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Functional calculus on non-homogeneous operators on nilpotent groups

Mattia Calzi[1] · **Fulvio Ricci**[1]

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Abstract

We study the functional calculus associated with a hypoelliptic left-invariant differential operator $\mathcal{L}$ on a connected and simply connected nilpotent Lie group G with the aid of the corresponding *Rockland* operator $\mathcal{L}_0$ on the 'local' contraction G_0 of G, as well as of the corresponding Rockland operator $\mathcal{L}_\infty$ on the 'global' contraction G_∞ of G. We provide asymptotic estimates of the Riesz potentials associated with $\mathcal{L}$ at 0 and at ∞, as well as of the kernels associated with functions of $\mathcal{L}$ satisfying Mihlin conditions of every order. We also prove some Mihlin–Hörmander multiplier theorems for $\mathcal{L}$ which generalize analogous results to the non-homogeneous case. Finally, we extend the asymptotic study of the density of the 'Plancherel measure' associated with $\mathcal{L}$ from the case of a quasi-homogeneous sub-Laplacian to the case of a quasi-homogeneous sum of even powers.

Keywords Nilpotent Lie groups · Hypoelliptic differential operators · Multiplier theorem · Heat kernel · Riesz potentials

Mathematics Subject Classification 43A22 · 22E30

1 Introduction

This paper deals with functional calculus on non-homogeneous left-invariant hypoelliptic self-adjoint differential operators on nilpotent Lie groups.

Functional calculus on self-adjoint Rockland operators (i.e., left-invariant, hypoelliptic and homogeneous) has been widely studied in the literature, in particular on sub-Laplacians (cf., for instance, [9,10,15,23–28]), but also in greater generality (cf., for instance, [7,8,16,17,

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✉ Fulvio Ricci
fulvio.ricci@sns.it

Mattia Calzi
mattia.calzi@sns.it

1 Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126 Pisa, Italy

20,22]). Also functional calculus on non-homogeneous sub-Laplacians has been considered (cf., for instance, [2,20,22,34]).

The approach introduced in [30] indicates that it is possible to transfer information on operators that are functions of a (positive) Rockland operator $\widetilde{\mathcal{L}}$ on a connected and simply connected graded group G, or on its convolution kernel, to analogous information relative to the projection of $\widetilde{\mathcal{L}}$ on a general connected and simply connected, but not necessarily homogeneous, quotient group.

Let $G = \widetilde{G}/I$ be the quotient group, where we assume that I is not dilation invariant to avoid trivialities. The one-parameter family of isomorphic quotient groups $G_s = \widetilde{G}/I_s$, where I_s is I dilated by $s \in \mathbb{R}_+$, admits two limits $G_0 = \widetilde{G}/I_0$ and $G_\infty = \widetilde{G}/I_\infty$ (no longer isomorphic to G), where I_0 and I_∞ are dilation invariant, so that G_0 and G_∞ admit induced gradations from $\widetilde{G}$.

Correspondingly, the operator $\widetilde{\mathcal{L}}$ induces a family $(\mathcal{L}_s)_{s\in[0,+\infty]}$, of projected operators on the different quotients. The limit operators $\mathcal{L}_0$, $\mathcal{L}_\infty$ are Rockland, while the other $\mathcal{L}_s$ lack homogeneity, remaining, however, hypoelliptic. More precisely, they are *weighted subcoercive*, according to the definition introduced in [37].[1]

The starting point in the analysis of [30] is a weighted generating family $X_1, \dots, X_n$ of the Lie algebra $\mathfrak{g}$ of G. The (Lie algebra of the) group G is then interpreted as the quotient of the free nilpotent Lie algebra $\mathcal{F}$ of sufficiently high step with generators $\widetilde{X}_1, \dots, \widetilde{X}_n$; the Lie algebra $\mathcal{F}$ is then endowed with the (unique) gradation obtained assigning to each $\widetilde{X}_j$ a degree equal to the weight of X_j. Thus, in the above notation, $\mathcal{F}$ is the Lie algebra of $\widetilde{G}$ and the quotient map is uniquely determined by the condition that each $\widetilde{X}_j$ is mapped onto X_j. A non-commutative homogeneous polynomial $\mathcal{P}$ in n indeterminates (endowed with the same weights of $X_1, \dots, X_n$) is then considered under the assumption that the operator $\widetilde{\mathcal{L}} = \mathcal{P}(\widetilde{X}_1, \dots \widetilde{X}_n)$ is hypoelliptic (hence Rockland). In particular, also the operator $\mathcal{L} = \mathcal{P}(X_1, \dots, X_n)$ is hypoelliptic; examples of such operators are the sums of even powers of generating vector fields.

It was proved in [30] that there is a fundamental solution K of $\mathcal{L}$ satisfying the asymptotic relations[2]

$$K(x) \sim P(x) + K_0(x) \quad \text{as } x \to 0\,, \qquad K(x) \sim K_\infty(x) \quad \text{as } x \to \infty\,,$$

where K_0 and K_∞ are fundamental solutions of $\mathcal{L}_0$ on G_0 and of $\mathcal{L}_\infty$ on G_∞, respectively, while P is a suitable polynomial on G_0.

The results of the present paper can be divided into four parts. The first part concerns the heat kernels associated with the operator $\mathcal{L}$, i.e., the kernels of the operators $\mathrm{e}^{-t\mathcal{L}}$. In Sect. 2, we recall the basic constructions of [30] and then we introduce a (somewhat redundant) family of left-invariant vector fields $X_{s,j}$ on each group G_s, $s \in [0, \infty]$, which behaves nicely under dilation (which can no longer be defined as automorphisms of the group G_s, but rather as isomorphisms between different G_s). We then introduce two moduli $|\cdot|_s$ and $|\cdot|_{s,*}$ on each G_s: the former behaves nicely under dilation and equals a homogeneous norm on G_0 near the identity e and a homogeneous norm on G_∞ near ∞, under suitable identifications; the latter, inspired by [20,22], is a compromise between the modulus $|\cdot|_s$ and the Riemannian distance

[1] Functional calculus on weighted subcoercive operators (or systems of operators) has been developed in [20–22]. In these works, the homogeneous limit G_0 mentioned above is used, at least for comparison with the homogeneous setting by a contraction argument.

[2] These formulas assume identifications, as manifolds, of G with G_0 and G_∞, respectively. This will be explained in the next section. More precisely, it is proved in [30] that $K(x)$ admits two infinite asymptotic expansions at 0 and ∞, with terms which are homogeneous of increasing and decreasing orders, respectively, relative to the dilations of the corresponding limit group.

from e associated with the vectors $X_{s,j}$. The importance of $|\cdot|_{s,*}$ lies in the fact that it grows much faster than $|\cdot|_s$ at ∞, in general, so that it leads to better multiplier theorems. In Sect. 3, we then make use of the vector fields $X_{s,j}$ and the moduli $|\cdot|_{s,*}$ to prove uniform 'Gaussian' estimates for the kernels $h_{s,t}$ of the $\mathrm{e}^{-t\mathcal{L}_s}$ (Theorem 3.1); we also consider estimates of the derivatives in s of the $h_{s,t}$, appropriately defined.

In the second part (Sect. 4), we extend the asymptotic estimates in [30] to general complex powers of $\mathcal{L}$ (Theorem 4.4), defined by analytic continuation in the same fashion of the Euclidean case. Even though it would be possible to use the same techniques employed in [30], we shall rely as much as possible on the estimates on $h_{s,t}$ provided in Theorem 3.1; in this way, we are able to describe more precisely also the higher-order terms of the obtained developments, in some specific situations (Theorem 4.7).

In the third part (Sect. 5) we give asymptotic estimates to kernels of more general multiplier operators (Theorem 5.12) and prove some multiplier theorems of Mihlin–Hörmander type (Theorems 5.15, 5.17). For what concerns the asymptotic estimates, here we consider more general functions of the operator $\mathcal{L}$—namely, functions satisfying Mihlin conditions of every order up to the multiplication by a fractional power. Even though these functions include the complex powers of $\mathcal{L}$, Theorem 3.1 is not completely contained in Theorem 5.12, since several terms of the developments obtained in the latter are only defined up to polynomials. We then pass to some multiplier theorems, which are generalization of some of the results presented in [22] to the non-homogeneous case. While Theorem 5.15 is stated in full generality and gives non-homogeneous Mihlin–Hörmander conditions on the multipliers in the fashion of [2,34], Theorem 5.17 makes use, in a quite more specific situation, of the techniques introduced in [15,16] and then systematically developed in [20,22] to lower the regularity threshold up to half the topological dimension of G (instead of half the growth of the volume of G as in Theorem 5.15). Optimality is achieved when G is a product of Métivier and abelian groups, and $\mathcal{L}$ is (any) hypoelliptic sub-Laplacian thereon.

The fourth part (Sect. 6) deals with the spectral Plancherel measure $\beta_{\mathcal{L}}$ and its comparison with $\beta_{\mathcal{L}_0}$ and $\beta_{\mathcal{L}_\infty}$ (Theorem 6.4), when $\mathcal{L}$ is 'quasi-homogeneous', following [34]. Here, we both extend the results of [34] to sums of even powers of generating homogeneous vector fields (instead of quasi-homogeneous sub-Laplacians), and we also observe that the estimates on the density of $\beta_{\mathcal{L}}$ with respect to the Lebesgue measure on $\mathbb{R}_+$ automatically improve to asymptotic expansions at 0 and at ∞.

2 General setting

In this section, we shall present the general framework in which we shall work in the sequel. It is basically the same as that of [30], except for the fact that we shall not require that the graded group $\widetilde{G}$ be a free nilpotent Lie group. We shall briefly repeat the basic constructions for the ease of the reader.

2.1 Contractions

Let $\widetilde{G}$ be a graded, connected, and simply connected Lie group with Lie algebra $\widetilde{\mathfrak{g}}$, with gradation $(\widetilde{\mathfrak{g}}_j)$; let pr_j be the projection of $\widetilde{\mathfrak{g}}$ onto $\widetilde{\mathfrak{g}}_j$ with kernel $\bigoplus_{j'\neq j}\widetilde{\mathfrak{g}}_{j'}$, and define $n := \max\left\{ j > 0 \colon \widetilde{\mathfrak{g}}_j \neq 0 \right\}$.

On $\widetilde{G}$ we introduce the dilations $x \mapsto r\cdot x$, $r \in \mathbb{R}_+ = (0,\infty)$, adapted to the given gradation, i.e., such that $r\cdot x = r^j x$ if $x \in \widetilde{\mathfrak{g}}_j$. We shall sometimes denote by ρ_r the dilation by r. A linear subspace $\mathfrak{v}$ of $\widetilde{\mathfrak{g}}$ is graded, i.e., $\mathfrak{v} = \bigoplus_j \mathfrak{v} \cap \widetilde{\mathfrak{g}}_j$, if and only if it is homogeneous, i.e., invariant under dilations. We say that a linear map from a graded subspace $\mathfrak{v}$ of $\widetilde{\mathfrak{g}}$ to $\widetilde{\mathfrak{g}}$ is

- *homogeneous* if it maps $\mathfrak{v} \cap \widetilde{\mathfrak{g}}_j$ into $\widetilde{\mathfrak{g}}_j$ for every j;
- *strictly subhomogeneous* if it maps $\mathfrak{v} \cap \widetilde{\mathfrak{g}}_j$ into $\bigoplus_{j'<j} \widetilde{\mathfrak{g}}_{j'}$ for every j;
- *strictly super-homogeneous* if it maps $\mathfrak{v} \cap \widetilde{\mathfrak{g}}_j$ into $\bigoplus_{j'>j} \widetilde{\mathfrak{g}}_{j'}$ for every j.

Now, let G be the quotient of $\widetilde{G}$ by a (not necessarily homogeneous) normal subgroup, and denote by π the corresponding projection; we shall assume that G is simply connected. Let $\mathfrak{i}$ be the kernel of $\mathrm{d}\pi$, and observe that $\ker \pi = \exp_{\widetilde{G}} \mathfrak{i}$ since G is simply connected. Then, define

$$\mathfrak{i}_0 := \bigoplus_{j=1}^{n} \mathrm{pr}_j \left(\mathfrak{i} \cap \left(\bigoplus_{j' \leq j} \widetilde{\mathfrak{g}}_{j'} \right) \right), \qquad \text{and} \qquad \mathfrak{i}_\infty := \bigoplus_{j=1}^{n} \mathrm{pr}_j \left(\mathfrak{i} \cap \left(\bigoplus_{j' \geq j} \widetilde{\mathfrak{g}}_{j'} \right) \right).$$

For $s \in (0, \infty)$, we define $\mathfrak{i}_s := s^{-1} \cdot \mathfrak{i}$.

The following result is basically a generalization of [30, Proposition 2, Lemma, and Corollary of § 2].

Proposition 2.1 *The vector spaces $\mathfrak{i}_0$ and $\mathfrak{i}_\infty$ are graded ideals of $\widetilde{\mathfrak{g}}$ and have the same dimension as $\mathfrak{i}$. In addition, there are two linear mappings $\psi_{0,1} \colon \mathfrak{i}_0 \to \widetilde{\mathfrak{g}}$ and $\psi_{\infty,1} \colon \mathfrak{i}_\infty \to \widetilde{\mathfrak{g}}$ such that*

- *$\psi_{0,1}$ is strictly subhomogeneous and $I + \psi_{0,1}$ is a bijection of $\mathfrak{i}_0$ onto $\mathfrak{i}$;*
- *$\psi_{\infty,1}$ is strictly super-homogeneous and $I + \psi_{\infty,1}$ is a bijection of $\mathfrak{i}_\infty$ onto $\mathfrak{i}$;*
- *defining, for $s \in \mathbb{R}_+$, $\psi_{0,s}$ and $\psi_{\infty,s}$ as $\psi_{0,s} = s^{-1} \cdot \psi_{0,1}(s\,\cdot)$ and $\psi_{\infty,s} = s^{-1} \cdot \psi_{\infty,1}(s\,\cdot)$, respectively, these maps are strictly sub- (resp. super-)homogeneous and*

$$\lim_{s \to 0^+} \psi_{0,s} = 0, \qquad \lim_{s \to \infty} \psi_{\infty,s} = 0\,;$$

- *if $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$ are graded complements of $\mathfrak{i}_0$ and $\mathfrak{i}_\infty$ in $\widetilde{\mathfrak{g}}$, respectively, then they are also algebraic complements of $\mathfrak{i}_s$ for every $s \in \mathbb{R}_+$.*

Proof It is clear that $\mathfrak{i}_0$ is a graded subspace of $\widetilde{\mathfrak{g}}$; let $(\mathfrak{i}_{0,j} = \mathfrak{i}_0 \cap \widetilde{\mathfrak{g}}_j)$ be its gradation. Take $x \in \widetilde{\mathfrak{g}}_{j_1}$ for some j_1 and $y \in \mathfrak{i}_{0,j_2}$ for some j_2; let us prove that $[x, y] \in \mathfrak{i}_{0,j_1+j_2}$. Now, there is $y' \in \mathfrak{i}$ such that $y - y' \in \bigoplus_{j'<j_2} \widetilde{\mathfrak{g}}_{j'}$, so that $[x, y] \in [x, y'] + \left(\bigoplus_{j'<j_2} \widetilde{\mathfrak{g}}_{j_1+j'} \right)$, whence $[x, y] = \mathrm{pr}_{j_1+j_2}([x, y']) \in \mathfrak{i}_{0,j_1+j_2}$. By the arbitrariness of x and y, it follows that $\mathfrak{i}_0$ is a graded ideal. In the same way, one proves that $\mathfrak{i}_\infty$ is a graded ideal.

Now, let us define $\psi_{0,1}$. Observe that, by induction, we may define a basis (e_k) of $\mathfrak{i}$ and an increasing sequence (k_j) such that $(e_k)_{k \leq k_j}$ is a basis of $\mathfrak{i} \cap \left(\bigoplus_{j' \leq j} \widetilde{\mathfrak{g}}_{j'} \right)$ for every j. Let us prove that, for every j, $(\mathrm{pr}_j(e_k))_{k_{j-1}<k\leq k_j}$ is a basis of $\mathfrak{i}_{0,j}$. Clearly, it will suffice to prove linear independence. Now, if $(\lambda_k)_{k_{j-1}<k\leq k_j}$ is a family of real numbers such that $\sum_k \lambda_k \,\mathrm{pr}_j(e_k) = 0$, then $\sum_{k_{j-1}<k\leq k_j} \lambda_k e_k \in \mathfrak{i} \cap \left(\bigoplus_{j'<j} \widetilde{\mathfrak{g}}_{j'} \right)$. Hence, there is a family $(\lambda_k)_{k \leq k_{j-1}}$ of real numbers such that

$$\sum_{k_{j-1}<k\leq k_j} \lambda_k e_k = \sum_{k \leq k_{j-1}} \lambda_k e_k,$$

whence $\lambda_k = 0$ for every $k = 1, \dots, k_j$. Then, we may simply define $\psi_{0,1}$ as the linear map such that

$$\psi_{0,1}\big(\mathrm{pr}_j(e_k)\big) = e_k - \mathrm{pr}_j(e_k) = \sum_{j'<j} \mathrm{pr}_{j'}(e_k),$$

for every $j = 1, \dots, n$ and for every $k = k_{j-1} + 1, \dots, k_j$. Then $\psi_{0,1}$ is strictly subhomogeneous and

$$(I + \psi_{0,1})\big(\operatorname{pr}_j(e_k)\big) = e_k\,,$$

showing that $I + \psi_{0,1}$ maps $\mathfrak{i}_0$ onto $\mathfrak{i}$ bijectively. It is also clear that

$$\psi_{0,s}(\operatorname{pr}_j(e_k)) = \sum_{j'<j} s^{j-j'} \operatorname{pr}_{j'}(e_k), \tag{1}$$

which tends to 0 as $s \to 0^+$.

In a similar way, one constructs $\psi_{\infty,1}$ and proves the corresponding properties. In particular, we see that $\mathfrak{i}$, $\mathfrak{i}_0$, and $\mathfrak{i}_\infty$ have the same dimension.

Let $\mathfrak{h}_0$ be a graded complement of $\mathfrak{i}_0$ in $\widetilde{\mathfrak{g}}$. Since the mapping $s \mapsto \mathfrak{i}_s$ is continuous on $[0, \infty]$ (with values in the Grassmannian of $(\dim \mathfrak{i})$-dimensional subspaces of $\widetilde{\mathfrak{g}}$), it follows that $\mathfrak{h}_0$ is an algebraic complement of $\mathfrak{i}_r$ for some $r > 0$. Therefore, $\mathfrak{h}_0 = (s^{-1}r) \cdot \mathfrak{h}_0$ is an algebraic complement of $\mathfrak{i}_s = (s^{-1}r) \cdot \mathfrak{i}_r$ for every $s \in (0, \infty)$.

The assertions concerning $\mathfrak{h}_\infty$ are proved in a similar way. □

Observe that, by (1) and its analogue for $\psi_{\infty,s}$, the linear mappings $\psi_{0,s}$ and $\psi_{\infty,1/s}$ depend polynomially on s.

For $s \in [0, \infty]$, consider the quotient Lie algebras $\mathfrak{g}_s = \widetilde{\mathfrak{g}}/\mathfrak{i}_s$. Dilation of $\widetilde{\mathfrak{g}}$ by $r > 0$ induces an isomorphism between $\mathfrak{g}_s$ and $\mathfrak{g}_{r^{-1}s}$; in particular, $\mathfrak{g}_s$ is isomorphic to $\mathfrak{g}_1$ for every $s \in (0, \infty)$, while $\mathfrak{g}_0$ and $\mathfrak{g}_\infty$ need not be isomorphic with any other $\mathfrak{g}_s$. We call $\mathfrak{g}_0$ and $\mathfrak{g}_\infty$ the local and the global contractions of $\mathfrak{g}_1$, respectively.

We fix once and for all two *graded* algebraic complements $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$ of $\mathfrak{i}_0$ and $\mathfrak{i}_\infty$, respectively. By Proposition 2.1, both $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$ are complementary to $\mathfrak{i}_s$ for all $s \in \mathbb{R}_+$.

Definition 2.2 For $s \in [0, \infty)$, let $P_{0,s}$ be the projection of $\widetilde{\mathfrak{g}}$ onto $\mathfrak{h}_0$ with kernel $\mathfrak{i}_s$ and, for $s \in (0, \infty]$, let $P_{\infty,s}$ be the projection of $\widetilde{\mathfrak{g}}$ onto $\mathfrak{h}_\infty$ with kernel $\mathfrak{i}_s$.

Lemma 2.3 *$P_{0,0}$ is homogeneous and, for every $s \in \mathbb{R}_+$, $P_{0,s} - P_{0,0}$ is strictly subhomogeneous; in addition, $P_{0,rs} = r^{-1} \cdot P_{0,s}(r\,\cdot\,)$ for every $r, s \in \mathbb{R}_+$.*

Analogously, $P_{\infty,\infty}$ is homogeneous and, for every $s \in \mathbb{R}_+$, $P_{\infty,s} - P_{\infty,\infty}$ is strictly super-homogeneous; in addition, $P_{\infty,rs} = r^{-1} \cdot P_{\infty,s}(r\,\cdot\,)$ for every $r, s \in \mathbb{R}_+$.

Proof The homogeneity of $P_{0,0}$ and $P_{\infty,\infty}$ is obvious, as well as the scaling properties of the projections. Thus, we may reduce ourselves to proving sub- (resp. super-)homogeneity of $P_{0,1}$ and $P_{\infty,1}$. Then, take $x \in \widetilde{\mathfrak{g}}_k$ for some k, and let us prove that $\operatorname{pr}_h(P_{0,1}(x) - P_{0,0}(x)) = 0$ for $h \geq k$. Indeed, assume that $\operatorname{pr}_h(P_{0,1}(x)) \neq 0$ for some $h \geq k$, and let h' be the maximum of such h. Then, $\operatorname{pr}_h(x - P_{0,1}(x)) = \operatorname{pr}_h(P_{0,1}(x)) = 0$ for every $h > h'$, so that $\operatorname{pr}_{h'}(x - P_{0,1}(x)) \in \mathfrak{i}_0$ since $x - P_{0,1}(x) \in \mathfrak{i}$ by the definition of $P_{0,1}$. Since $\operatorname{pr}_{h'}(P_{0,1}(x)) \in \mathfrak{h}_0$ and $\mathfrak{h}_0 \cap \mathfrak{i}_0 = \{\,0\,\}$, we then deduce that $h' = k$ and that $x - \operatorname{pr}_k(P_{0,1}(x)) \in \mathfrak{i}_0$, so that $\operatorname{pr}_k(P_{0,1}(x)) = P_{0,0}(x)$.

One proves analogously that $P_{\infty,1} - P_{\infty,\infty}$ is strictly super-homogeneous. □

Each map $P_{0,s}$ (resp. $P_{\infty,s}$) induces a Lie algebra structure on $\mathfrak{h}_0$ (resp. $\mathfrak{h}_\infty$); we denote by $[\,\cdot\,,\,\cdot\,]_{0,s}$ (resp. $[\,\cdot\,,\,\cdot\,]_{\infty,s}$) the corresponding Lie bracket. In other words,

$$[x, y]_{0,s} = P_{0,s}[x, y], \quad \forall\, x, y \in \mathfrak{h}_0, \qquad [x, y]_{\infty,s} = P_{\infty,s}[x, y], \quad \forall\, x, y \in \mathfrak{h}_\infty. \tag{2}$$

Notice that, for $r, s \in \mathbb{R}_+$,

$$r \cdot [x, y]_{0,s} = [r \cdot x, r \cdot y]_{0,r^{-1}s}, \forall x, y \in \mathfrak{h}_0, \quad r \cdot [x, y]_{\infty,s} = [r \cdot x, r \cdot y]_{\infty,r^{-1}s}, \forall x, y \in \mathfrak{h}_\infty. \tag{3}$$

We use the Baker–Campbell–Hausdorff products induced by the Lie brackets in (2) to realize either $\mathfrak{h}_0$, if $s \in [0, \infty)$, or $\mathfrak{h}_\infty$, if $s \in (0, \infty]$, as the underlying manifold[3] of the group $G_s := \widetilde{G}/\exp_{\widetilde{G}} \mathfrak{i}_s$. We call G_0 and G_∞ the local and the global contractions of G_1, respectively.

Notice that

$$\begin{aligned} [x, y]_{0,s} &= [x, y]_{0,0} + O(s), \ s \to 0, \ \text{uniformly for } x, y \text{ bounded in } \mathfrak{h}_0, \\ [x, y]_{\infty,s} &= [x, y]_{\infty,\infty} + O(1/s), \ s \to \infty, \ \text{uniformly for } x, y \text{ bounded in } \mathfrak{h}_\infty, \end{aligned} \tag{4}$$

and the analogous formulae for products in G_s.

We denote by π_s, $s \in [0, \infty]$, the canonical projection of $\widetilde{G}$ onto G_s and by $\mathrm{d}\pi_s : \widetilde{\mathfrak{g}} \to \mathfrak{g}_s$ its differential. By an abuse of language, we shall keep the same notation whenever G_s, or $\mathfrak{g}_s$, is identified with either $\mathfrak{h}_0$ or $\mathfrak{h}_\infty$.

Since the ideals $\mathfrak{i}_0$ and $\mathfrak{i}_\infty$ are graded, the corresponding quotients $\mathfrak{g}_0$ and $\mathfrak{g}_\infty$ inherit a gradation and the corresponding dilations. These dilations coincide with the restriction to $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$, respectively, of the dilations of $\widetilde{\mathfrak{g}}$.

For $s, r \in \mathbb{R}_+$, it follows from (3) that dilation by r on either $\mathfrak{h}_0$ or $\mathfrak{h}_\infty$ induces an isomorphism of G_s onto $G_{r^{-1}s}$ for every s. Notice that the so-induced mappings $G_s \to G_{r^{-1}s}$ do not depend on the chosen identifications (cf. *(ii)* and *(iv)* of Proposition 2.4).

We denote by $\widetilde{Q}$, Q_0, and Q_∞ the homogeneous dimensions of $\widetilde{G}$, G_0, and G_∞, respectively.

The following result generalizes [30, Proposition 3 and p. 264].

Proposition 2.4 *For every $s \in \mathbb{R}_+$, define $\lambda_s := (P_{\infty,s})_{|\mathfrak{h}_0} : \mathfrak{h}_0 \to \mathfrak{h}_\infty$; let N be a homogeneous norm on $\widetilde{\mathfrak{g}}$. Then,*

(i) *λ_s is the unique linear mapping such that $x - \lambda_s(x) \in \mathfrak{i}_s$ for every $x \in \mathfrak{h}_0$; in addition, λ_s is invertible and its inverse $\lambda_s^{-1} = (P_{0,s})_{|\mathfrak{h}_\infty}$ is the unique linear mapping such that $x - \lambda_s^{-1}(x) \in \mathfrak{i}_s$ for every $x \in \mathfrak{h}_\infty$;*

(ii) *λ_s intertwines the two identifications of $\mathfrak{g}_s$ with $\mathfrak{h}_0$, resp. $\mathfrak{h}_\infty$, i.e.,*

$$\lambda_s[x, y]_{0,s} = [\lambda_s x, \lambda_s y]_{\infty,s}$$

for all $x, y \in \mathfrak{h}_0$;

(iii) *$\lambda_s - I$ is strictly super-homogeneous and $\lambda_s^{-1} - I$ is strictly subhomogeneous;*

(iv) *$\lambda_{rs} = r^{-1} \cdot \lambda_s(r\,\cdot)$ for every $r > 0$.*

(v) *$Q_\infty \geq Q_0$;*

(vi) *$N(\lambda_s(x)) = O(N(x))$ for $x \to \infty$ in $\mathfrak{h}_0$ and $N(\lambda_s^{-1}(x)) = O(N(x))$ for $x \to 0$ in $\mathfrak{h}_\infty$.*

Proof By the definition of $P_{0,s}$, $P_{\infty,s}$, the two cosets $x + \mathfrak{i}_s$ and $\lambda_s(x) + \mathfrak{i}_s$ coincide for $x \in \mathfrak{h}_0$. This gives (i) and (ii); (iii) and (iv) follow directly from Lemma 2.3.

[3] In principle, we shall privilege the realization of G_s on $\mathfrak{h}_0$ for s close to 0 and that on $\mathfrak{h}_\infty$ for s close to ∞. We prefer anyhow to keep the double realization for every $s \in \mathbb{R}_+$ in order to avoid apparent discontinuities in s at some finite point, on the one hand, and a priori quantifications of 'closeness' to 0 or ∞, on the other.

To prove (v), we argue as in the proof of [30, Proposition 3]. Define $\mathfrak{h}_{0,j} := \widetilde{\mathfrak{g}}_j \cap \mathfrak{h}_0$ and $\mathfrak{h}_{\infty,j} := \widetilde{\mathfrak{g}}_j \cap \mathfrak{h}_\infty$, and set $s = 1$. Since λ_1 is super-homogeneous, we see that, for every $k = 1, \dots, n$,

$$\bigoplus_{j<k} \lambda_1(\mathfrak{h}_{0,j}) + \bigoplus_{j \geq k} \mathfrak{h}_{\infty,j} = \mathfrak{h}_\infty,$$

so that

$$\sum_{j<k} \dim(\mathfrak{h}_{0,j}) + \sum_{j \geq k} \dim(\mathfrak{h}_{\infty,j}) \geq \dim(\mathfrak{h}_\infty).$$

Summing up all these inequalities, we see that

$$n \dim(\mathfrak{h}_0) - Q_0 + Q_\infty \geq n \dim(\mathfrak{h}_\infty),$$

whence $Q_\infty \geq Q_0$.

For what concerns (vi), fix a norm $\|\cdot\|$ on $\widetilde{\mathfrak{g}}$ and observe that there is a constant $C \geq 1$ such that

$$\frac{1}{C} \max_k \|\mathrm{pr}_k(x)\|^{1/k} \leq N(x) \leq C \max_k \|\mathrm{pr}_k(x)\|^{1/k}$$

for every $x \in \widetilde{\mathfrak{g}}$. Further, by [10, Proposition 1.6] we see that there is a constant $C' > 0$ such that, for $x \in \mathfrak{h}_0$,

$$\begin{aligned} N(\lambda_s(x)) &\leq C' \sum_k N(\lambda_s(\mathrm{pr}_k(x))) \\ &\leq C C' \sum_k \max\left\{\|\lambda_s(\mathrm{pr}_k(x))\|^{1/k}, \|\lambda_s(\mathrm{pr}_k(x))\|^{1/n}\right\}. \end{aligned}$$

Therefore, there is a constant $C'' > 0$ such that

$$N(\lambda_s(x)) \leq C'' \sum_k \max\left\{\|\mathrm{pr}_k(x)\|^{1/k}, \|\mathrm{pr}_k(x)\|^{1/n}\right\} \leq n C'' + C'' \sum_k \|\mathrm{pr}_k(x)\|^{1/k},$$

so that $N(\lambda_s(x)) = O(N(x))$ for $x \to \infty$ in $\mathfrak{h}_0$.

The second part is proved similarly. □

2.2 Invariant vector fields

We now pass to the approximation of differential operators, following [30, § 4].

Definition 2.5 Let V be a homogeneous vector space, with dilations ρ_r, $r \in \mathbb{R}_+$. If T is a distribution on V, we define $\rho_r^* T$ and $T \circ \rho_r$ by

$$\left\langle \rho_r^* T, \varphi \right\rangle := \left\langle T, \varphi \circ \rho_r^{-1} \right\rangle \qquad \text{and} \qquad \langle T \circ \rho_r, \varphi \rangle := \left\langle T, r^{-Q} \varphi \circ \rho_r^{-1} \right\rangle$$

for every $\varphi \in C_c^\infty(V)$, where Q is the homogeneous dimension of V. We also define $(\rho_r)_* T := (\rho_r^{-1})^* T$.

We say that a function (or a distribution) f is *homogeneous of degree* $d \in \mathbb{C}$ if $f \circ \rho_r = r^d f$ for all $r \in \mathbb{R}_+$. We say that f is *log-homogeneous of degree* $d \in \mathbb{N}$ if there are a homogeneous polynomial P of degree d on V and a homogeneous norm N such that $f - P \log N$ is homogeneous of degree d.[4]

We say that a continuous linear operator $X\colon C^\infty(V) \to C^\infty(V)$ (for example a (linear) differential operator) is *homogeneous of order d* if $X(f \circ \rho_r) = r^d (Xf) \circ \rho_r$ for all $r > 0$.

Notice that, if X is a left-invariant differential operator under a homogeneous Lie group structure on V, then $Xf = f * (X\delta_0)$ and X is homogeneous of order d if and only if the distribution $X\delta_0$ is homogeneous of degree $-Q - d$.

In addition, if f is a function of class C^∞ on V and M_f is the operator of multiplication by f, then f is homogeneous of degree d if and only if M_f is homogeneous of order $-d$.

As a consequence, if X is a homogeneous differential operator of order d and f is a homogeneous function of degree d' and of class C^∞, then fX is a homogeneous differential operator of order $d - d'$.

Finally, observe that, if an element X of the enveloping algebra of $\widetilde{G}$ is homogeneous of degree d, then the corresponding left-invariant differential operator is homogeneous of order d, and conversely. Similar statements hold for $\widetilde{G}$, G_0, and G_∞.

Now, observe that $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$ are graded subspaces of $\widetilde{\mathfrak{g}}$, so that also $\mathfrak{h}_0 \cap \mathfrak{h}_\infty$ is a graded subspace of $\widetilde{\mathfrak{g}}$. Hence, we may complete a homogeneous basis of $\mathfrak{h}_0 \cap \mathfrak{h}_\infty$ to homogeneous bases of $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$, and then complete the union of the two (which is a homogeneous basis of $\mathfrak{h}_0 + \mathfrak{h}_\infty$) to a homogeneous basis of $\widetilde{\mathfrak{g}}$. Consequently, we may state the following definition.

Definition 2.6 We denote by $(\widetilde{X}_j)_{j\in J}$ a *homogeneous* basis of $\widetilde{\mathfrak{g}}$ such that there are two subsets J_0 and J_∞ of J such that $(\widetilde{X}_j)_{j\in J_0}$ is a basis of $\mathfrak{h}_0$, while $(\widetilde{X}_j)_{j\in J_\infty}$ is a basis of $\mathfrak{h}_\infty$. We denote by d_j the degree of $\widetilde{X}_j$ (as an element of the graded Lie algebra $\widetilde{\mathfrak{g}}$, so that $\widetilde{X}_j$ is homogeneous of order d_j as a differential operator). Fixing coordinates on $\widetilde{\mathfrak{g}}$ associated with the basis $(\widetilde{X}_j)_{j\in J}$, we denote by $(\partial_j)_{j\in J}$ the corresponding partial derivatives.

Define $X_{s,j} := \mathrm{d}\pi_s(\widetilde{X}_j)$ for every $j \in J$ and for every $s \in [0,\infty]$, so that

$$(r\,\cdot)_* X_{s,j} = r^{\mathrm{d}_j} X_{r^{-1}s,j}$$

for every $s \in [0,\infty]$, for every $r > 0$, and for every $j \in J$.

Finally, fix a total ordering on J and define, for every $\gamma \in \mathbb{N}^J$,

$$\widetilde{\mathbf{X}}^\gamma = \prod_{j\in J} \widetilde{X}_j^{\gamma_j},$$

$$\widetilde{\mathbf{X}}^\gamma = \prod_{j\in J} \widetilde{X}_j^{\gamma_j},$$

so that $\widetilde{\mathbf{X}}^\gamma$ is homogeneous of order $\mathrm{d}_\gamma := \sum_{j\in J} \gamma_j \mathrm{d}_j$. Define ∂^γ and $\mathbf{X}_s^\gamma$, for every $s \in [0,\infty]$, in a similar way. To simplify the notation, we shall identify $\mathbb{N}^{J_0}$ and $\mathbb{N}^{J_\infty}$ with subsets of $\mathbb{N}^J$; when $\gamma \in \mathbb{N}^{J_0}$ (resp. $\gamma \in \mathbb{N}^{J_\infty}$), we shall also write ∂_0^γ (resp. ∂_∞^γ) instead of ∂^γ.

The following result is a simple generalization of [30, Propositions 4 and 5]. Observe that, even though in [30, Propositions 4 and 5] the polynomials $p_{0,\gamma,\gamma'}$ and $p_{\infty,\gamma,\gamma'}$ were constructed comparing the products on G_0, G_∞, and G_1, if one tries to define the matrix

[4] If N' is another homogeneous norm, then $f - P\log N' = f - P\log N + P\log(N/N')$ is still homogeneous of degree d.

$(\delta_{\gamma,\gamma'} + p'_{0,\gamma,\gamma'})$ as the inverse of $(\delta_{\gamma,\gamma'} + p_{0,\gamma,\gamma'})$, then one would only prove that the $p'_{0,\gamma,\gamma'}$ are (everywhere defined) rational functions. Consequently, we shall present a different proof.

Proposition 2.7 *For every $\gamma \in \mathbb{N}^J$, there are two* unique *finite families*

$$(p_{0,\gamma,\gamma'})_{\gamma' \in \mathbb{N}^{J_0}} \qquad \text{and} \qquad (p'_{0,\gamma,\gamma'})_{\gamma' \in \mathbb{N}^{J_0}}$$

of polynomials on $\mathfrak{h}_0$ such that, identifying G_s and G_0 with $\mathfrak{h}_0$ for every $s \in [0, \infty)$,

$$\mathbf{X}_s^\gamma = \mathbf{X}_0^\gamma + \sum_{\gamma'} s^{\mathrm{d}_\gamma - \mathrm{d}_{\gamma'}} p_{0,\gamma,\gamma'}(s\,\cdot)\mathbf{X}_0^{\gamma'}, \quad \mathbf{X}_0^\gamma = \mathbf{X}_s^\gamma + \sum_{\gamma'} s^{\mathrm{d}_\gamma - \mathrm{d}_{\gamma'}} p'_{0,\gamma,\gamma'}(s\,\cdot)\mathbf{X}_s^{\gamma'}.$$

In addition, $p_{0,\gamma,\gamma'}$ and $p'_{0,\gamma,\gamma'}$ are sums of homogeneous polynomials of degrees strictly greater than $\mathrm{d}_{\gamma'} - \mathrm{d}_\gamma$.

Analogously, there are two unique *finite families*

$$(p_{\infty,\gamma,\gamma'})_{\gamma' \in \mathbb{N}^{J_\infty}, \mathrm{d}_{\gamma'} > \mathrm{d}_\gamma} \qquad \text{and} \qquad (p'_{\infty,\gamma,\gamma'})_{\gamma' \in \mathbb{N}^{J_\infty}, \mathrm{d}_{\gamma'} > \mathrm{d}_\gamma}$$

of polynomials on $\mathfrak{h}_\infty$ such that, identifying G_s and G_∞ with $\mathfrak{h}_\infty$ for every $s \in (0, \infty]$,

$$\mathbf{X}_s^\gamma = \mathbf{X}_\infty^\gamma + \sum_{\gamma'} s^{\mathrm{d}_\gamma - \mathrm{d}_{\gamma'}} p_{\infty,\gamma,\gamma'}(s\,\cdot)\mathbf{X}_\infty^{\gamma'}, \quad \mathbf{X}_\infty^\gamma = \mathbf{X}_s^\gamma + \sum_{\gamma'} s^{\mathrm{d}_\gamma - \mathrm{d}_{\gamma'}} p'_{\infty,\gamma,\gamma'}(s\,\cdot)\mathbf{X}_s^{\gamma'}.$$

In addition, $p_{\infty,\gamma,\gamma'}$ and $p'_{\infty,\gamma,\gamma'}$ are sums of homogeneous polynomials of degrees strictly smaller than $\mathrm{d}_{\gamma'} - \mathrm{d}_\gamma$.[5]

In particular,

$$\lim_{s \to 0} \mathbf{X}_s^\gamma = \mathbf{X}_0^\gamma, \qquad \lim_{s \to \infty} \mathbf{X}_s^\gamma = \mathbf{X}_\infty^\gamma.$$

Notice that, when $\gamma \in \mathbb{N}^{J_0}$, it may happen that $p_{0,\gamma,\gamma} \neq 0$ and $p'_{0,\gamma,\gamma} \neq 0$. Nonetheless, it is always true that both $p_{0,\gamma,\gamma}$ and $p'_{0,\gamma,\gamma}$ vanish at 0.

For example, consider the case in which $\widetilde{G}$ is the free two-step nilpotent Lie group on three generators $\widetilde{X}_1$, $\widetilde{X}_2$, and $\widetilde{X}_3$ (and the standard dilations), and define $\mathfrak{i}$ as the vector space generated by $[\widetilde{X}_1, \widetilde{X}_2] - \widetilde{X}_1 - \widetilde{X}_3$, $[\widetilde{X}_1, \widetilde{X}_3]$, and $[\widetilde{X}_2, \widetilde{X}_3] - \widetilde{X}_1 - \widetilde{X}_3$. Then, $\mathfrak{i}_0 = [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ and G_1 is isomorphic to the three-dimensional Heisenberg group, while G_0 is isomorphic to $\mathbb{R}^3$. Fix coordinates (x_1, x_2, x_3) on G_1 corresponding to the basis $(X_{1,1}, X_{1,2}, X_{1,3})$, so that $X_{0,j} = \partial_{x_j}$ under the identification of G_0 and G_1 with $\mathfrak{h}_0$. Then, simple computations show that

$$\begin{aligned} X_{1,1} &= X_{0,1} - \frac{x_2}{2} X_{0,1} - \frac{x_2}{2} X_{0,3} \\ X_{1,2} &= X_{0,2} + \frac{x_1 - x_3}{2} X_{0,1} + \frac{x_1 - x_3}{2} X_{0,3} \\ X_{1,3} &= X_{0,3} + \frac{x_2}{2} X_{0,1} + \frac{x_2}{2} X_{0,3}, \end{aligned}$$

[5] By the general theory, it is also clear that $\sum_j \gamma'_j \leq \sum_j \gamma_j$ if $p_{0,\gamma,\gamma'} \neq 0$, $p'_{0,\gamma,\gamma'} \neq 0$, $p_{\infty,\gamma,\gamma'} \neq 0$, or $p'_{\infty,\gamma,\gamma'} \neq 0$.

while

$$X_{0,1} = X_{1,1} + \frac{x_2}{2} X_{1,1} + \frac{x_2}{2} X_{1,3}$$
$$X_{0,2} = X_{1,2} - \frac{x_1 - x_3}{2} X_{1,1} - \frac{x_1 - x_3}{2} X_{1,3}$$
$$X_{0,3} = X_{1,3} - \frac{x_2}{2} X_{1,1} - \frac{x_2}{2} X_{1,3},$$

whence our assertion.

Proof Observe first that [40, Theorem 1.1.2] shows that there are two (unique) finite families $(p_{0,\gamma,\gamma'})$ and $(p'_{0,\gamma,\gamma'})$ of C^∞ functions on $\mathfrak{h}_0$ such that

$$\mathbf{X}_1^\gamma - \mathbf{X}_0^\gamma = \sum_{\gamma'} p_{0,\gamma,\gamma'} \mathbf{X}_0^{\gamma'} = -\sum_{\gamma'} p'_{0,\gamma,\gamma'} \mathbf{X}_1^{\gamma'}.$$

Applying the dilation by s^{-1}, we then get

$$\mathbf{X}_s^\gamma - \mathbf{X}_0^\gamma = \sum_{\gamma'} s^{\mathrm{d}_\gamma - \mathrm{d}_{\gamma'}} p_{0,\gamma,\gamma'}(s\,\cdot) \mathbf{X}_0^{\gamma'} = -\sum_{\gamma'} s^{\mathrm{d}_\gamma - \mathrm{d}_{\gamma'}} p'_{0,\gamma,\gamma'}(s\,\cdot) \mathbf{X}_s^{\gamma'}$$

for every $s \in (0, \infty)$. Now, let us prove that the $p_{0,\gamma,\gamma'}$ and the $p'_{0,\gamma,\gamma'}$ are polynomials.

Observe that $((\mathbf{X}_s^{\gamma'})_0)_{\gamma' \in \mathbb{N}^{J_0}, \sum_j \gamma'_j \le k}$ is a basis of the space of distributions on $\mathfrak{h}_0$ supported at 0 and of order at most k, for every $k \in \mathbb{N}$ and for every $s \in [0, \infty)$. Therefore, there are two families $(S_{0,\gamma'})$ and $(S'_{0,\gamma'})$ of polynomials on $\mathfrak{h}_0$ such that $(\mathbf{X}_0^{\gamma''} S_{0,\gamma'})(0) = \delta_{\gamma',\gamma''}$ and $(\mathbf{X}_1^{\gamma''} S'_{0,\gamma'})(0) = \delta_{\gamma',\gamma''}$ for every $\gamma'' \in \mathbb{N}^{J_0}$. Then, define $S_{x,\gamma'}(y) := S_{0,\gamma'}(x^{-1} \cdot_{G_0} y)$ and $S'_{x,\gamma'}(y) := S'_{0,\gamma'}(x^{-1} \cdot_{G_1} y)$ for every $x, y \in \mathfrak{h}_0$, so that

$$(\mathbf{X}_0^{\gamma''} S_{x,\gamma'})(x) = (\mathbf{X}_1^{\gamma''} S'_{x,\gamma'})(x) = \delta_{\gamma',\gamma''}$$

for every $\gamma', \gamma'' \in \mathbb{N}^{J_0}$. Therefore,

$$p_{0,\gamma,\gamma'}(x) = (\mathbf{X}_1^\gamma - \mathbf{X}_0^\gamma)(S_{x,\gamma'})(x) = (\mathbf{X}_1^\gamma S_{x,\gamma'})(x) - \delta_{\gamma,\gamma'}$$
$$p'_{0,\gamma,\gamma'}(x) = (\mathbf{X}_0^\gamma - \mathbf{X}_1^\gamma)(S'_{x,\gamma'})(x) = (\mathbf{X}_0^\gamma S'_{x,\gamma'})(x) - \delta_{\gamma,\gamma'}$$

for every $\gamma' \in \mathbb{N}^{J_0}$ and for every $x \in \mathfrak{h}_0$. Now, it is clear that the mappings $\mathfrak{h}_\infty \ni (x, y) \mapsto S_{x,\gamma'}(x \cdot_{G_1} y) = S_{0,\gamma}(x^{-1} \cdot_{G_0} (x \cdot_{G_1} y))$ and $\mathfrak{h}_\infty \ni (x, y) \mapsto S'_{x,\gamma'}(x \cdot_{G_0} y) = S'_{0,\gamma}(x^{-1} \cdot_{G_1} (x \cdot_{G_0} y))$ are polynomials; therefore, it is easily seen that $p_{0,\gamma,\gamma'}$ and $p'_{0,\gamma,\gamma'}$ are polynomials.

Finally, let us prove that $p_{0,\gamma,\gamma'}$ and $p'_{0,\gamma,\gamma'}$ are sums of homogeneous polynomials of degrees strictly greater than $\mathrm{d}_{\gamma'} - \mathrm{d}_\gamma$. Indeed, observe that the continuity of $P_{0,s}$ and $[\,\cdot\,,\,\cdot\,]_{0,s}$ in s at 0 shows that

$$0 = \lim_{s \to 0^+} (\mathbf{X}_0^\gamma - \mathbf{X}_s^\gamma) = \lim_{s \to 0^+} \sum_{\gamma'} s^{\mathrm{d}_\gamma - \mathrm{d}_{\gamma'}} p_{0,\gamma,\gamma'}(s\,\cdot) \mathbf{X}_0^{\gamma'}$$
$$= \lim_{s \to 0^+} \sum_{\gamma'} s^{\mathrm{d}_\gamma - \mathrm{d}_{\gamma'}} p'_{0,\gamma,\gamma'}(s\,\cdot) \mathbf{X}_s^{\gamma'}.$$

Since the $\mathbf{X}_s^{\gamma'}$ are pointwise linearly independent for every s and converge to the $\mathbf{X}_0^{\gamma'}$, we must have $p_{0,\gamma,\gamma'}(s \cdot x), p'_{0,\gamma,\gamma'}(s \cdot x) = o(s^{\mathrm{d}_{\gamma'} - \mathrm{d}_\gamma})$ for $s \to 0^+$, for every $x \in \mathfrak{h}_0$. The assertion follows in this case.

The properties of the families $(p_{\infty,\gamma,\gamma'})$ and $(p'_{\infty,\gamma,\gamma'})$ are proved even more easily. □

2.3 Moduli

Here, we construct some control moduli on $\widetilde{G}$ and the G_s, following [21, § 2.3]. Cf. also [31,37] for more details on 'weighted' control distances.

Definition 2.8 For every $x \in \widetilde{G}$, we define $|x|$ (resp. $|x|_*$) as the greatest lower bound of the set of $\varepsilon > 0$ such that there are an absolutely continuous curve $\gamma\colon [0,1] \to \widetilde{G}$ and some measurable functions $a_j\colon [0,1] \to \mathbb{R}$ such that $\|a_j\|_\infty \leq \varepsilon^{\mathrm{d}_j}$ (resp. $\|a_j\|_\infty \leq \min(\varepsilon, \varepsilon^{\mathrm{d}_j})$) for every $j \in J$, such that $\gamma(0) = e$ and $\gamma(1) = x$, and such that

$$\gamma'(t) = \sum_{j\in J} a_j(t)(\widetilde{X}_j)_{\gamma(t)}$$

for almost every $t \in [0,1]$. We define $B(r)$ (resp. $B_*(r)$) as the set of $x \in \widetilde{G}$ such that $|x| < r$ (resp. $|x|_* < r$), for every $r > 0$.

Proposition 2.9 *The following hold:*

- $|\cdot|$ *and* $|\cdot|_*$ *are finite, symmetric, and proper maps which vanish only at* e*;*
- $|z_1 z_2| \leq |z_1| + |z_2|$ *and* $|z_1 z_2|_* \leq |z_1|_* + |z_2|_*$ *for every* $z_1, z_2 \in \widetilde{G}$*;*
- $|z| = |z|_*$ *for every* $z \in \widetilde{G}$ *such that* $|z|_* \leq 1$ *(or, equivalently,* $|z| \leq 1$*); in addition,* $|z| \leq |z|_* \leq |z|^n$ *for every* $z \in \widetilde{G}$ *such that* $|z|_* \geq 1$*;*
- $B_*(1)^h \subseteq B_*(r) \subseteq B_*(1)^{h+1}$ *for every* $r \in [h, h+1]$ *and for every* $h \in \mathbb{N}$*;*
- $|\cdot|$ *is a homogeneous norm.*

Recall that a continuous function f between two topological spaces X and Y is said to be proper if it maps closed subsets of X onto closed subsets of Y and if its fibres are quasi-compact. In this case, saying that $|\cdot|$ and $|\cdot|_*$ are proper is equivalent to saying that $|\cdot|$ and $|\cdot|_*$ are continuous and that the associated closed balls $\left\{ z \in \widetilde{G} \colon |z| \leq r \right\}$ and $\left\{ z \in \widetilde{G} \colon |z|_* \leq r \right\}$ are compact for every $r > 0$.

The proof is simple and is omitted.

In order to provide some more insight into the moduli $|\cdot|$ and $|\cdot|_*$, let us introduce some more notation. First, we define $|x|'_R$ as the greatest lower bound of the set of $\varepsilon > 0$ such that there are an absolutely continuous curve $\gamma\colon [0,1] \to \widetilde{G}$ and some measurable functions $a_j\colon [0,1] \to \mathbb{R}$ such that $\|a_j\|_\infty \leq \varepsilon$ for every $j \in J$, such that $\gamma(0) = e$ and $\gamma(1) = x$, and such that

$$\gamma'(t) = \sum_{j\in J} a_j(t)(\widetilde{X}_j)_{\gamma(t)}$$

for almost every $t \in [0,1]$. Then, it is not hard to see that the following hold:

- $|x|_* = |x|'_R$ for every $x \in \widetilde{G}$ such that $|x|_* \geq 1$ or, equivalently, $|x|'_R \geq 1$;
- $|x|_*^n \leq |x|'_R \leq |x|_*$ for every $x \in \widetilde{G}$ such that $|x|_* \leq 1$ or, equivalently, $|x|'_R \leq 1$;
- $|x|_* = \max(|x|, |x|'_R)$ for every $x \in \widetilde{G}$.

In addition, if we denote by d_R the (left-invariant) Riemannian distance associated with the (left-invariant) Riemannian metric for which $(\widetilde{X}_j)_{j\in J}$ is an orthonormal basis, then $|x|'_R \leq d_R(0,x) \leq \dim \widetilde{G}\, |x|'_R$ for every $x \in \widetilde{G}$. Consequently, $|\cdot|_*$ is a reasonable compromise between a homogeneous norm (locally) and a Riemannian distance (globally).

Definition 2.10 For every $s \in [0, \infty]$ and for every $x \in G_s$, we define

$$|x|_s := \inf_{\pi_s(z)=x} |z| \quad \text{and} \quad |x|_{s,*} := \inf_{\pi_s(z)=x} |z|_*.$$

We define $B_s(r)$ (resp. $B_{s,*}(r)$) as the set of $x \in \widetilde{G}_s$ such that $|x|_s < r$ (resp. $|x|_{s,*} < r$), for every $r > 0$.

One may prove that the moduli $|\cdot|_s$ and $|\cdot|_{s,*}$ can be defined in the same fashion of the moduli $|\cdot|$ and $|\cdot|_*$. We leave the details to the reader.

Proposition 2.11 *The following hold:*

1. $|\cdot|_s$ *and* $|\cdot|_{s,*}$ *are symmetric, subadditive, and proper maps which vanish only at* e*;*
2. $|x|_s = |x|_{s,*}$ *for every* $s \in [0, \infty]$ *and for every* $x \in G_s$ *such that* $|x|_{s,*} \leq 1$*; in addition,* $|x|_s \leq |x|_{s,*} \leq |x|_s^n$ *for every* $x \in G_s$ *such that* $|x|_{s,*} \geq 1$*;*
3. $B_{s,*}(1)^h \subseteq B_{s,*}(r) \subseteq B_{s,*}(1)^{h+1}$ *for every* $s \in [0, \infty]$*, for every* $r \in [h, h+1]$*, and for every* $h \in \mathbb{N}$*;*
4. $|r \cdot x|_s = r|x|_{rs}$ *for every* $s \in [0, \infty]$*, for every* $r > 0$*, and for every* $x \in G_{rs}$*;*
5. *the mappings* $[0, \infty] \times \widetilde{G} \ni (s, z) \mapsto |\pi_s(z)|_s$ *and* $[0, \infty] \times \widetilde{G} \ni (s, z) \mapsto |\pi_s(z)|_{s,*}$ *are continuous;*
6. *there is a constant* $C > 0$ *such that*

$$\frac{1}{C} \min(|P_{0,s}(z)|, |P_{\infty,s}(z)|) \leq |\pi_s(z)|_s \leq C \min(|P_{0,s}(z)|, |P_{\infty,s}(z)|)$$

for every $s \in (0, \infty)$ *and for every* $z \in \widetilde{\mathfrak{g}}$.

Proof **1–4.** These assertions follow from the corresponding ones of Proposition 2.9.

5. Fix $z \in \widetilde{\mathfrak{g}}$ and observe that, since $|\cdot|$ is proper, for every $s \in [0, \infty]$ there is $y_s \in \mathfrak{i}_s$ such that $|z + y_s| = |\pi_s(z)|_s$. In particular, $|z + y_s| \leq |z|$, so that the set $\{ y_s : s \in [0, \infty] \}$ is relatively compact in $\widetilde{\mathfrak{g}}$. Then, fix $s' \in [0, \infty]$ and observe that there is a sequence (s_k) of elements of $[0, \infty]$ converging to s' such that $\lim_{k\to\infty} |\pi_{s_k}(z)|_{s_k} = \liminf_{s\to s'} |\pi_s(z)|_s$. Notice that we may assume that (y_{s_k}) converges to some y' in $\widetilde{\mathfrak{g}}$, so that $y' \in \mathfrak{i}_{s'}$. Therefore,

$$|\pi_{s'}(z)|_{s'} \leq |z + y'| = \lim_{k\to\infty} |z + y_{s_k}| = \lim_{k\to\infty} |\pi_{s_k}(z)|_{s_k} = \liminf_{s\to s'} |\pi_s(z)|_s.$$

Conversely, take a sequence (s'_k) of elements of $[0, \infty]$ converging to s' such that $\lim_{k\to\infty} |\pi_{s'_k}(z)|_{s'_k} = \limsup_{s\to s'} |\pi_s(z)|_s$, and observe that we may take $y'_{s'_k} \in \mathfrak{i}_{s'_k}$, for every $k \in \mathbb{N}$, in such a way that the sequence $(y'_{s'_k})$ converges to $y_{s'}$. Therefore,

$$|\pi_{s'}(z)|_{s'} = |z + y_{s'}| = \lim_{k\to\infty} |z + y'_{s'_k}| \geq \lim_{k\to\infty} |\pi_{s'_k}(z)|_{s'_k} = \limsup_{s\to s'} |\pi_s(z)|_z,$$

whence the first assertion. The second assertion is proved similarly.

6. The assertion follows from Proposition 2.4 and from **4** and **5**. □

Definition 2.12 For every $s \in [0, \infty]$, we define ν_{G_s} as the unique Haar measure on G_s such that $\nu_{G_s}(B_s(1)) = 1$. We define D_s, the volume growth of G_s, in such a way that $\nu_{G_s}(U^k) \asymp k^{D_s}$ for $k \to \infty$ for every compact neighbourhood U of e (cf., for instance, [14, Theorem II.1]).

Notice that $\nu_{G_s}(B_{s,*}(r)) \asymp r^{D_s}$ as $r \to +\infty$, for every $s \in [0, \infty]$, thanks to **3** of Proposition 2.11.

Corollary 2.13 *The following hold:*

1. $D_s = D_1 \geq \max(D_0, D_\infty)$ *for every* $s \in (0, \infty)$*;*
2. $D_1 \leq Q_\infty$*;*
3. $D_0 \leq Q_0$ *(resp.* $D_\infty \leq Q_\infty$*), with equality if and only if* G_0 *(resp.* G_∞*) is stratified.*

Notice that it may happen that either $D_0 > D_\infty$, or $D_0 < D_\infty$, or $D_1 > \max(D_0, D_\infty)$. Indeed, consider the case $\widetilde{G} = \mathbb{H}^1 \times \mathbb{R}$, $G = \mathbb{H}^1$, where $\mathbb{H}^1$ is the three-dimensional Heisenberg group; denote by X, Y, T, U a basis of $\widetilde{\mathfrak{g}}$ such that $[X, Y] = T$, while the other commutators vanish, and endow $\widetilde{G}$ with coordinates such that $((z, t), u)$ corresponds to $\exp(\mathrm{Re}\, zX + \mathrm{Im}\, zY + tT + uU)$; endow G with similar coordinates and define $\pi((z, t), u) := (z, t+u)$. Define dilations on $\widetilde{G}$ so that X, Y, T, U have degrees 1, 1, 2, 3, respectively. Then, $\mathfrak{i} = (T - U)\mathbb{R}$, $\mathfrak{i}_0 = U\mathbb{R}$, and $\mathfrak{i}_\infty = T\mathbb{R}$, so that $G_0 \cong \mathbb{H}^1$, and $G_\infty \cong \mathbb{R}^3$. Hence, in this case, $D_0 = 4 > 3 = D_\infty$.

If, in the same example considered above, we choose dilations on $\widetilde{G}$ in such a way that X, Y, T, U have degrees 1, 1, 2, 1, respectively, then $\mathfrak{i}_0 = T\mathbb{R}$ and $\mathfrak{i}_\infty = U\mathbb{R}$. Consequently, $D_0 = 3 < 4 = D_\infty$.

Finally, if we consider $\widetilde{G}$, π, and G as the products of the ones in the preceding examples, then clearly $D_1 = 8 > 7 = D_0 = D_\infty$.

Proof 1. Since G_s is isomorphic to G_1, for $s \in (0, \infty)$, it is clear that $D_s = D_1$. In addition, denote by $\mathfrak{g}_s$ the Lie algebra of G_s, and define inductively $\mathfrak{g}_{s,[1]} := \mathfrak{g}_s$ and $\mathfrak{g}_{s,[j+1]} := [\mathfrak{g}_s, \mathfrak{g}_{s,[j]}]$ for every $j \geq 1$. Then, $D_s = \sum_{j \geq 1} \dim \mathfrak{g}_{s,[j]}$ (cf., for example, [14, Theorem II.1]). Now, since $\lim_{s \to 0^+} [x, y]_s = [x, y]_0$ for every $x, y \in \mathfrak{h}_0$, it is easily seen that $\dim \mathfrak{g}_{0,[j]} \leq \dim \mathfrak{g}_{1,[j]}$ for every $j \in \mathbb{N}$, whence $D_0 \leq D_1$. In the same way one proves that $D_\infty \leq D_1$.

2. Indeed, Proposition 2.11 and the above remarks imply that

$$r^{D_1} \asymp \nu_{G_1}(B_{1,*}(r)) \leq \nu_{G_1}(B_1(r)) \asymp r^{Q_\infty}$$

as $r \to +\infty$. The assertion follows.

3. This follows easily from the formula for D_s used in **1**. □

Here is a simple result which will be useful later on. The proof, which is a simple modification of that of [41, VIII.1.1], is omitted.

Lemma 2.14 *For every* $s \in [0, \infty]$*, for every* $p \in [1, \infty]$*, for every* $f \in C^1(G_s)$*, and for every* $x \in G_s$*,*

$$\| f(\cdot\, x) - f \|_p \leq \sum_{j \in J} |x|_s^{\mathrm{d}_j} \| X_{s,j} f \|_p.$$

We conclude this subsection with some uniform estimates on the growth of the volume of the balls associated with the $|\cdot|_{s,*}$. Indeed, observe that the preceding facts prove that for every $s \in [0, \infty]$ there is a constant $C_s > 0$ such that $\nu_{G_s}\left(B_{s,*}(r)\right) \leq C_s r^{D_s}$ for every $r \geq 1$; however, we shall need to know that one may take the C_s to be *independent* of s. Actually, we shall prove a finer result, showing how the growth of the volume of balls decreases as s approaches 0 or ∞.

Proposition 2.15 *There are constant* $C > 0$ *and two integers* $N_0, N_\infty \geq 1$ *such that*

$$\nu_{G_s}\left(B_{s,*}(r)\right) \leq C \begin{cases} \max(r^{D_0}, s^{N_0} r^{D_1}) & \text{if } s \in [0, 1] \\ \max(r^{D_\infty}, s^{-N_\infty} r^{D_1}) & \text{if } s \in [1, \infty] \end{cases}$$

for every $s \in [0, \infty]$ *and for every* $r \geq 1$. *In addition, when* $\widetilde{G}$ *is stratified, so that* $Q_0 = D_0$ *and* $Q_\infty = D_\infty = D_1$ *by Corollary 2.13, one may take* $N_0 = Q_\infty - Q_0$.

In particular, for every $\varepsilon > 0$ *there is a constant* $C_\varepsilon > 0$, *independent of* s, *such that*

$$\|\min((1 + |\cdot|_{s,*})^{-D_0-\varepsilon}, s^{-N_0}(1 + |\cdot|_{s,*})^{-D_1-\varepsilon})\|_1 \leq C_\varepsilon$$

for every $s \in [0, 1]$, *while*

$$\|\min((1 + |\cdot|_{s,*})^{-D_\infty-\varepsilon}, s^{N_\infty}(1 + |\cdot|_{s,*})^{-D_1-\varepsilon})\|_1 \leq C_\varepsilon$$

for every $s \in [1, \infty]$.

Notice that, when $\widetilde{G}$ is not stratified, then (the optimal) N_0 and N_∞ may be smaller or larger than $D_1 - D_0$ and $D_1 - D_\infty$, respectively.

Let F_k be the Lie group whose Lie algebra has a basis $X, Y_1, \dots, Y_k$ such that $Y_{j+1} = [X, Y_j]$ for every $j = 1, \dots, k-1$, while the other commutators vanish. Consider $\widetilde{G} := \mathbb{R} \times F_k$, with basis of the corresponding Lie algebra $U, X, Y_1, \dots, Y_k$. Fix $d, d' \in \mathbb{N}^*$ such that $d < k + d' - 1$. Give degree 1 to X, degree $j + d' - 1$ to Y_j ($j = 1, \dots, k$), and degree d to U. Define $\mathfrak{i}_1 := \langle Y_k - U \rangle$, so that $\mathfrak{i}_s = \left\langle Y_k - s^{k+d'-1-d} U \right\rangle$ for every $s \in [0, \infty)$. Then, define $\mathfrak{h}_0 = \langle X, Y_1, \dots, Y_{k-1}, U \rangle$ and fix a neighbourhood of the identity $Q := [-1, 1]^{k+1}$ (in the coordinates associated with the basis $X, Y_1, \dots, Y_{k-1}, U$). Then, the Baker–Campbell–Hausdorff formula shows that, for every $s \in [0, \infty]$,

$$\begin{aligned}(x_1, y_1, u_1) \cdot_{G_s} \cdots \cdot_{G_s} (x_h, y_h, u_h) \\ = (P_h(x_1, y_1; \dots; x_h, y_h), u_1 + \cdots + u_h + s^{k+d'-1-d} R_h(x_1, y_1; \dots; x_h, y_h))\end{aligned}$$

for every $(x_j, y_j, u_j) \in Q$ (with $y_j \in [-1, 1]^{k-1}$), $j = 1, \dots, h$, where P_h and R_h are suitable polynomial mappings (independent of s). Integrating in (x, y) first and then in u, we see that

$$\begin{aligned}\nu_{\mathfrak{h}_0}(Q^{\cdot_{G_s} h}) &= (1 - s^{k+d'-1-d}) \nu_{\mathfrak{h}_0}(Q^{\cdot_{G_0} h}) + s^{k+d'-1-d} \nu_{\mathfrak{h}_0}(Q^{\cdot_{G_1} h}) \\ &\asymp h^{D_0} + s^{k+d'-1-d} h^{D_1}\end{aligned}$$

for $h \to \infty$, uniformly for $s \in [0, 1]$, where $\nu_{\mathfrak{h}_0}$ denotes Lebesgue measure on $\mathfrak{h}_0$. Now, it is not hard to see that this quantity is comparable with $\nu_{G_s}\left(B_{s,*}(h)\right)$ (uniformly for $s \in [0, 1]$ and $h \geq 1$), so that $N_0 = k + d' - 1 - d$, which may be either smaller or larger than $k - 1 = D_1 - D_0$.

Choosing $d > k + d' - 1$, one may then obtain examples with N_∞ either smaller or larger than $D_1 - D_\infty$. Taking products, examples with both $N_0 - (D_1 - D_0) \neq 0$ and $N_\infty - (D_1 - D_\infty) \neq 0$ (with all combinations of signs) may be produced.

Proof We shall divide the proof into several steps.

1. We consider only the case $s \in [0, 1]$, since the case $s \in [1, \infty]$ is completely analogous (or almost trivial when $\widetilde{G}$ is stratified, see **4**). Define $\widetilde{\mathfrak{g}}_{[1]} := \widetilde{\mathfrak{g}}$ and, by induction, $\widetilde{\mathfrak{g}}_{[k+1]} := [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}_{[k]}]$, so that $(\widetilde{\mathfrak{g}}_{[k]})$ is a decreasing sequence of graded ideals of $\widetilde{\mathfrak{g}}$ (the lower central series). Notice that, arguing as in the proof of Proposition 2.1, one may prove that $\mathfrak{i}_s \cap \widetilde{\mathfrak{g}}_{[k]}$ converges to some limit $\mathfrak{i}_{0,[k]} \subseteq \mathfrak{i}_0 \cap \widetilde{\mathfrak{g}}_{[k]}$ as $s \to 0^+$, for every $k \in \mathbb{N}^*$. Then, for every $k \in \mathbb{N}^*$ choose a graded complement V_k of $(\mathfrak{i}_0 \cap \widetilde{\mathfrak{g}}_{[k]}) + \widetilde{\mathfrak{g}}_{[k+1]}$ in $\widetilde{\mathfrak{g}}_{[k]}$ and a graded complement W_k of $\mathfrak{i}_{0,k}$ in $\mathfrak{i}_0 \cap \widetilde{\mathfrak{g}}_{[k]}$. Observe that $\bigoplus_{k' \geq k} V_{k'}$ is a graded complement of $\mathfrak{i}_0 \cap \widetilde{\mathfrak{g}}_{[k]}$ in $\widetilde{\mathfrak{g}}_{[k]}$ for every $k \in \mathbb{N}^*$, so that we may assume that $\mathfrak{h}_0 = \bigoplus_k V_k$. Analogously, observe that $W_k \oplus (\bigoplus_{k' \geq k} V_{k'})$ is a graded complement of $\mathfrak{i}_{0,k}$ in $\widetilde{\mathfrak{g}}_{[k]}$; arguing as in the Proof of Proposition 2.1, we then see that $W_k \oplus (\bigoplus_{k' \geq k} V_{k'})$ is a graded complement of $\mathfrak{i}_s \cap \widetilde{\mathfrak{g}}_{[k]}$ in $\widetilde{\mathfrak{g}}_{[k]}$ for every $s \in (0, \infty)$.

Then, we may find a family $(k_j)_{j\in J_0}$ of positive integers and a homogeneous basis $(\widetilde{Y}_j)_{j\in J_0}$ of $\mathfrak{h}_0$ such that $(\widetilde{Y}_j)_{k_j=k}$ is a basis of V_k, for every $k\in\mathbb{N}^*$, and such that $\widetilde{Y}_j$ has degree d_j for every $j\in J_0$. Choose, in addition, a homogeneous basis $(\widetilde{Y}_j)_{j\in J_k}$ of W_k for every $k\in\mathbb{N}^*$ (to make the notation consistent, we assume that the J_k, for $k\in\mathbb{N}$, are mutually disjoint); we define $k_j:=k$ and we denote by d_j the degree of $\widetilde{Y}_j$ for every $j\in J_k$. Define $\widetilde{Y}^{(s)}_j:=P_{0,s}(\widetilde{Y}_j)$ for every $j\in\widetilde{J}:=\bigcup_{k\in\mathbb{N}}J_k$; observe that $\widetilde{Y}^{(s)}_j=\widetilde{Y}_j$ for some (hence every) $s\in[0,1]$ if and only if $j\in J_0$, and that $\widetilde{Y}^{(0)}_j=0$ if and only if $j\in\widetilde{J}\setminus J_0$.

2. Observe that **5** of Proposition 2.11 shows that there is a constant $C_1>0$ such that, under the identification of G_s with $\mathfrak{h}_0$,

$$B_{s,*}(1)\subseteq\sum_{j\in J_0}[-C_1,C_1]\widetilde{Y}_j=:Q_s$$

for every $s\in[0,1]$. In addition, denoting by $\nu_{\mathfrak{h}_0}$ the (fixed) Lebesgue measure on $\mathfrak{h}_0$, again by **5** of Proposition 2.11 we see that there is a constant $C_2>0$ such that

$$\nu_{G_s}\leq C_2\nu_{\mathfrak{h}_0}$$

under the identification of G_s with $\mathfrak{h}_0$. Thanks to **3** of Proposition 2.11, it will then suffice to estimate $\nu_{\mathfrak{h}_0}(Q_s^{\cdot_{G_s}h})$ for every $h\in\mathbb{N}^*$ and for every $s\in[0,1]$.

3. Now, observe that, arguing as in the proof of [6, Theorem 2 of Chapter II, § 6, No. 4], we see that, for every $j_1,\dots,j_h\in J_0$,

$$\widetilde{Y}_{j_1}\cdots\widetilde{Y}_{j_h}=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\sum_{|\ell_1|,\dots,|\ell_m|\geq 1}\frac{1}{\ell_1!\cdots\ell_m!}[\widetilde{Y}_{j_1}^{\ell_{1,1}}\cdots\widetilde{Y}_{j_h}^{\ell_{m,h}}],$$

where

$$[\widetilde{Y}_{j_1}^{\ell'_{1,1}}\cdots\widetilde{Y}_{j_h}^{\ell'_{m,h}}]=(\mathrm{ad}(\widetilde{Y}_{j_1})^{\ell'_{1,1}}\cdots\mathrm{ad}(\widetilde{Y}_{j_h})^{\ell'_{1,h}})\cdots(\mathrm{ad}(\widetilde{Y}_{j_1}^{\ell'_{m,1}})\cdots\mathrm{ad}(\widetilde{Y}_{j_h})^{\ell'_{m,h}})\widetilde{Y}_{j_{\bar h}},$$

where $\bar h:=\max\left\{h'\colon\ell_{m,h'}\neq 0\right\}$, $\ell'_{m'}:=\ell_{m'}$ for every $m'=1,\dots,m-1$, and $\ell'_m=\ell_m-(\delta_{h',\bar h})_{h'}$. Then, taking **1** into account, we see that

$$[Y_{j_1}^{\ell_{1,1}}\cdots Y_{j_h}^{\ell_{m,h}}]\in\left\langle(\widetilde{Y}_j)_{k_j\geq|\ell_1+\cdots+\ell_m|,\,\mathrm{d}_j\leq\ell_{1,1}\mathrm{d}_{j_1}+\cdots+\ell_{m,h}\mathrm{d}_{j_h}}\right\rangle+\mathfrak{i}_1$$

for every $\ell_1,\dots,\ell_m$ with $|\ell_1|,\dots,|\ell_m|\geq 1$ and for every $m\in\mathbb{N}^*$. Therefore, there is a constant $C_3>0$ such that

$$Q_s^{\cdot_{G_s}h}\subseteq\sum_{j\in\widetilde{J}}[-C_3h^{k_j},C_3h^{k_j}]\widetilde{Y}^{(s)}_j$$

for every $s\in[0,1]$ and for every $h\in\mathbb{N}^*$. Now, arguing by induction on $\mathrm{Card}(\widetilde{J})\geq\mathrm{Card}(J_0)$, we see that

$$\sum_{j\in\widetilde{J}}[-C_3h^{k_j},C_3h^{k_j}]\widetilde{Y}^{(s)}_j=\bigcup_{\substack{J',J''\subseteq\widetilde{J}\\ J'\cap J''=\emptyset\\ \mathrm{Card}(J'')=\mathrm{Card}(J_0)}}\ \bigcup_{\substack{\varepsilon'\in\{-1,1\}^{J'}\\ \varepsilon''\in\{-1,1\}^{J''}}}\Bigg(\sum_{j'\in J'}C_3h^{k_{j'}}\varepsilon'_{j'}\widetilde{Y}^{(s)}_{j'}$$
$$+\sum_{j''\in J''}[0,1]C_3\varepsilon''_{j''}h^{k_{j''}}\widetilde{Y}^{(s)}_{j''}\Bigg).$$

Therefore,

$$\begin{aligned} \nu_{\mathfrak{h}_0}(Q_s^{\cdot G_s h}) &\le 2^{2\,\mathrm{Card}(\widetilde{J})-\mathrm{Card}(J_0)} \sum_{J'\subseteq \widetilde{J}, \mathrm{Card}(J')=\mathrm{Card}(J_0)} \nu_{\mathfrak{h}_0}\left(\sum_{j\in J'} [0,1] C_3 h^{k_j} \widetilde{Y}_j^{(s)}\right)\\ &= 2^{2\,\mathrm{Card}(\widetilde{J})} \left(\frac{C_3}{2}\right)^{\mathrm{Card}(J_0)} \sum_{J'\subseteq \widetilde{J}, \mathrm{Card}(J')=\mathrm{Card}(J_0)} h^{k_{J'}} \det\left(\widetilde{Y}_{J'}^{(s)}\right), \end{aligned}$$

where $k_{J'} := \sum_{j\in J'} k_j$ and $\det\left(\widetilde{Y}_{J'}^{(s)}\right)$ is the determinant of the basis $\left(\widetilde{Y}_j^{(s)}\right)_{j\in J'}$ of $\mathfrak{h}_0$ with respect to the measure $\nu_{\mathfrak{h}_0}$ (that is, with respect to any basis whose fundamental parallelotope has measure 1).

Now, take $J' \subseteq \widetilde{J}$ such that $\mathrm{Card}(J') = \mathrm{Card}(J_0)$ and $\det\left(\widetilde{Y}_{J'}^{(s)}\right) \ne 0$ for some (hence every) $s \in (0,1]$. Observe that, since $\det\left(\widetilde{Y}_{J'}^{(s)}\right) = \det\left(\widetilde{Y}_{J'}^{(0)}\right) + O(s)$, the first assertion will be established if we prove that $k_{J'} \le D_1$ for every such J', and that $k_{J'} = D_0$ if $\det\left(\widetilde{Y}_{J'}^{(0)}\right) \ne 0$. Then, for every $k \in \mathbb{N}^*$ define J'_k as the set of $j \in J'$ such that $k_j = k$, and observe that $(\widetilde{Y}_j)_{j\in\bigcup_{k'\ge k} J'_{k'}}$ is the basis of a graded subspace of $\widetilde{\mathfrak{g}}_{[k]}$ whose intersection with $\mathfrak{i}_s \cap \widetilde{\mathfrak{g}}_{[k]}$ is 0 for every $s \in (0,1]$, since $(\widetilde{Y}_j^{(s)})_{j\in\bigcup_{k'\ge k} J'_{k'}}$ is the basis of a subspace of $\widetilde{\mathfrak{g}}_{[k]} + \mathfrak{i}_s$ whose intersection with $\mathfrak{i}_s$ is 0 and $\widetilde{Y}_j - \widetilde{Y}_j^{(s)} \in \mathfrak{i}_s$ for every $s \in (0,1]$ and for every $j \in J'$. Therefore, $\sum_{k'\ge k} \mathrm{Card}(J'_{k'}) \le \dim[(\mathfrak{g}_{[k]} + \mathfrak{i}_s)/\mathfrak{i}_s]$ so that, summing over $k \in \mathbb{N}^*$,

$$k_{J'} = \sum_{k\in\mathbb{N}^*} k\,\mathrm{Card}(J'_k) \le \sum_{k\in\mathbb{N}^*} \dim[(\mathfrak{g}_{[k]} + \mathfrak{i}_s)/\mathfrak{i}_s] = D_s$$

for every $s \in (0,1]$, where the last equality follows from [14, Theorem II.1].

Finally, assume that $\det\left(\widetilde{Y}_{J'}^{(0)}\right) \ne 0$. Then, by **1** we see that $J' = J_0$, so that the assertion follows by the same argument used above.

4. Now, assume that $\widetilde{G}$ is stratified. Then, it is clear that $\widetilde{\mathfrak{g}}_{[k]} = \bigoplus_{q\ge k} \widetilde{\mathfrak{g}}_q$, so that the assertion for $s \in [1,\infty]$ is trivial. Then, consider the preceding construction for $s \in [0,1]$ and observe that $k_j \le \mathrm{d}_j$ for every $j \in \widetilde{J}$, with equality when $j \in J_0$. Take J' as in **3**.

Observe that we may construct, by induction on $k = 1,\dots,n$, mutually disjoint subsets J'_k of J' such that $J' \cap J_0 \subseteq J'_k$ and such that $(\widetilde{Y}_j^{(s)})_{j\in\bigcup_{k'\ge k} J'_{k'}}$ is the basis of a graded complement of $\left(\bigoplus_{q<k} \widetilde{\mathfrak{g}}_q\right) \cap \mathfrak{h}_0$ in $\mathfrak{h}_0$, for every $k = 1,\dots,n$. Define $k'_j := k$ for every $j \in J'_k$ and for every $k = 1,\dots,n$, and observe that $\mathrm{d}_j \ge k'_j$ for every $j \in J'$ thanks to Proposition 2.7. Furthermore, define $\widetilde{Y}_j^{J'}$ as the homogeneous component of degree k'_j of $\widetilde{Y}_j^{(1)}$ for every $j \in J'$, and observe that $(\widetilde{Y}_j^{J'})_{j\in J'}$ is a basis of $\mathfrak{h}_0$. In addition, arguing as in **3** above we see that $\sum_{k=1}^n k\,\mathrm{Card}(J'_k) = \sum_{k=1}^n k \dim(\mathfrak{h}_0 \cap \widetilde{\mathfrak{g}}_k) = Q_0 = D_0$ since G_0 is a stratified group, so that

$$h^{k_{J'}} \det\left(\widetilde{Y}_{J'}^{(s)}\right) = h^{D_0} (hs)^{k_{J'}-D_0} s^{\sum_{j\in J'} \mathrm{d}_j - k_{J'}} \det\left((s^{k'_j - \mathrm{d}_j} \widetilde{Y}_j^{(s)})_{j\in J'}\right).$$

Now, observe that $\sum_{j\in J'} \mathrm{d}_j - k_{J'} \ge 0$, so that our assertion will be established if we prove that $\det\left((s^{k'_j-\delta_j} \widetilde{Y}_j^{(s)})_{j\in J'}\right)$ is independent of $s \in [0,1]$, and hence equal to $\det\left((\widetilde{Y}_j^{J'})_{j\in J'}\right)$. To prove this fact, one may use Gauss elimination to the family $(s^{k'_j-\delta_j} \widetilde{Y}_j^{(s)})$ (more precisely, to the matrix of the coordinates of the vectors $\widetilde{Y}_j^{(s)}$ with respect to the basis $(\widetilde{Y}_j)_{j\in J_0}$) and observe that, by homogeneity arguments, the resulting family is (linearly independent and) independent of s. □

3 Estimates of the heat kernel

We now introduce the operators in which we shall be mainly interested. Fix a homogeneous left-invariant differential operator $\widetilde{\mathcal{L}}$ on $\widetilde{G}$ such that $\widetilde{\mathcal{L}}+\widetilde{\mathcal{L}}^*$ is a positive Rockland operator of degree δ; then, we define $\mathcal{L}_s := \mathrm{d}\pi_s(\widetilde{\mathcal{L}})$ for every $s\in[0,\infty]$. We shall sometimes write $\mathcal{L}$ instead of $\mathcal{L}_1$ to simplify the notation.[6] Then, the operators $\widetilde{\mathcal{L}}$, $\mathcal{L}$, and $\mathcal{L}_s$ are weighted subcoercive, hence hypoelliptic (cf. [21, Theorem 2.3]).

Denote by $(\widetilde{h}_t)_{t>0}$ the heat kernel of $\widetilde{\mathcal{L}}$, which we shall consider as a semi-group of *measures* on $\widetilde{G}$. In addition, for every $s\in[0,\infty]$ and for every $t>0$, we shall define $h_{s,t} := (\pi_s)_*(\widetilde{h}_t)$, so that $(h_{s,t})_{t>0}$ is the heat kernel of $\mathcal{L}_s$. Observe that

$$h_{rs,t} = (r^{-1}\cdot)_* h_{s,r^\delta t}$$

for every $r>0$, for every $s\in[0,\infty]$, and for every $t>0$.

We fix a Lebesgue measure on $\widetilde{\mathfrak{g}}$ and identify $\widetilde{h}_t$ with its density. With the $h_{s,t}$ we shall be more careful, though. Indeed, for $s\in(0,\infty)$ the group G_s can be identified with both $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$, and it is not possible to find Lebesgue measures on $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$ which induce the same measure on G_s for all $s\in(0,\infty)$. Therefore, we shall fix two Lebesgue measures on $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$ and define two densities $h_{0,s,t}$ and $h_{\infty,s,t}$ of $h_{s,t}$ accordingly.

Precisely, for $s\in[0,\infty)$, we define $h_{0,s,t}$ as the density of $(P_{0,s})_*(\widetilde{h}_t)$ with respect to the fixed Lebesgue measure on $\mathfrak{h}_0$; in this way, $h_{0,s,t}$ becomes the (density of) $h_{s,t}$, under the identification of $\mathfrak{g}_s$ (hence of G_s) with $\mathfrak{h}_0$ given in Definition 2.2. Observe that, with these choices (and with a suitable Lebesgue measure on $\mathfrak{i}_0$, independent of s),

$$h_{0,s,t}(x) = \int_{\mathfrak{i}_0} \widetilde{h}_t(x+y+\psi_{0,s}(y))\,\mathrm{d}y$$

for every $s\in[0,\infty)$, for every $t>0$, and for every $x\in\mathfrak{h}_0$.

Analogously, for $s\in(0,\infty]$ we shall define $h_{\infty,s,t}$ as the density of $(P_{\infty,s})_*(\widetilde{h}_t)$ with respect to the fixed Lebesgue measure on $\mathfrak{h}_\infty$. Similar remarks apply.

We now prove some uniform estimates on $h_{0,s,t}$ and $h_{\infty,s,t}$ and their derivatives which cannot be derived from the general estimates for weighted subcoercive operators.

Theorem 3.1 *Fix $c>0$ and $d\in\mathbb{R}$, and let X_0 and X_∞ be two homogeneous differential operators with continuous coefficients on $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$, respectively, of order d. Then, for every $k\in\mathbb{N}$ there are two constants $C, b>0$ (independent of s) such that the following hold:*

1. *for every $s\in[0,\infty)$, for every $x\in\mathfrak{h}_\infty$, and for every $t>cs^\delta$,*

$$|X_\infty \partial_s^k h_{\infty,1/s,t}(x)| \le \frac{C}{t^{\frac{Q_\infty+d+k}{\delta}}}\mathrm{e}^{-b|\pi_{1/s}(t^{-1/\delta}\cdot x)|_{1/s,*}^{\frac{\delta}{\delta-1}}} \le \frac{C}{t^{\frac{Q_\infty+d+k}{\delta}}}\mathrm{e}^{-b\left(\frac{|\pi_{1/s}(x)|_{1/s}}{t^{1/\delta}}\right)^{\frac{\delta}{\delta-1}}};$$

[6] Notice that in [30] the operator $\widetilde{\mathcal{L}}$ is only required to be Rockland; nonetheless, since we are interested in the corresponding heat kernels, additional restrictions have to be imposed.

2. *for every* $s \in [0, \infty)$*, for every* $x \in \mathfrak{h}_0$*, and for every* $t \in (0, cs^{-\delta}]$,

$$|X_0 \partial_s^k h_{0,s,t}(x)| \leq \frac{C}{t^{\frac{Q_0+d-k}{\delta}}} \mathrm{e}^{-b|\pi_s(t^{-1/\delta}\cdot x)|_{s,*}^{\frac{\delta}{\delta-1}}} \leq \frac{C}{t^{\frac{Q_0+d-k}{\delta}}} \mathrm{e}^{-b\left(\frac{|\pi_s(x)|_s}{t^{1/\delta}}\right)^{\frac{\delta}{\delta-1}}}.$$

Proof **1.** Consider the first assertion; notice that we may reduce to the case in which $X_\infty = f\partial_\infty^\alpha$, where f is a continuous homogeneous function on $\mathfrak{h}_\infty$ of degree $\mathrm{d}_\alpha - d$; notice that $\mathrm{d}_\alpha - d > 0$ since f is continuous. Now, observe that, with a change of variables,

$$\begin{aligned} h_{\infty,1/s,t}(x) &= \int_{\mathfrak{i}_\infty} \widetilde{h}_t(x + y + \psi_{\infty,1/s}(y))\,\mathrm{d}y \\ &= t^{-\frac{Q_\infty}{\delta}} \int_{\mathfrak{i}_\infty} \widetilde{h}_1(t^{-1/\delta}\cdot x + y + \psi_{\infty,(t^{-1/\delta}s)^{-1}}(y))\,\mathrm{d}y \end{aligned}$$

for every $x \in \mathfrak{h}_\infty$, for every $s \in [0, \infty)$, and for every $t > 0$. Therefore, Faà di Bruno's formula shows that

$$\begin{aligned} X_\infty \partial_s^k h_{\infty,1/s,t}(x) = \frac{f(x)}{t^{\frac{Q_\infty+\mathrm{d}_\alpha+k}{\delta}}} \sum_{\sum_{\ell=1}^k \ell|\gamma_\ell|=k} \frac{k!}{\gamma!} \int_{\mathfrak{i}_\infty} \partial_\infty^\alpha \partial^{\gamma_1+\cdots+\gamma_k} \widetilde{h}_1(t^{-1/\delta}\cdot x + y \\ + \psi_{\infty,(t^{-1/\delta}s)^{-1}}(y)) \prod_{\ell=1}^k \left(\frac{1}{\ell!}\partial_{s'}^\ell\big|_{s'=t^{-1/\delta}s} \psi_{\infty,1/s'}(y)\right)^{\gamma_\ell} \,\mathrm{d}y \end{aligned}$$

for every $s \in [0, \infty)$, for every $t > 0$, for every $k \in \mathbb{N}$, and for every $x \in \mathfrak{h}_\infty$. In addition, observe that $\partial_{s'}^\ell \left(\mathrm{pr}_j \circ \psi_{\infty,1/s'}\right)$ is a (linear) polynomial of degree at most $j - \ell$ for every $j = 2, \dots, n$ and for every $\ell = 1, \dots, j - 1$, and is 0 otherwise. Therefore, there are $C_1, \widetilde{b} > 0$ such that $|X_\infty \partial_s^k h_{\infty,1/s,t}(x)|$ is less than

$$\frac{C_1}{t^{\frac{Q_\infty+\mathrm{d}_\alpha+k}{\delta}}} |x|^{\mathrm{d}_\alpha - d} \sum_{\sum_{\ell=1}^k \ell|\gamma_\ell|=k} \mathrm{e}^{-\widetilde{b}|t^{-1/\delta}\cdot x + y + \psi_{\infty,(t^{-1/\delta}s)^{-1}}(y))|_*^{\frac{\delta}{\delta-1}}} (1 + |y|)^{\mathrm{d}_\gamma - k}$$

for every $(x, y) \in \mathfrak{h}_\infty \oplus \mathfrak{i}_\infty$, for every $s \in [0, \infty)$, and for every $t > cs^\delta$ (cf. [21, Theorem 2.3 (e)]). Now,

$$|t^{-1/\delta}\cdot(x + y + \psi_{\infty,1/s}(y))|_* \geq \frac{1}{2}|\pi_{1/s}(t^{-1/\delta}\cdot x)|_{1/s,*} + \frac{1}{2}|t^{-1/\delta}\cdot(x + y + \psi_{\infty,1/s}(y))|$$

for every $(x, y) \in \mathfrak{h}_\infty \oplus \mathfrak{i}_s$, for every $t > 0$, and for every $s \in (0, \infty]$, with some abuses of notation. Therefore,

$$\begin{aligned} |X_\infty \partial_s^k h_{\infty,1/s,t}(x)| \leq \frac{C_1}{t^{\frac{Q_\infty+d+k}{\delta}}} \mathrm{e}^{-2^{\frac{\delta}{1-\delta}}\widetilde{b}|\pi_{1/s}(t^{-1/\delta}\cdot x)|_{1/s,*}^{\frac{\delta}{\delta-1}}} |t^{-1/\delta}\cdot x|^{\mathrm{d}_\alpha - d} \times \\ \times \sum_{\sum_{\ell=1}^k \ell|\gamma_\ell|=k} \int_{\mathfrak{i}_\infty} \mathrm{e}^{-2^{\frac{\delta}{1-\delta}}\widetilde{b}\left|t^{-1/\delta}\cdot x + y + \psi_{\infty,(t^{-1/\delta}s)^{-1}}(y)\right|^{\frac{\delta}{\delta-1}}} (1 + |y|)^{\mathrm{d}_\gamma - k}\,\mathrm{d}y. \end{aligned}$$

Now, fix a norm $\|\cdot\|$ on $\widetilde{\mathfrak{g}}$, and recall that $\psi_{\infty,{}^1\!/_{s'}}$ is strictly super-homogeneous, so that there are two constants $C_2, C_2' > 0$ such that

$$\begin{aligned}\left|\psi_{\infty,{}^1\!/_{s'}}(y)\right| &\leq C_2 \sum_{j=2}^{n} \left\| \left(\mathrm{pr}_j \circ \psi_{\infty,{}^1\!/_{s'}}\right)(y_1 + \cdots + y_{j-1}) \right\|^{\frac{1}{j}} \\ &\leq C_2 \sum_{j=2}^{n} \left\| \mathrm{pr}_j \circ \psi_{\infty,{}^1\!/_{s'}} \right\| \left(\|y_1\| + \ldots + \left\|y_{j-1}\right\|\right)^{\frac{1}{j}} \\ &\leq C_2' \max\left(|y|^{\frac{n-1}{n}}, |y|^{\frac{1}{n}}\right),\end{aligned}$$

for every $y \in \mathfrak{i}_\infty$ with homogeneous components $y_1, \ldots, y_n$, and for every $s' \in [0, c^{1/\delta}]$. In addition, observe that all homogeneous norms on $\widetilde{G}$ are equivalent and that both $\mathfrak{h}_\infty$ and $\mathfrak{i}_\infty$ are homogeneous subspaces of $\widetilde{\mathfrak{g}}$, so that there is a constant $C_3 > 0$ such that $|z_1 + z_2| \leq C_3(|z_1| + |z_2|)$ for every $z_1, z_2 \in \widetilde{\mathfrak{g}}$, and such that $|x + y| \geq \frac{1}{C_3}(|x| + |y|)$ for every $(x, y) \in \mathfrak{h}_\infty \oplus \mathfrak{i}_\infty$. In addition, since $\frac{\delta}{\delta-1} \geq 1$, there is a constant $C_4 \geq 1$ such that

$$a_1^{\frac{\delta}{\delta-1}} + a_2^{\frac{\delta}{\delta-1}} \leq (a_1 + a_2)^{\frac{\delta}{\delta-1}} \leq C_4 \left(a_1^{\frac{\delta}{\delta-1}} + a_2^{\frac{\delta}{\delta-1}}\right).$$

Then, for every $x \in \mathfrak{h}_\infty$, for every $y \in \mathfrak{i}_\infty$ and for every $t > cs^\delta$,

$$\begin{aligned}&\left|t^{-1/\delta} \cdot x + y + \psi_{\infty,(t^{-1/\delta}s)^{-1}}(y)\right|^{\frac{\delta}{\delta-1}} \\ &\quad\geq \frac{1}{C_3^{\frac{\delta}{\delta-1}} C_4} \left|t^{-1/\delta} \cdot x + y\right|^{\frac{\delta}{\delta-1}} - \left|\psi_{\infty,(t^{-1/\delta}s)^{-1}}(y)\right|^{\frac{\delta}{\delta-1}} \\ &\quad\geq \frac{1}{C_3^{\frac{2\delta}{\delta-1}} C_4} \left(|t^{-1/\delta} \cdot x|^{\frac{\delta}{\delta-1}} + |y|^{\frac{\delta}{\delta-1}}\right) - C_2' \max\left(|y|^{\frac{n-1}{n}}, |y|^{\frac{1}{n}}\right).\end{aligned}$$

Therefore, there is a constants $C_5 > 0$ such that

$$2^{\frac{\delta}{1-\delta}} \widetilde{b} \left|t^{-1/\delta} \cdot x + y + \psi_{\infty,(t^{-1/\delta}s)^{-1}}(y)\right|^{\frac{\delta}{\delta-1}} \geq \frac{1}{C_5} \left(|t^{-1/\delta} \cdot x|^{\frac{\delta}{\delta-1}} + |y|^{\frac{\delta}{\delta-1}}\right) - C_5$$

for every $(x, y) \in \mathfrak{h}_\infty \oplus \mathfrak{i}_\infty$, for every $s \in [0, \infty)$, and for every $t > cs^\delta$.

Hence, there is a constant $C_6 > 0$ which is greater than

$$|t^{-1/\delta} \cdot x|^{\mathrm{d}_\alpha - d} \sum_{\sum_{\ell=1}^{k} \ell|\gamma_\ell| = k} \int_{\mathfrak{i}_\infty} \mathrm{e}^{-2^{\frac{\delta}{1-\delta}} \widetilde{b} \left|t^{-1/\delta} \cdot x + y + \psi_{\infty,(t^{-1/\delta}s)^{-1}}(y)\right|^{\frac{\delta}{\delta-1}}} (1 + |y|)^{\mathrm{d}_\gamma - k} \,\mathrm{d}y$$

for every $x \in \mathfrak{h}_\infty$, for every $s \in [0, \infty)$, and for every $t > cs^\delta$, so that

$$|X_\infty \partial_s^k h_{\infty,{}^1\!/_s,t}(x)| \leq \frac{C_1 C_6}{t^{\frac{Q_\infty + d + k}{\delta}}} \mathrm{e}^{-2^{\frac{\delta}{1-\delta}} \widetilde{b} |\pi_{{}^1\!/_s}(t^{-1/\delta} \cdot x)|_{{}^1\!/_s,*}^{\frac{\delta}{\delta-1}}}.$$

2. Consider, now, the second assertion. Observe that we may assume that $X_0 = f \partial_{\mathfrak{h}_0}^\alpha$ for some α and some continuous homogeneous function f on $\mathfrak{h}_0$ with degree $\mathrm{d}_\alpha - d$. Notice that $\mathrm{d}_\alpha - d > 0$ since f is continuous. Then, Faà di Bruno's formula shows that

$$X_0 \partial_s^k h_{0,s,t}(x) = \frac{f(x)}{t^{\frac{Q_0+\mathrm{d}_\alpha - k}{\delta}}} \sum_{\sum_{\ell=1}^k \ell|\gamma_\ell|=k} \frac{k!}{\gamma!} \int_{\mathfrak{i}_0} \partial_0^\alpha \partial^\gamma \widetilde{h}_1(t^{-1/\delta}\cdot x + y + \psi_{0,t^{1/\delta}s}(y)) \cdot \\ \cdot \prod_{\ell=1}^k \left(\frac{1}{\ell!}\partial_{s'}^\ell|_{s'=t^{1/\delta}s}\psi_{0,s'}(y)\right)^{\gamma_\ell} \,\mathrm{d}y$$

for every $s \in [0,1]$, for every $t > 0$, for every $k \in \mathbb{N}$, and for every $x \in \mathfrak{h}_0$. In addition, observe that $\partial_{s'}^\ell \left(\mathrm{pr}_j \circ \psi_{0,s'}\right)$ is a (linear) polynomial of degree at most n and of homogeneous order at least $j + \ell$ for every $j = 1, \dots, n-1$ and for every $\ell = 1, \dots, n-j$, and is 0 otherwise. Therefore, there are $C_1, \widetilde{b} > 0$ such that $|X_0 \partial_s^k h_{0,s,t}(x)|$ is less than

$$\frac{C_1}{t^{\frac{Q_0+\mathrm{d}_\alpha-k}{\delta}}}|x|^{\mathrm{d}_\alpha - d} \sum_{\sum_{\ell=1}^k \ell|\gamma_\ell|=k} \mathrm{e}^{-\widetilde{b}|t^{-1/\delta}\cdot x + y + \psi_{0,t^{1/\delta}s}(y)|_*^{\frac{\delta}{\delta-1}}} |y|^{\mathrm{d}_\gamma+k}(1+|y|)^{n|\gamma|-\mathrm{d}_\gamma-k}$$

for every $(x,y) \in \mathfrak{h}_0 \oplus \mathfrak{i}_0$, for every $s \in [0,\infty)$, and for every $t > 0$ (cf. [21, Theorem 2.3]). Therefore, arguing as in **1** we see that

$$|X_0\partial_s^k h_{0,s,t}(x)| \leq \frac{\mathrm{e}^{-2^{\frac{\delta}{1-\delta}}\widetilde{b}|\pi_s(t^{-1/\delta}\cdot x)|_{s,*}^{\frac{\delta}{\delta-1}}}}{t^{\frac{Q_0+d-k}{\delta}}} C_1 |t^{-1/\delta}\cdot x|^{\mathrm{d}_\alpha-d} \times \\ \times \sum_{\sum_{\ell=1}^k \ell|\gamma_\ell|=k} \int_{\mathfrak{i}_0} \mathrm{e}^{-2^{\frac{\delta}{1-\delta}}\widetilde{b}|t^{-1/\delta}\cdot x + y + \psi_{0,t^{1/\delta}s}(y)|^{\frac{\delta}{\delta-1}}} |y|^{\mathrm{d}_\gamma+k}(1+|y|)^{n|\gamma|-\mathrm{d}_\gamma-k}\,\mathrm{d}y$$

for every $x \in \mathfrak{h}_0$, for every $s \in [0,\infty)$, and for every $t > 0$.

Now, observe that there is a constant $C_2 \geq 1$ such that

$$\frac{1}{C_2}\min(\|z\|, \|z\|^{1/n}) \leq |z| \leq C_2 \max(\|z\|, \|z\|^{1/n})$$

for every $z \in \widetilde{\mathfrak{g}}$. In addition, observe that the linear mapping $L_{s'}\colon x+y \mapsto x+y+\psi_{0,s'}(y)$ is an automorphism of $\widetilde{\mathfrak{g}}$ for every $s' \in [0,\infty)$, and that the mapping $[0,\infty) \ni s' \mapsto L_{s'} \in \mathcal{L}(\widetilde{\mathfrak{g}})$ is continuous. Therefore, there is a constant $C_3 > 0$ such that $\|L_{s'}^{-1}\| \leq C_3$ for every $s' \in [0, c^{1/\delta}]$. In particular, assuming that $\|x+y\| = \|x\| + \|y\|$ for every $(x,y) \in \mathfrak{h}_0 \oplus \mathfrak{i}_0$ for simplicity,

$$\begin{aligned}\frac{2}{C_2 C_3^{1/n}} + |t^{-1/\delta}\cdot x + y + \psi_{0,t^{1/\delta}s}(y)| &\geq \frac{1}{C_2}\left(C_3^{-1/n} + \|t^{-1/\delta}\cdot x + y + \psi_{0,t^{1/\delta}s}(y)\|^{1/n}\right)\\ &\geq \frac{1}{2C_2C_3^{1/n}}\left(2 + \|t^{-1/\delta}\cdot x\|^{1/n} + \|y\|^{1/n}\right)\\ &\geq \frac{1}{2C_2^{\frac{n+1}{n}}C_3^{1/n}}\left(|t^{-1/\delta}\cdot x|^{1/n} + |y|^{1/n}\right)\end{aligned}$$

for every $(x,y) \in \mathfrak{h}_0 \oplus \mathfrak{i}_0$, for every $s \in [0,\infty)$, and for every $t \in (0, cs^{-\delta}]$. Hence, there is a constant $C_4 > 0$ such that

$$|t^{-1/\delta}\cdot x|^{\mathrm{d}_\alpha-d} \sum_{\sum_{\ell=1}^k \ell|\gamma_\ell|=k} \int_{\mathfrak{i}_0} \mathrm{e}^{-2^{\frac{\delta}{1-\delta}}\widetilde{b}|t^{-1/\delta}\cdot x + y + t\psi_{0,t^{1/\delta}s}(y)|^{\frac{\delta}{\delta-1}}} \times \\ \times |y|^{\mathrm{d}_\gamma+k}(1+|y|)^{n|\gamma|-\mathrm{d}_\gamma-k}\,\mathrm{d}y \leq C_4$$

for every $(x, y) \in \mathfrak{h}_0 \oplus \mathfrak{i}_0$, for every $s \in [0, 1]$, and for every $t \in (0, cs^{-\delta}]$, so that

$$|X_0 \partial_s^k h_{0,s,t}(x)| \leq C_1 C_4 \frac{e^{-2^{\frac{\delta}{1-\delta}} \widetilde{b} |\pi_s(t^{-1/\delta} \cdot x)|_{s,*}^{\frac{\delta}{\delta-1}}}}{t^{\frac{Q_0+d-k}{\delta}}}$$

$(x, y) \in \mathfrak{h}_0 \oplus \mathfrak{i}_0$, for every $s \in [0, \infty)$, and for every $t \in (0, cs^{-\delta}]$. The proof is complete. □

Proposition 3.2 *For every $c > 0$ and for every γ, there are $C > 0$ and $\omega > 0$ such that for every $s \in [0, \infty]$ and for every $t > 0$,*

$$\left\| \mathbf{X}_s^\gamma h_{s,t} e^{c|\cdot|_{s,*}} \right\|_1 \leq \frac{C}{t^{\frac{d_\gamma}{\delta}}} e^{\omega t}.$$

Proof Observe that [21, Theorem 2.3 (f)] implies that there are C and ω such that

$$\left\| \widetilde{\mathbf{X}}^\gamma \widetilde{h}_t e^{c|\cdot|_*} \right\|_1 \leq \frac{C}{t^{\frac{d_\gamma}{\delta}}} e^{\omega t}$$

for every $t > 0$. Therefore,

$$\left\| \mathbf{X}_s^\gamma h_{s,t}\, e^{c|\cdot|_{s,*}} \right\|_1 = \left\| (\pi_s)_* \left(\widetilde{\mathbf{X}}^\gamma \widetilde{h}_t\, e^{c(|\cdot|_{s,*} \circ \pi_s)} \right) \right\|_1 \leq \left\| \widetilde{\mathbf{X}}^\gamma \widetilde{h}_t\, e^{c|\cdot|_*} \right\|_1 \leq \frac{C}{t^{\frac{d_\gamma}{\delta}}} e^{\omega t}$$

for every $t > 0$ and for every $s \in [0, \infty]$. □

4 Riesz potentials

We keep the notation of the preceding section. Here, we generalize the asymptotic study of the fundamental solutions made in [30] to the complex powers of $\mathcal{L}_s$. Notice first that, while the convolution kernels of $\mathcal{L}_s^{-\frac{\alpha}{\delta}}$ (the Riesz potentials) are easily defined when $\operatorname{Re} \alpha < Q_\infty$, in order to define them also for $\operatorname{Re} \alpha \geq Q_\infty$ we shall need to argue by analytic continuation.

4.1 Definition and (log-)homogeneity of Riesz potentials

In the following statement, functions on $\mathfrak{h}_0$ (resp. $\mathfrak{h}_\infty$) are identified with distributions by means of the *fixed* Lebesgue measure on $\mathfrak{h}_0$ (resp. $\mathfrak{h}_\infty$).

Proposition 4.1 *For every $s \in (0, \infty]$ there is a unique meromorphic $\mathcal{S}'(G_s)$-valued mapping $\alpha \mapsto I_{s,\alpha}$ on $\mathbb{C}$, with poles of order at most 1 at the elements of $Q_\infty + \mathbb{N}$, such that the following hold:*

1. *if $\alpha \in \mathbb{C}$, $-\delta k_1 < \operatorname{Re} \alpha < Q_\infty + k_2$ for some $k_1, k_2 \in \mathbb{N}$, and $\alpha \notin Q_\infty + \mathbb{N}$, then*

$$\begin{aligned} I_{\infty,s,\alpha} &= \frac{1}{\Gamma(\frac{\alpha}{\delta})} \int_0^1 t^{\frac{\alpha}{\delta}} \left(h_{\infty,s,t} - \sum_{j<k_1} (-\mathcal{L}_s)^j \delta_0 \frac{t^j}{j!} \right) \frac{dt}{t} \\ &+ \sum_{j<k_1} \frac{1}{j!(\frac{\alpha}{\delta}+j)\Gamma(\frac{\alpha}{\delta})} (-\mathcal{L}_s)^j \delta_0 \\ &+ \frac{1}{\Gamma(\frac{\alpha}{\delta})} \int_1^{+\infty} t^{\frac{\alpha}{\delta}} \left(h_{\infty,s,t} - \sum_{d_\gamma<k_2} \partial_\infty^\gamma h_{\infty,s,t}(0) \frac{(\cdot)^\gamma}{\gamma!} \right) \frac{dt}{t} + P_{s,\alpha,k_2}, \end{aligned}$$

where P_{s,α,k_2} is a sum of homogeneous polynomials on $\mathfrak{h}_\infty$ of degree at most k_2-1;

2. *$I_{\infty,s,\alpha}\in L^1_{\mathrm{loc}}(\mathfrak{h}_\infty)$ when $\operatorname{Re}\alpha>0$;*
3. *the restriction of $I_{\infty,s,\alpha}$ to $\mathfrak{h}_\infty\setminus\{0\}$ has a density of class C^∞;*
4. *$I_{\infty,s,-\delta k}=\mathcal{L}_s^k\delta_0$ for every $k\in\mathbb{N}$.*

Similar assertions hold for $s=0$, replacing $\mathfrak{h}_\infty$ with $\mathfrak{h}_0$, ∂_∞ with ∂_0, and Q_∞ with Q_0.

Proof Fix $s\in(0,\infty]$. In addition, fix $k_1,k_2\in\mathbb{N}$ and observe that, if $0<\operatorname{Re}\alpha<\frac{Q_\infty}{\delta}$, then

$$
\begin{aligned}
I_{\infty,s,\alpha}&=\frac{1}{\Gamma\left(\frac{\alpha}{\delta}\right)}\int_0^\infty t^{\frac{\alpha}{\delta}}h_{\infty,s,t}\,\frac{\mathrm{d}t}{t}\\
&=\frac{1}{\Gamma(\frac{\alpha}{\delta})}\int_0^1 t^{\frac{\alpha}{\delta}}\left(h_{\infty,s,t}-\sum_{j<k_1}(-\mathcal{L}_s)^j\delta_0\frac{t^j}{j!}\right)\frac{\mathrm{d}t}{t}\\
&\quad+\sum_{j<k_1}\frac{1}{j!(\frac{\alpha}{\delta}+j)\Gamma(\frac{\alpha}{\delta})}(-\mathcal{L}_s)^j\delta_0\\
&\quad+\frac{1}{\Gamma(\frac{\alpha}{\delta})}\int_1^{+\infty}t^{\frac{\alpha}{\delta}}\left(h_{\infty,s,t}-\sum_{\mathrm{d}_\gamma<k_2}\partial_\infty^\gamma h_{\infty,s,t}(0)\frac{(\,\cdot\,)^\gamma}{\gamma!}\right)\frac{\mathrm{d}t}{t}+P_{s,\alpha,k_2},
\end{aligned}
$$

where

$$
P_{s,\alpha,k_2}(x):=\frac{1}{\Gamma(\frac{\alpha}{\delta})}\sum_{\mathrm{d}_\gamma<k_2}\frac{x^\gamma}{\gamma!}\int_1^{+\infty}t^{\frac{\alpha}{\delta}}\partial_\infty^\gamma h_{\infty,s,t}(0)\,\frac{\mathrm{d}t}{t}
$$

for every $x\in\mathfrak{h}_\infty$. Taking into account Lemmas 7.5, 7.6, and 7.7, it suffices to prove that the mapping $\alpha\mapsto P_{s,\alpha,k_2}$ extends to a meromorphic mapping on $\mathbb{C}$ with poles of order at most 1 at the elements of $Q_\infty+\mathbb{N}$. Indeed,

$$
\begin{aligned}
\partial_\infty^\gamma h_{\infty,s,t^{-\delta}}(0)&=\int_{\mathfrak{i}_\infty}\partial_\infty^\gamma\widetilde{h}_{t^{-\delta}}\left(y+\psi_{\infty,s}(y)\right)\mathrm{d}y\\
&=t^{Q_\infty+\mathrm{d}_\gamma}\int_{\mathfrak{i}_\infty}\partial_\infty^\gamma\widetilde{h}_1\left(y+t\cdot\psi_{\infty,s}(t^{-1}\cdot y)\right)\mathrm{d}y
\end{aligned}
$$

for every $x\in G$ and for every $t>0$. In addition, since $\psi_{\infty,s}$ is linear and strictly super-homogeneous,

$$
t\cdot\psi_{\infty,s}(t^{-1}\cdot y)=\sum_{\ell<j}t^{j-\ell}\operatorname{pr}_j(\psi_{\infty,s}(y_\ell))
$$

for every $t>0$ and $y\in\mathfrak{i}_\infty$. As a consequence, the mapping $t\mapsto\partial_\infty^\gamma h_{\infty,s,t^{-\delta}}(0)$ extends to a mapping of class C^∞ on $\mathbb{R}$. Let $\sum_{j\geq Q_\infty+\mathrm{d}_\gamma}b_{s,\gamma,j}t^j$ be its Taylor development at the origin.

Now, fix $N\in\mathbb{N}$ and observe that, for $\operatorname{Re}\alpha<N+1$,

$$
\begin{aligned}
\int_1^{+\infty}t^{\frac{\alpha}{\delta}}\partial_\infty^\gamma h_{\infty,s,t}(0)\,\frac{\mathrm{d}t}{t}&=\delta\int_0^1 t^{-\alpha}\partial_\infty^\gamma h_{\infty,t^{-\delta}}(0)\,\frac{\mathrm{d}t}{t}\\
&=\delta\sum_{Q_\infty+\mathrm{d}_\gamma\leq j\leq N}b_{s,\gamma,j}\int_0^1 t^{-\alpha+j}\,\frac{\mathrm{d}t}{t}+\int_0^1 t^{-\alpha}O\left(t^{N+1}\right)\frac{\mathrm{d}t}{t}\\
&=\delta\sum_{Q_\infty+\mathrm{d}_\gamma\leq j\leq N}\frac{b_{s,\gamma,j}}{j-\alpha}+\int_0^1 O\left(t^{-\operatorname{Re}\alpha+N}\right)\mathrm{d}t.
\end{aligned}
$$

By the arbitrariness of N, it follows that the mapping $\alpha \mapsto \int_1^{+\infty} t^{\frac{\alpha}{\delta}} \partial^\gamma_{\mathfrak{h}_\infty} h_{\infty,s,t}(0) \frac{\mathrm{d}t}{t}$ extends to a meromorphic mapping on $\mathbb{C}$ with poles of order at most 1 at every element of $Q_\infty + \mathrm{d}_\gamma + \mathbb{N}$. Summing up all these facts, it follows that the mapping $\alpha \mapsto I_{\infty,s,\alpha}$ extends to a meromorphic mapping on $\mathbb{C}$, with poles of order at most 1 at every element of $Q_\infty + \mathbb{N}$. Finally, it is clear that $I_{\infty,s,-k} = \mathcal{L}_s^k \delta_0$.

The case $s = 0$ is treated similarly. □

Definition 4.2 Fix $s \in (0, \infty]$. For every $\alpha \in \mathbb{C}$ such that $\alpha \notin Q_\infty + \mathbb{N}$, we define $I_{s,\alpha}$ as the distribution on G_s induced by the distribution $I_{\infty,s,\alpha}$ of Proposition 4.1 under the identification of G_s with $\mathfrak{h}_\infty$ (in other words, $I_{s,\alpha} = (\pi_s)_*(I_{\infty,s,\alpha})$ by an abuse of notation). We define $I_{s,\alpha}$, for $\alpha \in Q_\infty + \mathbb{N}$, as the zeroth-order term of the Laurent expansion of the mapping $\alpha' \mapsto I_{s,\alpha'}$ at α.[7]

We denote by $I_{0,s,\alpha}$ the distribution on $\mathfrak{h}_0$ induced by $I_{s,\alpha}$ under the identification of G_s with $\mathfrak{h}_0$, for every $s \in (0, \infty)$; $I_{0,0,\alpha}$ is defined as in Proposition 4.1. We define $I_{0,\alpha} := (\pi_0)_*(I_{0,0,\alpha})$, with the same abuse of notation used above.

Proposition 4.3 *For every $s \in (0, \infty]$, for every $r > 0$, and for every $\alpha \in \mathbb{C}$, the following hold:*

1. *$(r\,\cdot)_* I_{s,\alpha} = r^{-\alpha} I_{r^{-1}s,\alpha}$ if $I_{s,\cdot}$ is regular at α (in which case also $I_{r^{-1}s,\cdot}$ is regular at α);*
2. *$(r\,\cdot)_* I_{s,\alpha} = r^{-\alpha} I_{r^{-1}s,\alpha} + r^{-\alpha} \log r\, P_{r^{-1}s,\alpha}$, where $P_{r^{-1}s,\alpha}$ is a polynomial such that $P_{r^{-1}s,\alpha}(x) = O(|x|_{r^{-1}s}^{-Q_\infty+\alpha})$ for $x \to \infty$, if $I_{s,\cdot}$ has a pole at α (in which case also $I_{r^{-1}s,\cdot}$ has a pole at α). In addition, $P_{\infty,\alpha}$ is homogeneous of degree $-Q_\infty + \alpha$.*

Analogous assertions hold for $s = 0$, replacing $\mathfrak{h}_\infty$ with $\mathfrak{h}_0$ and Q_∞ with Q_0.

In particular, $I_{\infty,\alpha}$ is homogeneous of degree $-Q_\infty + \alpha$ when $I_{\infty,\cdot}$ regular at α, while $I_{\infty,\alpha}$ is log-homogeneous of degree $-Q_\infty + \alpha$ otherwise (cf. Definition 2.2).

Analogous statements hold for $I_{0,\alpha}$, with the obvious modifications.

Proof The first assertion for $0 < \operatorname{Re}\alpha < Q_\infty$ follows easily from the equality $(r\,\cdot)_* h_{s,t} = h_{r^{-1}s, r^\delta t}$, which holds for every $r > 0$, for every $s \in [0, \infty]$, and for every $t > 0$. The general statement then holds by holomorphy.

For what concerns the second assertion, take $s \in (0, \infty]$ and a pole α of $I_{s,\cdot}$, so that, in particular, $\alpha \in Q_\infty + \mathbb{N}$. Then, for every $\alpha' \neq \alpha$ in a neighbourhood of α, and for every $r > 0$

$$(r\,\cdot)_* I_{s,\alpha'} = r^{-\alpha'} I_{r^{-1}s,\alpha'},$$

so that, taking the zeroth-order term of the Laurent expansions of both sides of the equality at α,

$$(r\,\cdot)_* I_{s,\alpha} = r^{-\alpha} I_{r^{-1}s,\alpha} - r^{-\alpha} \log r \lim_{\alpha' \to \alpha} (\alpha' - \alpha) I_{r^{-1}s,\alpha'}.$$

Now, with the notation of Proposition 4.1, it is easily seen that, chosen $k_1 = 0$ and $k_2 = -Q_\infty + \alpha + 1$,

$$\lim_{\alpha' \to \alpha} (\alpha' - \alpha) I_{\infty,r^{-1}s,\alpha}(x) = \lim_{\alpha' \to \alpha} (\alpha' - \alpha) P_{r^{-1}s,\alpha,k_2}(x).$$

By inspection of the Proof of Proposition 4.1, we see that $\lim_{\alpha' \to \alpha} (\alpha' - \alpha) P_{\infty,\alpha',k_2}$ is a homogeneous polynomial of degree $-Q_\infty + \alpha$, whence the result. □

[7] Notice that the mapping $\alpha' \mapsto I_{s,\alpha'}$ *may* be regular at α.

4.2 Asymptotic expansions

We keep the notation of the preceding sections. We prove some asymptotic developments of the I_α generalizing those proved in [30] for the fundamental solutions. Even though the procedure of [30] may be generalized to the present setting, we prefer to give a different proof, which is shorter and gives a little more insight into the meaning of the further terms of the development. We then present, under rather restrictive assumptions, another proof which describes quite explicitly the terms of the development.

Theorem 4.4 *Take and $\alpha \in \mathbb{C}$ and $s \in \mathbb{R}_+$. Then, the following hold:*

1. *there is a sequence of log-homogeneous functions $(I^{(k)}_{\infty,\alpha})$ of class C^∞ on $\mathfrak{h}_\infty \setminus \{0\}$ such that $I^{(0)}_{\infty,\alpha} = I_{\infty,\infty,\alpha}$, such that $I^{(k)}_{\infty,\alpha}$ has degree $-Q_\infty + \alpha - k$, and such that for every $N \in \mathbb{N}$ and for every γ there is a constant $C_{N,\gamma} > 0$ such that, for every $s \in [1,\infty)$,*

$$\left|\partial^\gamma_\infty \left(I_{\infty,s,\alpha} - \sum_{k<N} s^{-k} I^{(k)}_{\infty,\alpha}\right)(x)\right| \leq \frac{C_{N,\gamma} s^{-N}}{|x|^{Q_\infty - \operatorname{Re}\alpha + \mathrm{d}_\gamma + N}} (1 + |\log|s\cdot x||)$$

   for every $x \in \mathfrak{h}_\infty$ such that $|x| \geq s^{-1}$; the factor $1 + |\log|s\cdot x||$ may be omitted if $\alpha \notin Q_\infty + \mathrm{d}_\gamma + N + \mathbb{N}$;

2. *there are a sequence $(P_{\alpha,k})$ of homogeneous polynomials on $\mathfrak{h}_0$ and a sequence $(I^{(k)}_{0,\alpha})$ of log-homogeneous functions of class C^∞ on $\mathfrak{h}_0 \setminus \{0\}$ such that $I^{(0)}_{0,\alpha} = I_{0,0,\alpha}$, such that $I^{(k)}_{0,\alpha}$ has degree $-Q_0 + \alpha + k$, such that $P_{0,\alpha,k}$ has degree k, and such that for every $N \in \mathbb{N}$ and for every γ there is a constant $C'_{N,\gamma} > 0$ such that, for every $s \in (0,1]$,*

$$\left|\partial^\gamma_0 \left(I_{0,s,\alpha} - \sum_{k<N} s^k I^{(k)}_{0,\alpha} - \sum_{k<-Q_0+\operatorname{Re}\alpha+N} s^{k-Q_0+\alpha} P_{\alpha,k}\right)(x)\right|$$
$$\leq \frac{C'_{N,\gamma} s^N}{|x|^{Q_0 - \operatorname{Re}\alpha + \mathrm{d}_\gamma - N}} (1 + |\log|s\cdot x||)$$

   for every $x \in \mathfrak{h}_0$ such that $0 \neq |x| \leq s^{-1}$; the factor $1 + |\log|s\cdot x||$ may be omitted if $\alpha \notin Q_0 + \mathrm{d}_\gamma - N + \mathbb{N}$.

Proof **1.** Define $H_\infty(s', t, x) := h_{\infty, 1/s', t}(x)$ for every $s' \in [0,\infty)$, for every $t > 0$, and for every $x \in \mathfrak{h}_\infty$, to simplify the notation. Take $\alpha \in \mathbb{C}$ such that $0 < \operatorname{Re}\alpha < Q_\infty$, and observe that a Taylor expansion of H_∞ in the first variable gives

$$I_{\infty,s,\alpha} = \frac{1}{\Gamma(\frac{\alpha}{\delta})} \int_0^1 t^{\frac{\alpha}{\delta}} h_{\infty,s,t} \frac{\mathrm{d}t}{t} + \sum_{k<N} \frac{s^{-k}}{k!\Gamma(\frac{\alpha}{\delta})} \int_1^{+\infty} \partial_1^k H_\infty(0,t,\,\cdot\,) \frac{\mathrm{d}t}{t} + s^{-N} R_{s,\alpha,N},$$

where

$$R_{s,\alpha,N} = \frac{1}{(N-1)!\Gamma(\frac{\alpha}{\delta})} \int_1^{+\infty} t^{\frac{\alpha}{\delta}} \int_0^1 \partial_1^N H_\infty\left(\frac{\theta}{s}, t,\,\cdot\,\right) (1-\theta)^{N-1} \,\mathrm{d}\theta \frac{\mathrm{d}t}{t}.$$

Now, Lemma 7.6 implies that the mapping $\alpha \mapsto \frac{1}{\Gamma(\frac{\alpha}{\delta})} \int_0^1 t^{\frac{\alpha}{\delta}} h_{\infty,s,t} \frac{\mathrm{d}t}{t}$ extends to an entire function with values in $\mathcal{E}'(\mathfrak{h}_\infty) + \mathcal{S}(\mathfrak{h}_\infty)$. Next, observe that

$$H_\infty(s', t, x) = t^{-\frac{Q_\infty}{\delta}} H_\infty(t^{-1/\delta} s', 1, t^{-1/\delta}\cdot x)$$

for every $s' \in [0, \infty)$, for every $t > 0$, and for every $x \in \mathfrak{h}_\infty$, so that

$$\partial_1^k H_\infty(0, t, x) = t^{-\frac{Q_\infty + k}{\delta}} \partial_1^k H_\infty(0, 1, t^{-1/\delta} \cdot x)$$

for every $x \in \mathfrak{h}_\infty$ and for every $t > 0$.

Therefore, Lemma 7.6 and the estimates of $\partial_1^k H_\infty(0, t, \cdot\,)$ provided in Theorem 3.1 show that, for $0 < \operatorname{Re} \alpha < Q_\infty + k$,

$$\frac{1}{k!\Gamma(\frac{\alpha}{\delta})} \int_1^{+\infty} t^{\frac{\alpha}{\delta}} \partial_1^k H_\infty(0, t, \cdot\,) \frac{\mathrm{d}t}{t} = \frac{1}{k!\Gamma(\frac{\alpha}{\delta})} \int_0^{+\infty} t^{\frac{\alpha}{\delta}} \partial_1^k H_\infty(0, t, \cdot\,) \frac{\mathrm{d}t}{t} + R'_{\alpha,k},$$

where $R'_{\alpha,k}$ is an entire function of α with values in $\mathcal{E}'(\mathfrak{h}_\infty) + \mathcal{S}(\mathfrak{h}_\infty)$. In addition, we also see that the mapping, initially defined for $0 < \operatorname{Re} \alpha < Q_\infty + k$,

$$\alpha \mapsto I^{(k)}_{\infty,\alpha} := \frac{1}{k!\Gamma(\frac{\alpha}{\delta})} \int_0^{+\infty} t^{\frac{\alpha}{\delta}} \partial_1^k H_\infty(0, t, \cdot\,) \frac{\mathrm{d}t}{t}$$

extends to a meromorphic function on $\mathbb{C}$ such that $I^{(k)}_{\infty,\alpha}$ is homogeneous of degree $-Q_\infty + \alpha - k$ for every α in the domain of holomorphy of $I^{(k)}_{\infty,\cdot}$ (argue as in the proof of Propositions 4.1 and 4.3). Log-homogeneity holds at poles, where $I^{(k)}_{\infty,\alpha}$ denotes the zeroth-order term of the Laurent expansion of $I^{(k)}_{\infty,\cdot}$ (argue as in the Proof of Proposition 4.3).

Finally, assume that $\operatorname{Re} \alpha < Q_\infty + N$. Observe that there is a constant $C > 0$ such that

$$|\pi_{s'}(x)|_{s'} \geq C(|x| - 1)$$

for every $s' \in [1, \infty]$ and for every $x \in \mathfrak{h}_\infty$, thanks to Proposition 2.11. Therefore, Theorem 3.1 and the preceding computations imply that $R_{s,\alpha,N}(x)$ is well defined for $x \neq 0$ and that there are there are two constants $C' > 0$ and $b > 0$ such that, for every γ,

$$|\partial^\gamma_\infty R_{s,\alpha,N}(x)| \leq C' \int_1^{+\infty} t^{\frac{\operatorname{Re}\alpha - Q_\infty - \mathrm{d}_\gamma - N}{\delta}} \mathrm{e}^{-b\left(\frac{|x|}{t^{1/\delta}}\right)^{\frac{\delta}{\delta-1}}} \frac{\mathrm{d}t}{t}.$$

Hence,

$$|\partial^\gamma_\infty R_{s,\alpha,N}(x)| \leq C' |x|^{\operatorname{Re}\alpha - Q_\infty - \mathrm{d}_\gamma - N} \int_0^{+\infty} t^{\frac{-\operatorname{Re}\alpha + Q_\infty + \mathrm{d}_\gamma + N}{\delta}} \mathrm{e}^{-bt^{\frac{1}{\delta - 1}}} \frac{\mathrm{d}t}{t}.$$

The assertion follows for s fixed. In order to get uniform estimates for $s \in [1, \infty)$, reduce to the case $s = 1$ by means of Proposition 4.3. Let us give some more details in the case in which $I_{s,\cdot}$ has a pole at α. Indeed, for every $s \in [1, \infty]$, $I_{\infty,s,\alpha} - \sum_{k<N} s^{-k} I^{(k)}_{\infty,\alpha}$ equals

$$s^{-\alpha}(s^{-1}\cdot)_*\left(I_{\infty,1,\alpha} - \sum_{k<N} I^{(k)}_{\infty,\alpha}\right) + s^{-\alpha}\log s\,(s^{-1}\cdot)_*\left(P_{1,\alpha} \circ \pi_1 - \sum_{k<N} P'_{\alpha,k}\right),$$

where $P_{1,\alpha}$ is defined in Proposition 4.3, while $P'_{\alpha,k}$ is a suitable homogeneous polynomial on $\mathfrak{h}_\infty$ of degree $-Q_\infty + \alpha - k$. Since the term

$$s^{-\alpha}(s^{-1}\cdot)_*\left(I_{\infty,1,\alpha} - \sum_{k<N} I^{(k)}_{\infty,\alpha}\right)$$

satisfies the estimates of the statement, all we need to prove is that $P_{1,\alpha} \circ \pi_1 - \sum_{k<N} P'_{\alpha,k}$ has degree at most $-Q_\infty + \alpha - N$. One may prove this by expressing $P_{1,\alpha}$ and the $P'_{\alpha,k}$ in terms of $h_{s,t}$ and its derivatives in s^{-1}. Nonetheless, since the above proof shows that

$I_{\infty,s,\alpha}-\sum_{k<N}s^{-k}I^{(k)}_{\infty,\alpha}$ satisfies the estimates of the statement (with constants depending on s), the same necessarily applies to $s^{-\alpha}\log s(s^{-1})_*\left(P_{1,\alpha}\circ\pi_1-\sum_{k<N}P'_{\alpha,k}\right)$, whence our claim.

2. Define $H_0(s,t,x):=h_{0,s,t}(x)$ for every $s\in[0,\infty)$, for every $t>0$, and for every $x\in\mathfrak{h}_0$, to simplify the notation. Take $\alpha\in\mathbb{C}$ such that $0<\operatorname{Re}\alpha<Q_0$, and observe that a Taylor expansion of H_0 in the first variable gives

$$I_{0,s,\alpha}=\sum_{k<N}s^k\frac{1}{k!\Gamma(\frac{\alpha}{\delta})}\int_0^1 t^{\frac{\alpha}{\delta}}\partial_1^k H_0(0,t,\cdot\,)\frac{\mathrm{d}t}{t}+s^N R_{s,\alpha,N}+\frac{1}{\Gamma(\frac{\alpha}{\delta})}\int_1^{+\infty}t^{\frac{\alpha}{\delta}}h_{0,s,t}\frac{\mathrm{d}t}{t},$$

where

$$R_{s,\alpha,N}=\frac{1}{\Gamma(\frac{\alpha}{\delta})(N-1)!}\int_0^1 t^{\frac{\alpha}{\delta}}\int_0^1\partial_1^N H_0(\theta s,t,\cdot\,)(1-\theta)^{N-1}\,\mathrm{d}\theta\,\frac{\mathrm{d}t}{t}.$$

Now, by means of Lemma 7.7 we see that the mapping $\alpha\mapsto\frac{1}{\Gamma(\frac{\alpha}{\delta})}\int_1^{+\infty}t^{\frac{\alpha}{\delta}}h_{0,s,t}\frac{\mathrm{d}t}{t}$ extends to a meromorphic function on $\mathbb{C}$ with values in $\mathcal{E}(G)$. In addition, as in **1** one may prove that

$$\partial_1^k H_0(0,t,x)=t^{-\frac{Q_0-k}{\delta}}\partial_1^k H_0(0,1,t^{-1/\delta}\cdot x)$$

for every $x\in\mathfrak{h}_0$ and for every $t>0$. Therefore, making use of the estimates of $\partial_1^k H_0(0,t,\cdot\,)$ provided in Theorem 3.1, we see that

$$R'_{s,\alpha,k}:=\frac{1}{k!\Gamma(\frac{\alpha}{\delta})}\int_1^{+\infty}t^{\frac{\alpha}{\delta}}\partial_1^k H_0(0,t,\cdot\,)\frac{\mathrm{d}t}{t}\in C^\infty(G_s),$$

initially defined for $\operatorname{Re}\alpha<Q_0-k$, extends to a meromorphic function on $\mathbb{C}$. In addition, we also see that the mapping, initially defined for $-k<\operatorname{Re}\alpha<Q_0-k$,

$$\alpha\mapsto I^{(k)}_{0,\alpha}:=\frac{1}{k!\Gamma(\frac{\alpha}{\delta})}\int_0^1 t^{\frac{\alpha}{\delta}}\partial_1^k H_0(0,t,\cdot\,)\frac{\mathrm{d}t}{t}+R'_{s,\alpha,k}$$

extends to a meromorphic mapping on $\mathbb{C}$.[8] Let us prove that $I^{(k)}_{0,\alpha}$ is homogeneous of degree $-Q_0+\alpha+k$ for α in the domain of holomorphy of $I^{(k)}_{0,\alpha}$. By analyticity, we may reduce to prove this fact for $-k<\operatorname{Re}\alpha<Q_0-k$, in which case

$$I^{(k)}_{0,\alpha}=\frac{1}{k!\Gamma(\frac{\alpha}{\delta})}\int_0^\infty t^{\frac{\alpha}{\delta}}\partial_1^k H_0(0,t,\cdot\,)\frac{\mathrm{d}t}{t},$$

so that the assertion is easily established. Log-homogeneity holds at the poles of $I^{(k)}_{0,\cdot}$, where $I^{(k)}_{0,\alpha}$ denote the zeroth-order term of the Laurent expansion of $I^{(k)}_{0,\cdot}$ at α.

Finally, take γ and assume that $\operatorname{Re}\alpha>Q_0+\mathrm{d}_\gamma-N$; in addition, define $|x|':=\inf_{s\in[0,1]}|\pi_s(x)|_s$ for every $x\in\mathfrak{h}_0$, and observe that $|x|'>0$ for every nonzero $x\in\mathfrak{h}_0$, and that there is a constant $C>0$ such that

$$\frac{1}{C}|x|\le|x|'\le C|x|$$

[8] Using the estimates of H_0 provided in Theorem 3.1, it is not hard to see that $\int_0^1 t^{\frac{\alpha}{\delta}}\partial_1^k H_0(0,t,\cdot\,)\frac{\mathrm{d}t}{t}\in L^1(G_s)$ for $\operatorname{Re}\alpha>-k$.

for every $x \in \mathfrak{h}_0$ such that $|x| \leq 1$ (cf. Proposition 2.11). In addition, Theorem 3.1 and the preceding computations imply that there are there are two constants $C' > 0$ and $b > 0$ such that

$$|\partial_0^\gamma R_{s,\alpha,N}(x)| \leq C' \int_0^1 t^{\frac{\operatorname{Re}\alpha - Q_0 - \mathrm{d}_\gamma + N}{\delta}} \mathrm{e}^{-b\left(\frac{|x|'}{t^{1/\delta}}\right)^{\frac{\delta}{\delta-1}}} \frac{\mathrm{d}t}{t},$$

so that $R_{s,\alpha,N}(x)$ is well defined for $x \neq 0$. In addition,

$$|\partial_0^\gamma R_{\alpha,N}(x)| \leq C' |x|'^{\operatorname{Re}\alpha - Q_0 - \mathrm{d}_\gamma + N} \int_0^{+\infty} t^{\frac{Q_0 + \mathrm{d}_\gamma - N - \operatorname{Re}\alpha}{\delta}} \mathrm{e}^{bt^{\frac{1}{\delta-1}}} \frac{\mathrm{d}t}{t}.$$

The assertion follows for s fixed. In order to get uniform estimates for $s \in (0, 1]$, reduce to the case $s = 1$ by means of Proposition 4.3 (argue as in **1**). □

Observe that, with the same techniques used to prove [30, Theorem 2], one may prove the following result.

Corollary 4.5 *Take $s \in (0, \infty)$, $\gamma \in \mathbb{N}^J$, and $\alpha \in \mathbb{C}$ such that $\operatorname{Re}\alpha \geq \mathrm{d}_\gamma$. Then, for every $p, q \in (1, \infty)$ such that $\frac{\operatorname{Re}\alpha - \mathrm{d}_\gamma}{Q_\infty} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\operatorname{Re}\alpha - \mathrm{d}_\gamma}{Q_0}$, convolution on the right with $\mathbf{X}_s^\gamma I_{s,\alpha}$ induces a bounded operator from $L^p(G_s)$ into $L^q(G_s)$.*

Notice that, if convolution on the right with $\mathbf{X}_s^\gamma I_{s,\alpha}$ induces a bounded operator T_s from $L^p(G_s)$ into $L^q(G_s)$ for some $p, q \in (1, \infty)$ and for some $s \in (0, \infty)$, then $\frac{\operatorname{Re}\alpha - \mathrm{d}_\gamma}{Q_\infty} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\operatorname{Re}\alpha - \mathrm{d}_\gamma}{Q_0}$. Indeed, take $r > 0$ and $f \in L^p(G_{r^{-1}s})$, and define $\rho_r(x) := r \cdot x$ for every $x \in G_{r^{-1}s}$. Then,

$$T_s(f \circ \rho_r) = (f \circ \rho_r) * (\mathbf{X}_s^\gamma I_{s,\alpha}) = [f * (\rho_r)_* (\mathbf{X}_s^\gamma I_{s,\alpha})] \circ \rho_r = r^{-\alpha + \mathrm{d}_\gamma} (T_{r^{-1}s} f) \circ \rho_r,$$

where $T_{r^{-1}s}$ is given by convolution on the right with $\mathbf{X}_{r^{-1}s}^\gamma I_{r^{-1}s,\alpha}$. Now, identify $G_{s'}$ with $\mathfrak{h}_0$ for every $s' \in [0, s]$. Observe that, denoting by $\nu_{\mathfrak{h}_0}$ the fixed Lebesgue measure on $\mathfrak{h}_0$, we have $\nu_{G_{s'}} = a_{0,s'} \nu_{\mathfrak{h}_0}$ for some $a_{0,s'} > 0$; in addition, the mapping $s' \mapsto a_{0,s'}$ is continuous on $[0, s]$ thanks to **5** of Proposition 2.11. Therefore, there is a constant $C > 0$ such that

$$\begin{aligned} C r^{-Q_0/p} \|f\|_{L^p(\mathfrak{h}_0)} = C \|f \circ \rho_r\|_{L^p(\mathfrak{h}_0)} &\geq r^{-\operatorname{Re}\alpha + \mathrm{d}_\gamma} \|(T_{r^{-1}s} f) \circ \rho_r\|_{L^q(\mathfrak{h}_0)} \\ &= r^{-\operatorname{Re}\alpha + \mathrm{d}_\gamma - Q_0/q} \|T_{r^{-1}s} f\|_{L^q(\mathfrak{h}_0)}. \end{aligned}$$

for every $f \in L^p(\mathfrak{h}_0)$ and for every $r \geq 1$. Now, $T_{r^{-1}s} f$ converges pointwise to $T_0 f$ as $r \to +\infty$ for every $f \in \mathcal{S}(\mathfrak{h}_0)$ with vanishing moments of all orders,[9] so that

$$\|T_0 f\|_{L^q(\mathfrak{h}_0)} \leq C \|f\|_{L^p(\mathfrak{h}_0)} \liminf_{r \to \infty} r^{\operatorname{Re}\alpha - \mathrm{d}_\gamma + Q_0\left(\frac{1}{q} - \frac{1}{p}\right)}$$

for every such f. Since $T_0 \neq 0$, it follows that $\operatorname{Re}\alpha - \mathrm{d}_\gamma \geq Q_0\left(\frac{1}{p} - \frac{1}{q}\right)$. The other inequality is proved similarly.

Proof We shall briefly indicate the procedure employed in [30], for the sake of completeness. When $p \neq q$, observe that $\mathbf{X}_s^\gamma I_{s,\alpha}$ belongs to weak L^r for every $r \in (1, \infty)$ such that $\frac{\operatorname{Re}\alpha - \mathrm{d}_\gamma}{Q_\infty} \leq \frac{1}{r'} \leq \frac{\operatorname{Re}\alpha - \mathrm{d}_\gamma}{Q_0}$ thanks to Theorem 4.4. Then, arguing as in the Proof of [10, Proposition 1.19], we see that weak L^r convolves $L^p(G_s)$ into $L^q(G_s)$ for $\frac{1}{p} - \frac{1}{q} = \frac{1}{r'}$ and $p, q, r \in (1, \infty)$.

[9] Notice that the set of such f is dense in $L^p(\mathfrak{h}_0)$, since $p \in (1, \infty)$.

When $p=q$, take $\tau\in C^\infty_c(G_s)$ so that τ equals 1 in a neighbourhood of e. We shall prove that $\tau\mathbf{X}_s^\gamma I_{s,\alpha}$ convolves $L^p(G_s)$ into itself; one may prove analogously that also $(1-\tau)\mathbf{X}_s^\gamma I_{s,\alpha}$ convolves $L^p(G_s)$ into itself and conclude the proof. Now, Proposition 2.7 and Theorem 4.4 show that $\tau\mathbf{X}_s^\gamma I_{s,\alpha}$ equals $\tau\mathbf{X}_0^\gamma I_{0,\alpha}$ up to an integrable function, under the identification of G_s with G_0 through $\mathfrak{h}_0$; consequently, it will suffice to show that $\tau\mathbf{X}_0^\gamma I_{0,\alpha}$ convolves $L^p(G_s)$ into itself (with respect to the convolution of G_s). Now, it is clear that there is a constant $C>0$ such that

$$|\mathbf{X}_0^{\gamma'}(\tau\mathbf{X}_0^\gamma I_{0,\alpha})(x)|\leq C|x|_s^{-Q_0-\mathrm{d}_{\gamma'}}$$

for every $x\in G_s$ and for every γ' with length at most 1, thanks to **6** of Proposition 2.11. By Lemma 2.14, we then see that we may take C in such a way that

$$|(\tau\mathbf{X}_0^\gamma I_{0,\alpha})(xy^{-1})-(\tau\mathbf{X}_0^\gamma I_{0,\alpha})(x)|\leq C\frac{|y|_s}{|x|_s^{Q_0+1}}$$

for every $x,y\in G_s$ such that $|x|_s>2|y|_s>0$. In addition, since $\mathbf{X}_0^\gamma I_{0,\alpha}$ is a homogeneous distribution of degree $-Q_0+\alpha-\mathrm{d}_\gamma$, it is clear that $\mathbf{X}_0^\gamma I_{0,\alpha}$ has zero mean on the unit sphere (relative to $|\cdot|_0$) *when* $\operatorname{Im}\alpha=0$. Similar remarks apply to $(\mathbf{X}_0^\gamma I_{0,\alpha})^*$. Taking into account [12, Lemma of Chapter III, § 3.1], it is not hard to see that we may apply [12, Theorem of Chapter III, § 4.3], so that $\tau\mathbf{X}_0^\gamma I_{0,\alpha}$ convolves $L^p(G_s)$ into itself for every $p\in(1,\infty)$. □

Remark 4.6 Observe that, if $G=\mathbb{R}^n$ and $\mathcal{L}=\Delta^2-\Delta$, then it is not hard to prove that $I_{1,\alpha}=I_\alpha^\Delta * J_\alpha$, where $J_\alpha=((1-\Delta)^{-\frac{\alpha}{2}})\delta_0$ and I_α^Δ is the kernel of $(-\Delta)^{-\frac{\alpha}{2}}$ defined by analytic continuation, for every $\alpha\in\mathbb{C}$. Then, observe that, for $\alpha\in(0,\infty)\setminus(n+\mathbb{N})$, I_α^Δ and J_α keep a constant sign (in particular, they vanish nowhere), so that

$$I_{1,\alpha}(0)=(J_\alpha * I_\alpha^\Delta)(0)=\int_{\mathbb{R}^n}J_\alpha(x)I_\alpha^\Delta(-x)\,\mathrm{d}x\neq 0$$

when $\alpha>n$. Hence, the polynomials appearing in the local expansion of I_α in Theorem 4.4 cannot be omitted, in general.

Theorem 4.7 *Take $\alpha\in\mathbb{C}$ and $s\in\mathbb{R}_+$. Then, the following hold:*

1. *assume that $G_1=G_\infty$ (under the identification through $\mathfrak{h}_\infty$) as Lie groups and that $[\mathcal{L}_1,\mathcal{L}_\infty]=0$. Let d_∞ be the least degree of the nonzero homogeneous components of $\mathcal{L}_1-\mathcal{L}_\infty$. Then, for every $N\in\mathbb{N}$ and for every γ there is a constant $C_{N,\gamma}>0$ such that, for every $s\in[1,\infty)$,*

$$\left|\mathbf{X}_\infty^\gamma\left(I_{s,\alpha}-\sum_{k<N}\binom{-\alpha/\delta}{k}(\mathcal{L}_s-\mathcal{L}_\infty)^k I_{\infty,\alpha+\delta k}\right)(x)\right|\leq\frac{s^{-N}C_{N,\gamma}}{|x|^{Q_\infty-\operatorname{Re}\alpha+\mathrm{d}_\gamma+N(d_\infty-\delta)}}(1+|\log|s\cdot x||)$$

for every $x\in\mathfrak{h}_\infty$ such that $|x|\geq s^{-1}$; the factor $1+\log|s\cdot x|$ may be omitted if $\alpha\notin Q_\infty+\mathrm{d}_\gamma+N(d_\infty-\delta)+\mathbb{N}$;
2. *assume that $G_1=G_0$ (under the identification through $\mathfrak{h}_0$) as Lie groups and that $[\mathcal{L}_1,\mathcal{L}_0]=0$. Let d_0 be the greatest degree of the nonzero homogeneous components of $\mathcal{L}_1-\mathcal{L}_0$. Then, there is a sequence $(P_{\alpha,k})$ of homogeneous polynomials on G_0 such that*

$P_{\alpha,k}$ has degree k for every $k \in \mathbb{N}$, and such that for every $N \in \mathbb{N}$ and for every γ there is a constant $C_{N,\gamma} > 0$ such that, for every $s \in (0, 1]$,

$$\left|\mathbf{X}_0^{\gamma}\left(I_{s,\alpha} - \sum_{k<N}\binom{-\alpha/\delta}{k}(\mathcal{L}_s - \mathcal{L}_0)^k I_{0,\alpha+\delta k} - \sum_{k<-Q_0+\operatorname{Re}\alpha+N(\delta-d_0)} s^{k-Q_0+\alpha} P_{\alpha,k}\right)(x)\right| \leq \frac{s^N C_{N,\gamma}(1+|\log|s\cdot x||)}{|x|^{Q_0-\operatorname{Re}\alpha+\mathrm{d}_\gamma-N(\delta-d_0)}}$$

for every $x \in \mathfrak{h}_0$ such that $0 \neq |x| \leq s^{-1}$; the factor $1 + \log|s \cdot x|$ may be omitted if $\alpha \notin Q_0 + \mathrm{d}_\gamma - N(\delta - d_0) + \mathbb{N}$.

Let us describe some examples. Take a two-step nilpotent Lie group G and a hypoelliptic sub-Laplacian $\mathcal{L}$ thereon. Then, we may endow G with the structure of a stratified group in such a way that $\mathcal{L} = \mathcal{L}_\infty + \mathcal{L}'$, where $\mathcal{L}_\infty$ and $\mathcal{L}'$ are homogeneous sums of squares of degrees 2 and 4, respectively. By means of the construction described in Introduction, we may choose a two-step stratified group $\widetilde{G}$ and a sub-Laplacian $\widetilde{\mathcal{L}}$ in such a way that $G_s = G_\infty$ as Lie groups[10] and $\mathcal{L}_s = \mathcal{L}_\infty + s^{-2}\mathcal{L}'$ for every $s \in (0, \infty]$. Thus, in this case the first part of Theorem 4.7 applies.

If, in the preceding example, we define $\mathcal{L}_1 = \mathcal{L}_\infty^k + \mathcal{L}'$ for some $k \geq 3$, then, applying an analogous construction, we get $G_s = G_0$ as Lie groups and $\mathcal{L}_s = \mathcal{L}_\infty^k + s^{2(k-1)}\mathcal{L}'$ for every $s \in [0, \infty)$, so that the second part of Theorem 4.7 applies.

Proof Assume that $G_1 = G_\infty$ as Lie groups and that $[\mathcal{L}_1, \mathcal{L}_\infty] = 0$. Define, for every $t > 0$, for every $s \in (0, \infty]$, and for every $\theta \in [0, 1]$,

$$h_{s,t}^{(\theta)} := h_{\infty,(1-\theta)t} * h_{s,\theta t};$$

observe that $(h_{s,t}^{(\theta)})_t$ is a semi-group under convolution and that the mapping $\theta \mapsto h_{s,t}^{(\theta)} \in \mathcal{S}'(G)$ is of class C^∞ on $[0, 1]$, with

$$\frac{\mathrm{d}^k}{\mathrm{d}\theta^k} h_{s,t}^{(\theta)} = (-t)^k(\mathcal{L}_s - \mathcal{L}_\infty)^k h_{s,t}^{(\theta)}$$

for every $t > 0$, for every $s \in (0, \infty]$, and for every $\theta \in [0, 1]$. Now, Proposition 2.11 and Theorem 3.1 imply that for every γ and for every $k \in \mathbb{N}$ there are two constants $C, b > 0$ such that

$$|\mathbf{X}_\infty^\gamma(\mathcal{L}_s - \mathcal{L}_\infty)^k h_{s,t}^{(\theta)}(x)| \leq \frac{C}{t^{\frac{Q_\infty+\mathrm{d}_\gamma+kd_\infty}{\delta}}} \mathrm{e}^{-b\left(\frac{|x|_s}{t^{1/\delta}}\right)^{\frac{\delta}{\delta-1}}}$$

for every $s \in [1, \infty]$, for every $t \geq 1$, for every $\theta \in [0, 1]$, and for every $x \in G_s$. Now, take $\alpha \in \mathbb{C}$ such that $0 < \operatorname{Re}\alpha < Q_\infty$, and observe that a Taylor expansion of $h_{s,t}^{(\theta)}$ in θ gives

$$I_{s,\alpha} = \frac{1}{\Gamma(\frac{\alpha}{\delta})}\int_0^1 t^{\frac{\alpha}{\delta}} h_{s,t}\,\frac{\mathrm{d}t}{t} + \sum_{k<N}\frac{(-1)^k}{k!\Gamma(\frac{\alpha}{\delta})}\int_1^{+\infty} t^{\frac{\alpha}{\delta}+k}(\mathcal{L}_s - \mathcal{L}_\infty)^k h_{\infty,t}\,\frac{\mathrm{d}t}{t} + R_{s,\alpha,N},$$

where

$$R_{s,\alpha,N} = \frac{(-1)^N}{\Gamma(\frac{\alpha}{\delta})(N-1)!}\int_1^{+\infty} t^{\frac{\alpha}{\delta}+N}\int_0^1 (\mathcal{L}_s - \mathcal{L}_\infty)^N h_{s,t}^{(\theta)}(1-\theta)^{N-1}\,\mathrm{d}\theta\,\frac{\mathrm{d}t}{t}.$$

[10] This is a general fact when $\widetilde{G}$ is a two-step stratified group, cf. the Proof of Theorem 5.17

The proof then proceeds as that of Theorem 4.4.

The case in which $G_s = G_0$ and $[\mathcal{L}_s, \mathcal{L}_0] = 0$ is treated similarly. □

5 Spectral measures and multipliers

In this section, we assume that $\widetilde{\mathcal{L}}$ is Rockland and formally self-adjoint, but not necessarily positive. Then $\mathcal{L}_s^2$ is weighted subcoercive, so that $(\mathcal{L}_s)$ is a weighted subcoercive system in the sense of [20,21]. Then, the operator $\mathcal{L}_s$, considered as an unbounded operator on $L^2(G_s)$ with initial domain $C_c^\infty(G_s)$, is essentially self-adjoint (cf. [21, Proposition 3.2]). We shall then denote by $\sigma(\mathcal{L}_s)$ the corresponding spectrum.

Now, if $m\colon \sigma(\mathcal{L}_s) \to \mathbb{C}$ is bounded and Borel measurable, then there is a unique distribution $\mathcal{K}_{\mathcal{L}_s}(m)$ on G_s such that

$$m(\mathcal{L}_s)\varphi = \varphi * \mathcal{K}_{\mathcal{L}_s}(m)$$

for every $\varphi \in C_c^\infty(G_s)$ (cf. [21, Subsection 3.2]). In addition, there is a unique positive Radon measure $\beta_{\mathcal{L}_s}$ on $\sigma(\mathcal{L}_s)$ such that $\mathcal{K}_{\mathcal{L}_s}$ extends to an isometry of $L^2(\beta_{\mathcal{L}_s})$ into $L^2(G_s)$ (cf. [21, Theorem 3.10]).

Lemma 5.1 *There is a constant $C > 0$ such that*

$$\beta_{\mathcal{L}_s}([-r, r]) \leq C \frac{\min\left((s^{-1} r^{1/\delta})^{Q_0}, (s^{-1} r^{1/\delta})^{Q_\infty}\right)}{\min(s^{-Q_0}, s^{-Q_\infty})}$$

for every $s \in [0, \infty]$ and for every $r > 0$. In particular, $\beta_{\mathcal{L}_s}(\{0\}) = 0$ for every $s \in [0, \infty]$.

For the proof, argue as in [34, § 2], using the estimates for the heat kernel associated with $\mathcal{L}_s^2$ provided in Theorem 3.1.

Proposition 5.2 *The following hold:*

1. *for every bounded Borel measurable function $m\colon \mathbb{R} \to \mathbb{C}$, for every $s \in [0, \infty]$, and for every $r > 0$,*

$$\mathcal{K}_{\mathcal{L}_{rs}}(m) = (r^{-1} \cdot)_* \mathcal{K}_{\mathcal{L}_s}(m(r^\delta \cdot));$$

2. *$\beta_{\mathcal{L}_{rs}} = \nu_{G_s}(B_s(r))(r^\delta \cdot)_*(\beta_{\mathcal{L}_s})$;*
3. *the mapping $[0, \infty] \ni s' \mapsto \beta_{\mathcal{L}_{s'}}$ is vaguely continuous.*

Proof Fix $s \in [0, \infty]$, $r > 0$, and $m \in \mathcal{S}(\mathbb{R})$, so that $\mathcal{K}_{\widetilde{\mathcal{L}}}(m) \in \mathcal{S}(\widetilde{G})$, where $\mathcal{K}_{\widetilde{\mathcal{L}}}(m)$ denotes the right convolution kernel of $m(\widetilde{\mathcal{L}})$ (cf. [20, Proposition 4.2.1]). Now, [21, Proposition 3.7], applied to the quasi-regular representation of $\widetilde{G}$ in $L^2(G_s)$, implies that

$$(\pi_s)_*(\mathcal{K}_{\widetilde{\mathcal{L}}}(m)) = \mathcal{K}_{\mathcal{L}_s}(m).$$

Now, $\pi_{rs} = (r^{-1} \cdot) \circ \pi_s \circ (r \cdot)$; in addition, since $(r \cdot)_* \widetilde{\mathcal{L}} = r^\delta \widetilde{\mathcal{L}}$ by homogeneity, we have

$$\mathcal{K}_{\mathcal{L}_{rs}}(m) = (r^{-1} \cdot)_*(\pi_s)_*(\mathcal{K}_{\widetilde{\mathcal{L}}}(m(r^\delta \cdot))) = (r^{-1} \cdot)_* \mathcal{K}_{\mathcal{L}_s}(m(r^\delta \cdot)).$$

The spectral calculus then shows that $\mathcal{K}_{\mathcal{L}_{rs}}(m) = (r^{-1} \cdot)_* \mathcal{K}_{\mathcal{L}_s}(m(r^\delta \cdot))$ for every bounded Borel measurable function $m\colon \mathbb{R} \to \mathbb{C}$. In addition, if $m \in \mathcal{S}(\mathbb{R})$, then, identifying $\mathcal{K}_{\mathcal{L}_{rs}}(m)$

and $\mathcal{K}_{\mathcal{L}_s}(m)$ with their densities with respect to $\nu_{G_{rs}}$ and ν_{G_s}, respectively,

$$\begin{aligned}\int_{\mathbb{R}} m\,\mathrm{d}\beta_{\mathcal{L}_{rs}}(\lambda) &= \mathcal{K}_{\mathcal{L}_{rs}}(m)(e)\\ &= \nu_{G_s}(B_s(r))\mathcal{K}_{\mathcal{L}_s}(m(r^\delta\,\cdot))(e)\\ &= \nu_{G_s}(B_s(r))\int_{\mathbb{R}} m\,\mathrm{d}(r^\delta\,\cdot)_*\beta_{\mathcal{L}_s}(\lambda),\end{aligned}$$

so that $\beta_{\mathcal{L}_{rs}} = \nu_{G_s}(B_s(r))(r^\delta\,\cdot)_*\beta_{\mathcal{L}_s}$ by the arbitrariness of m. In addition, it is easily seen that $\mathcal{K}_{\mathcal{L}_{s'}}(m)(e)$ converges to $\mathcal{K}_{\mathcal{L}_s}(m)(e)$ as $s' \to s$ (see also Lemma 5.4). Thanks to Lemma 5.1 and the preceding remarks, this is sufficient to prove that $\beta_{\mathcal{L}_{s'}}$ converges vaguely to $\beta_{\mathcal{L}_s}$ as $s' \to s$. □

5.1 Asymptotic developments

Definition 5.3 For every $m \in \mathcal{S}(\mathbb{R})$ and for every $s \in [0,\infty]$, we denote by $\mathcal{K}_{0,s}(m)$ (for $s \neq \infty$) and $\mathcal{K}_{\infty,s}(m)$ (for $s \neq 0$) the densities of the measures corresponding to $\mathcal{K}_{\mathcal{L}_s}(m)$ on $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$, respectively, under the usual identifications.

Lemma 5.4 *The mappings*

$$[0,\infty) \ni s \mapsto \mathcal{K}_{0,s} \in \mathcal{L}(\mathcal{S}(\mathbb{R}); \mathcal{S}(\mathfrak{h}_0)) \quad \textit{and} \quad [0,\infty) \ni s \mapsto \mathcal{K}_{\infty,{}^1\!/_s} \in \mathcal{L}(\mathcal{S}(\mathbb{R}); \mathcal{S}(\mathfrak{h}_\infty))$$

are of class C^∞.

Proof We prove only the first assertion. Observe that $\mathcal{K}_{\mathcal{L}_s} = (\pi_s)_* \circ \mathcal{K}_{\widetilde{\mathcal{L}}}$; since $\mathcal{K}_{\widetilde{\mathcal{L}}} \in \mathcal{L}(\mathcal{S}(\mathbb{R}); \mathcal{S}(\widetilde{G}))$ by [20, Proposition 4.2.1], it will suffice to prove that the mapping $[0,\infty) \ni s \mapsto (P_{0,s})_* \in \mathcal{L}(\mathcal{S}(\widetilde{\mathfrak{g}}); \mathcal{S}(\mathfrak{h}_0))$ is of class C^∞. Now, for every $s \in [0,\infty)$, denote by L_s the automorphism $x + y \mapsto x + y + \psi_{0,s}(y)$ of (the vector space) $\widetilde{\mathfrak{g}} \cong \mathfrak{h}_0 \oplus \mathfrak{i}_0$, and observe that L_s depends polynomially on s, so that we may define L_s for every $s \in \mathbb{R}$. With this modification, it is readily verified that L_s is still a measure-preserving automorphism of $\widetilde{\mathfrak{g}}$ for every $s \in \mathbb{R}$, since $\psi_{0,s}$ is strictly subhomogeneous. Then, observe that $P_{0,s} = P_{0,0} \circ L_s^{-1}$ for every $s \in [0,\infty)$; since $(P_{0,0})_* \in \mathcal{L}(\mathcal{S}(\widetilde{\mathfrak{g}}); \mathcal{S}(\mathfrak{h}_0))$, it will suffice to prove that the mapping $\mathbb{R} \ni s \mapsto (L_s^{-1})_* \in \mathcal{L}(\mathcal{S}(\widetilde{\mathfrak{g}}))$ is of class C^∞. However, this last assertion is an easy consequence of the fact that the mapping $\mathbb{R} \ni s \mapsto L_s \in \mathcal{L}(\widetilde{\mathfrak{g}})$ is of class C^∞. □

Definition 5.5 For every $k \in \mathbb{N}$, for every $s \in [0,\infty)$, and for every $m \in \mathcal{S}(\mathbb{R})$, define

$$\mathcal{K}_{0,s}^{(k)}(m) = \left.\frac{\mathrm{d}^k}{\mathrm{d}s'^k}\right|_{s'=s} \mathcal{K}_{0,s'}(m),$$

and

$$\mathcal{K}_{\infty,{}^1\!/_s}^{(k)}(m) = \left.\frac{\mathrm{d}^k}{\mathrm{d}s'^k}\right|_{s'=s} \mathcal{K}_{\infty,{}^1\!/_{s'}}(m).$$

Lemma 5.6 *Take $k \in \mathbb{N}$ and $m \in \mathcal{S}(\mathbb{R})$. Then, for every $s \in [0,\infty)$ and for every $r > 0$,*

$$\begin{aligned}\mathcal{K}_{0,s}^{(k)}(m(r^\delta\,\cdot)) &= r^k (r\,\cdot)_* \mathcal{K}_{0,rs}^{(k)}(m)\\ \mathcal{K}_{\infty,{}^1\!/_s}^{(k)}(m(r^\delta\,\cdot)) &= r^{-k} (r\,\cdot)_* \mathcal{K}_{\infty,{}^r\!/_s}^{(k)}(m)\end{aligned}$$

Proof The assertion follows from Proposition 5.2 by differentiation. □

Definition 5.7 Take $r \in \mathbb{R}$, and define $\mathcal{M}_r(\mathbb{R}^*)$ as the set of $m \in C^\infty(\mathbb{R}^*)$ such that for every $k \in \mathbb{N}$ there is a constant $C_k > 0$

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} m(\lambda) \right| \leq C_k |\lambda|^{r-k}$$

for every $\lambda \in \mathbb{R}^*$. We endow $\mathcal{M}_r(\mathbb{R}^*)$ with the corresponding semi-norms.

Definition 5.8 Take $r \in \mathbb{R} \cup \{ \infty \}$ and $s \in [0, \infty]$. We define $\mathcal{S}'_r(G_s)$ as the dual of the set $\mathcal{S}_r(G_s)$ of $\varphi \in \mathcal{S}(G_s)$ such that $\int_{G_s} \varphi(x) P(x) \, \mathrm{d}x = 0$ for every polynomial P such that $P(x) = O(|x|_s^r)$ for $x \to \infty$; we thus identify $\mathcal{S}'_r(G_s)$ with the quotient of $\mathcal{S}'(G_s)$ by the set of the polynomials as above. Similar definitions replacing G_s with $\mathfrak{h}_0$ or $\mathfrak{h}_\infty$ and $|\cdot|_s$ with $|\cdot|$.

Observe that, if $s \in (0, \infty]$, then $\mathcal{S}_r(G_s) = \mathcal{S}_r(\mathfrak{h}_\infty)$ under the identification of G_s with $\mathfrak{h}_\infty$ (cf. **6** of Proposition 2.11).

Definition 5.9 Take $r \in \mathbb{R}$, and define $\mathcal{CZ}_r(\mathfrak{h}_0)$ as the set of $K \in \mathcal{S}'_{-Q_0-r}(\mathfrak{h}_0)$, such that the following hold:

– for every α such that $\mathrm{d}_\alpha > -Q_0 - r$, $\partial_0^\alpha K$ has a density of class C^∞ on $\mathfrak{h}_0 \setminus \{ 0 \}$ and there is a constant $C_\alpha > 0$ such that

$$\left| \partial_0^\alpha K \right| \leq \frac{C_\alpha}{|x|^{Q_0 + r + \mathrm{d}_\alpha}}$$

for every nonzero $x \in \mathfrak{h}_0$;
– there is a constant $C > 0$ such that

$$|\langle K, \varphi(\theta \, \cdot) \rangle| \leq C\theta^r$$

for every $\varphi \in \mathcal{S}_{-Q_0-r}(\mathfrak{h}_0)$ such that $\operatorname{Supp} \varphi \subseteq B(1)$, and $\|\partial_0^\alpha \varphi\|_\infty \leq 1$ for every α with length at most $([r] + 1)_+$.

We endow $\mathcal{CZ}_r(\mathfrak{h}_0)$ with the corresponding semi-norms.

We define $\mathcal{CZ}_r(\mathfrak{h}_\infty)$ in a similar way. For every $s \in (0, \infty)$, we define $\mathcal{CZ}_r(G_s)$ as the set of $K \in \mathcal{S}'_{-Q_\infty-r}(G_s)$ such that there are $K_0 \in \mathcal{E}'(G_s) + \mathcal{S}(G_s)$ and $K_\infty \in C^\infty(G_s) \cap \mathcal{S}'_{-Q_\infty-r}(G_s)$ such that $K = K_0 + K_\infty$, and such that the distributions on $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$ corresponding to K_0 and K_∞ belong to $\mathcal{CZ}_r(\mathfrak{h}_0)$ and $\mathcal{CZ}_r(\mathfrak{h}_\infty)$, respectively. We endow $\mathcal{CZ}_r(G_s)$ with the corresponding topology.

Finally, we denote by $\nu_{\mathbb{R}_+}$ the Haar measure on $(\mathbb{R}_+, \cdot)$ such that $\int_0^\infty f \, \mathrm{d}\nu_{\mathbb{R}_+} = \int_0^\infty f(x) \frac{\mathrm{d}x}{x}$ for every $f \in C_c(\mathbb{R}_+)$.

Proposition 5.10 *Take $r \in \mathbb{R}$, a set B and a bounded family $(\varphi_{t,b})_{t \in (0,\infty), b \in B}$ of elements of $\mathcal{S}_r(\mathfrak{h}_0)$. Then, the mapping $t \mapsto t^{-r}(t \, \cdot)_* \varphi_{t,b} \in \mathcal{S}'_{-Q_0-r}(\mathfrak{h}_0)$ is $\nu_{\mathbb{R}_+}$-integrable and the set of*

$$K_b := \int_0^{+\infty} t^{-r}(t \, \cdot)_* \varphi_{t,b} \, \frac{\mathrm{d}t}{t},$$

as b runs through B, is bounded in $\mathcal{CZ}_r(\mathfrak{h}_0)$. In addition, K_b has a representative $\widetilde{K}_b$, for every $b \in B$, such that for every α there is a constant $C_\alpha > 0$ such that

$$|\partial_0^\alpha \widetilde{K}_b(x)| \leq \frac{C_\alpha}{|x|^{Q_0 + r + \mathrm{d}_\alpha}} (1 + |\log|x||)$$

for every $x \in \mathfrak{h}_0 \setminus \{0\}$*, and for every* $b \in B$*; the factor* $1 + |\log|x||$ *may be omitted if* $-Q_0 - r - \mathrm{d}_\alpha \notin \mathbb{N}$*. In addition, if L is a bounded subset of* $C_c^{([r]+1)_+}(\mathfrak{h}_0)$*, then there is a constant* $C' > 0$ *such that*

$$\left|\langle \widetilde{K}_b, \psi(\theta\,\cdot)\rangle\right| \leq C'\theta^r(1 + |\log\theta|)$$

for every $\theta > 0$*, for every* $\psi \in L$*, and for every* $b \in B$*; the factor* $1 + |\log\theta|$ *may be omitted if* $-Q_0 - r \notin \mathbb{N}$.

Notice that, arguing in the spirit of [32, Theorem 2.2.1], where the case $r = 0$ is essentially considered, one may prove that for every bounded family $(K_b)_{b\in B}$ of elements of $\mathcal{CZ}_r(\mathfrak{h}_0)$ there is a bounded family $(\varphi_{t,b})_{t>0,b\in B}$ of elements of $\mathcal{S}_\infty(\mathfrak{h}_0)$ such that $K_b = \int_0^{+\infty} t^{-r}(t\,\cdot)_*\varphi_{t,b}\,\frac{\mathrm{d}t}{t}$ for every $b \in B$.

Analogous statements hold for $\mathfrak{h}_\infty$.

Proof We shall divide the proof into several steps.

1. Let L be a subset of $\mathcal{S}_{-Q_0-r}(\mathfrak{h}_0)$ which is bounded in $C_c^{([r]+1)_+}(\mathfrak{h}_0)$. For every $\psi \in L$, denote by P_ψ the Taylor polynomial of ψ of degree $[r]$ about 0. Then, there is a constant $C_1 > 0$ such that

$$|(\psi - P_\psi)(x)| \leq C_1|x|^{([r]+1)_+}$$

for every $x \in \mathfrak{h}_0$. Since $\varphi_{t,b} \in \mathcal{S}_r(\mathfrak{h}_0)$ for every $t > 0$ and for every $b \in B$,

$$\begin{aligned}\int_0^{1/\theta} t^{-r}|\langle (t\,\cdot)_*\varphi_{t,b}, \psi(\theta\,\cdot)\rangle|\,\frac{\mathrm{d}t}{t} &= \int_0^{1/\theta} t^{-r}|\langle \varphi_{t,b}, (\psi - P_\psi)(t\theta\,\cdot)\rangle|\,\frac{\mathrm{d}t}{t}\\ &\leq \theta^r C_1 \int_0^1 t^{([r]+1)_+-r}\left\||\cdot|^{([r]+1)_+}\varphi_{t,b}\right\|_1 \frac{\mathrm{d}t}{t}\end{aligned}$$

for every $\theta > 0$, for every $\psi \in L$, and for every $b \in B$. On the other hand, denote by $P_{t,b}$ the Taylor polynomial of $\varphi_{t,b}$ of degree $-Q_0 + [-r]$ about 0, for every $t > 0$ and for every $b \in B$, and observe that there is a constant $C_2 > 0$ such that

$$|(\varphi_{t,b} - P_{t,b})(x)| \leq C_2|x|^{(-Q_0+[-r]+1)_+},$$

for every $t > 0$ and for every $b \in B$. Then, since $\psi \in \mathcal{S}_{-Q_0-r}(\mathfrak{h}_0)$,

$$\begin{aligned}\int_{1/\theta}^{+\infty} t^{-r}|\langle (t\,\cdot)_*\varphi_{t,b}, \psi(\theta\,\cdot)\rangle|\,\frac{\mathrm{d}t}{t} &= \int_{1/\theta}^{+\infty} t^{-r}|\langle (t\theta)_*(\varphi_{t,b} - P_{t,b}), \psi\rangle|\,\frac{\mathrm{d}t}{t}\\ &\leq \theta^r C_2\left\||\cdot|^{(-Q_0+[-r]+1)_+}\psi\right\|_1 \int_1^{+\infty} t^{-Q_0-r-(-Q_0+[-r]+1)_+}\,\frac{\mathrm{d}t}{t}.\end{aligned}$$

Next, take α such that $\mathrm{d}_\alpha > -Q_0 - r$, and observe that there is a constant $C_{3,\alpha} > 0$ such that

$$|\partial_0^\alpha \varphi_{t,b}(x)| \leq \frac{C_{3,\alpha}}{(1 + |x|)^{Q_0+r+\mathrm{d}_\alpha+1}}$$

for every $x \in \mathfrak{h}_0$, for every $t > 0$, and for every $b \in B$. Then, fix a nonzero $x \in \mathfrak{h}_0$, and observe that

$$\begin{aligned}\int_0^{+\infty} t^{-r}\left|\partial_0^\alpha (t\,\cdot)_*\varphi_{t,b}(x)\right|\frac{\mathrm{d}t}{t} &= \int_0^{+\infty}\left(\frac{t}{|x|}\right)^{Q_0+r+\mathrm{d}_\alpha}\left|\partial_0^\alpha\varphi_{\frac{|x|}{t},b}\left(\frac{t}{|x|}\cdot x\right)\right|\frac{\mathrm{d}t}{t}\\ &\leq C_{3,\alpha}|x|^{-Q_0-r-\mathrm{d}_\alpha}\int_0^{+\infty}\frac{t^{Q_0+r+\mathrm{d}_\alpha}}{(1+t)^{Q_0+r+\mathrm{d}_\alpha+1}}\,\frac{\mathrm{d}t}{t}\end{aligned}$$

for every $b \in B$.

Taking into account all the preceding inequalities, we see that the mapping $t \mapsto t^{-r}(t\,\cdot)_*\varphi_{t,b} \in \mathcal{S}'_{-Q_0-r}(\mathfrak{h}_0)$ is $\nu_{\mathbb{R}_+}$-integrable and that the set of K_b, as b runs through B, is bounded in $\mathcal{CZ}_r(\mathfrak{h}_0)$.

2. Keep the notation of **1**, and denote by $P_{t,b,j}$ the homogeneous component of $P_{t,b}$ of degree j, for every $j = 0, \dots, -Q_0 + [-r]$; define $P'_{t,b} := \sum_{j<-Q_0-r} P_{t,b,j}$. Then, the arguments of **1** show that

$$\widetilde{K}_b := \int_0^1 t^{-r}(t\,\cdot)_*(\varphi_{t,b} - P'_{t,b})\,\frac{\mathrm{d}t}{t} + \int_1^{+\infty} t^{-r}(t\,\cdot)_*(\varphi_{t,b} - P_{t,b})\,\frac{\mathrm{d}t}{t}$$

defines a representative of K_b in $\mathcal{S}'(\mathfrak{h}_0)$ (treat $\int_0^1 t^{-r}(t\,\cdot)_* P'_{t,b}\,\frac{\mathrm{d}t}{t}$ separately). In addition, arguing as in **1** we see that, for every α,

$$|x|^{Q_0+r+\mathrm{d}_\alpha}\left[\int_0^{|x|} t^{-r}\partial_0^\alpha(t\,\cdot)_*\varphi_{t,b}(x)\,\frac{\mathrm{d}t}{t} + \int_{|x|}^{+\infty} t^{-r}\partial_0^\alpha(t\,\cdot)_*(\varphi_{t,b} - P_{t,b})(x)\,\frac{\mathrm{d}t}{t}\right]$$

is uniformly bounded as x runs through $\mathfrak{h}_0 \setminus \{0\}$, and b runs through B. Now, take $j \in \mathbb{N}$ such that $j \le -Q_0 + [-r]$. If $j < -Q_0 - r$, then clearly

$$|x|^{r+Q_0+\mathrm{d}_\alpha}\left|\int_0^{|x|} t^{-r}\partial_0^\alpha(t\,\cdot)_* P_{t,b,j}(x)\,\frac{\mathrm{d}t}{t}\right|$$

is bounded as x runs through $\mathfrak{h}_0 \setminus \{0\}$, and b runs through B. Finally, if $j = -Q_0 - r$, then clearly

$$\frac{|x|^{r+Q_0+\mathrm{d}_\alpha}}{1+|\log|x||}\left|\int_1^{|x|} t^{-r}\partial_0^\alpha(t\,\cdot)_* P_{t,b,j}(x)\,\frac{\mathrm{d}t}{t}\right|$$

is bounded as x runs through $\mathfrak{h}_0 \setminus \{0\}$, and b runs through B. The other estimates are proved in a similar way. Thus, $\widetilde{K}_b$ is the required representative of K_b. □

Corollary 5.11 *Take $s \in (0,\infty)$, $r \in \mathbb{R}$, a set B, and a family $(\varphi_{t,b})_{t>0,b\in B}$ such that $\varphi_{t,b} \in \mathcal{S}_r(G_{st})$ for every $t > 0$ and for every $b \in B$, and such that for every $k \in \mathbb{N}$ there is a constant $C_k > 0$ such that*

$$|\mathbf{X}_{st}^\gamma \varphi_{t,b}(x)| \le \frac{C_k}{(1+|x|_{st})^k}$$

for every γ such that $\mathrm{d}_\gamma \le k$, for every $b \in B$, for every $t > 0$, and for every $x \in G_{st}$. Then, the mapping $t \mapsto t^{-r}(t\,\cdot)_*\varphi_{t,b} \in \mathcal{S}'_{-Q_\infty-r}(G_s)$ is $\nu_{\mathbb{R}_+}$-integrable for every $b \in B$, and the set of

$$K_b := \int_0^{+\infty} t^{-r}(t\,\cdot)_*(\varphi_{t,b}\nu_{G_{st}})\,\frac{\mathrm{d}t}{t},$$

as b runs through B, is bounded in $\mathcal{CZ}_r(G_s)$.

Proof By an abuse of notation, we shall identify $G_{s'}$ with $\mathfrak{h}_0$ if $s' \in (0,1)$ and with $\mathfrak{h}_\infty$ if $s' \in (1,\infty)$. In addition, we shall identify the measures $\varphi_{t,b}\nu_{G_{st}}$ with its density $\widetilde{\varphi}_{t,b}$ with respect to the fixed Lebesgue measure of $\mathfrak{h}_0$, for $t < s^{-1}$, or to the fixed Lebesgue measure of $\mathfrak{h}_\infty$, for $t > s^{-1}$. Then, $\widetilde{\varphi}_{t,b}$ differs from $\varphi_{t,b}\circ\pi_{st}$ by a multiplicative constant which stays bounded as t runs through $\mathbb{R}_+$.

Observe first that, using Proposition 2.4 and **6** of Proposition 2.11, it is not difficult to show that there is a constant $C > 0$ such that

$$1 + |x|_{s'} \geq C(1 + |x|^{1/n})$$

for every $x \in \mathfrak{h}_0$ and for every $s' \in (0, 1)$, while

$$1 + |x|_{s'} \geq C(1 + |x|)$$

for every $x \in \mathfrak{h}_\infty$ and for every $s' \in (1, \infty)$.

Hence, the set of $\widetilde{\varphi}_{t,b}$, as t runs through $(0, s^{-1})$ and b runs through B, is bounded in $\mathcal{S}(\mathfrak{h}_0)$, while the set of $\widetilde{\varphi}_{t,b}$, as t runs through (s^{-1}, ∞) and b runs through B, is bounded in $\mathcal{S}(\mathfrak{h}_\infty)$. Consequently, Proposition 5.10 and its proof imply that the

$$K_{b,0} := \int_0^{s^{-1}} t^{-r}(t\,\cdot)_* \varphi_{t,b} \,\frac{\mathrm{d}t}{t}$$

are well-defined elements of $\mathcal{S}'(G_s)$ and stay bounded in $\mathcal{CZ}_r(\mathfrak{h}_0)$; analogously, the

$$K_{b,\infty} := \int_{s^{-1}}^{+\infty} t^{-r}(t\,\cdot)_* \varphi_{t,b} \,\frac{\mathrm{d}t}{t}$$

are well-defined elements of $\mathcal{S}'_{-Q_0-r}(G_s)$, and stay bounded in $\mathcal{CZ}_r(\mathfrak{h}_\infty)$.

It will then suffice to prove that $K_{b,0}$ equals a Schwartz function in a neighbourhood of ∞, and that (every representative of) $K_{b,\infty}$ is of class C^∞ on the whole of G_s (with the required boundedness).

On the one hand, take $k \geq 1$ and α, and observe that there is a constant $C_{k,\alpha} > 0$ such that

$$|\partial_0^\alpha \widetilde{\varphi}_{t,b}(x)| \leq \frac{C_{k,\alpha}}{(1 + |x|)^{k+Q_0+r+\mathrm{d}_\alpha}}$$

for every $x \in \mathfrak{h}_0$, for every $t \in (0, s^{-1})$, and for every $b \in B$. Then,

$$|\partial_0^\alpha K_{b,0}(x)| \leq C_{k,\alpha} \int_0^{s^{-1}} \frac{t^{-Q_0-r-\mathrm{d}_\alpha}}{(1 + |t^{-1}\cdot x|)^{k+Q_0+r+\mathrm{d}_\alpha}} \,\frac{\mathrm{d}t}{t} \leq \frac{C_{k,\alpha} s^{-k}}{k|x|^{k+Q_0+r+\mathrm{d}_\alpha}}$$

for every nonzero $x \in \mathfrak{h}_0$ and for every $b \in B$. By the arbitrariness of k and α, it follows that the $(1 - \tau)K_{b,0}$ stay in a bounded subset of $\mathcal{S}(\mathfrak{h}_0)$ as b runs through B, where τ is an element of $C_c^\infty(\mathfrak{h}_0)$ which equals 1 on a neighbourhood of 0.

On the other hand, denote by $P_{t,b,j}$ the homogeneous component of degree j of the Taylor series of $\widetilde{\varphi}_{t,b} \in \mathcal{S}(\mathfrak{h}_\infty)$ about 0, for every $t > 0$, for every $b \in B$, and for every $j \in \mathbb{N}$. If $k > -Q_\infty - r$, then

$$\int_{s^{-1}}^{+\infty} t^{-r}(t\,\cdot)_* P_{t,b,k} \,\frac{\mathrm{d}t}{t}$$

is a well-defined homogeneous polynomial of degree k, while for every α there is a constant $C'_{k,\alpha} > 0$ such that

$$\left| \int_{s^{-1}}^{+\infty} t^{-r} \partial_\infty^\alpha (t\,\cdot)_* \left(\widetilde{\varphi}_{t,b} - \sum_{j<k} P_{t,b,j} \right)(x) \,\frac{\mathrm{d}t}{t} \right| \leq C'_{k,\alpha} |x|^{(k-\mathrm{d}_\alpha)_+}$$

for every $x \in \mathfrak{h}_\infty$, and for every $b \in B$. Define

$$\widetilde{K}_{\infty,b} := \int_{s^{-1}}^{+\infty} t^{-r}(t\,\cdot)_* \left(\widetilde{\varphi}_{t,b} - \sum_{j \leq -Q_\infty - r} P_{t,b,j}\right) \frac{\mathrm{d}t}{t},$$

so that $\widetilde{K}_{b,\infty}$ is a well-defined representative of $K_{b,\infty}$ (under the identification of G_s with $\mathfrak{h}_\infty$) by the Proof of Proposition 5.10. Then, for every $k \in \mathbb{N}$, the

$$\widetilde{K}_{\infty,b} = \int_{s^{-1}}^{+\infty} t^{-r}(t\,\cdot)_* \left(\widetilde{\varphi}_{t,b} - \sum_{j \leq k} P_{t,b,j}\right) \frac{\mathrm{d}t}{t} + \sum_{-Q_\infty - r < j \leq k} \int_1^{+\infty} t^{-r}(t\,\cdot)_* P_{t,b,j} \frac{\mathrm{d}t}{t}$$

stay bounded in $C^k(\mathfrak{h}_\infty)$. By the arbitrariness of k, it follows that the $\widetilde{K}_{\infty,b}$ stay bounded in $C^\infty(\mathfrak{h}_\infty)$. □

Theorem 5.12 *Take $r \in \mathbb{R}$. Then, the following hold:*

- *for every $k \in \mathbb{N}$, the continuous linear map $\mathcal{K}_{0,0}^{(k)} \colon C_c^\infty(\mathbb{R}^*) \to \mathcal{CZ}_{r\delta-k}(\mathfrak{h}_0)$ induces a unique continuous linear map $\mathcal{K}_{0,0}^{(k)} \colon \mathcal{M}_r(\mathbb{R}^*) \to \mathcal{CZ}_{r\delta-k}(\mathfrak{h}_0)$ such that, if $\mathfrak{F}$ is a bounded filter on $\mathcal{M}_r(\mathbb{R}^*)$ which converges pointwise to some m in $\mathcal{M}_r(\mathbb{R}^*)$, then $\mathcal{K}_{0,0}^{(k)}(\mathfrak{F})$ converges to $\mathcal{K}_{0,0}^{(k)}(m)$ in $\mathcal{S}'_{-Q_0-r\delta+k}(\mathfrak{h}_0)$;*
- *for every $k \in \mathbb{N}$, the continuous linear map $\mathcal{K}_{\infty,\infty}^{(k)} \colon C_c^\infty(\mathbb{R}^*) \to \mathcal{CZ}_{r\delta+k}(\mathfrak{h}_\infty)$ induces a unique continuous linear map $\mathcal{K}_{\infty,\infty}^{(k)} \colon \mathcal{M}_r(\mathbb{R}^*) \to \mathcal{CZ}_{r\delta+k}(\mathfrak{h}_\infty)$ such that, if $\mathfrak{F}$ is a bounded filter on $\mathcal{M}_r(\mathbb{R}^*)$ which converges pointwise to some m in $\mathcal{M}_r(\mathbb{R}^*)$, then $\mathcal{K}_{\infty,\infty}^{(k)}(\mathfrak{F})$ converges to $\mathcal{K}_{\infty,\infty}^{(k)}(m)$ in $\mathcal{S}'_{-Q_\infty-r\delta-k}(\mathfrak{h}_\infty)$;*
- *for every $s \in (0,\infty)$, the continuous linear map $\mathcal{K}_{\mathcal{L}_s} \colon C_c^\infty(\mathbb{R}^*) \to \mathcal{CZ}_{r\delta}(G_s)$ induces a unique continuous linear map $\mathcal{K}_{\mathcal{L}_s} \colon \mathcal{M}_r(\mathbb{R}^*) \to \mathcal{CZ}_{r\delta}(G_s)$ such that, if $\mathfrak{F}$ is a bounded filter on $\mathcal{M}_r(\mathbb{R}^*)$ which converges pointwise to some m in $\mathcal{M}_r(\mathbb{R}^*)$, then $\mathcal{K}_{\mathcal{L}_s}(\mathfrak{F})$ converges to $\mathcal{K}_{\mathcal{L}_s}(m)$ in $\mathcal{S}'_{-Q_\infty-r\delta}(G_s)$.*

In addition, let M be a bounded subset of $\mathcal{M}_r(\mathbb{R}^*)$, and take $\tau_0 \in C_c^\infty(\mathfrak{h}_0)$ and $\tau_\infty \in C_c^\infty(\mathfrak{h}_\infty)$ such that τ_0 and τ_∞ equal 1 in a neighbourhood of 0. Then, the following hold:[11]

- *for every $N \in \mathbb{N}$, there is a bounded family $(K_{0,m,N,s})_{m\in M, s\in(0,1]}$ of elements of $\mathcal{CZ}_{r\delta-N}(\mathfrak{h}_0)$ such that*

$$\tau_0 \left(\mathcal{K}_{0,s}(m) - \sum_{k<N} s^k \mathcal{K}_{0,0}^{(k)}(m)\right) = s^N \tau_0 K_{0,m,N,s}$$

in $\mathcal{S}'_{-Q_0-\delta r+N}(\mathfrak{h}_0)$, for every $m \in M$ and for every $s \in (0,1]$;
- *for every $N \in \mathbb{N}$, there is a bounded family $(K_{\infty,m,N,s})_{m\in M, s\in[1,\infty)}$ of elements of $\mathcal{CZ}_{r\delta+N}(\mathfrak{h}_\infty)$ such that*

$$(1-\tau_\infty)\left(\mathcal{K}_{\infty,s}(m) - \sum_{k<N} s^{-k} \mathcal{K}_{\infty,\infty}^{(k)}(m)\right) = s^{-N}(1-\tau_\infty) K_{\infty,m,N,s}$$

in $\mathcal{S}'_{-Q_\infty-\delta r}(\mathfrak{h}_0)$, for every $m \in M$ and for every $s \in [1,\infty)$.

[11] We denote by $\mathcal{K}_{0,s}(m)$ and $\mathcal{K}_{\infty,s}(m)$ the distributions on $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$, respectively, corresponding to $\mathcal{K}_{\mathcal{L}_s}(m)$ under the usual identifications.

Proof Let M be a bounded subset of $\mathcal{M}_r(\mathbb{R}^*)$ and fix a positive function $\varphi \in C_c^\infty(\mathbb{R}^*)$ such that $\int_0^\infty \varphi(y^\delta \lambda) \frac{\mathrm{d}y}{y} = 1$ for every $\lambda \in \mathbb{R}^*$. Let us prove that the family $(y^{\delta r} m(y^{-\delta} \cdot)\varphi)_{y>0, m\in M}$ is bounded in $\mathcal{S}(\mathbb{R})$. Indeed, take h and observe that there is a constant $C_h > 0$ such that

$$\left|\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} m(\lambda)\right| \leq C_h |\lambda|^{r-p}$$

for every $\lambda \in \mathbb{R}^*$, for every $m \in M$, and for every $p = 0, \dots, h$. Then,

$$y^{\delta r}\left|\frac{\mathrm{d}^h}{\mathrm{d}\lambda^h}\left[m(y^{-\delta}\cdot)\varphi\right](\lambda)\right| \leq \sum_{h_1+h_2=h} \frac{h!}{h_1! h_2!} C_{h_1} |\lambda|^{r-h_1} \|\varphi\|_{W^{h_2,\infty}(\mathbb{R})} \chi_{\mathrm{Supp}(\varphi)}(\lambda)$$

for every $\lambda \in \mathbb{R}^*$, for every $y > 0$, and for every $m \in M$, whence the assertion. Next, let us prove that $\mathcal{K}^{(k)}_{0,s}(m')$ (and analogously $\mathcal{K}^{(k)}_{\infty,s}(m')$) has all vanishing moments for $m' \in C_c^\infty(\mathbb{R}^*)$. It will suffice to prove our assertion for $k = 0$, hence for $\mathcal{K}_{\mathcal{L}_s}(m')$. Now, for every $h \in \mathbb{N}$ we have $m'_h := (\cdot)^{-h} m' \in C_c^\infty(\mathbb{R})$, so that $\mathcal{K}_{\mathcal{L}_s}(m') = \mathcal{L}_s^h \mathcal{K}_{\mathcal{L}_s}(m'_h)$. Since every polynomial is $\mathcal{L}_s^h$-harmonic for sufficiently large h (use Proposition 2.7 or observe that a similar property applies to $\widetilde{\mathcal{L}}$ by homogeneity arguments), the assertion follows by (sesquilinear) transposition.

Therefore, for every $k \in \mathbb{N}$, the family $(y^{\delta r}\mathcal{K}^{(k)}_{0,s}(m(y^{-\delta}\cdot)\varphi))_{y>0, s\in[0,1], m\in M}$ is bounded in $\mathcal{S}_\infty(\mathfrak{h}_0)$, while the family $(y^{\delta r}\mathcal{K}^{(k)}_{\infty,s}(m(y^{-\delta}\cdot)\varphi))_{y>0, s\in[1,\infty], m\in M}$ is bounded in $\mathcal{S}_\infty(\mathfrak{h}_\infty)$. Hence, Propositions 5.10 and 5.11 show that the

$$\mathcal{K}^{(k)}_{0,0}(m) := \int_0^{+\infty} y^{-r\delta+k} (y\,\cdot)_* \mathcal{K}^{(k)}_{0,0}(m(y^{-\delta}\cdot)\varphi) \frac{\mathrm{d}y}{y}$$

are well defined and stay in a bounded subset of $\mathcal{K}_{r\delta-k}(\mathfrak{h}_0)$ for every $k \in \mathbb{N}$, that the

$$\mathcal{K}^{(k)}_{\infty,\infty}(m) := \int_0^{+\infty} y^{-r\delta-k} (y\,\cdot)_* \mathcal{K}^{(k)}_{\infty,\infty}(m(y^{-\delta}\cdot)\varphi) \frac{\mathrm{d}y}{y}$$

are well defined and stay in a bounded subset of $\mathcal{K}_{r\delta+k}(\mathfrak{h}_\infty)$ for every $k \in \mathbb{N}$, and that the

$$\mathcal{K}_{\mathcal{L}_s}(m) := \int_0^{+\infty} y^{-r\delta} (y\,\cdot)_* \mathcal{K}_{\mathcal{L}_{sy}}(m(y^{-\delta}\cdot)\varphi) \frac{\mathrm{d}y}{y}$$

are well defined and stay in a bounded subset of $\mathcal{K}_{r\delta}(G_s)$. Therefore, the so-defined linear mappings $\mathcal{K}^{(k)}_{0,0}$, $\mathcal{K}^{(k)}_{\infty,\infty}$, and $\mathcal{K}_{\mathcal{L}_s}$ are continuous; in addition, by Proposition 5.2 and Lemma 5.6 and the choice of φ, they agree with their previous definition on $\mathcal{S}(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, and $\mathcal{M}_r(\mathbb{R}^*) \cap \ell^\infty(\mathbb{R}^*)$, respectively.

Now, if $\mathfrak{F}$ is a filter on M which converges to some m_0 pointwise on $\mathbb{R}^*$, then $y^{\delta r}\mathfrak{F}(y^{-\delta}\cdot)\varphi$ converges pointwise to $y^{\delta r} m_0(y^{-\delta}\cdot)\varphi$, hence in $\mathcal{S}(\mathbb{R})$. As a consequence, also $\mathcal{K}^{(k)}_{0,0}(y^{\delta r}\mathfrak{F}(y^{-\delta}\cdot)\varphi)$ converges to $\mathcal{K}^{(k)}_{0,0}(y^{\delta r} m_0(y^{-\delta}\cdot)\varphi)$ in $\mathcal{S}(\mathfrak{h}_0)$ for every $k \in \mathbb{N}$. Hence, $\mathcal{K}^{(k)}_{0,0}(\mathfrak{F})$ converges to $\mathcal{K}^{(k)}_{0,0}(m_0)$ in $\mathcal{S}'_{-Q_0-r\delta+k}(\mathfrak{h}_0)$ for every $k \in \mathbb{N}$. The analogous assertions concerning $\mathcal{K}_{\mathcal{L}_s}$, for $s \in (0,\infty)$, and $\mathcal{K}^{(k)}_{\infty,\infty}$, for $k \in \mathbb{N}$, are proved similarly. The first three assertions of the statement are therefore established.

Now, observe that, for every $N \in \mathbb{N}$,

$$\mathcal{K}_{0,sy}(m(y^{-\delta}\cdot)\varphi) = \sum_{k<N} \mathcal{K}^{(k)}_{0,0}(m(y^{-\delta}\cdot)\varphi)\frac{(sy)^k}{k!} + (sy)^N \int_0^1 \mathcal{K}^{(N)}_{0,sy\theta}(m(y^{-\delta}\cdot)\varphi)\frac{(1-\theta)^{N-1}}{(N-1)!}\,\mathrm{d}\theta$$

for $y \in (0, s^{-1})$, while

$$\mathcal{K}_{\infty,sy}(m(y^{-\delta}\cdot)\varphi) = \sum_{k<N} \mathcal{K}^{(k)}_{\infty,\infty}(m(y^{-\delta}\cdot)\varphi)\frac{(sy)^{-k}}{k!} + (sy)^{-N} \int_0^1 \mathcal{K}^{(N)}_{\infty,{}^{sy}/_{\theta}}(m(y^{-\delta}\cdot)\varphi)\frac{(1-\theta)^{N-1}}{(N-1)!}\,\mathrm{d}\theta$$

for $y \in (s^{-1}, \infty)$. In addition, the $\int_0^1 \mathcal{K}^{(N)}_{0,ys\theta}(m(y^{-\delta}\cdot)\varphi)\frac{(1-\theta)^{N-1}}{(N-1)!}\,\mathrm{d}\theta$ are bounded in $\mathcal{S}_\infty(\mathfrak{h}_0)$ as y run through $(0, s^{-1})$, while the $\int_0^1 \mathcal{K}^{(N)}_{\infty,{}^{sy}/_{\theta}}(m(y^{-\delta}\cdot)\varphi)\frac{(1-\theta)^{N-1}}{(N-1)!}\,\mathrm{d}\theta$ are bounded in $\mathcal{S}_\infty(\mathfrak{h}_\infty)$ as y runs through (s^{-1}, ∞). In addition, observe that, if V is a finite-dimensional vector space, F is a closed subset of V with non-empty interior, and $\mathcal{P}$ is a linearly independent finite set of polynomials on V, then the mapping $\mathcal{S}(F) \ni \varphi \mapsto (\int \varphi(x)P(x)\,\mathrm{d}x) \in \mathbb{C}^{\mathcal{P}}$ is onto, where $\mathcal{S}(F)$ is the set of $\varphi \in \mathcal{S}(V)$ supported in F, with the topology induced by $\mathcal{S}(V)$. Applying [4, Proposition 12 of Chapter II, § 4, No. 7], we see that, if B_0 and B_∞ are bounded subsets of $C^\infty(\mathfrak{h}_0)$ and $\mathcal{S}(\mathfrak{h}_\infty)$, respectively, then there are bounded subsets B'_0 and B'_∞ of $\mathcal{S}_{r\delta-N}(\mathfrak{h}_0)$ and $\mathcal{S}_{r\delta+N}(\mathfrak{h}_\infty)$ such that $\tau_0 B_0 = \tau_0 B'_0$ and $(1-\tau_\infty)B_\infty = (1-\tau_\infty)B'_\infty$. Hence, Corollary 5.11 (and its proof) again implies that the

$$\tau_0\left(\mathcal{K}_{0,s}(m) - \sum_{k<N} s^k \mathcal{K}^{(k)}_{0,0}(m)\right)$$

stay bounded in $\tau_0\mathcal{K}_{r\delta-N}(\mathfrak{h}_0)$ as m runs through M and s is fixed, while the

$$(1-\tau_\infty)\left(\mathcal{K}_{\infty,s}(m) - \sum_{k<N} s^{-k} \mathcal{K}^{(k)}_{\infty,\infty}(m)\right)$$

stay bounded in $(1-\tau_\infty)\mathcal{K}_{r\delta+N}(\mathfrak{h}_\infty)$ as m runs through M and s is fixed. In order to establish uniform boundedness for general s as in the statement, it suffices to reduce to the case $s = 1$, taking into account Proposition 5.2 and Lemma 5.6. The proof is therefore complete. □

5.2 Multiplier theorems

Here, we shall repeat the arguments of [20, § 4.1] in order to provide a multiplier theorem for the operators $\mathcal{L}_s$, which will imply some sort of continuity for the mapping $s \mapsto \mathcal{K}_{\mathcal{L}_s}(m)$ for more general m. Even though the following results hold when $\widetilde{\mathcal{L}}$ is self-adjoint, in order to avoid some technical issues we shall assume that $\widetilde{\mathcal{L}}$ is positive.

In this section, when μ is a measure on G_s which is absolutely continuous with respect to the Haar measure, we shall write $\|\mu\|_{L^p(\nu_{G_s})}$ to denote the L^p norm of its density with respect to ν_{G_s}, $p \in [1, \infty]$.

We recall the definition of some Besov spaces on $\mathbb{R}$ (cf. [39, Theorem of Section 2.6.1] and [3, Section 5]).

Definition 5.13 Take $\alpha > 0$. Then, $B^{\alpha}_{\infty,\infty}(\mathbb{R})$ is the space of $f \in L^\infty(\mathbb{R})$ such that

$$\sup_{x\neq 0} \frac{\|\Delta_x^{([\alpha]+1)} f\|_\infty}{|x|^\alpha} < \infty,$$

where $\Delta_x^{([\alpha]+1)} f := \sum_{j=0}^{[\alpha]+1} (-1)^{[\alpha]+1-j} \binom{[\alpha]+1}{j} f(\,\cdot\, + jx)$ for every $x \in \mathbb{R}$, endowed with the corresponding topology. We denote by $b^{\alpha}_{\infty,\infty}(\mathbb{R})$ the closure of $B^{\alpha+1}_{\infty,\infty}(\mathbb{R})$ in $B^{\alpha}_{\infty,\infty}(\mathbb{R})$.

We also denote by $H^\alpha(\mathbb{R})$ the classical Sobolev space of $f \in L^2(\mathbb{R})$ such that $\mathcal{F}^{-1}((1 + |\cdot|^2)^{\alpha/2}\mathcal{F} f) \in L^2$, where $\mathcal{F}$ denotes the Fourier transform.

Recall that $\nu_{\mathbb{R}_+}$ is a Haar measure on the multiplicative group $\mathbb{R}_+$.

Proposition 5.14 *For every $r > 0$, for every γ, and for every $\alpha_1, \alpha_2 \geq 0$ such that $\alpha_2 > \alpha_1$ there is a constant $C > 0$ such that*

$$\|\mathbf{X}_s^\gamma \mathcal{K}_{\mathcal{L}_s}(m)(1 + |\cdot|_{s,*})^{\alpha_1}\|_{L^2(\nu_{G_s})} \leq C\|m\|_{B^{\alpha_2}_{\infty,\infty}(\mathbb{R})}$$

for every $m \in B^{\alpha_2}_{\infty,\infty}(\mathbb{R})$ with $\operatorname{Supp} m \subseteq [-r, r]$, *and for every $s \in [0, \infty]$.*

If $\beta_{\mathcal{L}_1}$ has a density with respect to $\nu_{\mathbb{R}_+}$ bounded by $\min[(\,\cdot\,)^{Q_0/\delta}, (\,\cdot\,)^{Q_\infty/\delta}]$, *then we may take C in such a way that*

$$\|\mathbf{X}_s^\gamma \mathcal{K}_{\mathcal{L}_s}(m)(1 + |\cdot|_{s,*})^{\alpha_1}\|_{L^2(\nu_{G_s})} \leq C\|m\|_{H^{\alpha_2}(\mathbb{R})}$$

for every $m \in H^{\alpha_2}(\mathbb{R})$ with $\operatorname{Supp} m \subseteq [-r, r]$, *and for every $s \in [0, \infty]$.*

Proof Proceed as in the proofs of [20, Lemma 4.1.1 to Theorem 4.1.6], taking into account the following modifications and remarks:

– define

$$E_\ell = \mathrm{e}^{i\ell \mathrm{e}^{-1-(\cdot)}} - 1 = \sum_{k=1}^{\infty} \frac{(i\ell)^k}{k!} \mathrm{e}^{-k-k(\cdot)},$$

for every $\ell \in \mathbb{Z}$;

– replace the references to [21, 2.3 (e) and (f)] with Theorem 3.1 and Proposition 3.2;

– $\sup_{s\in[0,\infty]} \beta_{\mathcal{L}_s}([-r, r])$ is finite for every $r > 0$ thanks to Lemma 5.1;

– if $\beta_{\mathcal{L}_1} \leq C' \min[(\,\cdot\,)^{Q_0/\delta}, (\,\cdot\,)^{Q_\infty/\delta}] \cdot \nu_{\mathbb{R}_+}$ for some $C' > 0$, then there is a constant $C'' > 0$ such that $\beta_{\mathcal{L}_s} \leq C'' \max[(\,\cdot\,)^{Q_0/\delta}, (\,\cdot\,)^{Q_\infty/\delta}] \cdot \nu_{\mathbb{R}_+}$ for every $s \in [0, \infty]$. □

Here, $L^{1,\infty}(\nu_{G_s})$ denotes the weak-L^1 space on the space G_s, endowed with the measure ν_{G_s}.

Theorem 5.15 *Take N_0 and N_∞ as in Proposition 2.15. In addition, take a nonzero $\psi \in C^\infty_c(\mathbb{R}_+)$ and $\alpha > 0$, and for every $s \in [0, \infty]$ denote by $\mathcal{M}_{\alpha,s}$ the space of $m \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that*

$$\|m\|_{\mathcal{M}_{\alpha,s}} := \sup_{t>s^\delta} \left(\|\psi\, m(t\,\cdot)\|_{B^{(D_0+\alpha)/2}_{\infty,\infty}(\mathbb{R})} + (t/s^\delta)^{-N_0/2\delta} \|\psi\, m(t\,\cdot)\|_{B^{(D_1+\alpha)/2}_{\infty,\infty}(\mathbb{R})}\right)$$
$$+ \sup_{0<t\leq s^\delta} \left(\|\psi\, m(t\,\cdot)\|_{B^{(D_\infty+\alpha)/2}_{\infty,\infty}(\mathbb{R})} + (t/s^\delta)^{N_\infty/2\delta} \|\psi\, m(t\,\cdot)\|_{B^{(D_1+\alpha)/2}_{\infty,\infty}(\mathbb{R})}\right)$$

is finite. Then, for every $p \in [1, \infty)$ there is a constant $C_p > 0$ such that

$$\|m(\mathcal{L}_s)\|_{\mathcal{L}(L^1(\nu_{G_s}); L^{1,\infty}(\nu_{G_s}))} \leq C_1 \|m\|_{\mathcal{M}_{\alpha,s}}$$

and such that, if $p > 1$,

$$\|m(\mathcal{L}_s)\|_{\mathcal{L}(L^p(\nu_{G_s}))} \leq C_p \|m\|_{\mathcal{M}_{\alpha,s}},$$

for every $m \in \mathcal{M}_{\alpha,s}$ and for every $s \in [0,\infty]$.

In addition, take $s_0 \in (0,\infty)$, two functions m_0, m_∞ such that

$$\sup_{0<s<s_0} \|m_0\|_{\mathcal{M}_{\alpha,s}}, \sup_{s>s_0} \|m_\infty\|_{\mathcal{M}_{\alpha,s}} < \infty,$$

and $f_s \in L^p(G_s)$, for every $s \in [0,\infty]$, such that f_s converges to f_0 in $L^p(\mathfrak{h}_0)$ as $s \to 0^+$, and such that f_s converges to f_∞ in $L^p(\mathfrak{h}_\infty)$ as $s \to +\infty$; then,

$$\lim_{s\to 0^+} (f_s * \mathcal{K}_{\mathcal{L}_s}(m_0)) = f_0 * \mathcal{K}_{\mathcal{L}_0}(m_0) \quad \text{and} \quad \lim_{s\to +\infty} (f_s * \mathcal{K}_{\mathcal{L}_s}(m_\infty)) = f_\infty * \mathcal{K}_{\mathcal{L}_\infty}(m_\infty),$$

in $L^p(\mathfrak{h}_0)$ and in $L^p(\mathfrak{h}_\infty)$, respectively.

Finally, assume $\beta_{\mathcal{L}_1}$ has a density with respect to $\nu_{\mathbb{R}_+}$ which is bounded by $\min[(\,\cdot\,)^{Q_0/\delta}, (\,\cdot\,)^{Q_\infty/\delta}]$; define $\mathcal{M}'_{\alpha,s}$ as the space of $m \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that

$$\begin{aligned}\|m\|_{\mathcal{M}'_{\alpha,s}} := \sup_{t>s^\delta} \Big(\|\psi\, m(t\,\cdot\,)\|_{H^{(D_0+\alpha)/2}(\mathbb{R})} + (t/s^\delta)^{-N_0/\delta} \|\psi\, m(t\,\cdot\,)\|_{H^{(D_1+\alpha)/2}(\mathbb{R})} \Big) \\ + \sup_{0<t\leq s^\delta} \Big(\|\psi\, m(t\,\cdot\,)\|_{H^{(D_\infty+\alpha)/2}(\mathbb{R})} + (t/s^\delta)^{N_\infty/\delta} \|\psi\, m(t\,\cdot\,)\|_{H^{(D_1+\alpha)/2}(\mathbb{R})} \Big)\end{aligned}$$

is finite. Then, $\mathcal{M}_{\alpha,s}$ may be replaced by $\mathcal{M}'_{\alpha,s}$ in the previous assertions.

Taking into account Theorem 6.4, this generalizes [2] for higher-order operators, and also [34, Theorem 2] for quasi-homogeneous sums of even powers of left-invariant vector fields on a homogeneous group, with $N_0 = D_1 - D_0$ at least when these powers are all equal. Notice that the proofs of [2, Theorem] and [34, Theorem 2], which are based on the property of finite speed of propagation of the wave equation, cannot be extended to the present setting.

Observe that multiplier theorems for higher-order positive operators, based on the properties of the associated heat kernels, have also been considered in the literature (cf., e.g., [11,35]). In [11], multiplier theorems for higher-order positive differential operators on doubling Riemannian manifolds, based on suitable Gaussian estimates, are developed using techniques which are essentially similar to those of [20,22]. In [35], multiplier theorems for positive operators in a very general setting, based on suitable Gaussian and Stein–Tomas restriction estimates, are developed with different techniques.

Let us now compare Theorem 5.15 with [35, Theorem 5.2]. Using the estimates provided in Theorem 3.1 and [35, Theorem 5.2], one finds the following result: 'if $p \in]1,\infty[$, $\alpha > Q_\infty \left|\frac{1}{2} - \frac{1}{p}\right|$, and ψ is a nonzero element of $C^\infty_c(\mathbb{R}_+)$, then there is a constant $C > 0$ such that, for every $m \in L^\infty(\mathbb{R})$,

$$\|m(\mathcal{L}_1)\|_{\mathcal{L}(L^p(\nu_{G_1}))} \leq C \sup_{t>0} \|\psi\, m(t\,\cdot\,)\|_{W^{\alpha,\infty}(\mathbb{R})}.\text{'}$$

Notice that the difference between using $B^\alpha_{\infty,\infty}(\mathbb{R})$ or $W^{\alpha,\infty}(\mathbb{R})$ is immaterial, since $B^{\alpha'}_{\infty,\infty}(\mathbb{R}) \subseteq W^{\alpha,\infty}(\mathbb{R}) \subseteq B^\alpha_{\infty,\infty}(\mathbb{R})$ for every $\alpha' > \alpha$.

The preceding result is weaker than Theorem 5.15 in at least three aspects. Firstly, the regularity threshold is $Q_\infty \left|\frac{1}{2} - \frac{1}{p}\right|$ instead of $D_1 \left|\frac{1}{2} - \frac{1}{p}\right|$, as one gets interpolating the result of Theorem 5.15 with the case $p = 2$. Secondly, no endpoint estimates are provided in [35, Theorem 5.2], while Theorem 5.15 provides uniform weak type (1, 1) estimates. Finally, it is not clear if [35, Theorem 5.2] provides results which are uniform in s.

Proof We shall divide the proof into two steps.

1. We shall denote by M_s the space $\mathcal{M}_{\alpha,s}$ under the first set of assumptions, and the space $\mathcal{M}'_{\alpha,s}$ under the second set of assumptions. Notice that we may assume that ψ is positive and chosen in such a way that $\sum_{j\in\mathbb{Z}}\psi(2^{-\delta j}\lambda)=1$ for every $\lambda>0$. Fix $\varepsilon\in(0,\alpha)$. Then, Propositions 5.14 and 2.15 imply that there is $p_0>1$ such that for every γ there is a constant $\widetilde{C}_\gamma>0$ such that

$$\int_{G_{2^{-j}s}}|\mathbf{X}^\gamma_{2^{-j}s}\mathcal{K}_{\mathcal{L}_{2^{-j}s}}(\psi\, m(2^{\delta j}\,\cdot))(x)|^p(1+|x|_{2^{-j}s})^\varepsilon\,\mathrm{d}x$$
$$\leq\int_{G_{2^{-j}s}}|\mathbf{X}^\gamma_{2^{-j}s}\mathcal{K}_{\mathcal{L}_{2^{-j}s}}(\psi\, m(2^{\delta j}\,\cdot))(x)|^p(1+|x|_{2^{-j}s,*})^\varepsilon\,\mathrm{d}x\leq\widetilde{C}_\gamma\|m\|_{M_s}$$

for every $p\in[1,p_0]$, for every $s\in[0,\infty]$, for every $m\in M_s$, and for every $j\in\mathbb{Z}$. In addition,

$$\int_{G_s}\mathcal{K}_{\mathcal{L}_s}(\psi\, m(2^{\delta j}\,\cdot))(x)\,\mathrm{d}x=0$$

for every $s\in[0,\infty]$, for every $m\in M_s$, and for every $j\in\mathbb{Z}$. Observe that, since $D_0,D_\infty\geq 1$, the space M_s embeds in $L^\infty(\mathbb{R})$ (cf. [20, Propositions 2.3.2 and 2.3.6]). On the other hand, observe that

$$\mathcal{K}_{\mathcal{L}_s}(\psi(2^{-\delta j}\,\cdot\,)\,m)=(2^{-j}\,\cdot)_*\mathcal{K}_{\mathcal{L}_{2^{-j}s}}(\psi\, m(2^{\delta j}\,\cdot))$$

for every $s\in[0,\infty]$, for every $m\in M_s$, for every γ, and for every $j\in\mathbb{Z}$.

Now, since m is the sum of the series $\sum_{j\in\mathbb{Z}}\psi(2^{-\delta j}\,\cdot)m$ pointwise on $(0,\infty)$, and since the partial sums of that series are uniformly bounded, we see that

$$\mathcal{K}_{\mathcal{L}_s}(m)=\sum_{j\in\mathbb{Z}}\mathcal{K}_{\mathcal{L}_s}(\psi(2^{-\delta j}\,\cdot\,)\,m)$$

in the space of (right) convolutors of $L^2(G_s)$, for every $s\in[0,\infty]$ and for every $m\in M_s$; in particular, in $\mathcal{S}'(G_s)$.

Let us first prove that the sum converges in $L^1_{\mathrm{loc}}(G_s\setminus\{\,e\,\})$. Indeed, take a compact subset L of $G_s\setminus\{\,e\,\}$, and observe that

$$\sum_{j\in\mathbb{Z}}\|\chi_L\mathcal{K}_{\mathcal{L}_s}(\psi(2^{-\delta j}\,\cdot)m)\|_1\leq\sum_{j\in\mathbb{Z}}\|\chi_{2^j\cdot L}\mathcal{K}_{\mathcal{L}_{2^{-j}s}}(\psi\, m(2^{\delta j}\,\cdot))\|_1$$
$$\leq\sum_{j\leq 0}\widetilde{C}_0^{\frac{1}{p_0}}\nu_{G_{2^{-j}s}}(2^j\cdot L)^{\frac{1}{p_0'}}+\sum_{j>0}\widetilde{C}_0\sup_{2^j\cdot L}|\cdot|^{-\varepsilon}_{2^{-j}s}$$
$$=\widetilde{C}_0^{\frac{1}{p_0}}\nu_{G_s}(L)^{\frac{1}{p_0'}}\sum_{j\leq 0}\nu_{G_{2^{-j}s}}\left(B_{2^{-j}s}(2^j)\right)^{\frac{1}{p_0'}}+\widetilde{C}_0\sup_L|\cdot|_s^{-\varepsilon}\sum_{j>0}2^{-\varepsilon j},$$

which is finite for every $s\in[0,\infty]$ and for every $m\in M_s$ with $\|m\|_{M_s}\leq 1$, since $\nu_{G_{2^{-j}s}}\left(B_{2^{-j}s}(2^j)\right)\asymp 2^{jQ_\infty}$ as $j\to-\infty$ for fixed $s\neq 0$ thanks to **6** of Proposition 2.11, while $\nu_{G_0}\left(B_0(2^j)\right)=2^{jQ_0}$ for every $j\in\mathbb{Z}$.

Next, let us prove that

$$\sup_{\substack{s\in[0,\infty]\\ \|m\|_{M_s}\leq 1\\ y\neq e}}\sum_{j\in\mathbb{Z}}\int_{|x|_s\geq 2|y|_s}|\mathcal{K}_{\mathcal{L}_s}(\psi(2^{-\delta j}\,\cdot\,)\,m)(y^{-1}x)-\mathcal{K}_{\mathcal{L}_s}(\psi(2^{-\delta j}\,\cdot)m)(x)|\,\mathrm{d}x<+\infty.$$

Indeed,

$$\begin{aligned}
&\sum_{2^j|y|_s\geq 1}\int_{|x|_s\geq 2|y|_s} |\mathcal{K}_{\mathcal{L}_s}(\psi(2^{-\delta j}\cdot)\, m)(y^{-1}x)-\mathcal{K}_{\mathcal{L}_s}(\psi(2^{-\delta j}\cdot)\, m)(x)|\,\mathrm{d}x\\
&\quad=\sum_{2^j|y|_s\geq 1}\int_{|x|_{2^{-j}s}\geq 2|2^j\cdot y|_{2^{-j}s}} |\mathcal{K}_{\mathcal{L}_{2^{-j}s}}(\psi\, m(2^{\delta j}\cdot))((2^j\cdot y)^{-1}x)\\
&\qquad-\mathcal{K}_{\mathcal{L}_{2^{-j}s}}(\psi\, m(2^{\delta j}\cdot))(x)|\,\mathrm{d}x\\
&\quad\leq 2\sum_{2^j|y|_s\geq 1}\int_{|x|_{2^{-j}s}\geq 2^j|y|_s} |\mathcal{K}_{\mathcal{L}_{2^{-j}s}}(\psi\, m(2^{\delta j}\cdot))(x)|\,\mathrm{d}x\\
&\quad\leq 2\widetilde{C}_0\sum_{2^j|y|_s\geq 1}\sup_{|x|_{2^{-j}s}\geq 2^j|y|_s}|x|_{2^{-j}s}^{-\varepsilon}\\
&\quad= 2\widetilde{C}_0\sum_{2^j|y|_s\geq 1}(2^j|y|)^{-\varepsilon},
\end{aligned}$$

which is uniformly bounded for $s\in[0,\infty]$, $\|m\|_{M_s}\leq 1$, and $y\in G_s\setminus\{e\}$.

Finally,

$$\begin{aligned}
&\sum_{2^j|y|_s< 1}\int_{|x|_s\geq 2|y|_s} |\mathcal{K}_{\mathcal{L}_s}(\psi(2^{-\delta j}\cdot)m)(y^{-1}x)-\mathcal{K}_{\mathcal{L}_s}(\psi(2^{-\delta j}\cdot)m)(x)|\,\mathrm{d}x\\
&\quad=\sum_{2^j|y|_s< 1}\int_{|x|_{2^{-j}s}\geq 2|2^j\cdot y|_{2^{-j}s}} |\mathcal{K}_{\mathcal{L}_{2^{-j}s}}(\psi\, m(2^{\delta j}\cdot))((2^j\cdot y)^{-1}x)\\
&\qquad-\mathcal{K}_{\mathcal{L}_{2^{-j}s}}(\psi\, m(2^{\delta j}\cdot))(x)|\,\mathrm{d}x\\
&\quad\leq\sum_{j'\in J}\widetilde{C}_{\mathbf{e}_{j'}}\sum_{2^j|y|_s<1}|2^j\cdot y|_s^{\mathrm{d}_{j'}},
\end{aligned}$$

which is uniformly bounded for $s\in[0,\infty]$, $\|m\|_{M_s}\leq 1$, and $y\in G_s\setminus\{e\}$ (here, Lemma 2.14 is applied to $\mathcal{K}_{\mathcal{L}_{2^{-j}s}}(\psi\overline{m(2^{\delta j}\cdot)})=\mathcal{K}_{\mathcal{L}_{2^{-j}s}}(\psi m(2^{\delta j}\cdot))^*$).

Observe, now, that, for every $s\in[0,\infty]$ and for every $t>0$,

$$\frac{\nu_{G_s}(B_s(2t))}{\nu_{G_s}(B_s(t))}=\frac{\nu_{G_{t^{-1}s}}(B_{t^{-1}s}(2))}{\nu_{G_{t^{-1}s}}(B_{t^{-1}s}(1))}=\nu_{G_{t^{-1}s}}(B_{t^{-1}s}(2)),$$

which is a bounded function of $t^{-1}s$ on $[0,\infty]$. Therefore, thanks to [36, Theorem 3 of Chapter 1] we see that for every $p\in[1,2]$ there is a constant $C_p>0$ such that

$$\|m(\mathcal{L}_s)\|_{\mathcal{L}(L^1(\nu_{G_s});L^{1,\infty}(\nu_{G_s}))}\leq C_1\|m\|_{\mathcal{M}_{\alpha,s}}$$

and such that, if $p>1$,

$$\|m(\mathcal{L}_s)\|_{\mathcal{L}(L^p(\nu_{G_s}))}\leq C_p\|m\|_{M_s}$$

for every $s\in[0,\infty]$ and for every $m\in M_s$. A similar assertion holds, by duality, also for $p\in(2,\infty)$.

2. Now, identify G_s with $\mathfrak{h}_0$ for every $s\in[0,s_0]$, and observe that

$$\lim_{s\to 0^+} m(\mathcal{L}_s)=m(\mathcal{L}_0)$$

in $\mathcal{L}(L^p(\mathfrak{h}_0))$ for every $m \in \mathcal{S}(\mathbb{R})$. Define $\widetilde{\mathcal{M}}_{\alpha,s}$ replacing the Besov spaces $B_{\infty,\infty}$ with the little Besov spaces $b_{\infty,\infty}$, and define $\widetilde{M}_s$ as $\widetilde{\mathcal{M}}_{\alpha/2,s}$ and $\mathcal{M}'_{\alpha,s}$ under the first and second set of assumptions, respectively. Observe that M_s embeds continuously into $\widetilde{M}_s$ (cf. [20, Proposition 2.3.2]), so that we may replace M_s with $\widetilde{M}_s$ in the assumptions. Then, by means of [20, Corollaries 2.3.10 and 2.3.7, and Proposition 2.3.13] we see that $\tau\mathcal{S}(\mathbb{R})$ is dense in $\tau\widetilde{M}_s$ for every $\tau \in C_c^\infty(\mathbb{R}^*)$, so that

$$\lim_{s\to 0^+}(\tau m_0)(\mathcal{L}_s) = (\tau m_0)(\mathcal{L}_0)$$

in $\mathcal{L}(L^p(\mathfrak{h}_0))$, since the $(\tau m_0)(\mathcal{L}_s)$ are equicontinuous on $L^p(\mathfrak{h}_0)$ thanks to **1** and the assumptions on m_0. Therefore, for every finite subset J of $\mathbb{Z}$,

$$\lim_{s\to 0^+}\sum_{j\in J}(\psi(2^{-\delta j}\cdot)m_0)(\mathcal{L}_s) = \sum_{j\in J}(\psi(2^{-\delta j}\cdot)m_0)(\mathcal{L}_0)$$

in $\mathcal{L}(L^p(\mathfrak{h}_0))$. Now, define $K_{m_0,s,k} := \sum_{-k<j\leq k}\mathcal{K}_{\mathcal{L}_s}(\psi(2^{-\delta j}\cdot)m_0)$ and $\widetilde{\psi} \in C_c^\infty(\mathbb{R})$ so that $\widetilde{\psi} = \sum_{j\leq 0}\psi(2^{-\delta j}\cdot)$ on $\mathbb{R}^*$. Then,

$$K_{m_0,s,k} = \mathcal{K}_{\mathcal{L}_s}(\widetilde{\psi}(2^{\delta k}\cdot) - \widetilde{\psi}(2^{-\delta k}\cdot)) * \mathcal{K}_{\mathcal{L}_s}(m_0)$$

for every $s \in [0, s_0]$ and for every $k \in \mathbb{N}$. Therefore, for every $\varphi \in C_c^\infty(\mathfrak{h}_0)$ we have, with some abuses of notation,

$$\begin{aligned}
&\limsup_{s\to 0^+}\|\varphi *_{G_s}\mathcal{K}_{\mathcal{L}_s}(m_0) - \varphi *_{G_0}\mathcal{K}_{\mathcal{L}_0}(m_0)\|_p\\
&\leq \limsup_{s\to 0^+}\big(\|\varphi *_{G_s}\mathcal{K}_{\mathcal{L}_s}(m) - \varphi *_{G_s} K_{m_0,s,k}\|_p\\
&\qquad + \|\varphi *_{G_s} K_{m_0,s,k} - \varphi *_{G_0} K_{m_0,0,k}\|_p + \|\varphi *_{G_0} K_{m_0,0,k} - \varphi *_{G_0}\mathcal{K}_{\mathcal{L}_0}(m_0)\|_p\big)\\
&\leq 2C''\sup_{s\in[0,s_0]}\|\varphi - \varphi *_{G_s}\mathcal{K}_{\mathcal{L}_s}(\widetilde{\psi}(2^{\delta k}\cdot) - \widetilde{\psi}(2^{-\delta k}\cdot))\|_p
\end{aligned}$$

where $C'' = \sup_{s\in[0,s_0]}\|m_0(\mathcal{L}_s)\|_{\mathcal{L}(L^p(G_s))}$. Now, since $\varphi \in C_c^\infty(\mathfrak{h}_0)$ and since the $\mathcal{K}_{\mathcal{L}_s}(\widetilde{\psi})$, as s runs through $[0, s_0]$, stay in a bounded subset of $\mathcal{S}(\mathfrak{h}_0)$, it is easily seen that

$$\lim_{k\to\infty}\sup_{s\in[0,s_0]}\|\varphi - \varphi *_{G_s}\mathcal{K}_{\mathcal{L}_s}(\widetilde{\psi}(2^{\delta k}\cdot) - \widetilde{\psi}(2^{-\delta k}\cdot))\|_p = 0,$$

whence

$$\lim_{s\to 0^+}\varphi *_{G_s}\mathcal{K}_{\mathcal{L}_s}(m_0) = \varphi *_{G_0}\mathcal{K}_{\mathcal{L}_0}(m_0)$$

in $L^p(\mathfrak{h}_0)$. Since the $m_0(\mathcal{L}_s)$, as s runs through $[0, s_0]$, induce equicontinuous endomorphisms of $L^p(\mathfrak{h}_0)$, the assertion in the statement follows. The case $s \to +\infty$ is treated similarly. □

Notice that the regularity threshold in Theorem 5.15 is not optimal, in general. We shall now present an improvement of Theorem 5.15, under more restrictive hypotheses, in the spirit of [15,16,22]. Let us briefly recall the notion of capacity introduced in [20,22]; we shall present it in a slightly simpler way *in the setting of two-step stratified groups*.

Definition 5.16 Let G' be a two-step stratified group with Lie algebra $\mathfrak{g}'$; let $(\mathfrak{g}'_1, \mathfrak{g}'_2)$ be the stratification of $\mathfrak{g}'$ and take $h \in \{0, \dots, \dim\mathfrak{g}'_2\}$. Endow $\mathfrak{g}'$ with a scalar product. Then, we say that G' is h-capacious if there is a linearly independent family $X_1, \dots, X_h$ of elements of $\mathfrak{g}_1'$ and a linearly independent family $T_1, \dots, T_h$ of elements of $\mathfrak{g}_2'$ such that

$$|\langle T|[X, \cdot\,]\rangle|_{\mathfrak{g}^*} \geq \sum_{j=1}^{h} |\langle X|X_j\rangle\langle T|T_j\rangle|$$

for every $X \in \mathfrak{g}_1'$ and for every $T \in \mathfrak{g}_2'$.

For instance, if G' is the product of a finite family of Métivier or abelian groups, then G' is $\dim[G', G']$-capacious (cf. [22, Proposition 3.9]), so that the following result applies (with a suitable choice of $\widetilde{G}$) when $\mathcal{L}_1$ has the form $\sum_{j\in J_1}(iX_j)^\alpha$, where $\alpha \in 2\mathbb{N}^*$ and (X_j) is a family of left-invariant vector fields on G_1 which generates its Lie algebra.

Notice that, when G' is an H-type group and $\mathcal{L}_1 = \mathcal{L}_1' - \sum_j T_j^2$, where $\mathcal{L}_1'$ is the standard (homogeneous) sub-Laplacian and the T_j stay in the centre of $\mathfrak{g}_1$, then Theorem 5.17 is a consequence of [29, Corollary 2.4].

Theorem 5.17 *Assume that $\widetilde{G}$ is a two-step stratified group and that G_∞ is h-capacious for some $h \in \mathbb{N}$. Then, there is $h' \geq (h - Q_\infty + Q_0)_+$ such that G_0 is h'-capacious. In addition, take a nonzero $\psi \in C_c^\infty(\mathbb{R}_+)$ and $\alpha > 0$, and for every $s \in [0, \infty]$ denote by $\mathcal{M}_{\alpha,s}$ the space of $m \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that*

$$\|m\|_{\mathcal{M}_{\alpha,s}} := \sup_{t>0}\Big(\|\psi m(t\,\cdot)\|_{B^{(Q_0-h'+\alpha)/2}_{\infty,\infty}(\mathbb{R})} + (1 + t/s^\delta)^{(Q_0-h'-Q_\infty+h)/(2\delta)} \|\psi m(t\,\cdot)\|_{B^{(Q_\infty-h+\alpha)/2}_{\infty,\infty}(\mathbb{R})}\Big)$$

is finite. Then, for every $p \in [1, \infty)$ there is a constant $C_p > 0$ such that

$$\|m(\mathcal{L}_s)\|_{\mathcal{L}(L^1(\nu_{G_s}); L^{1,\infty}(\nu_{G_s}))} \leq C_1 \|m\|_{\mathcal{M}_{\alpha,s}}$$

and such that, if $p > 1$,

$$\|m(\mathcal{L}_s)\|_{\mathcal{L}(L^p(\nu_{G_s}))} \leq C_p \|m\|_{\mathcal{M}_{\alpha,s}},$$

for every $m \in \mathcal{M}_{\alpha,s}$ and for every $s \in [0, \infty]$.

An analogue of the convergence results for $s \to 0^+$ and $s \to +\infty$ of Theorem 5.15 can be proved, with the same techniques, also under the assumptions of Theorem 5.17. We leave the details to the reader.

Notice that, when G is a product of Métivier group (so that one may take $h = \dim[G_\infty, G_\infty]$), then $Q_0 - h'$ and $Q_\infty - h$ both equal the Euclidean dimension $\dim G_1$ of G_1, so that

$$\|m\|_{\mathcal{M}_{\alpha,s}} = \sup_{t>0} \|\psi\, m(t\,\cdot)\|_{B^{(\dim G_1+\alpha)/2}_{\infty,\infty}(\mathbb{R})}.$$

Therefore, at least when $\mathcal{L}_1$ is a sub-Laplacian, the regularity threshold of this result is optimal (cf. [19,25]). Cf. [1,15,16,20,22–25,28] and the references therein for other results in this direction.

Proof We shall divide the proof into two steps.

1. Observe first that $G_s = G_\infty$ as Lie groups for every $s \in (0, \infty]$, under the identification with $\mathfrak{h}_\infty$. Indeed, it suffices to observe that, if $X, Y \in \mathfrak{h}_\infty$, then $[X, Y] \in \widetilde{\mathfrak{g}}_2$, so that $[X, Y]_s =$

$P_{\infty,s}[X,Y] = P_{\infty,\infty}[X,Y] = [X,Y]_\infty$ thanks to Lemma 2.3. Fix scalar products on $\mathfrak{h}_\infty$ and $\mathfrak{h}_0$ such that the bases $(\widetilde{X}_j)_{j\in J_\infty}$ and $(\widetilde{X}_j)_{j\in J_0}$ are orthonormal. Observe that, since G_∞ is h-capacious, there are two linearly independent families $(Y_j)_{j=1,\dots,h}$ of elements of $\mathfrak{h}_\infty \cap \widetilde{\mathfrak{g}}_1$ and $(T_j)_{j=1,\dots,h}$ of elements of $\mathfrak{h}_\infty \cap \widetilde{\mathfrak{g}}_2$ such that

$$|\langle T|[X,\,\cdot\,]_\infty\rangle|_{\mathfrak{h}_\infty^*} \geq \sum_{j=1}^h |\langle X|Y_j\rangle\langle T|T_j\rangle|$$

for every $X \in \mathfrak{h}_\infty \cap \widetilde{\mathfrak{g}}_1$ and for every $T \in \mathfrak{h}_\infty \cap \widetilde{\mathfrak{g}}_2$. By the preceding remarks, we also have

$$|\langle T|[X,\,\cdot\,]_s\rangle|_{\mathfrak{h}_\infty^*} \geq \sum_{j=1}^h |\langle X|Y_j\rangle\langle T|T_j\rangle|$$

for every $s \in (0,\infty]$, for every $X \in \mathfrak{h}_\infty \cap \widetilde{\mathfrak{g}}_1$, and for every $T \in \mathfrak{h}_\infty \cap \widetilde{\mathfrak{g}}_2$. Then, repeating the arguments of [22, Section 3] with minor modifications, we see that for every $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_2 < \frac{1}{2}$ and $\alpha_3 > \alpha_1$ there is a constant $C_1 > 0$ such that

$$\left\| \mathbf{X}_s^\gamma \mathcal{K}_{\mathcal{L}_s}(m)(1+|\cdot|_s)^{\alpha_1} \prod_{j=1}^h (1+|\langle\,\cdot\,|Y_j\rangle|)^{\alpha_2} \right\|_{L^2(G_s)} \leq C_1 \|m\|_{B^{\alpha_3}_{\infty,\infty}(\mathbb{R})}$$

for every $s \in (0,\infty]$, for every γ with length at most 1, and for every $m \in B^{\alpha_3}_{\infty,\infty}(\mathbb{R})$ with support in $[-1,1]$. Then, arguing as in the proof of [22, Theorem 3.11], we see that for every $\alpha > \frac{Q_\infty - h}{2}$ there are $\varepsilon > 0$, $p_0 > 1$, and a constant $C_2 > 0$ such that

$$\left\| \mathbf{X}_s^\gamma \mathcal{K}_{\mathcal{L}_s}(m)(1+|\cdot|_s)^{\varepsilon} \right\|_{L^p(G_s)} \leq C_2 \|m\|_{B^{\alpha}_{\infty,\infty}(\mathbb{R})}$$

for every $p \in [1,p_0]$, for every $s \in [1,\infty]$, for every γ with length at most 1, and for every $m \in B^\alpha_{\infty,\infty}(\mathbb{R})$ with support in $[-1,1]$.

2. Take (Y_j) and (T_j) as in **1**, and observe that (Y_j) is the basis of an algebraic complement of $\mathrm{pr}_1\, \mathfrak{i}_1$ in $\widetilde{\mathfrak{g}}_1$; in particular, $\langle (Y_j)\rangle \cap (\mathfrak{i}_1 \cap \widetilde{\mathfrak{g}}_1) = 0$. Since $\mathfrak{i}_1 \cap \widetilde{\mathfrak{g}}_1 = \mathfrak{i}_0 \cap \widetilde{\mathfrak{g}}_1$ by definition, we may assume that $Y_j \in \mathfrak{h}_0$ for every $j = 1,\dots,h$. Now, define $h' := \dim[(\langle (T_j)\rangle + \mathfrak{i}_0)/\mathfrak{i}_0]$, so that $h' \geq h - Q_\infty + Q_0$; then, we may assume that $T_1,\dots,T_{h'}$ belong to $\mathfrak{h}_0$, so that $P_{0,0}T_j \in \langle (T_{j'})_{j'=1,\dots,h'}\rangle$ for every $j = h'+1,\dots,h$. Since the $P_{0,1}T_j$, $j = 1,\dots,h$, are linearly independent, and since $\mathrm{pr}_1(P_{0,1}T_j) = (P_{0,1} - P_{0,0})T_j$, we see that the $\mathrm{pr}_1(P_{0,1}T_j)$, for $j = h'+1,\dots,h$, are linearly independent. More precisely, we see that the Y_j, $j = 1,\dots,h$, and the $\mathrm{pr}_1(P_{0,1}T_{j'})$, $j' = h'+1,\dots,h$, are linearly independent.

Therefore, there is a constant $C_1 > 0$ such that

$$|\langle T|[X,\,\cdot\,]_1\rangle|_{\mathfrak{h}_0^*} \geq C_1 \sum_{j=1}^h |\langle X|Y_j\rangle\langle T|P_{0,1}T_j\rangle|$$

for every $X \in \mathfrak{h}_0 \cap \widetilde{\mathfrak{g}}_1$ and for every $T \in \mathfrak{h}_0 \cap \widetilde{\mathfrak{g}}_2$. Observe that the dilations are self-adjoint with respect to the chosen scalar product on $\mathfrak{h}_0$, so that

$$|\langle T|[X,\,\cdot\,]_s\rangle|_{\mathfrak{h}_0^*} \geq C_1 \sum_{j=1}^h |\langle X|Y_j\rangle\langle T|P_{0,s}T_j\rangle|$$

for every $s \in [0,\infty)$, for every $X \in \mathfrak{h}_0 \cap \widetilde{\mathfrak{g}}_1$, and for every $T \in \mathfrak{h}_0 \cap \widetilde{\mathfrak{g}}_2$. In particular, for $s = 0$ we infer that G_0 is h'-capacious. Then, repeating the arguments of [22, Section 3] with minor modifications, we see that for every $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_2 < \frac{1}{2}$ and $\alpha_3 > \alpha_1$ there is a constant $C_2 > 0$ such that

$$\left\| \mathbf{X}_s^{\gamma} \mathcal{K}_{\mathcal{L}_s}(m)(1 + |\cdot|_s)^{\alpha_1} \prod_{j=1}^{h} (1 + |P_{0,s} T_j| |\langle \cdot | Y_j \rangle|)^{\alpha_2} \right\|_{L^2(G_s)} \leq C_2 \|m\|_{B^{\alpha_3}_{\infty,\infty}(\mathbb{R})}$$

for every $s \in [0, \infty)$, for every γ with length at most 1, and for every $m \in B^{\alpha_3}_{\infty,\infty}(\mathbb{R})$ with support in $[-1, 1]$. Therefore, we need to prove that

$$\sup_{s \in [0,1]} \int_{\mathfrak{h}_0} (1 + |X|_s^{\alpha_1'} + s^{\alpha_1 - \alpha_1'} |X|_s^{\alpha_1})^{-1} \prod_{j=1}^{h} (1 + |P_{0,s} T_j| |\langle X | Y_j \rangle|)^{-\alpha_2} \, \mathrm{d}X < \infty$$

whenever $0 < \alpha_2 < 1$, $\alpha_1' + h'\alpha_2 > Q_0$, and $\alpha_1 + h\alpha_2 > Q_\infty$. Notice that it will suffice to prove the preceding assertion when α_2 is sufficiently close to 1, so that we shall also assume that $\alpha_1' > Q_0 - h'\alpha_2 + (1 - \alpha_2)(h - h')$.

Notice that the preceding arguments imply that there are a homogeneous basis $(Z_j)_{j \in J_0}$ of $\mathfrak{h}_0$, a partition $(J_{0,1}, J_{0,2}, J_{0,3})$ of J_0, and two maps $\kappa, \kappa' \colon \{1, \dots, h\} \to J_0$ such that the following hold:

- $(Z_j)_{j \in J_{0,1}}$ is a basis of $\widetilde{\mathfrak{g}}_2 \cap \mathfrak{h}_0$ and $Z_{\kappa(j)} = T_j$ for every $j = 1, \dots, h'$;
- $(Z_j)_{j \in J_{0,2}}$ is the basis of $\mathrm{pr}_1(P_{0,1}(V))$, where V is an algebraic complement of $\widetilde{\mathfrak{g}}_2 \cap (\mathfrak{h}_0 + \mathfrak{i}_\infty) = (\widetilde{\mathfrak{g}}_2 \cap \mathfrak{h}_0) \oplus (\widetilde{\mathfrak{g}}_2 \cap \mathfrak{i}_\infty)$ in $\widetilde{\mathfrak{g}}_2$ and $Z_{\kappa(j)} = \mathrm{pr}_1(P_{0,1}(T_j))$ for every $j = h'+1, \dots, h$;
- $(Z_j)_{j \in J_{0,3}}$ is the basis of an algebraic complement $\mathrm{pr}_1(P_{0,1}(V)) + (\widetilde{\mathfrak{g}}_1 \cap \mathfrak{i}_0)$ in $\widetilde{\mathfrak{g}}_1$ and $Z_{\kappa'(j)} = Y_j$ for every $j = 1, \dots, h$.

Now, take $j \in \left\{ h' + 1, \dots, h \right\}$ and observe that $\langle \mathrm{pr}_1(P_{0,1} T_j) | \mathrm{pr}_2(P_{0,1} T_j) \rangle = 0$, so that $|P_{0,s} T_j| \geq s |Z_{\gamma(j)}|$ for every $s \in [0, \infty)$. In addition, using **6** of Lemma 2.14, it is not hard to prove that there is $C_3 > 0$ such that

$$|X|_s \geq C_3 \begin{cases} |\langle X | Z_j \rangle|^{1/2} & \text{for every } j \in J_{0,1} \\ \min(|\langle X | Z_j \rangle|, |s^{-1} \langle X | Z_j \rangle|^{1/2}) & \text{for every } j \in J_{0,2} \\ |\langle X | Z_j \rangle| & \text{for every } j \in J_{0,3} \end{cases}$$

for every $X \in \mathfrak{h}_0$. Denote by $p_{s,j}(X)$ the right-hand side of the preceding inequality.

Now, observe that our assumptions on α_1 and α_2 show that we may find $\beta_j > 0$ and β_j' for every $j \in J_0$ such that the following hold: $\alpha_1 = \sum_{j \in J_0} \beta_j$ and $\alpha_1' = \sum_{j \in J_0} \beta_j'$; $\beta_j = \beta_j' > 2$ for every $j \in J_{0,1} \setminus \kappa(\left\{ 1, \dots, h' \right\})$; $\beta_j > 2$ and $\beta_j' > 1$ for every $j \in J_{0,2} \setminus \kappa(\left\{ h' + 1, \dots, h \right\})$; $\beta_j = \beta_j' > 2 - \alpha_2$ for $j \in \kappa(\{ 1, \dots, h \})$; $\beta_j = \beta_j' > 1$ for every $j \in J_{0,3}$. Therefore, it will suffice to prove that

$$\begin{aligned} \sup_{s \in [0,1]} \int_{\mathfrak{h}_0} \prod_{j \in J} (1 + p_{s,j}(X)^{\beta_j'} + s^{\beta_j - \beta_j'} p_{s,j}(X)^{\beta_j})^{-1} \prod_{j=1}^{h'} \left(1 + \left|\langle X | Z_{\kappa(j)} \rangle\right|\right)^{-\alpha_2} \times \\ \times \prod_{j=h'+1}^{h} \left(1 + s\left|\langle X | Z_{\kappa(j)} \rangle\right|\right)^{-\alpha_2} \, \mathrm{d}X < \infty. \end{aligned}$$

Now, use Tonelli's theorem to integrate separately each coordinate with respect to the basis (Z_j). We shall prove that the integrals of the factors corresponding to $Z_{\kappa(j)}$ for $j = h' +$

$1,\dots,h$ are uniformly bounded for $s\in[0,1]$; the other factors are easier and left to the reader. Then, we have to prove that

$$\sup_{s\in[0,1]}\int_0^\infty (1+\min(x,\sqrt{x/s}))^{-\beta}(1+sx)^{\alpha_2}\,\mathrm{d}x<\infty,$$

where $\beta>2-\alpha_2(>1)$. Now, on the one hand,

$$\int_0^{1/s}(1+\min(x,\sqrt{x/s}))^{-\beta}(1+sx)^{-\alpha_2}\,\mathrm{d}x\le\int_1^{1/s}(1+x)^{-\beta}\,\mathrm{d}x\le\frac{1}{\beta-1},$$

for every $s\in[0,1]$ since $\beta>1$. On the other hand,

$$\int_{1/s}^{+\infty}(1+\min(x,\sqrt{x/s}))^{-\beta}(1+sx)^{-\alpha_2}\,\mathrm{d}x\le s^\beta\int_1^{+\infty}x^{-\frac{\beta}{2}-\alpha_2}\,\mathrm{d}x\le\frac{1}{\frac{\beta}{2}+\alpha_2-1},$$

for every $s\in[0,1]$ since $\frac{\beta}{2}+\alpha_2>1+\frac{\alpha_2}{2}>1$ and $\beta>1$. The proof is then completed as that of Theorem 5.15. □

6 Quasi-homogeneous operators

We shall now investigate further the properties of the Plancherel measures $\beta_{\mathcal{L}_s}$ in some specific situations: following [34], we shall prove that, when $\mathcal{L}_s$ is 'quasi-homogeneous' in a suitable sense, then $\beta_{\mathcal{L}_s}$ has a density of class C^∞ with respect to $\nu_{\mathbb{R}_+}$, with complete and almost explicit asymptotic expansions at 0 and at ∞.

In addition to the assumptions of Sects. 2 and 4, we assume now that there is a finite family $(\widetilde{\mathcal{L}}_\ell)_{\ell\in L}$ of self-adjoint, positive, homogeneous, left-invariant differential operators on $\widetilde{G}$ with the same degree δ such that $\widetilde{\mathcal{L}}=\sum_{\ell\in L}\widetilde{\mathcal{L}}_\ell$. We also assume that G_1 is endowed with the structure of a homogeneous group of homogeneous dimension Q, and that $\mathrm{d}\pi_1(\widetilde{\mathcal{L}}_\ell)$ is homogeneous of degree δ_ℓ for every $\ell\in L$.

Before proceeding further, let us describe an example.

Example 6.1 Let $(X'_\ell)_{\ell\in L}$ be a (finite) generating family of *homogeneous* elements of the Lie algebra of G_1, and define $\mathcal{L}_1=\sum_{\ell\in L}(iX'_\ell)^{\alpha_\ell}$, where $\alpha_\ell\in 2\mathbb{N}^*$ for every $\ell\in L$. In addition, let $\widetilde{G}$ be the free nilpotent group with L generators and the same step as G; denote by $(\widetilde{X}'_\ell)_{\ell\in L}$ the generators of its Lie algebra. We endow $\widetilde{G}$ with the unique gradation for which $\widetilde{X}'_\ell$ is homogeneous of degree $\prod_{\ell'\neq\ell}\alpha_{\ell'}$ for every $\ell\in L$. Let $\pi_1\colon\widetilde{G}\to G_1$ be the unique homomorphism of Lie groups such that that $\mathrm{d}\pi_1(\widetilde{X}'_\ell)=X'_\ell$ for every $\ell\in L$. In this context, we may define $\widetilde{\mathcal{L}}_\ell:=(i\widetilde{X}'_\ell)^{\alpha_\ell}$, $\delta:=\prod_{\ell\in L}\alpha_\ell$, and $\delta_\ell:=\mathrm{d}'_\ell\alpha_\ell$, where d'_ℓ is the degree of X'_ℓ, for every $\ell\in L$.

Now, for every $\theta\in(0,\pi]$ define $\Sigma_\theta:=\left\{\mathrm{e}^{x+iy}\colon x\in\mathbb{R},\,y\in]-\theta,\theta[\right\}$, and for every $a\in\mathbb{C}^L$ define

$$\widetilde{\mathcal{L}}_a:=\sum_{\ell\in L}a_\ell\widetilde{\mathcal{L}}_\ell;$$

the reader may easily verify that $\widetilde{\mathcal{L}}_a+\widetilde{\mathcal{L}}_a^*=\widetilde{\mathcal{L}}_{\mathrm{Re}\,a}$ is a positive Rockland operator for every $a\in\Sigma_{\pi/2}^L$. We define $\mathcal{L}_{s,a}:=\mathrm{d}\pi_s(\widetilde{\mathcal{L}}_a)$ for every $a\in\Sigma_{\pi/2}^L$ and for every $s\in[0,\infty]$; observe that $\mathcal{L}_{s,a}$ is weighted subcoercive, so that we may denote by $(h_{s,a,t})_{t>0}$ its heat kernel. In addition, we define $t\cdot a:=(t^{\delta_\ell}a_\ell)_\ell$ for every $a\in\mathbb{C}^L$ and for every $t\in\mathbb{C}\setminus\mathbb{R}_-$; we still denote by ra the multiplication of a by the scalar r for every $a\in\mathbb{C}^L$ and for every $r\in\mathbb{C}$.

Proposition 6.2 *Denote by Ω the set of $(t,a)\in\mathbb{C}\times\mathbb{C}^L$ such that $ta\in\Sigma^L_{\pi/2}$, and observe that $h_{1,t,a}$ is defined for every $(t,a)\in\Omega$. In addition, the following hold:*

- *the mapping $\Omega\ni(t,a)\mapsto h_{1,t,a}\in C^\infty(G_1)$ is holomorphic;*
- *$h_{1,t,ra}=h_{1,rt,a}$ whenever $(t,ra),(rt,a)\in\Omega$;*
- *$h_{1,t,r\cdot a}(e)=r^{-Q}h_{1,t,a}(e)$ whenever $(t,r\cdot a),(t,a)\in\Omega$.*

Proof Let us prove that, for every $p\in\mathbb{N}$ and for every $(t,a)\in\Omega$, $\operatorname{dom}(\mathcal{L}^p_{1,ta})$ is the space W^p of $f\in L^2(G_1)$ such that $\mathbf{X}^\gamma_1 f\in L^2(G_1)$ for every γ such that $\mathrm{d}_\gamma\le\delta p$, endowed with the topology induced by the Hilbertian norm $f\mapsto\left(\sum_{\mathrm{d}_\gamma\le\delta p}\|\mathbf{X}^\gamma_1 f\|_2^2\right)^{1/2}$. On the one hand, arguing as in the proof of Corollary 4.5, we see that $\mathbf{X}^\gamma_1(I+\mathcal{L}^p_{1,ta})^{-1}$ induces a bounded operator on $L^2(G_1)$ for every such γ, so that $\operatorname{dom}(\mathcal{L}^p_{1,ta})$ embeds continuously into W^p. On the other hand, it is easily seen that $C^\infty_c(G_1)$, which is contained (and dense) in $\operatorname{dom}(\mathcal{L}^p_{1,ta})$, is contained and dense in W^p, whence the asserted equality.

Now, it is clear that, if $f\in W^1$, then the mapping $\Omega\ni(t,a)\mapsto\mathcal{L}_{1,ta}f\in L^2(G_1)$ is holomorphic, so that $(\mathcal{L}_{1,ta})_{(t,a)\in\Omega}$ is an analytic family of type (A) in the sense of [18] (more precisely, the restriction of $(\mathcal{L}_{1,ta})_{(t,a)\in\Omega}$ to every complex line is an analytic family of type (A)). In addition, $\mathcal{L}_{1,ta}$ is weighted subcoercive thanks to the preceding remarks, so that it is the generator of a holomorphic semi-group by [37, Theorem 8.2]. Therefore, [18, Theorem and 2.6 of Chapter 9] implies that the mapping $\Omega\ni(t,a)\mapsto\mathrm{e}^{-\mathcal{L}_{1,ta}}\in\mathcal{L}(L^2(G_1))$ is holomorphic.[12] Therefore, taking the derivatives in t we see that, for every $p\in\mathbb{N}$, the mapping

$$\Omega\ni(t,a)\mapsto\mathcal{L}^p_{1,ta}\mathrm{e}^{-\mathcal{L}_{1,ta}}\in\mathcal{L}(L^2(G_1))$$

is holomorphic, so that the mapping

$$\Omega\ni(t,a)\mapsto\mathrm{e}^{-\mathcal{L}_{1,ta}}\in\mathcal{L}(L^2(G_1);W^p)$$

is holomorphic. By the arbitrariness of p, this implies that the mapping

$$\Omega\ni(t,a)\mapsto\mathrm{e}^{-\mathcal{L}_{1,ta}}\in\mathcal{L}(L^2(G_1);W^\infty)$$

is holomorphic, where W^∞ is the intersection of the W^p, endowed with the corresponding topology. Since $\mathcal{L}^*_{1,ta}=\mathcal{L}_{\overline{ta}}$, and since Ω is conjugate-symmetric, by (sesquilinear) transposition we see that the mapping

$$\Omega\ni(t,a)\mapsto\mathrm{e}^{-\mathcal{L}_{1,ta}}\in\mathcal{L}(W^{-\infty};L^2(G_1))$$

is holomorphic, where $W^{-\infty}$ is the strong dual of W^∞.[13] Finally, arguing again as above we see that the mapping

$$\Omega\ni(t,a)\mapsto\mathrm{e}^{-\mathcal{L}_{1,ta}}\in\mathcal{L}(W^{-\infty};W^\infty)$$

[12] First apply [18, Theorem and 2.6 of Chapter 9] to the intersection of every complex line with Ω, and then, recall that a mapping from Ω into the Banach space $\mathcal{L}(L^2(G_1))$ is holomorphic if and only if it is holomorphic on every line.

[13] In principle, we should endow $\mathcal{L}(W^{-\infty};L^2(G_1))$ with the topology of uniform convergence on the equicontinuous subsets of $W^{-\infty}$, instead of the topology of bounded convergence. However, W^∞ is a reflexive Fréchet space since it is isomorphic to a closed subspace of the reflexive Fréchet space $L^2(G_1)^{\mathbb{N}^{\dim G_1}}$ (cf. [4, Propositions 14 and 15 of Chatper IV, § 1, No. 5 and Corollary to Theorem 1 of Chapter IV, § 2, No. 2]), so that $W^{-\infty}$ is bornological by [4, Proposition 4 of Chapter IV, § 3, No. 4]; therefore, a subset of $W^{-\infty}$ is bounded if and only if it is equicontinuous on W^∞ by [4, Propositions 9 and 10 of Chapter III, § 3, No. 7].

is holomorphic. Now, the Sobolev embeddings easily show that the inclusion $W^\infty \subseteq C^\infty(G_1)$ is continuous; consequently, also the canonical mapping $\mathcal{L}(W^{-\infty}; W^\infty) \to \mathcal{L}(\mathcal{E}'(G_1); C^\infty(G_1))$ is continuous. Now, $\mathcal{L}(\mathcal{E}'(G_1); C^\infty(G_1))$ is canonically isomorphic to $C^\infty(G_1 \times G_1)$ by the Schwartz's kernel theorem (cf. [38, Proposition 50.5]), so that the mapping

$$\Omega \ni (t, a) \mapsto h_{1,t,a} \in C^\infty(G_1)$$

is holomorphic.

The second assertion is trivial, while, for what concerns the third one, just observe that $(\rho_r^{G_1})_* \mathcal{L}_{1,ta} = \mathcal{L}_{r\cdot(ta),1}$ for every $(t, a) \in \Omega$ and for every $r > 0$, where $\rho_r^{G_1}$ denotes the dilation by r in G_1 (not to be confused with the mapping $r\cdot : G_1 \to G_{r^{-1}}$ of the preceding sections); the general assertion follows by holomorphy. ◻

Corollary 6.3 *Take $a \in \mathbb{R}_+^L$. Then, there is $\varepsilon > 0$ such that the mapping $t \mapsto h_{1,t,a}(e)$ extends to a holomorphic mapping $H_a : \Sigma_{\pi/2+\varepsilon} \to \mathbb{C}$. In addition, for every $k \in \mathbb{N}$ there is a constant $C_k > 0$ such that, for every $t \in \mathbb{R}^*$,*

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}t^k} H_a(it) \right| \leq C_k \min\left(|t|^{-\frac{Q_0}{\delta}-k}, |t|^{-\frac{Q_\infty}{\delta}-k} \right).$$

The proof is similar to that of [34, Lemma 4] and is omitted.

Theorem 6.4 *Take $a \in \mathbb{R}_+^L$. Then, $\beta_{\mathcal{L}_{1,a}}$ has a density f_a of class C^∞ with respect to $\nu_{\mathbb{R}_+}$. In addition, there are two constants $C_0, C_\infty > 0$ such that, for every $k \in \mathbb{N}$,*

$$f_a^{(k)}(\lambda) \sim C_0 \left(\frac{Q_0}{\delta} \right)_k \lambda^{\frac{Q_0}{\delta}-k}$$

as $\lambda \to 0^+$, while

$$f_a^{(k)}(\lambda) \sim C_\infty \left(\frac{Q_\infty}{\delta} \right)_k \lambda^{\frac{Q_\infty}{\delta}-k}$$

as $\lambda \to +\infty$, where $x_k := x(x-1)\cdots(x-k+1)$ for every $x \in \mathbb{R}$.

In particular, in this situation we may apply the second part of Theorem 5.15, thus extending [34, Theorem 2], which corresponds to the case $\alpha_\ell = 2$ for every $\ell \in L$ in the situation of Example 6.1.

Proof Observe that, with the notation of Corollary 6.3,

$$H_a(t) = \int_{[0,\infty)} \mathrm{e}^{-t\lambda} \, \mathrm{d}\beta_{\mathcal{L}_{1,a}}(\lambda)$$

for every $t > 0$, so that

$$\mathcal{F}(\mathrm{e}^{-\varepsilon\,\cdot}\, \beta_{\mathcal{L}_{1,a}})(t) = H_a(\varepsilon + it)$$

for every $\varepsilon > 0$ and for every $t \in \mathbb{R}$. Passing to the limit for $\varepsilon \to 0^+$, we see that the restriction of $\mathcal{F}(\beta_{\mathcal{L}_{1,a}})$ to $\mathbb{R} \setminus \{0\}$ has a density of class C^∞, and that

$$\mathcal{F}(\beta_{\mathcal{L}_{1,a}})(t) = H_a(it)$$

for every $t \neq 0$. Then, Corollary 6.3 and [34, Proposition 1] show that $\beta_{\mathcal{L}_{1,a}}$ has a density f_a of class C^∞ with respect to $\nu_{\mathbb{R}_+}$ such that for every $k \in \mathbb{N}$ there is a constant $C_k > 0$ such that

$$|f_a^{(k)}(\lambda)| \leq C_k \min\left(\lambda^{\frac{Q_0}{\delta}-k}, \lambda^{\frac{Q_\infty}{\delta}-k}\right)$$

for every $\xi > 0$. Now, Proposition 5.2 shows that $s^{Q_0} f_a(s^{-\delta}\,\cdot)\nu_{\mathbb{R}_+}$ converges vaguely to the measure $\beta_{\mathcal{L}_{0,a}}$ as $s \to 0^+$. In addition, by homogeneity it is easily seen that there is a constant $C_0 > 0$ such that $\beta_{\mathcal{L}_{0,a}} = C_0(\,\cdot\,)^{\frac{Q_0}{\delta}}\nu_{\mathbb{R}_+}$. Finally, the preceding estimates show that the $s^{Q_0} f_a(s^{-\delta}\,\cdot)$ stay bounded in $C^\infty(\mathbb{R}_+)$, so that they converge to $C_0\lambda^{\frac{Q_0}{\delta}}$ in $C^\infty(\mathbb{R}_+)$. The first assertion follows; the second one is proved similarly. □

7 Appendix: Technical lemmas

In this section, we consider a homogeneous vector space V, endowed with a homogeneous basis ∂ of translation-invariant vector fields and a homogeneous norm $|\cdot|$; for every γ, we denote by d_γ the degree of ∂^γ. We fix $\varepsilon > 0$, $\eta \in \mathbb{R}$ and $\eta' \in \mathbb{R}^{\mathbb{N}}$.

Recall that $\mathcal{S}(V)$ denotes the Schwartz space, $\mathcal{S}'(V)$ the space of tempered distributions, $\mathcal{D}'(V)$ the space of distributions, and $\mathcal{E}'(V)$ the space of distributions with compact support on V. We denote by $\mathcal{M}^1$ the space of bounded (Radon) measures.

Definition 7.1 Define $\mathcal{H}_{\varepsilon,\eta}(V)$ as the space of $H \in C(\mathbb{R}_+ \times V)$ such that the set of $t^{Q\varepsilon+\eta} H(t, t^\varepsilon\,\cdot)$, as t runs through $\mathbb{R}_+$, is bounded in $\mathcal{S}(V)$.

We endow $\mathcal{H}_{\varepsilon,\eta}(V)$ with the topology induced by the norms

$$H \mapsto \sup_{t>0}\sup_{x\in V}(1+|x|)^h \sum_{\mathrm{d}_\gamma \leq h} t^{Q\varepsilon+\eta+\varepsilon \mathrm{d}_\gamma}|\partial_2^\gamma H(t, t^\varepsilon \cdot x)|,$$

for $h \in \mathbb{N}$, so that $\mathcal{H}_{\varepsilon,\eta}(V)$ becomes a metrizable locally convex space (actually, a Fréchet space).

Lemma 7.2 *Take $H \in \mathcal{H}_{\varepsilon,\eta}(V)$. Then, $[t \mapsto H(t,\,\cdot)] \in C(\mathbb{R}_+; \mathcal{S}(V))$. In particular, the function $\partial_2^\gamma H$ is continuous for every γ.*

Proof Take $t_0 > 0$, and observe that the set of $H(t,\,\cdot)$, as t runs through $[\frac{t_0}{2}, 2t_0]$, is bounded, hence relatively compact, in $\mathcal{S}(V)$. Therefore, $H(t,\,\cdot)$ has at least one cluster point in $\mathcal{S}(V)$ as $t \to t_0$. On the other hand, each cluster point of $H(t,\,\cdot)$ in $\mathcal{S}(V)$ as $t \to t_0$ is also a cluster point of $H(t,\,\cdot)$ in $\mathcal{E}^0(V)$ as $t \to t_0$, so that it must equal $H(t_0,\,\cdot)$ by the continuity of H. The assertion follows. □

Lemma 7.3 *Take $H \in \mathcal{H}_{\varepsilon,\eta'_0}(V)$ such that $\partial_1^k H \in \mathcal{H}_{\varepsilon,\eta'_k}(V)$ for every $k \in \mathbb{N}$. Then, the following conditions are equivalent:*

1. *the mapping $\mathbb{R}_+ \ni t \mapsto \partial_1^k H(t,\,\cdot) \in \mathcal{D}'(V)$ extends by continuity to $[0,\infty)$, for every $k \in \mathbb{N}$;*
2. *the mapping $\mathbb{R}_+ \ni t \mapsto \partial_1^k H(t,\,\cdot) \in \mathcal{E}'(V) + \mathcal{S}(V)$ extends by continuity to $[0,\infty)$, for every $k \in \mathbb{N}$;*
3. *there is $\tau \in C_c^\infty(V)$ such that τ equals 1 on a neighbourhood of 0 and such that the mapping $\mathbb{R}_+ \ni t \mapsto \tau\partial_1^k H(t,\,\cdot) \in \mathcal{E}'(V)$ extends by continuity to $[0,\infty)$, for every $k \in \mathbb{N}$;*

4. *for every* $\tau \in C^\infty_c(V)$ *such that* τ *equals* 1 *on a neighbourhood of* 0*, the mapping* $\mathbb{R}_+ \ni t \mapsto \tau \partial_1^k H(t, \cdot) \in \mathcal{E}'(V)$ *extends by continuity to* $[0, \infty)$*, for every* $k \in \mathbb{N}$.

Proof It is clear that **1** implies **4**, that **4** implies **3**, and that **2** implies **1**. Let us then prove that **3** implies **2**. Then, take τ as in **3**, and observe that it will suffice to prove that the mapping $\mathbb{R}_+ \ni t \mapsto (1-\tau)\partial_1^k H(t, \cdot) \in \mathcal{S}(V)$ extends by continuity to $[0, \infty)$, for every $k \in \mathbb{N}$. Then, take $h, k \in \mathbb{N}$, and observe that, since $\partial_1^k H \in \mathcal{H}_{\varepsilon,\eta'_k}(V)$, for every $N \in \mathbb{N}$ there is a constant $C_N > 0$ such that

$$\sup_{\mathrm{d}_\gamma \leq h} |\partial_1^k \partial_2^\gamma H(t,x)| \leq \frac{C_N}{(1+|t^{-\varepsilon}\cdot x|)^N t^{Q\varepsilon + \eta'_k + \varepsilon \mathrm{d}_\gamma}}$$

for every $(t,x) \in \mathbb{R}_+ \times V$. Then, for every $(t,x) \in \mathbb{R}_+ \times V$

$$\begin{aligned}&\sum_{\mathrm{d}_\gamma \leq h} |\partial_1^k \partial_2^\gamma [((1-\tau)\circ \mathrm{pr}_2)H](t,x)| \\ &\quad\leq \chi_{\mathrm{Supp}(1-\tau)}(x) \sum_{\mathrm{d}_\gamma \leq h} \sum_{\gamma'+\gamma''=\gamma} \frac{\gamma!}{\gamma'!\gamma''!} \|\partial^{\gamma'}\tau\|_\infty \frac{C_N}{|x|^N t^{Q\varepsilon+\eta'_k+\varepsilon \mathrm{d}_{\gamma''}-\varepsilon N}};\end{aligned}$$

therefore, there is $C'_N > 0$ such that, for every $(t,x) \in (0,1] \times V$,

$$\sum_{\mathrm{d}_\gamma \leq h} |\partial_1^k \partial_2^\gamma [((\chi_V - \tau)\circ \mathrm{pr}_2)H](t,x)| \leq C'_N \frac{t^{\varepsilon N - Q\varepsilon - \eta'_k - \varepsilon h}}{(1+|x|)^N},$$

which tends to 0 as $t \to 0^+$ provided that $N > Q + \frac{\eta'_k}{\varepsilon} + h$. The assertion follows by the arbitrariness of k, h, and N. □

Definition 7.4 We define $\widetilde{\mathcal{H}}_{\varepsilon,\eta'}(V)$ as the space of H satisfying the equivalent conditions of Lemma 7.3. We endow $\widetilde{\mathcal{H}}_{\varepsilon,\eta'}(V)$ with the topology induced by the norms of $\mathcal{H}_{\varepsilon,\eta'_k}(V)$ applied to $\partial_1^k H$ ($k \in \mathbb{N}$), and by the semi-norms

$$H \mapsto \sup_{t\in(0,1]} \sup_{\varphi \in B} \left| \left\langle \tau \partial_1^k H(t, \cdot), \varphi \right\rangle \right|$$

as k runs through $\mathbb{N}$, τ is an element of $C^\infty_c(V)$ which equals 1 on a neighbourhood of 0, and B runs through the bounded subsets of $C^\infty(V)$.

Lemma 7.5 *Take* $\tau \in C^\infty_c(V)$ *such that* $\tau - 1$ *vanishes of order* ∞ *at* 0*, and fix* $p \in \mathbb{N}$*. In addition, let* M_p *be the set of (Radon) measures* μ *on* $\mathbb{R}_+$ *such that* $\int_0^1 t^p \,\mathrm{d}|\mu|(t) < +\infty$*, and such that* $\int_1^{+\infty} t^k \,\mathrm{d}|\mu|(t) < +\infty$ *for every* $k \in \mathbb{N}$*. Endow* M_p *with the corresponding topology.*

Then, for every $\mu \in M_p$ *and for every* $H \in \mathcal{H}_{\varepsilon,\eta}(V)$*, the mapping* $t \mapsto (1-\tau)H(t, \cdot) \in \mathcal{S}(V)$ *is* μ*-integrable. In addition, the bilinear mapping*

$$M_p \times \mathcal{H}_{\varepsilon,\eta}(V) \ni \mu \mapsto \int_0^{+\infty} (1-\tau)H(t, \cdot)\,\mathrm{d}\mu(t) \in \mathcal{S}(V)$$

is continuous.

Proof Indeed, take $\mu \in M_p$ and $H \in \mathcal{H}_{\varepsilon,\eta}(V)$. Observe that, for every $N \in \mathbb{N}$, there is a continuous semi-norm ρ_N on $\mathcal{H}_{\varepsilon,\eta}(V)$ such that

$$|\partial_2^\gamma H(t,x)| \leq \frac{\|H\|_{\rho_N}}{t^{\varepsilon(Q+\mathrm{d}_\gamma)+\eta}(1+|t^{-\varepsilon}\cdot x|)^N}$$

for every $x \in V$ and for every γ such that $\mathrm{d}_\gamma \leq N$; fix $k \in \mathbb{N}$ and γ. Then, apply Leibniz's rule and, for $t \leq 1$, estimate the derivatives of $1-\tau$ with $|\cdot|^{N-k}$ for some fixed $N \geq \max(\mathrm{d}_\gamma, Q+\mathrm{d}_\gamma+\frac{p+\eta}{\varepsilon})$; we then see that there is a constant $C'>0$ such that

$$|x|^k|\partial^\gamma[(1-\tau)H(t,\,\cdot)](x)| \leq C' t^{\varepsilon(N-Q-\mathrm{d}_\gamma)-\eta}\|H\|_{\rho_N} \leq C' t^p \|H\|_{\rho_N}$$

for every $x \in V$. On the other hand, if $t \geq 1$, then simply estimate the derivatives of $1-\tau$ with $\chi_{\mathrm{Supp}(1-\tau)}$; we then see that there is a constant $C''>0$ such that

$$|x|^k|\partial^\gamma[(1-\tau)H(t,\,\cdot)](x)| \leq C'' t^{\varepsilon(k-Q)-\eta}\|H\|_{\rho_{\max(k,\mathrm{d}_\gamma)}}$$

for every $x \in V$. Therefore,

$$\int_0^{+\infty} |x|^k|\partial^\gamma[(\chi_V-\tau)H(t,\,\cdot)](x)|\,\mathrm{d}|\mu|(t)$$
$$\leq C'\|H\|_{\rho_N}\int_{(0,1]} t^p\,\mathrm{d}|\mu|(t) + C''\|H\|_{\rho_{\max(k,\mathrm{d}_\gamma)}}\int_{[1,+\infty)} t^{\varepsilon(k-Q)-\eta}\,\mathrm{d}|\mu|(t).$$

By the arbitrariness of k and γ, the assertion follows. □

Lemma 7.6 *For every $\mu \in \mathcal{M}^1((0,1])$ and for every $H \in \widetilde{\mathcal{H}}_{\varepsilon,\eta'}(V)$, the mapping*

$$t \mapsto t^{-k}\left(H(t,\,\cdot) - \sum_{j<k}\partial_1^j H(0,\,\cdot)\frac{t^j}{j!}\right) \in \mathcal{E}'(V)+\mathcal{S}(V)$$

is scalarly μ-integrable and its integral belongs to $\mathcal{E}'(V)+\mathcal{S}(V)$. In addition, the bilinear mapping

$$(\mu, H) \mapsto \int_0^1 t^{-k}\left(H(t,\,\cdot) - \sum_{j<k}\partial_1^j H(0,\,\cdot)\frac{t^j}{j!}\right)\mathrm{d}\mu(t)$$

is continuous from $\mathcal{M}^1((0,1]) \times \widetilde{\mathcal{H}}_{\varepsilon,\eta'}(V)$ into $\mathcal{E}'(V)+\mathcal{S}(V)$.

Proof Take some $\tau \in C_c^\infty(V)$ which equals 1 in a neighbourhood of 0, and let us prove that the mapping $t \mapsto t^{-k}\tau\left(H(t,\,\cdot) - \sum_{j<k}\partial_1^j H(0,\,\cdot)\frac{t^j}{j!}\right) \in \mathcal{E}'(V)$ is scalarly μ-integrable and that its integral belongs to $\mathcal{E}'(V)$. Observe that, since $\mathcal{E}'(V)$ is quasi-complete, by [5, Proposition 8 of Chapter VI, § 1, No. 2] it will suffice to prove that the mapping $t \mapsto t^{-k}\tau\left(H(t,\,\cdot) - \sum_{j<k}\partial_1^j H(0,\,\cdot)\frac{t^j}{j!}\right) \in \mathcal{E}'(V)$ is continuous and bounded. However, Taylor's formula implies that

$$t^{-k}\tau\left(H(t,\,\cdot) - \sum_{j<k}\partial_1^j H(0,\,\cdot)\frac{t^j}{j!}\right) = \int_0^1 \tau\partial_1^k H(ts,\,\cdot)\frac{(1-s)^{k-1}}{(k-1)!}\,\mathrm{d}s.$$

Now, for every bounded subset B of $C^\infty(V)$ there is a continuous semi-norm ρ_B of $\widetilde{\mathcal{H}}_{\varepsilon,\eta'}(V)$ such that

$$\left|\left\langle \tau\partial_1^k H(t,\,\cdot), \varphi\right\rangle\right| \leq \|H\|_{\rho_B}$$

for every $\varphi \in B$ and for every $t \in (0,1]$. Hence,

$$t^{-k}\sup_{\varphi\in B}\left|\left\langle H(t,\,\cdot) - \sum_{j<k}\partial_1^j H(0,\,\cdot)\frac{t^j}{j!}, \tau\varphi\right\rangle\right| \leq \frac{\|H\|_{\rho_B}}{k!},$$

whence our claim (cf. also Lemma 7.3). The assertion then follows by means of Lemma 7.5. □

Recall that $\mathcal{O}_C(V)$ is the set of $f \in C^\infty(V)$ such that there is $k \in \mathbb{N}$ such that $\partial^\alpha f(x) = O\big(|x|^k\big)$ as $x \to \infty$ for every α; $\mathcal{O}_C(V)$ can then be identified with the dual of the space $\mathcal{O}'_C(V)$ of convolutors of $\mathcal{S}(V)$ and carries the corresponding strong dual topology (cf. [33, pp. 244 and 245] and [13, Chapter II, § 4, No. 4]).

Lemma 7.7 *For every $k \geq 0$, for every $\mu \in \mathcal{M}^1([1, +\infty))$, and for every $H \in \mathcal{H}_{\varepsilon,\eta}(V)$, the mapping*

$$t \mapsto t^{\varepsilon(Q+k)+\eta}\left(H(t,\,\cdot\,) - \sum_{\mathrm{d}_\gamma < k} \partial_2^\gamma H(t,0)\frac{(\,\cdot\,)^\gamma}{\gamma!}\right) \in \mathcal{O}_C(V)$$

is scalarly μ-integrable, and its integral belongs to $\mathcal{O}_C(V)$. In addition, the bilinear mapping

$$(\mu, H) \mapsto \int_1^{+\infty} t^{\varepsilon(Q+k)+\eta}\left(H(t,\,\cdot\,) - \sum_{\mathrm{d}_\gamma < k} \partial_2^\gamma H(t,0)\frac{(\,\cdot\,)^\gamma}{\gamma!}\right)\mathrm{d}\mu(t)$$

is continuous from $\mathcal{M}^1([1, +\infty)) \times \mathcal{H}_{\varepsilon,\eta}(V)$ into $\mathcal{O}_C(V)$.

Proof Observe that [5, Proposition 8 of Chapter VI, § 1, No. 2] implies that it will suffice to prove that the mapping

$$t \mapsto t^{\varepsilon(Q+k)+\eta}\left(H(t,\,\cdot\,) - \sum_{\mathrm{d}_\gamma < k} \partial_2^\gamma H(t,0)\frac{(\,\cdot\,)^\gamma}{\gamma!}\right) \in \mathcal{O}_C(V)$$

is continuous on $[1, +\infty)$ and takes values in an equicontinuous subset of $\mathcal{O}_C(V)$ (considered as the strong dual of $\mathcal{O}'_C(V)$). Now, continuity is clear. In addition, fix γ' and observe that [10, Theorem 1.37] implies that there is a constant $C_{\gamma'} > 0$ such that

$$\begin{aligned}&\left|\partial_2^{\gamma'} H(t,x) - \sum_{\mathrm{d}_\gamma < k} \partial_2^{\gamma+\gamma'} H(t,0)\frac{x^\gamma}{\gamma!}\right|\\ &\qquad\leq C_{\gamma'} \sum_{\substack{\sum_j \gamma_j \leq \left[\frac{k}{d}\right]+1\\ \mathrm{d}_\gamma \geq k}} |x|^{\mathrm{d}_\gamma} \sup_{|x'| \leq C_{\gamma'}|x|} \left|\partial_2^{\gamma+\gamma'} H(t,x')\right|,\end{aligned}$$

where d is the minimum degree of the nonzero homogeneous elements of V, for every $x \in V$ and for every $t > 0$. Therefore, there is a continuous semi-norm $\rho_{\gamma'}$ on $\mathcal{H}_{\varepsilon,\eta}(V)$ such that

$$\left|\partial_2^{\gamma'} H(t,x) - \sum_{\mathrm{d}_\gamma < k} \partial_2^{\gamma+\gamma'} H(t,0)\frac{x^\gamma}{\gamma!}\right| \leq t^{-\varepsilon(Q+k+\mathrm{d}_{\gamma'})-\eta}(1+|x|)^{D\left(\left[\frac{k}{d}\right]+1\right)}\|H\|_{\rho_{\gamma'}},$$

where D is the maximum degree of the nonzero homogeneous elements of V, for every $x \in V$ and for every $t \geq 1$. The assertion follows easily. □

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References

1. Ahrens, J., Cowling, M.G., Martini, A., Müller, D.: Quaternionic spherical harmonics and a sharp multiplier theorem on quaternionic spheres. Math. Z. **294**, 1659–1686 (2020)
2. Alexopoulos, G.: Spectral multipliers on Lie groups of polynomial growth. Proc. Am. Math. Soc. **120**, 973–979 (1994)
3. Amann, H.: Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications. Math. Nachr. **186**, 5–56 (1997)
4. Bourbaki, N.: Topological Vector Spaces (Elements of Mathematics). Springer, Berlin (2003)
5. Bourbaki, N.: Integration I. Chapter 1–6 (Elements of Mathematics). Springer, Berlin (2004)
6. Bourbaki, N.: Groupes et algèbres del Lie. Chapter 2–3 (Éléments de Mathématique). Springer, Berlin (2006)
7. Calzi, M.: Spectral multipliers on 2-step stratified groups, II. J. Geom. Anal. (2019). https://doi.org/10.1007/s12220-019-00208-0
8. Calzi, M.: Spectral multipliers on 2-step stratified groups, I. J. Fourier Anal. Appl. **26**, 35 (2020). https://doi.org/10.1007/s00041-020-09740-y
9. Christ, M.: L^p bounds for spectral multipliers on nilpotent groups. Trans. Am. Math. Soc. **328**, 73–81 (1991)
10. Folland, G.B., Stein, E.M.: Hardy Spaces on Homogeneous Group. Princeton University Press, Princeton (1982)
11. Georgiadis, A.: Mikhlin-Hörmander theorem for higher order differential operators on Riemannian manifolds. J. Math. Sci. Adv. Appl. **7**, 115–132 (2011)
12. Goodman, R.W.: Nilpotent Lie Groups: Structure and Applications to Analysis. Springer, Berlin (1976)
13. Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires. Memb. Am. Math. Soc. **16** (1966)
14. Guivarc'h, Y.: Croissance polynomiale et périodes des fonctions harmoniques. Bull. Soc. Math. Fr. **101**, 333–379 (1973)
15. Hebisch, W.: Multiplier theorem on generalized Heisenberg groups. Colloq. Math. **65**, 231–239 (1993)
16. Hebisch, W., Zienkiewicz, J.: Multiplier theorem on generalized Heisenberg groups, II. Colloq. Math. **69**, 29–36 (1995)
17. Hulanicki, A.: A functional calculus for Rockland operators on nilpotent Lie groups. Stud. Math. **78**, 253–266 (1984)
18. Kato, T.: Perturbation Theory for Linear Operators. Springer, Berlin (1980)
19. Martini, A., Müller, D., Nicolussi Golo, S.: Spectral multipliers and wave equation for sub-laplacians: lower regularity bounds of euclidean type, preprint (2018). arXiv:1812.02671 [math.AP]
20. Martini, A.: Algebras of differential operators on lie groups and spectral multipliers. Ph.D. thesis, Scuola Normale Superiore (2010). arXiv:1007.1119v1 [math.FA]
21. Martini, A.: Spectral theory for commutative algebras of differential operators on Lie groups. J. Funct. Anal. **260**, 2767–2814 (2011)
22. Martini, A.: Analysis of joint spectral multipliers on lie groups of polynomial growth. Ann. I. Fourier **62**, 1215–1263 (2012)
23. Martini, A., Müller, D.: L^p spectral multipliers on the free group $N_{3,2}$. Studia Math. **217**, 41–55 (2013)
24. Martini, A., Müller, D.: Spectral multiplier theorems of Euclidean type on new classes of two-step stratified groups. Proc. Lond. Math. Soc. **109**, 1229–1263 (2014)
25. Martini, A., Müller, D.: Spectral multipliers on 2-step groups: topological versus homogeneous dimension. Geom. Funct. Anal. **26**, 680–702 (2016)

26. Martini, A., Ricci, F., Tolomeo, L.: Convolution kernels versus spectral multipliers for sub-Laplacians on groups of polynomial growth. J. Funct. Anal. **277**, 1603–1638 (2019)
27. Mauceri, G., Meda, S.: Vector-valued multipliers on stratified groups. Rev. Mat. Iberoam. **6**, 141–154 (1990)
28. Müller, D., Stein, E.M.: On spectral multipliers for Heisenberg and related groups. J. Math. Pures Appl. **73**, 413–440 (1994)
29. Müller, D., Ricci, F., Stein, E.M.: Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups, II. Math. Z. **221**, 267–291 (1996)
30. Nagel, A., Ricci, F., Stein, E.M.: Harmonic analysis and fundamental solutions on nilpotent Lie groups. Analysis and partial differential equations. Lecture Notes in Pure and Appl. Math., vol. 122, pp. 249–275, Dekker, New York (1990)
31. Nagel, A., Stein, E.M., Wainger, S.: Balls and metrics defined by vector fields I: basic properties. Acta Math. **155**, 103–147 (1985)
32. Nagel, A., Ricci, F., Stein, E.M.: Singular integrals with flag kernels and analysis on quadratic CR manifolds. J. Funct. Anal. **181**, 29–118 (2001)
33. Schwartz, L.: Théorie des Distributions. Hermann, Paris (1978)
34. Sikora, A.: On the $L^2 \to L^\infty$ norms of spectral multipliers of "Quasi-Homogeneous" operators on homogeneous groups. Trans. Am. Math. Soc. **351**(9), 3743–3755 (1999)
35. Sikora, A., Yan, L., Yao, X.: Sharp spectral multipliers for operators satisfying generalized Gaussian estimates. J. Funct. Anal. **266**, 368–409 (2014)
36. Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton (1993)
37. ter Elst, A.F.M., Robinson, D.W.: Weighted subcoercive operators on Lie groups. J. Funct. Anal. **157**, 88–163 (1998)
38. Treves, F.: Topological Vector Spaces, Distributions and Kernels. Academic Press, Cambridge (1967)
39. Triebel, H.: Theory of Function Spaces, II. Birkhäuser Verlag, Basel (1992)
40. Varadarajan, V.S.: Lie Groups, Lie Algebras, and Their Representations. Springer, Berlin (1974)
41. Varopoulos, N.T., Saloff-Coste, L., Coulhon, T.: Analysis and Geometry on Groups. Cambridge University Press, Cambridge (1992)