functor $\mathsf{H} \circ \mathrm{Res}_{\mathrm{id}\otimes\mathsf{I}}$ is right adjoint to $\mathrm{Ind}_{\mathrm{id}\otimes\mathsf{I}}$ at the level of h-projective dg modules (see Lemma 3.8.3) and the latter is a quasi-equivalence. □

Now, recall that $\tilde{\Phi}^{dg}_{\beta}(-)$ was defined to be the dg functor $\mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}(\beta)$ for any element $\beta \in \mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))$ and recall that we have defined $\mathcal{E}_{\mathsf{F}} := \mathrm{Ind}_{\mathrm{id}\otimes\mathsf{F}'}(\alpha')$.

Thanks to Lemma 4.2.4, Lemma 4.2.5 and Proposition 4.2.6 we can start writing, in **Hqe**:

$$\begin{aligned}\tilde{\Phi}^{dg}_{\mathcal{E}_{\mathsf{F}}}(-) :&= \mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}} \circ \mathrm{Ind}_{\mathrm{id}\otimes\mathsf{F}'}(\alpha') = \mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_{\mathrm{id}\otimes\mathsf{F}'} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}(\alpha') \\ &= \mathrm{Ind}_{\mathsf{F}'} \circ \mathrm{Res}_{\pi_2'} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}(\alpha') = \mathrm{Ind}_{\mathsf{F}'} \circ \mathrm{Res}_{\pi_2'} \circ \mathsf{H} \circ \mathrm{Res}_{\mathrm{id}\otimes\mathsf{I}} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}(\alpha) \\ &= \mathrm{Ind}_{\mathsf{F}'} \circ \mathsf{H} \circ \mathrm{Res}_{\pi_2'} \circ \mathrm{Res}_{\mathrm{id}\otimes\mathsf{I}} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}(\alpha) \\ &= \mathrm{Ind}_{\mathsf{F}'} \circ \mathsf{H} \circ \mathrm{Res}_{\mathsf{I}} \circ \mathrm{Res}_{\pi_2} \circ \mathrm{Ind}_{\mathsf{K}_{(-)}\otimes\mathrm{id}}(\alpha)\end{aligned}$$

where the last two equalities come, respectively, from Remark 4.1.8 and from the obvious equality $(\mathrm{id} \otimes \mathsf{I}) \circ \pi_2' = \pi_2 \circ \mathsf{I}$.

We spend some effort now in order to better describing $\alpha \simeq \mathsf{H} \circ \mathrm{Res}_{\mathsf{K}} \circ Y_{\mathcal{U}}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))$. Let us denote $W := \mathrm{Res}_{\mathsf{K}} \circ Y_{\mathcal{U}}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)) = \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(- \otimes =, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))$. It is an element of $\mathrm{dgMod}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U}))$ and we have a canonical isomorphism, see (2.6):

$$\eta : \mathrm{dgMod}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U})) \longrightarrow \underline{Hom}(\mathcal{P}(\mathcal{U})^{\circ}, \mathrm{dgMod}(\mathcal{P}(\mathcal{U})))$$

In our particular situation, thanks to the following result, we can give a concrete description of the image η_W of W in $\underline{Hom}(\mathcal{P}(\mathcal{U})^{\circ}, \mathrm{dgMod}(\mathcal{P}(\mathcal{U})))$.

Lemma 4.2.7. *For any pair of bounded complexes of vector bundles P and Q on X we have a homotopy equivalence:*

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(P), \mathcal{H}om(\mathcal{C}_{\mathcal{U}}(Q), \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))) \longrightarrow \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(P) \otimes \mathcal{C}_{\mathcal{U}}(Q), \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)).$$

Proof. First of all we observe that the sheaf complex $\mathcal{H}om(Q, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))$ lies in $\mathcal{P}(\mathcal{U})$. Indeed, if $j : V \longrightarrow X$ is the inclusion of an open subset we have the natural isomorphisms (see also Remark C.6 of [31])

$$j_* j^* \mathcal{H}om_X(Q, \mathcal{O}_X) \simeq j_* \mathcal{H}om_V(j^* Q, j^* \mathcal{O}_X) \simeq \mathcal{H}om_X(Q, j_* j^* \mathcal{O}_X)$$

and hence we obtain the isomorphism of complexes $\mathcal{C}_{\mathcal{U}}(\mathcal{H}om(Q, \mathcal{O}_X)) \simeq \mathcal{H}om(Q, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))$. Note that $\mathcal{H}om(Q, \mathcal{O}_X)$ is a bounded complex of locally free sheaves if Q is such.

Now, by Lemma C.7 of [31] we get that the morphism

$$\mathcal{H}om(\mathcal{C}_{\mathcal{U}}(Q), \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)) \longrightarrow \mathcal{H}om(Q, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))$$

induced by the Čech resolution is actually a homotopy equivalence.

We can therefore write the following chain of maps

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(P), \mathcal{H}om(\mathcal{C}_{\mathcal{U}}(Q), \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))) & & \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(P) \otimes \mathcal{C}_{\mathcal{U}}(Q), \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)) \\
\Big\downarrow \text{h.eq.} & & \Big\downarrow \text{h.eq.} \\
\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(P), \mathcal{H}om(Q, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\mathcal{C}_{\mathcal{U}}(P) \otimes Q, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))
\end{array}$$

where the object $\mathcal{C}_{\mathcal{U}}(P) \otimes Q$ lives in $\mathcal{P}(\mathcal{U})$ since it is isomorphic to $\mathcal{C}_{\mathcal{U}}(P \otimes Q)$ (see the Lemma below). The first and the last maps are homotopy equivalences because we can actually define dg endofunctors $\mathrm{Hom}_{C_{dg}(\mathrm{Sh}(X))}(\mathcal{C}_{\mathcal{U}}(P), -)$ and $\mathrm{Hom}_{C_{dg}(\mathrm{Sh}(X))}(-, \mathcal{C}_{\mathcal{U}}(P))$ of the category $C_{dg}(\mathrm{Sh}(X))$ and so they preserve homotopy equivalences (note that $\mathcal{C}_{\mathcal{U}}(P) \otimes \mathcal{C}_{\mathcal{U}}(Q)$ and $\mathcal{C}_{\mathcal{U}}(P) \otimes Q$ are homotopy equivalent by Corollary 3.6.3 and by the Lemma below) as well. The middle map is the isomorphism obtained by taking global sections of the functorial isomorphism of complexes of sheaves of modules (see [42, Tag 0A8M])

$$\mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})) \longrightarrow \mathcal{H}om(\mathcal{F} \otimes \mathcal{G}, \mathcal{H})$$

that we have for any $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ complexes of sheaves of modules. □

Lemma 4.2.8. *Let A and B bounded complexes of vector bundles on X. Then we have the isomorphism of complexes $\mathcal{C}_{\mathcal{U}}(A \otimes B) \simeq \mathcal{C}_{\mathcal{U}}(A) \otimes B$.*

Proof. On one hand we have, for any integer m:

$$\mathcal{C}_{\mathcal{U}}(A \otimes B)^m = \bigoplus_{p+q=m} \bigoplus_{i_0,..,i_q} j_{q*} j_q{}^* (\bigoplus_{r+s=p} A^r \otimes B^s) = \bigoplus_{p+q=m} \bigoplus_{r+s=p} \bigoplus_{i_0,..,i_q} j_{q*} j_q{}^* (A^r \otimes B^s) \simeq$$
$$\simeq \bigoplus_{r+s+q=m} \bigoplus_{i_0,..,i_q} j_{q*}(j_q{}^* A^r \otimes j_q{}^* B^s) \simeq \bigoplus_{r+s+q=m} \bigoplus_{i_0,..,i_q} (j_{q*} j_q{}^* A^r) \otimes B^s$$

where the last isomorphism comes from the projection formula (see Proposition 1.3.1). On the other hand:

$$(\mathcal{C}_{\mathcal{U}}(A) \otimes B)^m = \bigoplus_{a+b=m} \mathcal{C}^a_{\mathcal{U}}(A) \otimes B^b = \bigoplus_{a+b=m} (\bigoplus_{h+k=a} \bigoplus_{i_0,..,i_k} j_{k*} j_k{}^* A^h) \otimes B^b \simeq$$
$$\simeq \bigoplus_{h+k+b=m} \bigoplus_{i_0,..,i_k} (j_{k*} j_k{}^* A^h) \otimes B^b$$

and hence the two sides are clearly isomorphic. □

Remark 4.2.9. Actually, Lemma 4.2.7 is true in greater generality: Lemma C.7 of [31] holds for a generic vector bundle in place of $\mathcal{O}_X$ (as it can be easily seen by its proof); moreover, it holds also for a bounded complex of vector bundles R. In fact $\mathcal{H}om(\mathcal{C}_{\mathcal{U}}(Q), \mathcal{C}_{\mathcal{U}}(R))$ still lies in $\mathcal{P}(\mathcal{U})$ thanks to the funtoriality of Čech resolutions and to the fact that, for any $B \in \mathcal{P}(\mathcal{U})$, $\mathcal{H}om(B, -)$ is a well defined dg functor on all $C_{dg}(\mathrm{Sh}(X))$. Now, since also $\mathcal{H}om(-, B)$ is a well defined dg functor on all $C_{dg}(\mathrm{Sh}(X))$, and since - as we already observed - any object of $\mathcal{P}(\mathcal{U})$ is homotopy equivalent to the Čech resolution of a bounded complex of vector bundles, we can state what follows: for any A, B and D objects of $\mathcal{P}(\mathcal{U})$ we have that

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A, \mathcal{H}om(B, D)) \longrightarrow \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(A \otimes B, D).$$

is a homotopy equivalence.

Now, by Proposition 3.12 of [13], for any dg categories $\mathcal{A}$ and $\mathcal{B}$ we have a bijection

$$\Lambda_{\mathcal{A},\mathcal{B}} : \mathrm{Iso}(\mathrm{H}^0(\text{h-proj}(\mathcal{A}^\circ \otimes \mathcal{B}))) \longleftrightarrow \mathbf{Hqe}(\mathcal{A}, \text{h-proj}(\mathcal{B}))$$

sending (the isomorphism class of) an element $D \in \mathrm{H}^0(\text{h-proj}(\mathcal{A}^\circ \otimes \mathcal{B}))$ to the image in **Hqe** of what we have here called η_D. On the other hand, following carefully the proof of such proposition, we can see that, if we are dealing with a honest dg functor $\mathsf{G} : \mathcal{A} \longrightarrow \text{h-proj}(\mathcal{B})$, it corresponds through $\Lambda_{\mathcal{A},\mathcal{B}}$ to (the isomorphism class in H^0 of) the h-projective resolution of the image of G via the inverse of η.

Coming back to our situation, recall that we have denoted by W the object $\mathrm{Res}_{\mathsf{K}} \circ Y_{\mathcal{U}}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))$ of $\mathrm{dgMod}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U}))$. Thanks to Lemma 4.2.7 and the discussion above we get that the image of the dg functor

$$\eta_W = \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(-, \mathcal{H}om(=, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))) \in \underline{Hom}(\mathcal{P}(\mathcal{U})^\circ, \text{h-proj}(\mathcal{P}(\mathcal{U})))$$

through $\Lambda_{\mathcal{P}(\mathcal{U})^\circ, \mathcal{P}(\mathcal{U})}$ is the isomorphism class in H^0 of $\alpha = \mathsf{H} \circ W$, the h-projective resolution of W. But it is clear that α is also the image of η_α via the same map; from this it follows that η_W and η_α are equal in **Hqe**.

Let us recall that we have:

$$\tilde{\Phi}^{dg}_{\mathcal{E}_{\mathsf{F}}}(-) = \mathrm{Ind}_{\mathsf{F}'} \circ \mathsf{H} \circ \mathrm{Res}_{\mathsf{I}} \circ \mathrm{Res}_{\pi_2} \circ \mathrm{Ind}_{\mathsf{K}_{(-)} \otimes \mathrm{id}}(\alpha).$$

Let us set $E(-) := \mathrm{Ind}_{\mathsf{K}_{(-)} \otimes \mathrm{id}}(\alpha)$. By the proof of Lemma 3.4 in [13] we have $\mathrm{Res}_{\pi_2}(E(-)) = \eta_{E(-)}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))$ and by the one of Proposition 3.8 (2), always in [13] $\eta_{E(-)} = \mathrm{Ind}_{\mathsf{K}_{(-)}} \circ \eta_\alpha = \mathrm{Ind}_{\mathsf{K}_{(-)}} \circ \eta_W$.

With this being said we can write:

$$\begin{aligned} \mathrm{Ind}_{\mathsf{K}_{(-)}} \circ \eta_W &= \mathrm{Ind}_{\mathsf{K}_{(-)}} \circ \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\sim, \mathcal{H}om(=, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))) \\ &= \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\sim, \mathsf{K}_{(-)}(\mathcal{H}om(=, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)))) \\ &= \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\sim, \mathcal{H}om(=, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)) \otimes -) \end{aligned}$$

and therefore

$$\begin{aligned} \mathrm{Res}_{\pi_2}(E(-)) = \eta_{E(-)}(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)) &= \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\sim, \mathcal{H}om(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)) \otimes -) \\ &= \mathrm{Hom}_{\mathcal{P}(\mathcal{U})}(\sim, -) = Y_{\mathcal{U}}(-) \end{aligned}$$

in **Hqe** since, by Lemma C.7 of [31], we have the homotopy equivalence between $\mathcal{H}om(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X), \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X))$ and $\mathcal{H}om(\mathcal{O}_X, \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)) \simeq \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)$ and the tensorization with $\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)$ is equal to the identity in **Hqe**. In fact, thanks to Corollary 3.6.3, we have that $\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X) \otimes \mathcal{C}_{\mathcal{U}}(A)$ is homotopy equivalent to $\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X \otimes A) \cong \mathcal{C}_{\mathcal{U}}(A)$ for any bounded complex of vector bundles. But since - as we have already observed many times - tensorization is a well defined dg functor on $C_{dg}(\mathrm{Sh}(X))$ we get that for any complex B that is homotopy equivalent to $\mathcal{C}_{\mathcal{U}}(A)$ (as any object of $\mathcal{P}(\mathcal{U})$ is) $\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X) \otimes B$ is homotopy equivalent to B.

In conclusion we can write, in **Hqe**:

$$\begin{aligned}\tilde{\Phi}^{dg}_{\mathcal{E}_{\mathsf{F}}}(-) &= \mathrm{Ind}_{\mathsf{F}'} \circ \mathsf{H} \circ \mathrm{Res}_{\mathsf{I}} \circ \mathrm{Res}_{\pi_2}(E(-)) = \mathrm{Ind}_{\mathsf{F}'} \circ \mathsf{H} \circ \mathrm{Res}_{\mathsf{I}} \circ Y_{\mathcal{U}}(-) \\ &= \mathrm{Ind}_{\mathsf{F}'} \circ \mathrm{Ind}_{\mathsf{I}}^{-1} \circ Y_{\mathcal{U}}(-) = \mathrm{Ind}_{\mathsf{F}'} \circ Y_{\mathcal{U}'} \circ \mathsf{I}^{-1}(-) \\ &= Y_{\mathcal{V}} \circ \mathsf{F}' \circ \mathsf{I}^{-1}(-) = Y_{\mathcal{V}} \circ \mathsf{F}(-)\end{aligned}$$

where, for the third equality, we have adopted the same reasoning as in the proof of Proposition 4.2.6, while the equalities following it are due to Proposition 2.5.2 (ii). This finally gives us a proof of Proposition 4.2.3.

4.3 Uniqueness of Fourier-Mukai kernels

We have seen in Section 1.4 that at the triangulated level it can happen to have different kernels giving rise to the same Fourier-Mukai functor. We prove that, moving to the dg level, this cannot happen anymore.

Before stating (and proving) Proposition 4.3.1 it is worth setting some notation that will be employed. In Section 1.4 we have defined the functor

$$\hat{\boxtimes} : \mathrm{Sh}(X \times X) \times \mathrm{Sh}(X \times Y) \longrightarrow \mathrm{Sh}(X \times X \times X \times Y).$$

In complete analogy with what happens for the ordinary boxtimes functor (see Section 3.2) it gives rise to a dg functor — that by abuse of notation will be denoted in the same way

$$\hat{\boxtimes} : \mathcal{P}(\mathcal{U} \times \mathcal{U}) \otimes \mathcal{P}(\mathcal{U} \times \mathcal{V}) \longrightarrow \mathcal{P}(\mathcal{W})$$

where we have abbreviated $\mathcal{W} := \mathcal{U} \times \mathcal{U} \times \mathcal{U} \times \mathcal{V}$. The dg category $\mathcal{P}(\mathcal{W})$ is defined analogously to $\mathcal{P}(\mathcal{U})$: it is the Čech dg enhancement of $\mathfrak{Perf}(X \times X \times X \times Y)$. We will denote by $\hat{\boxtimes}_{|}$ the restriction of $\hat{\boxtimes}$ to $\mathcal{P}_*(\mathcal{U} \times \mathcal{U}) \otimes \mathcal{P}_*(\mathcal{U} \times \mathcal{V})$.

We define

$$\pi_{\mathcal{U} \times \mathcal{U}} : \mathcal{P}(\mathcal{U} \times \mathcal{U}) \longrightarrow \mathcal{P}(\mathcal{U} \times \mathcal{U}) \otimes \mathcal{P}(\mathcal{U} \times \mathcal{V})$$

and

$$\pi_{\mathcal{U} \times \mathcal{V}} : \mathcal{P}(\mathcal{U} \times \mathcal{V})) \longrightarrow \mathcal{P}(\mathcal{U} \times \mathcal{U}) \otimes \mathcal{P}(\mathcal{U} \times \mathcal{V})$$

by sending an object $A \in \mathcal{P}(\mathcal{U} \times \mathcal{U})$ to $(A, \mathcal{C}_{\mathcal{U} \times \mathcal{V}}(\mathcal{O}_{X \times Y}))$ and $B \in \mathcal{P}(\mathcal{U} \times \mathcal{V})$ to $(\mathcal{C}_{\mathcal{U} \times \mathcal{U}}(\mathcal{O}_{X \times X}), B)$, respectively. Moreover $Y_{\mathcal{W}}$ is the dg Yoneda embedding

$$Y^{dg}_{\mathcal{P}(\mathcal{W})} : \mathcal{P}(\mathcal{W}) \longrightarrow \mathrm{Perf}(\mathcal{P}(\mathcal{W}))$$

and $\iota_{\mathcal{U} \times \mathcal{U}}$ (respectively $\iota_{\mathcal{U} \times \mathcal{V}}$) is the inclusion, that is actually a quasi-equivalence, of $\mathcal{P}_*(\mathcal{U} \times \mathcal{U})$ into $\mathcal{P}(\mathcal{U} \times \mathcal{U})$ (respectively of $\mathcal{P}_*(\mathcal{U} \times \mathcal{V})$ into $\mathcal{P}(\mathcal{U} \times \mathcal{V})$).

With that said, for any $\mathcal{F} \in \mathfrak{Perf}(X \times X \times X \times Y)$, we can define the dg functor $\Psi_{\mathcal{F}}$ as the composition

$$\begin{array}{ccccc}
\mathcal{P}(\mathcal{U}\times\mathcal{U}) \xrightarrow{\pi_{\mathcal{U}\times\mathcal{U}}} \mathcal{P}(\mathcal{U}\times\mathcal{U}) \otimes \mathcal{P}(\mathcal{U}\times\mathcal{V}) & & \mathrm{Perf}(\mathcal{P}(\mathcal{W})) & \xrightarrow{\quad \mathsf{S} \quad} & \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \\
\Big\downarrow \hat{\boxtimes} & & \Big\uparrow Y_{\mathcal{W}} & & \\
\mathcal{P}(\mathcal{W}) & \longrightarrow & \mathcal{P}(\mathcal{W}) & &
\end{array}$$

where the lower horizontal arrow is the tensorization with $\mathcal{C}_{\mathcal{W}}(F)$ for a bounded complex of vector bundles F quasi-isomorphic to $\mathcal{F}$ and $\mathsf{S} := \mathrm{Res}_{\pi_{\mathcal{U}\times\mathcal{V}}} \circ \mathrm{Ind}_{\iota_{\mathcal{U}\times\mathcal{U}} \otimes \iota_{\mathcal{U}\times\mathcal{V}}} \circ \mathrm{Res}_{\hat{\boxtimes}_{|}}$. Thanks to what we proved in Chapter 3 we have that $\Psi_{\mathcal{F}}$ is a dg lift of the Fourier-Mukai functor $\Phi_{\mathcal{F}} : \mathrm{Perf}(X \times X) \longrightarrow \mathfrak{Perf}(X \times Y)$, defined as

$$\Phi_{\mathcal{F}}(A) := Rr_{2*}(\mathcal{F} \otimes^{L} r_1^{*} A)$$

where we have employed the notation of the diagram below, that contains also all the different projections that will be used later

$$\begin{array}{ccccccc}
 & & & X \times X \times X \times Y & & & \\
 & & \swarrow^{r_1} & & \searrow^{r_2} & & \\
 & X \times X & & & & X \times Y & \\
 \swarrow^{r_{11}} & & \searrow^{r_{12}} & & \swarrow^{r_{21}} & & \searrow^{r_{22}} \\
X & & & X \quad X & & & Y
\end{array}$$

We are now ready for the following

Proposition 4.3.1. *Assume that we have* $\Phi^{dg}_{\mathcal{E}_1} = \Phi^{dg}_{\mathcal{E}_2}$ *in* **Hqe**. *Then* $\mathcal{E}_1 \cong \mathcal{E}_2$ *in* $\mathfrak{Perf}(X \times Y)$.

Proof. We start by fixing a bounded complex D of vector bundles that is quasi-isomorphic to $\mathcal{O}_{\Delta} := \Delta_* \mathcal{O}_X$ (we know it exists since by Lemma 4.1.5 $\mathcal{O}_{\Delta}$ lies in $\mathrm{Perf}(X \times X)$); we fix moreover a bounded complex E of vector bundles that is quasi-isomorphic to $\mathcal{E} \in \mathrm{Perf}(X \times Y)$. We claim the commutativity in **Hqe** of the following diagram:

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U}) & \xrightarrow{\quad\quad \boxtimes \quad\quad} & \mathcal{P}(\mathcal{U}\times\mathcal{U}) \\
\Big\downarrow \Phi^{dg}_{\mathcal{O}_{\Delta}} \otimes \Phi^{dg}_{\mathcal{E}} & & \Big\downarrow \Psi_{\mathcal{E}} \\
\mathrm{Perf}(\mathcal{P}(\mathcal{U})) \otimes \mathrm{Perf}(\mathcal{P}(\mathcal{V})) & (\star) & \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \\
\Big\downarrow \bar{Y}_{\mathcal{U}\times\mathcal{V}} & & \Big\downarrow \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_{|}} \\
\mathrm{Perf}(\mathrm{Perf}(\mathcal{P}(\mathcal{U})) \otimes \mathrm{Perf}(\mathcal{P}(\mathcal{V}))) & \xrightarrow{\mathrm{Res}_{Y_{\mathcal{U}} \otimes Y_{\mathcal{V}}}} & \mathrm{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})).
\end{array}$$

where $\bar{Y}_{\mathcal{U}\times\mathcal{V}}$ is the Yoneda embedding and, by a slightly abuse of notation, $\Psi_{\mathcal{E}} := \Psi_{\mathcal{O}_{\Delta} \hat{\boxtimes} \mathcal{E}}$ denotes the dg functor we have above defined that is a dg lift of $\Phi_{\mathcal{O}_{\Delta} \hat{\boxtimes} \mathcal{E}}$.

The existence of the lower horizontal arrow is guaranteed by Corollary 2.5.8. Observe that $D \hat{\boxtimes} E$ is a bounded complex of vector bundles quasi-isomorphic to $\mathcal{O}_{\Delta} \hat{\boxtimes} \mathcal{E}$.

Now, given an element $(\mathcal{C}_{\mathcal{U}}(A), \mathcal{C}_{\mathcal{U}}(B)) \in \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{U})$, we have:

$$
\begin{aligned}
&\mathrm{Res}_{Y_{\mathcal{U}} \times Y_{\mathcal{V}}} \circ \bar{Y}_{\mathcal{U} \times \mathcal{V}} \circ (\Phi^{dg}_{\mathcal{O}_\Delta} \otimes \Phi^{dg}_{\mathcal{E}})(\mathcal{C}_{\mathcal{U}}(A), \mathcal{C}_{\mathcal{U}}(B)) = \\
&\mathrm{Res}_{Y_{\mathcal{U}} \times Y_{\mathcal{V}}} \circ \bar{Y}_{\mathcal{U} \times \mathcal{V}}(\Phi^{dg}_{\mathcal{O}_\Delta}(\mathcal{C}_{\mathcal{U}}(A)), \Phi^{dg}_{\mathcal{E}}(\mathcal{C}_{\mathcal{U}}(B))) = \\
&\mathrm{Res}_{Y_{\mathcal{U}} \times Y_{\mathcal{V}}} \circ \mathrm{Hom}\big((-,=), (\Phi^{dg}_{\mathcal{O}_\Delta}(\mathcal{C}_{\mathcal{U}}(A)), \Phi^{dg}_{\mathcal{E}}(\mathcal{C}_{\mathcal{U}}(B)))\big) = \\
&\mathrm{Hom}\big((Y_{\mathcal{U}}(-), Y_{\mathcal{V}}(=)), (\Phi^{dg}_{\mathcal{O}_\Delta}(\mathcal{C}_{\mathcal{U}}(A)), \Phi^{dg}_{\mathcal{E}}(\mathcal{C}_{\mathcal{U}}(B)))\big) = \\
&\mathrm{Hom}\big(Y_{\mathcal{U}}(-), \Phi^{dg}_{\mathcal{O}_\Delta}(\mathcal{C}_{\mathcal{U}}(A))\big) \otimes_{\Bbbk} \mathrm{Hom}\big(Y_{\mathcal{V}}(=), \Phi^{dg}_{\mathcal{E}}(\mathcal{C}_{\mathcal{U}}(B))\big) \cong \\
&\Phi^{dg}_{\mathcal{O}_\Delta}(\mathcal{C}_{\mathcal{U}}(A))(-) \otimes_{\Bbbk} \Phi^{dg}_{\mathcal{E}}(\mathcal{C}_{\mathcal{U}}(B))(=)
\end{aligned}
$$

where the Hom complexes are in the obvious categories and the last isomorphism is a consequence of dg Yoneda Lemma - see the isomorphism (2.2) of Section 2.2. Now, from the commutativity of the diagram

$$
\begin{array}{ccc}
\mathcal{P}_*(\mathcal{V}) & \xrightarrow{\ \bar{\pi}_{\mathcal{V}}\ } & \mathcal{P}_*(\mathcal{U}) \otimes \mathcal{P}_*(\mathcal{V}) \\
{\scriptstyle \iota_{\mathcal{V}}}\downarrow & & \downarrow{\scriptstyle L} \\
\mathcal{P}(\mathcal{V}) & \xrightarrow{\ \pi_{\mathcal{V}}\ } & \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})
\end{array}
$$

where we recall that $\iota_{\mathcal{V}}$ is the inclusion and $\bar{\pi}_{\mathcal{V}}$ is the restricted version of $\pi_{\mathcal{V}}$ (see Section 3.3) we obtain the equality $\mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_L = \mathrm{Ind}_{\iota_{\mathcal{V}}} \circ \mathrm{Res}_{\bar{\pi}_{\mathcal{V}}}$ in **Hqe** when we consider them as dg functors between perfect dg modules. We can write:

$$
\begin{aligned}
\Phi^{dg}_{\mathcal{E}}(\mathcal{C}_{\mathcal{U}}(B)) &= \mathrm{Res}_{\pi_{\mathcal{V}}} \circ \mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U} \times \mathcal{V}} \circ K_{\mathcal{C}_{\mathcal{U} \times \mathcal{V}}(E)} \circ \boxtimes \circ \pi_{\mathcal{U}}(\mathcal{C}_{\mathcal{U}}(B)) \\
&= \mathrm{Ind}_{\iota_{\mathcal{V}}} \circ \mathrm{Res}_{\bar{\pi}_{\mathcal{V}}} \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U} \times \mathcal{V}} \circ K_{\mathcal{C}_{\mathcal{U} \times \mathcal{V}}(E)}(\mathcal{C}_{\mathcal{U}}(B) \boxtimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)) \\
&= \mathrm{Ind}_{\iota_{\mathcal{V}}} \circ \mathrm{Res}_{\bar{\pi}_{\mathcal{V}}} \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U} \times \mathcal{V}} \circ K_{\mathcal{C}_{\mathcal{U} \times \mathcal{V}}(E)}(\mathcal{C}_{\mathcal{U}}(B) \boxtimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)) \\
&= \mathrm{Ind}_{\iota_{\mathcal{V}}} \circ \mathrm{Res}_{\bar{\pi}_{\mathcal{V}}} \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U} \times \mathcal{V}}((\mathcal{C}_{\mathcal{U}}(B) \boxtimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)) \otimes \mathcal{C}_{\mathcal{U} \times \mathcal{V}}(E)) \\
&= \mathrm{Ind}_{\iota_{\mathcal{V}}} \circ \mathrm{Res}_{\bar{\pi}_{\mathcal{V}}} \circ \mathrm{Res}_{\boxtimes_|} \circ \mathrm{Hom}_{\mathcal{P}(\mathcal{U} \times \mathcal{V})}\Big(-, ((\mathcal{C}_{\mathcal{U}}(B) \boxtimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)) \otimes \mathcal{C}_{\mathcal{U} \times \mathcal{V}}(E))\Big) \\
&= \mathrm{Ind}_{\iota_{\mathcal{V}}} \bigg(\mathrm{Hom}_{\mathcal{P}(\mathcal{U} \times \mathcal{V})}\Big(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X) \boxtimes -, ((\mathcal{C}_{\mathcal{U}}(B) \boxtimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)) \otimes \mathcal{C}_{\mathcal{U} \times \mathcal{V}}(E))\Big)\bigg)
\end{aligned}
$$

Now, let us write more explicitly:

$$
\begin{aligned}
\Theta_1 :&= \mathrm{Hom}_{\mathcal{P}(\mathcal{U} \times \mathcal{V})}\Big(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X) \boxtimes =, ((\mathcal{C}_{\mathcal{U}}(B) \boxtimes \mathcal{C}_{\mathcal{V}}(\mathcal{O}_Y)) \otimes \mathcal{C}_{\mathcal{U} \times \mathcal{V}}(E))\Big) \\
&\overset{\text{h.eq.}}{\simeq} \mathrm{Hom}_{\mathcal{P}(\mathcal{U} \times \mathcal{V})}\Big(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X) \boxtimes =, \mathcal{C}_{\mathcal{U} \times \mathcal{V}}(r^*_{21}(B) \otimes E)\Big); \\
\Theta_2 :&= \mathrm{Hom}_{\mathcal{P}(\mathcal{U} \times \mathcal{U})}\Big(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X) \boxtimes -, ((\mathcal{C}_{\mathcal{U}}(A) \boxtimes \mathcal{C}_{\mathcal{U}}(\mathcal{O}_X)) \otimes \mathcal{C}_{\mathcal{U} \times \mathcal{U}}(D))\Big) \\
&\overset{\text{h.eq.}}{\simeq} \mathrm{Hom}_{\mathcal{P}(\mathcal{U} \times \mathcal{U})}\Big(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X) \boxtimes -, \mathcal{C}_{\mathcal{U} \times \mathcal{U}}(r^*_{11}(A) \otimes D)\Big).
\end{aligned}
$$

From essentially the same argument as the one of Proposition 3.2.7 we have a homotopy equivalence between

$$
\mathrm{Hom}_{\mathcal{P}(\mathcal{U} \times \mathcal{U})}\Big(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X) \boxtimes -, \mathcal{C}_{\mathcal{U} \times \mathcal{U}}(r^*_{11}(A) \otimes D)\Big) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{P}(\mathcal{U} \times \mathcal{V})}\Big(\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X) \boxtimes =, \mathcal{C}_{\mathcal{U} \times \mathcal{V}}(r^*_{21}(B) \otimes E)\Big)
$$

and

$$
\mathrm{Hom}_{\mathcal{P}(\mathcal{W})}\Big((\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X) \boxtimes -) \hat{\boxtimes} (\mathcal{C}_{\mathcal{U}}(\mathcal{O}_X) \boxtimes =), \mathcal{C}_{\mathcal{U} \times \mathcal{U}}(r^*_{11}(A) \otimes D) \hat{\boxtimes} \mathcal{C}_{\mathcal{U} \times \mathcal{V}}(r^*_{21}(B) \otimes E)\Big). \quad (4.3)
$$

Summarising we have the following chain of homotopy equivalences:

$$\mathrm{Res}_{Y_{\mathcal{U}}\times Y_{\mathcal{V}}}\circ \bar{Y}_{\mathcal{U}\times\mathcal{V}}\circ(\Phi^{dg}_{\mathcal{O}_\Delta}\otimes\Phi^{dg}_{\mathcal{E}})(\mathcal{C}_{\mathcal{U}}(A),\mathcal{C}_{\mathcal{U}}(B))=\Phi^{dg}_{\mathcal{O}_\Delta}(\mathcal{C}_{\mathcal{U}}(A))(-)\otimes_{\Bbbk}\Phi^{dg}_{\mathcal{E}}(\mathcal{C}_{\mathcal{U}}(B))(=)$$
$$\simeq \mathrm{Ind}_{\iota_{\mathcal{U}}}(\Theta_2)\otimes_{\Bbbk}\mathrm{Ind}_{\iota_{\mathcal{V}}}(\Theta_1)=\mathrm{Ind}_L(\Theta_2\otimes_{\Bbbk}\Theta_1)\simeq\mathrm{Ind}_L\big((4.3)\big)$$

when the equality between $\mathrm{Ind}_{L:=\iota_{\mathcal{U}}\otimes\iota_{\mathcal{V}}}$ and $\mathrm{Ind}_{\iota_{\mathcal{U}}}\otimes\mathrm{Ind}_{\iota_{\mathcal{V}}}$ can be easily verified.

On the other hand — by applying an argument analogous to the one we used above for $\Phi^{dg}_{\mathcal{E}}(\mathcal{C}_{\mathcal{U}}(B))$ — we have, in **Hqe**

$$\Psi_{\mathcal{E}}\circ\boxtimes(\mathcal{C}_{\mathcal{U}}(A),\mathcal{C}_{\mathcal{U}}(B))=\Psi_{\mathcal{E}}(\mathcal{C}_{\mathcal{U}}(A)\boxtimes\mathcal{C}_{\mathcal{U}}(B))=\mathrm{Ind}_{\iota_{\mathcal{U}\times\mathcal{V}}}(\Theta_{12})$$

where $\Theta_{12}\in\mathrm{Perf}(\mathcal{P}_*(\mathcal{U}\times\mathcal{V}))$ is

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{W})}\Big(\mathcal{C}_{\mathcal{U}\times\mathcal{U}}(\mathcal{O}_{X\times X})\hat{\boxtimes}-,\,((\mathcal{C}_{\mathcal{U}}(A)\boxtimes\mathcal{C}_{\mathcal{U}}(B))\hat{\boxtimes}\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(\mathcal{O}_{X\times Y}))\otimes\mathcal{C}_{\mathcal{W}}(D\hat{\boxtimes}E)\Big)$$

which is homotopy equivalent to

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{W})}\Big(\mathcal{C}_{\mathcal{U}\times\mathcal{U}}(\mathcal{O}_{X\times X})\hat{\boxtimes}-,\mathcal{C}_{\mathcal{W}}(((A\boxtimes B)\hat{\boxtimes}\mathcal{O}_{X\times Y})\otimes(D\hat{\boxtimes}E))\Big).\qquad(4.4)$$

But now, $\mathrm{Ind}_{\iota_{\mathcal{U}\times\mathcal{V}}}(\Theta_{12})$ is just Θ_{12} viewed as a dg module over $\mathcal{P}(\mathcal{U}\times\mathcal{V})$ and we can see by tedious computations (see Lemma 4.3.2 below) that if we apply $\mathrm{Res}_{\boxtimes_|}$ to (4.4) it is homotopy equivalent to (4.3). It is not hard to see that such a homotopy equivalence holds for any objects of $\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{U})$ therefore we have a termwise homotopy equivalence that yields by Proposition 2.5.10, in **Hqe**, the equality

$$\mathrm{Res}_{Y_{\mathcal{U}}\otimes Y_{\mathcal{V}}}\circ\bar{Y}_{\mathcal{U}\times\mathcal{V}}\circ(\Phi^{dg}_{\mathcal{O}_\Delta}\otimes\Phi^{dg}_{\mathcal{E}})=\mathrm{Ind}_L\circ\mathrm{Res}_{\boxtimes_|}\circ\Psi_{\mathcal{E}}\circ\boxtimes$$

that is exactly the commutativity we were looking for.

Now, if we apply the functor Ind to the diagram ($\star$), we still get a commutative one. Moreover, if we restrict to perfect dg modules, most of the dg functors involved are actually quasi-equivalences. In fact:

- $\mathrm{Ind}_{\boxtimes}:\mathrm{Perf}(\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V}))\longrightarrow\mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))$ is a quasi-equivalence by Lemma 3.2.14;
- $\mathrm{Res}_{Y_{\mathcal{U}}\otimes Y_{\mathcal{V}}}:\mathrm{Perf}(\mathrm{Perf}(\mathcal{P}(\mathcal{U}))\otimes\mathrm{Perf}(\mathcal{P}(\mathcal{V})))\longrightarrow\mathrm{Perf}(\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V}))$ is a quasi-equivalence by Corollary 2.5.8;
- the dg functor Ind_F is a quasi-equivalence between perfect dg modules anytime F is such - see Proposition 2.5.3. Therefore the dg functors $\mathrm{Ind}_{\mathrm{Ind}_L\circ\mathrm{Res}_{\boxtimes_|}}$ and $\mathrm{Ind}_{\mathrm{Res}_{Y_{\mathcal{U}}\otimes Y_{\mathcal{V}}}}$ are quasi-equivalences;
- In order to prove that also $\mathrm{Ind}_{\bar{Y}_{\mathcal{U}\times\mathcal{V}}}$ is a quasi-equivalence we use a more general reasoning. For any dg category $\mathcal{A}$ we have the dg Yoneda functor $Y_{\mathcal{A}}:\mathcal{A}\longrightarrow\mathrm{Perf}(\mathcal{A})$ and therefore also - by Proposition 2.5.2 (ii) - a commutative diagram

$$\begin{array}{ccc}\mathrm{Perf}(\mathcal{A}) & \xrightarrow{\mathrm{Ind}_{Y_{\mathcal{A}}}} & \mathrm{Perf}(\mathrm{Perf}(\mathcal{A}))\\ {\scriptstyle Y_{\mathcal{A}}}\uparrow & & \uparrow{\scriptstyle Y_{\mathrm{Perf}(\mathcal{A})}}\\ \mathcal{A} & \xrightarrow{Y_{\mathcal{A}}} & \mathrm{Perf}(\mathcal{A})\end{array}$$

This implies $\mathrm{Ind}_{Y_\mathcal{A}} \circ Y_\mathcal{A} \simeq Y_{\mathrm{Perf}(\mathcal{A})} \circ Y_\mathcal{A}$. But now, by the so called "derived Morita theory" (see for example Corollary 4.2 of [13]) we have $\mathrm{Ind}_{Y_\mathcal{A}} = Y_{\mathrm{Perf}(\mathcal{A})}$ in **Hqe**. In conclusion $\mathrm{Ind}_{Y_\mathcal{A}}$ is a quasi-equivalence since $Y_{\mathrm{Perf}(\mathcal{A})}$ is such: in fact $\mathrm{Perf}(\mathcal{A})$ is always a perfect dg category (see Proposition 2.4.4 and the discussion above it).

Hence what we obtain is the fact that $\mathrm{Ind}_{\varphi_\mathcal{E}}$ and $\mathrm{Ind}_{\Psi_\mathcal{E}}$ can be identified in **Hqe** up to invertible elements, where we have set $\varphi_\mathcal{E} := \Phi^{dg}_{\mathcal{O}_\Delta} \otimes \Phi^{dg}_\mathcal{E}$.

Now assume that $\Phi^{dg}_{\mathcal{E}_1} = \Phi^{dg}_{\mathcal{E}_2}$ in **Hqe**. Then we have

$$\mathrm{Ind}_{\varphi_{\mathcal{E}_1}} = \mathrm{Ind}_{\varphi_{\mathcal{E}_2}} \qquad \text{in } \mathbf{Hqe}$$

and, by the commutativity of the diagram obtained by applying the functor Ind to the commutative diagram ($\star$) and restricting to perfect dg modules, also $\mathrm{Ind}_{\Psi_{\mathcal{E}_1}}$ and $\mathrm{Ind}_{\Psi_{\mathcal{E}_2}}$ are equal in **Hqe**. From the commutative diagrams below, for $i \in \{1, 2\}$

$$\begin{array}{ccc} \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{U})) & \xrightarrow{\mathrm{Ind}_{\Psi_{\mathcal{E}_i}}} & \mathrm{Perf}(\mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V}))) \\ \uparrow & & \uparrow \\ \mathcal{P}(\mathcal{U}\times\mathcal{U}) & \xrightarrow{\Psi_{\mathcal{E}_i}} & \mathrm{Perf}(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \end{array}$$

we deduce that the same happens for $\Psi_{\mathcal{E}_1}$ and $\Psi_{\mathcal{E}_2}$. If we apply $\mathrm{H}^0(-)$, we get therefore that $\Phi_{\mathcal{O}_\Delta \hat{\boxtimes} \mathcal{E}_1}$ and $\Phi_{\mathcal{O}_\Delta \hat{\boxtimes} \mathcal{E}_2}$ can be identified as exact functors, up to isomorphism. Whence we can conclude, as claimed:

$$\mathcal{E}_1 \cong \Phi_{\mathcal{O}_\Delta \hat{\boxtimes} \mathcal{E}_1}(\mathcal{O}_\Delta) \cong \Phi_{\mathcal{O}_\Delta \hat{\boxtimes} \mathcal{E}_2}(\mathcal{O}_\Delta) \cong \mathcal{E}_2$$

where the first and the last isomorphisms come from Lemma 1.4.3. □

Lemma 4.3.2. *For any $\mathcal{C}_\mathcal{U}(M) \in \mathcal{P}_*(\mathcal{U})$ and $\mathcal{C}_\mathcal{V}(N) \in \mathcal{P}_*(\mathcal{V})$ we have that the object*

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{W})}\Big((\mathcal{C}_\mathcal{U}(\mathcal{O}_X) \boxtimes \mathcal{C}_\mathcal{U}(M)) \hat{\boxtimes} (\mathcal{C}_\mathcal{U}(\mathcal{O}_X) \boxtimes \mathcal{C}_\mathcal{V}(N)), \mathcal{C}_{\mathcal{U}\times\mathcal{U}}(r_{11}^*(A) \otimes D) \hat{\boxtimes} \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(r_{21}^*(B) \otimes E)\Big)$$

is homotopy equivalent to

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{W})}\Big(\mathcal{C}_{\mathcal{U}\times\mathcal{U}}(\mathcal{O}_{X\times X}) \hat{\boxtimes} (\mathcal{C}_\mathcal{U}(M) \boxtimes \mathcal{C}_\mathcal{V}(N)), \mathcal{C}_\mathcal{W}(((A \boxtimes B) \hat{\boxtimes} \mathcal{O}_{X\times Y}) \otimes (D \hat{\boxtimes} E))\Big).$$

Proof. Since all the functors $\otimes$, $\boxtimes$, $\hat{\boxtimes}$ and the (co- and contro-)variant Hom are well-defined dg functors on the dg category of quasi-coherent sheaves, they behave well with respect to classes of homotopy equivalent objects, as we already observed. Therefore our claim follows from the following facts:

- $(\mathcal{C}_\mathcal{U}(\mathcal{O}_X) \boxtimes \mathcal{C}_\mathcal{U}(M)) \hat{\boxtimes} (\mathcal{C}_\mathcal{U}(\mathcal{O}_X) \boxtimes \mathcal{C}_\mathcal{V}(N)) \overset{\text{h.eq.}}{\simeq} \mathcal{C}_{\mathcal{U}\times\mathcal{U}}(\mathcal{O}_X \boxtimes M) \hat{\boxtimes} \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(\mathcal{O}_X \boxtimes N) \simeq \mathcal{C}_{\mathcal{U}\times\mathcal{U}}(r_{12}^* M) \hat{\boxtimes} \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(r_{22}^* N) \overset{\text{h.eq.}}{\simeq} \mathcal{C}_\mathcal{W}(r_{12}^* M \hat{\boxtimes} r_{22}^* N) \simeq \mathcal{C}_\mathcal{W}((r_{12} \circ r_1)^* M \otimes (r_{22} \circ r_2)^* N)$;
- $\mathcal{C}_{\mathcal{U}\times\mathcal{U}}(\mathcal{O}_{X\times X}) \hat{\boxtimes} (\mathcal{C}_\mathcal{U}(M) \boxtimes \mathcal{C}_\mathcal{V}(N)) \overset{\text{h.eq.}}{\simeq} \mathcal{C}_{\mathcal{U}\times\mathcal{U}}(\mathcal{O}_{X\times X}) \hat{\boxtimes} \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(M \boxtimes N) \overset{\text{h.eq.}}{\simeq} \mathcal{C}_\mathcal{W}(\mathcal{O}_{X\times X} \hat{\boxtimes} (r_{21}^* M \otimes r_{22}^* N)) \simeq \mathcal{C}_\mathcal{W}((r_{21} \circ r_2)^* M \otimes (r_{22} \circ r_2)^* N)$;
- $\mathcal{C}_{\mathcal{U}\times\mathcal{U}}(r_{11}^* A \otimes D) \hat{\boxtimes} \mathcal{C}_{\mathcal{U}\times\mathcal{V}}(r_{21}^* B \otimes E) \overset{\text{h.eq.}}{\simeq} \mathcal{C}_\mathcal{W}((r_{11}^* A \otimes D) \hat{\boxtimes} (r_{21}^* B \otimes E)) \simeq \mathcal{C}_\mathcal{W}(r_1^*(r_{11}^* A \otimes D) \otimes r_2^*(r_{21}^* B \otimes E)) \simeq \mathcal{C}_\mathcal{W}((r_{11} \circ r_1)^* A \otimes r_1^* D \otimes (r_{21} \circ r_2)^* B \otimes r_2^* E)$;

- $\mathcal{C}_{\mathcal{W}}(((A \boxtimes B)\hat{\boxtimes}\mathcal{O}_{X\times Y})\otimes(D\hat{\boxtimes}E)) \simeq \mathcal{C}_{\mathcal{W}}(((r_{11}^*A\otimes r_{12}^*B)\hat{\boxtimes}\mathcal{O}_{X\times Y})\otimes(r_1^*D\otimes r_2^*E)) \simeq \mathcal{C}_{\mathcal{W}}((r_{11}\circ r_1)^*A\otimes(r_{12}\circ r_1)^*B\otimes r_1^*D\otimes r_2^*E)$;
- there exists a canonical isomorphism interchanging the two middle factors of $\mathcal{U}\times\mathcal{U}\times\mathcal{U}\times\mathcal{V}$.

□

4.4 Proof of Theorem 4.4.1

We are now ready to collect what we have done in order to define the explicit bijection and showing that it yields in a straightforward way the proof of the claim we were looking for. Let us fix $\mathfrak{Perf}^{dg}(X) := \mathcal{P}(\mathcal{U})$ and $\mathfrak{Perf}^{dg}(Y) := \mathcal{P}(\mathcal{V})$; we know they are two dg enhancements of $\mathfrak{Perf}(X)$ and of $\mathfrak{Perf}(Y)$ respectively.

Theorem 4.4.1. *Let X and Y be two smooth proper schemes over a field. Then there exists a bijective map*

$$\gamma : \mathrm{Iso}(\mathfrak{Perf}(X\times Y)) \xrightarrow{1:1} \mathbf{Hqe}(\mathfrak{Perf}^{dg}(X), \mathfrak{Perf}^{dg}(Y))$$

compatible with Fourier-Mukai kernels; i.e. such that, for any $\mathcal{E} \in \mathfrak{Perf}(X\times Y)$, we have $\mathrm{H}^0(\gamma(\mathcal{E})) \simeq \Phi_{\mathcal{E}}$.

Proof. Consider now the map

$$\begin{array}{ccc} \mathrm{Iso}(\mathfrak{Perf}(X\times Y)) & \xrightarrow{\gamma} & \mathbf{Hqe}(\mathcal{P}(\mathcal{U}), \mathcal{P}(\mathcal{V})) \\ [\mathcal{E}]_{\mathrm{iso}} & \longmapsto & Y_{\mathcal{V}}^{-1}\circ\Phi_{\mathcal{E}}^{dg} \end{array}$$

In the following steps we are going to show that it satisfies all the requests we need in order to prove the theorem.

It is well-defined: remember that, Remark 3.7.1 tells us that the class in **Hqe** of $\Phi_{\mathcal{E}}^{dg}$ does not depend on the choice of the bounded complex of vector bundles quasi-isomorphic to $\mathcal{E}$. Moreover, if we have $\mathcal{E}\simeq\mathcal{F}$ we can apply the same argument and get that $\Phi_{\mathcal{E}}^{dg}$ and $\Phi_{\mathcal{F}}^{dg}$ are termwise homotopy equivalent; hence they represent the same morphism in **Hqe**. It follows that $\gamma([\mathcal{E}]_{\mathrm{iso}}) = \gamma([\mathcal{F}]_{\mathrm{iso}})$.

Injectivity: if $\gamma([\mathcal{E}]_{\mathrm{iso}}) = \gamma([\mathcal{F}]_{\mathrm{iso}})$ in **Hqe** then certainly we have $\Phi_{\mathcal{E}}^{dg} = \Phi_{\mathcal{F}}^{dg}$ in **Hqe**, too. But this, from Proposition 4.3.1, implies $\mathcal{E}\simeq\mathcal{F}$ in $\mathfrak{Perf}(X\times Y)$.

Surjectivity: let F be an element of $\mathbf{Hqe}(\mathcal{P}(\mathcal{U}), \mathcal{P}(\mathcal{V}))$. We know from Proposition 4.2.3 that $Y_{\mathcal{V}}\circ\mathsf{F}$ is equal to $\tilde{\Phi}_{\mathcal{E}_{\mathsf{F}}}^{dg}$ in **Hqe** with $\mathcal{E}_{\mathsf{F}}\in\mathrm{Perf}(\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V}))$. Now, since we have a quasi-equivalence

$$\mathrm{Ind}_L\circ\mathrm{Res}_{\boxtimes_|}\circ Y_{\mathcal{U}\times\mathcal{V}} : \mathcal{P}(\mathcal{U}\times\mathcal{V}) \longrightarrow \mathrm{Perf}(\mathcal{P}(\mathcal{U})\otimes\mathcal{P}(\mathcal{V}))$$

there exists an object $\mathcal{E}'_{\mathsf{F}} \in \mathrm{H}^0(\mathcal{P}(\mathcal{U}\times\mathcal{V})) \simeq \mathfrak{Perf}(X\times Y)$ such that $\mathrm{H}^0(\mathrm{Ind}_L\circ\mathrm{Res}_{\boxtimes_|}\circ Y_{\mathcal{U}\times\mathcal{V}})(\mathcal{E}'_{\mathsf{F}}) \simeq \mathcal{E}_{\mathsf{F}}$. Now, let E' be a bounded complex of vector bundles on $X\times Y$

quasi-isomorphic to $\mathcal{E}'_{\mathsf{F}}$. It follows that $\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E')$ is isomorphic to $\mathcal{E}'_{\mathsf{F}}$ in $\mathrm{H}^0(\mathcal{P}(\mathcal{U}\times\mathcal{V}))$ and therefore $\mathrm{H}^0(\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U}\times\mathcal{V}})(\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E')) \simeq \mathcal{E}_{\mathsf{F}}$. In conclusion, recalling the relationship between Φ^{dg} and $\tilde{\Phi}^{dg}$ (see the discussion before Proposition 4.2.3), we can write the following equalities in **Hqe**

$$\gamma([\mathcal{E}'_{\mathsf{F}}]_{\mathrm{iso}}) = Y_{\mathcal{V}}^{-1} \circ \Phi^{dg}_{\mathcal{E}'_{\mathsf{F}}} = Y_{\mathcal{V}}^{-1} \circ \tilde{\Phi}^{dg}_{\mathrm{Ind}_L \circ \mathrm{Res}_{\boxtimes_|} \circ Y_{\mathcal{U}\times\mathcal{V}}(\mathcal{C}_{\mathcal{U}\times\mathcal{V}}(E'))} = Y_{\mathcal{V}}^{-1} \circ \tilde{\Phi}^{dg}_{\mathcal{E}_{\mathsf{F}}} = \mathsf{F}.$$

and get the surjectivity of γ.

Compatibility with Fourier-Mukai kernels: it comes form the definition of γ. In fact, for any $\mathcal{E} \in \mathfrak{Perf}(X \times Y)$, we have

$$\mathrm{H}^0(\gamma(\mathcal{E})) = \mathrm{H}^0(Y_{\mathcal{V}}^{-1} \circ \Phi^{dg}_{\mathcal{E}}) = \mathrm{H}^0(Y_{\mathcal{V}}^{-1}) \circ \mathrm{H}^0(\Phi^{dg}_{\mathcal{E}}).$$

Recall now that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{P}(\mathcal{U})) & \xrightarrow{\mathrm{H}^0(\Phi^{dg}_{\mathcal{E}})} & \mathrm{Perf}(\mathcal{P}(\mathcal{V})) \\ \Big\downarrow{\scriptstyle \omega_X} & & \Big\downarrow{\scriptstyle \omega_Y \circ \mathrm{H}^0(Y_{\mathcal{V}})^{-1}} \\ \mathfrak{Perf}(X) & \xrightarrow{\Phi_{\mathcal{E}}} & \mathfrak{Perf}(Y) \end{array}$$

where ω_X and ω_Y are exact equivalences. It follows that $\mathrm{H}^0(\gamma(\mathcal{E}))$ and $\Phi_{\mathcal{E}}$ can be identified up to equivalences. □

The importance of Theorem 4.4.1 relies also on the following fact. Suppose we have a triangulated functor $f : \mathfrak{Perf}(X) \longrightarrow \mathfrak{Perf}(Y)$ that is dg liftable: we know by theoretical reasons - at least in the case of smooth proper schemes over a field - that f is of Fourier-Mukai type. Now, the way we have costructed the bijective map γ allow us to explicitly compute a kernel $\mathcal{E}_f$ of f.

4.5 Possible generalizations

In the conclusion of this thesis we want to point out that a consistent part of the results proved in Chapter 4 does not depend on the fact that we are working over a field and is indeed true when the schemes satisfy the assumption $(**)$.

By looking carefully at Chapter 4, it can be realized that — if we are under the milder assumption $(**)$ — the only relevant difference is that all our categories are just h-flat (thaks to Proposition 3.8.1) instead of being h-projective. In practice, this translates in the fact that the dg functor $\mathrm{Res}_{\pi_{\mathcal{V}}}$ (and its "relatives" Res_{π_2}, etc.) do not restrict to h-projective dg modules and hence we need to compose it with the *h-projective resolution* quasi-functor H, as we have done in Section 3.8. For this reason we belive that it will be possible to extend all the proofs of Chapter 4 — and therefore the one of Theorem 4.4.1 — also in the generality of $(**)$.

Actually we are close to a proof of Proposition 4.2.3, which is essentially the sujectivity of our bijection, but still some details has to be fixed. This, together with a generalized version of Proposition 4.3.1, is something that might be investigated in the future.

Remark 4.5.1. We just want to point out here that, if our $\Bbbk$ happens to be a perfect (commutative) ring, then the whole proof of Theorem 4.4.1 readily applies. In fact perfect (commutative) rings can be characterized (see Theorem 24.25 of [27]) as the rings such that flat modules over them are projective. This clearly implies that — under this hypotesis — all our dg categories are also h-projective and therefore, by what we said above, all the results of Chapter 4 remain true.

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