Normal Forms for Answer Sets Programming

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submitted 11 July 2001; revised 11 December 2003. 15 March 2004; accepted 30 September 2004

Abstract

Normal forms for logic programs under stable/answer set semantics are introduced. We argue that these forms can simplify the study of program properties, mainly consistency. The first normal form, called the kernel of the program, is useful for studying existence and number of answer sets. A kernel program is composed of the atoms which are undefined in the Well-founded semantics, which are those that directly affect the existence of answer sets. The body of rules is composed of negative literals only. Thus, the kernel form tends to be significantly more compact than other formulations. Also, it is possible to check consistency of kernel programs in terms of colorings of the Extended Dependency Graph program representation which we previously developed. The second normal form is called 3-kernel. A 3-kernel program is composed of the atoms which are undefined in the Well-founded semantics. Rules in 3-kernel programs have at most two conditions, and each rule either belongs to a cycle, or defines a connection between cycles. 3-kernel programs may have positive conditions. The 3-kernel normal form is very useful for the static analysis of program consistency, i.e., the syntactic characterization of existence of answer sets. This result can be obtained thanks to a novel graph-like representation of programs, called Cycle Graph which presented in the companion article (Costantini 2004b).

KEYWORDS: answer set programming, program transformation, normal forms.

1 Introduction

Answer Set Programming is based on the Answer Set semantics (Gelfond and Lifschitz 1988) (Gelfond and Lifschitz 1991), which gives a declarative meaning to negation as failure and establishes a direct connection to Reiter’s Default logic and other relevant non-monotonic reasoning formalism. In Answer Set Programming (from now on, ASP), solutions to a problem are represented by answer sets, and not by variable substitutions produced in response to a query, like in traditional logic programming. The Answer Set semantics deals correctly with cyclic negative def-
initions, by selecting some of the classical models of the theory (in particular, those that are minimal and supported). In contrast, in the Well-founded Semantics (Van Gelder et al. 1990) all atoms involved in negative cycles are deemed undefined.

Our long-term research objectives, to which this article contributes the foundations, are:

1. to study the property of consistency (existence of an answer set) of a program;
2. to develop some practical model theory for ASP, e.g., finding the syntactic features that affect existence and number of answer sets;
3. to investigate the relationship between the restricted syntax of normal forms and the development of fast algorithms for ASP computation.

In this article we propose two normal forms for logic programs. Both normal forms have a uniform restricted syntax, i.e., no facts and few positive conditions. Hence the representation of programs is rather simple.

First, we define the kernel normal form, which forbids positive conditions, facts (and their direct consequences) and undefined atoms, i.e., atoms that can be considered irrelevant from the point of view of program consistency.

The study of program consistency, as well as actual model computation can be made much easier by the 3-kernel normal form, where the length of rule bodies is limited to two literals, and some further simplification of irrelevant atoms and rules is made (at the expense of reintroducing some positive conditions). We define a simple transformation from kernel to 3-kernel programs.

The results presented here build on our earlier results on program representation and analysis under the answer set semantics. First, [Brignoli et al. 1999] proposed an approach where programs are represented by directed graphs, and deduction algorithms are given in terms of graph coloring. The 3-kernel normal form will make the corresponding graphs have a regular and simple structure. Such feature greatly helps when checking consistency, as shown in [Costantini 2004b], which gives the first purely-syntactic and complete characterization of consistent (w.r.t. Answer Sets semantics) logic programs.

We believe that several, alternative normal forms can be introduced and studied to improve the body of technical results and useful transformations for ASP programs. The contributions of this article are, in summary:

1. the definition of the kernel format;
2. a representation theorem for the kernel form;
3. the definition of 3-kernel normal form;
4. a normalization algorithm for 3-kernel form.

2 The kernel Normal Form

We assume the standard definitions of (propositional) general logic program (henceforth, program) and of Well-founded (Van Gelder et al. 1990), stable model (Gelfond and Lifschitz 1988).
Normal Forms for Answer Sets Programming

and answer set semantics (Gelfond and Lifschitz 1991). Whenever we mention consistency (or stability) conditions, or the Gelfond-Lifschitz transformation, we refer to those introduced in (Gelfond and Lifschitz 1988).

Let \( \Pi \) be a logic program, we denote by \( WFS(\Pi) \) the well-founded model of \( \Pi \). \( \Gamma(\Pi, S) \) denotes the application of the Gelfond-Lifschitz operator to \( \Pi \) w.r.t. the set of literals \( S \).

**Definition 1**

A program \( \Pi \) is WFS-irreducible if and only if \( WFS(\Pi) = \langle \emptyset, \emptyset \rangle \).

That is, in WFS-irreducible programs all the atoms have truth value *undefined* under the Well-founded semantics.

The atoms that are relevant for deciding whether answer sets exist and finding them (Costantini 1995) are exactly those that are deemed undefined under the Well-founded semantics. Therefore, we aim for normal forms that are WFS-irreducible. Below is the definition of kernel normal form.

**Definition 2**

A logic program \( \Pi \) is in kernel normal form (or, equivalently, \( \Pi \) is a kernel program) if and only if the following conditions hold.

1. \( \Pi \) is WFS-irreducible;
2. every rule has its body composed of negative literals only;
3. every atom occurring in \( \Pi \) appears in the body of some rule;

It is easy to see that: (a) in kernel programs there are no facts and (b) every atom occurs as the head of some rule and also, by definition, in the body of some rule (possibly the same one). Moreover, one can observe that all atoms are either part of a cyclic definition or are defined by using atoms that are part of a cycle. In other words, all atoms somewhat depend on cycles. This notion is made precise and developed in the work of (Costantini 2004b).

**Example 1**

Consider the problem of 3-colorability of a graph. Given a graph \( G \), we need to show a complete assignment of nodes to one of three colors (in this case green, red and blue) such that no adjacent nodes are assigned to the same color. The program below solves the 3-colorability problem and is in kernel normal form.

Let us name nodes with integers. For each node (here we consider node 0) the program must contain these 3 rules, which impose the assignment of a color to the considered node:

\[
\begin{align*}
\text{color}(0, \text{red}) &\leftarrow \neg \text{color}(0, \text{blue}), \neg \text{color}(0, \text{green}). \\
\text{color}(0, \text{blue}) &\leftarrow \neg \text{color}(0, \text{red}), \neg \text{color}(0, \text{green}). \\
\text{color}(0, \text{green}) &\leftarrow \neg \text{color}(0, \text{blue}), \neg \text{color}(0, \text{red}).
\end{align*}
\]
The next set of rules is used to record the color assignments that have not been chosen:

\[ n_{\text{color}}(0, \text{red}) \leftarrow \neg \text{color}(0, \text{red}). \]
\[ n_{\text{color}}(0, \text{green}) \leftarrow \neg \text{color}(0, \text{green}). \]
\[ n_{\text{color}}(0, \text{blue}) \leftarrow \neg \text{color}(0, \text{blue}). \]

Finally, for each edge of \( G \) we need to add to the program the following set of rules (the sample set below is for an edge between vertex 0 and vertex 1):

\[ \text{edge}_{\text{ok}}(0, 1) \leftarrow \neg \text{edge}_{\text{ko}}(0, 1). \]
\[ \text{edge}_{\text{ok}}(0, 1) \leftarrow \neg \text{edge}_{\text{ko}}(0, 1). \]
\[ \text{edge}_{\text{ko}}(0, 1) \leftarrow \neg n_{\text{color}}(0, \text{red}), \neg n_{\text{color}}(1, \text{red}). \]
\[ \text{edge}_{\text{ko}}(0, 1) \leftarrow \neg n_{\text{color}}(0, \text{green}), \neg n_{\text{color}}(1, \text{green}). \]
\[ \text{edge}_{\text{ko}}(0, 1) \leftarrow \neg n_{\text{color}}(0, \text{blue}), \neg n_{\text{color}}(1, \text{blue}). \]

The first two rules are used to impose the truth of \( \text{edge}_{\text{ok}}(0, 1) \) and, equivalently, the falsity of \( \text{edge}_{\text{ko}}(0, 1) \). It remains easy to show that each and all answer sets of the program defined as in the schema above contain a set of \( \text{color} \) atoms, one for each node, from which a solution to the original problem, i.e., a suitable 3-coloration of the graph, can be read out. Similarly, we can show that if the considered graph admits a 3-coloring then the correspondent kernel program defined using the schema above must admit a corresponding answer set.

We are now ready to state and prove the main technical result, i.e., that kernel is a normal form.

### 3 The proof of normality

The kernel form is normal in the sense that any logic program under answer set semantics admits an equivalent kernel program, i.e., one which has the same answer sets, modulo some projection.

This is a consequence of the following representation Theorem[1] An alternative, elegant proof has been recently suggested by W. Marek and J. Remmel in a personal communication. It consists in showing that their rational reconstruction of Turing machines [Marek and Remmel 2001] can be kernelized, thus showing that kernel programs can encode all problems in the NP class.

In order to state the Theorem, let us recall the following:

**Fact 1**

The answer sets of any logic program form an anti-chain.\(^2\)

---

1 Normally, ASP solvers extend the language of logic programs with abbreviations that allow to specify concisely a constraint on the value of a formula. So, the two rules above are often abbreviated by the formula \( \leftarrow \text{edge}_{\text{ko}}(0, 1) \).

2 A collection of sets is an anti-chain if no component is subset of another.
This fact follows directly from minimality of answer sets. Consistency of the given
program is immaterial here. As point out, logic
programs can be seen as a compact representation of anti-chains. The representation
theorem follows. For simplicity, it is worded for normal programs, i.e., with no
explicit negation (¬).

**Theorem 1**

Let \( H = \{a_1, \ldots, a_n\} \) be a set of atoms and let \( P(H) \) be its powerset. Let \( A \subseteq P(H) \)
be an arbitrary anti-chain over \( H \). There exists a kernel logic program that has
exactly the collection of answer sets \( A \), modulo projection over \( H \).

**Proof**

The proof is by construction.

First, suppose that \( H \) contains neither \( m \) nor \( \bot \); these are special atoms used in
the construction below. Each component, say \( A \in A \) is a set \( \{a_1, \ldots, a_r\} \subseteq H \). Let
\( \{n_1, \ldots, n_s\} = H \setminus A \) be the set of atoms not belonging to \( A \).

Second, we complete \( H \) by adding to it, for each atom \( h_i \in H \) a fresh atom \( \overline{h_i} \).

Now, the kernel \( \pi_A \) is built as follows:

i) for each atom \( h_i \in H \) we put in \( \pi_A \)

\[
\begin{align*}
    h_i & \leftarrow \text{not } \overline{h_i}, \\
    \overline{h_i} & \leftarrow \text{not } h_i.
\end{align*}
\]

ii) for each component, \( A = \{a_1, \ldots, a_r\} \), of \( A \) we put in \( \pi_A \) the following rule

\[
m \leftarrow \text{not } \overline{a_1}, \ldots, \text{not } \overline{a_r}, \text{not } n_1, \ldots, \text{not } n_s.
\]

iii) Finally, we put the consistency axiom:

\[
\bot \leftarrow \text{not } \bot, \text{not } m.
\]

The intuitive reading is: in order for \( \pi_A \) to be consistent, \( m \) must be true, so at
least one of the rules defining \( m \) must have all its conditions true. These conditions
describe exactly one of the answer sets. It is easy to see that, by the anti-chain
property which is enforced by the introduction of the \( \overline{a_i} \) atoms, no two rules for \( m \)
can fire under the same answer set.

In fact, suppose that \( I \) is an answer set for \( \pi_A \) s.t. two rules of \( \pi_A \) fire, say

\[
m \leftarrow \text{not } \overline{a_1}, \ldots, \text{not } \overline{a_r}, \text{not } n_1, \ldots, \text{not } n_s.
\]

\[
m \leftarrow \text{not } \overline{a'_1}, \ldots, \text{not } \overline{a'_r}, \text{not } n'_1, \ldots, \text{not } n'_s.
\]

where we primed the second rule to distinguish its atoms. Now, by repeated application
of Marek and Subrahmanian lemma [Gelfond and Lifschitz 1991], we conclude that \( a_1, \ldots, a_r \in I \) and \( a'_1, \ldots, a'_r \in I \). By the antichain property, \( \{a_1, \ldots, a_r\} \)
and \( \{a'_1, \ldots, a'_t\} \) are disjoint. Thus, for each \( a_i \) we have \( a_i \not \in \{a'_1, \ldots, a'_t\} \) and then by construction \( \overline{a_i} \in I \); but this means that condition not \( n_x \) with \( n_x \equiv a_i \) is false. Therefore, the second rule cannot fire. A similar argument applies for an arbitrary \( a'_j \not \in \{a_1, \ldots, a_r\} \).

So, each answer set corresponds to the truth of exactly one body of the definition of \( m \). In turn, each such body corresponds, up to \( \mathcal{H} \) (i.e., in all but the special atom \( m \)), to a component of \( A \), as intended.

It remains easy to check that for any \( A \), the resulting \( \pi_A \) is in kernel format. This concludes the proof.

**Corollary 1**

Any logic program \( \Pi \) admits an equivalent —w.r.t. answer set semantics— program in kernel format.

The theorem above establish that kernel program is a normal form, but it does not suggest a straightforward, efficient computational mechanism for answer set computing. The main reason is that what is taken into account, i.e., anti-chains over \( \mathcal{H} \), may be of cardinality exponential w.r.t. that of \( \mathcal{H} \). However, it is important as far as establishing that kernel equivalents always do exist, and hence that Kernel is a normal form.

### 4 The 3-kernel normal form

In this Section we define a variation of the kernel normal form that we call 3-kernel normal form, where rules have at most two conditions. The more restrictive 3-kernel form has in our view two advantages, i.e., it

1. allows the study of program consistency in a mathematically elegant way and
2. might help to discover efficient algorithms and heuristics for answer set computation.

Also, we hope that it will become a useful program representation in the search for new algorithms/heuristics. We expect the 3-kernel normal form to play for ASP the same rôle that 3SAT plays for propositional logic, i.e., a streamlined formula presentation that is generally used for the input of satisfiability solvers and model checkers. In our related work (Brignoli et al. 1999) (Costantini 2004b), we have argued that the existence of answer sets depends on both the cycles which are present in the program, and their connections. Indeed, the 3-kernel form is defined with the aim of making cyclic definitions and their connections thereof explicit. Before defining the 3-kernel normal form, it is useful to report some definitions from (Costantini 2004a).

**Definition 3**
A set of rules \( C \) is called a cycle if it has the following form:

\[
\begin{align*}
\lambda_1 & \leftarrow \text{not } \lambda_2, \Delta_1 \\
\lambda_2 & \leftarrow \text{not } \lambda_3, \Delta_2 \\
& \ldots \\
\lambda_n & \leftarrow \text{not } \lambda_1, \Delta_n
\end{align*}
\]

where \( \lambda_1, \ldots, \lambda_n \) are distinct atoms. Each \( \Delta_i, i \leq n \), is a possibly empty conjunction \( \delta_{i_1}, \ldots, \delta_{i_h} \) of conditions (positive or negative), where \( i_h \geq 0 \) and for each literal \( \delta_{i_j} \in \Delta_i, i_j \leq i_h, \delta_{i_j} \neq \lambda_i \) and \( \delta_{i_j} \neq \text{not } \lambda_i \). The \( \Delta_i \)'s are called AND handles of the cycle.

We say that \( \Delta_i \) from Definition above is an AND handle for atom \( \lambda_i \), or, equivalently, an AND handle referring to \( \lambda_i \). Cycles of length \( n = 1 \) correspond to self-loops \( \lambda_1 \leftarrow \text{not } \lambda_1, \Delta_1 \), which are critical to determine consistency. We will say that \( C \) has size \( n \) and it is even (respectively odd) if \( n = 2k, k \geq 1 \) (respectively, \( n = 2k + 1, k \geq 0 \)). The \( \lambda_i \)'s are the atoms involved in cycle \( C \), or, equivalently, the composing atoms of the cycle. Rules belonging to a cycle \( C \) are said to be involved in, or belong to, or form the cycle \( C \).

**Definition 4**

A rule is called an auxiliary rule of cycle \( C \) (or, equivalently, to cycle \( C \)) if it is of this form:

\[
\lambda_i \leftarrow \Delta
\]

where \( \lambda_i \) is involved in cycle \( C \), and \( \Delta \) is a non-empty conjunction \( \delta_{i_1}, \ldots, \delta_{i_h} \) of literals (either positive or negative), where \( i_h \geq 1 \) and for each \( \delta_{i_j}, i_j \leq i_h, \delta_{i_j} \neq \lambda_i \) and \( \delta_{i_j} \neq \text{not } \lambda_i \). \( \Delta \) is called an OR handle of cycle \( C \) (more specifically, an OR handle for \( \lambda_i \) or, equivalently, and OR handle referring to \( \lambda_i \)).

A cycle may have several auxiliary rules. Hence, a cycle may have some AND handles, occurring in one or more of the rules that form the cycle itself, and also some OR handles, occurring in its auxiliary rules. Handles are seen as connections between cycles. In order to make these connections explicit, it is useful that:

- each handle be composed of just one atom, not belonging to the cycle, and
- bridges should be as short as possible, even at the expense of re-introducing some positive literals.

These requirements are fulfilled by the definition of 3-kernel normal form.

**Definition 5**

A logic program \( \Pi \) is in 3-kernel normal form (or, equivalently, \( \Pi \) is a 3-kernel program) if the following conditions hold.

1. \( \Pi \) is WFS-irreducible;
2. every atom occurring in \( \Pi \) is involved in some cycle;
3. each rule of $\Pi$ is either involved in a cycle, or is an auxiliary rule of some cycle, or both;

4. the body of each rule of $\Pi$ which is involved in a cycle consists of either one or two literals;

5. each atom occurring in the AND handle of a cycle is not involved in that cycle;

6. the body of each rule of $\Pi$ which is an auxiliary rule of some cycle consists of exactly one literal.

In fact, as mentioned above all atoms in a kernel program are either part of a cyclic definition or defined (directly or indirectly) using atoms that are part of a cycle. However, allowing handles composed of several atoms means that a handle constitutes a connection between the cycle it refers to, and several other cycles. Also, connections between cycles can be defined indirectly, by means of a chain of dependencies.

In the view of studying program consistency in terms of a Cycle Graph \cite{Costantini2004b}, where vertexes are cycles and edges are connections between cycles, the 3-kernel form guarantees that each handle consists of just one condition, and thus it corresponds exactly to a connection between cycle $C$ and one other cycle $C'$. This leads to a much cleaner representation of the program.

### 4.1 3-kernelization

The aim of this section is to show how to transform a kernel program into a 3-kernel program. Given a kernel program, every rule with non-unit body can be eliminated, by transforming it into a cycle. Consider for instance program $\pi_5$:

\begin{verbatim}
p ← not p.
p ← not a, not c.
a ← not b.
b ← not a.
c ← not d.
d ← not c.
\end{verbatim}

Program $\pi_5$ would be in 3-kernel format but for the second rule. In fact, it is an auxiliary rule of cycle $\{p ← not p\}$, but it does not match the definition of 3-kernel as its body consists of two literals instead of one. This rule is then transformed as follows:
Normal Forms for Answer Sets Programming

\[ p \leftarrow \text{not} \, p. \]
\[ p \leftarrow \text{not} \, h_1. \]
\[ h_1 \leftarrow \text{not} \, h_2. \]
\[ h_2 \leftarrow \text{not} \, h_3, \text{not} \, a. \]
\[ h_3 \leftarrow \text{not} \, h_4. \]
\[ h_4 \leftarrow \text{not} \, h_5, \text{not} \, c. \]
\[ h_5 \leftarrow \text{not} \, p. \]
\[ a \leftarrow \text{not} \, b. \]
\[ b \leftarrow \text{not} \, a. \]
\[ c \leftarrow \text{not} \, d. \]
\[ d \leftarrow \text{not} \, c. \]

The latter program is equivalent to \( \pi_5 \) up to the language of \( \pi_5 \) itself. The long auxiliary rule has been replaced by a new cycle. More generally, the following transformation can be applied.

**Definition 6** (*Long Rules simplification*)

Let \( \Pi \) be a program in kernel normal form. The Long Rule simplification \( \Pi' \) of \( \Pi \) is created as follows. Each rule

\[ \rho: \ h \leftarrow \text{not} \, b_1, \ldots, \text{not} \, b_j \]

occurring in \( \Pi \) that is either

1. auxiliary to a cycle \( C \), with \( j > 1 \), or
2. involved in a cycle \( C \), and \( j > 2 \)

is substituted in \( \Pi' \) by the set of rules (that form a new cycle):

\[ h \leftarrow \text{not} \, h_1. \]
\[ h_1 \leftarrow \text{not} \, h_2. \]
\[ h_2 \leftarrow \text{not} \, h_3, \text{not} \, b_1. \]
\[ h_3 \leftarrow \text{not} \, h_4. \]
\[ h_4 \leftarrow \text{not} \, h_3, \text{not} \, b_2. \]
\[ \ldots \]
\[ h_{2j} \leftarrow \text{not} \, h_{2j+1}, \text{not} \, b_j. \]
\[ h_{2j+1} \leftarrow \text{not} \, h. \]

where \( h_1, \ldots, h_{2j+1} \) are fresh atoms, i.e., do not appear in \( \Pi \).

By discarding the-truth value of the fresh atoms, one can check that the truth conditions for \( h \) remain the same as before.

As far as complexity is concerned, we notice that a long rule has at most \( n = |\mathcal{A}| \) literals in the body, where \( |\mathcal{A}| \) is the number of atoms occurring originally in \( \Pi \). For each condition appearing in the original rule we introduce two new atoms and two new rules. Then we introduce a final rule to close the new cycle. Thus, for each long rule we add at most \( 2n \) atoms and \( 2n + 1 \) rules. In the worst case, i.e., when we have to apply the transformation to all rules of \( \Pi \), we add \( n \cdot (2 \cdot n) \) new atoms and \( n \cdot ((2 \cdot n) + 1) \) new rules.
Since every atom occurring in Π has at least one defining rule, then $A$ is less than or equal to $m = |Π|$, i.e., to the number of rules of Π. Hence, the new program after Long rule simplification has at most $2m^2$ new atoms and $2m^2 + m$ new rules.

4.2 Bridge elimination and other useful equivalences

Even after performing the previous transformation, the program may still not be in 3-kernel form. In fact, it may contain rules that do not belong to any cycle nor are they auxiliary to a cycle. Such rules are said to from bridges, seen as paths connecting cycles. Bridges can be eliminated without affecting the semantics, at the price of dropping some atoms. The truth values of the dropped atoms can be reconstructed at a later stage since it can be proved that the truth value of any single atom of a bridge determines the truth values of the each other atom of the bridge. Elimination of bridges will now be discussed by case analysis. In cases (i) and (ii) we discuss how to eliminate bridges that originate in OR handles (called OR-bridges). In cases (iii) and (iv) we discuss how to eliminate bridges that originate in AND handles (call AND-bridges).

(i) Consider a set of rules of the form:

\[
\begin{align*}
p & \leftarrow \text{not } p. \\
p & \leftarrow \text{not } e. \\
e & \leftarrow \text{not } f. \\
f & \leftarrow \text{not } a. \\
a & \leftarrow \text{not } b. \\
b & \leftarrow \text{not } a.
\end{align*}
\]

that corresponds to a bridge between cycles \{p \leftarrow \text{not } p\} and \{a \leftarrow \text{not } b, b \leftarrow \text{not } a\} via an OR handle. In fact, $p$ depends on $\text{not } e$ (first rule, or first step, of the bridge), $e$ depends on $\text{not } f$ (second step), $f$ depends on $\text{not } a$ (third step). Since the bridge originates in an OR handle, it will be called an OR-bridge. The bridge involves three rules, i.e. three dependencies. Apart from $p$, but it involves two atoms, namely $e$ and $f$. Based on the number of atoms, we say that it is an OR-bridge of even length. To the extent of checking consistency, this set of rules can be substituted by the following set.

\[
\begin{align*}
p & \leftarrow \text{not } p. \\
p & \leftarrow \text{not } a. \\
a & \leftarrow \text{not } b. \\
b & \leftarrow \text{not } a.
\end{align*}
\]

Clearly, the latter program is equivalent to the former up to the language. In the latter one however there is no bridge, rather a direct connection between the two cycles via the OR handle $\text{not } a$. The truth value of atoms $e$ and $f$ can be easily reconstructed from each answer set of the latter program. Moreover, the transformation seen above can be applied whenever an even number of atoms are used to form a bridge.

(ii) Consider a set of rules of the form:
where we can see an OR-bridge of odd-length between cycles \( \{ p \leftarrow \neg p \} \) and \( \{ a \leftarrow \neg b, b \leftarrow \neg a \} \) involving atoms \( e, f, \) and \( g \). To the extent of checking consistency, this set can be substituted by the following:

\[
\begin{align*}
p &\leftarrow \neg p. \\
p &\leftarrow a. \\
a &\leftarrow \neg b. \\
b &\leftarrow \neg a.
\end{align*}
\]

The two set of rules, seen as programs, are equivalent. The price we pay to ridding programs of this type of rules is the introduction of a positive condition. The formal definition of OR-bridge follows.

**Definition 7 (OR-bridge)**
An OR-bridge is a set of rules

\[
\begin{align*}
p &\leftarrow \neg q. \\
p &\leftarrow \neg \lambda_1. \\
\lambda_1 &\leftarrow \neg \lambda_2. \\
\vdots \\
\lambda_n &\leftarrow \neg a.
\end{align*}
\]

where the first rule belongs to some cycle \( C \) where \( p \) and \( q \) are defined, the second rule is auxiliary to \( C \) and \( a \) is defined in some other cycle. An OR-bridge is of even (resp. odd) length, (even OR-bridge for short) if \( n \) is even (resp. odd).

**Definition 8 (even OR-bridge simplification)**
An even OR-bridge can be substituted by the set of rules:

\[
\begin{align*}
p &\leftarrow \neg q. \\
p &\leftarrow \neg a.
\end{align*}
\]

**Definition 9 (odd OR-bridge simplification)**
An odd OR-bridge can be substituted by the set of rules:

\[
\begin{align*}
p &\leftarrow \neg q. \\
p &\leftarrow a.
\end{align*}
\]

(iii) Consider a set of rules of the form:
that corresponds to a bridge between cycles \{p \leftarrow \text{not } p, e \leftarrow \text{not } f, f \leftarrow \text{not } a, a \leftarrow \text{not } b, b \leftarrow \text{not } a.\} via an AND handle. In fact, \( p \) depends on not \( e \) (first rule, or first step, of the bridge), \( e \) depends on not \( f \) (second step), \( f \) depends on not \( a \) (third step). Since the bridge originates in an AND handle, it will be called an OR-bridge. The bridge involves three rules, i.e. three dependencies. Apart from \( p \), it involves two atoms, namely \( e \) and \( f \). Based on the number of atoms, we say it is an AND-bridge of even length. In an AND-bridge, the first dependency that forms the bridge occurs in a rule belonging to a cycle, rather than in an auxiliary rule like for an OR-bridge. To the extent of checking consistency, this set can be substituted by the following set.

\[
\begin{align*}
p & \leftarrow \text{not } p, \text{not } a. \\
a & \leftarrow \text{not } b. \\
b & \leftarrow \text{not } a. 
\end{align*}
\]

Clearly, the transformation is equivalence-preserving up to the language, while the truth value of omitted atoms (namely the intermediate atoms of the bridge: \( e \) and \( f \)) can be deterministically obtained from the previous ones. In the latter program however there is no bridge, rather a direct connection between the two cycles via the AND handle not \( a \).

(iv) Consider a set of rules of the form:

\[
\begin{align*}
p & \leftarrow \text{not } p, e \\
e & \leftarrow \text{not } f. \\
f & \leftarrow \text{not } g. \\
g & \leftarrow \text{not } a. \\
a & \leftarrow \text{not } b. \\
b & \leftarrow \text{not } a. 
\end{align*}
\]

where we can see an AND-bridge of odd length between cycles \{p \leftarrow \text{not } p\} and \{a \leftarrow \text{not } b, b \leftarrow \text{not } a.\}, involving atoms \( e, f, g \). To the extent of checking consistency, this set of rules can be substituted by the following set.

\[
\begin{align*}
p & \leftarrow \text{not } p, a. \\
a & \leftarrow \text{not } b. \\
b & \leftarrow \text{not } a. 
\end{align*}
\]

Let us make intuition about AND bridges formal.

\textit{Definition 10 (AND-bridge)}

An AND-bridge is a set of rules
Normal Forms for Answer Sets Programming

\[
p \leftarrow \neg q, \neg \lambda_1.
\]
\[
\lambda_1 \leftarrow \neg \lambda_2.
\]
\[
\lambda_n \leftarrow \neg a.
\]

where the first rule belongs to cycle \( C \), the second rule is auxiliary to \( C \) and atom \( a \) is defined in another cycle \( C_2 \). An AND-bridge is of \textit{even} (resp. \textit{odd}) \textit{length} (even AND-bridge, for short) if \( n \) is even (resp. odd).

**Definition 11 (even AND-bridge simplification)**

An even AND-bridge can be substituted by the set of rules:

\[
p \leftarrow \neg q, \neg a.
\]

**Definition 12 (odd AND-bridge simplification)**

An odd AND-bridge can be substituted by the set of rules:

\[
p \leftarrow \neg q, a.
\]

The OR and AND bridge simplification can clearly be done in reasonable time, and have the effect of decreasing the size of the program. After applying the above transformations, we have obtained the 3-kernel normal form of the original program \( \Pi \). From the above reasoning, we are lead to the following general conclusions.

**Theorem 2**

For a given program \( \Pi \), its 3-kernel normal form \( 3\ker(\Pi) \) is obtained from its kernel normal form \( \ker(\Pi) \) via the application of (i) long rule simplification and (ii) bridge simplification.

**Theorem 3**

For any given program \( \Pi \), the answer sets of the 3-kernel normal form \( 3\ker(\Pi) \) correspond, up to the language, to those of \( \Pi \).

Therefore, the answer sets of an arbitrary program \( \Pi \) can be obtained by applying 3-kernelization and then expanding each answer set over the language of \( \Pi \). Notice that the bridge simplification reintroduces positive atoms. The 3-kernel form in fact admits positive handles. The different kinds of bridges between cycles that a program may contain after the simplification introduced above, are illustrated by the following example.

**Example 2**

Let \( \pi_6 \) be the 3-kernel program:

\[
a \leftarrow \neg b.
\]
\[
b \leftarrow \neg a.
\]
\[
p \leftarrow \neg p, \neg b.
\]
\[
q \leftarrow \neg q.
\]
\[
q \leftarrow \neg a.
\]
We can distinguish the even cycle $EC_1 = \{a \leftarrow \text{not } b, b \leftarrow \text{not } a\}$ and the odd cycles $OC_1 = \{p \leftarrow \text{not } p\}$ and $OC_2 = \{q \leftarrow \text{not } q\}$. There is an AND-bridge from $EC_1$ and $OC_1$, corresponding to the AND handle $\text{not } b$ of $OC_1$. There is an OR-bridge from $EC_1$ and $OC_2$, corresponding to the OR handle $\text{not } a$ of $OC_2$. In these handles, that form bridges, is the key of the existence of the answer set $\{b, q\}$. In fact, the odd cycles if taken alone would be inconsistent. The even cycle instead has the two answer sets $\{a\}, \{b\}$. By choosing $\{b\}$, the odd cycles are constrained by the handles and become consistent.

5 Discussion

The study of consistency of ASP programs led us to the definition of kernel and 3-kernel normal forms. Together with the introduction of the Extended Dependency Graph (EDG) program representation in [Brignoli et al. 1999], these normal forms, and the formal results presented here, provide a suitable theoretical framework for the study of program properties and the development of an ASP model theory.

It could be noticed that stating 3-colorability, an NP-complete problem, as a kernel program (whose size is polynomial in the size of the considered graph) already amounts to a proof of NP completeness for kernel. However, such equivalence is indirect and not suggestive of the connection existing between a general programs and its kernel counterparts.

Indeed, the kernel normal form may yield advantages in terms of design and implementation of ASP computations. First, regularities are exploited in a more concise program representation. This fact has been exploited, for instance, in the application of neuro-genetic approximation methods to ASP computation in [Bertoni et al., 2001]. Second, existing polynomial-time program simplifications are efficiently integrated in the answer set computation process and this process is led by semantics considerations throughout. Finally, the 3-kernel form, is reminiscent of predicate binarization, a parallelization technique that has been in studied in the context of parallel Prolog program execution, and suggests an approach to the parallel computation of answer sets, an issue that only recently has received attention.

A subject of future research is to investigate whether relevant open problems, such as program equivalence, and safe program composition, can be usefully rephrased in terms of 3-kernel programs.

References


Web location of the most known ASP solvers.

Cmodels: http://www.cs.utexas.edu/users/yuliya/
Aspaps: http://www.cs.uky.edu/ai/aspaps/
DLV: http://www.dbai.tuwien.ac.at/proj/dlv/
NoMoRe: http://www.cs.uni-potsdam.de/~linke/nomore/
Smmodels: http://www.tcs.hut.fi/Software/smodels/