FREE BOUNDED ARCHIMEDEAN ℓ-ALGEBRAS

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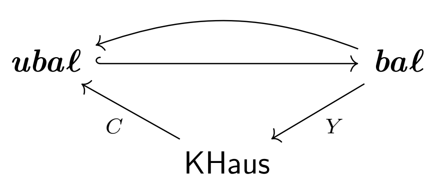
Abstract. We show that free objects on sets do not exist in the category $\boldsymbol{ba\ell}$ of bounded archimedean ℓ-algebras. On the other hand, we introduce the category of weighted sets and prove that free objects on weighted sets do exist in $\boldsymbol{ba\ell}$. We conclude by discussing several consequences of this result.

1. Introduction

The category $\boldsymbol{ba\ell}$ of bounded archimedean ℓ-algebras plays an important role in the study of Gelfand duality as algebraic counterparts of compact Hausdorff spaces live in $\boldsymbol{ba\ell}$. Indeed, for each compact Hausdorff space X, the ℓ-algebra $C(X)$ of continuous real-valued functions on X is an object of $\boldsymbol{ba\ell}$, and these algebras can be characterized as uniformly complete objects of $\boldsymbol{ba\ell}$ (see Section 2 for details). This yields a contravariant functor C from the category KHaus of compact Hausdorff spaces to $\boldsymbol{ba\ell}$. The functor C has a contravariant adjoint $Y : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ sending each $A \in \boldsymbol{ba\ell}$ to the Yosida space Y_A of maximal ℓ-ideals of A (more details are given in Section 2). This yields a contravariant adjunction between $\boldsymbol{ba\ell}$ and KHaus that restricts to a dual equivalence between KHaus and the reflective subcategory $\boldsymbol{uba\ell}$ of $\boldsymbol{ba\ell}$ consisting of uniformly complete objects of $\boldsymbol{ba\ell}$. The reflector $\boldsymbol{ba\ell} \to \boldsymbol{uba\ell}$ is the uniform completion functor. We thus arrive at the following commutative diagram.



Gelfand duality can be thought of as a generalization to KHaus of Stone duality between the categories BA of boolean algebras and Stone of Stone spaces. By Tarski duality, the category CABA of complete and atomic boolean algebras and complete boolean homomorphisms is dually equivalent to the category Set of sets and functions (see, e.g., [15, VI.4.6(a)]). A version of Tarski duality was established in [8] between Set and a (non-full) subcategory $\boldsymbol{balg}$ of $\boldsymbol{ba\ell}$ whose objects are Dedekind complete objects of $\boldsymbol{ba\ell}$ whose boolean algebra of idempotents is atomic (see Section 4 for details). As we will see in Section 4, $\boldsymbol{balg}$ is a reflective subcategory of $\boldsymbol{ba\ell}$, and the reflector is the canonical extension functor developed in [7].

In this article we study free objects in $\boldsymbol{ba\ell}$ as well as in $\boldsymbol{uba\ell}$ and $\boldsymbol{balg}$. We first show that the forgetful functor $\boldsymbol{ba\ell} \to \mathsf{Set}$ does not have a left adjoint, and hence free objects do not

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exist in $\boldsymbol{ba\ell}$ in the usual sense. We next introduce the category WSet of weighted sets and prove that the forgetful functor $\boldsymbol{ba\ell} \to \mathsf{WSet}$ does indeed have a left adjoint $F : \mathsf{WSet} \to \boldsymbol{ba\ell}$, thus showing that free objects do exist in $\boldsymbol{ba\ell}$ in this modified sense. As a consequence, we obtain that F composed with the uniform completion functor is left adjoint to the forgetful functor $\boldsymbol{uba\ell} \to \mathsf{WSet}$, and that F composed with the canonical extension functor is left adjoint to the forgetful functor $\boldsymbol{balg} \to \mathsf{WSet}$. Thus, free objects also exist in $\boldsymbol{uba\ell}$ and $\boldsymbol{balg}$ in this modified sense.

2. Preliminaries

We start by recalling some basic facts about lattice-ordered rings and algebras. We use Birkhoff's book [9, Ch. XIII and onwards] as our main reference. All rings we consider are assumed to be commutative and unital.

Definition 2.1. A ring A with a partial order $\leq$ is a *lattice-ordered ring*, or an *ℓ-ring* for short, provided

- $(A, \leq)$ is a lattice;
- $a \leq b$ implies $a + c \leq b + c$ for each c;
- $0 \leq a, b$ implies $0 \leq ab$.

An ℓ-ring A is an *ℓ-algebra* if it is an $\mathbb{R}$-algebra and for each $0 \leq a \in A$ and $0 \leq r \in \mathbb{R}$ we have $0 \leq r \cdot a$.

It is well known and easy to see that the conditions defining ℓ-algebras are equational, and hence ℓ-algebras form a variety. We denote this variety and the corresponding category of ℓ-algebras and unital ℓ-algebra homomorphisms by $\boldsymbol{\ell alg}$.

Definition 2.2. Let A be an ℓ-ring.

- A is *bounded* if for each $a \in A$ there is $n \in \mathbb{N}$ such that $a \leq n \cdot 1$ (that is, 1 is a *strong order unit*).
- A is *archimedean* if for each $a, b \in A$, whenever $n \cdot a \leq b$ for each $n \in \mathbb{N}$, then $a \leq 0$.

Let $\boldsymbol{ba\ell}$ be the full subcategory of $\boldsymbol{\ell alg}$ consisting of bounded archimedean ℓ-algebras. It is easy to see that $\boldsymbol{ba\ell}$ is not a variety (it is closed under neither products nor homomorphic images).

Definition 2.3. Let $A \in \boldsymbol{\ell alg}$. For $a \in A$, define the *absolute value* of a by

$$|a| = a \vee (-a).$$

If in addition $A \in \boldsymbol{ba\ell}$, define the *norm* of a by

$$||a|| = \inf\{r \in \mathbb{R} \mid |a| \leq r \cdot 1\}.$$

Then A is *uniformly complete* if the norm is complete.

Remark 2.4. Since $A \in \boldsymbol{ba\ell}$ is bounded, $\|\cdot\|$ is well defined, and $\|\cdot\|$ is a norm since A is archimedean.

Let $\boldsymbol{uba\ell}$ be the full subcategory of $\boldsymbol{ba\ell}$ consisting of uniformly complete ℓ-algebras.

Theorem 2.5 (Gelfand duality)**.** *There is a dual adjunction between* $\boldsymbol{ba\ell}$ *and* KHaus *which restricts to a dual equivalence between* KHaus *and* $\boldsymbol{uba\ell}$*.*

Remark 2.6. Gelfand duality is also known as Gelfand-Naimark-Stone duality (see, e.g., [6]). This duality was established by Gelfand and Naimark [13] between KHaus and the category of commutative C^*-algebras. Gelfand and Naimark worked with complex-valued functions and associated with each $X \in \mathsf{KHaus}$ the C^*-algebra of all continuous complex-valued functions on X. On the other hand, Stone [20] worked with real-valued functions and associated with each $X \in \mathsf{KHaus}$ the ℓ-algebra of all continuous real-valued functions on X. In this respect, Theorem 2.5 is more closely related to Stone's work. Nevertheless, we follow Johnstone [15, Sec. IV.4] in calling this result Gelfand duality. The Gelfand-Naimark and Stone approaches are equivalent in that the complexification functor establishes an equivalence between $\boldsymbol{uba\ell}$ and the category of commutative C^*-algebras (see [6, Sec. 7] for details).

We briefly describe the functors $C : \mathsf{KHaus} \to \boldsymbol{ba\ell}$ and $Y : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ establishing the dual adjunction of Theorem 2.5; for details see [6, Sec. 3] and the references therein. For a compact Hausdorff space X let $C(X)$ be the ring of (necessarily bounded) continuous real-valued functions on X. For a continuous map $\varphi : X \to Y$ let $C(\varphi) : C(Y) \to C(X)$ be defined by $C(\varphi)(f) = f \circ \varphi$ for each $f \in C(Y)$. Then $C : \mathsf{KHaus} \to \boldsymbol{ba\ell}$ is a well-defined contravariant functor.

For $A \in \boldsymbol{\ell alg}$, we recall that an ideal I of A is an *ℓ-ideal* if $|a| \leq |b|$ and $b \in I$ imply $a \in I$, and that ℓ-ideals are exactly the kernels of ℓ-algebra homomorphisms. If $A \in \boldsymbol{ba\ell}$, then we can associate to A a compact Hausdorff space as follows. Let Y_A be the space of maximal ℓ-ideals of A, whose closed sets are exactly sets of the form

$$Z_\ell(I) = \{M \in Y_A \mid I \subseteq M\},$$

where I is an ℓ-ideal of A. As follows from the work of Yosida [21], $Y_A \in \mathsf{KHaus}$. The space Y_A is often referred to as the *Yosida space* of A. We set $Y(A) = Y_A$, and for a morphism α in $\boldsymbol{ba\ell}$ we let $Y(\alpha) = \alpha^{-1}$. Then $Y : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ is a well-defined contravariant functor, and the functors C and Y yield a contravariant adjunction between $\boldsymbol{ba\ell}$ and KHaus.

Moreover, for $X \in \mathsf{KHaus}$ we have that $\varepsilon_X : X \to Y_{C(X)}$ is a homeomorphism where

$$\varepsilon_X(x) = \{f \in C(X) \mid f(x) = 0\}.$$

Furthermore, for $A \in \boldsymbol{ba\ell}$ define $\zeta_A : A \to C(Y_A)$ by $\zeta_A(a)(M) = r$ where r is the unique real number satisfying $a + M = r + M$. Then ζ_A is a monomorphism in $\boldsymbol{ba\ell}$ separating points of Y_A. Therefore, by the Stone-Weierstrass theorem, $\zeta_A : A \to C(Y_A)$ is the uniform completion of A. Thus, if A is uniformly complete, then ζ_A is an isomorphism. Consequently, the contravariant adjunction restricts to a dual equivalence between $\boldsymbol{uba\ell}$ and KHaus, yielding Gelfand duality. Another consequence of these considerations is the following well-known result.

Proposition 2.7. $\boldsymbol{uba\ell}$ *is a full reflective subcategory of* $\boldsymbol{ba\ell}$*, and the reflector assigns to each* $A \in \boldsymbol{ba\ell}$ *its uniform completion* $C(Y_A) \in \boldsymbol{uba\ell}$*.*

3. Free Objects in $\boldsymbol{ba\ell}$

As we pointed out in Section 2, $\boldsymbol{\ell alg}$ is a variety, hence has free algebras by Birkhoff's theorem (see, e.g., [11, Thm. 10.12]). Since $\boldsymbol{ba\ell}$ is not a subvariety of $\boldsymbol{\ell alg}$, it does not follow immediately that $\boldsymbol{ba\ell}$ has free algebras. In fact, we show that free algebras on sets do not exist in $\boldsymbol{ba\ell}$. In other words, we show that the forgetful functor $U : \boldsymbol{ba\ell} \to \mathsf{Set}$ does not have a left adjoint.

Let $A \in \boldsymbol{\ell alg}$. If $A \neq 0$, then sending $r \in \mathbb{R}$ to $r \cdot 1 \in A$ embeds $\mathbb{R}$ into A, and we identify $\mathbb{R}$ with a subalgebra of A. By this identification, if $A, B \neq 0$ and $\alpha : A \to B$ is a $\boldsymbol{\ell alg}$-morphism, then $\alpha(r) = r$ for each $r \in \mathbb{R}$.

Lemma 3.1. *Let $A, B \in \boldsymbol{ba\ell}$ and $\alpha : A \to B$ be a $\boldsymbol{ba\ell}$-morphism. Then for each $a \in A$ we have $\alpha(|a|) = |\alpha(a)|$ and $\|\alpha(a)\| \leq \|a\|$.*

Proof. Let $a \in A$. Then $\alpha(|a|) = \alpha(a \vee -a) = \alpha(a) \vee -\alpha(a) = |\alpha(a)|$. For the second statement it is sufficient to assume $A, B \neq 0$. Since $|a| \leq \|a\|$, we have $\alpha(|a|) \leq \alpha(\|a\|) = \|a\|$. Therefore, $|\alpha(a)| = \alpha(|a|) \leq \|a\|$ and hence $\|\alpha(a)\| \leq \|a\|$. $\square$

Theorem 3.2. *The forgetful functor $U : \boldsymbol{ba\ell} \to \mathsf{Set}$ does not have a left adjoint.*

Proof. If U has a left adjoint, then for each $X \in \mathsf{Set}$, there is $F(X) \in \boldsymbol{ba\ell}$ and a function $f : X \to F(X)$ such that for each $A \in \boldsymbol{ba\ell}$ and each function $g : X \to A$ there is a unique $\boldsymbol{ba\ell}$-morphism $\alpha : F(X) \to A$ satisfying $\alpha \circ f = g$.

$$\begin{array}{ccc} X & \xrightarrow{f} & F(X) \\ & \searrow^{g} & \downarrow{\scriptstyle \alpha} \\ & & A \end{array}$$

Let X be a nonempty set. Pick $x \in X$, choose $r \in \mathbb{R}$ with $r > \|f(x)\|$, and define $g : X \to \mathbb{R}$ by setting $g(y) = r$ for each $y \in X$. There is a (unique) $\boldsymbol{ba\ell}$-morphism $\alpha : F(X) \to \mathbb{R}$ with $\alpha \circ f = g$, so $\alpha(f(x)) = r$. But if $a \in F(X)$, then $\|\alpha(a)\| \leq \|a\|$ by Lemma 3.1. Therefore,

$$r = \|\alpha(f(x))\| \leq \|f(x)\| < r.$$

The obtained contradiction proves that $F(X)$ does not exist. Thus, U does not have a left adjoint. $\square$

The key reason for nonexistence of a left adjoint to the forgetful functor $U : \boldsymbol{ba\ell} \to \mathsf{Set}$ can be explained as follows. The norm on A provides a weight function on the set A, and each $\boldsymbol{ba\ell}$-morphism α respects this weight function due to the inequality $\|\alpha(a)\| \leq \|a\|$. The forgetful functor $U : \boldsymbol{ba\ell} \to \mathsf{Set}$ forgets this, which is the obstruction to the existence of a left adjoint as seen in the proof of Theorem 3.2. We repair this by working with weighted sets.

Definition 3.3.

- A *weight function* on a set X is a function w from X into the nonnegative real numbers.
- A *weighted set* is a pair (X, w) where X is a set and w is a weight function on X.

- Let WSet be the category whose objects are weighted sets and whose morphisms are functions $f : (X_1, w_1) \to (X_2, w_2)$ satisfying $w_2(f(x)) \le w_1(x)$ for each $x \in X$.

Lemma 3.4. *There is a forgetful functor $U : \boldsymbol{ba\ell} \to \mathsf{WSet}$.*

Proof. If $A \in \boldsymbol{ba\ell}$, then $(A, \|\cdot\|) \in \mathsf{WSet}$. Moreover, if $\alpha : A \to B$ is a $\boldsymbol{ba\ell}$-morphism, then $\|\alpha(a)\| \le \|a\|$ by Lemma 3.1. Therefore, α is a WSet-morphism. Thus, the assignment $A \mapsto (A, \|\cdot\|)$ defines a forgetful functor $U : \boldsymbol{ba\ell} \to \mathsf{WSet}$. $\square$

Definition 3.5. Let $A \in \boldsymbol{\ell alg}$. Call $a \in A$ *bounded* if there is $n \in \mathbb{N}$ with $-n \cdot 1 \le a \le n \cdot 1$. Let A^* be the set of bounded elements of A.

Let $A \in \boldsymbol{\ell alg}$. If $a, b \in A^*$, then there are $n, m \in \mathbb{N}$ with $-n \cdot 1 \le a \le n \cdot 1$ and $-m \cdot 1 \le b \le m \cdot 1$. Therefore, $-(n+m) \cdot 1 \le a \pm b \le (n+m) \cdot 1$. Similar facts hold for join, meet, and multiplication. Thus, we have the following:

Lemma 3.6. *Let $A \in \boldsymbol{\ell alg}$. Then A^* is a subalgebra of A, and hence A^* is a bounded ℓ-algebra. Therefore, if A is archimedean, then $A^* \in \boldsymbol{ba\ell}$.*

Let $A \in \boldsymbol{\ell alg}$. As we pointed out in Section 2, ℓ-ideals are kernels of ℓ-algebra homomorphisms. However, if I is an ℓ-ideal of A, then the quotient A/I may not be archimedean even if A is archimedean.

Definition 3.7. We call an ℓ-ideal I of $A \in \boldsymbol{\ell alg}$ *archimedean* if A/I is archimedean.

Remark 3.8. Archimedean ℓ-ideals were studied by Banaschewski (see [3, App. 2], [4]) in the category of archimedean f-rings.

It is easy to see that the intersection of archimedean ℓ-ideals is archimedean. Therefore, we may talk about the archimedean ℓ-ideal of A generated by $S \subseteq A$.

Theorem 3.9 (Main result)**.** *The forgetful functor $U : \boldsymbol{ba\ell} \to \mathsf{WSet}$ has a left adjoint.*

Proof. It is enough to show that there is a free object in $\boldsymbol{ba\ell}$ on each $(X, w) \in \mathsf{WSet}$ (see, e.g., [1, Ex. 18.2(2)]). Let $G(X)$ be the free object in $\boldsymbol{\ell alg}$ on X and let $g : X \to G(X)$ be the corresponding map. We next quotient $G(X)$ by an archimedean ℓ-ideal I so that $-w(x) \le g(x) + I \le w(x)$ for each $x \in X$. Let I be the archimedean ℓ-ideal of $G(X)$ generated by

$$\{g(x) - ((g(x) \vee -w(x)) \wedge w(x)) \mid x \in X\},$$

and set $F(X, w) = G(X)/I$. Let $\pi : G(X) \to F(X, w)$ be the canonical projection. Clearly $F(X, w)$ is an archimedean ℓ-algebra. We show that $F(X, w)$ is bounded, and hence that $F(X, w) \in \boldsymbol{ba\ell}$. Let $G(X)^*$ be the bounded subalgebra of $G(X)$ (see Lemma 3.6). Since $G(X)$ is generated by $\{g(x) \mid x \in X\}$, we have that $G(X)/I$ is generated by $\{\pi g(x) \mid x \in X\}$. Now,

$$\pi g(x) = \pi((g(x) \vee -w(x)) \wedge w(x))$$

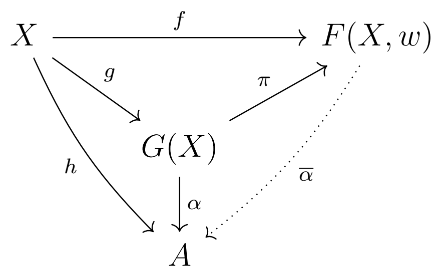
since $g(x) - ((g(x) \vee -w(x)) \wedge w(x)) \in I$. We have $-w(x) \le (g(x) \vee -w(x)) \wedge w(x) \le w(x)$, so $(g(x) \vee -w(x)) \wedge w(x) \in G(X)^*$. This shows that the generators of $F(X, w)$ lie in $\pi[G(X)^*]$, so $F(X, w) \cong G(X)^*/(I \cap G(X)^*)$ is a quotient of $G(X)^*$. Thus, $F(X, w)$ is bounded.

Let $f : X \to F(X, w)$ be given by $f(x) = \pi g(x)$. Since $f(x) = \pi((g(x) \vee -w(x)) \wedge w(x))$, we have $-w(x) \le f(x) \le w(x)$, so $\|f(x)\| \le w(x)$. Therefore, f is a WSet-morphism.

Let $A \in \boldsymbol{ba\ell}$ and $h : X \to A$ be a WSet-morphism, so $\|h(x)\| \le w(x)$ for each $x \in X$. There is an ℓ-algebra homomorphism $\alpha : G(X) \to A$ with $\alpha \circ g = h$. Because A is archimedean, $G(X)/\ker(\alpha)$ is archimedean, so $\ker(\alpha)$ is an archimedean ℓ-ideal of $G(X)$. We show that $I \subseteq \ker(\alpha)$. It suffices to show that $g(x) - ((g(x) \vee -w(x)) \wedge w(x)) \in \ker(\alpha)$ for each $x \in X$ since $\ker(\alpha)$ is an archimedean ℓ-ideal. Because $\|h(x)\| \le w(x)$, we have $-w(x) \le h(x) \le w(x)$. Therefore,

$$\begin{aligned}\alpha((g(x) \vee -w(x)) \wedge w(x)) &= (\alpha g(x) \vee -w(x)) \wedge w(x)\\ &= (h(x) \vee -w(x)) \wedge w(x)\\ &= h(x)\\ &= \alpha g(x),\end{aligned}$$

and hence $\alpha(g(x) - ((g(x) \vee -w(x)) \wedge w(x))) = 0$. Thus, $I \subseteq \ker(\alpha)$, so there is a well-defined ℓ-algebra homomorphism $\overline{\alpha} : F(X, w) \to A$ satisfying $\overline{\alpha} \circ \pi = \alpha$. Consequently, $\overline{\alpha} \circ f = \overline{\alpha} \circ \pi \circ g = \alpha \circ g = h$.



It is left to show uniqueness of $\overline{\alpha}$. Let $\gamma : F(X, w) \to A$ be a $\boldsymbol{ba\ell}$-morphism satisfying $\gamma \circ f = h$. If $\alpha' = \gamma \circ \pi$, then $\alpha' : G(X) \to A$ is an $\boldsymbol{\ell alg}$-morphism and $\alpha' \circ g = \gamma \circ \pi \circ g = \gamma \circ f = h$. Since $G(X)$ is a free object in $\boldsymbol{\ell alg}$ and $\alpha' \circ g = h = \alpha \circ g$, uniqueness implies that $\alpha' = \alpha$. From this we get $\gamma \circ \pi = \alpha = \overline{\alpha} \circ \pi$. Because π is onto, we conclude that $\gamma = \overline{\alpha}$. ☐

Remark 3.10. If $(X, w) \in \mathsf{WSet}$, then $\|f(x)\| = w(x)$. To see this, since $w : (X, w) \to (\mathbb{R}, |\cdot|)$ is a WSet-morphism, by Theorem 3.9, there is a $\boldsymbol{ba\ell}$-morphism $\alpha : F(X, w) \to \mathbb{R}$ with $\alpha \circ f = w$. Because f is a weighted set morphism, by Lemma 3.1 we have $w(x) = \|\alpha(f(x))\| \le \|f(x)\| \le w(x)$. Thus, $\|f(x)\| = w(x)$.

We next show that the Yosida space $Y_{F(X,w)}$ of $F(X, w)$ is homeomorphic to a power of $[0, 1]$, and that $F(X, w)$ embeds into the ℓ-algebra of piecewise polynomial functions on $Y_{F(X,w)}$. For a set Z we let $PP([0, 1]^Z)$ be the ℓ-algebra of piecewise polynomial functions on $[0, 1]^Z$. If Z is finite, then the definition of $PP([0, 1]^Z)$ is standard (see, e.g., [12, p. 651]). If Z is infinite, we define $PP([0, 1]^Z)$ as the direct limit of $\{PP([0, 1]^Y) \mid Y \text{ a finite subset of } Z\}$. It is straightforward to see that $PP([0, 1]^Z) \in \boldsymbol{ba\ell}$.

For each $A \in \boldsymbol{ba\ell}$ and $M \in Y_A$ it is well known that $A/M \cong \mathbb{R}$ (see, e.g., [14, Cor. 2.7]). This allows us to identify the Yosida space Y_A with the space $\hom_{\boldsymbol{ba\ell}}(A, \mathbb{R})$ of $\boldsymbol{ba\ell}$-morphisms from A to $\mathbb{R}$, by sending $\alpha : A \to \mathbb{R}$ to $\ker(\alpha)$ and $M \in Y_A$ to the natural homomorphism

$A \to \mathbb{R}$. The topology on $\hom_{\boldsymbol{ba\ell}}(A, \mathbb{R})$ is the subspace topology of the product topology on $\mathbb{R}^A$.

Theorem 3.11. *Let* $(X, w) \in \mathsf{WSet}$ *and let* $X' = \{x \in X \mid w(x) > 0\}$.
(1) *The Yosida space of* $F(X, w)$ *is homeomorphic to* $[0, 1]^{X'}$.
(2) $F(X, w)$ *embeds into* $PP([0, 1]^{X'})$.

Proof. (1). We identify $Y_{F(X,w)}$ with $\hom_{\boldsymbol{ba\ell}}(F(X, w), \mathbb{R})$ as in the paragraph before the theorem. From the universal mapping property, we see that there is a homeomorphism between $\hom_{\boldsymbol{ba\ell}}(F(X, w), \mathbb{R})$ and $\hom_{\mathsf{WSet}}((X, w), (\mathbb{R}, |\cdot|))$. If $g : X \to \mathbb{R}$ is a WSet-morphism, then $|g(x)| \le w(x)$, so $-w(x) \le g(x) \le w(x)$. Therefore, $\hom_{\mathsf{WSet}}((X, w), (\mathbb{R}, |\cdot|)) = \Pi_{x \in X}[-w(x), w(x)]$. If $x \in X'$, then $[-w(x), w(x)]$ is homeomorphic to $[0, 1]$, and if $x \notin X'$, then $[-w(x), w(x)] = \{0\}$. Thus, $\Pi_{x \in X}[-w(x), w(x)]$ is homeomorphic to $[0, 1]^{X'}$, and hence $Y_{F(X,w)}$ is homeomorphic to $[0, 1]^{X'}$.

(2). Let $\varphi : Y_{F(X,w)} \to \Pi_{x \in X'}[-w(x), w(x)]$ be the homeomorphism from the proof of (1) and let $\tau_x : [0, 1] \to [-w(x), w(x)]$ be the homeomorphism given by $\tau_x(a) = 2w(x)a - w(x)$. If τ is the product of the τ_x, then $\tau : [0, 1]^{X'} \to \Pi_{x \in X'}[-w(x), w(x)]$ is a homeomorphism, and so $\rho := \tau^{-1} \circ \varphi$ is a homeomorphism from $Y_{F(X,w)}$ to $[0, 1]^{X'}$. Therefore, $C(\rho) : C(Y_{F(X,w)}) \to C([0, 1]^{X'})$ is a $\boldsymbol{ba\ell}$-isomorphism. Since $F(X, w)$ is generated by $f[X]$, it is sufficient to show that $C(\rho)(f(x)) \in PP([0, 1]^{X'})$. Let $x \in X$. If $w(x) = 0$, then since $\|f(x)\| = w(x)$ (see Remark 3.10), $f(x) = 0$, so $C(\rho)(f(x)) = 0 \in PP([0, 1]^{X'})$. Suppose that $w(x) > 0$. Then $C(\rho)(f(x)) = 2w(x)p_x - w(x) \in PP([0, 1]^{X'})$, completing the proof. □

Remark 3.12. We compare our results with those in the vector lattice literature. Recall (see, e.g., [16, p. 48]) that the definition of a vector lattice, or Riesz space, is the same as that of an ℓ-algebra except that multiplication is not present in the signature, and so in vector lattices there is no analogue of the multiplicative identity.

(1) Let VL be the category of vector lattices and vector lattice homomorphisms. Then VL is a variety, so free vector lattices exist by Birkhoff's theorem. Therefore, the forgetful functor $U : \mathsf{VL} \to \mathsf{Set}$ has a left adjoint.
(2) Let a *pointed vector lattice* be a vector lattice with a prescribed element, and a pointed vector lattice homomorphism a vector lattice homomorphism preserving the prescribed element. The associated category pVL is a variety, so the forgetful functor $U : \mathsf{pVL} \to \mathsf{Set}$ has a left adjoint.
(3) If we consider the full subcategory uVL of pVL consisting of pointed vector lattices whose prescribed element is a strong order-unit, then Birkhoff's theorem does not apply since uVL is not a variety. In fact, an argument similar to the proof of Theorem 3.2 shows that the forgetful functor $U : \mathsf{uVL} \to \mathsf{Set}$ does not have a left adjoint. However, a small modification of the proof of Theorem 3.9 yields that the forgetful functor $U : \mathsf{uVL} \to \mathsf{WSet}$ does have a left adjoint.
(4) Baker [2, Thm. 2.4] showed that the free vector lattice $F(X)$ on a set X embeds in the vector lattice $PL(\mathbb{R}^X)$ of piecewise linear functions on $\mathbb{R}^X$. In fact, Baker shows that $F(X)$ is isomorphic to the vector sublattice of $PL(\mathbb{R}^X)$ generated by the projection functions. Theorem 3.11(2) is an analogue of Baker's result since the proof shows that

$F(X, w)$ is isomorphic to the subalgebra of $PP([0,1]^{X'})$ generated by the projection functions. Beynon [5, Thm. 1] showed that if X is finite, then $F(X) = PL(\mathbb{R}^X)$. The analogue of Beynon's result for ℓ-algebras is related to the famous Pierce-Birkhoff conjecture [10, p. 68] (see also [19, 18]).

4. Some Consequences

The proof of Theorem 3.2 also yields that the forgetful functor $\boldsymbol{ubal} \to \mathsf{Set}$ does not have a left adjoint. On the other hand, since the forgetful functor $\boldsymbol{bal} \to \mathsf{WSet}$ has a left adjoint, if $\mathcal{C}$ is a reflective subcategory of $\boldsymbol{bal}$, then the forgetful functor $\mathcal{C} \to \mathsf{WSet}$ also has a left adjoint (because the composition of adjoints is an adjoint). Consequently, since $\boldsymbol{ubal}$ is a reflective subcategory of $\boldsymbol{bal}$, we obtain:

Proposition 4.1. *The forgetful functor* $U : \boldsymbol{ubal} \to \mathsf{WSet}$ *has a left adjoint.*

Since taking uniform completion is the reflector $\boldsymbol{bal} \to \boldsymbol{ubal}$, the left adjoint of Proposition 4.1 is obtained as the uniform completion of $F(X, w)$ for each $(X, w) \in \mathsf{WSet}$.

We next turn to describing a left adjoint to the forgetful functor $\boldsymbol{balg} \to \mathsf{WSet}$. We recall that an ℓ-algebra A is *Dedekind complete* if each subset of A that is bounded above has a least upper bound (and hence each subset bounded below has a greatest lower bound) in A. We also recall that if A is a commutative ring with 1, then the set $\mathrm{Id}(A)$ of idempotents of A is a boolean algebra under the operations

$$e \vee f = e + f - ef, \quad e \wedge f = ef, \quad \neg e = 1 - e.$$

Definition 4.2. [8, Def. 3.6] We call $A \in \boldsymbol{bal}$ a *basic algebra* if A is Dedekind complete and the boolean algebra $\mathrm{Id}(A)$ is atomic.

Let A, B be basic algebras. Following [16, Def. 18.12], we call a $\boldsymbol{bal}$-morphism $\alpha : A \to B$ a *normal homomorphism* if it preserves all existing joins and meets. Let $\boldsymbol{balg}$ be the category of basic algebras and normal homomorphisms. Then $\boldsymbol{balg}$ is a non-full subcategory of $\boldsymbol{bal}$. The category $\boldsymbol{balg}$ was introduced in [8] where it was shown that $\boldsymbol{balg}$ is dually equivalent to Set, hence providing a ring-theoretic version of Tarski duality. Thus, $\boldsymbol{balg}$ plays a similar role in $\boldsymbol{bal}$ to that of CABA in BA.

The functors $B : \mathsf{Set} \to \boldsymbol{balg}$ and $X : \boldsymbol{balg} \to \mathsf{Set}$ establishing the dual equivalence between Set and $\boldsymbol{balg}$ are defined as follows. For a set X let $B(X)$ be the ℓ-algebra of all bounded real-valued functions, and for a map $\varphi : X \to Y$ let $B(\varphi) : B(Y) \to B(X)$ be given by $B(\varphi)(f) = f \circ \varphi$ for $f \in B(Y)$. Then $B : \mathsf{Set} \to \boldsymbol{balg}$ is a well-defined contravariant functor.

For $A \in \boldsymbol{balg}$ let X_A be the set of atoms of $\mathrm{Id}(A)$. We then set $X(A) = X_A$, and for a $\boldsymbol{balg}$-morphism $\alpha : A \to B$ we let $X(\alpha) : X_B \to X_A$ be given by

$$X(\alpha)(x) = \bigwedge \{a \in \mathrm{Id}(A) \mid x \leq \alpha(a)\}$$

for $x \in X_A$. Then $X : \boldsymbol{balg} \to \mathsf{Set}$ is a well-defined contravariant functor, and the functors B and X yield a dual equivalence of $\boldsymbol{balg}$ and Set. The natural isomorphisms $\eta : 1_{\mathsf{Set}} \to X \circ B$

and $\vartheta : 1_{\boldsymbol{balg}} \to B \circ X$ are defined by letting $\eta_X(x)$ be the characteristic function of $\{x\}$ for each $x \in X$, and

$$\vartheta_A(a)(x) = \zeta_A(a)((1-x)A) \text{ for each } a \in A \text{ and } x \in X_A,$$

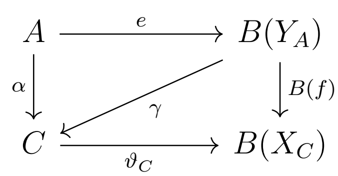
where $(1-x)A$ is the ℓ-ideal of A generated by $1-x$ (it is maximal since x is an atom of $\mathrm{Id}(A)$).

As was shown in [7], for $A \in \boldsymbol{ba\ell}$, the ℓ-algebra $B(Y_A)$ together with $\zeta_A : A \to B(Y_A)$ is the (unique up to isomorphism) canonical extension of A, where we recall that a *canonical extension* of A is $A^\sigma \in \boldsymbol{balg}$ together with a $\boldsymbol{ba\ell}$-monomorphism $e : A \to A^\sigma$ satisfying:

(1) (Density) Each $x \in A^\sigma$ is a join of meets of elements of $e[A]$.
(2) (Compactness) For $S, T \subseteq A$ and $0 < \varepsilon \in \mathbb{R}$, from $\bigwedge e[S] + \varepsilon \leq \bigvee e[T]$ it follows that $\bigwedge e[S'] \leq \bigvee e[T']$ for some finite $S' \subseteq S$ and $T' \subseteq T$.

Theorem 4.3. $(\cdot)^\sigma : \boldsymbol{ba\ell} \to \boldsymbol{balg}$ *is a reflector, so* $\boldsymbol{balg}$ *is a (non-full) reflective subcategory of* $\boldsymbol{ba\ell}$*.*

Proof. Let $A \in \boldsymbol{ba\ell}$, $C \in \boldsymbol{balg}$ and $\alpha : A \to C$ be a $\boldsymbol{ba\ell}$-morphism. By [17, p. 89], it suffices to show that there is a unique $\boldsymbol{balg}$-morphism $\gamma : A^\sigma \to C$ with $\gamma \circ e = \alpha$. Since α is a $\boldsymbol{ba\ell}$-morphism, $Y(\alpha) : Y_C \to Y_A$ is a continuous map. Let $f : X_C \to Y_A$ be given by $f(x) = Y(\alpha)((1-x)C)$ for each $x \in X_C$. In other words, if we identify X_C with a subset of Y_C (by sending x to $(1-x)C$), then f is the restriction of $Y(\alpha)$ to X_C. This induces a $\boldsymbol{ba\ell}$-morphism $B(f)$ from $A^\sigma = B(Y_A)$ to $B(X_C)$. Since $\vartheta_C : C \to B(X_C)$ is an isomorphism, we have a $\boldsymbol{balg}$-morphism $\gamma := \vartheta_C^{-1} \circ B(f) : B(Y_A) \to C$.



We show that $\gamma \circ e = \alpha$. For this it suffices to show that $B(f) \circ e = \vartheta_C \circ \alpha$. Let $x \in X_C$ and $a \in C$. Then $B(f)(e(a)) = e(a) \circ f$ sends x to $\zeta_A(a)(\alpha^{-1}((1-x)C))$, which is equal to the unique $r \in \mathbb{R}$ satisfying $a + \alpha^{-1}((1-x)C) = r + \alpha^{-1}((1-x)C)$. On the other hand,

$$(\vartheta_C \circ \alpha)(a)(x) = \vartheta_C(\alpha(a))(x) = \zeta_C(\alpha(a))((1-x)C),$$

which is the unique $s \in \mathbb{R}$ satisfying $\alpha(a) + (1-x)C = s + (1-x)C$. Since $a - r \in \alpha^{-1}((1-x)C)$, we have $\alpha(a-r) \in (1-x)C$. Therefore, $\alpha(a) - r \in (1-x)C$, so $\alpha(a) + (1-x)C = r + (1-x)C$. Thus, $r = s$, and hence $B(f) \circ e(a)$ and $(\vartheta_C \circ \alpha)(a)$ agree for each $x \in X_C$. Since $a \in C$ was arbitrary, we conclude that $B(f) \circ e = \vartheta_C \circ \alpha$.

For uniqueness, suppose that $\gamma' : A^\sigma \to C$ satisfies $\gamma' \circ e = \alpha$. Then $\gamma'|_{e[A]} = \gamma|_{e[A]}$. Since γ and γ' are $\boldsymbol{balg}$-morphisms and $e[A]$ is dense in A^σ, we conclude that $\gamma' = \gamma$. ☐

The following is now an immediate consequence of Theorems 3.9 and 4.3.

Proposition 4.4. *The forgetful functor* $U : \boldsymbol{balg} \to \mathsf{WSet}$ *has a left adjoint.*

This left adjoint is obtained as the canonical extension of $F(X,w)$ for each $(X,w) \in \mathsf{WSet}$. On the other hand, the proof of Theorem 3.2 shows that the forgetful functor $\boldsymbol{balg} \to \mathsf{Set}$ does not have a left adjoint.

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