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Research Article

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Posted Date: January 11th, 2023

DOI: https://doi.org/10.21203/rs.3.rs-1874417/v1

Bipolar argumentative semantics for t-norm based fuzzy logics

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Abstract

In this paper, we introduce argumentative sound and complete semantics for the three main t-norm based fuzzy logics, *Łukasiewicz* (**Ł**), *Gödel* (**G**) and *product* logic (**Π**). We understand the arguments as complex entities and instantiate the link between the support and the claim of an argument with the three t-norm consequence relations corresponding to **Ł**, **G** and **Π**. The argumentative frameworks considered are *bipolar* frames. Thus, the arguments interact through two different relations: the attack and the support relation. By introducing argumentative principles defined over the two relations and using the notion of argumentative immunity, the semantics are recovered.

Keywords: Logical Argumentation Theory, T-norm-based Argumentation Theory, Fuzzy Logic, T-norm, Argumentative Semantics, Bipolar Argumentation Frameworks.

1 Introduction

In his seminal work [22], Dung introduces abstract argumentation frameworks. Dung understands arguments as abstract entities that relate with each other through the attack relation. In the present work, we consider a variation of Dung's original argumentative setting. Many are the works aimed at deductively formalising Dung's work [1–3, 13, 24, 27, 33]. Following the same approach of the papers just cited, we understand the arguments as complex entities made of three parts: the support, the claim and the method of inference that makes explicit how the claim follows from the support. As method of inference, we consider the consequence relations relative to the main three t-norm based fuzzy logics [18]: *Łukasiewicz* (**Ł**), *Gödel* (**G**) and *product* logic (**Π**). In addition, we work not only with the attack relation but also the support relation. Argumentation frames with both the attack and the support relations are referred to as *bipolar* [5, 17]. In these new frameworks, we introduce *argumentative principles* defined using both relations and that, as the attack principles in [19–21], refine the existence (or non-existence) of the attack or support relations once the arguments involved share, in their claims, some or even all atomic propositional formulas. Following the same approach as [20], we define *argumentative immunity* in bipolar t-norm based argumentative frameworks. Thus, through the notion of argumentative immunity jointly with specific sets of argumentative principles we recover sound and complete semantics for **Ł**, **G** and **Π**.

Acknowledgements

The author acknowledges the funding by the Department of Philosophy "Piero Martinetti" of the University of Milan under the Project "Departments of Excellence 2018-2022" awarded by the Ministry of Education, University and Research (MIUR).

The present work aims at contributing to the literature on alternative semantics for fuzzy logics (see e.g. [12, 26, 31, 32, 34]). Possible applications of the argumentative frameworks and principles introduced remain an open line of research for future investigations.

The paper is organised as follows. In Section 2, we recall some notions of abstract and logical argumentation theory. In Section 3, we give a glimpse of the three main t-norm based fuzzy logics. The first contribution of the paper can be found in Section 4 with the introduction of bipolar t-norm based logical argumentation frameworks and bipolar principles. Finally, in Section 5, we prove completeness theorems for **Ł**, **G** and **Π**.

2 Abstract and Logical Argumentation Theory

Dung's abstract argumentation frameworks are defined as a set of arguments and a binary relation defined over them understood as the attack relation [22]. Thus, abstract argumentation frameworks can be depicted by directed graphs where the nodes represent the arguments and the edges the attack relation. In Dung's setting arguments have no inner structure, they are abstract entities and also the attack relation is not instantiated in any specific way. These very general frames can be used to model a wide range of scenarios in Artificial Intelligence and related fields. To enlarge the possible applications of abstract argumentation theory, frames with also a positive relation among arguments, called *support*, have been investigated [5, 17, 29, 35]; these are the *bipolar* argumentative frameworks. In some specific situations, support and attack are dependent notions. For example, if in a given argumentation frame there are three arguments A, B and C such that A attacks B, and B attacks C ($A \longrightarrow B \longrightarrow C$), then the argument A defends C and we infer that A *supports* C ($A \Longrightarrow C$). However, whenever the arguments are instantiated with actual propositions, this way of reasoning does not always hold as the following example shows.

Example 1 ([16]) There is a family that would like to go hiking. They prefer sunny weather to a cloudy one and cloudy weather to a rainy one. If it is rainy, then they will not go hiking. Otherwise, they will go. However, clouds could be a sign of rain. They look at the sky early in the morning, and it is cloudy.

The following exchange of informal arguments occurs:

A: Today it is a holiday, and we go hiking.

B: The weather will be cloudy, clouds are a sign of rain, we should cancel the hiking.

C: These clouds are early patches of mist; the day will be sunny without clouds, and the weather will be not cloudy.

D: Clouds will not grow. Thus, the weather will be cloudy, but not rainy.

In Example 1, we have that $D \longrightarrow C \longrightarrow B \longrightarrow A$. Even just by looking at the graphical representation of the example, we see that the argument D defends B and C defends A. However, both C and D support the idea of going on a hike. Therefore, the idea of considering chains of arguments and counter-arguments, counting the number of links between them and assess the odd ones as attacks and the even ones as supports is an oversimplification for the definition of support. This is one of the reasons why, in bipolar argumentation frames, the definition of the attack relation is independent from the definition of the support relation. However, through the argumentative principles, as shown in Section 5, it is possible to enforce some constraints to the existence of both relations in a given frame.

Definition 1 (Bipolar Argumentation Frame (BAF)) A bipolar argumentation framework is a triplet $A = \langle Ar, \mathscr{R}_{\rightarrow}, \mathscr{R}_{\Rightarrow} \rangle$ such that:

Ar is a set of arguments,

$\mathscr{R}_{\rightarrow}$ is the *attack* relation and it is a binary relation over Ar ($\mathscr{R}_{\rightarrow} \subseteq Ar \times Ar$),

$\mathscr{R}_{\Rightarrow}$ is the *support* relation and it is a binary relation over Ar ($\mathscr{R}_{\Rightarrow} \subseteq Ar \times Ar$).

Instead of writing $(A, B) \in \mathscr{R}_{\rightarrow}$ or $(A, B) \in \mathscr{R}_{\Rightarrow}$, we use the more intuitive notation $A \longrightarrow B$ and $A \Longrightarrow B$ and also use $\longrightarrow$ and $\Longrightarrow$ to denote $\mathscr{R}_{\rightarrow}$ and $\mathscr{R}_{\Rightarrow}$, respectively. The symbols $\not\longrightarrow$ and $\not\Longrightarrow$ stand for "not attacking" and "not supporting".

In logical argumentation theory, both the arguments and the relations among them are instantiated. The following definition of *logical argument* has been largely used in the literature [4, 13, 27].

Definition 2 (Deductive Argument [14]) Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic ($\mathcal{L}$ stands for the propositional language and $\vdash$ for a Tarskian consequence relation over $\mathcal{L}$). Let Δ be a finite set of propositional formulas in $\mathcal{L}$ (the database). An *argument* based on $\mathfrak{L}$ and Δ is a pair $\langle \Gamma, \varphi \rangle$ s.t. $\Gamma \subseteq \Delta$ and:

$\Gamma \vdash \varphi$

Γ is consistent

Γ is a minimal subset of Δ satisfying (1).

$\mathsf{S}(\langle \Gamma, \varphi \rangle)$ denotes the support of the argument and $\mathsf{C}(\langle \Gamma, \varphi \rangle)$ its claim. For all $S \subseteq \Delta$, $Arg(S)$ denotes the set of all arguments that can be built from S.

An alternative definition of logical argument has been introduced in [6] and widely investigated by Arieli and Straßer in [7–11]. In Arieli and Straßer's approach, arguments are understood as *sequents*, as introduced by Gentzen [25] and in their definition, the assumptions about minimality and consistency of the support are omitted.

In logical argumentation theory, the attack relation can be instantiated in several ways.

Definition 3 (Attack Relations [15, 27]) Let Δ be a database on the underlying logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and $\langle \Gamma_1, \psi_1 \rangle$ and $\langle \Gamma_2, \psi_2 \rangle$ two deductive (or sequent-based) arguments in $Arg(\Delta)$. We define the following attack relations by listing the conditions under which $\langle \Gamma_1, \psi_1 \rangle$ attacks $\langle \Gamma_2, \psi_2 \rangle$. On the left side, we list the symbol for each attack relation.

[Def] $\langle \Gamma_1, \psi_1 \rangle$ is a *defeater* of $\langle \Gamma_2, \psi_2 \rangle$ if $\psi_1 \vdash \neg \bigwedge \Gamma_2$.

[Ucut] $\langle \Gamma_1, \psi_1 \rangle$ is a *direct undercut* of $\langle \Gamma_2, \psi_2 \rangle$ if $\psi_1 \vdash \neg \bigwedge {\Gamma'}_2$ and $\neg \bigwedge {\Gamma'}_2 \vdash \psi_1$ for some ${\Gamma'}_2 \subseteq \Gamma_2$.

[Reb] $\langle \Gamma_1, \psi_1 \rangle$ is a *rebuttal* of $\langle \Gamma_2, \psi_2 \rangle$ if $\psi_1 \vdash \neg \psi_2$ and $\neg \psi_2 \vdash \psi_1$.

[C-Reb-1] $\langle \Gamma_1, \psi_1 \rangle$ is an *compact rebuttal 1*[1] of $\langle \Gamma_2, \psi_2 \rangle$ if $\Gamma_1 \vdash \neg \psi_2$.

[D-Reb] $\langle \Gamma_1, \psi_1 \rangle$ is an *defeat rebuttal* of $\langle \Gamma_2, \psi_2 \rangle$ if $\psi_1 \vdash \neg \psi_2$.

[I-Reb] $\langle \Gamma_1, \psi_1 \rangle$ is an *indirect rebuttal* of $\langle \Gamma_2, \psi_2 \rangle$ if there is $\varphi \in \mathcal{L}$ such that $\psi_1 \vdash \varphi$ and $\psi_2 \vdash \neg \varphi$.

A logical argumentation framework is defined as follows.

[1] In the existing literature, additional attack relations are defined, but in the present paper, we only consider some of them. To avoid confusion, we keep the same notation of the above-cited papers. Thus, even though in the sequel of the paper there is no reference to the attack relation of *compact rebuttal 2*, we have decided not to change the name of this attack relation to simply *compact rebuttal*.

Definition 4 (Logical Argumentation Framework (LAF)) Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic, Δ a database over $\mathcal{L}$-formulas and $\mathscr{A}$ a set of attack relations. A *logical argumentation framework* over Δ is a pair $A = \langle Arg(\Delta), Attack(\mathscr{A}) \rangle$ where $Arg(\Delta)$ is the set of all arguments generated by Δ and $Attack(\mathscr{A}) \subseteq Arg(\Delta) \times Arg(\Delta)$ is an attack relation such that $(A_1, A_2) \in Attack(\mathscr{A})$ iff there is some $\mathscr{R} \in \mathscr{A}$ such that A_1 $\mathscr{R}$-attacks A_2.

In logical argumentation theory, the support relation can be instantiated in several ways. The many definitions reflect the different kinds of support among arguments. Here below, we only mansion the definition of *direct support* that can be seen as the positive counterpart of *defeat rubuttal*.

Definition 5 (Direct Support - [D-Sup]) Let $\langle \Gamma_1;\ \psi_1 \rangle$ and $\langle \Gamma_2;\ \psi_2 \rangle$ two logical arguments in $Arg(\Delta)$. We say that $\langle \Gamma_1, \psi_1 \rangle$ *directly supports* $\langle \Gamma_2, \psi_2 \rangle$ ($\langle \Gamma_1, \psi_1 \rangle \xrightarrow{[D\text{-}Sup]} \langle \Gamma_2, \psi_2 \rangle$) if $\psi_1 \vdash \psi_2$.

In [19, 20], the authors have been working on an intermediary level of abstraction between Dung's and logical argumentation frameworks. Semi-abstract argumentation frames (SAFs) are logical argumentation frames where the arguments are still understood as complex entities, but the claims are the only part instantiated. In SAFs, *attack principles* (see [19]) have been defined and they can be intuitively seen as rules that refine the existence of attack relations whenever the arguments involved share some atomic propositions.

Definition 6 (Semi-Abstract Argumentation Frame (SAF)) A *semi-abstract argumentation frame (SAF)* is a pair $S = \langle \mathsf{C}(\mathcal{AF}), \mathscr{R}_{\rightarrow} \rangle$, where $\mathsf{C}(\mathcal{AF})$ is a set formulas representing claims of arguments of a logical argumentation frame $A = \langle Arg(\Delta), Attack(\mathscr{A}) \rangle$, i.e., $\mathsf{C}(\mathcal{AF}) = \{\varphi \mid \langle \Gamma, \varphi \rangle \in Arg(\Delta)\}$, and $\mathscr{R}_{\rightarrow}$ is an attack relation defined over $\mathsf{C}(\mathcal{AF}) \times \mathsf{C}(\mathcal{AF})$ such that $(\varphi_1, \varphi_2) \in \mathscr{R}_{\rightarrow}$ iff there is some $\langle \Gamma_1, \varphi_1 \rangle$, $\langle \Gamma_2, \varphi_2 \rangle$ in $Arg(\Delta)$ and $\mathscr{R} \in \mathscr{A}$ such that $\langle \Gamma_1, \varphi_1 \rangle$ $\mathscr{R}$-attacks $\langle \Gamma_2, \varphi_2 \rangle$.

In this *semi-abstract* setting, $X \longrightarrow A$ can be interpreted as "there exists an argument with claim X that attacks an argument with claim A."

(A.$\wedge$) If $X \longrightarrow A$ or $X \longrightarrow B$ then $X \longrightarrow A \wedge B$.
(A.$\vee$) If $X \longrightarrow A \vee B$ then $X \longrightarrow A$ and $X \longrightarrow B$.
(A.$\supset$) If $X \longrightarrow B$ and $X \not\longrightarrow A$ then $X \longrightarrow A \supset B$.
(A.$\neg$) If $X \longrightarrow A$ then $X \not\longrightarrow \neg A$.

(C.$\wedge$) If $X \longrightarrow A \wedge B$ then $X \longrightarrow A$ or $X \longrightarrow B$.
(C.$\vee$) If $X \longrightarrow A$ and $X \longrightarrow B$ then $X \longrightarrow A \vee B$.
(C.$\supset$) If $X \longrightarrow A \supset B$ then $X \longrightarrow B$ and $X \not\longrightarrow A$.
(C.$\neg$) If $X \not\longrightarrow A$ then $X \longrightarrow \neg A$.

In a fully instantiated logical argumentation framework, e.g., the attack principle (**A**.$\wedge$) can be read as follows. If in a given frame there exists an argument with claim X such that it attacks (using a specific attack relation) an argument with claim A, then there are arguments with claim $A \wedge B$ that are also attacked by the argument with claim X. Thus, the attack principles can be defined also in fully instantiated frames.

3 T-norm Based Fuzzy Logics

Łukasiewicz (**Ł**), Gödel (**G**) and product logic (**Π**) are the three fundamental t-norm based fuzzy logics [28].

Definition 7 (t-norm) A *t-norm* (or *triangular-norm*) is a binary function $*$ on $[0,1]$ such that

- $*$ is commutative and associative.
- $*$ is non-decreasing in both arguments.
- $1 * x = x$ and $0 * x = 0$ for all $x \in [0,1]$.

A t-norm $*$ is a *continuous t-norm* if it is continuous on $[0,1]^2$. For an extensive study of t-norms and continuous t-norms see [30].

The Łukasiewicz t-norm is $x *_{\mathbf{Ł}} y = \max\{0, x+y-1\}$, the Gödel t-norm is $x *_{\mathbf{G}} y = \min\{x,y\}$ and the product t-norm is $x *_{\mathbf{\Pi}} y = x \cdot y$. If we consider continuous t-norms as truth functions for conjunction, then we can define the corresponding truth function for implication in a unique way. The truth function for implication is called *residuum*.

Let $*$ be a continuous t-norm. Then, for any $x, y, z \in [0,1]$, the operation $x \supset_* y = \max\{\, z \mid x * z \leq y \,\}$ is the unique operation satisfying the condition $(x * z) \leq y$ iff $z \leq (x \supset_* y)$. Here below, we recall the residuum of $*_{\mathbf{Ł}}$, $*_{\mathbf{G}}$ and $*_{\mathbf{\Pi}}$.

$$x \supset_{*_{\mathbf{Ł}}} y = \min\{1, 1-x+y\}$$

$$x \supset_{*_{\mathbf{G}}} y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$$

$$x \supset_{*_{\mathbf{\Pi}}} y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise.} \end{cases}$$

Since for any continuous t-norm $*$ the function min and max can be defined in terms of $*$ and $\supset_*$[2], and they extend the bivalent truth-tables for classical conjunction and disjunction on $\{0,1\}$, respectively, we can consider them the truth-functions of *weak conjunction* ($\wedge$) and *weak disjunction* ($\vee$) while the one interpreted by the t-norm is referred as *strong conjunction* (&). In the case of Gödel logic the two conjunctions coincide, but they differ for all other t-norms. Negation can be defined as $\neg_* x := x \supset_* 0$, the fuzzy interpretation of *reductio ad absurdum*.

The (residual) negation of the three main continuous t-norms is defined as follows:

$$\neg_{*_{\mathbf{G}}} x = \neg_{*_{\mathbf{\Pi}}} x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\neg_{*_{\mathbf{Ł}}} x = 1 - x.$$

The functions $*$, $\supset_*$, min and max equip the interval $[0,1]$ with an algebraic structure that can be used for a standard definition of algebraic semantics for fuzzy logic. These algebras are called *t-algebras* (see [18]) and we denote them by $[0,1]^*$. The *t-algebra* of $*$ is the algebra

$$[0,1]^* = \langle [0,1], *, \supset_*, \min, \max, 0, 1 \rangle.$$

For any set of K of continuous t-norms we denote the corresponding set of t-algebras by $\mathbb{K}$, and vice versa. Now we can define the syntax and standard semantics of logics based on continuous t-norms as follows.

Definition 8 (Syntax and Standard Semantics of Logics of Continuous T-norms [18]) The language $\mathcal{L}$ of the propositional fuzzy logic L_K of continuous t-norms consists of the propositional variables p, q, r, $\ldots$, the binary propositional connectives & (strong conjunctions), $\supset$ (implication), $\wedge$ (weak conjunction), $\vee$ (weak disjunction), the unary propositional connective $\neg$ (negation) and the propositional constants $\bar{1}$ (truth) and $\bar{0}$ (falsity).

The set of propositional variables of $\mathcal{L}$ will be denoted by Var and the set of all formulas of $\mathcal{L}$ by $Fm_{\mathcal{L}}$.

[2] $\min\{x,y\} = x * (x \supset_* y)$ and $\max\{x,y\} = \min\{(x \supset_* y) \supset_* y, (y \supset_* x) \supset_* x\}$

Capital letters denote formulas and upper case greek letters denote sets of formulas.

A $[0,1]$-evaluation of propositional variables is a mapping $e_* : Var \to [0,1]$. The evaluation of propositional variables extends uniquely the $*$-evaluation to all formulas by the following recursive definition. For any proposition variable p and any formula A, B:

$$e_*(p) = e(p) \qquad e_*(\bar{0}) = 0$$
$$e_*(\bar{1}) = 1 \qquad e_*(\neg A) = \neg_*(e_*(A))$$
$$e_*(A\&B) = e_*(A) * e_*(B)$$
$$e_*(A \supset B) = e_*(A) \supset_* e_*(B)$$
$$e_*(A \wedge B) = \min\{e_*(A), e_*(B)\}$$
$$e_*(A \vee B) = \max\{e_*(A), e_*(B)\}$$

The *standard consequence relation* of L_K is defined as follows.

Definition 9 (Standard Consequence Relation of L_K [18]) A $*$-evaluation e_* is a $*$-model of a set Γ of formulas if $e(A) = 1$ for all $A \in \Gamma$. A formula A is a standard consequence of Γ in L_K ($\Gamma \models_K A$) if for each $* \in K$, all $*$-models e_* of Γ are $*$-models of A.

A formula A is L_K-valid, $\models_K A$, iff $e_*(A) = 1$ for all $*$-evaluations e_*.

In general it is not possible to axiomatize the consequence relation $\models_K$ by rules with finitely many premises, but its finitary version[3] $\models_K^{fin}$ is finitely axiomatizable for any set $\mathbb{K}$ of t-algebras, with modus ponens as the only deduction rule.

Definition 10 The logic of a set K of continuous t-norms (or equivalently the logic of $\mathbb{K}$) is identified with the finitary consequence relation $\models_K^{fin}$ and denoted by L_K.

The logics of $*_{\mathbf{Ł}}$, $*_{\mathbf{G}}$ and $*_{\mathbf{\Pi}}$ are, respectively, *Lukasiewicz* (**Ł**), *product* (**Π**) and *Gödel logic* (**G**). The logic of all continuous t-norms is the *basic logic* (**BL**).

In Definition 9, "1" is the only designated truth-value and the consequence relation is defined semantically by the truth-preserving paradigm. It is possible to define an alternative consequence relation, the *order-based consequence relation*, that preserves not only truth in its maximal degree, but all available degrees. Refer to [23] for a detailed introduction to the two consequence relations.

Definition 11 (Order-Based Consequence Relation) A formula A is a *order-based consequence* of Γ in L_K ($\Gamma \models_K^{\leq} A$) if for all evaluation e_* $\inf_{\gamma\in\Gamma} e_*(\gamma) \leq e_*(A)$.

For all three logics **Ł**, **G** and **Π** we can define a *standard consequence relation* and an *order-based consequence relation*. If we only consider validity, then the two consequence relations coincide, i.e. $\models_K A$ iff $\models_K^{\leq} A$. Note that, *modus ponens* is not sound with the order-based consequence relation.

Axiomatic systems for logics of continuous t-norms can be defined. A Hilbert-style axiomatisation sound and complete for Łukasiewicz Logic is the following.

$$
\begin{aligned}
[Tr] &: (F \supset G) \supset ((G \supset H) \supset (F \supset H))\\
[We] &: F \supset (G \supset F)\\
[Ex] &: (F \supset (G \supset H)) \supset (G \supset (F \supset H))\\
[\wedge\text{-}1] &: (F \wedge G) \supset F\\
[\wedge\text{-}2] &: (F \wedge G) \supset G\\
[\wedge\text{-}3] &: (H \supset F) \supset ((H \supset G) \supset (H \supset (F \wedge G)))\\
[\vee\text{-}1] &: F \supset (F \vee G)\\
[\vee\text{-}2] &: G \supset (F \vee G)\\
[\vee\text{-}3] &: (G \supset F) \supset ((H \supset F) \supset ((G \vee H) \supset F))\\
[Lin] &: (F \supset G) \vee (G \supset F)\\
[\bot] &: \bot \supset F\\
[Waj] &: ((F \supset G) \supset G) \supset ((G \supset F) \supset F)
\end{aligned}
$$

together with the deduction rule of *modus ponens*.

$[MP]$: from F and $F \supset G$, infer G.

Theorem 1 *The above Hilbert-style system is sound and complete for Łukasiewicz logic. In other words: a formula F is derivable in the system iff F is* **Ł***-valid.*

As axiomatic system for Gödel logic we consider $[Tr]$, $[We]$, $[Ex]$, $[\wedge$-1$]$, $[\wedge$-2$]$, $[\wedge$-3$]$, $[\vee$-1$]$, $[\vee$-2$]$, $[\vee$-3$]$, $[Lin]$, $[\bot]$ plus $[Con]$ and *modus ponens* as inference rule.

$$[Con] : (F \supset (F \supset G)) \supset (F \supset G)$$

[3] $\Gamma \models_K^{fin} A$ iff there is a finite set $\Gamma' \subseteq \Gamma$ such that $\Gamma \models_K A$.

As axiomatic system for product logic we consider

$$\begin{aligned}
[BL1] &: (F \supset B) \supset ((G \supset H) \supset (F \supset H)) \\
[BL4] &: F\&(F \supset G) \supset G\&(G \supset F) \\
[BL5a] &: (F\&G \supset H) \supset (F \supset (G \supset H)) \\
[BL5b] &: (F \supset (G \supset H)) \supset (F\&G \supset H) \\
[BL6] &: ((F \supset G) \supset H) \supset (((G \supset F) \supset H) \supset H) \\
[BL7] &: \bot \supset F \\
[P] &: \neg F \vee ((F \supset F\&G) \supset G)
\end{aligned}$$

together with *modus ponens*.

4 Bipolar T-norm Based Argumentative Frameworks

The consequence relations recalled in Definition 9 and Definition 11 can be considered as the method of inference between a support set and a claim. Thus, we can recover two definitions of *t-norm based arguments*. In these definitions, as in [8–10], we do not make any assumption on the support set except that it is a subset of formulas in $Fm_{\mathcal{L}}$.

Definition 12 (Argument based on $\models_K$) An argument based on $\models_K$ is a pair $\langle \Gamma_X, X \rangle$ such that $\Gamma_X \cup \{X\} \subseteq Fm_{\mathcal{L}}$ and $\Gamma_X \models_K X$.

Definition 13 (Argument based on $\models^{\leq}_K$) An argument based on $\models^{\leq}_K$ is a pair $\langle \Gamma_X, X \rangle$ such that $\Gamma_X \cup \{X\} \subseteq Fm_{\mathcal{L}}$ and $\Gamma_X \models^{\leq}_K X$.

To better understand the argumentative characteristics of these two ways of defining a t-norm based argument, we consider several attack relations and explore which attack principles are justified, and which are not. Since the attack principles are defined in terms of the claims of the arguments, for this analysis, we consider attack relations defined, at least partially, in terms of the claims of the arguments as well. The attack relations we consider are the following: *defeat*, *compact rebuttal 1*, *defeating rebuttal* and *indirect rebuttal*.

Following Definition 12, we have that the argument $\langle \Gamma_A, A \rangle$ attacks the argument $\langle \Gamma_B, B \rangle$ using the *defeat* attack relation iff $A \models_K \neg \bigwedge \Gamma_B$, i.e. for every evaluation e such that $e(A) = 1$, then $e(\neg \bigwedge \Gamma_B) = 1$. By Definition 13, the argument $\langle \Gamma_A, A \rangle$ attacks the argument $\langle \Gamma_B, B \rangle$ using the *defeat* attack relation whenever $A \models^{\leq}_K \neg \bigwedge \Gamma_B$, i.e. for every evaluation e $\inf_{\gamma \in \Gamma} e(A) \leq e(\neg \bigwedge \Gamma_B)$.

The attack principles are implications between (disjunction or conjunction) of assertions of the form $X \longrightarrow A$ or $X \not\longrightarrow A$. Thus, in each attack principle we can distinguish the *premise* and the *conclusion*. For example, the premise of (**A**.$\wedge$) is "$X \longrightarrow A$ or $X \longrightarrow B$", while its conclusion is "$X \longrightarrow A \wedge B$". We say that an attack principle is *justified* if the attacking condition of the conclusion of the principle *logically follows* (in L_K) from the attacking condition of the premise. We now analyse the attack principles considering *defeating rebuttal* as attacking relation and both definitions of **Ł**-based arguments. The other cases can be found in Appendix A.

Analysis of the attack principles considering [D-Reb] and $\models_{\text{Ł}}$.

We assume to work in fully instantiated argumentation frames such that the arguments mentioned in the attack principles belong to the set of arguments.

(**A**.$\wedge$) If $\langle \Gamma_X, X \rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_A, A \rangle$, then $X \models_{\text{Ł}} \neg A$, i.e. for any evaluation e s.t. $e(X) = 1$, $e(\neg A) = 1 - e(A) = 1$. Since for any evaluation e $e(\neg(A \wedge B)) = 1 - e(A \wedge B) = 1 - \min\{e(A), e(B)\} \geq 1 - e(A)$, if $1 - e(A) = 1$, then also $1 - e(A \wedge B) = 1$, i.e. $X \models_{\text{Ł}} \neg(A \wedge B)$ and the principle is justified.

(**C**.$\wedge$) If $\langle \Gamma_X, X \rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_{A \wedge B}, A \wedge B \rangle$, then for any evaluation e such that $e(X) = 1$, $e(\neg(A \wedge B)) = 1$ i.e. $1 - e(A \wedge B) = 1 - \min\{e(A), e(B)\}$=1. Therefore, either $e(A) = 0$ or $e(B) = 0$, i.e. either $\langle \Gamma_X, X \rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_A, A \rangle$, or $\langle \Gamma_X, X \rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_B, B \rangle$ and the principle is justified.

(**A**.$\vee$) If $\langle \Gamma_X, X \rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_{A \vee B}, A \vee B \rangle$, then $X \models_{\text{Ł}} \neg(A \vee B)$, i.e. for any evaluation e s.t. $e(X) = 1$, $e(\neg(A \vee B)) = 1 - e(A \vee B) = 1 - \max\{e(A), e(B)\} = 1$. Therefore, $\max\{e(A), e(B)\} = 0$, which implies both $e(A) = 0$ and $e(B) = 0$, i.e. $\langle \Gamma_X, X \rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_A, A \rangle$ and $\langle \Gamma_X, X \rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_B, B \rangle$. The principle is justified.

(**C**.$\vee$) If $\langle \Gamma_X, X \rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_A, A \rangle$ and $\langle \Gamma_X, X \rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_B, B \rangle$, then for any evaluation e s.t $e(X) = 1$, we have that $e(\neg A) = 1$ and $e(\neg B) = 1$, i.e. $e(A) = 0$ and $e(B) = 0$. Therefore, $e(\neg(A \vee B)) = 1 - \max\{e(A), e(B)\} = 1$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_{A\vee B}, A \vee B\rangle$ and the principle is justified.

(**A**.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \not\xrightarrow{[\text{D-Reb}]} \langle \Gamma_A, A\rangle$, then $X \models_{\text{Ł}} \neg B$ and $X \not\models_{\text{Ł}} \neg A$, i.e. for any evaluation e s.t. $e(X) = 1$, then $e(\neg B) = 1 - e(B) = 1$, which implies $e(B) = 0$. Moreover, there exists at least one evaluation e' s.t. $e'(X) = 1$ and $e'(A) > 0$. To have the principle hold we would need that for any evaluation e s.t. $e(X) = 1$, then $e(\neg(A \supset B)) = 1$, i.e. $1 - e(A \supset B) = 1$ from which it would follow that $e(A \supset B) = 0$. Since $e(A \supset B) = \min\{1, 1 - e(A) + e(B)\}$, $e(A \supset B) = 0$ only if $1 - e(A) + e(B) = 0$, i.e. $e(A) = 1$ and $e(B) = 0$, but this does not follow from the premises and the principle is not justified.

(**C**.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_{A\supset B}, A \supset B\rangle$, then $X \models_{\text{Ł}} \neg(A \supset B)$, i.e. whenever there is an evaluation e s.t. $e(X) = 1$, then $e(\neg(A \supset B)) = 1 - e(A \supset B) = 1 - \min\{1, 1 - e(A) + e(B)\} = 1$. Thus, $1 - e(A) + e(B) = 0$, i.e. $e(A) = 1$ and $e(B) = 0$. From $e(B) = 0$ and $e(A) = 1$ it follows that $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \not\xrightarrow{[\text{D-Reb}]} \langle \Gamma_A, A\rangle$. Thus, the attack principle is justified. From the premise it follows also that $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_{\neg A}, \neg A\rangle$.

(**A**.$\neg$) If $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_A, A\rangle$, then $X \models_{\text{Ł}} \neg A$, i.e. for any evaluation e s.t. $e(X) = 1$, $e(\neg A) = 1 - e(A) = 1$ i.e. $e(A) = 0$. We want to show that there exists an evaluation e' s.t. $e'(X) = 1$ and $e'(\neg(\neg A)) < 1$, i.e. $e'(A) < 1$. From the premise we have that $e(A) = 0$. Therefore, $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_{\neg A}, \neg A\rangle$ and the principle is justified.

(**C**.$\neg$) If $\langle \Gamma_X, X\rangle \not\xrightarrow{[\text{D-Reb}]} \langle \Gamma_{\neg A}, \neg A\rangle$, then for some evaluation e' s.t. $e'(X) = 1$, $e'(\neg\neg A) = e'(A) < 1$. However, from this hypothesis it does not follow that for any evaluation e s.t. $e(X) = 1$. $e(\neg A) = 1 - e(A) = 1$, i.e. $e(A) = 0$. Thus, the principle is not justified.

Analysis of the attack principles considering [D-Reb] and $\models^{\leq}_{\text{Ł}}$.

(**A**.$\wedge$) If $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_B, A\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg A$, i.e. for every evaluation e, $e(X) \leq e(\neg A) = 1 - e(A)$. Since for every evaluation e $e(A \wedge B) = \min\{e(A), e(B)\} \leq e(A)$, $1 - e(A) \leq 1 - e(A \wedge B)$. Therefore, $e(X) \leq 1 - e(A \wedge B)$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_{A\wedge B}, A \wedge B\rangle$ and the principle is justified.

(**C**.$\wedge$) If $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_{A\wedge B}, A \wedge B\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg(A \wedge B)$ i.e. for every evaluation e $e(X) \leq e(\neg(A \wedge B)) = 1 - \min\{e(A), e(B)\}$. It could be that there exists an evaluation e_i s.t. $e_i(A \wedge B) = e_i(A)$ and other evaluation e_j for which $e_j(A \wedge B) = e_j(B)$. Therefore, we cannot deduce that for every evaluation e either $e(X) \leq e(\neg A)$ or $e(X) \leq e(\neg B)$. The principle is not justified.

(**A**.$\vee$) If $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_{A\vee B}, A \vee B\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg(A \vee B)$ i.e. for every evaluation e, $e(X) \leq e(\neg(A \vee B)) = 1 - e(A \vee B) = 1 - \max\{e(A), e(B)\}$. Since for every evaluation e, $e(A \vee B) \geq e(A)$ and $e(A \vee B) \geq e(B)$, it follows that $1 - e(A \vee B) \leq 1 - e(A)$ and $1 - e(A \vee B) \leq 1 - e(B)$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_B, B\rangle$. The principle is justified.

(**C**.$\vee$) If $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_B, B\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg A$ and $X \models^{\leq}_{\text{Ł}} \neg B$, i.e. for every evaluation e, $e(X) \leq 1 - e(A)$ and $e(X) \leq 1 - e(B)$. Since $e(A \wedge B) = \max\{e(A), e(B)\} = e(A)$ or $e(A \wedge B) = \max\{e(A), e(B)\} = e(B)$, $e(X) \leq 1 - \max\{e(A), e(B)\}$ i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_{A\vee B}, A \vee B\rangle$ and the principle is justified.

(**A**.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[\text{D-Reb}]} \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \not\xrightarrow{[\text{D-Reb}]} \langle \Gamma_A, A\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg B$ and $X \not\models^{\leq}_{\text{Ł}} \neg A$, i.e. for every evaluation e, $e(X) \leq 1 - e(B)$ and for some evaluation e^*, $e^*(X) > 1 - e^*(A)$. We would need to show that for every evaluation e, $e(X) \leq 1 - \min\{1, 1 - e(A) + e(B)\}$, but this does not follow from the premises and the principle is not justified.

(**C**.⊃) If $\langle \Gamma_X, X\rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_{A\supset B}, A\supset B\rangle$, then for every evaluation e, $X \models^{\leq}_{\mathbf{Ł}} \neg(A \supset B)$, i.e. $e(X) \leq 1 - e(A \supset B) = 1 - \min\{1, 1 - e(A) + e(B)\}$. Since for every evaluation e, $\min\{1, 1-e(A)+e(B)\} \geq e(B)$, it follows that $1 - \min\{1, 1 - e(A) + e(B)\} \leq 1 - e(B)$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_B, B\rangle$. However, the other part of the claim, $\langle \Gamma_X, X\rangle \not\xrightarrow{\text{[D-Reb]}} \langle \Gamma_A, A\rangle$, does not follow from the premise and the principle is not justified.

(**A**.¬) If $\langle \Gamma_X, X\rangle \xrightarrow{\text{[D-Reb]}} \langle \Gamma_A, A\rangle$, then $X \models^{\leq}_{\mathbf{Ł}} \neg A$, i.e. for every evaluation e, $e(X) \leq 1 - e(A)$. To have the principle justified, we need to find some evaluation e' s.t. $e'(X) > e'(A)$, but this does not follow from the premise and the principle is not justified.

(**C**.¬) If $\langle \Gamma_X, X\rangle \not\xrightarrow{\text{[D-Reb]}} \langle \Gamma_{\neg A}, \neg A\rangle$, then $X \not\models^{\leq}_{\mathbf{Ł}} \neg\neg A$, i.e. there exists an evaluation e' s.t. $e'(X) > e'(\neg\neg A) = e'(A)$. However, from this premise it does not follow that for every evaluation e, $e(X) \leq e(\neg A) = 1 - e(A)$. Thus, the principle is not justified.

Whenever the attack relation is instantiated with *direct rebuttal* and the arguments are defined using Definition 12, the attack principles justified are $\mathsf{AP}^{\text{[D-Reb]}}_{\models_{\mathbf{Ł}}} = \{$(**A**.∧), (**C**.∧), (**A**.∨), (**C**.∨), (**C**. ⊃), (**A**.¬)$\}$. If Definition 13 is the one used for the definition of t-norm based arguments, then the attack principles justified are $\mathsf{AP}^{\text{[D-Reb]}}_{\models^{\leq}_{\mathbf{Ł}}} = \{$(**A**.∧), (**A**.∨), (**C**.∨)$\}$.

In Table 1, 2 and 3, we summarise the attack principles justified by the *defeat*, *compact rebuttal 1* and *indirect rebuttal* attack relation considering the two definitions of t-norm based argument. For the case of *defeat*, we have indicated which additional conditions the arguments need to satisfy to have the attack principle hold. Interestingly, the attack principles justified by most of the attack relations under Definition 12, are the same satisfied by the modal interpretation of the attack relation[4] introduced in [19]. This is not the case whenever the order-based definition is considered. In this case, none of the attack relations considered satisfy either (**C**.⊃) or (**A**.¬). The attack principle (**C**.¬) is problematic with both definitions. However, this is not surprising since it is a very demanding principle. Again, we have a confirmation that some of the attack principles are more acceptable, i.e. easier to justify, than others. In particular, those attack principles satisfied by the modal interpretation of the attack relation are justified in different scenarios: in sequent-based argumentation frames and now also in t-norm based ones.

As shown in [20], semi-abstract argumentation frames are not adequate to characterise fuzzy logics. In [20], a complete argumentative semantics for **Ł**,

[Def]	$\models_{\mathbf{Ł}}$	$\models^{\leq}_{\mathbf{Ł}}$
(**A**.∧)	$\Gamma_A \subseteq \Gamma_{A\wedge B}$	$\Gamma_A \subseteq \Gamma_{A\wedge B}$
(**A**.∨)	$\Gamma_{A\vee B} \subseteq \Gamma_A$ and $\Gamma_{A\vee B} \subseteq \Gamma_B$	$\Gamma_{A\vee B} \subseteq \Gamma_A$ and $\Gamma_{A\vee B} \subseteq \Gamma_B$
(**A**.⊃)	$\Gamma_B \subseteq \Gamma_{A\supset B}$	$\Gamma_B \subseteq \Gamma_{A\supset B}$
(**A**.¬)	✗	✗
(**C**.∧)	$\Gamma_{A\wedge B} \subseteq \Gamma_A$ or $\Gamma_{A\wedge B} \subseteq \Gamma_B$	$\Gamma_{A\wedge B} \subseteq \Gamma_A$ or $\Gamma_{A\wedge B} \subseteq \Gamma_B$
(**C**.∨)	$\Gamma_A \subseteq \Gamma_{A\vee B}$ or $\Gamma_B \subseteq \Gamma_{A\vee B}$	$\Gamma_A \subseteq \Gamma_{A\vee B}$ or $\Gamma_B \subseteq \Gamma_{A\vee B}$
(**C**.⊃)	$\Gamma_{A\supset B} \subseteq \Gamma_B$, $\Gamma_A \subset \Gamma_{A\supset B}$ and $\gamma^*_i \notin \Gamma_A$	✗
(**C**.¬)	✗	✗

Table 1: Attack principles justified by the *defeat* attack relation

[C-Reb-1]	$\models_{\mathbf{Ł}}$	$\models^{\leq}_{\mathbf{Ł}}$
(**A**.∧)	✓	✓
(**A**.∨)	✓	✓
(**A**.⊃)	✗	✗
(**A**.¬)	✓	✗
(**C**.∧)	✗	✗
(**C**.∨)	✓	✗
(**C**.⊃)	✓	✗
(**C**.¬)	✗	✗

Table 2: Attack principles justified by the *compact rebuttal 1* attack relation

[I-Reb]	$\models_{\mathbf{Ł}}$	$\models^{\leq}_{\mathbf{Ł}}$
(**A**.∧)	✓	✓
(**A**.∨)	✓	✓
(**A**.⊃)	✗	✗
(**A**.¬)	✗	✗
(**C**.∧)	✗	✗
(**C**.∨)	✓	✓
(**C**.⊃)	✗	✗
(**C**.¬)	✗	✗

Table 3: Attack principles justified by the *indirect rebuttal* attack relation

[4]The attack principles justified by this modal interpretation of the attack relations are: $\mathsf{MAP} = \{$ (**A**.∧), (**A**.∨), (**C**.∨), (**C**.⊃), (**A**.¬) $\}$.

G and **Π** is recovered considering *weighted* semi-abstract argumentation frames, i.e. argumentative frameworks where the attack relation is graded. An alternative way to recover complete semantics for the above mentioned logics is to consider *bipolar* t-norm based argumentative frameworks. From now on, as t-norm based argument, we only consider those defined as per Definition 12.

Definition 14 (Bi-**Ł**AF) A *Bipolar Ł-based Argumentation Frame* is a triplet $AF = \langle Ar, Attack(\mathscr{A}), Support(\mathscr{S})\rangle$ such that:

- Ar is a set of **Ł**-based arguments, i.e. $\langle\Gamma,\psi\rangle \in Ar$ only if $\Gamma \models_{\mathbf{Ł}} \psi$.
- $\mathscr{A}$ is a set of attack rules and $\mathscr{S}$ a set of support rules.
- $(\langle\Gamma_A, A\rangle, \langle\Gamma_B, B\rangle) \in Attack(\mathscr{A})$ iff there is some $\mathscr{R} \in \mathscr{A}$ such that $\langle\Gamma_A, A\rangle$ $\mathscr{R}$-attacks $\langle\Gamma_B, B\rangle$.
- $(\langle\Gamma_A, A\rangle, \langle\Gamma_B, B\rangle) \in Support(\mathscr{S})$ iff there is some $\mathscr{R} \in \mathscr{S}$ such that $\langle\Gamma_A, A\rangle$ $\mathscr{S}$-supports $\langle\Gamma_B, B\rangle$.

The definition of Bipolar **G**-based and Bipolar **Π**-based Argumentation Frameworks is similar. In bipolar argumentation frames, we can define (bipolar) *principles* that are similar to the attack principles, but in these, both relations are used. For example, for implication we can introduce the following principle.

($\mathbf{C}_{Bi}.\supset$) If $\langle\Gamma_X, X\rangle \longrightarrow \langle\Gamma_{A\supset B}, A \supset B\rangle$, then $\langle\Gamma_X, X\rangle \longrightarrow \langle\Gamma_B, B\rangle$ and $\langle\Gamma_X, X\rangle \Longrightarrow \langle\Gamma_A, A\rangle$.

In words: If an argument with claim X attacks an argument with claim $A \supset B$, then the argument with claim X also attacks an argument with claim B and supports an argument with claim A.

If we consider a Bi-**Ł**AF argumentation frame such that $\mathscr{A} = \{$[D-Reb]$\}$ and $\mathscr{S} = \{[D\text{-}Sup]\}$, the principle ($\mathbf{C}_{Bi}.\supset$) is justified. In fact, if $X \models_{\mathbf{Ł}} \neg(A \supset B)$, then whenever $e(X) = 1$, we have that $e(\neg(A \supset B)) = 1 - e(A \supset B) = 1$. Thus, $e(A \supset B) = 0$ holds. Since $e(A \supset B) = \min\{1, 1 - e(A) + e(B)\}$, if $e(A \supset B) = 0$, then $1 - e(A) + e(B) = 0$, i.e. $e(A) = 1$ and $e(B) = 0$. Therefore for any evaluation e such that $e(X) = 1$, $e(A) = 1$ and $e(B) = 0$, i.e. $X \models_{\mathbf{Ł}} A$ and $X \models_{\mathbf{Ł}} \neg B$.

In the same setting, also the following principles hold.

($\mathbf{A}_S.\vee$) If $\langle\Gamma_X, X\rangle \Longrightarrow \langle\Gamma_{A\vee B}, A \vee B\rangle$, then $\langle\Gamma_X, X\rangle \Longrightarrow \langle\Gamma_A, A\rangle$ or $\langle\Gamma_X, X\rangle \Longrightarrow \langle\Gamma_B, B\rangle$.

If $X \models_{\mathbf{Ł}} A \vee B$, then whenever $e(X) = 1$, we have that $e(A \vee B) = \min\{e(A), e(B)\} = 1$. Thus, either $e(A) = 1$ or $e(B) = 1$, i.e. either $X \models_{\mathbf{Ł}} A$ or $X \models_{\mathbf{Ł}} B$.

($\mathbf{A}_S.\supset$) If $\langle\Gamma_X, X\rangle \Longrightarrow \langle\Gamma_A, A\rangle$ and $\langle\Gamma_X, X\rangle \Longrightarrow \langle\Gamma_{A\supset B}, A \supset B\rangle$, then $\langle\Gamma_X, X\rangle \Longrightarrow \langle\Gamma_B, B\rangle$.

If $X \models_{\mathbf{Ł}} A$ and $X \models_{\mathbf{Ł}} A \supset B$, then whenever $e(X) = 1$ we have that $e(A) = 1$ and $e(A \supset B) = 1$. Since $e(A \supset B) = \min\{1, 1 - e(A) + e(B)\}$ and $e(A) = 1$, $e(A \supset B) = \min\{1, e(B)\} = e(B)$. Thus, $e(B) = 1$ and $X \models_{\mathbf{Ł}} B$.

($\mathbf{A}_S.\neg$) If $\langle\Gamma_X, X\rangle \Longrightarrow \langle\Gamma_A, A\rangle$, then $\langle\Gamma_X, X\rangle \not\Longrightarrow \langle\Gamma_{\neg A}, \neg A\rangle$.

If $X \models_{\mathbf{Ł}} A$, then whenever $e(X) = 1$ we have that $e(A) = 1$. Thus, $e(\neg A) = 0$ and $X \not\models_{\mathbf{Ł}} \neg A$.

($\mathbf{C}_{Bi}.\neg$) If $\langle\Gamma_X, X\rangle \longrightarrow \langle\Gamma_A, A\rangle$, then $\langle\Gamma_X, X\rangle \Longrightarrow \langle\Gamma_{\neg A}, \neg A\rangle$.

If $X \models_{\mathbf{Ł}} \neg A$, then whenever $e(X) = 1$ we have that $e(\neg A) = 1$, i.e. $\langle\Gamma_X, X\rangle \Longrightarrow \langle\Gamma_{\neg A}, \neg A\rangle$.

($\mathbf{A}.\top$) $\langle\Gamma_X, X\rangle \Longrightarrow \langle\Gamma_\top, \top\rangle$ for every argument $\langle\Gamma_X, X\rangle$.

$X \models_{\mathbf{Ł}} \top$ for every X.

5 Characterising Ł, G and Π

To relate fuzzy logics to the realm of all possible t-norm based argumentation frames that satisfy certain principles like the one discussed in the previous section, we define a closure operation on t-norm based argumentation frames similar to the one introduced in [20].

Definition 15 (Syntactic Closure of Bi-**Ł**AFs) Given Δ a finite set of propositional formulas, a Bi-**Ł**AF AF is *syntactically closed* with respect to Δ if all formulas and subformulas of formulas in Δ occur as claims of some argument in AF.

Definition 15 extends to Bi-**G**AFs and Bi-**Π**AFs in the obvious way and we will suppress the explicit reference to Δ whenever the context makes clear what formulas are expected to be available as claims of arguments.

The following notion of *immunity*, as explained in [20], provides a new view on logical validity not based on Tarski-style semantics, but rather only refers to claims of arguments and to the attacks between them.

Definition 16 Let P be a set of principles. A formula F is P-*argumentatively immune* (shortly: P-*immune*) if in all syntactically closed Bi-**Ł**AFs (with respect to F) that satisfy the principles in P F is not attacked.

Let us denote with **Ł**P the following set of principles. Some of them are attack principles, the others are the ones just introduced.

ŁP $= \{(\mathbf{A}.\wedge), (\mathbf{C}.\wedge), (\mathbf{A}.\vee), (\mathbf{C}.\vee), (\mathbf{A}.\top), (\mathbf{C}_{Bi}.\supset), (\mathbf{A}_S.\supset), (\mathbf{A}_S.\vee), (\mathbf{C}_{Bi}.\neg)\}$.

As proved in the following proposition, **Ł**P-immune arguments are closed over *modus ponens*.

Proposition 2 (Closure of **Ł**P-immune arguments over Modus Ponens) *If A and $A \supset B$ are argumentatively* **Ł**P*-immune, then also B is argumentatively* **Ł**P*-immune.*

Proof Since both A and $A \supset B$ are **Ł**P-immune, then for any X in a syntactically closed Bi-**Ł**AF frame $X \not\models_{\mathbf{Ł}} \neg A$ and $X \not\models_{\mathbf{Ł}} \neg(A \supset B)$. Thus, there is an evaluation e_1 s.t. $e_1(X) = 1$ and $e_1(\neg A) < 1$ and an evaluation e_2 s.t. $e_2(X) = 1$ and $e_2(\neg(A \supset B)) < 1$. Our claim is that $X \not\models_{\mathbf{Ł}} \neg B$ for any X in the frame, i.e. there exists an evaluation e_3 s.t. $e_3(X) = 1$ and $e_3(\neg B) < 1$. Since A belongs to the frame, also $X \wedge A$ is in the frame and from the hypothesis it must be that $X \wedge A \not\models_{\mathbf{Ł}} \neg A$ and $X \wedge A \not\models_{\mathbf{Ł}} \neg(A \supset B)$, i.e. there is an evaluation e_4 s.t. $e_4(X \wedge A) = 1$ and $e_4(\neg(A \supset B)) < 1$. Since $e_4(X \wedge A) = \min\{e_4(X), e_4(A)\} = 1$, then $e_4(A) = 1$. Moreover, since $e_4(\neg(A \supset B)) = 1 - e_4(A \supset B) = 1 - \min\{1, 1 - e_4(A) + e_4(B)\}$ and $e_4(A) = 1$, we have that $e_4(\neg(A \supset B)) = 1 - \min\{1, e_4(B)\} = 1 - e_4(B)$. Thus, from $e_4(\neg(A \supset B)) < 1$, it follows $e_4(B) > 0$ and $e_4(\neg B) < 1$. Therefore for any argument X in the frame we can find an evaluation showing that $X \not\models_{\mathbf{Ł}} \neg B$. □

Theorem 3 (Bipolar Argumentative Soundness of **Ł**) *Every* **Ł***-valid formula is argumentatively* **Ł**P*-immune.*

Proof By Theorem 1 and Proposition 2, it remains to check that the axioms of the axiomatic system for Łukasiewicz logic recalled in Section 3 are **Ł**P-immune. In the following, we implicitly assume that all arguments occurr in a Bi-**Ł**AF that is syntactically closed with respect to the axiom in question. In each case we argue indirectly and we derive a contradiction from the assumtpion that there is an argument X that attacks the axiom in question.

- [Tr] If $X \longrightarrow (F \supset G) \supset ((G \supset H) \supset (F \supset H))$, then by $(\mathbf{C}_{Bi}.\supset)$ we have that $X \Longrightarrow F \supset (G \supset H)$ and $X \longrightarrow ((F \supset G) \supset (F \supset H))$. Again by $(\mathbf{C}_{Bi}.\supset)$ we have $X \Longrightarrow F \supset G$ and $X \longrightarrow F \supset H$ from which it follows, by $(\mathbf{C}_{Bi}.\supset)$, $X \longrightarrow H$ and $X \Longrightarrow F$. Since $X \Longrightarrow F$ and $X \Longrightarrow F \supset (G \supset H)$, by $(\mathbf{A}_S.\supset)$, we have $X \Longrightarrow G \supset H$ and from $X \Longrightarrow F \supset G$ and $X \Longrightarrow F$, by $(\mathbf{A}_S.\supset)$, it follows $X \Longrightarrow G$. Again by $(\mathbf{A}_S.\supset)$, from $X \Longrightarrow G \supset H$ and $X \Longrightarrow G$ it follows $X \Longrightarrow H$ which goes against $X \longrightarrow H$.
- [We] If $X \longrightarrow F \supset (G \supset F)$, by $(\mathbf{C}_{Bi}.\supset)$ it follows $X \longrightarrow F \supset G$ and $X \Longrightarrow F$. From $X \longrightarrow F \supset G$ it follows by $(\mathbf{C}_{Bi}.\supset)$ $X \Longrightarrow G$ and $X \longrightarrow F$ which is incompatible with $X \Longrightarrow F$.
- [Ex] If $X \longrightarrow (F \supset (G \supset H)) \supset (G \supset (F \supset H))$ by $(\mathbf{C}_{Bi}.\supset)$ it follows that $X \longrightarrow (G \supset (F \supset H))$ and $X \Longrightarrow (F \supset (G \supset H))$. Again, by $(\mathbf{C}_{Bi}.\supset)$, we have that $X \longrightarrow F \supset H$ and $X \Longrightarrow G$. Since $X \longrightarrow F \supset H$ it follows also $X \longrightarrow H$ and $X \Longrightarrow F$. From $X \Longrightarrow (F \supset (G \supset H))$ and $X \Longrightarrow F$ by $(\mathbf{A}_S.\supset)$ it follows $X \Longrightarrow G \supset H$ and by $X \Longrightarrow G \supset H$ and $X \Longrightarrow G$ it follows $X \Longrightarrow H$ which is incompatible with $X \longrightarrow H$.
- [$\wedge$-1] If $X \longrightarrow (F \wedge G) \supset F$ by $(\mathbf{C}_{Bi}.\supset)$ it follows $X \longrightarrow F$, $X \Longrightarrow F \wedge G$ and $X \not\longrightarrow F \wedge G$. Moreover by $(\mathbf{A}.\wedge)$ we have $X \not\longrightarrow F$ and $X \not\longrightarrow G$ which contradicts $X \longrightarrow F$.
- [$\wedge$-2] See the previous case.
- [$\wedge$-3] If $X \longrightarrow (H \supset F) \supset ((H \supset G) \supset (H \supset (F \wedge G)))$ by $(\mathbf{C}_{Bi}.\supset)$ it follows $X \longrightarrow (H \supset G) \supset (H \supset (F \wedge G))$ and $X \Longrightarrow H \supset F$ and again, by the same principle we have $X \longrightarrow H \supset (F \wedge G)$ and $X \Longrightarrow H \supset G$. Since $X \longrightarrow H \supset (F \wedge G)$ it follows $X \longrightarrow F \wedge G$ and $H \Longrightarrow H$. From $X \longrightarrow F \wedge G$ by $(\mathbf{C}.\wedge)$ it follows that $X \longrightarrow F$ or $X \longrightarrow G$. Since $X \Longrightarrow H \supset F$ and $X \Longrightarrow H$, by $(\mathbf{A}_S.\supset)$ we have $X \Longrightarrow F$ and by $X \Longrightarrow H$ and $X \Longrightarrow H \supset G$ it follows $X \Longrightarrow G$, but this is in contradiction with $X \longrightarrow F$ or $X \longrightarrow G$.

[∨-1] If $X \longrightarrow F \supset (F \vee G)$ by ($\mathbf{C}_{Bi}.\supset$) we have $X \longrightarrow F \vee G$ and $X \Longrightarrow F$. From $X \longrightarrow F \vee G$ by ($\mathbf{A}.\vee$) it follows $X \longrightarrow G$ and $X \longrightarrow F$ which is in contradiction with $X \Longrightarrow F$.

[∨-2] See the previous case.

[∨-3] If $X \longrightarrow (G \supset F) \supset ((H \supset F) \supset ((G \vee H) \supset F))$ by ($\mathbf{C}_{Bi}.\supset$) we have $X \longrightarrow (H \supset F) \supset ((G \vee H) \supset F)$ and $X \Longrightarrow G \supset F$. Again, by ($\mathbf{C}_{Bi}.\supset$), we have also $X \longrightarrow (G \vee H) \supset F$ and $X \Longrightarrow H \supset F$. From $X \longrightarrow (G \vee H) \supset F$ it follows also $X \longrightarrow F$ and $X \Longrightarrow G \vee H$.

Since $X \Longrightarrow G \vee H$, by ($\mathbf{A}_S.\vee$) either (a) $X \Longrightarrow G$ or (b) $X \Longrightarrow H$. In the (a)-case, since $X \Longrightarrow G$ and $X \Longrightarrow G \supset F$, by ($\mathbf{A}_S.\supset$) we have that $X \Longrightarrow F$. However from $X \longrightarrow F$ and ($\mathbf{C}_{Bi}.\neg$) it follows $X \Longrightarrow \neg F$ and this is in contradiction with $X \Longrightarrow F$.

In the (b)-case we reach the contradiction in the same way.

[*Lin*] If $X \longrightarrow (F \supset G) \vee (G \supset F)$ from ($\mathbf{A}.\vee$) it follows $X \longrightarrow F \supset G$ and $X \longrightarrow G \supset F$. From $X \longrightarrow F \supset G$, by ($\mathbf{C}_{Bi}.\supset$), it follows $X \longrightarrow G$ and $X \Longrightarrow F$ while from $X \longrightarrow G \supset F$ it follows $X \longrightarrow F$ and $X \Longrightarrow G$ and we have reached a contradiction.

[⊥] If $X \longrightarrow \bot \supset F$ by ($\mathbf{C}_{Bi}.\supset$) it follows $X \longrightarrow F$ and $X \Longrightarrow \bot$. Thus by ($\mathbf{A}_S.\neg$) and ($\mathbf{A}.\top$) we reach a contradiction.

[*Waj*] If $X \longrightarrow ((F \supset G) \supset G) \supset ((G \supset F) \supset F)$ by ($\mathbf{C}_{Bi}.\supset$) we have $X \longrightarrow (G \supset F) \supset F$ and $X \Longrightarrow (F \supset G) \supset G$. From $X \longrightarrow (G \supset F) \supset F$ again by ($\mathbf{C}_{Bi}.\supset$) we have $X \longrightarrow F$ and $X \Longrightarrow G \supset F$. Therefore (1) $X \models_{\text{Ł}} \neg F$, (2) $X \models_{\text{Ł}} G \supset F$ and (3) $X \models_{\text{Ł}} (F \supset G) \supset G$. From (1) it follows that whenever $e(X) = 1$, $e(\neg F) = 1 - e(F) = 1$, i.e. $e(F) = 0$. From (2) it follows that whenever $e(X) = 1$, $e(G \supset F) = \min\{1, 1 - e(G) + e(F)\} = 1$ and given $e(F) = 0$ this is satisfied only if $e(G) = 0$. From (3) we have that whenever $e(X) = 1$ for some evaluation e, $e((F \supset G) \supset G) = 1$, i.e. $\min\{1, 1 - e(F \supset G) + e(G)\} = 1$. Since $e(F \supset G) = \min\{1, 1 - e(F) + e(G)\}$, given $e(F) = 0$ and $e(G) = 0$, we have $e(F \supset G) = 1$. Therefore $\min\{1, 1 - e(F \supset G) + e(G)\} = 0$ while it should have been 1.

In Table 4 we summarise which principles have been used in the corresponding section of the proof.

□

Theorem 4 (Bipolar Argumentative Completeness of **Ł**) *Every argumentatively* **Ł**P*-immune formula is* **Ł***-valid .*

Axiom	Principles used in the proof
[Tr]	($\mathbf{C}_{Bi}.\supset$), ($\mathbf{A}_S.\supset$)
[We]	($\mathbf{C}_{Bi}.\supset$)
[Ex]	($\mathbf{C}_{Bi}.\supset$), ($\mathbf{A}_S.\supset$)
[∧-1]	($\mathbf{C}_{Bi}.\supset$), ($\mathbf{A}.\wedge$)
[∧-2]	($\mathbf{C}_{Bi}.\supset$), ($\mathbf{A}.\wedge$)
[∧-3]	($\mathbf{C}_{Bi}.\supset$), ($\mathbf{A}_S.\supset$), ($\mathbf{C}.\wedge$)
[∨-1]	($\mathbf{C}_{Bi}.\supset$), ($\mathbf{A}.\vee$)
[∨-2]	($\mathbf{C}_{Bi}.\supset$), ($\mathbf{A}.\vee$)
[∨-3]	($\mathbf{C}_{Bi}.\supset$), ($\mathbf{A}_S.\supset$), ($\mathbf{A}_S.\vee$), ($\mathbf{C}_{Bi}.\neg$)
[Lin]	($\mathbf{C}_{Bi}.\supset$), ($\mathbf{A}.\vee$)
[⊥]	($\mathbf{C}_{Bi}.\supset$), ($\mathbf{A}_S.\neg$), ($\mathbf{A}.\top$)
[Waj]	($\mathbf{C}_{Bi}.\supset$)

Table 4: Principles used in Theorem 5

Proof We have to show that if F is not a **Ł**-valid valid formula, then it is not **Ł**P-immune. If F is not a **Ł**-valid valid formula, then that there is an evaluation e such that $e(F) < 1$. Since $e(\neg F) = 1 - e(F)$, any formula F is attacked by its negation, i.e. $\neg F \longrightarrow F$.

□

Concerning the attack principles needed to prove a completeness theorem with **G**-based argumentation frames, we first need to verify that the interpretation of ($\mathbf{C}.\wedge$), ($\mathbf{A}.\vee$), and ($\mathbf{C}.\supset$) are justified in in Bi-**G**AFs where $\mathscr{A} = \{$[D-Reb]$\}$ and $\mathscr{S} = \{[D\text{-}Sup]\}$.

($\mathbf{C}.\wedge$) If $X \models_{\mathbf{G}} \neg(A \wedge B)$, then for any evaluation e such that $e(X) = 1$, we have that $e(\neg(A \wedge B)) = 1$. Thus, $\min\{e(A), e(B)\} = 0$ which implies either $e(A) = 0$ or $e(B) = 0$. Therefore, either $X \models_{\mathbf{G}} \neg A$, or $X \models_{\mathbf{G}} \neg B$.

($\mathbf{A}.\vee$) If $X \models_{\mathbf{G}} \neg(A \vee B)$, then for any evaluation e such that $e(X) = 1$, then $e(\neg(A \vee B)) = 1$. Therefore $\max\{e(A), e(B)\} = 0$, which implies $e(A) = 0$ and $e(B) = 0$, i.e. $X \models_{\mathbf{G}} \neg B$ and $X \models_{\mathbf{G}} \neg A$.

($\mathbf{C}.\supset$) If $X \models_{\mathbf{G}} \neg(A \supset B)$, then for any evaluation e such that $e(X) = 1$, then also $e(\neg(A \supset B)) = 1$ and this happens only if $e(A \supset B) = 0$. The only case in which the implication has value 0 is whenever $e(B) = 0$ and $e(A) > 0$. Therefore $e(\neg B) = 1$ and $e(\neg A) = 0$, i.e. $X \models_{\mathbf{G}} \neg B$ and $X \not\models_{\mathbf{G}} \neg A$.

In addition, we also need the following principles.

($\mathbf{C}'_{Bi}.\supset$) If $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_{A \supset B}, A \supset B\rangle$, then $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \Longrightarrow \langle \Gamma_A, A\rangle$

If $X \models_{\mathbf{G}} \neg(A \supset B)$, then whenever there is an evaluation e such that $e(X) = 1$, then

$e(\neg(A \supset B)) = 1$. Therefore, $e(A \supset B) = 0$, from which it follows $e(B) = 0$ and $e(A) > 0$. From $e(B) = 0$ it follows $e(\neg B) = 1$ and from $e(A) > 0$ it follows $e(\neg A) = 0$ and $e(\neg\neg A) = 1$. Conclusively, $X \models_{\mathbf{G}} \neg B$ and $X \models_{\mathbf{G}} \neg\neg A$.

($\mathbf{A}.\bot$) $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_\bot, \bot\rangle$ for every argument $\langle \Gamma_X, X\rangle$.

$X \models_{\mathbf{G}} \neg\bot$ for every X.

Let us define the set **G**P of principles as follows.

$\mathbf{G}\mathsf{P} = \{(\mathbf{C}.\wedge),\ (\mathbf{A}.\vee),\ (\mathbf{C}.\supset),\ (\mathbf{A}.\bot),\ (\mathbf{C}'_{Bi}.\supset)\}$

Proposition 5 (Closure of **G**P-immune arguments over Modus Ponens) *If A and $A \supset B$ are argumentatively **G**P-immune, then also B is argumentatively **G**P-immune.*

Proof We have to show that if (a) $X \not\models_{\mathbf{G}} \neg A$ and (b) $X \not\models_{\mathbf{G}} \neg(A \supset B)$, then $X \not\models_{\mathbf{G}} \neg B$. From (a) it follows that there is an evaluation e_1 such that $e_1(X) = 1$ and $e_1(\neg A) < 1$, from which it follows $e_1(\neg A) = 0$ and $e_1(A) > 0$. From (b) it follows there is an evaluation e_2 such that $e_2(X) = 1$ and $e_2(\neg(A \supset B)) < 1$, i.e. $e_2(\neg(A \supset B)) = 0$ and $e_2(A \supset B) > 0$. If we now consider an evaluation e^* such that $e^*(X) = 1$, $e^*(A) = 1$ and $e^*(A \supset B) = 1$. This evaluation e^* is compatible with both hypothesis (a) and (b). Since $e^*(A \supset B) = 1$, we have that $e^*(A) \leq e^*(B)$ and having $e^*(A) = 1$.Thus, $e^*(B) = 1$ and $e^*(\neg B) = 0$, i.e. $X \not\models_{\mathbf{G}} \neg B$. □

Theorem 6 (Bipolar Argumentative Soundness of **G**) *Any formula F is **G**-valid iff it is **G**P-immune.*

Proof (⇒) Given Proposition 5, it remain to show that every **G**-axiom is argumentatively **G**P-immune.

[Tr] If there is an argument X such that $X \longrightarrow (F \supset G) \supset ((G \supset H) \supset (F \supset H))$, then, by $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow (G \supset H) \supset (F \supset H)$ (1) and $X \Longrightarrow \neg\neg(F \supset G)$ (2). From (1) and $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow F \supset H$ (3) and $X \Longrightarrow \neg\neg G \supset H$ (4). From (3) and $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow H$ (7) and $X \Longrightarrow \neg\neg F$ (8). From (8) it follows that whenever there is an evaluation e such that $e(X) = 1$, then $e(\neg\neg F) = 1$. Therefore $e(\neg F) = 0$ and $e(F) > 0$ (9). From (4) it follows that whenever there is an evaluation e such that $e(X) = 1$, then $e(\neg\neg(G \supset H)) = 1$, i.e. $e(\neg(G \supset H)) = 0$ and $e(G \supset H) > 0$. Since $e(H) = 0$, it follows from (7), $e(G) = 0$. From (2) it follows that $e(\neg\neg(F \supset G)) = 1$, $e(\neg(F \supset G)) = 0$ and $e(F \supset G) > 0$. Since $e(G) = 0$, it follows that $e(F) = 0$, but this is in contradiction with (9).

[We] If there is some argument X such that $X \longrightarrow F \supset (G \supset F)$, then, by $(\mathbf{C}'_{Bi}.\supset)$ we have that $X \longrightarrow G \supset F$ (1) and $X \Longrightarrow \neg\neg F$ (2). From (1) and $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow F$ (3) and $X \Longrightarrow \neg\neg G$. From (2) it follows that whenever there is an evaluation e such that $e(X) = 1$, $e(\neg F) = 1$, i.e. $e(F) = 0$. From (3) it follows that whenever there is an evaluation e such that $e(X) = 1$, $e(\neg\neg F) = 1$, i.e. $e(\neg F) = 0$ and $e(F) > 0$, but this is in contradiction with $e(F) = 0$.

[Ex] If there is some argument X such that $X \longrightarrow (F \supset (G \supset H)) \supset (G \supset (F \supset H))$, then, by $(\mathbf{C}'_{Bi}.\supset)$ we have $X \longrightarrow G \supset (F \supset H)$ (1) and $X \Longrightarrow \neg\neg(F \supset (G \supset H))$ (2). From (1) and $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow F \supset H$ (3) and $X \Longrightarrow \neg\neg G$ (4). From (3) and $(\mathbf{C}'_{Bi}.\supset)$ we have $X \longrightarrow H$ (5) and $X \Longrightarrow \neg\neg F$ (6). From (5) it follows that whenever there is an evaluation e such that $e(X) = 1$, $e(H) = 0$, from (4) it follows that $e(F) > 0$ and from (4) it follows $e(G) > 0$. From (2) it follows $e(\neg\neg(F \supset (G \supset H))) = 1$. Therefore $e(F \supset (G \supset H)) > 0$. This last inequality holds if either (i) $e(G \supset H) > 0$ or (ii) $e(F) = e(G \supset H)$. From (i) and $e(H) = 0$, it follows $e(G) = 0$, but this is in contradiction with $e(G) > 0$. From (ii), $e(G) > 0$ and $e(H) = 0$ it follows $e(G \supset H) = 0$. Therefore $e(F) = 0$, but this is in contradiction with $e(F) = 0$.

[∧-1] If there is some argument X such that $X \longrightarrow (F \wedge G) \supset F$, then, by $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow F$ and $X \Longrightarrow \neg\neg(F \wedge G)$. Therefore whenever there is an evaluation e such that $e(X) = 1$, then $e(F) = 0$ and $e(\neg\neg(F \wedge G)) = 1$ from which it follows $\min\{e(F), e(G)\} > 0$, in particular $e(F) > 0$ that is in contradiction with $e(F) = 0$.

[∧-2] This case is similar to the previous one.

[∧-3] If there is some argument X such that $X \longrightarrow (H \supset F) \supset ((H \supset G) \supset (H \supset (F \wedge G)))$, then, by $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow (H \supset G) \supset (H \supset (F \wedge G))$ (1) and $X \Longrightarrow \neg\neg(H \supset F)$ (2). From (1) and $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow H \supset (F \wedge G)$ (3) and $X \Longrightarrow \neg\neg(H \supset G)$ (4). From (3) and $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow (F \wedge G)$ (5) and $X \Longrightarrow \neg\neg H$ (6). From (7) and $(\mathbf{C}.\wedge)$ it follows either (i) $X \longrightarrow F$, or (ii) $X \longrightarrow G$. In the case (i) we have that whenever there is an evaluation e such that $e(X) = 1$, then $e(F) = 0$. From (6) it follows that $e(H) > 0$ and from (2) we have that

$e(\neg\neg(H \supset F)) = 1$. Therefore $e(H \supset F) > 0$. However from $e(H) > 0 = e(F)$ it follows $e(H \supset F) = 0$ and we have found a contradiction. In the case (ii) we can reach a contradiction in a very similar way.

[∨-1] If there is some argument X such that $X \longrightarrow F \supset (F \vee G)$, then, by $(\mathbf{C}'_{Bi}.\supset)$ we have $X \longrightarrow F \vee G$ (1) and $X \Longrightarrow \neg\neg F$ (2). From (1) and $(\mathbf{A}.\vee)$ it follows $X \longrightarrow F$ and $X \longrightarrow G$. Therefore whenever there is an evaluation e such that $e(X) = 1$, $e(F) = 0$ and $e(G) = 0$, but this is in contradiction with what follows from (2): $e(F) > 0$.

[∨-2] This case is similar to the previous one.

[∨-3] If there is some argument X such that $X \longrightarrow (G \supset F) \supset ((H \supset F) \supset ((G \vee H) \supset F))$, then, by $(\mathbf{C}'_{Bi}.\supset)$ we have $X \longrightarrow (H \supset F) \supset ((G \vee H) \supset F)$ (1) and $X \Longrightarrow \neg\neg(G \supset F)$ (2). From (1) and $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow (G \vee H) \supset F$ (3) and $X \Longrightarrow \neg\neg(H \supset F)$ (4). From (3) and $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow F$ (5) and $X \Longrightarrow \neg\neg(G \vee H)$ (6). From (5) it follows that whenever there is an evaluation e such that $e(X) = 1$, $e(F) = 0$. From (6) it follows $e(G \vee H) > 0$, i.e. $\max\{e(G), e(H)\} > 0$. Therefore either (i) $e(G) > 0$ or (ii) $e(H) > 0$. In the first case we have that $e(G) > e(F) = 0$ from which it follows $e(G \supset F) = e(F) = 0$, but this is in contradiction with what follows from (2): $e(G \supset F) > 0$. In the case (ii) we reach a contradiction form (4) as in the previous point.

[Lin] If there is some argument X such that $X \longrightarrow (F \supset G) \vee (G \supset F)$, by $(\mathbf{C}'_{Bi}.\supset)$ we have $X \longrightarrow F \supset G$ (1) and $X \longrightarrow G \supset F$ (2). From (1) and $(\mathbf{C}.\supset)$ it follows $X \longrightarrow G$ and $X \not\longrightarrow F$, but this is in contradiction with what follows from (2) and $(\mathbf{C}.\supset)$: $X \longrightarrow F$ and $X \not\longrightarrow G$.

[⊥] If there is some argument X such that $X \longrightarrow \bot \supset F$, then, by $(\mathbf{C}.\supset)$ we have $X \longrightarrow F$ and $X \not\longrightarrow \bot$. However, this it cannot be since every argument is supposed to attack *falsum* as $(\mathbf{A}.\bot)$ requires.

[Con] If there is some argument X such that $X \longrightarrow (F \supset (F \supset G)) \supset (F \supset G)$, by $(\mathbf{C}'_{Bi}.\supset)$ we have $X \longrightarrow F \supset G$ (1) and $X \Longrightarrow \neg\neg(F \supset (F \supset G))$ (2). From (1) and $(\mathbf{C}'_{Bi}.\supset)$ it follows $X \longrightarrow G$ (3) and $X \Longrightarrow \neg\neg F$ (4). From (3) and (4) it follows that whenever there is an evaluation e such that $e(X) = 1$, then $e(G) = 0$ and $e(F) > 0$. Therefore, since $e(F) > e(G) = 0$, $e(F \supset G) = 0$ and $e(F \supset (F \supset G)) = 0$, but this is in contradiction with what follows from (2): $e(F \supset (F \supset G)) > 0$. □

Axiom	Principles used in the proof
[Tr]	$(\mathbf{C}'_{Bi}.\supset)$
[We]	$(\mathbf{C}'_{Bi}.\supset)$
[Ex]	$(\mathbf{C}'_{Bi}.\supset)$
[∧-1]	$(\mathbf{C}'_{Bi}.\supset)$
[∧-2]	$(\mathbf{C}'_{Bi}.\supset)$
[∧-3]	$(\mathbf{C}'_{Bi}.\supset)$, $(\mathbf{C}.\wedge)$
[∨-1]	$(\mathbf{C}'_{Bi}.\supset)$, $(\mathbf{A}.\vee)$
[∨-2]	$(\mathbf{C}'_{Bi}.\supset)$
[∨-3]	$(\mathbf{C}'_{Bi}.\supset)$
[Lin]	$(\mathbf{C}.\supset)$
[⊥]	$(\mathbf{C}.\supset)$, $(\mathbf{A}.\bot)$
[Con]	$(\mathbf{C}'_{Bi}.\supset)$

Table 5: Principles used in Theorem 6

Theorem 7 (Bipolar Argumentative Completeness of **G**) *Every argumentatively **G**P-immune formula is **G**-valid .*

Proof We prove the theorem indirectly. Suppose there is a formula F that is not **G**-valid, we will show that F is not argumentatively **G**P-immune. Since F that is not **G**-valid, , then there is an evaluation e such that $e(F) < 1$. We distinguish two possibilities: (a) $e(F) = 0$, and (b) $e(F) \in (0, 1)$.

(a) If $e(F) = 0$, then $e(\neg F) = 1$ and $\neg F \models_{\mathbf{G}} \neg F$, i.e. $\neg F \longrightarrow F$.

(b) If $e(F) \in (0, 1)$ then $e(\neg F) = 0$. Therefore $e(\neg F) \leq e(F)$ and $e(F \supset \neg F) = 0$, i.e. also the formula $F \supset \neg F$ is not **G**-valid and $\neg(F \supset \neg F) \longrightarrow (F \supset \neg F)$.

□

In Table 5 we summarise which attack principle has been used in the corresponding sub-case of Theorem 6.

To recover a complete semantics for product logic, we procede in a similar way. The attack principles needed are: $(\mathbf{A}.\vee)$, $(\mathbf{A}.\neg)$, $(\mathbf{C}.\supset)$, $(\mathbf{A}.\supset)$ together with two additional attack principles regarding strong conjunction: $(\mathbf{A}.\&)$ and $(\mathbf{C}.\&)$.

$(\mathbf{A}.\&)$ If $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_A, A\rangle$ or $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_B, B\rangle$, then $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_{A\&B}, A\&B\rangle$.

$(\mathbf{C}.\&)$ If $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_{A\&B}, A\&B\rangle$, then $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_A, A\rangle$ or $\langle \Gamma_X, X\rangle \longrightarrow \langle \Gamma_B, B\rangle$.

Let us now verify that the above mentioned principles are justified in Bi-**Π**AFs where $\mathscr{A} = \{[\text{D-Reb}]\}$ and $\mathscr{S} = \{[D\text{-}Sup]\}$.

($\mathbf{A}.\neg$) If $X \models_{\mathbf{\Pi}} \neg\neg A$, then for any evaluation e such that $e(X) = 1$, $e(\neg\neg A) = 1$. Therefore $e(\neg A) = 0$, i.e. $X \not\models_{\mathbf{\Pi}} \neg A$

($\mathbf{A}.\vee$) If $X \models_{\mathbf{\Pi}} \neg(A \vee B)$, , then for any evaluation e such that $e(X) = 1$, then $e(\neg(A \vee B)) = 1$. Therefore $e(A \vee B) = \max\{e(A), e(B)\} = 0$, from which it follows $e(A) = 0$ and $e(B) = 0$, i.e. $X \models_{\mathbf{\Pi}} \neg A$ and $X \models_{\mathbf{\Pi}} \neg B$.

($\mathbf{A}.\&$) If $X \models_{\mathbf{\Pi}} \neg A$ or $X \models_{\mathbf{\Pi}} \neg B$, then for any evaluation e such that $e(X) = 1$, then either $e(\neg A) = 1$, i.e. $e(A) = 0$, or $e(\neg B) = 1$, i.e. $e(B) = 0$. Therefore $e(A) \cdot e(B) = 0$, i.e. $X \models_{\mathbf{\Pi}} \neg(A\&B)$.

($\mathbf{C}.\&$) If $X \models_{\mathbf{\Pi}} \neg(A\&B)$, then for any evaluation e such that $e(X) = 1$, then $e(\neg(A\&B)) = 1$. Therefore $e(A\&B)$=0 which implies either $e(A) = 0$, or $e(B) = 0$.

($\mathbf{A}.\supset$) If $X \models_{\mathbf{\Pi}} \neg B$ and $X \not\models_{\mathbf{\Pi}} \neg A$, then, whenever there is an evaluation e such that $e(X) = 1$, then $e(\neg B) = 1$ and $e(\neg A) < 1$. Therefore $e(B) = 0$ and $e(A) > 0$ which implies $e(A \supset B) = 0$ and $e(\neg(A \supset B)) = 1$.

($\mathbf{C}.\supset$) If $X \models_{\mathbf{\Pi}} \neg(A \supset B)$, then for any evaluation e such that $e(X) = 1$, then $e(\neg(A \supset B)) = 1$. Therefore $e(A \supset B) = 0$ and this happens only if $e(B) = 0$ and $e(A) > 0$. From this it follows $e(\neg B) = 1$ and $e(\neg A) = 0$, i.e. $X \models_{\mathbf{\Pi}} \neg B$ and $X \not\models_{\mathbf{\Pi}} \neg A$.

Let us define the set $\mathbf{\Pi}$P of principles as follows.

$\mathbf{\Pi}\mathsf{P} = \{(\mathbf{A}.\neg), (\mathbf{A}.\vee), (\mathbf{A}.\bot), (\mathbf{A}.\&), (\mathbf{C}.\&), (\mathbf{A}.\supset)\}$

Proposition 8 (Closure of $\mathbf{\Pi}$P-immune arguments over Modus Ponens) *If A and $A \supset B$ are argumentatively $\mathbf{\Pi}$P-immune, then also B is argumentatively $\mathbf{\Pi}$P-immune.*

Proof We have to show that if, for any X in the frame, (a) $X \not\models_{\mathbf{\Pi}} \neg A$ and (b) $X \not\models_{\mathbf{\Pi}} \neg(A \supset B)$, then $X \not\models_{\mathbf{\Pi}} \neg B$. From (a) it follows that there is an evaluation e_1 such that $e_1(X) = 1$ and $e_1(\neg A) < 1$, from which it follows $e_1(A) > 0$. From (b) it follows there is an evaluation e_2 such that $e_2(X) = 1$ and $e_2(\neg(A \supset B)) < 1$, i.e. $e_2(\neg(A \supset B)) = 0$ and $e_2(A \supset B) > 0$. In particular, we have that $e_2(A \supset B) > 0$ if either (i) $e_2(A) \leq e_2(B)$, from which it follows $e_2(A \supset B) = 0$, or (ii) $e_2(A) > e_2(B)$ and $e_2(B) > 0$, from which it follows $e_2(A \supset B) = \frac{e(B)}{e(A)}$. If (ii) happens we have already identified an evaluation such that $e(X) = 1$ and $e(\neg B) = 0$ from which it follows $X \not\models_{\mathbf{\Pi}} \neg B$. Otherwise, since both X and A belong to the frame, also $X \wedge A$ it does and from the hypothesis we have $X \wedge A \not\models_{\mathbf{\Pi}} \neg A$ and $X \wedge A \not\models_{\mathbf{\Pi}} \neg(A \supset B)$, i.e. there is an evaluation e_3 such that $e_3(X \wedge A) = 1$ (from which it follows $e_3(X) = 1$ and $e_3(A) = 1$) and $e_3(\neg(A \supset B)) < 1$. Therefore $e_3(A \supset B) > 0$ and if $e_3(A \supset B) = 1$, $1 = e_3(A) \leq e_3(B)$ which implies $e_3(B) = 1$; if $e_3(A \supset B) \in (0;\ 1)$, then $e_3(A) > e_3(B) > 0$. Conclusively in both cases $e_3(\neg B) = 0$. □

Theorem 9 (Adequateness Theorem for **G**) *Any formula F is **G**-valid iff it is **G**P-immune.*

Proof Given Proposition 8, it remain to show that every axiom is argumentatively $\mathbf{\Pi}$P-immune.

[BL1] Suppose there is an argument X such that $X \longrightarrow (F \supset B) \supset ((G \supset H) \supset (F \supset H))$. By ($\mathbf{C}.\supset$) we have that $X \longrightarrow (G \supset H) \supset (F \supset H)$ (1) and $X \not\longrightarrow F \supset B$ (2). From (1) and ($\mathbf{C}.\supset$) it follows $X \longrightarrow F \supset H$(3) and $X \not\longrightarrow G \supset H$ (4). From (3) and ($\mathbf{C}.\supset$) it follows $X \longrightarrow H$ (5) and $X \not\longrightarrow F$ (6). From (2) and ($\mathbf{A}.\supset$) it follows $X \not\longrightarrow G$ (7) or $X \longrightarrow F$ (8). From (4) and ($\mathbf{A}.\supset$) it follows $X \not\longrightarrow H$ or $X \longrightarrow G$. Since (5) holds, then also $X \longrightarrow G$ and $X \longrightarrow F$, but this is in contradiction with (6).

[BL4] Suppose there is an argument X such that $X \longrightarrow F\&(F \supset G) \supset G\&(G \supset F)$. By ($\mathbf{C}.\supset$) we have $X \longrightarrow G\&(G \supset F)$ (1) and $X \not\longrightarrow F\&(F \supset G)$ (2). By (1) and ($\mathbf{C}.\&$) it follows $X \longrightarrow G$ either $X \longrightarrow G$ (3) or $X \longrightarrow G \supset F$ (4). From (4) and ($\mathbf{C}.\supset$) it follows $X \longrightarrow F$ (5) and $X \not\longrightarrow G$ (6). From (2) and ($\mathbf{A}.\&$) it follows $X \not\longrightarrow F$ (7) and $X \not\longrightarrow F \supset G$ (8). From (8) and ($\mathbf{A}.\supset$) it follows either $X \longrightarrow F$ (9) or $X \not\longrightarrow G$ (10). Since (7) holds, then (9) does not hold and $X \not\longrightarrow G$ it does. Therefore (3) does not hold and (5) it does, but this is in contradiction with (7).

[BL5a] Suppose there is an argument X such that $X \longrightarrow (F\&G \supset H) \supset (F \supset (G \supset H))$. Therefore by ($\mathbf{C}.\supset$) we have $X \longrightarrow F \supset (G \supset H)$ (1) and $X \not\longrightarrow F\&G \supset H$ (2). From (1) and ($\mathbf{C}.\supset$) we have that $X \longrightarrow G \supset H$(3) and $X \not\longrightarrow F$ (4). From (3) and ($\mathbf{C}.\supset$) it follows $X \longrightarrow H$ (5) and $X \not\longrightarrow G$ (6). From (2) and ($\mathbf{A}.\supset$) it follows either $X \longrightarrow F\&G$ (7), or $X \longrightarrow G$ (8). Since (5) holds, (8) does not and from (7) and ($\mathbf{C}.\&$) we have wither $X \longrightarrow F$, that cannot be because of (4), or $X \longrightarrow G$ that is in contradiction with (6).

[BL5b] Suppose there is an argument X such that $X \longrightarrow (F \supset (G \supset H)) \supset (F\&G \supset H)$. By

($\mathbf{C}.\supset$) we have $X \longrightarrow (F\&G) \supset H$ (1) and $X \not\longrightarrow F \supset (G \supset H)$ (2). From (1) and ($\mathbf{C}.\supset$) it follows $X \longrightarrow H$ (3) and $X \not\longrightarrow F\&G$ (4). From (4) and ($\mathbf{A}.\&$) it follows $X \not\longrightarrow F$ (5) and $X \not\longrightarrow G$ (6). From (2) and ($\mathbf{A}.\supset$) it follows $X \not\longrightarrow G \supset H$ (7) or $X \longrightarrow F$ (8) and from (7) and ($\mathbf{A}.\supset$) we have $X \not\longrightarrow H$ (9) or $X \longrightarrow G$ (10). Since (5) holds, (8) does not. Since (3) holds, (9) does not. Therefore should hold both $X \not\longrightarrow G$ (6) and $X \longrightarrow G$ (10), and this cannot happen.

[BL6] Suppose there is an argument X such that $X \longrightarrow ((F \supset G) \supset H) \supset (((G \supset F) \supset H) \supset H)$. Therefore, by ($\mathbf{C}.\supset$) we have $X \longrightarrow ((G \supset F) \supset H) \supset H$ (1) and $X \not\longrightarrow (F \supset G) \supset H$ (2). From (1) and ($\mathbf{C}.\supset$) it follows $X \longrightarrow H$ (3) and $X \not\longrightarrow (G \supset F) \supset H$ (4). From (2) and ($\mathbf{A}.\supset$) we have either $X \not\longrightarrow H$ (5) or $X \longrightarrow F \supset G$ (6). From (6) and ($\mathbf{C}.\supset$) it follows $X \longrightarrow G$ (7) and $X \not\longrightarrow F$ (8). From (4) and ($\mathbf{A}.\supset$) it follows either $X \not\longrightarrow H$ (9) or $X \longrightarrow G \supset F$ (10), from which it follows, by ($\mathbf{C}.\supset$), $X \longrightarrow F$ and $X \not\longrightarrow G$. Since (3) holds, (5) and (9) do not and (6) it does. However, both (7) and (8) are in contradiction with what follows from (10), and this it cannot happen.

[BL7] Suppose there is an argument X such that $X \longrightarrow \bot \supset F$. Therefore by ($\mathbf{C}.\supset$) we have $X \longrightarrow F$ and $X \not\longrightarrow \bot$, but this it cannot be since every argument is supposed to attack $\bot$ as ($\mathbf{A}.\bot$) requires.

[P] Suppose there is an argument X such that $X \longrightarrow \neg F \vee ((F \supset F\&G) \supset G)$. Therefore by ($\mathbf{A}.\vee$) we have $X \longrightarrow \neg F$ (1) and $X \longrightarrow (F \supset (F\&G)) \supset F$ (2). From (1) and ($\mathbf{A}.\neg$), it follows $X \not\longrightarrow F$ (3). From (2) and ($\mathbf{C}.\supset$) it follows $X \longrightarrow F$ and $X \not\longrightarrow F \supset (F\&G)$. However $X \longrightarrow F$ is in contradiction with (3).

□

Theorem 10 (Bipolar Argumentative Completeness of P) *Every argumentatively* $\mathbf{\Pi}$P*-immune formula is* P*-valid .*

Proof We prove the theorem indirectly. Suppose there is a formula F that is not P-valid, we will show that F is not argumentatively $\mathbf{\Pi}$P-immune. Since F that is not P-valid, , then there is an evaluation e such that $e(F) < 1$. We distinguish two possibilities: (a) $e(F) = 0$, and (b) $e(F) \in (0, 1)$.

(a) If $e(F) = 0$, then $e(\neg F) = 1$ and $\neg F \models_{\mathbf{\Pi}} \neg F$, i.e. $\neg F \longrightarrow F$.

(b) If $e(F) \in (0, 1)$ then $e(\neg F) = 0$. Therefore $e(\neg F) < e(F)$ and $e(F \supset \neg F) = e(\neg F) = 0$, i.e. also the formula $F \supset \neg F$ is not P-valid and $\neg(F \supset \neg F) \longrightarrow (F \supset \neg F)$.

□

6 Conclusions and Future Work

In the present paper, we have introduced new semantics for the main three t-norm based fuzzy logics based on t-norm based arguments and bipolar frames. We have focused on a specific instantiation of both attack and support relation, namely *direct rebuttal* and *direct support*. This choice is related to the fact that the principles considered are defined in terms of the claims of the arguments. Considering argumentative principles defined using also the supports of the arguments, it would be interesting to investigate whether it is possible to recover additional complete semantics for the same logics based on different instantiations of the attack and support relation. The overall methodology used to prove our results is similar to the one used in [20]. What changes is the underlying framework both on the level of definition of arguments, we have introduced t-norm based arguments, and on the possible relations among them, we consider instantiations of both the attack and the support relation.

Some of the principles introduced are easier to justify than others. This is not surprising and it has already been the case in the characterisation of classical and fuzzy logics in [19] and [20]. The specificity of some principles might be seen as a way to look at the characteristics of the various logics from a new perspective and answer to the question “what is needed on the level of argumentative frames in order to characterise, e.g., Łukasiewicz logic?”

As future work, we would like to explore which types of applications may benefit from the argumentation-based semantics of fuzzy logics suggested here.

Compliance with Ethical Standards

This article does not contain any studies with human participants or animals performed by any of the authors. Furthermore the authors declare that there are no conflicts of interest.

Appendix A Justification of the attack principles considering Ł-Based Arguments

A.0.1 [Def] and $\models_{\text{Ł}}$-Based Arguments

(**A**.$\wedge$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$, then $X \models_{\text{Ł}} \neg\bigwedge\Gamma_A$, i.e. for any evaluation e s.t. $e(X) = 1$, $e(\neg\bigwedge\Gamma_A) = 1$. For every evaluation e we have $e(\neg\bigwedge\Gamma_A) = 1 - e(\bigwedge\Gamma_A) = 1 - \min_{\gamma\in\Gamma_A}(e(\gamma))$. Therefore, considering the argument $\langle\Gamma_{A\wedge B}, A\wedge B\rangle$, if $\Gamma_A \subseteq \Gamma_{A\wedge B}$, we have $1 - \min_{\gamma\in\Gamma_A}(e(\gamma)) \leq 1 - \min_{\gamma\in\Gamma_{A\wedge B}}(e(\gamma))$ and if $1 - \min_{\gamma\in\Gamma_A}(e(\gamma)) = 1$ also $1 - \min_{\gamma\in\Gamma_{A\wedge B}}(e(\gamma)) = 1$ i.e. $X \models_{\text{Ł}} \neg\bigwedge\Gamma_{A\wedge B}$. The attack principle is justified.

(**C**.$\wedge$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\wedge B}, A\wedge B\rangle$, then $X \models_{\text{Ł}} \neg\bigwedge\Gamma_{A\wedge B}$, i.e. for any evaluation e s.t. $e(X) = 1$, $e(\neg\bigwedge\Gamma_{A\wedge B}) = 1 - \min_{\gamma\in\Gamma_{A\wedge B}}(e(\gamma)) = 1$. Therefore $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ or $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$ only if either $\Gamma_{A\wedge B} \subseteq \Gamma_A$ or $\Gamma_{A\wedge B} \subseteq \Gamma_B$.

(**A**.$\vee$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\vee B}, A\vee B\rangle$, then $X \models_{\text{Ł}} \neg\bigwedge\Gamma_{A\vee B}$, i.e. for any evaluation e s.t. $e(X) = 1$, $e(\neg\bigwedge\Gamma_{A\vee B}) = 1 - \min_{\gamma\in\Gamma_{A\vee B}}(e(\gamma)) = 1$. Therefore, $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ and $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$ only if $\Gamma_{A\vee B} \subseteq \Gamma_A$ and $\Gamma_{A\vee B} \subseteq \Gamma_B$.

(**C**.$\vee$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ and $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$, then $X \models_{\text{Ł}} \neg\bigwedge\Gamma_A$ and $X \models_{\text{Ł}} \neg\bigwedge\Gamma_B$, i.e. for any evaluation e s.t. $e(X) = 1$ $e(\neg\bigwedge\Gamma_A) = 1 - \min_{\gamma\in\Gamma_A}(e(\gamma)) = e(\neg\bigwedge\Gamma_B) = 1 - \min_{\gamma\in\Gamma_B}(e(\gamma)) = 1$. Therefore, $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\vee B}, A\vee B\rangle$ only if either $\Gamma_A \subseteq \Gamma_{A\vee B}$ or $\Gamma_B \subseteq \Gamma_{A\vee B}$.

(**A**.$\supset$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\supset B}, A\supset B\rangle$, then for any evaluation e s.t. $e(X) = 1$, then $e(\neg\bigwedge\Gamma_{A\supset B}) = 1 - \min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma)) = 1$. If $\Gamma_{A\supset B} \subseteq \Gamma_B$, then $\min_{\gamma\in\Gamma_B}(e(\gamma)) \leq \min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma)$ and $1 - \min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma)) \leq 1 - \min_{\gamma\in\Gamma_B}(e(\gamma))$. Therefore, $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$. To show that $\langle\Gamma_X, X\rangle \not\xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ follows from the hypothesis, we need to find a specific evaluation e^* s.t. $e^*(X) = 1$, but $1 - \min_{\gamma\in\Gamma_A}(e^*(\gamma)) < 1$ while $1 - \min_{\gamma\in\Gamma_{A\supset B}}(e^*(\gamma)) = 1$. Given an evaluation e s.t. $e(X) = 1$, if we indicate with γ_i^* the elements of $\Gamma_{A\supset B}$ s.t. $e(\gamma_i^*) = \min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma))$, if $\Gamma_A \subset \Gamma_{A\supset B}$ and $\gamma_i^* \notin \Gamma_A$ for any i, then $\min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma)) < \min_{\gamma\in\Gamma_A}(e(\gamma))$ from which it follows $1 - \min_{\gamma\in\Gamma_A}(e(\gamma)) < 1 - \min_{\gamma\in\Gamma_{A\supset B}}(e(\gamma))$.

(**C**.$\supset$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$ and $\langle\Gamma_X, X\rangle \not\xrightarrow{[Def]} \langle\Gamma_A, A\rangle$, then for any evaluation e s.t. $e(X) = 1$, $e(\neg\bigwedge\Gamma_B) = 1 - \min_{\gamma\in\Gamma_B}(e(\gamma)) = 1$ and there is an evaluation e^* s.t. $e^*(X) = 1$, but $e^*(\neg\bigwedge\Gamma_A) = 1 - \min_{\gamma\in\Gamma_A}(e^*(\gamma)) < 1$. Therefore, $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\supset B}, A\supset B\rangle$ only if $\Gamma_B \subseteq \Gamma_{A\supset B}$.

(**A**.$\neg$) The principle is not justified.

(**C**.$\neg$) The principle is not justified.

A.0.2 [Def] and $\models^{\leq}_{\text{Ł}}$-Based Arguments

(**A**.$\wedge$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$, then $X \models^{\leq}_{\text{Ł}} \neg\bigwedge\Gamma_A$, i.e. for every evaluation e, $e(X) \leq e(\neg\bigwedge\Gamma_A) = 1 - \min_{\gamma\in\Gamma_A}(e(\gamma))$. If we consider the argument $\langle\Gamma_{A\wedge B}, A\wedge B\rangle$, if $\Gamma_A \subseteq \Gamma_{A\wedge B}$, we have $1 - \min_{\gamma\in\Gamma_A}(e(\gamma)) \leq 1 - \min_{\gamma\in\Gamma_{A\wedge B}}(e(\gamma))$. Therefore, for every e we have $e(X) \leq e(\neg\bigwedge\Gamma_{A\wedge B})$, i.e. $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\wedge B}, A\wedge B\rangle$ and the principle is justified.

(**C**.$\wedge$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\wedge B}, A\wedge B\rangle$, then it follows either $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ or $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$ if either $\Gamma_{A\wedge B} \subseteq \Gamma_A$ or $\Gamma_{A\wedge B} \subseteq \Gamma_B$, respectively.

(**A**.$\vee$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\vee B}, A\vee B\rangle$, then it follows $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ and $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$ only if $\Gamma_{A\vee B} \subseteq \Gamma_A$ and $\Gamma_{A\vee B} \subseteq \Gamma_B$.

(**C**.$\vee$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_A, A\rangle$ and $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_B, B\rangle$, then it follows $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\vee B}, A\vee B\rangle$ only if either $\Gamma_A \subseteq \Gamma_{A\vee B}$ or $\Gamma_B \subseteq \Gamma_{A\vee B}$.

(**A**.$\supset$) If $\langle\Gamma_X, X\rangle \xrightarrow{[Def]} \langle\Gamma_{A\supset B}, A\supset B\rangle$, then for any evaluation e $e(X) \leq e(\neg\bigwedge\Gamma_{A\supset B})$, therefore if $\Gamma_{A\supset B} \subseteq \Gamma_B$ $\langle\Gamma_X, X\rangle \xrightarrow{[Def]}$

$\langle \Gamma_B, B\rangle$. However from the hypothesis we cannot deduce that $\langle \Gamma_X, X\rangle \xrightarrow{[Def]}\!\!\!\!\!\!\!\!/\;\;\; \langle \Gamma_A, A\rangle$. The principle is not justified.

(**C**.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[Def]}\!\!\!\!\!\!\!\!/\;\;\; \langle \Gamma_A, A\rangle$, then it follows $\langle \Gamma_X, X\rangle \xrightarrow{[Def]} \langle \Gamma_{A\supset B}, A\supset B\rangle$ only if $\Gamma_B \subseteq \Gamma_{A\supset B}$.

(**A**.$\neg$) The principle is not justified.

(**C**.$\neg$) The principle is not justified.

A.0.3 [C-Reb-1] and $\models_{\L}$-Based arguments

(**A**.$\wedge$) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$, then $\Gamma_X \models_{\L} \neg A$, i.e. for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$, $e(\neg A) = 1 - e(A) = 1$. Since for every evaluation e $e(A) \geq (A \wedge B)$, if $e(A) = 0$ also $e(A \wedge B) = 0$ and $e(\neg(A \wedge B)) = 1$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\wedge B}, A\wedge B\rangle$.

(**C**.$\wedge$) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\wedge B}, A\wedge B\rangle$, then for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$, $e(\neg(A \wedge B)) = 1-e(A\wedge B) = 1$, i.e. $\min(e(A), e(B)) = 0$. However, this is different from having either $e(A) = 0$ or $e(B) = 0$. The principle is not justified.

(**A**.$\vee$) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\wedge B}, A\vee B\rangle$, then for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$, we have $e(\neg(A \vee B)) = 1$, i.e. $1 - e(A \vee B) = 1 - \max(e(A), e(B)) = 1$ from which it follows $\max(e(A), e(B)) = 0$. Since $e(A) \leq \max(e(A), e(B))$ and $e(B) \leq \max(e(A), e(B))$ for any evaluation e, if $\max(e(A), e(B)) = 0$ also $e(A) = 0$ and $e(B) = 0$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B\rangle$.

(**C**.$\vee$) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B\rangle$, then for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$ we have $e(\neg A) = 1 - e(A) = 1$ and $e(\neg B) = 1 - e(B) = 1$, i.e. $e(A) = 0$ and $e(B) = 0$. Therefore $\max(e(A), e(B)) = 0, 1-\max(e(A), e(B)) = 1$ and $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\vee B}, A\vee B\rangle$.

(**A**.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\supset B}, A\supset B\rangle$, then for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$, $e(\neg(A \supset B)) = 1 - e(A \supset B) = 1$, i.e. $e(A \supset B) = 0$. Since $e(A \supset B) = \min(1, 1 - e(A) + e(B))$, $1 - e(A) + e(B) = 0$, i.e. $1 + e(B) = e(A)$ which implies $e(B) = 0$ and $e(A) = 1$. Therefore whenever an evaluation e is such that $e(\bigwedge \Gamma_X) = 1$, then $e(\neg B) = 1 - e(B) = 1$ and $e(\neg A) = 1 - e(A) = 0$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]}\!\!\!\!\!\!\!\!\!\!\!\!\!/\;\;\;\;\;\; \langle \Gamma_A, A\rangle$. The principle is justified.

(**C**.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]}\!\!\!\!\!\!\!\!\!\!\!\!\!/\;\;\;\;\;\; \langle \Gamma_A, A\rangle$, then whenever an evaluation e is s.t. $e(\bigwedge \Gamma_X) = 1$, $e(\neg B) = 1 - e(B) = 1$, i.e. $e(B) = 0$. Since $e(B) = 0$ $e(A \supset B) = \min(1, 1 - e(A) + e(B)) = \min(1, 1 - e(A)) = 1 - e(A)$. In order to have $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\supset B}, A\supset B\rangle$ we need to show that $e(A \supset B) \leq e(B)$, i.e. $1 - e(A) \leq e(B)$. In fact, if this last inequality holds we would have $1 = 1 - e(B) \leq 1 - 1 + e(A) = 1 - e(A \supset B)$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A\supset B}, A\supset B\rangle$. However, we are under the hypothesis that $e(B) = 0$, therefore we would need $e(A) = 1$ for any e while from the hypothesis that $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]}\!\!\!\!\!\!\!\!\!\!\!\!\!/\;\;\;\;\;\; \langle \Gamma_A, A\rangle$ we can only have that for some e^* s.t. $e^*(\bigwedge \Gamma_X) = 1$, $e^*(A) > 0$. The principle is not justified.

(**A**.$\neg$) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$, then for any evaluation e s.t. $e(\bigwedge \Gamma_X) = 1$, $e(\neg A) = 1 - e(A) = 1$ i.e. $e(A) = 0$. Therefore $e(\neg\neg A) = 1 - 1 + e(A) = 0$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]}\!\!\!\!\!\!\!\!\!\!\!\!\!/\;\;\;\;\;\; \langle \Gamma_B, \neg A\rangle$. The principle is justified.

(**C**.$\neg$) If $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]}\!\!\!\!\!\!\!\!\!\!\!\!\!/\;\;\;\;\;\; \langle \Gamma_B, \neg A\rangle$, then there is some evaluation e^* s.t. $e^*(\bigwedge \Gamma_X) = 1$ and $e^*(\neg\neg A) = e^*(A) < 1$. In order to have $\langle \Gamma_X, X\rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A\rangle$ we would need $e^*(\neg A) = 1 - e^*(A) = 1$ i.e. $e^*(A) = 0$, but from the hypothesis we can only infer $e^*(A) < 1$. Therefore, the principle is not justified.

A.0.4 [C-Reb-1] and $\models^{\leq}_{\L}$-based arguments

(**A**.$\wedge$) The principle is justified for the same reason (**A**.$\wedge$) holds using Definition 12.

(**C**.$\wedge$) The principle is not justified because from $e(\bigwedge \Gamma_X) \leq 1 - e(A \wedge B)$ we cannot deduce neither $e(\bigwedge \Gamma_X) \leq 1 - e(A)$ nor $e(\bigwedge \Gamma_X) \leq$

$1 - e(B)$ since we only know that $1 - e(A) \leq 1 - \min(e(A), e(B))$.

(**A**.$\vee$) The principle is justified for the same reason (**A**.$\vee$) holds using Definition 12.

(**C**.$\vee$) The principle is not justified.

(**A**.$\supset$) If $\langle \Gamma_X, X \rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A \supset B}, A \supset B \rangle$, then for every evaluation e $e(\bigwedge \Gamma_X) \leq 1 - e(A \supset B)$. Since $e(A \supset B) = \min(1, 1 - e(A) + e(B))$, if $1 - e(A) + e(B) < 1$, then $e(A \supset B) = 1 - e(A) + e(B)$. Therefore $e(B) \leq e(A \supset B)$. If $1 - e(A) + e(B) \geq 1$ $e(A \supset B) = 1$ and also in this case $e(B) \leq e(A \supset B)$. In any case $\langle \Gamma_X, X \rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B \rangle$. However, from the hypothesis we cannot deduce that $\langle \Gamma_X, X \rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A \rangle$. Thus, the principle is not justified, but its shorter version (**A**.$\supset$) holds.

(**C**.$\supset$) If $\langle \Gamma_X, X \rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, B \rangle$, then for every evaluation e $e(\bigwedge \Gamma_X) \leq 1 - e(B)$ and in order to show that $\langle \Gamma_X, X \rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_{A \supset B}, A \supset B \rangle$ we would need $1 - e(B) \leq 1 - e(A \supset B)$, but this it cannot be because for every evaluation e, $e(B) \leq e(A \supset B)$. The principle is not justified.

(**A**.$\neg$) If $\langle \Gamma_X, X \rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A \rangle$, then for any evaluation e $e(\bigwedge \Gamma_X) \leq e(\neg A) = 1 - e(A)$. In order to show that $\langle \Gamma_X, X \rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, \neg A \rangle$ we would need $e^*(\bigwedge \Gamma_X) > 1 - e^*(\neg A) = e^*(A)$ for some evaluation e^*, but it does not follow from the hypothesis and the principle is not justified.

(**C**.$\neg$) If $\langle \Gamma_X, X \rangle \not\xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_B, \neg A \rangle$, then there exists some evaluation e^* s.t. $e^*(\bigwedge \Gamma_X) > e^*(A)$ and in order to have $\langle \Gamma_X, X \rangle \xrightarrow{[C\text{-}Reb\text{-}1]} \langle \Gamma_A, A \rangle$ we would need that for any evaluation e $e(\bigwedge \Gamma_X) \leq 1 - e(A)$, but this cannot be deduced from the hypothesis. Therefore, the principle is not justified.

A.0.5 [I-Reb] and $\models_{\text{Ł}}$-Based Arguments

(**A**.$\wedge$) If $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A \rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$ and $A \models_{\text{Ł}} \neg\varphi$. Therefore whenever there is an evaluation e s.t. $e(X) = 1$ $e(\varphi) = 1$ and whenever $e(A) = 1$, $e(\neg\varphi) = 1$. Since $e(A \wedge B) = \min\{e(A), e(B)\}$, whenever $e(A \wedge B) = 1$, both $e(A) = 1$ and $e(B) = 1$. From the hypothesis we have that from $e(A) = 1$ it follows $e(\neg\varphi) = 1$. Conclusively $A \wedge B \models_{\text{Ł}} \neg\varphi$, i.e. $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A \wedge B}, A \wedge B \rangle$ and the principle is justified.

(**C**.$\wedge$) If $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A \wedge B}, A \wedge B \rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$ and $A \wedge B \models_{\text{Ł}} \neg\varphi$. Therefore, whenever there is an evaluation e s.t. $e(X) = 1$ $e(\varphi) = 1$ and whenever $e(A \wedge B) = 1$ we have $e(\neg\varphi) = 1$. If $e(A \wedge B) = 1$, then $\min\{e(A), e(B)\} = 1$, i.e. both $e(A) = 1$ and $e(B) = 1$ and from this we cannot deduce that having just, for example, $e(A) = 1$ is enough to conclude $e(\neg\varphi) = 1$. Therefore the principle is not justified.

(**A**.$\vee$) If $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A \wedge B}, A \vee B \rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$ and $A \vee B \models_{\text{Ł}} \neg\varphi$. Therefore whenever there is an evaluation e s.t. $e(X) = 1$, then $e(\varphi) = 1$ and whenever $e(A \vee B) = 1$, $e(\neg\varphi) = 1$. If $e(A \vee B) = 1$, then $\max\{e(A), e(B)\} = 1$. Conclusively whenever there is an evaluation e s.t. $e(A) = 1$, $e(A \vee B) = 1$ and $e(\neg\varphi) = 1$. The same holds with B. This implies that both $A \models_{\text{Ł}} \neg\varphi$ and $B \models_{\text{Ł}} \neg\varphi$, i.e. $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A \rangle$ and $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B \rangle$ and the principle is justified.

(**C**.$\vee$) If $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A \rangle$ and $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B \rangle$, then there are φ and φ' in $Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$, $X \models_{\text{Ł}} \varphi'$, $A \models_{\text{Ł}} \neg\varphi$ and $B \models_{\text{Ł}} \neg\varphi'$. Therefore whenever there is an evaluation e s.t. $e(X) = 1$, then $e(\varphi) = 1$ and $e(\varphi') = 1$, which implies $e(\varphi \wedge \varphi') = \min\{e(\varphi), e(\varphi')\} = 1$. Whenever there is an evaluation e s.t. $e(A \vee B) = 1$, , then $\max\{e(A), e(B)\} = 1$. This implies that at least one between $e(A)$ and $e(B)$ is 1. From the hypothesis it follows either $e(\neg\varphi) = 1$ or $e(\neg\varphi') = 1$, i.e. $e(\varphi) = 0$ or $e(\varphi') = 0$. Conclusively we have that whenever there is an evaluation e s.t. $e(X) = 1$, $e(\varphi \wedge \varphi') = 1$ and whenever $e(A \wedge B) = 1$, $e(\neg(\varphi \wedge \varphi')) = 1 - e(\varphi \wedge \varphi') = 1$, i.e. $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A \vee B}, A \vee B \rangle$.

(**A**.$\supset$) If $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A \supset B}, A \supset B \rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$ and $A \supset B \models_{\text{Ł}} \neg\varphi$, i.e. whenever there is an evaluation e s.t. $e(X) = 1$, $e(\varphi) = 1$ and whenever

$e(A \supset B) = 1$, then $e(\neg\varphi) = 1$. Our first claim is that $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$. Therefore we have to show that there is $\varphi' \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi'$ and $B \models_{\text{Ł}} \neg\varphi$. Whenever there is an evaluation e s.t. $e(B) = 1$, since $e(A \supset B) = \min\{1, 1 - e(A) + e(B)\}$, also $e(A \supset B) = 1$ and from the hypothesis we have that $e(\neg\varphi) = 1$. Therefore $X \models_{\text{Ł}} \varphi$ and $B \models_{\text{Ł}} \neg\varphi$, i.e. $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$. However from the hypothesis it does not follows that $\langle \Gamma_X, X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$ and the principle is not justified.

(**C**.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$ and $B \models_{\text{Ł}} \neg\varphi$. If $\langle \Gamma_X, X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$, then for any formula $\varphi' \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi'$ $A \not\models_{\text{Ł}} \neg\varphi'$, i.e. there is at least an evaluation e^* s.t. $e^*(A) = 1$ and $e^*(\neg\varphi') = 1 - e^*(\varphi') < 1$, which implies $e^*(\varphi') > 0$. Therefore whenever $e(A \supset B) = 1$, then $\min\{1, 1 - e(A) + e(B)\} = 1$ and this happens whenever $e(B) \geq e(A)$. However for some evaluations e^*, $e^*(A) = 1$ and $e^*(\varphi) > 0$. At the same time, since we are under the assumption that $e^*(A \supset B) = 1$, we have $e^*(B) \geq e^*(A) = 1$. Therefore $e^*(B) = 1$ and from the first hypothesis $e^*(\varphi) = 0$, but this is a contradiction and the principle is not justified.

(**A**.$\neg$) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models_{\text{Ł}} \varphi$ and $A \models_{\text{Ł}} \neg\varphi$, i.e. whenever there is an evaluation e s.t. $e(X) = 1$, $e(\varphi) = 1$ and whenever $e(A) = 1$, then $e(\neg\varphi) = 1$. We should then show that there is an evaluation e^* s.t. $e^*(\neg A) = 1$ and $e^*(\neg\varphi) < 1$, but this does not follow from the hypothesis and the principle is not justified.

(**C**.$\neg$) The attack principle is not justified.

A.0.6 [I-Reb] and $\models^{\leq}_{\text{Ł}}$-Based Arguments

(**A**.$\wedge$) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models^{\leq}_{\text{Ł}} \varphi$ and $A \models^{\leq}_{\text{Ł}} \neg\varphi$. Therefore for any evaluation e $e(X) \leq e(\varphi)$ and $e(A) \leq e(\neg\varphi)$. Since $e(A \wedge B) = \min\{e(A), e(B)\}$, $e(A \wedge B) \leq e(A)$ from which it follows $e(A \wedge B) \leq e(A) \leq e(\neg\varphi)$, i.e. $A \wedge B \models^{\leq}_{\text{Ł}} \neg\varphi$ and the principle is justified.

(**C**.$\wedge$) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\wedge B}, A \wedge B\rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models^{\leq}_{\text{Ł}} \varphi$ and $A \wedge B \models^{\leq}_{\text{Ł}} \neg\varphi$, i.e. for any evaluation e $e(X) \leq e(\varphi)$ and $e(A \wedge B) \leq e(\neg\varphi)$. However, since $e(A \wedge B) = \min\{e(A), e(B)\} \leq e(A)$ and $e(A \wedge B) = \min\{e(A), e(B)\} \leq e(B)$ from the hypothesis we cannot conclude neither $A \models^{\leq}_{\text{Ł}} \neg\varphi$ or $B \models^{\leq}_{\text{Ł}} \neg\varphi$ and the principle is not justified.

(**A**.$\vee$) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\vee B}, A \vee B\rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models^{\leq}_{\text{Ł}} \varphi$ and $A \vee B \models^{\leq}_{\text{Ł}} \neg\varphi$, i.e. for every evaluation e $e(X) \leq e(\varphi)$ and $e(A \vee B) \leq e(\neg\varphi)$. Since $e(A \vee B) = \max\{e(A), e(B)\}$ we have both $e(A) \leq e(A \vee B) \leq e(\neg\varphi)$ and $e(B) \leq e(A \vee B) \leq e(\neg\varphi)$, i.e. $A \models^{\leq}_{\text{Ł}} \neg\varphi$ and $B \models^{\leq}_{\text{Ł}} \neg\varphi$ and the principle is justified.

(**C**.$\vee$) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$ and $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$, then there are φ and φ' in $Fm_{\text{Ł}}$ s.t. $X \models^{\leq}_{\text{Ł}} \varphi$, $X \models^{\leq}_{\text{Ł}} \varphi'$, $A \models^{\leq}_{\text{Ł}} \neg\varphi$ and $B \models^{\leq}_{\text{Ł}} \neg\varphi'$. Therefore for every evaluation e $e(X) \leq e(\varphi)$, $e(A) \leq e(\neg\varphi)$, $e(X) \leq e(\varphi')$ and $e(B) \leq e(\neg\varphi')$. Since $e(X) \leq e(\varphi)$ and $e(X) \leq e(\varphi')$, $e(x) \leq e(\varphi \wedge \varphi')$. In fact, whenever $e(\varphi \wedge \varphi') = \min\{e(\varphi), e(\varphi')\} = e(\varphi)$ from $e(X) \leq e(\varphi)$ we have $e(X) \leq e(\varphi \wedge \varphi')$ and the same holds whenever $e(\varphi \wedge \varphi') = e(\varphi')$. Since $e(A \vee B) = \max\{e(A), e(B)\}$, whenever $e(A \vee B) = e(A)$, $e(A \vee B) = e(A) \leq 1 - e(\varphi) \leq 1 - e(\varphi \wedge \varphi')$ and the same holds if $e(A \vee B) = e(B)$. Therefore in both cases we have that $e(A \wedge B) \leq 1 - e(\varphi \wedge \varphi')$ and the principle is justified.

(**A**.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_{A\supset B}, A \supset B\rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models^{\leq}_{\text{Ł}} \varphi$ and $A \supset B \models^{\leq}_{\text{Ł}} \neg\varphi$, i.e. for every evaluation e $e(X) \leq e(\varphi)$ and $e(A \supset B) \leq e(\neg\varphi)$. Since for every evaluation e $e(B) \leq 1 - e(A) + e(B)$ and $e(A \supset B) = \min\{1, 1 - e(A) + e(B)\}$, $e(B) \leq e(A \supset B)$ and from the hypothesis $e(B) \leq e(A \supset B) \leq e(\neg\varphi)$, i.e. $B \models^{\leq}_{\text{Ł}} \neg\varphi$ and $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$. However from the hypothesis we cannot deduce $\langle \Gamma_X, X\rangle \not\xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A\rangle$ and the principle is not justified.

(**C**.$\supset$) If $\langle \Gamma_X, X\rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_B, B\rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models^{\leq}_{\text{Ł}} \varphi$ and $B \models^{\leq}_{\text{Ł}} \neg\varphi$, i.e. for every evaluation e $e(X) \leq e(\varphi)$ and $e(B) \leq e(\neg\varphi)$. If $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]}\!\!\!\!\!\!\!\!/\;\;\;\; \langle \Gamma_A, A \rangle$, then $\varphi' \in Fm_{\text{Ł}}$ s.t. $X \models^{\leq}_{\text{Ł}} \varphi'$ and $A \not\models^{\leq}_{\text{Ł}} \neg\varphi$, i.e. for some evaluation e^* $e^*(X) \leq e^*(\varphi)$ and $e^*(A) > e^*(\neg\varphi)$. We would need to show that there is a formula $\varphi'' \in Fm_{\text{Ł}}$ s.t. $X \models^{\leq}_{\text{Ł}} \varphi''$ and $A \supset B \models^{\leq}_{\text{Ł}} \neg\varphi$, i.e. for every evaluation e $e(X) \leq e(\varphi)$ and $e(A \supset B) \leq e(\neg\varphi'')$, but this does not follows from the hypothesis and the principle is not justified.

(**A**.$\neg$) If $\langle \Gamma_X, X \rangle \xrightarrow{[I\text{-}Reb]} \langle \Gamma_A, A \rangle$, then there is a formula $\varphi \in Fm_{\text{Ł}}$ s.t. $X \models^{\leq}_{\text{Ł}} \varphi$ and $A \models^{\leq}_{\text{Ł}} \neg\varphi$, i.e. for every evaluation e $e(X) \leq e(\varphi)$ and $e(A) \leq e(\neg\varphi)$. We would need to show that for some evaluation e^*, $e^*(\neg A) > e^*(\neg\varphi)$ and the principle is not justified.

(**C**.$\neg$) The principle is not justified.

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