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Equilibrium price in intraday electricity markets

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Abstract

We formulate an equilibrium model of intraday trading in electricity markets. Agents face balancing constraints between their customers consumption plus intraday sales and their production plus intraday purchases. They have continuously updated forecast of their customers consumption at maturity. Forecasts are prone to idiosyncratic noise and common noise (weather). Agents production capacities are subject to independent random outages, which are each modeled by a Markov chain. The equilibrium price is defined as the price that minimizes trading cost plus imbalance cost of each agent and satisfies the usual market clearing condition. Existence and uniqueness of the equilibrium are proved, and we show that the equilibrium price and the optimal trading strategies are martingales. The main economic insights are the following: (i) when there is no uncertainty on generation, it is shown that the market price is a convex combination of forecasted marginal cost of each agent, with deterministic weights. Furthermore, the equilibrium market price is consistent with Almgren and Chriss's model, and we identify the fundamental part and the permanent market impact. It turns out that heterogeneity across agents is a necessary condition for Samuelson's effect to hold. We show that when heterogeneity lies only on costs, Samuelson's effect holds true. A similar result stands when heterogeneity lies only on market access quality. (ii) When there is production uncertainty only, we provide an approximation of the equilibrium for large number of players. The resulting price exhibits increasing volatility with time.

KEYWORDS
coupled forward–backward SDE with jumps, equilibrium model, intraday electricity markets, martingale optimality principle, Samuelson's effect

1 INTRODUCTION

Because electricity cannot be stored, the development of competitive electricity markets has led to the introduction of intraday markets. The purpose of these markets whose time-horizon does not exceed more than 36 h is to allow electricity market players to balance their position between their customers consumption and their generation for each hour of the day and avoid expensive imbalance costs. In the last decade, due to the high uncertainty of newly built wind generation production, the need of short-term balancing mechanism has increased. For electric utilities and intraday market players, the main problem is to find the optimal trading strategy that would minimize trading and imbalance costs. Indeed, they face the alternative of either building a precautionary position to cope with potential adverse and costly situation (anticipation strategy) or simply adapt their position when a new situation occurs (adaptation strategy). From a financial research point of view, intraday electricity markets offer a remarkable case of short-term physical futures market. They present convergent stylized facts about liquidity and volatility of intraday prices; liquidity depth and volatility both increase with time closer to delivery (see Kiesel and Paraschiv (2017), Balardy (2018), Kremer et al. (2020), Glas et al. (2020), Deschatre and Gruet (2021)).

This paper is a contribution to the understanding of the optimal trading strategy and of the drivers of the volatility and the liquidity of intraday markets. We develop an equilibrium model of intraday trading for a fixed hour of delivery T that allows to explain and understand the optimal trading strategies of market players and the market price dynamics. We consider that the market is composed of N agents i having each a forecast D_t^i at time $t \in [0, T]$ of their customers consumption at time T. We suppose that the volatility σ_t^i of the demand forecast is deterministic. The demand forecast of each agent is affected both by an individual Brownian noise and a collective Brownian noise with correlation ρ_i, reflecting the dependence of market players to weather and global economic conditions. Further, each agent is endowed with a generation capacity with linear marginal cost of coefficient β_t^i. Further, β_t^i evolves following a Markov chain, capturing by this way the possibility of power plant outages driving the marginal cost of an agent from a low to a high cost (and vice versa). Agents can buy or sell power for delivery at time T at the market price plus a liquidity premium $\gamma_i q_t^i$ proportional to the trade q_t^i. This premium translates the potential different market access cost of agents. The objective of each agent is to minimize the expected total trading cost plus the costs of imbalance $\eta_i(D_T^i - X_T^i - \xi_T^i)^2$ where η_i represents agent's i own perception of the cost of imbalance, X_T^i and ξ_T^i are the inventory and the production of the agent at T. Although the cost of imbalance is fixed by the electricity network operator and is the same for any market player, we allow for different evaluation of the cost of imbalance, translating the possibility that some players may have strong reluctance for imbalances while other might not care as much. All information regarding the state of the system is considered public. This point is particularly relevant in the case of intraday electricity markets: under REMIT financial regulation, market players have to report any unplanned outages to the market before a trading position can be taken. A market equilibrium is defined as trading strategies and a market price such that each agent has minimized her criteria and the market clears for the market price. This model owes agent's features to Aïd et al. (2016) and Tan and Tankov (2018) (see Edoli et al. (2017) for a variant without market impact but with a risk-averse agent). However, compared to the single-agent model in Aïd et al. (2016), we consider here a market equilibrium with N agents, and incorporate uncertainty in the production capacity. Moreover, since we aim to find an equilibrium, the price has to be endogenously determined, which means that we have to work a priori in a non-Markovian context. Therefore, instead of using the Bellman equation as in Aïd et al. (2016), we shall employ techniques from backward stochastic differential equations (SDEs) with jumps.

Moreover, this framework allows us to analyze the time variation of the market price volatility in the context of an equilibrium under perfect competition and full information. Since the seminal paper by Samuelson (1965) where the increase of volatility closer to maturity was posited and a first explanation was proposed, a whole stream of financial economics literature has developed to test, study, and explain this effect, named after its author, the *Samuelson's effect* (or *Samuelson's hypothesis* or *maturity effect*, or in the present context of this paper *the effect*). The increase in futures prices volatility closer to maturity has been statistically demonstrated to hold for many commodities. The effect has been found for agricultural commodities (Bessembinder et al. (1996), Duong and Kalev (2008)), whereas only mixed results were shown for metals (see Ng and Pirrong (1994) for a positive effect and Brooks (2020) for a negative one). In the case of crude oil, most studies find no effect (Galloway and Kolb (1996), Duong and Kalev (2008)). In the case of electricity, the effect has been obtained for other maturities than the intraday: for example, Jaeck and Lautier (2016) finds the effect for monthly contract delivery on several markets.

The literature proposes several potential reasons why or conditions under which the effect should hold or not. The most discussed theories are the *the flow of information*, the *state-variable hypothesis*, and the *cost of carry*. Explanations based on the flow of information rely on the observation that far from maturity, the increase of information from one day to another on market conditions at maturity evolves slower than closer to maturity. This point was raised in Samuelson's initial paper who then derived the effect in a model based on two hypothesis: the spot price is mean-reverting and the futures price is the conditional expectation of the spot price. Hong (2000) added to this theory that symmetry of information between players should hold for the effect to be observed or equivalently information asymmetry could break the maturity effect. The state-variable hypothesis was formulated by Anderson and Danthine (1983) and Anderson (1985). It states that the monotonicity (if any) of the volatility of futures prices depends on the way uncertainty on the equilibrium between demand and supply is resolved. In particular, in the case where there is no supply uncertainty and volatility of demand uncertainty decreases with time, this hypothesis predicts that the futures price volatility may decrease. The cost of carry (or storage cost) condition has been formulated by Bessembinder et al. (1996). It states that a necessary condition for the effect to hold is that the cost of carry[1] should be negatively correlated with the spot price. Routledge et al. (2000) showed that the effect could break down when storage is high, which is consistent with the cost of carry condition. Brooks (2020) developed this condition by making the difference between futures markets where carry arbitrage is still possible and those where it is not, and by stating that the effect holds only on the former. To our knowledge, no necessary and sufficient condition has been proposed so far.

In our framework, we obtain the following results. We prove existence and uniqueness of the equilibrium. The proof is based on the martingale optimality principle in stochastic control, and existence of solution to backward stochastic differential equations (backward SDEs) with jumps, for which we provide a complementary existence result to Becherer (2002). We show that both the equilibrium price and the optimal trading strategies are martingales, and they are characterized in terms of a coupled system of forward-backward SDE that we solve with explicit formulae. While the machinery of backward SDEs for tackling stochastic control is not new, our main contribution is to show how it can be applied in the context of market equilibrium to obtain not only an existence/uniqueness result, but also to find explicitly solutions, leading to quantitative economic insight. Let us mention the recent paper by Escauriaza et al. (2020), which also uses BSDE techniques (without jumps) for proving the existence of a Radner equilibrium.

In the case where there is no production cost uncertainty, we observe that the market price is a convex combination of the forecasted marginal cost of each agent where the weights are deterministic functions of time. The optimal trading rate of each agent consists in comparing her forecasted marginal cost to the market price and to take position accordingly, that is, to sell (resp. to buy) if it is lower (resp. higher). Thus, we find that the adaptation strategy, which is the most common strategy implemented in intraday trading desks is optimal, in the sense that, at equilibrium, a market player cannot hope to do much better. This result is in line with the papers by Aïd et al. (2016) and Tan and Tankov (2018), and complements their findings. Further, we show that the equilibrium price has the form of Almgren and Chriss model (Almgren and Chriss (2001)). We identify the fundamental part of the price as the average forecasted marginal cost to satisfy the demands and identify the market permanent impact of each agents. Permanent market impacts are deterministic function of time with a monotony depending on the agent. If all agents are identical, the market equilibrium reduces to its fundamental component because of the market clearing condition. This result closes the unanswered question of the origin of the fundamental price in Aïd et al. (2016), Tan and Tankov (2018), and more recently Féron et al. (2020).

The closed-form expression derived for the price and the trading strategies allows us to provide insight on the dynamics of the price volatility defined as the quadratic variation of the price. Contrary to the recent work of Féron et al. (2020) where the Samuelson's effect is obtained in a mean-field Nash equilibrium model of intraday trading, we do not need to resolve to strategic interactions to find conditions for the effect. We show that if all agents are identical, the price volatility monotonicity is fully determined by the volatility of the demand forecasts. In our case, the demand forecasts volatilities are decreasing in time (see Nedellec et al. (2014)). Thus, it implies that if Samuelson's effect holds true, agents must be heterogeneous. As a consequence, the Anderson and Danthine state variable hypothesis is not sufficient to explain increasing price volatility in a context of decreasing demand forecast volatility. We provide numerical illustrations where the mixing of agents of two different types allows to have decreasing or increasing volatility functions depending on the proportion of the agent's type. Because heterogeneity between agents can take complicated combinations, we provide two situations where the heterogeneity with respect to one single parameter can sustain the maturity effect. Consider that agents have the same constant demand volatility and the same correlation to common noise, we show that if they differ either by their marginal cost or by their market access quality, the Samuelson's effect does hold.

Although beyond the scope of our theoretical work, this result opens the possibility of testable predictions due to the observability of these parameters.

In the case where there is only production cost uncertainty, we show that if the number of agents is large, the equilibrium tends to the case of no production cost uncertainty by the independence of jumps in the Markov chains between agents. This result enables to simulate an approximated equilibrium price process and the corresponding approximated trading strategies. To provide a vivid illustration of the possible dynamics, we pick a situation where the market is composed of two large populations each of them having only two potential states for their cost (good or bad). Further, the first population enjoys high but stable marginal costs (low transition probability between states) whereas the other enjoys either zero or very high marginal cost with high transition probability. These features are intended to capture the high volatility of generation of intermittent sources of energy like windpower. The resulting equilibrium price process exhibits the Samuelson's effect. More specifically, this pure jump model succeeds in reproducing one important feature of the observed intraday price. During the first 60% of the period, the price does not change and trading rates are low whereas in the last 10% of the period, the price experiences large swings and trading rates are high. Besides, this behavior of the price admits a simple explanation in line with the information flow explanation of Samuelson (1965) and Hong (2000). Since there is a large number of players in the second population, there are always some players experiencing a change of marginal cost from good to bad (or vice versa). Far from maturity, this change has not much consequence as the player can still have the hope to experience the return to the other state or still have time to compensate its position by taking the appropriate trading position. But closer to maturity, when some players of the second population switch from good to bad, they suddenly switch from being long to being short and have little time to adjust their positions and thus, need to trade at a high speed.

The paper is structured as follows. Section 2 describes precisely the model. We provide in Section 3 the main results in terms of optimal strategies of each player for a given price process. Section 4 gives the market equilibrium characterization by solving explicitly the coupled system of forward–backward SDE, and the martingale properties of the equilibrium price. We describe in Section 5 the market equilibrium in the case of no production uncertainty while Section 6 provides the result in the other case.

2 | THE EQUILIBRIUM MODEL

We consider an economy with $N \in \mathbb{N}\backslash\{0\}$ power producers, which can buy/sell energy on an intraday electricity market. Their purpose is to satisfy the demand of their customers at a given fixed time T, minimizing trading costs.

2.1 | Single-agent optimal execution problem

Following Aïd et al. (2016), we formulate the optimization problem of a single agent $i \in \{1, 2, \dots, N\}$ in the economy on a finite time horizon $T > 0$. We begin introducing some notations.

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a finite set $E \subset (0, +\infty)$ of cardinality M, where M is a positive integer. We fix the following quantities at the initial time $t = 0$:

- the initial demand forecasts of the agents $(d_0^1, d_0^2, \dots, d_0^N) \in \mathbb{R}^N$;

- the initial production capacities $(e_0^1, e_0^2, \dots, e_0^N) \in E^N$;
- the initial net positions of the agents of sales/purchases of electricity in the intraday electricity market $(x_0^1, x_0^2, \dots, x_0^N) \in \mathbb{R}^N$.

On $(\Omega, \mathcal{F}, \mathbb{P})$, we consider $N+1$ independent real-valued Brownian motions $(W_t^0)_{t\geq 0}, (W_t^1)_{t\geq 0}, \dots, (W_t^N)_{t\geq 0}$ and N independent continuous-time homogenous Markov chains $(\beta_t^1)_{t\geq 0}, \dots, (\beta_t^N)_{t\geq 0}$. We assume that $(W^0, W^1, \dots, W^N)$ and $(\beta^1, \dots, \beta^N)$ are independent. Moreover, every Markov chain β^i is supposed to have finite state space E, starting point e_0^i at time $t=0$: it represents the uncertainty over time on the production capacity of agent i. We denote by $\Lambda_i = (\lambda_i(e, e') : e, e' \in E)$ the intensity matrix of β^i. We also denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the augmentation of the filtration generated by $(W^0, W^1, \dots, W^N)$ and $(\beta^1, \dots, \beta^N)$. Finally, $\mathcal{P}$ denotes the predictable σ-algebra on $\Omega \times [0,T]$ associated with $\mathbb{F}$.

The demand forecast D^i of agent i evolves on $[0,T]$ according to the equation

$$D_t^i = d_0^i + \mu_i\, t + \int_0^t \sigma_i(s) \left(\rho_i\, dW_s^0 + \sqrt{1-\rho_i^2}\, dW_s^i \right), \tag{1}$$

where $\mu_i \in \mathbb{R}$, $\rho_i \in [-1,1]$ and $\sigma_i : [0,T] \to \mathbb{R}$ is a deterministic function of time. Further, the dynamics of D^i takes into account the potential common dependence of realized demands to weather conditions. In order to satisfy the terminal demand D_T^i, agent i have the two following possibilities:

- *Power production*. The agent can choose to product a quantity ξ_i, facing at the terminal time T the cost

$$c_i(\xi_i) = \frac{1}{2}\, \beta_T^i\, \xi_i^2. \tag{2}$$

- *Trading in intraday electricity market*. Let X_t^{i,q^i} denote the agent net position of sales/purchases of electricity at time $t \in [0,T]$, delivered at the terminal date T, which is given by

$$X_t^{i,q^i} = x_0^i + \int_0^t q_s^i\, ds, \tag{3}$$

  where q^i, called the trading rate, is chosen by the agent.

We define an admissible pair of controls for each agent i as a pair (q, ξ) in $\mathcal{A}^q \times \mathcal{A}^{\xi,+}$, where

$$\mathcal{A}^q = \left\{ q = (q_t)_{0\leq t\leq T} : q \text{ is a real-valued } \mathbb{F}\text{-adapted process such that } \mathbb{E} \int_0^T q_t^2\, dt < +\infty \right\},$$

$$\mathcal{A}^{\xi,+} = \{\xi : \Omega \to [0,+\infty) : \xi \text{ is an } \mathcal{F}_T\text{-measurable random variable}\}. \tag{4}$$

The expected total cost for agent i is given by

$$J_i(q^i, \xi_i) = \mathbb{E}\left[\int_0^T q_t^i \left(P_t + \gamma_i\, q_t^i \right) dt + c_i(\xi_i) + \frac{\eta_i}{2} (D_T^i - X_T^{i,q^i} - \xi_i)^2 \right], \tag{5}$$

where γ_i and η_i are positive constants, while P denotes the intraday electricity quoted price, which will be endogenously determined in the following class of processes:

$\mathbf{L}^2(0,T)$ = the set of all $\mathbb{F}$-adapted processes $P = (P_t)_{0\leq t\leq T}$ such that

$$\mathbb{E}\left[\int_0^T |P_t|^2 dt\right] < \infty. \tag{6}$$

The agent's i optimization problem consist in trading at minimal cost to achieve a given terminal target, taking into account the liquidity cost of her sales or purchases. We take a potentially different impact parameter per agent γ_i, capturing here the potential different liquidity cost faced by market players. In this sense, we deviate from Almgren and Chriss (2001) and Aïd et al. (2016), in the sense that there is no permanent market impact in agent's i problem. Further, although in intraday electricity market, the same penalty cost is applied by the transmission system operator to any market player, we capture the idea that agents may have different appreciation of the cost of imbalances by using different imbalance cost parameter η_i. Thus, each agent i is characterized by her cost function with Markov chain β^i, her valuation of imbalances η_i, her liquidity access γ_i, her demand forecast error function σ^i, and her correlation with the common noise ρ_i.

The optimization problem of agent i consists in minimizing the expected total cost (Equation (5)) over all admissible pairs of controls (q,ξ) in $\mathcal{A}^q \times \mathcal{A}^{\xi,+}$. In order to solve such an optimization problem, we begin noting that we can easily find the optimal $\xi_i^{*,+} \in \mathcal{A}^{\xi,+}$ for agent i. As a matter of fact, in the expected total cost (Equation (5)), the control ξ_i appears only at the terminal time T. Then, the optimal $\xi_i^{*,+}$ is a nonnegative $\mathcal{F}_T$-measurable random variable minimizing the quantity

$$\mathbb{E}\left[c_i(\xi_i) + \frac{\eta_i}{2}\left(D_T^i - X_T^{i,q^i} - \xi_i\right)^2\right]. \tag{7}$$

It is then easy to see that $\xi_i^{*,+}$ is given by

$$\xi_i^{*,+} = \frac{\eta_i}{\eta_i + \beta_T^i}\left(D_T^i - X_T^{i,q^i}\right)^+ = \begin{cases} \dfrac{\eta_i}{\eta_i + \beta_T^i}\left(D_T^i - X_T^{i,q^i}\right), & D_T^i \geq X_T^{i,q^i}, \\ 0, & D_T^i < X_T^{i,q^i}. \end{cases} \tag{8}$$

Remark 2.1. The optimization problem in Equation (5) shares some similarities with optimal execution problems in equity markets. The key differences are the presence of a stochastic demand and an uncertain capacity production, which are specific to power trading.

2.2 | Auxiliary optimal execution problem

In the present section, inspired by Aïd et al. (2016), we consider a relaxed version of the optimization problem for agent i, where the control ξ_i is not constrained to be nonnegative, but it belongs to the set $\mathcal{A}^\xi$ defined as

$$\mathcal{A}^\xi = \left\{\xi : \Omega \to \mathbb{R} : \xi \text{ is an } \mathcal{F}_T\text{-measurable random variable}\right\}. \tag{9}$$

The optimization problem of agent i now consists in minimizing the expected total cost (Equation (5)) over all admissible pairs of controls (q,ξ) in $\mathcal{A}^q \times \mathcal{A}^\xi$. From the expression of J_i in Equation (5), it is straightforward to see that the optimal control ξ_i^* is given by

$$\xi_i^* = \frac{\eta_i}{\eta_i + \beta_T^i}\left(D_T^i - X_T^{i,q^i}\right). \tag{10}$$

Plugging ξ_i^* into J_i, we find (to alleviate notation, we still denote by J_i the new expected total cost, that now depends only on the control q^i)

$$J_i(q^i) := J_i(q^i,\xi_i^*) = \mathbb{E}\left[\int_0^T q_t^i\left(P_t + \gamma_i\, q_t^i\right)dt + \frac{1}{2}\frac{\eta_i\,\beta_T^i}{\eta_i + \beta_T^i}\left(D_T^i - X_T^{i,q^i}\right)^2\right]. \tag{11}$$

In conclusion, the optimization problem of agent i consists in minimizing (Equation (11)) over all controls $q^i \in \mathcal{A}^q$. Because of the presence of the stochastic process P, we cannot solve such an optimization problem by means of the Bellman optimality principle, and, in particular, via PDE methods. For this reason, we rely on the martingale optimality principle, which can be implemented using only probabilistic techniques, based in particular on the theory of backward SDEs. More specifically, we solve the optimization problem of every agent finding N optimal trading rates $\hat{q}^{1,P},\dots,\hat{q}^{N,P}$, which depend on the price process P. Given the exogenous demands $(D^i)_i$ and production capacities $(\beta^i)_i$, the equilibrium price $\hat{P} = (\hat{P}_t)_{0\le t\le T}$ is then obtained imposing the *equilibrium condition*

$$\sum_{i=1}^N \hat{q}_t^{i,\hat{P}} = 0, \qquad \text{for all } 0 \le t \le T. \tag{12}$$

Remark 2.2. Despite the homogeneous description of market players, the market model above allows to take into account a diversity of agents like pure retailers, pure producers, or pure traders. Pure retailers have uncertain terminal demand D_T^i but no generation plant. They can be represented taking a constant Markov chain β^i taking a large value e^i. Pure producers have no demand D_T^i to satisfy and are represented by the Markov chain of their generation cost. Finally, pure traders have neither a demand to satisfy nor generation asset, but only an initial inventory position.

3 | MARTINGALE OPTIMALITY PRINCIPLE AND OPTIMAL TRADING RATES

The aim of this section is to find an optimal trading rate $\hat{q}^{i,P}$ of agent i for every fixed price process P. In order to do it in the present non-Markovian framework (the non-Markovian feature is due to the presence of the process P), we consider a value process $V^{i,q_i} = (V_t^{i,q_i})_{0\le t\le T}$ given by

$$V_t^{i,q^i} = \int_0^t q_s^i\,(P_s + \gamma_i\, q_s^i)ds + \left(D_t^i - X_t^{i,q^i}\right)^2 Y_t^{2,i} + \left(D_t^i - X_t^{i,q^i}\right) Y_t^{1,i} + Y_t^{0,i}, \tag{13}$$

for all $0 \le t \le T$, with $Y^{2,i}$, $Y^{1,i}$, $Y^{0,i}$ satisfying suitable backward SDEs, namely Equations (23), (32), and (43) below. Then, the optimal trading rate $\hat{q}^{i,P}$ is obtained using the martingale optimality principle, namely imposing that for such a $\hat{q}^{i,P}$, the value process $V^{i,\hat{q}^{i,P}}$ is a true martingale, while it has to be a submartingale for any other trading rate q^i (for more details on the martingale optimality principle see items (i)-(ii)-(iii) in the proof of Theorem 3.6 below).

The present section is organized as follows. We firstly consider the three building blocks of formula (13), namely Equations (23), (32), and (43) (whose forms are chosen in order to satisfy the martingale/submartingale requirements of the value process) and prove an existence and uniqueness result for each of them. Then, exploiting the properties of the value process V^{i,q^i}, we prove the main result of this section, namely Theorem 3.6.

3.1 | Notations and preliminary results

First of all, we introduce some notations. We denote by π^i the jump measure of the Markov chain β^i, which is given by $\pi^i = \sum_{t:\beta^i_t \neq \beta^i_{t-}} \delta_{(t,\beta^i_t)}$, where $\delta_{(t,\beta^i_t)}$ is the Dirac delta at (t, β^i_t). We also denote by ν^i the compensator of π^i, which has the following form (see for instance Section 8.3 and, in particular, Theorem 8.4 in Darling and Norris (2008)):

$$\nu^i(dt, \{e\}) = \lambda_i(\beta^i_{t-}, e)\, 1_{\{\beta^i_{t-} \neq e\}}\, dt, \qquad \forall\, e \in E. \tag{14}$$

In addition to the set $\mathbf{L}^2(0,T)$ previously defined, we introduce the following sets:

- $\mathbf{S}^\infty(0,T)$: *the set of all bounded càdlàg* $\mathbb{F}$*-adapted processes on* $[0,T]$.
- $\mathbf{S}^2(0,T)$: *the set of all càdlàg* $\mathbb{F}$*-adapted processes* $Y = (Y_t)_{0\le t\le T}$ *satisfying* $\mathbb{E}[\sup_{0\le t\le T} |Y_t|^2] < \infty$.
- $\mathbf{L}^2_{\text{Pred}}(0,T)$: *the set of all* $\mathbb{F}$*-predictable processes* $Z = (Z_t)_{0\le t\le T}$ *satisfying* $\mathbb{E}[\int_0^T |Z_t|^2 dt] < \infty$.
- $\mathbf{L}^2_{\beta^i}(0,T)$: *the set of all* $\mathcal{P} \otimes \mathcal{B}(E)$*-measurable maps* $U : \Omega \times [0,T] \times E \to \mathbb{R}$ *satisfying*

$$\mathbb{E}\left[\int_0^T \sum_{e\in E} |U_t(e)|^2 \lambda_i(\beta^i_{t-}, e) 1_{\{\beta^i_{t-}\neq e\}} dt\right] < \infty. \tag{15}$$

Here $\mathcal{B}(E)$ *denotes the Borel* σ*-algebra of* E*, which turns out to be equal to the power set of* E*, since* E *is a finite subset of* $(0, +\infty)$.

Construction of $Y^{2,i}$

Let us construct the first building block of formula (13), namely $Y^{2,i}$. First of all, for every $i = 1, \dots, N$, consider the following system of M (recall that the set E has cardinality M) coupled ordinary differential equations of Riccati type on the time interval $[0,T]$:

$$y'_{i,\bar{e}}(t) = \frac{1}{\gamma_i}|y_{i,\bar{e}}(t)|^2 - \sum_{e\in E} y_{i,e}(t)\lambda_i(\bar{e}, e), \qquad y_{i,\bar{e}}(T) = \frac{1}{2}\frac{\eta_i\, \bar{e}}{\eta_i + \bar{e}}, \tag{16}$$

for every $\bar{e} \in E$.

Lemma 3.1. *For every* $i = 1, \dots, N$*, there exists a unique continuously differentiable solution* $\mathbf{y}_i = (y_{i,e})_{e\in E} : [0,T] \to \mathbb{R}^M$ *to the system of equations (Equation (16)). Moreover, every component* $y_{i,e}$ *of* $\mathbf{y}_i$ *is nonnegative on the entire interval* $[0,T]$.

Proof. For simplicity of notation, we fix $i \in \{1, \dots, N\}$ and denote $\mathbf{y}_i = (y_{i,e})_{e\in E}$ simply by $\mathbf{y} = (y_e)_{e\in E}$. Notice that system (16) can be equivalently rewritten in forward form as follows:

$$\hat{y}'_{\bar{e}}(t) = -\frac{1}{\gamma_i}|\hat{y}_{\bar{e}}(t)|^2 + \sum_{e\in E} \hat{y}_e(t)\lambda_i(\bar{e}, e), \qquad \hat{y}_{\bar{e}}(0) = \frac{1}{2}\frac{\eta_i\,\bar{e}}{\eta_i + \bar{e}}, \tag{17}$$

with $\hat{y}_e(t) = y_e(T - t)$, for all $0 \le t \le T$. By the classical Picard–Lindelöf theorem (see for instance Theorem II.1.1 in Hartman (2002)), it follows that there exists an interval $[0, \delta) \subset [0, +\infty)$ on which system (16) admits a unique solution denoted by $\hat{\mathbf{y}} = (\hat{y}_e)_{e\in E}$. Let us prove that such a solution can be extended to the entire interval $[0, +\infty)$, so that, in particular, $\hat{\mathbf{y}}$ is defined on $[0, T]$.

According to standard extension theorems for ordinary differential equations (see for instance Corollary II.3.1 in Hartman (2002)), it is enough to prove that the solution $\hat{\mathbf{y}}$ does not blow up in finite time. This holds true for system (17) as a consequence of the two following properties:

(1) every component $\hat{y}_e$ of $\hat{\mathbf{y}}$ is nonnegative on the entire interval $[0, +\infty)$;
(2) the sum $\sum_{e\in E} \hat{y}'_e(t)$ is bounded from above by a constant independent of $t \in [0, +\infty)$.

We begin proving item (1). Define $t_0 = \inf\{t \ge 0 : \min_{e\in E} \hat{y}_e(t) \le 0\}$, with $\inf \emptyset = +\infty$. We prove that every $\hat{y}_e$, $e \in E$, is strictly positive on $[0, t_0)$ and identically equal to zero on $[t_0, +\infty)$ (in the case $t_0 = +\infty$, every $\hat{y}_e$ is strictly positive on the entire interval $[0, \infty)$). If $t_0 = +\infty$, there is nothing to prove. Therefore, suppose that $t_0 < +\infty$, so that there exists $e_0 \in E$ such that $\hat{y}_{e_0}(t_0) = 0$. Since for every $e \in E$ we have $\hat{y}_e(0) > 0$, then $t_0 > 0$ and, by continuity, every component $\hat{y}_e$ is strictly positive on the interval $[0, t_0)$. It remains to prove that every $\hat{y}_e$ is identically equal to zero on $[t_0, +\infty)$. Using Equation (17), this latter property follows if we prove that every $\hat{y}_e$ is equal to zero at t_0 (as a matter of fact, if this is true, then from Equation (17) we deduce that every $\hat{y}_e$ remains at zero for all $t > t_0$). In order to prove that every y_e is zero at t_0, we proceed by contradiction and assume that there exists $e_1 \in E$ such that $\hat{y}_{e_1}(t_0) > 0$. Then, it follows from Equation (17) that $\hat{y}'_{e_0}(t_0) > 0$. This is in contradiction with the fact that $\hat{y}_{e_0}$ is strictly positive on $[0, t_0)$ (which implies that $\hat{y}'_{e_0}(t_0) \le 0$). This concludes the proof of item (1).

Let us now prove item (2). Taking the sum over $\bar{e} \in E$ in Equation (17), we obtain

$$\sum_{\bar{e}\in E} \hat{y}'_{\bar{e}}(t) = -\frac{1}{\gamma_i}\sum_{\bar{e}\in E}|\hat{y}_{\bar{e}}(t)|^2 + \sum_{\bar{e},e\in E} \hat{y}_e(t)\lambda_i(\bar{e}, e). \tag{18}$$

By Young's inequality ($ab \le a^2/(2\gamma_i) + \gamma_i\, b^2/2$) we find

$$\sum_{\bar{e},e\in E} \hat{y}_e(t)\lambda_i(\bar{e}, e) = \sum_{e\in E} \hat{y}_e(t)\left(\sum_{\bar{e}\in E}\lambda_i(\bar{e}, e)\right) \le \frac{1}{2\gamma_i}\sum_{e\in E}|\hat{y}_e(t)|^2 + \hat{C}, \tag{19}$$

with

$$\hat{C} := \frac{\gamma_i}{2}\sum_{e\in E}\left|\sum_{\bar{e}\in E}\lambda_i(\bar{e}, e)\right|^2. \tag{20}$$

Plugging Equation (19) into Equation (18), we end up with

$$\sum_{\bar{e}\in E} \hat{y}'_{\bar{e}}(t) \;\leq\; -\frac{1}{2\gamma_i}\sum_{\bar{e}\in E} |\hat{y}_{\bar{e}}(t)|^2 + \hat{C} \;\leq\; \hat{C}, \tag{21}$$

which concludes the proof of item (2). □

By Lemma 3.1, we know that there exists a unique C^1-solution $\mathbf{y}_i = (y_{i,e})_{e\in E}$ to system (16). Then, define the stochastic process

$$Y_t^{2,i} = y_{i,\beta_t^i}(t), \quad \text{for all } 0 \leq t \leq T. \tag{22}$$

As it will be proved in Proposition 3.2 below, $Y^{2,i}$ solves the following backward SDE on $[0, T]$, driven by the Markov chain β^i, with quadratic growth in the component $Y^{2,i}$:

$$Y_t^{2,i} = \frac{1}{2}\frac{\eta_i\,\beta_T^i}{\eta_i + \beta_T^i} + \int_t^T f_s^{2,i}\,ds - \int_{(t,T]\times E} U_s^{2,i}(e)\,(\pi^i - \nu^i)(ds, de), \tag{23}$$

for all $0 \leq t \leq T$, where

$$f_t^{2,i} = -\frac{1}{\gamma_i}|Y_t^{2,i}|^2, \tag{24}$$

and

$$U_t^{2,i}(e) = y_{i,e}(t) - y_{i,\beta_{t-}^i}(t). \tag{25}$$

Proposition 3.2. *For every $i = 1, \dots, N$, the backward Equation (23) admits a unique solution $(Y^{2,i}, U^{2,i}) \in \mathbf{S}^\infty(0,T)\times \mathbf{L}^2_{\beta^i}(0,T)$ given by Equations (22) and (25). Moreover, $Y^{2,i}$ is nonnegative.*

Proof. Let $(Y^{2,i}, U^{2,i})$ be the pair given by Equations (22) and (25). Notice that $(Y^{2,i}, U^{2,i})$ belongs to $\mathbf{S}^\infty(0,T)\times \mathbf{L}^2_{\beta^i}(0,T)$. As a matter of fact

$$\sup_{0\leq t\leq T} \left|Y_t^{2,i}\right| = \sup_{0\leq t\leq T} |y_{i,\beta_t^i}(t)| \;\leq\; \sup_{0\leq t\leq T}\max_{e\in E} |y_{i,e}(t)| < +\infty, \tag{26}$$

and

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \sum_{e\in E}\left|U_t^{2,i}(e)\right|^2 \lambda_i(\beta_{t-}^i, e)dt\right] &= \mathbb{E}\left[\int_0^T \sum_{e\in E}\left|y_{i,e}(t) - y_{i,\beta_{t-}^i}(t)\right|^2 \lambda_i(\beta_{t-}^i, e)dt\right] \\
&\leq \mathbb{E}\left[\int_0^T \sum_{e\in E}\left(2\left|y_{i,e}(t)\right|^2 + 2\left|y_{i,\beta_{t-}^i}(t)\right|^2\right)\lambda_i(\beta_{t-}^i, e)dt\right] \\
&\leq 4M\int_0^T \max_{e\in E}|y_{i,e}(t)|^2\left(\sum_{e\in E}\lambda_i\left(\beta_{t-}^i, e\right)\right)dt < +\infty,
\end{aligned} \tag{27}$$

where recall that M is the cardinality of the set E. It remains to prove that $(Y^{2,i}, U^{2,i})$ solves Equation (23). Applying Itô's formula to $y_{i,\beta^i_\cdot}(\cdot)$ between $t \in [0,T)$ and T, we find

$$y_{i,\beta^i_T}(T) = y_{i,\beta^i_t}(t) + \int_t^T y'_{i,\beta^i_s}(s)ds + \sum_{t<s\le T}\left(y_{i,\beta^i_s}(s) - y_{i,\beta^i_{s-}}(s)\right). \tag{28}$$

Now, since $\sum_{e\in E}\lambda_i(\bar{e},e) = 0$, Equation (16) can be rewritten as follows:

$$y'_{i,\bar{e}}(t) = \frac{1}{\gamma_i}|y_{i,\bar{e}}(t)|^2 - \sum_{e\in E}\big(y_{i,e}(t) - y_{i,\bar{e}}(t)\big)\lambda_i(\bar{e},e). \tag{29}$$

Therefore

$$\int_t^T y'_{i,\beta^i_s}(s)ds = -\int_t^T \hat{f}^{2,i}_s\,ds - \int_t^T\int_E U^{2,i}_s(e)\,\nu^i(ds,de). \tag{30}$$

On the other hand, we have

$$\sum_{t<s\le T}\left(y_{i,\beta^i_s}(s) - y_{i,\beta^i_{s-}}(s)\right) = \int_{(t,T]\times E} U^{2,i}_s(e)\,\pi^i(ds,de). \tag{31}$$

Hence, plugging Equations (30) and (31) into Equation (28), we obtain Equation (23). □

Construction of $Y^{1,i}$

Let us construct the second ingredient of formula (13), namely $Y^{1,i}$, which will be denoted by $Y^{1,i,P}$ to emphasize its dependence on P. For every $i = 1,\dots,N$ and any $P \in \mathbf{L}^2(0,T)$, consider the following linear backward SDE on $[0,T]$, driven by the Brownian motions $W^0, W^1, \dots, W^N$ and the Markov chains $\beta^1,\dots,\beta^N$:

$$Y^{1,i,P}_t = \int_t^T f^{1,i,P}_s ds - \sum_{j=0}^N \int_t^T Z^{1,i,j,P}_s dW^j_s - \sum_{j=1}^N \int_{(t,T]\times E} U^{1,i,j,P}_s(e)\,(\pi^j - \nu^j)(ds,de), \tag{32}$$

where

$$f^{1,i,P}_t = 2\mu^i_t Y^{2,i}_t + \frac{1}{\gamma_i} Y^{2,i}_t\left(P_t - Y^{1,i,P}_t\right). \tag{33}$$

Notice that Equation (32) has zero terminal condition at time T: $Y^{1,i,P}_T = 0$. We also observe that the generator depends linearly on the component $Y^{1,i,P}$ and it is random (as it depends on $Y^{2,i}$ and P). We now address the problem of existence and uniqueness of a solution to Equation (32), for which we need the following martingale representation result.

Lemma 3.3. *For every square-integrable real-valued $\mathcal{F}_T$-measurable random variable ζ, there exist $Z^0, Z^1, \dots, Z^N \in \mathbf{L}^2_{Pred}(0,T)$, $U^1 \in \mathbf{L}^2_{\beta^1}(0,T)$, ..., $U^N \in \mathbf{L}^2_{\beta^N}(0,T)$ such that*

$$\zeta = \mathbb{E}[\zeta] + \sum_{j=0}^N \int_0^T Z^j_s dW^j_s + \sum_{j=1}^N \int_{(0,T]\times E} U^j_s(e)\,(\pi^j - \nu^j)(ds,de). \tag{34}$$

Proof. The result is standard and follows for instance from Example 2.1-(2) in Becherer (2002). For completeness, we report the main steps of the proof. Denote by $\mathbb{F}^W = (\mathcal{F}_t^W)_{t\geq 0}$ (resp. $\mathbb{F}^{\beta^i} = (\mathcal{F}_t^{\beta^i})_{t\geq 0}$) the augmentation of the filtration generated by $(W^0, W^1, \dots, W^N)$ (resp. β^i). It is well-known that if ζ is $\mathcal{F}_T^W$-measurable (resp. $\mathcal{F}_T^{\beta^i}$-measurable) then representation (34) holds; indeed, in this case representation (34) is such that $U^1, \dots, U^N$ (resp. $Z^0, Z^1, \dots, Z^N$ and U^j, $j \neq i$) are equal to zero.

It is then easy to see that representation (34) also holds for every ζ of the form $\zeta_0\zeta_1 \cdots \zeta_N$, with ζ_0 and ζ_i, $i \in \{1, \dots, N\}$, being respectively $\mathcal{F}_T^W$-measurable and $\mathcal{F}_T^{\beta^i}$-measurable. The claim follows from the fact that the linear span of the random variables of the form $\zeta_0\zeta_1 \cdots \zeta_N$ is dense in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ (the space of square-integrable real-valued $\mathcal{F}_T$-measurable random variables). □

Proposition 3.4. *For every $i = 1, \dots, N$ and any $P \in \mathbf{L}^2(0,T)$, the backward Equation (32) admits a unique solution $(Y^{1,i,P}, Z^{1,i,0,P}, Z^{1,i,1,P}, \dots, Z^{i,1,N,P}, U^{1,i,1,P}, \dots, U^{1,i,N,P}) \in \mathbf{S}^2(0,T) \times \mathbf{L}^2_{Pred}(0,T) \times \cdots \times \mathbf{L}^2_{Pred}(0,T) \times \mathbf{L}^2_{\beta^1}(0,T) \times \cdots \times \mathbf{L}^2_{\beta^N}(0,T)$. Moreover, $Y^{1,i,P}$ is given by*

$$Y_t^{1,i,P} = \frac{1}{\Gamma_t^i}\mathbb{E}\left[\int_t^T \Gamma_s^i Y_s^{2,i}\left(2\mu_i + \frac{1}{\gamma_i}P_s\right)ds\middle|\mathcal{F}_t\right], \qquad \mathbb{P}\text{-}a.s., \tag{35}$$

for all $0 \leq t \leq T$, where $\Gamma_t^i = e^{-\frac{1}{\gamma_i}\int_0^t Y_s^{2,i}ds} = e^{-\frac{1}{\gamma_i}\int_0^t y_{i,\beta_s^i}(s)ds}$.

Proof. Existence. Fix $i \in \{1, \dots, N\}$, $P \in \mathbf{L}^2(0,T)$ and define (to alleviate notation, we write ζ^i rather than $\zeta^{i,P}$ as P is fixed throughout the proof; we adopt the same convention for all the other quantities involved in the proof)

$$\zeta^i = \int_0^T \Gamma_s^i Y_s^{2,i}\left(2\mu_i + \frac{1}{\gamma_i}P_s\right)ds. \tag{36}$$

Since ζ^i is a square-integrable real-valued $\mathcal{F}_T$-measurable random variable, we can apply Lemma 3.3 from which we deduce the existence of $\hat{Z}^{1,i,0}, \hat{Z}^{1,i,1}, \dots, \hat{Z}^{1,i,N} \in \mathbf{L}^2_{\mathrm{Pred}}(0,T)$, $\hat{U}^{1,i,1} \in \mathbf{L}^2_{\beta^1}(0,T)$, ..., $\hat{U}^{1,i,N} \in \mathbf{L}^2_{\beta^N}(0,T)$ such that

$$\zeta^i = \mathbb{E}[\zeta^i] + \sum_{j=0}^N \int_0^T \hat{Z}_s^{1,i,j} dW_s^j + \sum_{j=1}^N \int_{(0,T]\times E} \hat{U}_s^{1,i,j}(e)(\pi^j - \nu^j)(ds, de). \tag{37}$$

Now, define $\hat{Y}^{1,i} = (\hat{Y}_t^{1,i})_{0\leq t\leq T}$ as (the càdlàg version of)

$$\left(\mathbb{E}\left[\int_t^T \Gamma_s^i Y_s^{2,i}\left(2\mu_i + \frac{1}{\gamma_i}P_s\right)ds\middle|\mathcal{F}_t\right]\right)_{0\leq t\leq T}. \tag{38}$$

Since $P \in \mathbf{L}^2(0,T)$, we see that $\hat{Y}^{1,i} \in \mathbf{S}^2(0,T)$. Moreover, taking the conditional expectation with respect to $\mathcal{F}_t$ in Equation (37), we obtain

$$\hat{Y}_t^{1,i} = \hat{Y}_0^{1,i} - \int_0^t \Gamma_s^i Y_s^{2,i}\left(2\mu_i + \frac{1}{\gamma_i}P_s\right)ds + \sum_{j=0}^N \int_0^t \hat{Z}_s^{1,i,j}\,dW_s^j + \sum_{j=1}^N \int_{(0,t]\times E} \hat{U}_s^{1,i,j}(e)\,(\pi^j - \nu^j)(ds,de). \tag{39}$$

Finally, we define $Y^{1,i} = (Y_t^{1,i})_{0\le t\le T}$ as $Y_t^{1,i} = \hat{Y}_t^{1,i}/\Gamma_t^i$. Then, noting that

$$d\Gamma_t^i = -\frac{1}{\gamma_i}Y_t^{2,i}\Gamma_t^i dt, \qquad \Gamma_0^i = 1, \tag{40}$$

applying Itô's formula to $\hat{Y}_t^{1,i}/\Gamma_t^i$, we get

$$Y_t^{1,i} = Y_0^{1,i} - \int_0^t Y_s^{2,i}\left(2\mu_i + \frac{1}{\gamma_i}P_s\right)ds + \sum_{j=0}^N \int_t^T Z_s^{1,i,j}\,dW_s^j + \sum_{j=1}^N \int_{(t,T]\times E} U_s^{1,i,j}(e)\,(\pi^j - \nu^j)(ds,de) + \frac{1}{\gamma_i}\int_0^t Y_s^{1,i}Y_s^{2,i}ds, \tag{41}$$

where

$$Z_t^{1,i,j} = \frac{\hat{Z}_t^{1,i,j}}{\Gamma_t^i}, \qquad U_t^{1,i,j}(e) = \frac{\hat{U}_t^{1,i,j}(e)}{\Gamma_t^i}. \tag{42}$$

This proves that $(Y^{1,i}, Z^{1,i,0}, Z^{1,i,1}, \dots, Z^{1,i,N}, U^{1,i,1}, \dots, U^{1,i,N})$ solves Equation (32); moreover, since $(\Gamma^i)^{-1} \in \mathbf{S}^\infty(0,T)$, it is easy to see that such a solution belongs to $\mathbf{S}^2(0,T) \times \mathbf{L}^2_{\mathrm{Pred}}(0,T) \times \cdots \times \mathbf{L}^2_{\mathrm{Pred}}(0,T) \times \mathbf{L}^2_{\beta^1}(0,T) \times \cdots \times \mathbf{L}^2_{\beta^N}(0,T)$.

Uniqueness. Fix $i \in \{1,\dots,N\}$ and let $(\tilde{Y}^{1,i}, \tilde{Z}^{1,i,0}, \tilde{Z}^{1,i,1}, \dots, \tilde{Z}^{1,i,N}, \tilde{U}^{1,i,1}, \dots, \tilde{U}^{1,i,N}) \in \mathbf{S}^2(0,T) \times \mathbf{L}^2_{\mathrm{Pred}}(0,T) \times \cdots \times \mathbf{L}^2_{\mathrm{Pred}}(0,T) \times \mathbf{L}^2_{\beta^1}(0,T) \times \cdots \times \mathbf{L}^2_{\beta^N}(0,T)$ be a solution to Equation (32). Applying Itô's formula to the product $\Gamma_t^i \tilde{Y}_t^{1,i}$, it is easy to see that $\tilde{Y}^{1,i}$ is given by Equation (35). This proves the uniqueness of the Y-component, which in turn implies the uniqueness of all other components and concludes the proof. □

Construction of $Y^{0,i}$

Let us finally construct the third and last ingredient of formula (13), namely $Y^{0,i,P}$, which will be denoted by $Y^{0,i,P}$ to emphasize its dependence on P. For every $i = 1,\dots,N$ and any $P \in \mathbf{L}^2(0,T)$, consider the following backward SDE on $[0,T]$, driven by the Brownian motions $W^0, W^1, \dots, W^N$ and the Markov chains $\beta^1, \dots, \beta^N$:

$$Y_t^{0,i,P} = \int_t^T f_s^{0,i,P}ds - \sum_{j=0}^N \int_t^T Z_s^{0,i,j,P}\,dW_s^j - \sum_{j=1}^N \int_{(t,T]\times E} U_s^{0,i,j,P}(e)\,(\pi^j - \nu^j)(ds,de), \tag{43}$$

where

$$f_t^{0,i,P} := |\sigma_t^i|^2 Y_t^{2,i} + \mu_i Y_t^{1,i,P} + \sigma_t^i \rho_i Z_t^{1,i,0,P} + \sigma_t^i \sqrt{1-\rho_i^2} Z_t^{1,i,i,P} - \frac{1}{4\gamma_i}\left(P_t - Y_t^{1,i,P}\right)^2. \tag{44}$$

Notice that Equation (43) has zero terminal condition at time T: $Y_T^{0,i,P} = 0$.

Proposition 3.5. *For every $i = 1, \dots, N$ and any $P \in \mathbf{L}^2(0,T)$, the backward Equation (43) admits a unique solution $(Y^{0,i,P}, Z^{0,i,0,P}, Z^{0,i,1,P}, \dots, Z^{0,i,N,P}, U^{0,i,1,P}, \dots, U^{0,i,N,P}) \in \mathbf{S}^2(0,T) \times \mathbf{L}^2_{Pred}(0,T) \times \cdots \times \mathbf{L}^2_{Pred}(0,T) \times \mathbf{L}^2_{\beta^1}(0,T) \times \cdots \times \mathbf{L}^2_{\beta^N}(0,T)$. Moreover, $Y^{0,i,P}$ is given by*

$$Y_t^{0,i,P} = \mathbb{E}\left[\int_t^T \left(|\sigma_s^i|^2 Y_s^{2,i} + \mu_i Y_s^{1,i,P} + \sigma_s^i \rho_i Z_s^{1,i,0,P} + \sigma_s^i \sqrt{1-\rho_i^2} Z_s^{1,i,i,P} - \frac{1}{4\gamma_i}\left(P_s - Y_s^{1,i,P}\right)^2\right) ds \middle| \mathcal{F}_t\right], \quad \mathbb{P}\text{-a.s.} \tag{45}$$

for all $0 \le t \le T$.

Proof. The result can be proved proceeding along the same lines as in the proof of Proposition 3.4, noting that the backward equation is still linear (in this case, the generator does not even depend on the unknowns). □

3.2 | Main result

We can finally state our main result, which provides the optimal trading rate of agent i given a fixed price process P.

Theorem 3.6. *For every $i = 1, \dots, N$ and any $P \in \mathbf{L}^2(0,T)$, there exists a unique (up to $\mathbb{P}$-indistinguishability) continuous process $\hat{X}^{i,P} = (\hat{X}_t^{i,P})_{0 \le t \le T}$ in $\mathbf{L}^2_{Pred}(0,T)$ satisfying the following equation:*

$$\hat{X}_t^{i,P} = x_0^i + \frac{1}{2\gamma_i}\int_0^t \left(2Y_s^{2,i}\left(D_s^i - \hat{X}_s^{i,P}\right) + Y_s^{1,i,P} - P_s\right) ds, \quad \textit{for all } 0 \le t \le T, \ \mathbb{P}\text{-a.s.} \tag{46}$$

with $Y^{2,i}$ and $Y^{1,i,P}$ given, respectively, by Equations (22) and (35). Define

$$\hat{q}_t^{i,P} = \frac{1}{2\gamma_i}\left(2Y_t^{2,i}\left(D_t^i - \hat{X}_t^{i,P}\right) + Y_t^{1,i,P} - P_t\right), \quad \textit{for all } 0 \le t \le T. \tag{47}$$

Then $\hat{X}^{i,P} \equiv X^{i,\hat{q}^{i,P}}$ and the following holds:

(1) $\hat{q}^{i,P}$ is an admissible control: $\hat{q}^{i,P} \in \mathcal{A}^q$;
(2) $\hat{q}^{i,P}$ is an optimal control for agent i.

Proof. Concerning Equation (46), notice that such an equation is deterministic with stochastic coefficients, so it can be solved pathwise. More precisely, Equation (46) is a first-order linear ordinary differential equation (with stochastic coefficients), so that it admits a unique solution, which can be written in explicit form. It is then clear that such a solution is continuous and $\mathbb{F}$-adapted, since all the coefficients are also $\mathbb{F}$-adapted.

It remains to prove items (1) and (2). To this end, fix $i = 1, \dots, N$ and $P \in \mathbf{L}^2(0,T)$ (to alleviate notation, in the sequel we do not explicitly report the dependence on P; so, for instance, we simply write $\hat{X}^i, Y^{1,i}, \hat{q}^i$ instead of $\hat{X}^{i,P}, Y^{1,i,P}, \hat{q}^{i,P}$). The admissibility of $\hat{q}^i$ follows directly from its definition, using the integrability properties of P, D^i, $\hat{X}^i$, $Y^{1,i}$, $Y^{2,i}$. Let us now prove item (2). In order to prove the optimality of $\hat{q}^i$, we implement the martingale optimality principle. More precisely, we construct a family of processes $(V_t^{i,q^i})_{0\le t\le T}$, for every $q^i \in \mathcal{A}^q$, satisfying the following properties:

(i) for every $q^i \in \mathcal{A}^q$, we have

$$V_T^{i,q^i} = \int_0^T q_t^i \left(P_t + \gamma_i\, q_t^i\right) dt + \frac{1}{2}\frac{\eta_i\, \beta_T^i}{\eta_i + \beta_T^i}\left(D_T^i - X_T^{i,q^i}\right)^2. \tag{48}$$

(ii) V_0^{i,q^i} is a constant independent of $q^i \in \mathcal{A}^q$.
(iii) V^{i,q^i} is a submartingale for all $q^i \in \mathcal{A}^q$, and $V^{i,\hat{q}^i}$ is a martingale when $q^i = \hat{q}^i$.

Notice that when q^i is given by $\hat{q}^i$ then $X^{i,\hat{q}^i} \equiv \hat{X}^i$, with $\hat{X}^i$ satisfying Equation (46). Suppose for a moment that we have already constructed a family of stochastic processes $(V_t^{i,q^i})_{0\le t\le T}$, $q^i \in \mathcal{A}^q$, satisfying points (i)-(ii)-(iii). Then, observe that, for any $q^i \in \mathcal{A}^q$, we have

$$\begin{aligned} J_i(q^i) &= \mathbb{E}\left[\int_0^T q_t^i \left(P_t + \gamma_i\, q_t^i\right) dt + \frac{1}{2}\frac{\eta_i\, \beta_T^i}{\eta_i + \beta_T^i}\left(D_T^i - X_T^{i,q^i}\right)^2\right] = \mathbb{E}\left[V_T^{i,q^i}\right] \ge V_0^{i,q^i} = V_0^{i,\hat{q}^i} \\ &= \mathbb{E}\left[V_T^{i,\hat{q}^i}\right] = \mathbb{E}\left[\int_0^T \hat{q}_t^i \left(P_t + \gamma_i\, \hat{q}_t^i\right) dt + \frac{1}{2}\frac{\eta_i\, \beta_T^i}{\eta_i + \beta_T^i}\left(D_T^i - \hat{X}_T^i\right)^2\right] = J_i(\hat{q}^i), \end{aligned} \tag{49}$$

which proves the optimality of $\hat{q}^i$. It remains to construct $(V_t^{i,q^i})_{0\le t\le T}$, $q^i \in \mathcal{A}^q$, satisfying (i)–(ii)–(iii). Given $q^i \in \mathcal{A}^q$, we take $(V_t^{i,q^i})_{0\le t\le T}$ as in Equation (13), namely

$$V_t^{i,q^i} = \int_0^t q_s^i\,(P_s + \gamma_i\, q_s^i) ds + \left(D_t^i - X_t^{i,q^i}\right)^2 Y_t^{2,i} + \left(D_t^i - X_t^{i,q^i}\right) Y_t^{1,i} + Y_t^{0,i}, \tag{50}$$

for all $0 \le t \le T$, with $Y^{2,i}$, $Y^{1,i}$, and $Y^{0,i}$ satisfying, respectively, Equations (23), (32), and (43).

From the definition of $(V_t^{i,q^i})_{0\le t\le T}$, it is clear that (i) holds. Moreover, since $\hat{Y}^{2,i}$, $\hat{Y}^{1,i}$, $\hat{Y}^{0,i}$ are independent of q^i, we see that (ii) holds as well. It remains to prove item (iii). By Itô's formula, we obtain $V_t^{i,q^i} = V_0^{i,q^i} + \int_0^t b_s^{i,q^i} ds +$ martingale, where

$$\begin{aligned} b_t^{i,q^i} &= q_t^i(P_t + \gamma_i q_t^i) - \left(D_t^i - X_t^{i,q^i}\right)^2 f_t^{2,i} - \left(D_t^i - X_t^{i,q^i}\right) f_t^{1,i} - f_t^{0,i} \\ &\quad + Y_t^{2,i}\left[2\left(D_t^i - X_t^{i,q^i}\right)(\mu_i - q_t^i) + |\sigma_i(t)|^2\right] + (\mu_i - q_t^i)Y_t^{1,i} + \sigma_i(t)\rho_i Z_t^{1,i,0} + \sigma_i(t)\sqrt{1-\rho_i^2}Z_t^{1,i,i}. \end{aligned} \tag{51}$$

It is easy to see that when $q^i = \hat{q}^i$, the drift $b^{i,\hat{q}^i}$ becomes zero. So, in particular, $V^{i,\hat{q}^i}$ is a true martingale. In order to conclude the proof, we need to prove that in general we have $b^{i,q^i} \geq 0$, that is $(V_t^{i,q^i})_{0\leq t\leq T}$ is a submartingale for any q^i. To this end, it is useful to rewrite b_t^{i,q^i} as a quadratic polynomial in the variable q_t^i

$$\begin{aligned} b_t^{i,q^i} &= \gamma_i |q_t^i|^2 + \left[P_t - 2Y_t^{2,i}\left(D_t^i - X_t^{i,q^i}\right) - Y_t^{1,i}\right] q_t^i - \left(D_t^i - X_t^{i,q^i}\right)^2 f_t^{2,i} \\ &\quad - \left(D_t^i - X_t^{i,q^i}\right) f_t^{1,i} - f_t^{0,i} + Y_t^{2,i}\left[2\left(D_t^i - X_t^{i,q^i}\right)\mu + |\sigma_i(t)|^2\right] \\ &\quad + \mu_i Y_t^{1,i} + \sigma_i(t)\rho_i Z_t^{1,i,0} + \sigma_i(t)\sqrt{1-\rho_i^2} Z_t^{1,i,i}. \end{aligned} \tag{52}$$

Since $\gamma_i > 0$, b_t^{i,q^i} is nonnegative for every value of q_t^i if and only if the discriminant is nonpositive. Notice, however, that the discriminant cannot be strictly negative, otherwise this would give a contradiction to the fact that $b^{i,\hat{q}^i}$ is zero. In conclusion, the discriminant has be identically equal to zero, namely

$$\begin{aligned} &4\gamma_i \Big\{ -\left(D_t^i - X_t^{i,q^i}\right)^2 f_t^{2,i} - \left(D_t^i - X_t^{i,q^i}\right) f_t^{1,i} - f_t^{0,i} + Y_t^{2,i}\left[2\left(D_t^i - X_t^{i,q^i}\right)\mu_i + |\sigma_i(t)|^2\right] \\ &+ \mu_i Y_t^{1,i} + \sigma_i(t)\rho_i Z_t^{1,i,0} + \sigma_i(t)\sqrt{1-\rho_i^2} Z_t^{1,i,i} \Big\} = \left[P_t - 2Y_t^{2,i}\left(D_t^i - X_t^{i,q^i}\right) - Y_t^{1,i}\right]^2. \end{aligned} \tag{53}$$

Rewriting it in terms of the variable $D_t^i - X_t^{i,q^i}$, we find

$$\begin{aligned} &4\left(-\gamma_i f_t^{2,i} - |Y_t^{2,i}|^2\right)\left(D_t^i - X_t^{i,q^i}\right)^2 + 4\left[\gamma_i\left(2\mu_i Y_t^{2,i} - f_t^{1,i}\right) + Y_t^{2,i}\left(P_t - Y_t^{1,i}\right)\right]\left(D_t^i - X_t^{i,q^i}\right) \\ &+ 4\gamma_i\left(|\sigma_i(t)|^2 Y_t^{2,i} + \mu_i Y_t^{1,i} + \sigma_i(t)\rho_i Z_t^{1,i,0} + \sigma_i(t)\sqrt{1-\rho_i^2} Z_t^{1,i,i} - f_t^{0,i}\right) - \left(P_t - Y_t^{1,i}\right)^2 = 0. \end{aligned} \tag{54}$$

Now, we see that $f^{2,i}$, $f^{1,i}$, and $f^{0,i}$ (defined in Equations (24), (33), and (44), respectively) are such that the above equality is always satisfied, regardless of the value of $D_t^i - X_t^{i,q^i}$. It follows that b^{i,q^i} is nonnegative, which implies that $(V_t^{i,q^i})_{0\leq t\leq T}$ is a submartingale and concludes the proof. □

Remark 3.7. In the particular case with one agent ($N = 1$), no uncertainty in the capacity production (β is constant), the price process P is an arithmetic Brownian motion, we recover the result in Aïd et al. (2016) that was derived by Bellman equation.

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In the present section, we use the explicit expression of $\hat{q}^{i,P}$ in Equation (47) together with the equilibrium condition (12) to find the equilibrium price process $\hat{P} = (\hat{P}_t)_{0\leq t\leq T}$ (Theorem 4.1). We also find the dynamics of the equilibrium price process (Theorem 4.2), and obtain notably the martingale property of the equilibrium price process.

Theorem 4.1. *There exists a unique solution* $(\hat{X}^i, \hat{Y}^{1,i}, \hat{Z}^{1,i,0}, \hat{Z}^{1,i,j}, \hat{U}^{1,i,j})_{i,j=1,\dots,N}$, *with* $\hat{X}^i \in \mathbf{L}^2_{Pred}(0,T)$ *being a continuous process,* $\hat{Y}^{1,i} \in \mathbf{S}^2(0,T)$, $\hat{Z}^{1,i,j} \in \mathbf{L}^2_{Pred}(0,T)$, $\hat{U}^{1,i,j} \in \mathbf{L}^2_{\beta^j}(0,T)$, *satisfying the following coupled forward–backward system of SDEs:*

$$\begin{cases} \hat{X}^i_t = x^i_0 + \frac{1}{2\gamma_i}\int_0^t \left(2Y^{2,i}_s(D^i_s - \hat{X}^i_s) + \hat{Y}^{1,i}_s - \hat{P}_s\right) ds, \quad 0 \le t \le T, & (55)\\ \hat{Y}^{1,i}_t = \int_t^T \left(2\mu_i Y^{2,i}_s + \frac{1}{\gamma_i} Y^{2,i}_s\left(\hat{P}_s - \hat{Y}^{1,i}_s\right)\right) ds - \sum_{j=0}^N \int_t^T \hat{Z}^{1,i,j}_s dW^j_s & (56)\\ \qquad - \sum_{j=1}^N \int_{(t,T]\times E} \hat{U}^{1,i,j}_s(e)(\pi^j - \nu^j)(ds,de), \quad 0 \le t \le T, \end{cases}$$

where

$$\hat{P}_t := \sum_{j=1}^N \frac{\bar{\gamma}}{\gamma_j}\left(2Y^{2,j}_t\left(D^j_t - \hat{X}^j_t\right) + \hat{Y}^{1,j}_t\right), \qquad \bar{\gamma} := \left(\sum_{i=1}^N \frac{1}{\gamma_i}\right)^{-1}, \quad \text{for all } 0 \le t \le T. \tag{57}$$

Moreover, $\hat{X}^i$ *coincides with* $\hat{X}^{i,\hat{P}}$ *of Equation (46), while* $\hat{Y}^{1,i}, \hat{Z}^{1,i,j}, \hat{U}^{1,i,j}$ *coincide with* $\hat{Y}^{1,i,\hat{P}}, \hat{Z}^{1,i,j,\hat{P}}, \hat{U}^{1,i,j,\hat{P}}$ *of Equation (32). Finally,* $\hat{P}$ *is the price process satisfying the equilibrium condition (12) with* $\hat{q}^{i,\hat{P}}$ *as in Equation (47).*

Proof. Existence and uniqueness for system (55)–(56) can be proved proceeding along the same lines as in the proof of Lemma 2.2 in Li and Wei (2014), the only difference being that π^j is a Poisson random measure in Li and Wei (2014). We also notice that, proceeding as in the proof of Proposition 3.1 in Li and Wei (2014), we obtain the following estimate:

$$\sum_{i=1}^N \mathbb{E}\left[\sup_{0\le t\le T}|\hat{X}^i_t|^2 + \sup_{0\le t\le T}|\hat{Y}^{1,i}_t|^2 + \sum_{j=0}^N \int_0^T |\hat{Z}^{1,i,j}_t|^2\, dt \right.$$
$$\left. + \sum_{j=1}^N \int_0^T \sum_{e\in E} |\hat{U}^{1,i,j}_t(e)|^2 \lambda_i(\beta^i_{t-}, e)\, 1_{\{\beta^i_{t-}\neq e\}}\, dt\right] \le \hat{C}, \tag{58}$$

where $\hat{C}$ is a positive constant depending only on $x^i_0, d^i_0, E, \mu_i, \sigma^i, \Lambda_i\, \gamma_i, \eta_i, T$.

Finally, regarding the last part of the statement, it is easy to see that $\hat{X}^i$ coincides with $\hat{X}^{i,\hat{P}}$ of Equation (46), while $\hat{Y}^{1,i}, \hat{Z}^{1,i,j}, \hat{U}^{1,i,j}$ coincide with $\hat{Y}^{1,i,\hat{P}}, \hat{Z}^{1,i,j,\hat{P}}, \hat{U}^{1,i,j,\hat{P}}$ of Equation (32). Finally, it is also clear that $\hat{P}$ is the equilibrium price process, as formula (57) follows directly from the equilibrium condition (12) and the definition of $\hat{q}^{i,\hat{P}}$ in Equation (47). □

Theorem 4.2. *The equilibrium price process* $\hat{P} = (\hat{P}_t)_{0\le t\le T}$ *is a martingale. More precisely, the dynamics of* $\hat{P}$ *is given by*

$$\hat{P}_t = \hat{P}_0 + \sum_{i=1}^N \int_0^t \frac{\bar{\gamma}}{\gamma_i} 2Y^{2,i}_s \sigma_i(s)\left(\rho_i\, dW^0_s + \sqrt{1-\rho_i^2}\, dW^i_s\right) + \sum_{i=0}^N \int_0^t \left(\sum_{j=1}^N \frac{\bar{\gamma}}{\gamma_j}\hat{Z}^{1,j,i}_s\right) dW^i_s \tag{59}$$
$$+ \sum_{i=1}^N \int_{(0,t]\times E}\left(\frac{\bar{\gamma}}{\gamma_i}2(D^i_s - \hat{X}^i_s)U^{2,i}_s(e) + \sum_{j=1}^N \frac{\bar{\gamma}}{\gamma_j}\hat{U}^{1,j,i}_s(e)\right)(\pi^i - \nu^i)(ds,de),$$

for all $0 \le t \le T$*. Similarly, the optimal trading strategies* $\hat{q}^{1,\hat{P}}, \dots, \hat{q}^{N,\hat{P}}$ *are martingales.*

Proof. Recall that

$$\hat{P}_t = \sum_{i=1}^{N} \frac{\bar{\gamma}}{\gamma_i}\left(2Y_t^{2,i}(D_t^i - \hat{X}_t^i) + \hat{Y}_t^{1,i}\right), \qquad 0 \le t \le T, \tag{60}$$

where $D^i, Y^{2,i}, \hat{X}^i, \hat{Y}^{1,i}$ satisfy, respectively, Equations (1), (23), (55), and (56). Then, an application of Itô's formula yields

$$\hat{P}_t = \hat{P}_0 + \int_0^t \hat{b}_s \, ds + \text{martingale}, \tag{61}$$

with the martingale term as in Equation (59) and

$$\hat{b}_t = \sum_{i=1}^{N} \frac{\bar{\gamma}}{\gamma_i}\left(2Y_t^{2,i}\left(\mu_i - \hat{q}_t^{i,\hat{P}}\right) - 2f_t^{2,i}\,(D_t^i - \hat{X}_t^i) - f_t^{1,i,\hat{P}}\right), \tag{62}$$

where recall that ($\hat{q}^i$ stands for $\hat{q}^{i,\hat{P}}$)

$$\begin{aligned} \hat{q}_t^i &= \frac{1}{2\gamma_i}\left(2Y_t^{2,i}(D_t^i - \hat{X}_t^i) + \hat{Y}_t^{1,i} - \hat{P}_t\right), \\ f_t^{2,i} &= -\frac{1}{\gamma_i}|Y_t^{2,i}|^2, \\ f_t^{1,i,\hat{P}} &= 2\mu_i Y_t^{2,i} + \frac{1}{\gamma_i} Y_t^{2,i}\left(\hat{P}_t - \hat{Y}_t^{1,i}\right). \end{aligned} \tag{63}$$

Hence, $\hat{b}_t$ can be rewritten as

$$\begin{aligned} \hat{b}_t &= \sum_{i=1}^{N} \frac{\bar{\gamma}}{\gamma_i}\left(-2\hat{q}_t^i Y_t^{2,i} + \frac{2}{\gamma_i}|Y_t^{2,i}|^2\,(D_t^i - \hat{X}_t^i) - \frac{1}{\gamma_i}Y_t^{2,i}\left(\hat{P}_t - \hat{Y}_t^{1,i}\right)\right) \\ &= \sum_{i=1}^{N} \frac{\bar{\gamma}}{\gamma_i}\left(-2\hat{q}_t^i Y_t^{2,i} + \frac{1}{\gamma_i}Y_t^{2,i}\left(2Y_t^{2,i}\,(D_t^i - \hat{X}_t^i) - \left(\hat{P}_t - \hat{Y}_t^{1,i}\right)\right)\right). \end{aligned} \tag{64}$$

By the expression of $\hat{q}_t^i$ in Equation (63), we find

$$\hat{b}_t = \sum_{i=1}^{N} \frac{\bar{\gamma}}{\gamma_i}\left(-2\hat{q}_t^i Y_t^{2,i} + 2\hat{q}_t^i Y_t^{2,i}\right) = 0, \tag{65}$$

which proves that $\hat{P}$ is a martingale. Finally, let us consider an optimal trading strategy $\hat{q}^i$. By formula (63), we have

$$\hat{q}^i_t = \hat{q}^i_0 + \int_0^t \hat{b}^i_s \, ds + \text{martingale}, \tag{66}$$

with

$$\hat{b}^i_t = \frac{1}{2\gamma_i}\Big(2Y^{2,i}_t\left(\mu_i - \hat{q}^i_t\right) - 2f^{2,i}_t\left(D^i_t - \hat{X}^i_t\right) - f^{1,i,\hat{P}}_t\Big), \tag{67}$$

where we used the fact that $\hat{P}$ is a martingale. Then, we see that proceeding along the same lines as for $\hat{P}$, we deduce that $\hat{q}^i$ is a martingale. □

We now provide a formula for the solution to the coupled forward–backward system of Equations (55)–(56). To this end, the following formula (68) for the Y-component turns out to be particularly useful, especially in the case without jumps (as it will be shown in the next section). In the general case, formula (68) provides compact expressions for both the equilibrium price and the forward process, see formulae (74) and (75) of Proposition 4.3. In particular, formula (74) for the equilibrium price allows in turn to find a more explicit formula for the optimal trading rates in the case $\mu_i = 0$ for every i (see Corollary 4.4).

Proposition 4.3. *The following formula holds (notice that $1 - \bar{\gamma}\,\theta_t \neq 0$, for every $0 \le t \le T$):*

$$\hat{\mathbf{Y}}^1_t = \frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}\boldsymbol{a}_t\,\mathbf{1}_N^{\intercal}\,\mathbf{J}\,(2\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t) + 2\,\tilde{\boldsymbol{a}}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t, \tag{68}$$

for all $0 \le t \le T$, where $\bar{\gamma}$ is as in Equation (57), $\theta_t = a^1_t/\gamma_1 + \cdots + a^N_t/\gamma_N$ with a^i_t as in Equation (70), $\mathbf{1}_N^{\intercal}$ denotes the transpose of the column vector $\mathbf{1}_N$ with all entries equal to one, while $\hat{\mathbf{Y}}^1_t$, $\boldsymbol{\Delta}_t$, $\boldsymbol{a}_t$, $\tilde{\boldsymbol{a}}_t$, $\boldsymbol{b}_t$ are column vectors of dimension N given by

$$\hat{\mathbf{Y}}^1_t = \begin{pmatrix}\hat{Y}^{1,1}_t\\ \vdots \\ \hat{Y}^{1,N}_t\end{pmatrix}, \quad \boldsymbol{\Delta}_t = \begin{pmatrix}Y^{2,1}_t\left(D^1_t - \hat{X}^1_t\right)\\ \vdots \\ Y^{2,N}_t\left(D^N_t - \hat{X}^N_t\right)\end{pmatrix}, \quad \boldsymbol{a}_t = \begin{pmatrix}a^1_t\\ \vdots \\ a^N_t\end{pmatrix}, \quad \tilde{\boldsymbol{a}}_t = \begin{pmatrix}\mu_1\gamma_1 a^1_t\\ \vdots \\ \mu_2\gamma_2 a^N_t\end{pmatrix}, \quad \boldsymbol{b}_t = \begin{pmatrix}b^1_t\\ \vdots \\ b^N_t\end{pmatrix}, \tag{69}$$

with

$$a^i_t = \frac{1}{\gamma_i}(T-t)\,Y^{2,i}_t, \tag{70}$$

$$b^i_t = \frac{1}{\gamma_i \Gamma^i_t}\int_t^T \left(\int_t^s \mathbb{E}\left[\Gamma^i_r \kappa^i_r \middle| \mathcal{F}_t\right] dr\right) ds = \frac{1}{\gamma_i \Gamma^i_t}\mathbb{E}\left[\int_t^T \Gamma^i_r \kappa^i_r\,(T-r)\,dr \middle| \mathcal{F}_t\right], \tag{71}$$

$$\kappa_t^i = \sum_{e\in E} U_t^{2,i}(e)\left(2(D_t^i - \hat{X}_t^i)U_t^{2,i}(e) + \sum_{j=1}^N \frac{\gamma_i}{\gamma_j}\hat{U}_t^{1,j,i}(e)\right)\lambda_i(\beta_{t-}^i, e)\,1_{\{\beta_{t-}^i\neq e\}}. \tag{72}$$

Moreover, the $N\times N$ matrices $\mathbf{A}_t$ and $\mathbf{J}$ are defined as

$$\mathbf{A}_t = \begin{pmatrix}\boldsymbol{a}_t & \boldsymbol{a}_t & \cdots & \boldsymbol{a}_t\end{pmatrix} = \begin{pmatrix} a_t^1 & a_t^1 & a_t^1 & \cdots & a_t^1\\ a_t^2 & a_t^2 & a_t^2 & \cdots & a_t^2\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_t^N & a_t^N & a_t^N & \cdots & a_t^N\end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix} \frac{1}{\gamma_1} & 0 & 0 & \cdots & 0\\ 0 & \frac{1}{\gamma_2} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{\gamma_N}\end{pmatrix}. \tag{73}$$

In addition, the equilibrium price is given by

$$\hat{P}_t = \frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}\mathbf{1}_N^\intercal\,\mathbf{J}\,(2\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t), \qquad \textit{for all } 0\le t\le T. \tag{74}$$

Finally, Equation (55) can be rewritten as follows:

$$d\hat{\boldsymbol{X}}_t = \frac{1}{2}\mathbf{J}\left(\mathbf{I} - \frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}(\mathbf{1}_{N\times N} - \mathbf{A}_t)\,\mathbf{J}\right)(2\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t)\,dt, \qquad 0\le t\le T, \tag{75}$$

with $\hat{\boldsymbol{X}}_0 = \boldsymbol{x}_0$, where $\mathbf{1}_{N\times N}$ denotes the $N\times N$ matrix with all entries equal to 1 and

$$\hat{\boldsymbol{X}}_t = \begin{pmatrix}\hat{X}_t^1\\ \vdots\\ \hat{X}_t^N\end{pmatrix}, \qquad \boldsymbol{x}_0 = \begin{pmatrix}x_0^1\\ \vdots\\ x_0^N\end{pmatrix}. \tag{76}$$

Proof. We split the proof into three steps. □

Proof of formula (68). We begin recalling from Equation (35) that $\hat{Y}^{1,i}$ is given by the following formula:

$$\begin{aligned}\hat{Y}_t^{1,i} &= \frac{1}{\Gamma_t^i}\mathbb{E}\left[\int_t^T \Gamma_s^i Y_s^{2,i}\left(2\mu_i + \frac{1}{\gamma_i}\hat{P}_s\right)ds\,\middle|\,\mathcal{F}_t\right]\\ &= \frac{1}{\Gamma_t^i}\int_t^T\left(2\mu_i\mathbb{E}\left[\Gamma_s^i Y_s^{2,i}\middle|\mathcal{F}_t\right] + \frac{1}{\gamma_i}\mathbb{E}\left[\Gamma_s^i Y_s^{2,i}\hat{P}_s\middle|\mathcal{F}_t\right]\right)ds,\end{aligned} \tag{77}$$

for all $0\le t\le T$, where $\Gamma_t^i = e^{-\frac{1}{\gamma_i}\int_0^t Y_s^{2,i}ds}$. Now, an application of Itô's formula yields that the process $\Gamma^i Y^{2,i}$ is a martingale and, in particular, it holds that

$$\Gamma_s^i Y_s^{2,i} = \Gamma_t^i Y_t^{2,i} + \int_{(t,s]\times E}\Gamma_r^i U_r^{2,i}(e)\,(\pi^i - \nu^i)(dr, de), \qquad 0\le t\le s\le T. \tag{78}$$

As a consequence, recalling that the dynamics of $\hat{P}$ is given by Equation (59), we see that

$$\mathbb{E}\left[\Gamma_s^i Y_s^{2,i}\hat{P}_s\middle|\mathcal{F}_t\right] = \Gamma_t^i Y_t^{2,i}\hat{P}_t + \mathbb{E}\left[\int_{(t,s]\times E}\Gamma_r^i U_r^{2,i}(e)\left(\frac{\bar{\gamma}}{\gamma_i}2(D_r^i - \hat{X}_r^i)U_r^{2,i}(e) + \sum_{j=1}^N \frac{\bar{\gamma}}{\gamma_j}\hat{U}_r^{1,j,i}(e)\right)\nu^i(dr,de)\middle|\mathcal{F}_t\right]$$
$$= \Gamma_t^i Y_t^{2,i}\hat{P}_t + \frac{\bar{\gamma}}{\gamma_i}\mathbb{E}\left[\int_t^s \Gamma_r^i \kappa_r^i\, dr\middle|\mathcal{F}_t\right], \tag{79}$$

where κ^i is given by Equation (72). Hence, by the martingale property of $\Gamma^i Y^{2,i}$ and Equation (79), we can rewrite formula (77) as follows:

$$\hat{Y}_t^{1,i} = \frac{1}{\Gamma_t^i}\int_t^T\left(2\mu_i\Gamma_t^i Y_t^{2,i} + \frac{1}{\gamma_i}\Gamma_t^i Y_t^{2,i}\hat{P}_t + \frac{\bar{\gamma}}{\gamma_i^2}\int_t^s \mathbb{E}\left[\Gamma_r^i\kappa_r^i\middle|\mathcal{F}_t\right]dr\right)ds = a_t^i\,\hat{P}_t + 2\mu_i\gamma_i\,a_t^i + \frac{\bar{\gamma}}{\gamma_i}\,b_t^i, \tag{80}$$

where a^i and b^i are given by Equations (70) and (71), respectively. Using formula (57) for $\hat{P}$, we find

$$\hat{Y}_t^{1,i} = a_t^i\sum_{j=1}^N\frac{\bar{\gamma}}{\gamma_j}\left(\hat{Y}_t^{1,j} + 2Y_t^{2,j}\left(D_t^j - \hat{X}_t^j\right)\right) + 2\mu_i\gamma_i\,a_t^i + \frac{\bar{\gamma}}{\gamma_i}\,b_t^i. \tag{81}$$

The latter can be written in matrix form as follows:

$$\hat{\mathbf{Y}}_t^1 = \bar{\gamma}\,\mathbf{A}_t\,\mathbf{J}\,\hat{\mathbf{Y}}_t^1 + 2\,\bar{\gamma}\,\mathbf{A}_t\,\mathbf{J}\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t, \tag{82}$$

where $\boldsymbol{\Delta}_t$ is the column vector of dimension N given in Equation (69). In order to solve for $\hat{\mathbf{Y}}^1$, we rewrite Equation (82) as follows:

$$(\mathbf{I} - \bar{\gamma}\,\mathbf{A}_t\,\mathbf{J})\hat{\mathbf{Y}}_t^1 = 2\,\bar{\gamma}\,\mathbf{A}_t\,\mathbf{J}\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t, \tag{83}$$

where $\mathbf{I}$ is the $N\times N$ identity matrix. Hence, we can solve for $\hat{\mathbf{Y}}^1$ if the matrix on the left-hand side of Equation (83) is invertible. We now prove that this holds true and the inverse matrix of $\mathbf{I} - \bar{\gamma}\,\mathbf{A}_t\,\mathbf{J}$ is given by

$$(\mathbf{I} - \bar{\gamma}\,\mathbf{A}_t\,\mathbf{J})^{-1} = \mathbf{I} + \frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}\mathbf{A}_t\,\mathbf{J}, \qquad \text{with } \theta_t = \sum_{i=1}^N\frac{a_t^i}{\gamma_i}. \tag{84}$$

Let us first check that $1 - \bar{\gamma}\,\theta_t \neq 0$, so that Equation (84) is well-defined. To this regard, notice that the ith element a_t^i, which is given by formula (70), can also be written as follows:

$$a_t^i = 1 - \mathbb{E}\left[e^{-\frac{1}{\gamma_i}\int_t^T Y_s^{2,i}ds}\middle|\mathcal{F}_t\right]. \tag{85}$$

Let us prove equality (85). By the definition of Γ^i, we have $\Gamma_T^i = \Gamma_t^i - \frac{1}{\gamma_i}\int_t^T \Gamma_s^i Y_s^{2,i}\,ds$. Taking the conditional expectation with respect to $\mathcal{F}_t$, we find

$$\mathbb{E}\left[e^{-\frac{1}{\gamma_i}\int_t^T Y_s^{2,i}ds}\middle|\mathcal{F}_t\right] = \frac{1}{\Gamma_t^i}\mathbb{E}\left[\Gamma_T^i\middle|\mathcal{F}_t\right] = 1 - \frac{1}{\gamma_i \Gamma_t^i}\int_t^T \mathbb{E}\left[\Gamma_s^i Y_s^{2,i}\middle|\mathcal{F}_t\right]ds. \tag{86}$$

By the martingale property of $\Gamma^i Y^{2,i}$, we see that

$$\mathbb{E}\left[e^{-\frac{1}{\gamma_i}\int_t^T Y_s^{2,i}ds}\middle|\mathcal{F}_t\right] = 1 - a_t^i, \tag{87}$$

from which Equation (85) follows. Now, multiplying the above equality by $1/\gamma_i$ and summing with respect to i, we obtain

$$\frac{1}{\bar\gamma} - \theta_t = \sum_{i=1}^N \mathbb{E}\left[e^{-\frac{1}{\gamma_i}\int_t^T Y_s^{2,i}ds}\middle|\mathcal{F}_t\right], \tag{88}$$

namely $1 - \bar\gamma\theta_t = \bar\gamma\sum_{i=1}^N \mathbb{E}[\exp(-\frac{1}{\gamma_i}\int_t^T Y_s^{2,i}ds)|\mathcal{F}_t]$. Recalling from Proposition 3.2 that $Y^{2,i}$ is nonnegative and belongs to $\mathbf{S}^\infty(0,T)$, we deduce that $1-\bar\gamma\theta_t$ is a strictly positive real number. This shows that Equation (84) is well-defined.

Let us now prove that the matrix on the right-hand side of Equation (84) is the inverse matrix of $\mathbf{I} - \bar\gamma\,\mathbf{A}_t\,\mathbf{J}$. To this end, notice that

$$(\mathbf{A}_t\,\mathbf{J})^2 = \theta_t\,\mathbf{A}_t\,\mathbf{J}, \tag{89}$$

where we recall that $\theta_t = a_t^1/\gamma_1 + \cdots + a_t^N/\gamma_N$, namely θ_t is the trace of the matrix $\mathbf{A}_t\,\mathbf{J}$. Then, by direct calculation, it is easy to see that

$$\left(\mathbf{I} + \frac{\bar\gamma}{1-\bar\gamma\,\theta_t}\mathbf{A}_t\,\mathbf{J}\right)(\mathbf{I} - \bar\gamma\,\mathbf{A}_t\,\mathbf{J}) = \mathbf{I}, \tag{90}$$

which shows the validity of Equation (84). This allows us to solve for $\hat{\boldsymbol{Y}}^1$ in Equation (83), so that we obtain

$$\hat{\boldsymbol{Y}}_t^1 = \left(\mathbf{I} + \frac{\bar\gamma}{1-\bar\gamma\,\theta_t}\mathbf{A}_t\,\mathbf{J}\right)(2\,\bar\gamma\,\mathbf{A}_t\,\mathbf{J}\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t + \bar\gamma\,\mathbf{J}\,\boldsymbol{b}_t). \tag{91}$$

By Equation (89) and the property $\mathbf{A}_t\,\mathbf{J}\,\boldsymbol{v} = (\sum_{i=1}^N v_i/\gamma_i)\boldsymbol{a}_t$, valid for every $\boldsymbol{v}\in\mathbb{R}^N$, we find

$$\hat{\boldsymbol{Y}}_t^1 = 2\,\bar\gamma\left(1 + \frac{\bar\gamma\,\theta_t}{1-\bar\gamma\,\theta_t}\right)\mathbf{A}_t\,\mathbf{J}\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t + \bar\gamma\,\mathbf{J}\,\boldsymbol{b}_t + \frac{\bar\gamma}{1-\bar\gamma\,\theta_t}\sum_{i=1}^N \frac{1}{\gamma_i}\left(2\mu_i\gamma_i a_t^i + \frac{\bar\gamma}{\gamma_i}b_t^i\right)\boldsymbol{a}_t. \tag{92}$$

Since $1+\frac{\bar{\gamma}\,\theta_t}{1-\bar{\gamma}\,\theta_t}=\frac{1}{1-\bar{\gamma}\,\theta_t}$ and $\mathbf{A}_t\,\mathbf{J}\,\boldsymbol{\Delta}_t=\boldsymbol{a}_t\sum_{i=1}^N Y_t^{2,i}(D_t^i-\hat{X}_t^i)/\gamma_i$, this yields

$$\hat{\boldsymbol{Y}}_t^1=\frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}\boldsymbol{a}_t\sum_{i=1}^N\frac{1}{\gamma_i}\left(2\,Y_t^{2,i}(D_t^i-\hat{X}_t^i)+2\,\mu_i\gamma_i a_t^i+\frac{\bar{\gamma}}{\gamma_i}b_t^i\right)+2\,\tilde{\boldsymbol{a}}_t+\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t. \tag{93}$$

Finally, noting that $\sum_{i=1}^N\frac{1}{\gamma_i}(2Y_t^{2,i}(D_t^i-\hat{X}_t^i)+2\mu_i\gamma_i a_t^i+\bar{\gamma}b_t^i/\gamma_i)=\mathbf{1}_N^\top\,\mathbf{J}\,(2\,\boldsymbol{\Delta}_t+2\,\tilde{\boldsymbol{a}}_t+\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t)$, we conclude that formula (68) holds. □

Proof of formula (74). Rewriting Equation (57) in matrix form, we obtain

$$\hat{P}_t=\bar{\gamma}\,\mathbf{1}_N^\top\,\mathbf{J}\left(2\boldsymbol{\Delta}_t+\hat{\boldsymbol{Y}}_t^1\right),\qquad \text{for all } 0\le t\le T. \tag{94}$$

Plugging formula (68) into the above equality, we find

$$\hat{P}_t=\bar{\gamma}\,\mathbf{1}_N^\top\,\mathbf{J}\left(2\,\boldsymbol{\Delta}_t+\frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}\boldsymbol{a}_t\,\mathbf{1}_N^\top\,\mathbf{J}\,(2\,\boldsymbol{\Delta}_t+2\,\tilde{\boldsymbol{a}}_t+\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t)+2\,\tilde{\boldsymbol{a}}_t+\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t\right). \tag{95}$$

Notice that $\mathbf{1}_N^\top\,\mathbf{J}\,\boldsymbol{a}_t=\theta_t$, so that

$$\begin{aligned}\hat{P}_t&=\bar{\gamma}\,\frac{\bar{\gamma}\,\theta_t}{1-\bar{\gamma}\,\theta_t}\mathbf{1}_N^\top\,\mathbf{J}\,(2\,\boldsymbol{\Delta}_t+2\,\tilde{\boldsymbol{a}}_t+\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t)+\bar{\gamma}\,\mathbf{1}_N^\top\,\mathbf{J}\,(2\,\boldsymbol{\Delta}_t+2\,\tilde{\boldsymbol{a}}_t+\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t)\\&=\frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}\mathbf{1}_N^\top\,\mathbf{J}\,(2\,\boldsymbol{\Delta}_t+2\,\tilde{\boldsymbol{a}}_t+\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t),\end{aligned} \tag{96}$$

which proves formula (74). □

Proof of formula (75). We recall from Theorem 4.1 that $\hat{X}^i$ solves the following ordinary differential equation with stochastic coefficients:

$$d\hat{X}_t^i=\frac{1}{2\gamma_i}\left(2Y_t^{2,i}\left(D_t^i-\hat{X}_t^i\right)+\hat{Y}_t^{1,i}-\hat{P}_t\right)dt,\qquad 0\le t\le T. \tag{97}$$

The latter can be written in matrix as follows:

$$d\hat{\boldsymbol{X}}_t=\frac{1}{2}\,\mathbf{J}\left(2\,\boldsymbol{\Delta}_t+\hat{\boldsymbol{Y}}_t^1-\mathbf{1}_N\,\hat{P}_t\right)dt. \tag{98}$$

Using the expressions of $\hat{\boldsymbol{Y}}_t^1$ and $\hat{P}_t$ in Equations (68) and (74), respectively, we find

$$d\hat{\boldsymbol{X}}_t=\frac{1}{2}\,\mathbf{J}\left(2\,\boldsymbol{\Delta}_t+2\,\tilde{\boldsymbol{a}}_t+\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t-\frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}\left(\mathbf{1}_N\,\mathbf{1}_N^\top-\boldsymbol{a}_t\,\mathbf{1}_N^\top\right)\mathbf{J}\,(2\,\boldsymbol{\Delta}_t+2\,\tilde{\boldsymbol{a}}_t+\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t)\right). \tag{99}$$

Noting that $\boldsymbol{a}_t\,\mathbf{1}_N^\top = \mathbf{A}_t$ and $\mathbf{1}_N\,\mathbf{1}_N^\top = \mathbf{1}_{N\times N}$ (where we recall that $\mathbf{1}_{N\times N}$ denotes the $N \times N$ matrix with all entries equal to 1), we obtain

$$\begin{aligned} d\hat{\boldsymbol{X}}_t &= \frac{1}{2}\,\mathbf{J}\left(2\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t - \frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}(\mathbf{1}_{N\times N} - \mathbf{A}_t)\,\mathbf{J}\,(2\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t)\right) \\ &= \frac{1}{2}\,\mathbf{J}\left(\mathbf{I} - \frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}(\mathbf{1}_{N\times N} - \mathbf{A}_t)\,\mathbf{J}\right)(2\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t), \end{aligned} \tag{100}$$

which corresponds to formula (75). □

Using Proposition 4.3, it is possible to provide more precise results for the optimal trading rates and the equilibrium price, when agents have no systematic bias on their forecasts ($\mu_i = 0$).

Corollary 4.4. *Suppose that $\mu_i = 0$ for every i.*

(i) The optimal trading rate of agent i is given by (we denote $\hat{q}^i := \hat{q}^{i,\hat{P}}$)

$$\hat{q}_t^i = \frac{1-a_t^i}{2\gamma_i}\left(\frac{2\,Y_t^{2,i}\,(D_t^i - \hat{X}_t^i) + \frac{\bar{\gamma}}{\gamma_i} b_t^i}{1-a_t^i} - \hat{P}_t\right), \tag{101}$$

where $1 - a_t^i = 1 - \frac{1}{\gamma_i}(T-t)Y_t^{2,i}$ is strictly positive, for every i, as it follows from equality (87), moreover b_t^i is given by Equation (71).

(ii) The equilibrium price is given by

$$\hat{P}_t = \sum_{i=1}^N \pi_t^i\,\frac{2\,Y_t^{2,i}\,(D_t^i - \hat{X}_t^i) + \frac{\bar{\gamma}}{\gamma_i} b_t^i}{1-a_t^i}, \quad \textit{where } \pi_t^i := \frac{\frac{1}{\gamma_i}(1-a_t^i)}{\sum_{j=1}^N \frac{1}{\gamma_j}(1-a_t^j)}. \tag{102}$$

Proof.

(i) When $\mu_i = 0$ for every i, the expression of $\hat{\boldsymbol{Y}}_t^1$ in Equation (68) reads (notice that the vector $\tilde{\boldsymbol{a}}_t$ in Equation (68) is equal to zero)

$$\hat{\boldsymbol{Y}}_t^1 = \frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}\boldsymbol{a}_t\,\mathbf{1}_N^\top\,\mathbf{J}\,(2\,\boldsymbol{\Delta}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t) + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t. \tag{103}$$

Similarly, the expression of $\hat{P}_t$ in Equation (74) becomes

$$\hat{P}_t = \frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}\mathbf{1}_N^\top\,\mathbf{J}\left(2\,\boldsymbol{\Delta}_t + \bar{\gamma}\,\mathbf{J}^2\,\boldsymbol{b}_t\right). \tag{104}$$

Then, it holds that

$$\hat{\boldsymbol{Y}}_t^1 = \boldsymbol{a}_t\,\hat{P}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t. \tag{105}$$

Thus, by Equations (109), (47), and (105), the optimal trading rate at equilibrium can be written as (we denote $\hat{q}^i := \hat{q}^{i,\hat{P}}$)

$$\hat{q}_t^i = \frac{1}{2\gamma_i}\left(2\,\Delta_t^i + \hat{Y}_t^{1,i} - \hat{P}_t\right) = \frac{1}{2\gamma_i}\left(2\,\Delta_t^i + \frac{\bar{\gamma}}{\gamma_i} b_t^i - (1 - a_t^i)\,\hat{P}_t\right), \tag{106}$$

which yields equality (101) recalling that $\Delta_t^i = Y_t^{2,i}(D_t^i - \hat{X}_t^i)$.

(ii) From the expression of $\hat{P}_t$ in Equation (74), we obtain (recalling that $\tilde{\boldsymbol{a}}_t = 0$)

$$\begin{aligned}\hat{P}_t &= \frac{\bar{\gamma}}{1 - \bar{\gamma}\,\theta_t}\mathbf{1}_N^\top \mathbf{J}\,(2\,\boldsymbol{\Delta}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t) = \frac{1}{\sum_{j=1}^N \frac{1}{\gamma_j}(1 - a_t^j)}\mathbf{1}_N^\top \mathbf{J}\,(2\,\boldsymbol{\Delta}_t + \bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t)\\
&= \frac{1}{\sum_{j=1}^N \frac{1}{\gamma_j}(1 - a_t^j)} \sum_{i=1}^N \frac{1}{\gamma_i}\left(2\,Y_t^{2,i}\,(D_t^i - \hat{X}_t^i) + \frac{\bar{\gamma}}{\gamma_i} b_t^i\right)\\
&= \sum_{i=1}^N \pi_t^i\,\frac{2\,Y_t^{2,i}\,(D_t^i - \hat{X}_t^i) + \frac{\bar{\gamma}}{\gamma_i} b_t^i}{1 - a_t^i},\end{aligned} \tag{107}$$

with π_t^i as in (102).

□

5 | THE CASE WITHOUT JUMPS

In the present section, we focus on the case where there are no jumps, so that the terminal condition of $Y^{2,i}$ is deterministic and given by $\frac{1}{2}\frac{\eta_i\,e_i}{\eta_i + e_i}$, for some fixed $e_i \in E$. We also assume that for all $i = 1, \dots, N$, $\mu_i = 0$, meaning that market players have unbiased forecasts of their terminal demand. In such a framework, $Y^{2,i}$ solves the following backward equation:

$$dY_t^{2,i} = \frac{1}{\gamma_i}|Y_t^{2,i}|^2\,dt, \qquad Y_T^{2,i} = \frac{1}{2}\,\frac{\eta_i\,e_i}{\eta_i + e_i} =: \frac{1}{2}\epsilon_i. \tag{108}$$

Hence, $Y^{2,i}$ is given by

$$Y_t^{2,i} = \frac{Y_T^{2,i}}{1 + \frac{1}{\gamma_i}Y_T^{2,i}(T - t)} =: \frac{1}{2}\,\frac{\epsilon_i}{1 + \frac{1}{2}\phi_i(T - t)}, \qquad 0 \le t \le T, \quad \text{with } \phi_i := \frac{\epsilon_i}{\gamma_i}. \tag{109}$$

5.1 | Main result

In the present framework, we can give more precise formulae, compared to Corollary 4.4, for the optimal trading rates and the equilibrium price when $\mu_i = 0$; moreover, we can provide a formula for the volatility of the equilibrium price. For sake of notations, we write $\tilde{W}_t^i := \rho_i W_t^0 + \sqrt{1-\rho_i^2} W_t^i$.

Corollary 5.1. *Suppose that $\mu_i = 0$ for every i.*

(i) The equilibrium price $\hat{P}_t$ is given by (recall that $c_i(\xi_i) = e_i \xi_i^2/2$, so in particular $c_i'(\hat{\xi}_t^i) = e_i \hat{\xi}_t^i$)

$$\hat{P}_t = \sum_{i=1}^N F_i(t) c_i'(\hat{\xi}_t^i), \quad \text{with } F_i(t) := \frac{G(t)}{f_i(t)}, \quad G(t) := \left(\sum_{i=1}^N 1/f_i(t)\right)^{-1},$$
$$f_i(t) := \gamma_i + \frac{1}{2}\epsilon_i(T-t), \quad \text{and} \quad \hat{\xi}_t^i := \frac{\eta_i}{e_i + \eta_i}(D_t^i - \hat{X}_t^i). \tag{110}$$

and the optimal trading rate $\hat{q}_t^i$ is given by

$$\hat{q}_t^i = \frac{1}{2}\frac{c_i'(\hat{\xi}_t^i) - \hat{P}_t}{f_i(t)}. \tag{111}$$

(ii) The dynamics of the equilibrium price writes

$$d\hat{P}_t = \sum_{i=1}^N F_i(t)\epsilon_i \sigma_i(t) d\tilde{W}_t^i. \tag{112}$$

(iii) In particular, the volatility $\zeta = (\zeta_t)_{t\in[0,T]}$ of the equilibrium price process is deterministic, and satisfies

$$\zeta_t^2 = \sum_{i=1}^N \left(1-\rho_i^2\right)\left(\epsilon_i F_i(t)\sigma_i(t)\right)^2 + \left(\sum_{i=1}^N \rho_i \epsilon_i F_i(t)\sigma_i(t)\right)^2. \tag{113}$$

(iv) Moreover, if $\sigma_i = \sigma$, $\gamma_i = \gamma$, $\rho_i = \rho$, for every i, then

$$\zeta_t^2 = \left(1-\rho^2\right)\sigma^2 G^2(t) \sum_{i=1}^N \left(\frac{2Y_T^{2,i}}{\gamma + Y_T^{2,i}(T-t)}\right)^2 + \rho^2 \sigma^2 G^2(t)\left(\sum_{i=1}^N \frac{2Y_T^{2,i}}{\gamma + Y_T^{2,i}(T-t)}\right)^2. \tag{114}$$

If in addition, all players have the same cost functions, namely $e_i = e$, for every i, then the volatility of the equilibrium price is constant equal to

$$\zeta_t^2 = \left[4\left(1-\rho^2\right)\frac{1}{N}|Y_T^2|^2 + 4\rho^2 |Y_T^2|^2\right]\sigma^2, \tag{115}$$

where $Y_T^2 = \frac{1}{2}\frac{\eta e}{\eta + e}$, for every i.

Proof. Item (i). By formula (101), we have in the case without jumps ($b^i = 0$)

$$\hat{q}_t^i = \frac{1}{2\gamma_i}\left(2\,Y_t^{2,i}\,(D_t^i - \hat{X}_t^i) - (1-a_t^i)\hat{P}_t\right) = \frac{1}{2\gamma_i}\left(\frac{c_i'(\hat{\xi}_t^i)}{1+\frac{1}{2}\phi_i(T-t)} - (1-a_t^i)\hat{P}_t\right), \tag{116}$$

where we used that

$$2Y_t^{2,i}(D_t^i - \hat{X}_t^i) = \frac{\epsilon_i(D_t^i - \hat{X}_t^i)}{1+\frac{1}{2}\phi_i(T-t)} = \frac{c_i'(\hat{\xi}_t^i)}{1+\frac{1}{2}\phi_i(T-t)}. \tag{117}$$

Noting $1 - a_t^i = 1/(1+\frac{1}{2}\phi_i(T-t))$, we get

$$\hat{q}_t^i = \frac{1}{2\gamma_i}\left(\frac{c_i'(\hat{\xi}_t^i) - \hat{P}_t}{1+\frac{1}{2}\phi_i(T-t)}\right). \tag{118}$$

Thus, summing up all the trading rates, we find

$$\hat{P}_t = \sum_{i=1}^N \left(\sum_{k=1}^N \left(\gamma_k\left(1+\frac{1}{2}\phi_k(T-t)\right)\right)^{-1}\right)^{-1} \frac{1}{\gamma_i(1+\frac{1}{2}\phi_i(T-t))} c_i'(\hat{\xi}_t^i), \tag{119}$$

with

$$F_i(t) \;:=\; \frac{G(t)}{\gamma_i(1+\frac{1}{2}\phi_i(T-t))}, \qquad G(t) \;:=\; \left(\sum_{k=1}^N \left(\gamma_k\left(1+\frac{1}{2}\phi_k(T-t)\right)\right)^{-1}\right)^{-1}. \tag{120}$$

Items (ii) and (iii). Using formula (110) and that the noise terms are only due to $D_t^1, \dots, D_t^N$, we find

$$d\hat{P}_t = \sum_{i=1}^N F_i(t)\epsilon_i\sigma_i(t)d\tilde{W}_t^i, \tag{121}$$

which corresponds to formula (112). From such a formula, we immediately get Equation (113).

Item (iv). Formula (114) is a direct consequence of Equation (113). Finally, if all cost functions are identical, namely $Y_T^{2,i} = Y_T^2 = \frac{1}{2}\frac{\eta e}{\eta+e}$, for every i, then by Equation (114), we immediately get Equation (115), which proves that the volatility is decreasing in t. □

Remark 5.2.

(i) The item (i) in Corollary 5.1 shows that the equilibrium price is a convex combination of the forecasted marginal cost to produce the quantity $D_t^i - \hat{X}_t^i$. The quantity $D_t^i - \hat{X}_t^i$ is the best estimator an agent can have on the quantity she will have to produce at time T. The weights of the convex combination are deterministic functions of time. Further, for any agent, the optimal trading strategy is simply to compare its forecasted marginal cost $c_i'(\hat{\xi}_t^i)$ to the equilibrium price $\hat{P}_t$. If the forecasted marginal cost is higher (resp. lower) than $\hat{P}_t$, she buys (resp. sell).

(ii) Using Equation (110), we can rewrite $\hat{P}$ as

$$\hat{P}_t = S_t - \sum_{i=1}^{N} F_i(t)\epsilon_i \hat{X}_t^i, \quad S_t := \sum_{i=1}^{N} F_i(t)\epsilon_i (D_t^i - x_0^i). \tag{122}$$

The process S_t is an uncontrolled process, which represents a *fundamental price*, and the factors $\epsilon_i F_i(t)$ reads as the *permanent market impact* of each agent. This decomposition is consistent with the famous Almgren and Chriss model of intraday trading (Almgren and Chriss (2001)). Note that if agents are identical, $\hat{P}_t$ reduces to its fundamental component because of the market clearing condition.

(iii) Under the assumptions for the validity of formula (115), we see that ζ_t^2 converges to zero as N goes to infinity when $\rho = 0$, while the limit is strictly positive when $\rho \neq 0$. This result translates in the following remark: in a market with no production shocks, prices move because agents face a common economic factor.

(iv) Rewriting Equation (110), it holds that

$$F_i(t) = \left[\sum_{k=1}^{N} \frac{\gamma_i + \frac{1}{2}\epsilon_i (T-t)}{\gamma_k + \frac{1}{2}\epsilon_k (T-t)} \right]^{-1}. \tag{123}$$

Hence, when there are no market frictions, that is, all the γ_i are zero, all the functions F_i are constant, and thus, the equilibrium price still exists.

5.2 | Samuelson's effect

We study here the monotonicity in time of the volatility function ζ_t given in the formula (113). We start by a remark. If the market players are homogeneous (same cost e_i, same penalization of imbalances η_i, same market access γ_i), all functions F_i are constant equal to $1/N$ and, if we further assume the same dependence on common shocks $\rho_i = \rho$, the volatility reduces to

$$\zeta_t^2 = \frac{\epsilon^2}{N^2} \left[(1-\rho^2) \sum_{i=1}^{N} \sigma_i^2(t) + \rho^2 \left(\sum_{i=1}^{N} \sigma_i(t) \right)^2 \right]. \tag{124}$$

In this homogeneous case, the monotonicity of the price volatility is fully determined by the monotonicity of the volatility of the demand forecasts. And as a consequence, in the case of a decrease of the demand uncertainty closer to maturity, which is the case of intraday electricity markets, the Samuelson's effect would not hold, contrary to the prediction of the state-variable hypothesis and Anderson and Danthine (1983). For the Samuelson's effect to be observed, some heterogeneity is somehow necessary. Moreover, heterogeneity can induce rich behaviors of the price volatility. Figure 1 illustrates this fact, in the case where two populations I and II of identical players are interacting, and when the volatility demand is in the form $\sigma_i^2(t) = \vartheta_I + \psi_I(T-t)$, i in group I, and $\sigma_i^2(t) = \vartheta_{II} + \psi_{II}(T-t)$, i in group II, for some nonnegative constants $\vartheta_I, \psi_I, \vartheta_{II}, \psi_{II}$. The param-

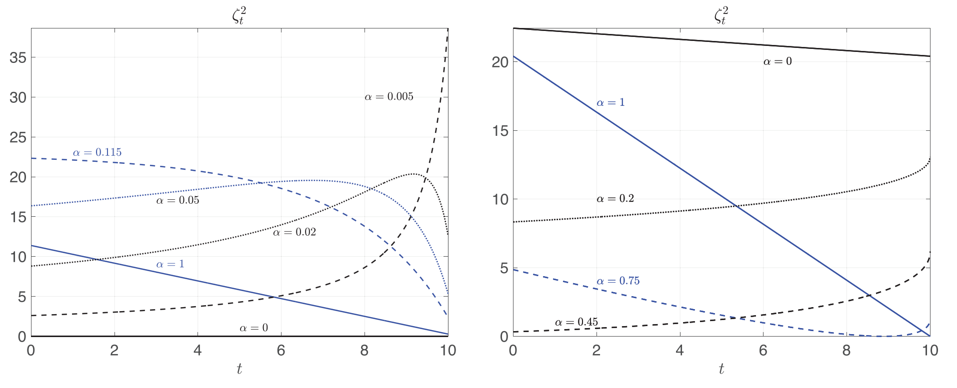

FIGURE 1 Price volatility function ζ_t^2 as a function of the proportion of type 1 agents α with parameters given in Table 1 [Color figure can be viewed at wileyonlinelibrary.com]

TABLE 1 Parameters value used for Figure 1

	Type I	Type II
Figure 1 (left)	$\psi_I = 20, \vartheta_I = 5, \rho = 0,$	$\psi_{II} = 0, \vartheta_{II} = 0, \rho = 0,$
	$e = 10, \gamma = 0.1\ \eta = 5.$	$e = 10, \gamma = 100, \eta = 5.$
Figure 1 (right)	$\psi_I = 1, \vartheta_I = 0, \rho = 1,$	$\psi_{II} = 0.1, \vartheta_{II} = 10, \rho = -1,$
	$e = 2, \gamma = 1\ \eta = 5.$	$e = 1, \gamma = 1\ \eta = 5.$

eters of each population are given in Table 1. We observe that a small fraction of population can make the monotonicity of the volatility go from nonincreasing to nondecreasing (Figure 1 (left) for $\alpha = 5\,10^{-3}$). Further, we also observe on Figure 1 (right) that it is also possible to produce the Samuelson's effect by mixing two populations who individually would lead a nonincreasing price volatility function.

Thus, because it is utterly difficult to isolate a necessary and sufficient condition on the whole set of parameters for the Samuelson's effect to hold, we focus on parameters, which would yield a positive result only by themselves, all other parameters being equal. From the analysis above, it is already clear that different correlations with the common noise cannot explain by themselves the Samuelson's effect. But, we show below that heterogeneity on market quality access only (γ_i) or only on production costs resumed by the parameter ϵ_i leads to a positive result. For simplicity, we assume in both cases that the demand volatility is constant $\sigma_i(t) = \sigma$.

Market access quality

We assume that all the market players have the same cost e_i, and penalization imbalance, hence the same $\epsilon_i = \epsilon, i = 1, \dots, N$ (recall Equation (108)). Firms only differ by their market access quality parameter γ_i, sorted in a nondecreasing order ($\gamma_i < \gamma_j$, for $i < j$). The (square) of the volatility is given by

$$\zeta_t^2 = (1-\rho^2)\epsilon^2\sigma^2 \sum_{i=1}^{N} F_i^2(t) + \rho^2\epsilon^2\sigma^2. \tag{125}$$

Recalling the expression of G and F_i in Equation (110), their derivatives are given by

$$G' = -\frac{\epsilon}{2}\frac{\sum_k 1/f_k^2}{\left(\sum_k 1/f_k\right)^2} = -\frac{\epsilon}{2}G^2\sum_{k=1}^{N}\frac{1}{f_k^2}, \qquad F_i' = \frac{G'}{f_i} - \frac{Gf_i'}{f_i^2} = \frac{\epsilon}{2}G^2\sum_{k=1}^{N}\frac{\gamma_k-\gamma_i}{f_k^2 f_i^2}. \tag{126}$$

It follows that the time derivative of ζ^2 is equal to

$$(\zeta^2)' = (1-\rho^2)\sigma^2\epsilon^2\sum_{i=1}^{N}2F_iF_i' = (1-\rho^2)\sigma^2\epsilon^3G^2\sum_{i=1}^{N}\frac{1}{f_i}\sum_{k=1}^{N}\frac{\gamma_k-\gamma_i}{f_k^2f_i^2}. \tag{127}$$

Now, by rearranging the sum of terms as

$$\begin{aligned}\sum_{i=1}^{N}\frac{1}{f_i}\sum_{k=1}^{N}\frac{\gamma_k-\gamma_i}{f_k^2f_i^2} &= \frac{1}{f_1}\left(\frac{\gamma_2-\gamma_1}{f_1^2f_2^2}+\frac{\gamma_3-\gamma_1}{f_1^2f_3^2}+\cdots\right)+\frac{1}{f_2}\left(\frac{\gamma_1-\gamma_2}{f_2^2f_1^2}+\frac{\gamma_3-\gamma_2}{f_2^2f_3^2}+\cdots\right)\\ &\quad+\frac{1}{f_3}\left(\frac{\gamma_1-\gamma_3}{f_3^2f_1^2}+\frac{\gamma_2-\gamma_3}{f_3^2f_2^2}+\cdots\right)+\cdots\\ &= \frac{\gamma_2-\gamma_1}{f_1^2f_2^2}\left[\frac{1}{f_1}-\frac{1}{f_2}\right]+\frac{\gamma_3-\gamma_1}{f_1^2f_3^2}\left[\frac{1}{f_1}-\frac{1}{f_3}\right]+\frac{\gamma_3-\gamma_2}{f_2^2f_3^2}\left[\frac{1}{f_2}-\frac{1}{f_3}\right]+\cdots,\end{aligned} \tag{128}$$

and noting that

$$\frac{1}{f_i}-\frac{1}{f_k} = \frac{\gamma_k-\gamma_i}{f_if_k} > 0, \quad i<k, \tag{129}$$

we deduce that

$$\sum_{i=1}^{N}\frac{1}{f_i}\sum_{k=1}^{N}\frac{\gamma_k-\gamma_i}{f_k^2f_i^2} > 0, \tag{130}$$

which implies that $(\zeta_t^2)' > 0$, that is, the Samuelson's effect holds true.

Production costs

We assume that all the market players have the same market impact $\gamma_i = \gamma$, $i = 1,\dots,N$, but differ through their cost e_i or imbalance η_i. We set $\epsilon_i < \epsilon_j$ for $i < j$. The (square) volatility is then given by

$$\zeta_t^2 = (1-\rho^2)\sigma^2\sum_{i=1}^{N}\epsilon_i^2F_i^2 + \rho^2\sigma^2\left(\sum_{i=1}^{N}\epsilon_iF_i\right)^2. \tag{131}$$

We now have

$$G' = -\frac{1}{2}G^2\sum_{k=1}^{N}\frac{\epsilon_k}{f_k^2}, \qquad F_i' = \frac{G'}{f_i}-\frac{Gf_i'}{f_i^2} = \frac{\gamma}{2}G^2\sum_{k=1}^{N}\frac{\epsilon_i-\epsilon_k}{f_k^2f_i^2}. \tag{132}$$

Denote by $v(t) := \sum_{i=1}^N \epsilon_i^2 F_i^2$ and $w(t) := (\sum_{i=1}^N \epsilon_i F_i)^2$. A straightforward calculation yields

$$v' = 2\sum_{i=1}^N \epsilon_i^2 F_i F_i' = \gamma G^3 \sum_{i=1}^N \frac{\epsilon_i^2}{f_i} \sum_{k=1}^N \frac{\epsilon_i - \epsilon_k}{f_k^2 f_i^2}, \tag{133}$$

and by rearranging again the sum of terms as

$$\begin{aligned}\sum_{i=1}^N \frac{\epsilon_i^2}{f_i} \sum_{k=1}^N \frac{\epsilon_i - \epsilon_k}{f_k^2 f_i^2} &= \frac{\epsilon_1^2}{f_1}\left(\frac{\epsilon_1-\epsilon_2}{f_1^2 f_2^2} + \frac{\epsilon_1-\epsilon_3}{f_1^2 f_3^2} + \cdots\right) + \frac{\epsilon_2^2}{f_2}\left(\frac{\epsilon_2-\epsilon_1}{f_2^2 f_1^2} + \frac{\epsilon_2-\epsilon_3}{f_2^2 f_3^2} + \cdots\right)\\ &\quad + \frac{\epsilon_3^2}{f_3}\left(\frac{\epsilon_3-\epsilon_1}{f_3^2 f_1^2} + \frac{\epsilon_3-\epsilon_2}{f_3^2 f_2^2} + \cdots\right) + \cdots\\ &= \frac{\epsilon_2-\epsilon_1}{f_2^2 f_1^2}\left[\frac{\epsilon_2^2}{f_2} - \frac{\epsilon_1^2}{f_1}\right] + \frac{\epsilon_3-\epsilon_1}{f_3^2 f_1^2}\left[\frac{\epsilon_3^2}{f_3} - \frac{\epsilon_1^2}{f_1}\right] + \frac{\epsilon_3-\epsilon_2}{f_3^2 f_2^2}\left[\frac{\epsilon_3^2}{f_3} - \frac{\epsilon_2^2}{f_2}\right] + \cdots,\end{aligned} \tag{134}$$

and noting that

$$\frac{\epsilon_i^2}{f_i} - \frac{\epsilon_k^2}{f_k} = \frac{\gamma(\epsilon_i^2 - \epsilon_k^2) + \frac{1}{2}\epsilon_i\epsilon_k(\epsilon_i - \epsilon_k)(T-t)}{f_i f_k} > 0, \quad k < i, \tag{135}$$

we deduce that $v' > 0$. Similarly, we have

$$w' = 2\left(\sum_{i=1}^N \epsilon_i F_i\right)\left(\sum_{i=1}^N \epsilon_i F_i'\right), \qquad \sum_{i=1}^N \epsilon_i F_i' = \frac{1}{2}\gamma G^2 \sum_{i=1}^N \epsilon_i \sum_{k=1}^N \frac{\epsilon_i - \epsilon_k}{f_k^2 f_i^2}, \tag{136}$$

and, observing again that

$$\sum_{i=1}^N \epsilon_i \sum_{k=1}^N \frac{\epsilon_i - \epsilon_k}{f_k^2 f_i^2} = \frac{\epsilon_2-\epsilon_1}{f_2^2 f_1^2}[\epsilon_2 - \epsilon_1] + \frac{\epsilon_3-\epsilon_1}{f_3^2 f_1^2}[\epsilon_3 - \epsilon_1] + \frac{\epsilon_3-\epsilon_2}{f_3^2 f_2^2}[\epsilon_3 - \epsilon_2] + \cdots > 0, \tag{137}$$

we deduce that $w' > 0$. It follows that the derivative of the volatility is positive, which means that the Samuelson's effect holds true.

6 | THE CASE WITH JUMPS

In the present section, we consider the case with jumps only and make the following assumptions:

(a) for every $i = 1, \dots, N$, the demand forecast is perfect and $\sigma^i = 0$;
(b) the set $E = \{g, b\} \subset (0, +\infty)$ consists of two states (g stands for *good* and b for *bad*), with $g < b$;

(c) for every $i = 1, \dots, N$, the Markov chain β^i has state space $E = \{g, b\}$, initial state g at time $t = 0$ and intensity matrix given by

$$\Lambda_i = \begin{pmatrix} -\lambda_i^g & \lambda_i^g \\ \lambda_i^b & -\lambda_i^b \end{pmatrix}, \tag{138}$$

where λ_i^g and λ_i^b are fixed strictly positive real numbers. In other words, each agent has an intensity rate λ_i^g to jump from the good state to the bad state and a λ_i^b intensity rate to jump from the bad state to the good state.

In this framework, for every $i = 1, \dots, N$, the Riccati type system of Equation (16) becomes

$$y'_{i,g}(t) = \frac{1}{\gamma_i} y_{i,g}(t)^2 + \lambda_i^g y_{i,g}(t) - \lambda_i^g y_{i,b}(t), \qquad y_{i,g}(T) = \frac{1}{2} \frac{\eta_i \, g}{\eta_i + g}, \tag{139}$$

$$y'_{i,b}(t) = \frac{1}{\gamma_i} y_{i,b}(t)^2 - \lambda_i^b y_{i,g}(t) + \lambda_i^b y_{i,b}(t), \qquad y_{i,b}(T) = \frac{1}{2} \frac{\eta_i \, b}{\eta_i + b}. \tag{140}$$

We recall that the backward SDE (23), driven by the Markov chain β^i, is such that

$$Y_t^{2,i} = y_{i,\beta_t^i}(t). \tag{141}$$

The case with N agents belonging to two groups

Suppose that there are two groups of agents, so that $N = N_I + N_{II}$, for some $N_I, N_{II} \in \mathbb{N}$, and

$$\begin{aligned} &\mu_i = \mu_I, \qquad \eta_i = \eta_I, \qquad \lambda_i = \lambda_I, \qquad i = 1, \dots, N_I, \\ &\mu_i = \mu_{II}, \qquad \eta_i = \eta_{II}, \qquad \lambda_i = \lambda_{II}, \qquad i = N_I + 1, \dots, N. \end{aligned} \tag{142}$$

Let $y_{I,g}, y_{I,b}$ (resp. $y_{II,g}, y_{II,b}$) denote the solutions to the system of Equations (139)–(140) with coefficients $\eta_I, \lambda_I, \eta_I$ (resp. $\eta_{II}, \lambda_{II}, \eta_{II}$). Moreover, for every $i = 1, \dots, N$, let β^i be the Markov chain with starting point e_0^i at time $t = 0$. Then, for all $t \in [0, T]$, it holds that

$$\begin{aligned} &Y_t^{2,i} = y_{I,\beta_t^i}(t), \qquad a_t^i = \frac{1}{\gamma_I}(T - t) Y_t^{2,i}, \qquad D_t^i = d_0^i + \mu_I \, t, \qquad i = 1, \dots, N_I, \\ &Y_t^{2,i} = y_{II,\beta_t^i}(t), \qquad a_t^i = \frac{1}{\gamma_{II}}(T - t) Y_t^{2,i}, \qquad D_t^i = d_0^i + \mu_{II} \, t, \qquad i = N_I + 1, \dots, N. \end{aligned} \tag{143}$$

Now, let

$$\hat{\boldsymbol{X}}_t = \begin{pmatrix} \hat{X}_t^1 \\ \vdots \\ \hat{X}_t^N \end{pmatrix}, \qquad \boldsymbol{x}_0 = \begin{pmatrix} x_0^1 \\ \vdots \\ x_0^N \end{pmatrix}. \tag{144}$$

In the case with jumps, Equation (75) for $\hat{\boldsymbol{X}}$ still depends on $\hat{\boldsymbol{Y}}^1$ through the following term:

$$\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t = \frac{1}{\frac{N_I}{\gamma_I} + \frac{N-N_I}{\gamma_{II}}} \begin{pmatrix} \frac{1}{\gamma_I} b_t^i \\ \vdots \\ \frac{1}{\gamma_I} b_t^{N_I} \\ \frac{1}{\gamma_{II}} b_t^{N_I+1} \\ \vdots \\ \frac{1}{\gamma_{II}} b_t^{N} \end{pmatrix}. \tag{145}$$

Recalling estimate (58), we see that $\sum_{i=1}^{N} \mathbb{E}[\int_0^T |b_t^i|^2 dt]$ is bounded by a constant, which is independent of N. As a consequence, the quantity $\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t$ can be neglected for N large enough, therefore obtaining the following system of ordinary differential equations with random coefficients (to simplify notation, we still denote by the same symbol, namely $\hat{\boldsymbol{X}}_t$, the solution to such a system):

$$d\hat{\boldsymbol{X}}_t = \frac{1}{2}\mathbf{J}\left(\mathbf{I} - \frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}(\mathbf{1}_{N\times N} - \mathbf{A}_t)\,\mathbf{J}\right)(2\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t)\,dt, \qquad 0 \le t \le T, \tag{146}$$

with $\hat{\boldsymbol{X}}_0 = \boldsymbol{x}_0$, where $\mathbf{I}$ denotes the identity matrix of order N, $\mathbf{1}_{N\times N}$ denotes the $N \times N$ matrix with all entries equal to 1, moreover

$$\bar{\gamma} = \frac{1}{\frac{N_I}{\gamma_I} + \frac{N_{II}}{\gamma_{II}}}, \quad \theta_t = \frac{1}{\gamma_I}\sum_{i=1}^{N_I} a_t^i + \frac{1}{\gamma_{II}}\sum_{i=N_I+1}^{N} a_t^i, \quad \boldsymbol{\Delta}_t = \begin{pmatrix} Y_t^{2,1}(D_t^1 - \hat{X}_t^1) \\ \vdots \\ Y_t^{2,N}(D_t^N - \hat{X}_t^N) \end{pmatrix}, \quad \tilde{\boldsymbol{a}}_t = \begin{pmatrix} \mu_I\gamma_I a_t^1 \\ \vdots \\ \mu_I\gamma_I a_t^{N_I} \\ \mu_{II}\gamma_{II} a_t^{N_I+1} \\ \vdots \\ \mu_{II}\gamma_{II} a_t^{N} \end{pmatrix}. \tag{147}$$

In addition, the $N \times N$ matrices $\mathbf{A}_t$ and $\mathbf{J}$ are defined as

$$\mathbf{A}_t = \begin{pmatrix} a_t^1 & a_t^1 & a_t^1 & \cdots & a_t^1 \\ a_t^2 & a_t^2 & a_t^2 & \cdots & a_t^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_t^N & a_t^N & a_t^N & \cdots & a_t^N \end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix} \frac{1}{\gamma_1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \ddots & 0 & 0 & 0 & \cdots \\ 0 & \cdots & \frac{1}{\gamma_I} & 0 & 0 & \cdots \\ 0 & \cdots & 0 & \frac{1}{\gamma_{II}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{\gamma_{II}} \end{pmatrix}, \tag{148}$$

with $\mathbf{J}$ being a block diagonal matrix, with diagonal blocks given by $\frac{1}{\gamma_I}\mathbf{I}_{N_I}$ and $\frac{1}{\gamma_{II}}\mathbf{I}_{N_{II}}$ (where $\mathbf{I}_{N_I}$ and $\mathbf{I}_{N_{II}}$ denote the identity matrices of order N_I and N_{II}, respectively).

Now, by Equation (74), still neglecting the term $\bar{\gamma}\,\mathbf{J}\,\boldsymbol{b}_t$, we see that the equilibrium price is approximately given by

$$\hat{P}_t \simeq \tilde{P}_t := \frac{\bar{\gamma}}{1-\bar{\gamma}\,\theta_t}\mathbf{1}_N^{\mathsf{T}}\,\mathbf{J}\,(2\,\boldsymbol{\Delta}_t + 2\,\tilde{\boldsymbol{a}}_t), \qquad 0 \le t \le T. \tag{149}$$

Finally, by Equations (47) and (80), we obtain the approximate optimal trading strategies

$$\hat{q}_t^i \simeq \tilde{q}_t^i := \begin{cases} \frac{1}{2\gamma_I}\left(2\,\Delta_t^i + (a_t^i - 1)\hat{P}_t + 2\tilde{a}_t^i\right), & i = 1, \dots, N_I, \\ \frac{1}{2\gamma_{II}}\left(2\,\Delta_t^i + (a_t^i - 1)\hat{P}_t + 2\tilde{a}_t^i\right), & i = N_I + 1, \dots, N, \end{cases} \tag{150}$$

for all $0 \le t \le T$.

Numerical illustration

We consider that the two populations have the following features: first population, $N_I = 100$, $\lambda_I^g = 0.1$, $\lambda_I^b = 0.1$, $\beta_I^g = 6$, $\beta_I^b = 7$, $\mu_I = 0$; second population, $N_{II} = 100$, $\lambda_{II}^g = 0.5$, $\lambda_{II}^b = 0.5$, $\beta_{II}^g = 0$, $\beta_{II}^b = 20$, $\mu_{II} = 0$. Both populations share the same market access cost of $\gamma = 10$ and the same $\eta = 10$.

The idea behind this setting is that the number of market participants in an intraday electricity markets like the German market is around 200 (see Balardy (2018)). We consider that half of them have production with low potentially variable costs (population I) while the second half of the population is made of producers with zero marginal cost but with high potential change in their costs.

Figure 2 shows in (a) the functions $y_{I,g}$, $y_{I,b}$, $y_{II,g}$ and $y_{II,b}$, in (b) the approximated market equilibrium price, and in (c) a measure of the trading activity given by $\frac{1}{N}\sum_i |\tilde{q}_t^i|$.

The functions $y_{i,e}$ captures the possible changes in marginal production costs. Indeed, recall that

$$\tilde{P}_t = \frac{\bar{\gamma}}{1 - \bar{\gamma}\theta_t} \sum_{i=1}^{N} 2y_{i,\beta_t^i}(t)(D_t^i - \hat{X}_t^i). \tag{151}$$

We observe that at initial time, the population II (red curves) enjoys a lower potential marginal cost than population I (blue curves). But, it experiences possible much larger change at terminal time. Between $t = 0$ and $t = 6$, $y_{II,g}$ and $y_{II,b}$ are undistinguishable and thus, any jump that would occur during this period will not trigger any change in the market price equilibrium nor in the trading activity. But, passed $t = 8$, a jump from the good state g to the bad state b will trigger a large change in the potential marginal cost at maturity and thus, will put the players in a situation where they suddenly need to readjust their positions, triggering thus large change in the trading activity and in the price. This mechanism explains the dynamics of the approximated equilibrium price shown on Figure 2b. During the first 60% of the period, the market price is stable, almost constant. Then we observe increasing swings of prices: in the last 5% of the period, the price loses and gains back 2 € in a few minutes. As a comparison, Figure 2d provides real trajectories of intraday prices on EEX for three different hours of delivery, which are extracted from Deschatre and Gruet (2021) with the courtesy of the authors. We clearly observe the increase of the volatility price closer to maturity and the similarity between our model simulation (b) and the behavior of the intraday price (d).

These phenomena of long period of inactivity followed by a period of frantic trading and exponentially increasing trading activity closer to maturity are highly documented in the mathematical finance literature on intraday electricity markets (see the references in the introduction). Thus, our model succeeds in capturing some important features of the observed behavior of the intraday electricity price while providing an explanation based on the fundamentals. More generally, it

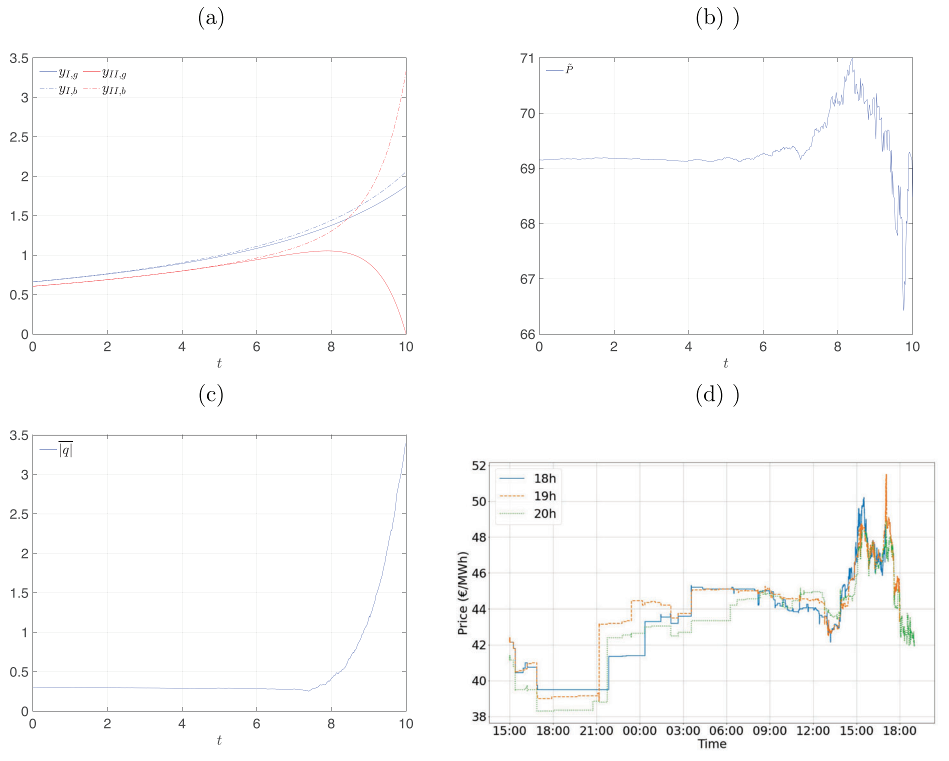

FIGURE 2 (a) The functions $y_{I,g}$, $y_{I,b}$, $y_{II,g}$, and $y_{II,b}$, in (b) the approximated market equilibrium price and in (c) a measure of the trading activity given by $\frac{1}{N}\sum_i |\hat{q}_t^i|$. Parameters value: $N_I = 100$, $\lambda_I^g = 0.1$, $\lambda_I^b = 0.1$, $\beta_I^g = 6$, $\beta_I^b = 7$, $\mu_I = 0$; $N_{II} = 100$, $\lambda_{II}^g = 0.5$, $\lambda_{II}^b = 0.5$, $\beta_{II}^g = 0$, $\beta_{II}^b = 20$, $\mu_{II} = 0$; $\gamma_i = \gamma = 10$, $\eta_i = \eta = 10$, $d_0^i = 20$. (d) Intraday mid-prices on EEX market on August, 30th, 2017 for deliveries at 18, 19, and 20 h up to 1 h before maturity, picture extracted from Deschatre and Gruet (2021) with the permission of the authors [Color figure can be viewed at wileyonlinelibrary.com]

shows that an equilibrium model of pure jumps affecting the production side succeeds in producing the Samuelson's effect. Besides, the reasons of the equilibrium price increasing volatility are consistent with the information flow argument of Samuelson (1965) and Hong (2000). In particular, it matches Hong's explanation of the damping effect of a production shock far from maturity compared to the same shock closer to delivery (see Hong (2000), p. 961).

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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ENDNOTE

[1] The cost of carry c_t is the process defined by $F_t(T) = S_t e^{r_t - c_t}$, where S_t is the spot price, $F_t(T)$ is the futures price for delivery at time T, and r_t is the risk-free rate.

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