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A NOTE ON A NEW NONLOCAL NONLINEAR DIFFUSION EQUATION: THE ONE DIMENSIONAL CASE

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Abstract

In this paper we prove a result on the existence and uniqueness of the solution of a new feature–preserving nonlinear nonlocal diffusion equation for signal denoising [1] for the one dimensional case. The partial differential equation is based on a novel diffusivity coefficient that uses a nonlocal automatically detected parameter related to the local bounded variation and the local oscillating pattern of the noisy input signal.

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1. Introduction

Nonlinear partial differential equations (PDEs) can be used in the analysis and processing of digital images or image sequences, for example to extract features and shapes or to filter out the noise in order to produce higher quality image (see e.g. [2, 3, 11, 12] and the references herein). Perhaps, the main application of PDEs based methods in this field is smoothing and restoration of images. From the mathematical point of view, the input (gray scale) image can be modeled by a real function $u_0(x)$, $u_0 : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^d$, represents the spatial domain. Typically this domain Ω is rectangular and $d = 1, 2, 3$. The function u_0 is considered as an initial data for a suitable evolution equation with some kind of boundary conditions. The simplest (and oldest) PDE method for smoothing images is to apply a linear diffusion process: the starting point is the simple observation that the so–called Gauss function is related to the fundamental solution of the linear diffusion (heat) equation.

The flow produced by the linear diffusion equation spreads the information equally in all directions. Although this property is good for a local noise reduction in the case of the additive noise, this filtering operation also destroys the image content as the boundaries of the objects and the subregions present in the image. This means that the Gaussian smoothing not only smooths noise, but also blurs important features in the signal.

Recently, a new anisotropic diffusion model was introduced in [9] in order to analyze experimental signals in neuroscience: the diffusivity coefficient uses a nonlocal

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parameter related to the local bounded variation and the local oscillating pattern of the noisy input signal. In [1] the model was extended to the multidimensional case with an analysis for the existence of the solution in the two-dimensional case (images) and the introduction of a suitable numerical scheme. In this note we focus on the one dimensional case providing a complete analysis of the nonlocal diffusion equation, complete analysis of the corresponding equation, including the uniqueness that was an open problem.

2. A $1D$ nonlocal nonlinear model

There is a vast literature concerning nonlinear anisotropic diffusions with application to image processing, which dates back to the seminal paper by Perona and Malik, who, in [10] consider a discrete version of the following equation

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (g(|\nabla u|)\nabla u) = 0, & \text{in} \quad \Omega_T = (0,T)\times\Omega, \\ u(x,0) = u_0(x) & \text{on} \quad \Omega \\ \frac{\partial u}{\partial \vec{n}}(x,t) = 0, & \text{on} \quad \Gamma\times(0,T), \end{cases} \tag{2.1}$$

where $\Gamma = \partial\Omega$, the image domain $\Omega \subset \mathbb{R}^2$ is an open regular set (typically a rectangle), $\vec{n}$ denotes the unit outer normal to its boundary Γ, $\nabla\cdot$ is the divergence operator, and $u(x,t)$ denotes the (scalar) image analysed at time (scale) t and point x. The initial condition $u_0(x)$ is, as in the linear case, the original image. In order to reduce smoothing at edges, the diffusivity g is chosen as a decreasing function of the "edge detector" $|\nabla u|$. Instead, we introduce a nonlocal diffusive coefficient that takes into account of the "monotonicity" of the signal. In other words, a high modulus of the gradient may lead to a small diffusion if the function is also locally monotone. At the same time we want to reduce the noise present as in the case of linear diffusion. We focus on the one dimensional case, more precisely let $u : [a,b] \to \mathbb{R}$ a real function defined on a bounded interval $[a,b]$, and a subinterval $[c,d] \subset [a,b]$. We define the *local variation* $LV_{[c,d]}(u)$ of u on the interval $[c,d]$ the value

$$LV_{[c,d]}(u) = |u(d) - u(c)|.$$

We also define the *total local variation* $TV_{[c,d]}(u)$ of u on the interval $[c,d]$ as follows

$$TV_{[c,d]}(u) = \sup_{\mathcal{P}} \sum_{i=0}^{n_P-1} |u(x_{i+1}) - u(x_i)|$$

where $\mathcal{P} = \{P = \{x_0,\dots,x_{n_P}\}\,|P \text{ is a partition of } [c,d]\}$ is the set of all possible finite partition of the interval $[c,d]$.

Let $\varepsilon \in \mathbb{R}^+$, $\varepsilon \ll 1$, $\varepsilon > 0$ and let $\delta \in \mathbb{R}^+$. We define the ratio,

$$R_{\delta,u} = \frac{\varepsilon + LV_{[x-\delta,x+\delta]}(u)}{\varepsilon + TV_{[x-\delta,x+\delta]}(u)} \tag{2.2}$$

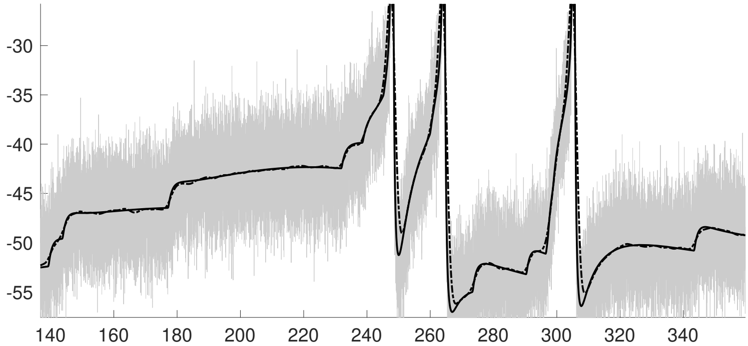

Figure 1. An example of signal denoising using the new nonlocal and nonlinear diffusion equation. Solid line: original signal; gray line: signal with noise; dotted line: reconstructed signal.

If the parameter δ is chosen appropriately we can distinguish between oscillations caused by noise contained in a range of amplitude δ. As in the Perona-Malik model (2.1), we adapt the diffusivity coefficient by using the above ratio $R_{\delta,u}$. For small values of the latter we have to reduce the noise, while for values close to 1, the upper bound of $R_{\delta,u}$, we have to preserve the signal variation (as the edges in the image). The resulting diffusivity coefficient $g(R_{\delta,u})$ becomes non-local. We assume that the $g : [0, +\infty) \to \mathbb{R}$ is a positive, nonincreasing, Lipschitz continuous function such that $g(0) = 1$, and $g(1) = \alpha > 0$. In the following we assume that the parameter $0 < \varepsilon \ll 1$ is fixed. In Figure 1 we show a simple example of a signal using our nonlocal and nonlinear diffusion filter.

In the following, $I = (a, b) \subset \mathbb{R}$ denotes a open bounded interval, $H^k(I)$, $k \in \mathbb{N}$, the Sobolev space of all function u defined in I such that u and its distributional derivatives of order $1, \ldots, k$ all belong to $L^2(I)$. Let D^s the distributional derivatives, H^k is a Hilbert space for the norm,

$$\|u\|_k = \|u\|_{H^k} = \left(\sum_{|s| \le k} \int_I |D^s u(x)|^2 \, dx \right)^{1/2}, \qquad \|u\|_0 = \|u\|_{L^2}.$$

Moreover, $L^p(0, T; H^k(I))$ is the set of all functions u, such that, for almost every t in $(0, T)$, $T > 0$, $u(t)$ belong to $H^k(I)$, $L^p(0, T; H^k(I))$ is a normed space for the norm

$$\|u\|_{L^p(0,T;H^k(I))} = \left(\int_0^T \|u\|_k^p dt \right)^{1/p}$$

$p \ge 1$ and $k \in \mathbb{N}$. Finally, we denote by $(\cdot, \cdot)$, the scalar product in $L^2(I)$.

We now establish our existence result, as initial condition we take the original signal u_0 but with some regularization obtained with a standard smoothing filter, e.g. a Gaussian filter, and we assume homogeneous Neumann condition at the boundary.

Theorem 2.1 (Existence). *Let* $u_0 \in H^1(I)$*, and* $T > 0$*,* $\delta > 0$*; then there exists* $u \in L^2(0,T;H^1(I)) \bigcap C^0([0,T];L^2(I))$*, and verifying* $u(x,0) = u_0(x)$ *on* I*,* $\frac{\partial u}{\partial x} = 0$ *at* $x = a,\ b$*, and*

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(g\left(R_{\delta,u}\right)\frac{\partial u}{\partial x}\right) = 0, \tag{2.3}$$

on $(0,T] \times I$ *in the distributional sense.*

Proof. We show the existence of a weak solution of (2.3) by a classical fixed point theorem of Schauder [6]. We introduce the space

$$V(0,T) = \left\{v \in L^2(0,T;H^1(I)),\ \frac{dv}{dt} \in L^2(0,T;(H^1(I))')\right\}. \tag{2.4}$$

The space $V(0,T)$ is a Hilbert space with the graph norm, let $v \in V(0,T) \bigcap L^\infty(0,T;L^2(I))$ such that

$$\|v\|_{L^\infty(0,T;L^2(I))} \leq \|u_0\|_{L^2(I)}. \tag{2.5}$$

We consider the problem (P_v) , here $\langle\cdot,\cdot\rangle$ represents duality product,

$$\left\langle\frac{\partial u}{\partial t}(t), w\right\rangle + \int_I g(R_{\delta,v})\frac{\partial u(t)}{\partial x}\frac{\partial w}{\partial x}dx = 0\ \forall w \in H^1(I) \text{ a.e. in } [0,T] \tag{2.6}$$

and $u(0) = u_0$.

A function $v \in H^1(I)$ has locally bounded variation [8], moreover is equal a.e. to an absolutely continuous function and v' exists a.e. and belongs to $L^2(I)$. Then the term $R_{\delta,v}$ can be stated as

$$R_{\delta,v} = \frac{\varepsilon + \left|\int_{x-\delta}^{x+\delta} u'(s)ds\right|}{\varepsilon + \int_{x-\delta}^{x+\delta} |u'(s)|\, ds} \tag{2.7}$$

and $0 < R_{\delta,v} \leq 1$. So, $g(R_{\delta,v}) \geq \alpha > 0$.

Using classical results about the parabolic equations [6, 7], the problem (P_v) has a unique solution $U(v)$ in $V(0,T)$. Then we can deduce the following estimates

$$\begin{aligned} &\|U(v)\|_{L^\infty(0,T;L^2(I))} \leq \|u_0\|_{L^2(I)},\\ &\|U(v)\|_{L^2(0,T;H^1(I))} \leq C_1,\\ &\|U(v)\|_{L^2(0,T;(H^1(I))')} \leq C_2, \end{aligned} \tag{2.8}$$

for suitable constants C_1, and C_2. Then, we introduce the subset V_0 of $V(0,T)$ defined by function $v \in V(0,T)$, such that the estimates are verified, and $v(0) = u_0$. The mapping U is a mapping from V_0 to V_0. Moreover, V_0 is a nonempty, convex, and weakly compact subset of $V(0,T)$.

In order to use the Schauder theorem, we have to prove that the mapping $v \to U(v)$ is weakly continuous from V_0 to V_0. Then, since $V(0,T)$ is contained in $L^2(0,T;L^2(I))$, with compact inclusion, this provide $u \in V_0$ such that $u = U(u)$.

Let (v_j) be a sequence in V_0 which converges weakly to $v \in V_0$ and $u_j = U(v_j)$. From the classical theorems of compact inclusion [6, 7] up to sub-sequences,

$$u_j \to u \text{ weakly in } L^2(0,T;H^1(I)),$$

$$\frac{du_j}{dt} \to \frac{du}{dt} \text{ weakly in } L^2(0,T;(H^1(I))'),$$

$$\frac{\partial u_j}{\partial x} \to \frac{\partial u}{\partial x} \text{ weakly in } L^2(0,T;L^2(I)).$$

Moreover, we have that $u_j \to u$ in $L^2(0,T;L^2(I))$ and a.e. on $I \times (0,T)$, and $u_j(0) \to u(0)$ in $(H^1(I))'$. For the (v_j), from (2.8), there is a sub-sequence such that $v_j \to v$ in $L^2(0,T;L^2(I))$, and, from the Rellich-Kodrachov Theorem [7] and the (2.7), $g(R_{\delta,v_j}) \to g(R_{\delta,v})$ in $L^2(0,T;L^2(I))$. By the uniqueness of the solution of (P_v), the whole sequence $u_j = U(v_j)$ converges weakly in $V(0,T)$. Then, the mapping U is weakly continuous from V_0 into V_0, so we apply the Schauder theorem. □

Remark A similar proof could be applied by considering a different measure of local variation. For example by considering the absolute value of the difference between maximum and minimum value in the sub-intervals of length 2δ.
If the same hypothesis of the Theorem 2.1 hold, we have the following uniqueness result.

Theorem 2.2 (Uniqueness). *The solution $u \in L^2(0,T;H^1(I)) \bigcap C^0([0,T];L^2(I))$ of the equation (2.3), with $u(0) \in H^1(I)$, and homogeneous Neumann conditions, is unique.*

Proof. Let $\bar{u}$ and $\hat{u}$ be two solutions of (2.3), then for a.e. t in $[0,T]$,

$$\frac{\bar{u}}{dt} - \frac{\partial}{\partial x}\left(g\left(R_{\delta,\bar{u}}\right)\frac{\partial \bar{u}}{\partial x}\right) = 0, \quad \bar{u}(0) = u_0; \tag{2.9}$$

$$\frac{\hat{u}}{dt} - \frac{\partial}{\partial x}\left(g\left(R_{\delta,\hat{u}}\right)\frac{\partial \hat{u}}{\partial x}\right) = 0, \quad \hat{u}(0) = u_0. \tag{2.10}$$

By subtracting (2.10) from (2.9),

$$\frac{d(\bar{u}-\hat{u})}{dt} - \frac{\partial}{\partial x}\left(g\left(R_{\delta,\bar{u}}\right)\frac{\partial \bar{u}}{\partial x}\right) + \frac{\partial}{\partial x}\left(g\left(R_{\delta,\hat{u}}\right)\frac{\partial \hat{u}}{\partial x}\right) = 0.$$

Adding and subtracting the quantity $\partial_x\left(g\left(R_{\delta,\bar{u}}\right)\partial_x \hat{u}\right)$, let $u = \bar{u} - \hat{u}$, we can rewrite the equation as

$$\frac{du}{dt} - \frac{\partial}{\partial x}\left(g\left(R_{\delta,\bar{u}}\right)\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\left[g\left(R_{\delta,\bar{u}}\right) - g\left(R_{\delta,\hat{u}}\right)\right]\frac{\partial \hat{u}}{\partial x}\right). \tag{2.11}$$

Multiplying the equation (2.11) by $u = (\bar{u} - \hat{u})$, integrating on the interval I, using the properties of the function g and the lower bound $g(1) = \alpha > 0$, the estimates (2.8), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(I)} + \alpha\left\|\frac{\partial}{\partial x}u(t)\right\|^2_{L^2(I)} \le C\|u(t)\|_{L^2(I)}\left\|\frac{\partial}{\partial x}\hat{u}(t)\right\|_{L^2(I)}\left\|\frac{\partial}{\partial x}u(t)\right\|_{L^2(I)}, \tag{2.12}$$

for a suitable constant C. The term on the right hand side can be written as

$$\frac{2}{\alpha}C^2\|u(t)\|^2_{L^2(I)}\|\frac{\partial}{\partial x}\hat{u}(t)\|^2_{L^2(I)} + \frac{\alpha}{2}\|\frac{\partial}{\partial x}u(t)\|^2_{L^2(I)},$$

then

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(I)} + \frac{\alpha}{2}\left\|\frac{\partial}{\partial x}u(t)\right\|^2_{L^2(I)} \leq C^*\,\|u(t)\|^2_{L^2(I)}\,. \tag{2.13}$$

Since $\bar{u}(0) = \hat{u}(0) = u_0$, by the inequality (2.13) and the Gronwall lemma [6] we obtain the uniqueness of the solution. □

Remark Similar nonlocal equation could be obtain as diffusive limit from different kinetic microscale description of the interactions of active particles (see e.g. [4, 5]).

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